GAME THEORY: Lecture 3

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Mixed Strategies

Strategic randomization

Mixed Strategy

- Let $S_i = \{s_{i1}, s_{i2}, \ldots, s_{ik}\}$ be the set of pure strategies for player i, then a mixed strategy is a probability distribution $P_i = (p_{i1}, p_{i2}, \ldots, p_{ik})$, where p_{ij} is the probability of player i selecting the pure strategy s_{ij}.
- Note that $0 \leq p_{ij} \leq 1$ and $\sum_{j=1}^{k} p_{ij} = 1$.
- **Completely mixed strategy:** If $p_{ij} > 0$ for all j. (If all pure strategies have non-zero probability of being played)
- **Degenerate mixed strategy:** If one $p_{ij} = 1$ (other strategies have 0 probability of being played)

Why do players use mixed strategies?

- If there is no NE in pure strategies, then it may be the only rational choice
- Keep the opponents indifferent between their pure strategies
- To prevent being exploited in strictly competitive games (special case of above)
- If players cannot coordinate or not know how (i.e. coordination games)
- Randomization by authorities (i.e. to ration limited capacity in towns)
- Randomization by nature (evolutionary game theory)

Solving Equilibrium in Mixed Strategies

- **Hint:** Use one player's payoffs to solve for other player's probability allocation in equilibrium

  - ❖Every finite strategic game has a mixed strategy equilibrium
  - ❖Every finite strategic game has a Nash equilibrium, possibly in mixed strategies

Matching Pennies

Player 2

Player 1		Heads	Tails
	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

No NE in pure strategies

Mixed Strategies in Matching Pennies

- Set of (pure) strategies, $S_i = \{ H, T\}$, $i = 1, 2$
- Let mixed strategy set for Player 1, $p_1 = (a , 1\text{-}a)$ Player 2, $p_2 = (b , 1\text{-}b)$
- Player 1 will choose a strategy to maximize "expected payoff".

Expected payoff from playing H against p_2:

$$E(u_1(H)) = b(-1) + (1-b)1 = 1-2b$$

Expected payoff from playing T against p_2:

$$E(u_1(T)) = b(1) + (1-b)(-1) = 2b-1$$

Best Response Function of Player 1

- Player 1 will choose:
  - Pure strategy H ($a=1$) if $1-2b > 2b-1$ $\Rightarrow$ if $b < ½$
  - Pure strategy T ($a=0$) if $1-2b < 2b-1$ $\Rightarrow$ if $b > ½$
  - Mixed strategy $(a,1-a)$ (i.e., indifferent between H & T) if $1-2b = 2b-1$ $\Rightarrow$ if $b = ½$
- Player 1's best response function:

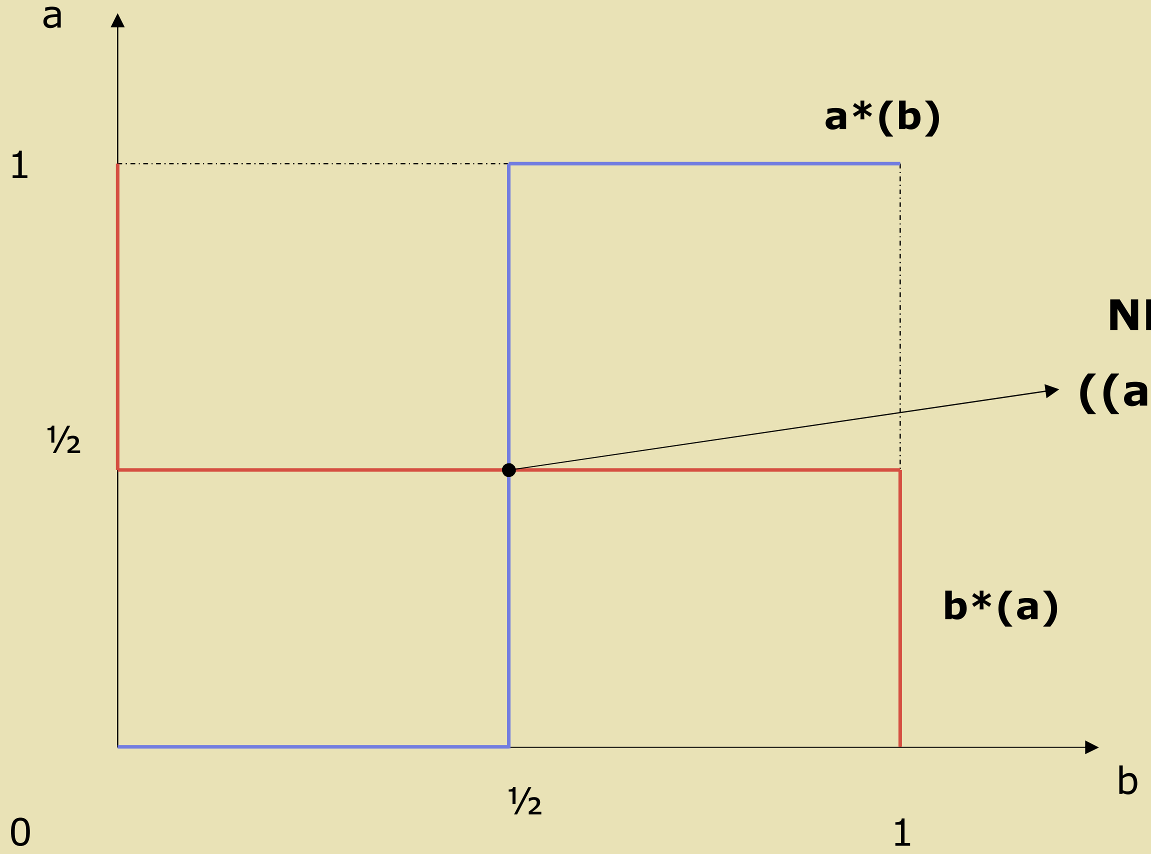
$$\mathbf{a^*}(b) = \begin{cases} a = 1 & \text{if } b < \frac{1}{2} \\ a = 0 & \text{if } b > \frac{1}{2} \\ 0 \le a \le 1 & \text{if } b = \frac{1}{2} \end{cases}$$

