CS1231S Discrete Structures

Tutorial 1: Propositional Logic and Proofs

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AY20/21 Semester 2

Admin Stuff

Safety Measures

- Masks on at all time
- Stay distanced
- Attendance taken

About Me

- Theodore Leebrant
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About This Tutorial

- Safe space - No stupid question policy
  - Questions, mistakes, comments are welcome.
- Expectations
  - **Come to tutorial prepared for discussion!**
  - (Try to) finish the tutorial questions.
  - If still cannot, at least read the questions.
- Workflow
  - We will go through the tutorial questions
  - I'll stay back for any questions if needed
  - Extra questions for practice (not in tutorial sheet) optional

Communication Channels

- LumiNUS announcements
  - Look out for this, profs will send general announcements
- Emails
  - For more official stuff, e.g. submitting MCs if you are absent.
- Telegram
  - (Official Module Group) Mostly clarifications, sometimes announcements.
  - (Tutorial Group) For less official stuff, e.g. asking questions or sharing memes.

Consultations

- PM me on Telegram (preferred) or drop an email
- Either group or 1-to-1 consultations are fine, keep it below 8 people.
- F2F (preferred) or online (through zoom/discord)
- Check for timing, at least 1 day ahead. Most free Wed-Fri.
- Amenable to open big consultations to discuss questions / explain specific concepts especially near exams

Teaching Style

- **Tend to be fast - feel free to interject!**
- Tell me if I'm going too fast or if you don't understand any parts of the explanation.
- You will be writing your answer on the board :D (don't worry, not graded)
- Will not go through extra questions unless requested

Learning Objectives

Learning Objectives

- Understanding the logical connectives $\wedge, \vee, \sim, \rightarrow, \leftrightarrow$. (Q1, Q5, Q6)
- Understanding the terms "necessary condition", "sufficient condition", "only if", "if and only if". (Q1)
- Applying laws of logical equivalences to simplify statements and to prove equivalence. (Q2, Q3)
- Knowing the negation, contrapositive, converse and inverse forms of a conditional statement and their logical relationship. (Q4)
- Using rules of inference. (Q5, Q6)
- Determining whether an argument is valid or invalid. (Q7)
- Solving knights and knaves problems. (Q8)
- Writing simple proofs. (Q9, Q10)

Tutorial Questions

Question 1a

What are the names of these logical connectives:

$\wedge, \vee, \sim, \rightarrow, \leftrightarrow$

Hint: Conditional, Biconditional, Conjunction, Disjunction, Negation

Answer for Question 1a

- $\sim$ is Negation (sometimes $\neg$)
- $\wedge$ is Conjunction
- $\vee$ is Disjunction
- $\rightarrow$ is Conditional
- $\leftrightarrow$ is Biconditional

Question 1b

Given the statement $p \rightarrow q$:

- p is the antecedent / hypotheses
- q is the consequent / conclusion

Question 1b

What is $\sim (p \rightarrow q)$?

Question 1b

What is $\sim(p \to q)$?

$$
\begin{aligned}
&\sim(p \to q) & \\
&\equiv\, \sim(\sim p \lor q) & \text{implication law} \\
&\equiv (\sim(\sim p)) \land (\sim q) & \text{De Morgan's law} \\
&\equiv p \land \sim q & \text{double negative law}
\end{aligned}
$$

Common mistakes: $p \to \sim q, \quad q \to p, \quad p \land \sim q$

Important Note

CS1231 is about proving.

Do not skip any steps - at the very least until midterm.

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge$ **true** $\equiv p$	$p \vee$ **false** $\equiv p$
5	Negation laws	$p \vee {\sim}p \equiv$ **true**	$p \wedge {\sim}p \equiv$ **false**
6	Double negative law	${\sim}({\sim}p) \equiv p$	
7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee$ **true** $\equiv$ **true**	$p \wedge$ **false** $\equiv$ **false**
9	De Morgan's laws	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **true** and **false**	~**true** $\equiv$ **false**	~**false** $\equiv$ **true**
12	Implication law	$p \rightarrow q \equiv {\sim}p \vee q$	

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q:

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \rightarrow q$

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \rightarrow q$
- p is a necessary condition for q:

