GAME THEORY: Lecture 1

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What is Game Theory?

- GT is an analytical tool for social sciences used to model strategic interactions.

  Strategic interaction:
  - When actions of a player influence payoffs to other players and vice versa.
  - Mutual awareness of this cross-effect

- GT models multi-person decision making

GT: science or art?

- GT is the science of rational behavior in interactive situations.
- Good strategists mix the science of GT with their own experience.
- How to use it:
  - Explanation
  - Prediction
  - Advice or prescription

Where can we use GT?

- Any situation that requires us to anticipate our rival's response to our action is a potential context for GT.
  - Games: chess, poker, tennis, soccer, etc.
  - Economics: imperfectly competitive markets; international trade; natural resource extraction; climate change mitigation; pollution abatement, etc.
  - Political science: war/peace (Cuban missile crisis)
  - Law: Designing laws that work (mechanism design)
  - Biology: animal behavior, evolution
  - Information systems: System competition/evolution

Where can we use GT? (cont.)

- Business:
  - Games against rival firms:
    - Pricing, advertising, marketing, auctions, R&D, joint ventures, investment, location, quality, take over etc.
  - Games against other players
    - Employee/employer, managers/stockholders
    - Supplier/buyer, producer/distributor, firm/government

Why is GT important?

- **Facilitates strategic thinking**.
- Provides a standard taxonomy for a scientific approach to analyzing strategic interactions.
- Helps confirm long-held beliefs.
- Provides new insights.

“To be literate in the modern age, you need to have a general understanding of GT.”

P. Samuelson

Some Terminology

- **Strategy:** choices available to players; complete plans of action
- **Payoff:** the number assigned to each possible outcome that reflects preferences
- **Rationality:** Maximizing own payoff
- **Common knowledge:** Mutual awareness of game rules
- **Equilibrium:** each player uses his/her best response to others' equilibrium strategies

Strategic (Normal) Form Games

Static Games of Complete and Imperfect Information

Normal Form Game
$G(I, S_i, u_i)$

1. A set of players (I)

I = {Player 1, Player 2, …, Player n}

2. A set of strategies for all players ($i \in I$)

$\mathbf{S_1, S_2, S_3, \ldots, S_n}$

3. A payoff function for all players ($i \in I$)

$\boldsymbol{u_i(s_1, s_2, \ldots, s_n)}$, for all $\boldsymbol{s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n}$

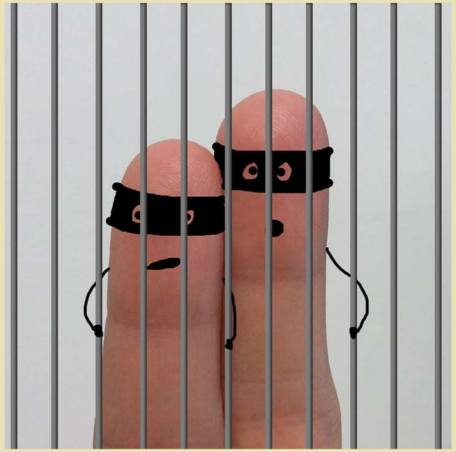
a payoff assigned to every contingency (every possible strategy profile as the outcome of the game)

Assumptions in Static Normal Form Games

- All players are rational.
- Rationality is common knowledge.
- No cooperation
- Players move simultaneously. (They do not know what the other player has chosen).
- Players have complete but imperfect information.
- One-shot game (no future, no history)

Prisoners' Dilemma: The story

- Two suspects are caught and put in different rooms (no communication). They are offered the following deal:
  - If both of you confess, you will both get 5 years in prison (-5 payoff)
  - If one of you confesses whereas the other does not confess, you will get 0 (0 payoff) and 10 (-10 payoff) years in prison respectively.
  - If neither of you confess, you both will get 2 years in prison (-2 payoff)



Prisoners' Dilemma: Normal form definition

1. I = {Prisoner 1, Prisoner 2}

2. S_1 = {Confess, Don't}, S_2 = {Confess, Don't}

3. u_1(Confess, Confess) = -5
   u_1(Confess, Don't) = 0
   u_1(Don't, Confess) = -10
   u_1(Don't, Don't) = -2

Players make their choices without seeing what the other has chosen, and the game ends.