Best Response Function of Player 2

- Player 2 will choose:
  - Pure strategy H (b=1) if $a(-1)+(1-a)(1) > a(1)+(1-a)(-1)$ $\Rightarrow$ if $a < ½$
  - Pure strategy T (b = 0) if $a(-1)+(1-a)(1) < a(1)+(1-a)(-1)$ $\Rightarrow$ if $a > ½$
  - Mixed strategy (b, 1-b) if $a(-1)+(1-a)(1) = a(1)+(1-a)(-1)$ $\Rightarrow$ if $a = ½$
- Player 2's best response function:

$$\mathbf{b^*}(a)= \begin{cases} b = 1 & \text{if } a < \tfrac{1}{2} \\ b = 0 & \text{if } a > \tfrac{1}{2} \\ 0 \le b \le 1 & \text{if } a = \tfrac{1}{2} \end{cases}$$

Best Response Functions



NE in mixed strategies:

((a*, 1-a*), (b*, 1-b*)) =

((½, ½), (½, ½))

Battle of the Sexes

Husband

Wife

	Rock & Roll	Movie
Rock & Roll	2, 1	0, 0
Movie	0, 0	1, 2

2 NE: (R&R, R&R) and (Movie, Movie)

Wife’s Expected Payoffs

- Let $p_{Wife} = (a, 1\text{-}a)$ be Wife’s and $p_{Husband} = (b, 1\text{-}b)$ be Husband’s mixed strategy
- Then for Wife

Expected payoff from playing a =1 against husband’s mix:

$$E\,(u_W(1, 0)) = 2b + 0(1\text{-}b) = 2b$$

Expected payoff from playing a =0 against husband’s mix:

$$E\,(u_W(0, 1)) = 0b + 1(1\text{-}b) = 1\text{-}b$$

Wife's Best Response Function

- Wife maximizes expected payoffs, so she
  - will choose $(a = 1)$ if $2b > 1-b$ $\Rightarrow$ if $b > 1/3$
  - will choose $(a = 0)$ if $2b < 1-b$ $\Rightarrow$ if $b < 1/3$
  - Will choose $(0 \leq a \leq 1)$ if $2b = 1-b$ $\Rightarrow$ if $b = 1/3$

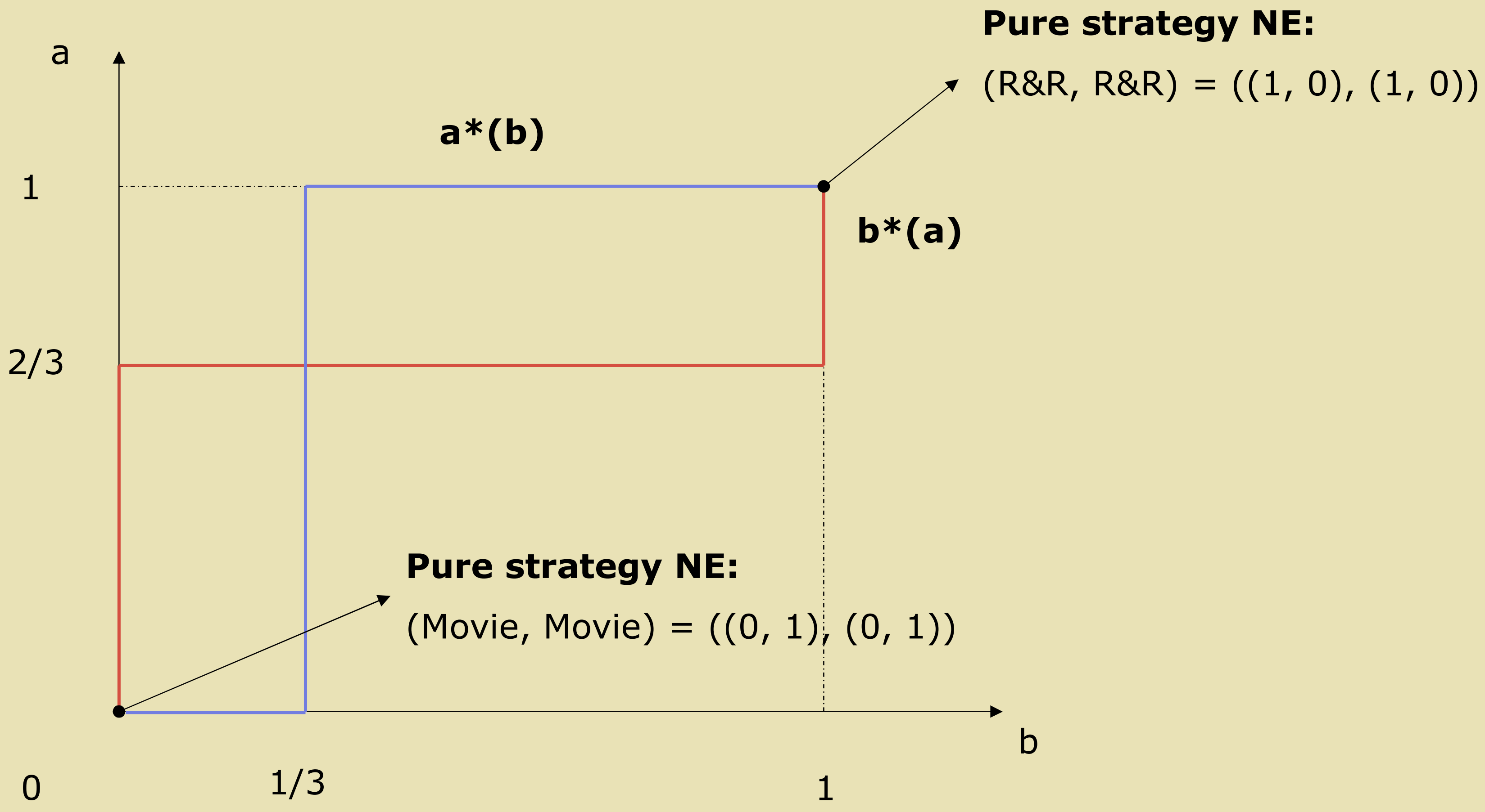
$$\mathbf{a^*}(b) = \begin{cases} a = 1 & \text{if } b > 1/3 \\ a = 0 & \text{if } b < 1/3 \\ 0 \leq a \leq 1 & \text{if } b = 1/3 \end{cases}$$