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \rightarrow q$
- p is a necessary condition for q: $q \rightarrow p$

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \rightarrow q$
- p is a necessary condition for q: $q \rightarrow p$
- p is a necessary and sufficient condition for q:

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \to q$
- p is a necessary condition for q: $q \to p$
- p is a necessary and sufficient condition for q: $p \leftrightarrow q$

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \to q$
- p is a necessary condition for q: $q \to p$
- p is a necessary and sufficient condition for q: $p \leftrightarrow q$
- p if q:

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \to q$
- p is a necessary condition for q: $q \to p$
- p is a necessary and sufficient condition for q: $p \leftrightarrow q$
- p if q: $q \to p$

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \rightarrow q$
- p is a necessary condition for q: $q \rightarrow p$
- p is a necessary and sufficient condition for q: $p \leftrightarrow q$
- p if q: $q \rightarrow p$
- p only if q:

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \rightarrow q$
- p is a necessary condition for q: $q \rightarrow p$
- p is a necessary and sufficient condition for q: $p \leftrightarrow q$
- p if q: $q \rightarrow p$
- p only if q: $p \rightarrow q$

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \to q$
- p is a necessary condition for q: $q \to p$
- p is a necessary and sufficient condition for q: $p \leftrightarrow q$
- p if q: $q \to p$
- p only if q: $p \to q$
- p if and only if q:

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \rightarrow q$
- p is a necessary condition for q: $q \rightarrow p$
- p is a necessary and sufficient condition for q: $p \leftrightarrow q$
- p if q: $q \rightarrow p$
- p only if q: $p \rightarrow q$
- p if and only if q: $p \leftrightarrow q$ or $q \leftrightarrow p$

Question 1c

Rewrite the following using logical connectives:

- p is a sufficient condition for q: $p \to q$
- p is a necessary condition for q: $q \to p$
- p is a necessary and sufficient condition for q: $p \leftrightarrow q$
- p if q: $q \to p$
- p only if q: $p \to q$
- p if and only if q: $p \leftrightarrow q$ or $q \leftrightarrow p$

One of my favourite example: "If I have a fever, I am sick."

Disclaimer

From after this slide onwards, I will use Prof's slides :D

Q2 (a) $\sim a \to \sim(b \vee \sim a)$

$$\sim a \to \sim(b \vee \sim a) \equiv \sim\big(a \to \sim(b \vee \sim a)\big)$$ ✗

$$\sim a \to \sim(b \vee \sim a) \equiv (\sim a) \to \sim(b \vee \sim a)$$ ✓

Steps? (Take down notes here)

Q2 (b) $\sim a \to \sim(b \vee \sim a)$

$\sim a \to \sim(b \vee \sim a)$

$\equiv \sim a \to \left(\sim b \wedge \sim(\sim a)\right)$ by De Morgan's law

$\equiv \sim a \to (\sim b \wedge a)$ by ~~De Morgan's law (step 1)~~ double negative law

$\equiv \sim(\sim a) \vee (\sim b \wedge a)$ by the implication law

$\equiv a \vee (\sim b \wedge a)$ by th~~e implication law (step 2)~~ double negative law

$\equiv a \vee (a \wedge \sim b)$ by the commutative law

$\equiv a$ by the absorption law (step 3)

Q2 (c) $(x \wedge x \vee y) \rightarrow z$

Ambiguous statement

Add parentheses to disambiguate:

$$\big((x \wedge x) \vee y\big) \rightarrow z$$

or

$$\big(x \wedge (x \vee y)\big) \rightarrow z$$

Q2 (d) $(p \wedge q) \rightarrow q$

$(p \wedge q) \rightarrow q$	
$\equiv \sim(p \wedge q) \vee q$	by the implication law
$\equiv (\sim p \vee \sim q) \vee q$	by De Morgan's law
$\equiv \sim p \vee (\sim q \vee q)$	by the associative law
$\equiv \sim p \vee (q \vee \sim q)$	by the commutative law
$\equiv \sim p \vee \mathbf{true}$	by the negation law
$\equiv \mathbf{true}$	by the universal bound law

Q2 (e) $(p \to q) \to r$

$(p \to q) \to r$	
$\equiv (\sim p \vee \ q) \to r$	by the implication law
$\equiv \big(\sim(\sim p \vee \ q)\big) \vee r$	by the implication law
$\equiv (\sim(\sim p) \wedge \sim q) \vee r$	by De Morgan's law
$\equiv (p \wedge \sim q) \vee r$	by the double negative law

Q3 Prove, or disprove:
$(p \to q) \to r$ is logically equivalent to $p \to (q \to r)$.