Prisoners' Dilemma: Payoff Matrix

Prisoner 2

Prisoner 1

	Confess	Don't Confess
Confess	-5, -5	0, -10
Don't Confess	-10, 0	-2, -2

Solutions for Static Normal Form Game

1. Dominant strategy equilibrium
2. Elimination of dominated strategies
3. Minimax strategies (for zero-sum games)
4. Nash equilibrium test
   i. Cell-by-cell inspection
   ii. Best response (quick assessment)

1. Strictly Dominant Strategy

- A strategy s_i^* is a strictly dominant strategy of player i if:

  $u_i(s_i^*, s_{-i}) > u_i(s_i', s_{-i})$ for all s_{-i} and s_i'
- A strictly dominant strategy always yields the highest payoff than other strategies, regardless of what the other players do.
  - Rational players will always play their strictly dominant strategies.
  - Dominant strategy equilibrium is also a Nash equilibrium

TL0 Greater or equal if weak dominant strategy.

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Prisoners' Dilemma

Dominant Strategy Equilibrium is (Confess, Confess)

Prisoner 2

Prisoner 1

	Confess	Don't Confess
Confess	-5, -5	0, -10
Don't Confess	-10, 0	-2, -2

Battle of the Bismark Sea: Weak dominance example

Japanese Navy

US Airforce

	North	South
North	2, -2	2, -2
South	1, -1	3, -3

Japanese have a (weakly) dominant strategy: North

2. Elimination of Strictly Dominated Strategies

- A strategy s_i' is a strictly dominated strategy if: $u_i(s_i', s_{-i}) < u_i(s_i, s_{-i})$ for all s_{-i} and for at least one s_i
- Iterated elimination **may** solve the game or reduce the size of the game

  - Rational players will never play their dominated strategies.
  - Nash equilibria survive the iterated elimination of strictly dominated strategies
  - Elimination of weakly dominated strategies may not work!

Prisoners’ Dilemma
Iterated Elimination Procedure yields (Confess, Confess)

Prisoner 2

Prisoner 1

	Confess
Confess	-5, -5

3x3 Game

Using Iterated Elimination

Player 2

Player 1

		Center
Top		1, 3
Bottom		2, 4

3. Minimax Strategies

- Assumes players are pessimistic
- Each player assumes the worst (minimum payoff) for each strategy and chooses the one with the highest (maximum of minima) payoff

  - A minimax equilibrium is also a Nash equilibrium
  - Not all games are minimax or dominance solvable.

American Football

Defense

Offense

	Run	Pass	Blitz	Minimum
Run	2, -2	5, -5	13, -13	2
Short pass	6, -6	5.6, -5.6	10.5, -10.5	**5.6**
Medium pass	6, -6	4.5, -4.5	1, -1	1
Long pass	10, -10	3, -3	-2, 2	-2
Minimum	-10	**-5.6**	-13	

Minimax equilibrium: (Short pass, Pass)

4. Nash Equilibrium Formal definition

The strategy vector $s^* = \{s_1^*, s_2^*, \ldots, s_n^*\}$ is a Nash equilibrium (NE) vector if for all $i \in I$:

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \text{ for all } s_i \in S_i$$

Where

$$s_{-i}^* = \{s_1^*, s_2^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*\}$$

Nash Equilibrium
Alternative definitions

- Nash Equilibrium (NE):
  - In equilibrium, neither player has an incentive to deviate from his/her strategy, given the equilibrium strategies of rival players.
  - Neither player can unilaterally change his/her strategy and increase his/her payoff, given other players’ strategies.
  - In equilibrium, each player’s strategy is the best response to other players’ strategies

Prisoners' Dilemma

Cell-by-cell Inspection

Prisoner 2

Prisoner 1

	Confess	Don't Confess
Confess	-5, -5	0, -10
Don't Confess	-10, 0	-2, -2

Prisoners' Dilemma

Best response

Prisoner 2

Prisoner 1

	Confess	Don't Confess
Confess	-5, -5	0, -10
Don't Confess	-10, 0	-2, -2

NE of Prisoners’ Dilemma

- The strategy profile {confess, confess} is the unique pure strategy NE where each player gets a payoff of –5.
- Paradox: (don’t confess, don’t confess) yields higher payoffs for both.
  - In PD-type games, there is a tension between cooperation and conflict (individual vs. group payoff maximization)

Possible PD-type games

- Trade barriers
- Fish stock
- Hard bargaining
- Climate mitigation
- Traffic rules
- Price competition (see example)
- Investment game (see example)

A Pricing Example

Firm 2

Firm 1

	High Price	Low Price
High Price	100, 100	-10, 140
Low Price	140, -10	0, 0

3 Player International Investment Game Example

- 3 Turkish firms investing in a Turkic Republic.
- A new law is being debated in the Turkic Republic and they all want the law to be favorable for Turkish firms. The president is very powerful. He promises to match the total donation made to a state university in terms of favorable tax cuts for Turkish firms.
- The 3 firms have to decide whether to contribute or not. The more they contribute the more favorable the law. Payoff matrixes are as follows:

International Investment Game

Firm 1 is the row player. Firm 2 is the column player. Firm 3 is the page player.