Husband’s Best Response Function

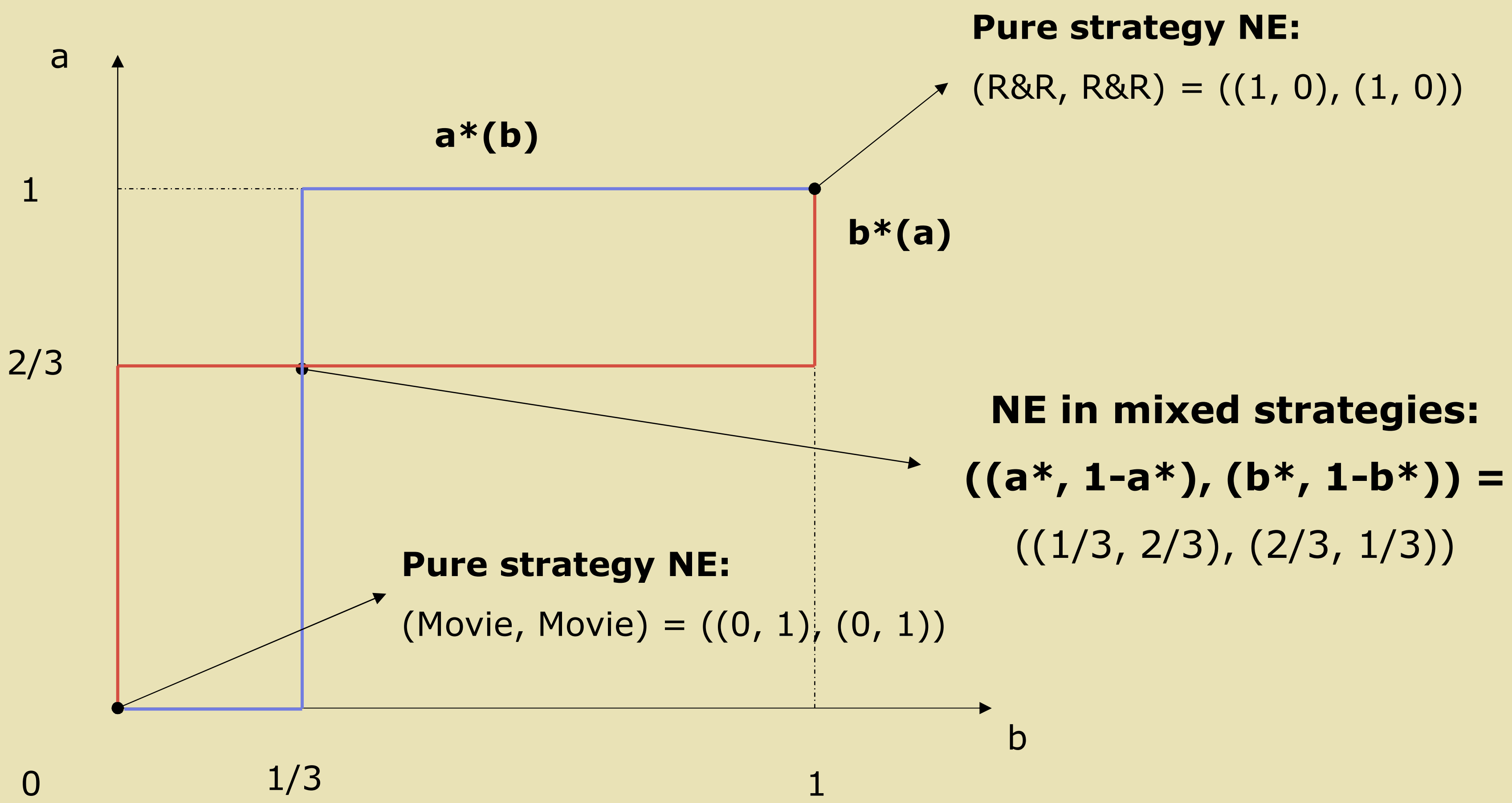
- Using Wife’s probabilities to assess Husband’s expected payoffs yields:

$$\mathbf{b^*}(a) = \begin{cases} b = 1 & \text{if } a > 2/3 \\ b = 0 & \text{if } a < 2/3 \\ 0 \le b \le 1 & \text{if } a = 2/3 \end{cases}$$

