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# Math 120A Homework 3

## Problem 1.3.2

Consider a regular curve $q(t)$ with arclength parameter $s$. Show that if $T$ is regular at $t_0$, then

$$1 = \lim_{t \to t_0} \frac{|\theta(t) - \theta(t_0)|}{\arccos(T(t) \cdot T(t_0))}$$

and

$$\kappa(t_0) = \lim_{t \to t_0} \frac{\arccos(T(t) \cdot T(t_0))}{|s(t) - s(t_0)|}.$$

Hint: Use exercise 1 from section 1.2.

*Proof.* Since $T$ is defined to be the unit tangent, $|T| = 1$. So, $\theta(t)$ is the arclength parameter. Then from Problem 1.2.1(e), we have the following:

$$1 \leq \frac{|\theta(t) - \theta(t_0)|}{\arccos(T(t) - T(t_0))} \leq \frac{|\theta(t) - \theta(t_0)|}{|T(t) - T(t_0)|}.$$

Now, we have that $\frac{dT}{dt}(t_0) = \left|\frac{dT}{dt}(t_0)\right| \neq 0$ since we are given that $T$ is regular at $t_0$. Thus, $\left|\frac{dT}{dt}(t_0)\right| >$. Then from Problem 1.2.1(d), we have the following:

$$1 = \lim_{t \to t_0} \frac{|\theta(t) - \theta(t_0)|}{|T(t) - T(t_0)|}.$$

So, we have the following:

$$1 = \lim_{t \to t_0} 1 \leq \lim_{t \to t_0} \frac{|\theta(t) - \theta(t_0)|}{\arccos(T(t) - T(t_0))} \leq \lim_{t \to t_0} \frac{|\theta(t) - \theta(t_0)|}{|T(t) - T(t_0)|} = 1.$$

Therefore, by the squeeze theorem,

$$1 = \lim_{t \to t_0} \frac{|\theta(t) - \theta(t_0)|}{\arccos(T(t) \cdot T(t_0))}.$$

Now, we show $\kappa(t_0)$. From the first part, we have that:

$$1 = \lim_{t \to t_0} \frac{|\theta(t) - \theta(t_0)|}{\arccos(T(t) \cdot T(t_0))} = \lim_{t \to t_0} \frac{\arccos(T(t) \cdot T(t_0))}{|\theta(t) - \theta(t_0)|}$$

By the definition of the derivative, we have:

$$\frac{d\theta}{dt}(t_0) = \left|\frac{d\theta}{dt}(t_0)\right| = \lim_{t \to t_0} \frac{|\theta(t) - \theta(t_0)|}{|t - t_0|}.$$

Now, we have that:

$$\frac{d\theta}{dt}(t_0)\cdot 1 = \lim_{t\to t_0}\frac{|\theta(t)-\theta(t_0)|}{|t-t_0|}\cdot\frac{\arccos(T(t)\cdot T(t_0))}{|\theta(t)-\theta(t_0)|} = \lim_{t\to t_0}\frac{\arccos(T(t)\cdot T(t_0))}{|t-t_0|}.$$

Then, by definition of curvature, $\kappa = \frac{d\theta}{ds} = \frac{dt}{ds}\frac{d\theta}{dt}$. So, we have the following:

$$\begin{aligned}\kappa(t_0) &= \frac{d\theta}{ds}(t_0)\\ &= \frac{dt}{ds}(t_0)\frac{d\theta}{dt}(t_0)\\ &= \lim_{t\to t_0}\frac{|t-t_0|}{|s(t)-s(t_0)|}\cdot\frac{\arccos(T(t)\cdot T(t_0))}{|t-t_0|}\\ &= \lim_{t\to t_0}\frac{\arccos(T(t)\cdot T(t_0))}{|s(t)-s(t_0)|}.\end{aligned}$$

□

# Problem 1.3.3

Show that for vectors $v, w \in \mathbb{R}^n$ we have

$$\begin{aligned}\text{area of parallelogram}(v,w) &= \sqrt{|v|^2|w|^2-(v\cdot w)^2}\\ &= |v||w|\sin\measuredangle(v,w).\end{aligned}$$

*Proof.* Let $v, w \in R^n$. By the definition of the area of a parallelogram, we have:

$$\text{area of parallelogram}(v,w) = |w|\left|v-(v\cdot w)\frac{w}{|w|^2}\right|$$