Definition of conditional statement $p \to q$:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Q3 Prove, or disprove:

$(p \rightarrow q) \rightarrow r$ is logically equivalent to $p \rightarrow (q \rightarrow r)$.

Full truth table:

p	q	r	$p \rightarrow q$	$(p \rightarrow q) \rightarrow r$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$
true	true	true	true	true	true	true
true	true	false	true	false	false	false
true	false	true	false	true	true	true
true	false	false	false	true	true	true
false	true	true	true	true	true	true
false	true	false	true	**false**	false	**true**
false	false	true	true	true	true	true
false	false	false	true	**false**	true	**true**

Prof's note: If you do not know if it is true or false, you may have to draw the truth table to find out:

Q3 Prove, or disprove:

$(p \to q) \to r$ is logically equivalent to $p \to (q \to r)$.

False. Counterexamples:

p = false; q = true; r = false

$$
\begin{aligned}
(p \to q) \to r &\\
&\equiv (F \to T) \to F\\
&\equiv T \to F \equiv F
\end{aligned}
$$

$$
\begin{aligned}
p \to (q \to r) &\\
&\equiv F \to (T \to F)\\
&\equiv F \to F \equiv T
\end{aligned}
$$

$p = q = r$ = false

$$
\begin{aligned}
(p \to q) \to r &\\
&\equiv (F \to F) \to F\\
&\equiv T \to F \equiv F
\end{aligned}
$$

$$
\begin{aligned}
p \to (q \to r) &\\
&\equiv F \to (F \to F)\\
&\equiv F \to F \equiv T
\end{aligned}
$$

Any of these two counterexamples suffices.

Q4

Given the conditional statement “If $12x - 7 = 29$, then $x = 3$”.

Negation: $12x - 7 = 29$ and $x \neq 3$.

Contrapositive: If $x \neq 3$, then $12x - 7 \neq 29$.

Converse: If $x = 3$, then $12x - 7 = 29$.

Inverse: If $12x - 7 \neq 29$, then $x \neq 3$.

Is the given conditional statement true? Yes.

Is the converse true? Yes.

In general, possible for converse to be true but inverse false? No.

Extra practice: prove that the converse and the inverse are equivalent.

Q5 $((p \to q) \land (q \to r)) \to (p \to r)$

p	q	$\boldsymbol{p \to_a q}$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$\boldsymbol{p \to_b q}$
T	T	T
T	F	F
F	T	T
F	F	F

p	q	$\boldsymbol{p \to_c q}$
T	T	T
T	F	F
F	T	F
F	F	T

A correct definition of conditional statement $p \to q$ would result in a tautology for $((p \to q) \land (q \to r)) \to (p \to r)$.

So, do the 3 versions above result in $((p \to q) \land (q \to r)) \to (p \to r)$ being a tautology if we substitute $\to$ with $\to_a$, $\to_b$, $\to_c$?

Q5 $\big((p \to q) \wedge (q \to r)\big) \to (p \to r)$

$(p \to_a q) \wedge (q \to_a r)$	$((p \to_a q) \wedge (q \to_a r)) \to_a (p \to_a r)$
true	true
false	false
false	false
false	false
false	false
false	false
false	false
false	false

p	q	$p \to_a q$
T	T	T
T	F	F
F	T	F
F	F	F

Conclusion:

$\big((p \to_a q) \wedge (q \to_a r)\big) \to_a (p \to_a r)$

is not a tautology.

Notice how the definition of $\to_a$ is actually the definition of $\wedge$

Q5 $((p \to q) \land (q \to r)) \to (p \to r)$

$(p \to_b q) \land (q \to_b r)$	$((p \to_b q) \land (q \to_b r)) \to_b (p \to_b r)$
true	true
false	**false**
false	true
false	**false**
true	true
false	**false**
false	true
false	**false**

p	q	$p \to_b q$
T	T	T
T	F	F
F	T	T
F	F	F

Conclusion:

$((p \to_b q) \land (q \to_b r)) \to_b (p \to_b r)$

is not a tautology.