Best Response Functions



Best Response Functions



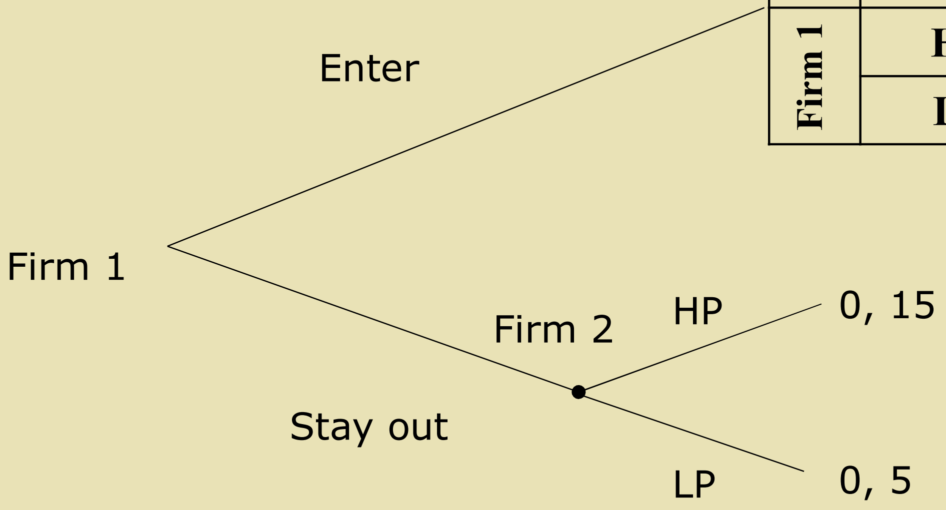
Games with Imperfect Information

Combining sequential and simultaneous move

Complete and Imperfect Information

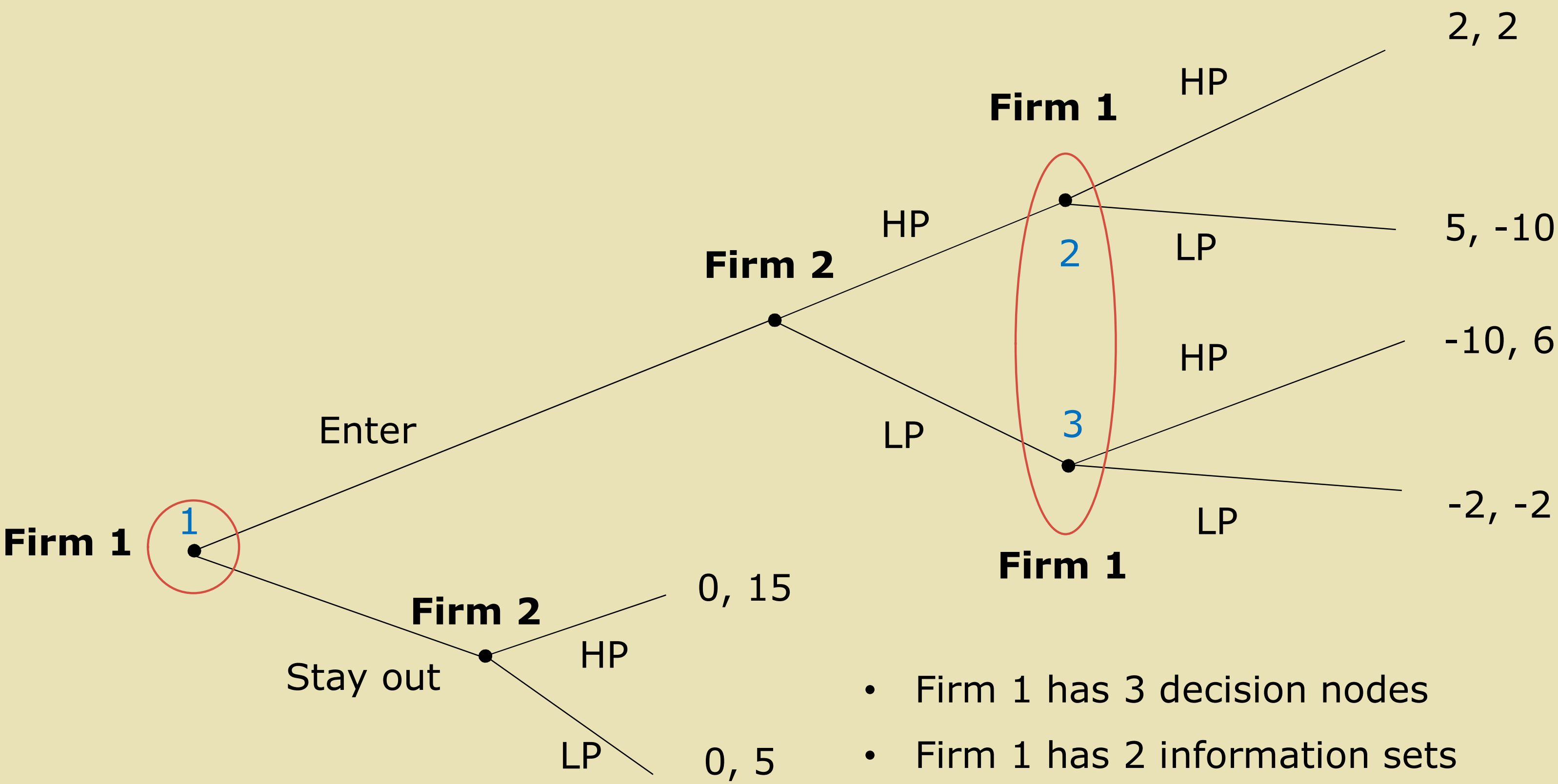
- **Complete information:** Players know each others moves and payoffs (payoff functions are common knowledge)
- **Perfect information:** Players know what other players have chosen
  - Complete and perfect info games: all information sets are singletons (e.g., chain store, Stackelberg duopoly)
  - Complete and imperfect info games: at least one information set is non-singleton (e.g., repeated PD, repeated Price War)