Square both sides and simplify:

$$\begin{aligned}\text{area}^2 &= |w|^2\left|v-(v\cdot w)\frac{w}{|w|^2}\right|^2\\ &= |w|^2\left(v-(v\cdot w)\frac{w}{|w|^2}\right)\cdot\left(v-(v\cdot w)\frac{w}{|w|^2}\right)\\ &= |w|^2\left(|v|^2-2\frac{(v\cdot w)^2}{|w|^2}+(v\cdot w)^2\frac{|w|^2}{|w|^4}\right)\\ &= |w|^2\left(|v|^2-2\frac{(v\cdot w)^2}{|w|^2}+\frac{(v\cdot w)^2}{|w|^2}\right)\\ &= |w|^2\left(|v|^2-\frac{(v\cdot w)^2}{|w|^2}\right)\\ &= |v|^2|w|^2-(v\cdot w)^2\end{aligned}$$

Thus, we have:

$$\boxed{\text{area of parallelogram}\,(v,w) = \sqrt{|v|^2\,|w|^2 - (v\cdot w)^2}}$$

Since $(v \cdot w)^2 = |v|\,|w|\cos\theta$, where $\theta = \measuredangle\,(v,w)$, we have:

$$\text{area} = \sqrt{|v|^2|w|^2 - (|v|\,|w|\cos\theta)^2}, \quad \theta = \measuredangle\,(v,w)$$

Simplify:

$$\begin{aligned}
\text{area} &= \sqrt{|v|^2\,|w|^2 - |v|^2\,|w|^2\cos^2\theta} \\
&= \sqrt{|v|^2\,|w|^2 - |v|^2\,|w|^2\left(1-\sin^2\theta\right)} \\
&= \sqrt{|v|^2\,|w|^2 - |v|^2\,|w|^2 + |v|^2\,|w|^2\sin^2\theta} \\
&= \sqrt{|v|^2\,|w|^2\sin^2\theta} \\
&= |v|\,|w|\sin\theta
\end{aligned}$$

Hence,

$$\boxed{\text{area} = |v|\,|w|\sin\measuredangle\,(v,w)}$$

□

# Problem 1.3.8

Give examples of regular curves $q(t) : (a,b) \to \mathbb{R}^n$ with $|q(t)| \geq R$ for all $t$, $|q(t_0)| = R$, and $\kappa(t_0) = c$ for any $c \geq 0$.

*Solution.* Suppose we have a smaller circle with radius $r = \frac{1}{c}$ that is (outside) tangential to a larger circle with radius $R$ ($|q(t)| \geq R$). Since the smaller circle is tangential to the larger circle, suppose the smaller circle intersects the larger circle such that $|q(t_0)| = R$. Then the curvature of this smaller circle is given by $\kappa\,(t_0) = \frac{1}{\text{radius}} = c > 0$. Now, let there be a tangential straight line outside the larger circle with radius $R$ that intersects the larger circle at $t_1$. Then $K(t_1) = 0$, and the conditions in the problem statement are satisfied.

# Problem 1.3.10

Let $q(t) = r(t)(\cos t, \sin t)$. Show that the speed is given by

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2$$

and the curvature

$$\kappa = \frac{\left|2\left(\frac{dr}{dt}\right)^2 + r^2 - r\frac{d^2r}{dt^2}\right|}{\left(\left(\frac{dr}{dt}\right)^2 + r^2\right)^{\frac{3}{2}}}.$$

*Proof.* Taking the derivative of $q(t)$ with respect to $t$:

$$\dot{q}(t) = \dot{r}\,(\cos t,\ \sin t) + r(-\sin t,\ \cos t).$$

Then, we have that

$$\left(\frac{ds}{dt}\right)^2 = |\dot{q}|^2 = (\dot{r})^2 + (r)^2 = \left(\frac{dr}{dt}\right)^2 + r^2.$$

Now, we proceed to showing the curvature. From Remark 1.3.4, we have the following formula for curvature:

$$\kappa = \frac{|v \times a|}{|v|^3} = \frac{|\dot{q} \times \ddot{q}|}{|\dot{q}|^3} \qquad \text{(eq. 1.3.10)}$$

Computing $\dot{q}$, we have:

$$\begin{aligned}\dot{q} &= \dot{r}\,(\cos t,\ \sin t,\ 0) + r\,(-\sin t,\ \cos t,\ 0)\\ &= (\dot{r}\cos t - r\sin t,\ \dot{r}\sin t + r\cos t,\ 0)\end{aligned}$$

Computing $\ddot{q}$, we have:

$$\begin{aligned}\ddot{q} &= (\ddot{r} - r)\,(\cos t,\ \sin t,\ 0) + 2\dot{r}\,(-\sin t,\ \cos t,\ 0)\\ &= ((\ddot{r} - r)\cos t - 2\dot{r}\sin t,\ (\ddot{r} - r)\sin t + 2\dot{r}\cos t,\ 0)\end{aligned}$$