Notice that $p \to_b q \equiv q$

Q5 $((p \to q) \land (q \to r)) \to (p \to r)$

$(p \to_c q) \land (q \to_c r)$	$((p \to_c q) \land (q \to_c r)) \to_c (p \to_c r)$
true	true
false	true
false	true
false	**false**
false	**false**
false	true
false	true
true	true

p	q	$p \to_c q$
T	T	T
T	F	F
F	T	F
F	F	T

Conclusion:

$$((p \to_c q) \land (q \to_c r)) \to_c (p \to_c r)$$

is not a tautology.

Notice how the definition of $\to_c$ is actually the definition of $\leftrightarrow$

Q6 (a) Prove $(p \leftrightarrow q) \rightarrow (p \rightarrow q)$

Recall that $(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ by definition of $\leftrightarrow$.

$(p \rightarrow q) \wedge (q \rightarrow p)$ (premise)

$\therefore p \rightarrow q$ (valid by specialization)

Q6

(b) Prove that biconditional is transitive:

$$\big((p \leftrightarrow q) \land (q \leftrightarrow r)\big) \rightarrow (p \leftrightarrow r)$$

$(p \leftrightarrow q) \land (q \leftrightarrow r)$

$\equiv (p \rightarrow q) \land (q \rightarrow p) \land (q \rightarrow r) \land (r \rightarrow q)$ by definition of $\leftrightarrow$

$\equiv (p \rightarrow q) \land (q \rightarrow r) \land (r \rightarrow q) \land (q \rightarrow p)$ by associativity/commutativity

$\rightarrow (p \rightarrow r) \land (r \rightarrow p)$ by D5 and transitivity of $\rightarrow$

$\equiv p \leftrightarrow r$ by definition of $\leftrightarrow$

D5: $\big((a \rightarrow x) \land (b \rightarrow y)\big) \rightarrow \big((a \land b) \rightarrow (x \land y)\big)$.

Q6

(c) In Q5, the third alternative definition $\rightarrow_c$ has been shown to be not transitive. Note that $\rightarrow_c$ is equivalent to $\leftrightarrow$. How do you reconcile the result in part (b) here and the result in Q5?

In Q5, we check whether $\rightarrow_c$ is transitive by using the alternative definition.

In part (b) here, we use the correct definition of $\rightarrow$ to check whether $\leftrightarrow$ is transitive.

Note that Q5(c) is actually showing that $\big((p \leftrightarrow q) \wedge (q \leftrightarrow r)\big) \leftrightarrow (p \leftrightarrow r)$ is not a tautology

Q7

(a) Given the following argument:

$$p \vee (q \wedge r)$$
$$\sim p$$
$$\therefore q \wedge r$$

Without actually drawing the truth table, determine the values of p, q and r in the critical row(s) of the truth table.

$$p \equiv false$$
$$q \equiv true$$
$$r \equiv true$$

$$q \wedge r \equiv true$$

Is the argument valid?

Yes

Q7

(b) Give a counterexample to show that the following argument is invalid:

$$p \vee (q \wedge r)$$
$$\sim(p \wedge q)$$
$$\therefore r$$

$$p \equiv true$$
$$q \equiv false$$
$$r \equiv false$$

Q7

(c) Determine whether the following argument is valid or invalid.

If I go to the beach, I will take my shades or my sunscreen.

I am taking my shades but not my sunscreen.

Therefore, I will go to the beach.

Let p = "I go to the beach";
q = "I take my shades";
r = "I take my sunscreen".

$p \to q \vee r$
$q \wedge \sim r$
$\therefore p$

Critical row: $p \equiv false$; $q \equiv true$; $r \equiv false$.

Conclusion: false.

Argument is **invalid**.