Sequential and Simultaneous moves combined



		Firm 2	
		HP	**LP**
Firm 1	**HP**	2, 2	-10, 5
	LP	5, -10	-2, -2

Sequential and Simultaneous moves combined (alternative display)



- Firm 1 has 3 decision nodes
- Firm 1 has 2 information sets

i1= {1} i2= {2, 3}

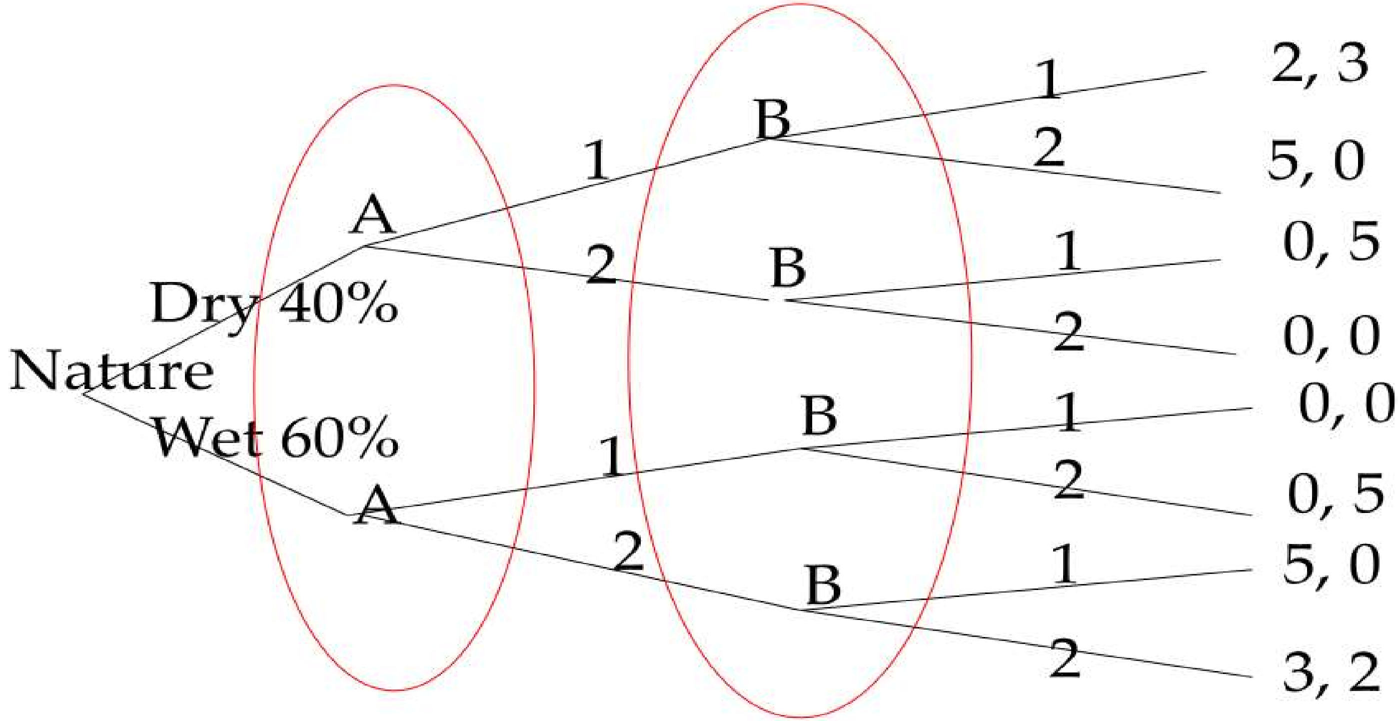
Exogenous Uncertainty

Two farmers decide at the beginning of the season what crop to plant. If the season is dry only type I crop will grow. If the season is wet only type II will grow. Suppose that the probability of a dry season is 40% and 60% for the wet weather. The following table describes the Farmers‘ payoffs.

Dry	Crop 1	Crop 2
Crop 1	2, 3	5, 0
Crop 2	0, 5	0, 0

Wet	Crop 1	Crop 2
Crop 1	0, 0	0, 5
Crop 2	5, 0	3, 2

Nature as a player



Payoff Matrix

- When A and B both choose Crop 1, with a 40% chance (Dry) that A, B will get 2 and 3 each, and a 60% chance (Wet) that A, B will get both 0.

  $E(u_A(\text{Crop I, Crop I}))$: 40%x2+60%x0=0.8

  $E(u_B(\text{Crop I, Crop I}))$: 40%x3+60%x0=1.2

- If they both choose Crop II, with 40% chance A gets 0 and B gets 0, and with 60% chance A gets 3 and B gets 2

  $E(u_A(\text{Crop II, Crop II}))$: 40%x0+60%x3=1.8

  $E(u_B(\text{Crop II, Crop II}))$: 40%x0+60%x2=1.2

	Crop I	**Crop II**
Crop I	0.8, 1.2	2, 3
Crop II	3, 2	1.8, 1.2

2 NE:

(Crop I, Crop II)

and

(Crop II, Crop I)

Static Games of Incomplete Information

Static Bayesian Games

Incomplete Information

- Payoff functions are not common knowledge. At least one player is uncertain about another player’s payoffs.
- **Harsanyi transformation:** Transform games with incomplete information into games with complete but imperfect information. It treats players with different payoffs as distinct types.