Firm 3 Donates

	Donate	Don’t
Donate	**5, 5, 5**	3, 6, 3
Don’t	**6, 3, 3**	4, 4, 1

Firm 3 does not Donate

	Donate	Don’t
Donate	**3, 3, 6**	**1, 4, 4**
Don’t	**4, 1, 4**	2, 2, 2

NE is: **(Don’t, Don’t, Don’t)**

Strategic (Normal) Form Games

Examples

A Strictly Competitive Game

Matching Pennies

Player 2

Player 1

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

No NE in pure strategies

No numerical payoffs
Principal Agent type Game

Employer

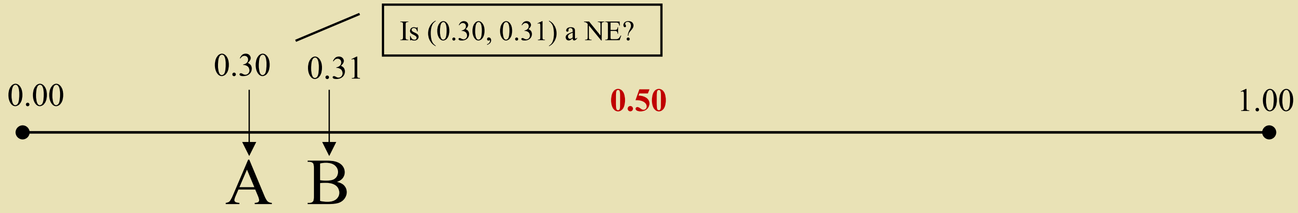
Worker

Assume: v>w>g>h>0	Inspect	Don't Inspect
Shirk	0, -h	w, -w
Work	w-g, v-w-h	w-g, v-w

No NE in pure strategies

No payoff matrix
Candidate positioning game

Assume a linear political spectrum from extreme left to extreme right. There are two candidates: A and B. Suppose candidate positions are captured by a univariate scale ranging from 0 to 1. 101 voters are placed at 0.00, 0.01, 0.02, …, 1.00. They vote for the candidate closest to their position (if candidates are equally distant, the voter flips a coin). The candidate with a majority of votes wins the election. Candidates choose their positions simultaneously. What is the NE?



Rational Pigs?!?

Dominant Pig

Small pig

	Press	Don’t Press
Press	1.5, 3.5	-0.5, 6
Don’t press	5, 0.5	0, 0

Strategic (Normal) Form Games with Multiple equilibria

Coordination games

Pure Coordination Game

Player 2

Player 1

	Left	**Right**
Up	1, 1	0, 0
Down	0, 0	1, 1

NE: (Up, Left) and (Down, Right)

Battle of the Sexes

Husband

		Rock & Roll	Movie
Wife	**Rock & Roll**	2, 1	0, 0
	Movie	0, 0	1, 2

NE: (R&R, R&R) and (Movie, Movie)

Battle of the Sexes:

After 30 Years of Marriage!

Husband

Wife		**Opera**	**Movie**
	Opera	3, 2	0, 0
	Movie	0, 0	1, 2

NE: (Opera, Opera) and (Movie, Movie)
Focal point prediction is (Opera, Opera)

Game of Assurance

USSR

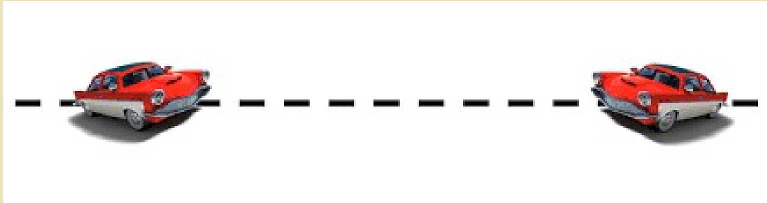
USA

	Refrain	Build
Refrain	4, 4	1, 3
Build	3, 1	2, 2

NE: (Refrain, Refrain) and (Build, Build)

Assurance needed for the efficient equilibrium

Game of Chicken



Dean

		Swerve (Chicken)	Straight (Tough)
James	Swerve	0, 0	-1, 1
	Straight	1, -1	-2, -2

NE: (Straight, Swerve) and (Swerve, Straight)

If both play tough, very bad outcome!