Q7 (c)

$p \to q \wedge r$

$q \wedge {\sim}r$

$\therefore p$

p	q	r	${\sim}r$	$q \vee r$	$p \to (q \vee r)$	$q \wedge {\sim}r$	p
T	T	T	F	T	T	F	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	F	T
T	F	F	T	F	F	F	T
F	T	T	F	T	T	F	F
F	T	F	T	T	T	T	F
F	F	T	F	T	T	F	F
F	F	F	T	F	T	F	F

Q7 (d) Determine whether the following argument is valid or invalid.

I will buy a new goat or a used Yugo.

If I buy both a new goat and a used Yugo, I will need a loan.

I bought a used Yugo but I don't need a loan.

Therefore, I didn't buy a new goat.

Let p = "I buy a new goat";
q = "I buy a used Yugo";
r = "I need a loan".

$p \vee q$
$(p \wedge q) \rightarrow r$
$q \wedge \sim r$
$\therefore \sim p$

Critical row: $p \equiv false;\ q \equiv true;\ r \equiv false.$

Conclusion: true.

Argument is **valid**.

Q7 (d)

$p \vee q$
$(p \wedge q) \rightarrow r$
$q \wedge \sim r$
$\therefore \sim p$

p	q	r	$\sim r$	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow r$	$q \wedge \sim r$	$\sim p$
T	T	T	F	T	T	T	F	F
T	T	F	T	T	T	F	T	F
T	F	T	F	F	T	T	F	F
T	F	F	T	F	T	T	F	F
F	T	T	F	F	T	T	F	T
F	T	F	T	F	T	T	T	T
F	F	T	F	F	F	T	F	T
F	F	F	T	F	F	T	F	T

Q8

(a) Two natives *A* and *B*:
A says: Both of us are knights.
B says: *A* is a knave.

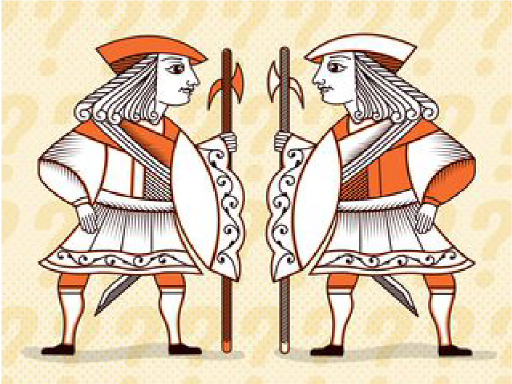
Tempting to say "Contradiction" right after 1.4. However, this is not valid because contradiction requires $p \wedge \sim p$, but 'knave' is not the negation of 'knight'.
Hence 1.5 is required before we arrive at the contradiction in 1.6.

Proof (by contradiction):

1. If *A* is a knight, then:
   - 1.1 What *A* says is true. (by definition of knight)
   - 1.2 ∴ *B* is a knight too. (that's what *A* says)
   - 1.3 ∴ What *B* says is true. (by definition of knight)
   - 1.4 ∴ *A* is a knave. (that's what *B* says)
   - 1.5 ∴ *A* is not a knight. (since *A* is either a knight or a knave, but not both)
   - 1.6 ∴ Contradiction to 1.
2. ∴ *A* is not a knight.
3. ∴ *A* is a knave. (since *A* is either a knight or a knave, but not both)
4. ∴ What *B* says is true.
5. ∴ *B* cannot be a knave. (as *B* has said something true)
6. ∴ *B* is a knight. (as there are only knights and knaves)
7. Conclusion: *A* is a knave and *B* is a knight.

Q8

(b) Two natives *C* and *D*:
C says: Both of us are knaves.
D says nothing.
What are *C* and *D*?



Proof:

1. If *C* is a knight, then:
   - 1.1 What *C* says is true. (by definition of knight)
   - 1.2 ∴ *C* is a knave. (that's what *C* says)
   - 1.3 This contradicts that *C* is a knight.
2. Therefore, *C* is not a knight, then:
   - 2.1 *C* is a knave. (one is a knight or a knave)
   - 2.2 ∴ what *C* says is false. (by definition of knave)
   - 2.3 ∴ not both of *C* and *D* are knaves. (*C* says both are knaves)
   - 2.4 So *D* cannot be a knave. (otherwise, both of them are knaves)
   - 2.5 ∴ *D* is a knight. (one is a knight or a knave)
3. Therefore, *C* is a knave and *D* is a knight.