Normal form definition

- Set of players, $I = \{1, 2, \ldots, n\}$
- Set of actions, $A= \{A_1, A_2,\ldots, A_n\}$, where $A_i=\{a_{i1}, a_{i2}, \ldots, a_{ik}\}$, $i\in I$
- Set of types, $T=\{T_1, T_2,\ldots,T_n\}$, where $T_i=\{t_{i1}, t_{i2}, \ldots, t_{ih}\}$ and h is number of types for player i
- Set of beliefs, $P = \{P_1, P_2, \ldots, P_n\}$
- Payoff functions, $U=\{u_1, u_2,\ldots, u_n\}$, where $u_i(a_1, a_2, \ldots, a_n; t_{ij})$, $i\in I$, $j=1,\ldots,h$
- Player i's type t_{ij} is privately known by i, but not others. (incomplete information)

Timing of static Bayesian game

- Harsanyi steps:

1. Nature draws a type vector, $t = (t_1, t_2, \ldots, t_n)$
2. Nature reveals t_i to player i, but not to others
3. Players simultaneously choose their actions
4. The payoffs are received

- Steps 1&2 transform the incomplete info game into an imperfect info game. (Players do not observe Nature's move, except own type)

Bayesian Nash Equilibrium

- A strategy profile s*={s1*(t1), s2*(t2), …, sn*(tn)} constitutes a Bayesian NE if for each $i \in I$ and $t_i \in T_i$

$$\mathbf{u_i(s_i{*};t_i)} \geq u_i(s_1{*}(t_1), \ldots, \mathbf{s_i'(t_i)}, \ldots, s_n{*}(t_n), t_i)$$

  - In a finite static Bayesian game (I and T are finite sets), there exists a Bayesian NE, possibly in mixed strategies

BoS Game with Incomplete Information

- I= {Lady, Man}
- A_i = {Bach, Stravinsky}, $i \in I$
- $T_L = \{x\}$, $T_M = \{\text{loving, hating}\}$
- $P_L(\text{loving} \mid x) = P_L(\text{hating} \mid x) = 0.5$ and $P_M(x \mid \text{loving}) = P_M(x \mid \text{hating}) = 1$
- Payoffs are given as follows:

BoS incomplete information payoffs

Loving type (0.5)

		Man	
		Bach	Stravinsky
Lady	Bach	2,1	0,0
	Stravinsky	0,0	1,2

Hating type (0.5)

		Man	
		Bach	Stravinsky
Lady	Bach	2,0	0,2
	Stravinsky	0,1	1,0

BoS Harsanyi steps

1. Nature draws a type vector: either (x,loving) or (x,hating)
2. Nature reveals the man his type but not to the lady
3. Players simultaneously choose Bach or Stravinsky
4. Payoffs are received

- When nature reveals the types individually, how does the expected payoff matrix look like?

Expected Payoffs for BoS

loving	B	S
B	2,1	0,0
S	0,0	1,2

hating	B	S
B	2,0	0,2
S	0,1	1,0

Man: We consider actions by and payoffs to loving and hating types

Lady: We consider all strategies and expected payoffs available to type x

For example: $u_L(B, (B,B); p_L) = 2*0.5 + 2*0.5 = 2$

$u_L(S, (B,S); p_L) = 0*0.5 + 1*0.5 = 0.5$

	(B,B)	(B,S)	(S,B)	(S,S)
B	2, (1,0)	1, (1,2)	1, (0,0)	0, (0,2)
S	0, (0,1)	0.5, (0,0)	0.5, (2,1)	1, (2,0)

Finding BNE in pure strategies

- Mark best responses for the Lady for each strategy pair for man types
- Mark best responses for loving man for each strategy of the lady
- Mark best responses for loving man for each strategy of the lady

	(B,B)	**(B,S)**	**(S,B)**	**(S,S)**
B	$\underline{2}$, ($\underline{1}$,0)	$\underline{1}$,($\underline{1}$,$\underline{2}$)	$\underline{1}$,(0,0)	0,(0,$\underline{2}$)
S	0, (0,$\underline{1}$)	0.5, (0,0)	0.5, ($\underline{2}$,$\underline{1}$)	$\underline{1}$,($\underline{2}$,0)

Bayesian NE in pure strategies: (B, (B,S))

Bayesian NE in mixed strategies

- Derive best response functions for Loving Man, Hating Man, and Lady separately
- Check if there can be an equilibrium in which both types mix (not in this game!)
- Check equilibrium where only Loving mixes
- Check equilibrium where only Hating mixes

- There are 3 equilibria:

1. {(1,0), ((1,0),(0,1))} (pure strategy BNE: (B, (B,S))
2. {(2/3,1/3), ((2/3,1/3),(0,1))}
3. {(1/3,2/3),((0,1),(2/3,1/3))}

Static Bayesian Game Applications

- A new interpretation of mixed strategies
  - A mixed strategy NE in a complete information game can be interpreted as a pure strategy Bayesian NE in a related game with incomplete information
- Private value first-price sealed bid auctions
- Private value second-price sealed bid auctions
- Oligopolistic competition where the firms choose quantities simultaneously, and cost functions are private information
- Job seeker and employer salary negotiations

Dynamic Games of Incomplete Information

Dynamic Bayesian Games

Recall: equilibrium concepts

- NE for static games of complete information
- SPE for dynamic games of complete information

(SPE refines NE: eliminates incredible threats)

- Bayesian NE for static games of incomplete information

+ **Perfect Bayesian NE** for dynamic games of incomplete information

Perfect Bayesian NE

- It consists of a strategy profile $s^*=\{s_1^*, \ldots, s_n^*\}$ and a belief profile $p^* = \{p_1^*, \ldots, p_n^*\}$ (a collection of probability assessments for each information set) such that