Now we compute $\dot{q} \times \ddot{q}$. Since the $z$-component of both $\dot{q}$ and $\ddot{q}$ are 0, then the $x$ and $y$ components of $\dot{q} \times \ddot{q}$ are 0. So, we need to compute the $z$-component of $\dot{q} \times \ddot{q}$. We have the following:

$$(\dot{r}\cos t - r\sin t)\,((\ddot{r} - r)\sin t + 2\dot{r}\cos t) - (\dot{r}\sin t + r\cos t)\,((\ddot{r} - r)\cos t - 2\dot{r}\sin t)$$

$$\rightsquigarrow (\dot{r}\cos t - r\sin t)\,(\ddot{r}\sin t - r\sin t + 2\dot{r}\cos t) - (\dot{r}\sin t + r\cos t)\,(\ddot{r}\cos t - r\cos t - 2\dot{r}\sin t)$$

$$\begin{aligned}\rightsquigarrow &\left(\ddot{r}\dot{r}\sin t\cos t - \dot{r}r\sin t\cos t + 2\dot{r}^2\cos^2 t - \ddot{r}r\sin^2 t + r^2\sin^2 t - 2\dot{r}r\sin t\cos t\right)\\ &- \left(\ddot{r}\dot{r}\sin t\cos t - \dot{r}r\sin t\cos t + 2\dot{r}^2\sin^2 t + \ddot{r}r\cos^2 t - r^2\cos^2 t - 2\dot{r}r\sin t\cos t\right)\end{aligned}$$

$$\rightsquigarrow 2\dot{r}^2\cos^2 t - \ddot{r}r\sin^2 t + r^2\sin^2 t + 2\dot{r}^2\sin^2 t - \ddot{r}r\cos^2 t + r^2\cos^2 t$$

$$\rightsquigarrow 2\dot{r}^2 - \ddot{r}r + r^2$$

So, for the numerator of (eq. 1.3.10), we have:

$$|\dot{q} \times \ddot{q}| = \left|2\dot{r}^2 + r^2 - r\ddot{r}\right|$$

Now, we compute the denominator of (eq. 1.3.10). From the first part of this problem, we found that $|\dot{q}|^2 = \dot{r}^2 + r^2$. Then, $|\dot{q}| = (\dot{r}^2 + r^2)^{1/2}$. So, we have for the denominator:

$$|\dot{q}|^3 = \left(\dot{r}^2 + r^2\right)^{3/2}.$$

Putting everything together, we have the following curvature:

$$\kappa = \frac{|\dot{q} \times \ddot{q}|}{|\dot{q}|^3} = \frac{|2\dot{r}^2 + r^2 - r\ddot{r}|}{(\dot{r}^2 + r^2)^{3/2}} = \frac{\left|2\left(\frac{dr}{dt}\right)^2 + r^2 - r\frac{d^2r}{dt^2}\right|}{\left(\left(\frac{dr}{dt}\right)^2 + r^2\right)^{\frac{3}{2}}}.$$

□

# Problem 1.3.12

Compute the curvature of the logarithmic spiral

$$ae^{bt}(\cos t, \sin t).$$

*Solution.* Let $q(t) = ae^{bt}(\cos t, \sin t)$. We use the following formula from Remark 1.3.4 to compute curvature:

$$\kappa = \frac{|v \times a|}{|v|^3} = \frac{|\dot{q} \times \ddot{q}|}{|\dot{q}|^3} \qquad \text{(eq. 1.3.12)}$$

Compute $\dot{q}$:

$$\dot{q} = abe^{bt}(\cos t, \sin t) + ae^{bt}(-\sin t, \cos t)$$
$$\rightsquigarrow \dot{q} = \left(abe^{bt}\cos t - ae^{bt}\sin t,\ abe^{bt}\sin t + ae^{bt}\cos t,\ 0\right)$$

Compute $\ddot{q}$:

$$\begin{aligned}\ddot{q} &= ab^2e^{bt}(\cos t, \sin t) + abe^{bt}(-\sin t, \cos t) + abe^{bt}(-\sin t, \cos t) + ae^{bt}(-\cos t, -\sin t)\\ &= ab^2e^{bt}(\cos t, \sin t) + 2abe^{bt}(-\sin t, \cos t) + ae^{bt}(-\cos t, -\sin t)\end{aligned}$$

$$\rightsquigarrow \ddot{q} = \left(ab^2e^{bt}\cos t - 2abe^{bt}\sin t - ae^{bt}\cos t,\ ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t,\ 0\right)$$