Q9 **Prove:**

The product of any two odd integers is an odd integer.

Definitions of even and odd integers are given in lecture (lect #1 slide 27)

Give justification

Direct proof:

1. Take any two odd integers n, m.
2. Then $n = 2k + 1$ and $m = 2p + 1$, for $k, p \in \mathbb{Z}$ (by definition of an odd integer)
3. Hence $nm = (2k + 1)(2p + 1)$
$$= \big(2k(2p + 1)\big) + (2p + 1)$$
$$= (4kp + 2k) + (2p + 1)$$
$$= 2(2kp + k + p) + 1 \text{ (by basic algebra)}$$
4. Let $q = 2kp + k + p$ which is an integer (by **closure** of integers under $+$ and $\times$).
5. Then $nm = 2q + 1$ which is odd (by definition of an odd integer)
6. Therefore, the product of any two odd integers is an odd integer.

Q10 Let n be a integer. Then n^2 is odd if and only if n is odd.

(a) Smart's attempt:

Missing justifications.

Proof (by contradiction) ?

1. Suppose n is an even integer.
2. Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k$. (by definition of an even integer)
3. Squaring both sides, we get $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
4. Since k is an integer, so is $2k^2$. (by closure of integers under $\times$)
5. Hence $n^2 = 2p$, with $p = 2k^2 \in \mathbb{Z}$.
6. Therefore, n^2 is even. (by definition of an even integer)
7. So, if n is even, then n^2 is even, which is the same as saying, if n^2 is odd, then n is odd.
8. Therefore, n^2 is odd if and only if n is odd.

Prove only one direction.

Note: I will show how to prove by contradiction and contrapositive.
Additional proof by exhaustion is in extra questions.

Q10 Let n be a integer. Then n^2 is odd if and only if n is odd.

(b) Write your own proof.

Proof:

1. (⇒) Proving the contraposition of "if n^2 is odd, then n is odd".
   1.1. Suppose n is even.
   1.2. Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k$. (by definition of an even integer)
   1.3. Squaring both sides, we get $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. (by basic algebra)
   1.4. Hence $n^2 = 2p$, with $p = 2k^2 \in \mathbb{Z}$. (by closure of integers under $\times$)
   1.5. Therefore, , n^2 is even. (by definition of an even integer)
   1.6. This proves that if n^2 is odd, then n is odd.
2. (⇐) If n is odd, then $n \times n = n^2$ is odd. (by question 9)
3. Therefore, n^2 is odd if and only if n is odd. (by lines 1.6 and 2)

Extra Questions

Question 1a

(T/F) Claim: Assuming $a \in \mathbb{R}$, the negation of $(1 < a < 5)$ is $(1 \geq a \geq 5)$.

Question 1a

(T/F) Claim: Assuming $a \in \mathbb{R}$, the negation of $(1 < a < 5)$ is $(1 \geq a \geq 5)$.

$$\begin{gathered} \sim (1 < a < 5) \\ \sim ((1 < a) \wedge (a < 5)) \\ (\sim (1 < a)) \vee (\sim (a < 5)) \\ (1 \geq a) \vee (a \geq 5) \end{gathered}$$

But $(1 \geq a \geq 5)$ means $(1 \geq a) \wedge (a \geq 5) \not\equiv (1 \geq a) \vee (a \geq 5)$.
Hence, (a) is **false**.

Question 1b

(T/F) Claim: The two statements are logically equivalent:

1. “he’s welcome to come along only if he behaves himself”
2. “if he behaves himself then he’s welcome to come along”

Question 1b

(T/F) Claim: The two statements are logically equivalent:

1. "he's welcome to come along only if he behaves himself"
2. "if he behaves himself then he's welcome to come along"

Let p be "he's welcome to come along" and q be "he behaves himself". Recall:

- p only if q means $p \rightarrow q$
- if p then q means $p \rightarrow q$

So Statement 1 $\equiv p \rightarrow q$
but Statement 2 $\equiv q \rightarrow p$.
Hence, (b) is **false**.