1. S* constitutes a NE, given p*
2. At each info set of player i, player's move maximizes u_i, given p_i^* at that info set (sequential rationality)
3. p_i^* can be derived from s* and common prior beliefs
4. p* are consistent with s* and Bayes' rule

Bayes’ Rule

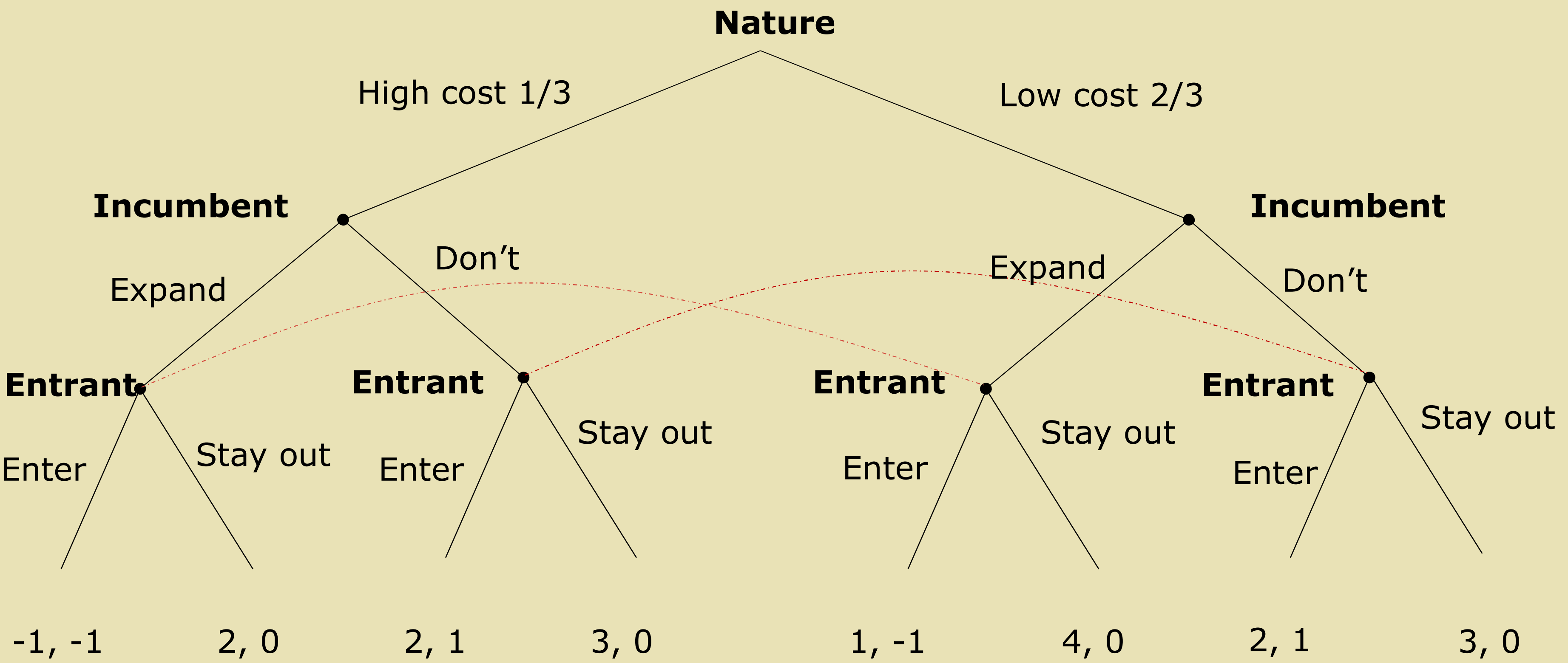
$$P(A|B)=\frac{P(B|A)P(A)}{P(B)}$$

where A and B are events and $P(B) \neq 0$

$P(A|B)$: conditional probability of event A given B is true

$P(A)$ and $P(B)$: a priori (marginal) probabilities of A and B

Entry Deterrence Game



- Incumbent’s strategy profile must specify a move for each of its types.