Strategic (Normal) Form Games with Infinite Strategy Sets

Oligopoly models

N player normal form game

- Each player i solves the following maximization problem:

Maximize $u_i(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$

Subject to $s_i \in S_i$

Then for $s^* = (s_1^*, \ldots, s_i^*, \ldots, s_n^*)$ to be a NE, it must solve the system of equations:

$$\frac{\partial u_i(s_1^*, \ldots, s_n^*)}{\partial s_i} = 0, \qquad i = 1, \ldots, n$$

Best Response Function

In an n-player game, the best response function of player i, is the function $R_i(s_{-i})$ that for given actions s_{-i} of other players it assigns an action $s_i = R_i(s_{-i})$ that maximizes player i's payoff, $u_i(s_i, s_{-i})$.

- ❖ If s_i^* is a NE strategy, then $s_i^* = R_i(s_{-i}^*)$.
- ❖ Players' Nash equilibrium strategies are mutual best responses to each other.

Cournot Duopoly

Two firms are competing in quantities. They have identical cost functions, $C_i = cq_i$ and face the inverse demand function $P = a - Q$, where $Q = q_1 + q_2$. What output level will they produce in (Nash) equilibrium?

Game representation:

1. $I = \{\text{Firm 1, Firm 2}\}$
2. $S_i = \{q_i \mid q_i \geq 0\}$, i= 1, 2
3. $u_i(q_i, q_j) = [a - (q_i+q_j)]q_i - cq_i$, i= 1, 2
4. They move simultaneously

Recall: $\pi = TR - TC$ where $TR = p.q$ and $TC = FC + VC$

Cournot Duopoly: Firm i's maximization problem

Maximize $u_i(q_i, q_j) = [a-(q_i+q_j)]q_i - cq_i$

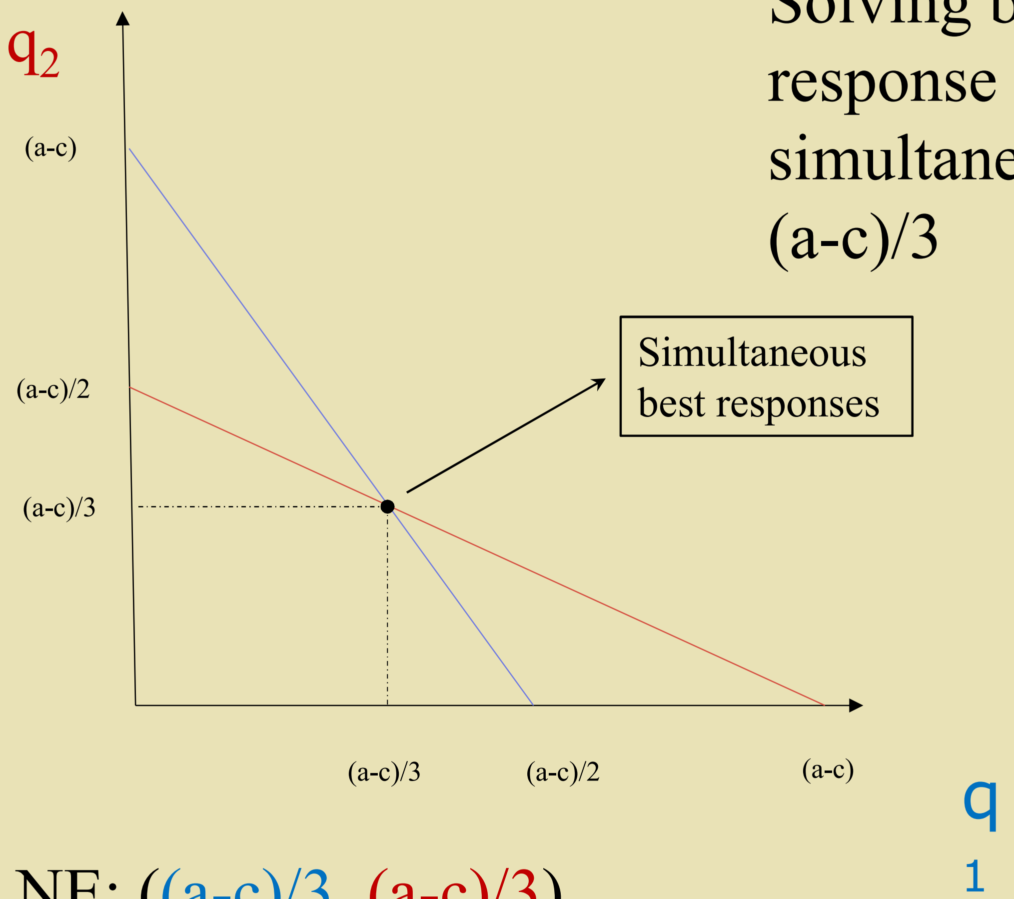
First order necessary condition (FOC):

$$\frac{\partial u_i(q_i,q_j)}{\partial q_i} = a - 2q_i - q_j - c = 0$$

Solving for q_i:

$$q_i = \frac{a-c-q_j}{2} = R_i(q_j), \quad i, j = 1, 2; i \neq j$$

Cournot best response functions



Solving best response functions simultaneously: (a-c)/3

NE: ((a-c)/3, (a-c)/3)