Question 2

a) Show that

$$\sim a \wedge (\sim a \rightarrow (a \wedge b))$$

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$$\sim a \wedge (\sim a \rightarrow (a \wedge b))$$

$\equiv\ \sim a \wedge (\sim (\sim a) \vee (a \wedge b))$	by implication law
$\equiv\ \sim a \wedge (a \vee (a \wedge b))$	by double negative law
$\equiv\ \sim a \wedge a$	by absorption law
$\equiv a \wedge \sim a$	by commutative law
$\equiv$ **false**	by negation law

Question 2

b) Show that

$$p \vee \sim q \to q$$

Question 2

b) Show that

$$p \vee \sim q \rightarrow q$$

$$
\begin{aligned}
&\equiv \sim (p \vee \sim q) \vee q && \text{by implication law} \\
&\equiv (\sim p \vee \sim (\sim q)) \vee q && \text{by De Morgan's law} \\
&\equiv (\sim p \vee q) \vee q && \text{by double negative law} \\
&\equiv q \vee (\sim p \vee q) && \text{by commutative law} \\
&\equiv q \vee (q \vee \sim p) && \text{by commutative law} \\
&\equiv q && \text{by absorption law}
\end{aligned}
$$

Question 2

c) Show that

$$\sim (p \vee \sim q) \vee (\sim p \wedge \sim q)$$

Question 2

c) Show that

$$\sim (p \vee \sim q) \vee (\sim p \wedge \sim q)$$

$$
\begin{aligned}
&\equiv (\sim p \vee \sim (\sim q)) \vee (\sim p \wedge \sim q) && \text{by De Morgan's Law} \\
&\equiv (\sim p \vee q) \vee (\sim p \wedge \sim q) && \text{by double negative law} \\
&\equiv \sim p \vee (q \wedge \sim q) && \text{by distributive law} \\
&\equiv \sim p \vee \textbf{true} && \text{by distributive law} \\
&\equiv \sim p && \text{by identity law}
\end{aligned}
$$

Question 2

d) Show that

$$(p \to q) \to r$$

Question 2

d) Show that

$$(p \to q) \to r$$

$$\begin{aligned}
&\equiv (\sim p \vee q) \to r && \text{by implication law} && (1)\\
&\equiv (\sim (\sim p \vee q)) \vee r && \text{by implication law} && (2)\\
&\equiv (\sim (\sim p) \wedge \sim q) \vee r && \text{by De Morgan's law} && (3)\\
&\equiv (p \wedge \sim q) \vee r && \text{by the double negative law} && (4)
\end{aligned}$$

Question 3

"The rule says that to qualify for the draw, SAFRA-DBS credit card holders must 'charge a minimum of S$50 nett to their card during the Qualifying Period', which is 1 July to 30 September 2017."

Let $C =$ "Charge a minimum of S$50 nett"
$P =$ "Charge during the Qualifying Period"
$W =$ "Win 100,000 AirAsia Miles"

(question continued in the following slide)

Question 3a

Write a conditional statement using C, P and W that describes the rule above.

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Write a conditional statement using C, P and W that describes the rule above.

Note that the qualifying conditions are **necessary but not sufficient** conditions. Thus, C and P are necessary for W, which translates to:

$$\text{if } W \text{ then } (C \wedge P)$$
$$W \rightarrow (C \wedge P)$$

Question 3b

Write the converse, inverse, contrapositive and negation forms of the statement in part (a).

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Write the converse, inverse, contrapositive and negation forms of the statement in part (a).

Statement:	$W \to (C \wedge P)$
Converse:	$(C \wedge P) \to W$
Inverse:	$\sim W \to \sim (C \wedge P)$
Contrapositive:	$\sim (C \wedge P) \to \sim W$
Negation:	$\sim (W \to (C \wedge P))$

Question 4a

Is the following argument valid?
(Extra extra practice: distinguish between sound and valid)
Sandra knows Java and Sandra knows C++.
$\therefore$ Sandra knows C++.

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Is the following argument valid?

(Extra extra practice: distinguish between sound and valid)

Sandra knows Java and Sandra knows C++.

$\therefore$ Sandra knows C++.

Let $p =$ "Sandra knows Java"

Let $q =$ "Sandra knows C++"

$$
\begin{aligned}
& p \wedge q && \text{(premise)} \\
& \therefore q && \text{(valid by specialisation)}
\end{aligned}
$$

Question 4b

Is the following argument valid?