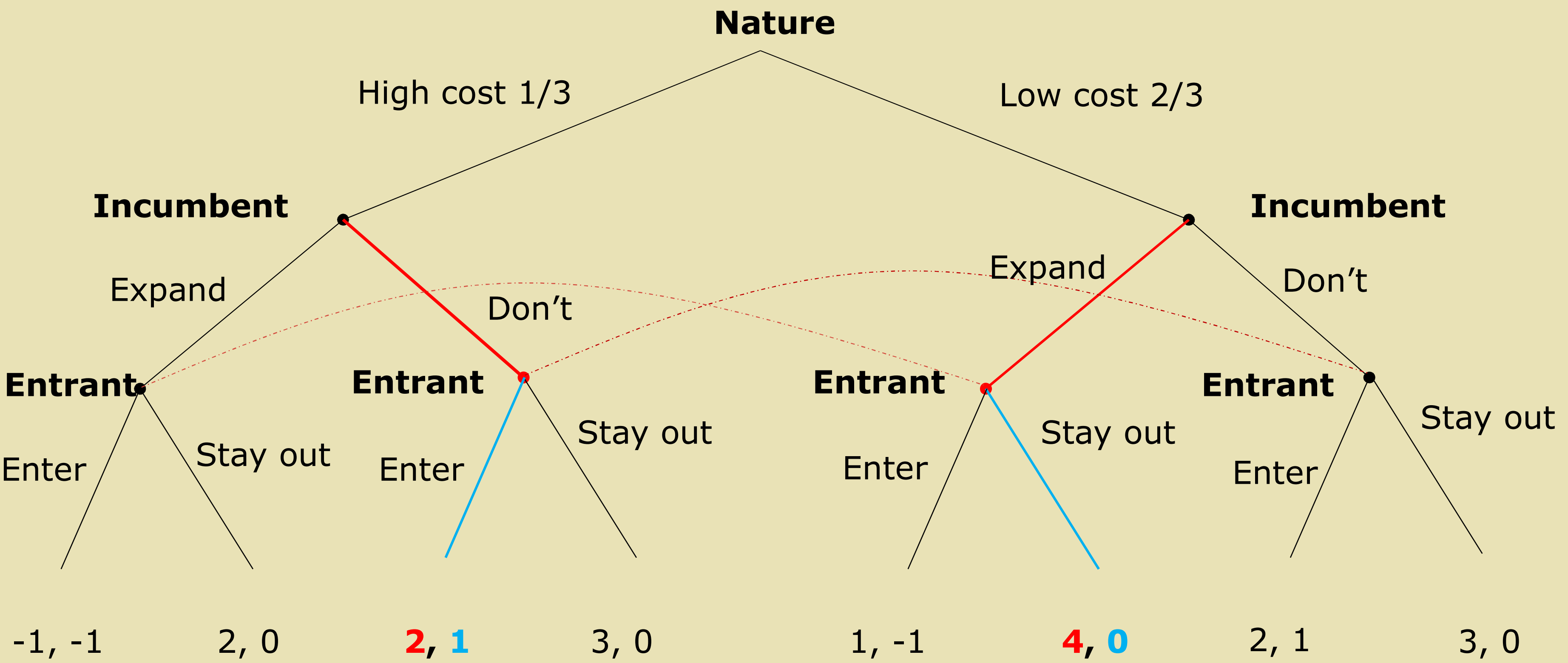
Example: (Expand | High cost, Don’t | Low cost)

- Entrant has 2 info sets, and strategy must specify a move for each info set

Example: (Enter | Expand, Stay out | Don’t)

Expected payoffs for a particular strategy profile:

{(Don't, Expand), (S, E)}



<u>{(Don't, Expand), (S, E)}</u>

$E(u_{entrant}) = 1(1/3) + 0(2/3) = 1/3$

$E(u_{HCincumbent}) = 2$

$E(u_{LCincumbent}) = 4$

Normal Form with conditional expected payoffs

{(Don’t, Expand), (S, E)} payoffs are {(2, 4), 1/3}

		Entrant (if expand, if don’t)			
		(Enter, Enter)	(Enter, Stay out)	(Stay out, Enter)	(Stay out, Stay out)
Incumbent (if HC, if LC)	(Expand, Expand)	(-1, 1), -1	(-1,1), -1/3	(1, 4), 0	(1, 4), 0
	(Expand, Don’t)	(-1, 2), 1/3	(-1, 3), -1/3	(1, 2), 2/3	(1, 3), 0
	(Don’t, Expand)	(2, 1), -1/3	(3, 1), -2/3	(2, 4), 1/3	(3, 4), 0
	(Don’t, Don’t)	(2, 2), 1	(3, 3), 0	(2, 2), 1	(3, 3), 0

2 pure strategy equilibria (candidates for PBNE):

A) {(Don’t, Expand), (Stay out, Enter)}

B) {(Don’t, Don’t), (Enter, Enter)} (implausible because it involves incredible threat)

We must check if belief profiles of the Entrant are consistent with A and B

Note: Entrant has 2 info sets therefore p* must have 2 probability assessments

Checking beliefs using Bayes' rule

Information set 1 (connecting Expand choice of incumbent)

- Suppose q = P(HC | Expand) (hence 1-q = P (LC | Expand)

$$q = \frac{(\frac{1}{3})P(Expand|HC)}{(\frac{1}{3})P(Expand|HC)+(\frac{2}{3})P(Expand|LC)}$$

- Evaluate PBNE candidates

(Don't, Don't): q = $\frac{\frac{1}{3}(0)}{(\frac{1}{3})(0)+(\frac{2}{3})(0)} = \frac{0}{0}$ indeterminate (there are no beliefs consistent with (Don't, Don't)

(Don't, Expand): q = $\frac{\frac{1}{3}(0)}{(\frac{1}{3})(0)+(\frac{2}{3})(1)} = 0$ so q= 0 and 1-q=1 is consistent with (Don't, Expand)

Continue checking beliefs using Bayes' rule

Information set 2 (connecting Don't choice of incumbent)

- Suppose p = P(HC | Don't) (hence 1-p = P(LC | Don't))

$$p = \frac{(\frac{1}{3})P(Don't|HC)}{\left(\frac{1}{3}\right)P(Don't|HC)+\left(\frac{2}{3}\right)P(Don't|LC)}$$

- Evaluate PBNE candidates

(Don't, Don't): p = $\frac{\frac{1}{3}(1)}{\left(\frac{1}{3}\right)(1)+(\frac{2}{3})(0)} = \frac{1}{3}$ but p= 1/3 and 1-p=2/3 is exactly the same with priori probabilities (not PBNE)

(Don't, Expand): p = $\frac{\frac{1}{3}(1)}{\left(\frac{1}{3}\right)(1)+(\frac{2}{3})(0)} = 1$ so p = 1 and 1-p=0 is consistent with (Don't, Expand).

PBNE of Entry Deterrence game

The unique perfect Bayesian Nash equilibrium constitutes

- Strategy profile:

{(Don’t | High cost, Expand | Low cost); (Stay out | Expand, Enter | Don’t}

- Belief profile:

{Expand: (0, 1), Don’t: (1, 0)}

Conclusion: You May Choose

- Which game to play
- With whom to play
- Which strategies are available to each player
- What payoff each outcome will yield

More importantly

- Whether to play or not