Bertrand Duopoly (with product differentiation)

Two firms compete in prices. They have identical cost functions, $C_i = cq_i$ and each face their own demand function $q_i(p_i, p_j) = a - p_i + bpj$. What price will they charge in (Nash) equilibrium?

Game representation:

1. $I = \{\text{Firm } 1, \text{Firm } 2\}$
2. $S_i = \{p_i \mid p_i \geq 0\}$, i= 1, 2
3. $u_i(p_i, p_j) = p_i[a - p_i + b_j] - c[a - p_i + b_j]$, i= 1, 2

OR $u_i(p_i, p_j) = [a - p_i + b_j](p_i - c)$

4. They move simultaneously

Bertrand Duopoly: Firm i's maximization problem

Maximize $u_i(p_i, p_j) = [a - p_i + b_j]\ (p_i - c)$

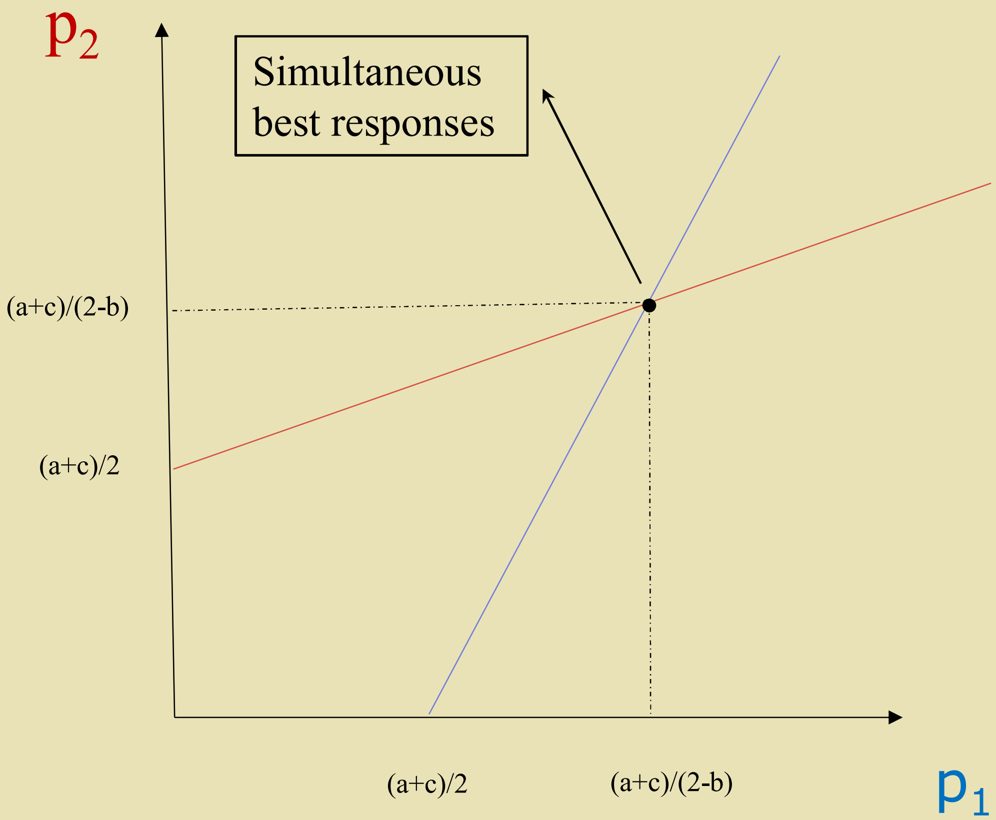
FOC:

$$\frac{\partial u_i(pi,pj)}{\partial p_i} = a - 2pi + bpj + c = 0$$

Solving for p_i:

$$p_i = \frac{a+bpj+c}{2} = \color{red}{R_i(p_j)}, \qquad i, j = 1, 2,\ i \neq j$$

Bertrand best response functions



Solving best response functions simultaneously: (a+c)/(2-b)

NE: ((a+c)/(2-b), (a+c)/(2-b))