If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.

Neither of these two numbers is divisible by 6.

$\therefore$ The product of these two numbers is not divisible by 6.

Question 4b

Is the following argument valid?

If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.

Neither of these two numbers is divisible by 6.

$\therefore$ The product of these two numbers is not divisible by 6.

Let $p =$ "the first number is divisible by 6"

Let $q =$ "the second number is divisible by 6"

Let $r =$ "the product of these two numbers is divisible by 6"

$$\begin{array}{lr} p \vee q \rightarrow r & \text{(premise)} \\ \sim p \wedge \sim q & \text{(premise)} \\ \therefore \sim r & \text{(invalid: inverse error)} \end{array}$$

Question 4c

Is the following argument valid?

If there are as many rational numbers as there are irrational numbers, then the set of all irrational numbers is infinite

The set of all irrational numbers is infinite.

$\therefore$ There are as many rational numbers as there are irrational numbers.

Question 4c

Is the following argument valid?
If there are as many rational numbers as there are irrational numbers, then the set of all irrational numbers is infinite
The set of all irrational numbers is infinite.
$\therefore$ There are as many rational numbers as there are irrational numbers.

Let $p =$ "there are as many rational numbers as there are irrational numbers"
Let $q =$ "the set of all irrational numbers is infinite"

$$\begin{aligned}
&p \to q && \text{(premise)} \\
&q && \text{(premise)} \\
&\therefore p && \text{(invalid: converse error)}
\end{aligned}$$

Question 4d

Is the following argument valid?

If I get a Christmas bonus, I'll buy a stereo.

If I sell my motorcycle, I'll buy a stereo.

$\therefore$ If I get a Christmas bonus or I sell my motorcycle, I'll buy a stereo.

Question 4d

Is the following argument valid?

If I get a Christmas bonus, I'll buy a stereo.

If I sell my motorcycle, I'll buy a stereo.

$\therefore$ If I get a Christmas bonus or I sell my motorcycle, I'll buy a stereo.

Let $p =$ "I get a Christmas bonus"

Let $q =$ "I sell my motorcycle"

Let $r =$ "I'll buy a stereo"

$$
\begin{aligned}
& p \rightarrow r && \text{(premise)} \\
& q \rightarrow r && \text{(premise)} \\
& (p \rightarrow r) \wedge (q \rightarrow r) && \text{(by conjunction)} \\
& (\sim p \vee r) \wedge (\sim q \vee r) && \text{(by implication law)} \\
& (\sim p \wedge \sim q) \vee r && \text{(by distributive law)} \\
& (p \vee q) \rightarrow r && \text{(by implication law)}
\end{aligned}
$$

Question 5

Prove that $\exists x, y, z \in \mathbb{Z}_{>10}$ such that $x^2 + y^2 = z^2$. What is your proof called? What are these values called?

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Prove that $\exists x, y, z \in \mathbb{Z}_{>10}$ such that $x^2 + y^2 = z^2$. What is your proof called? What are these values called?

Proof

1. Let $x = 11, y = 60, z = 61$.
2. Then $x, y, z \in \mathbb{Z}_{>10}$ and
   $x^2 + y^2 = 11^2 + 60^2 = 121 + 3600 = 3721 = 61^2$.
3. Thus $\exists x, y, z \in \mathbb{Z}_{>10}$ such that $x^2 + y^2 = z^2$. $\square$

This is proof by construction. The values are called Pythagorean triples.

Question 6

We are back to the island with knights and knaves.

Two natives C and D speak to you:

- C says: D is a knave.
- D says: C is a knave.

What are C and D?

Question 6 (cont.)

Proof (by exhaustion)

1. If C is a knight:
    1.1 What C says is true. (by definition of knight)
    1.2 $\therefore D$ is a knave. (what C says)
2. If C is not a knight:
    2.1 Then C is a knave. (one is either a knight or a knave)
    2.2 $\therefore$ what C says is false. (by definition of knave)
    2.3 $\therefore D$ is not a knave. (negation of what C says)
    2.4 $\therefore D$ is a knight (one is either a knight or a knave)
3. In both cases, there is one knight and one knave. $\square$