Now we compute $\dot{q} \times \ddot{q}$. Since the $z$-component of both $\dot{q}$ and $\ddot{q}$ are 0, then the $x$ and $y$ components of $\dot{q} \times \ddot{q}$ are 0. So, we need to compute the $z$-component of $\dot{q} \times \ddot{q}$. We have the following:

$$\left(abe^{bt}\cos t - ae^{bt}\sin t\right)\left(ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t\right)$$
$$- \left(abe^{bt}\sin t + ae^{bt}\cos t\right)\left(ab^2e^{bt}\cos t - 2abe^{bt}\sin t - ae^{bt}\cos t\right)$$

$$\rightsquigarrow (a^2b^3e^{2bt}\sin t\cos t + 2a^2b^2e^{2bt}\cos^2 t - a^2be^{bt}\sin t\cos t - a^2b^2e^{2bt}\sin^2 t$$
$$- 2a^2be^{2bt}\sin t\cos t + a^2e^{2bt}\sin^2 t)$$
$$- (a^2b^3e^{2bt}\sin t\cos t - 2a^2b^2e^{2bt}\sin^2 t - a^2be^{bt}\sin t\cos t - a^2b^2e^{2bt}\cos^2 t$$
$$- 2a^2be^{2bt}\sin t\cos t - a^2e^{2bt}\cos^2 t)$$

$$\rightsquigarrow 2a^2b^2e^{2bt}\cos^2 t - a^2b^2e^{2bt}\sin^2 + a^2e^{2bt}\sin^2 t + a^2e^{2bt}\sin^2 t$$
$$+ 2a^2b^2e^{2bt}\sin^2 t - a^2b^2e^{2bt}\cos^2 t + a^2e^{2bt}\cos^2 t$$

$$\rightsquigarrow 2a^2b^2e^{2bt} - a^2b^2e^{2bt} + a^2e^{2bt}$$
$$\rightsquigarrow a^2b^2e^{2bt} + a^2e^{2bt}$$
$$\rightsquigarrow a^2e^{2bt}\left(b^2+1\right)$$

So, for the numerator of (eq. 1.3.12), we have:

$$|\dot{q} \times \ddot{q}| = a^2e^{2bt}\left(b^2+1\right)$$

Now, we compute the denominator of (eq. 1.3.12). Since

$$\dot{q} = abe^{bt}\left(\cos t, \sin t\right) + ae^{bt}\left(-\sin t, \cos t\right),$$

$$|\dot{q}|^2 = a^2b^2e^{2bt} + a^2e^{2bt} = a^2e^{2bt}(b^2+1).$$

Thus, $|\dot{q}| = ae^{bt}\sqrt{b^2+1}$. So, we have for the denominator:

$$|\dot{q}|^3 = a^3e^{3bt}\left(b^2+1\right)^{3/2}.$$

Putting everything together, we have the following curvature:

$$\kappa = \frac{|\dot{q} \times \ddot{q}|}{|\dot{q}|^3} = \frac{a^2e^{2bt}\left(b^2+1\right)}{a^3e^{3bt}\left(b^2+1\right)^{3/2}} = \boxed{\frac{1}{ae^{bt}\sqrt{b^2+1}}}.$$

# Problem 2.1.6

Show that a planar curve is part of a circle if all its normal lines pass through a fixed point.

*Proof.* If all the normal lines of a planar curve pass through a fixed point, then the unit normal is radial. Since $T \perp N$, $T$ is perpendicular to the radial directions. Then, the curve lies on a sphere, and by Problem 1.1.13 from Homework 1, the curve lies on a circle. □

## Problem 2.1.7

Show that $\kappa_{\pm}\frac{ds}{dt} = \det\begin{bmatrix} T & \frac{dT}{dt} \end{bmatrix}$.

*Proof.* From Theorem 2.1.5, we have:

$$\frac{dT}{dt} = \kappa_{\pm}\frac{ds}{dt}N_{\pm}.$$

So, after substitution, we get:

$$\det\begin{bmatrix} T & \kappa_{\pm}\frac{ds}{dt}N_{\pm} \end{bmatrix}.$$

$\kappa_{\pm}\frac{ds}{dt}$ is a scalar, so using properties of the determinant, we have:

$$\kappa_{\pm}\frac{ds}{dt}\det\begin{bmatrix} T & N_{\pm} \end{bmatrix}.$$

Since $T$ and $N_{\pm}$ are orthogonal to each other, $\det\begin{bmatrix} T & N_{\pm} \end{bmatrix} = 1$. Thus, we have $\kappa_{\pm}\frac{ds}{dt}$. So,

$$\det\begin{bmatrix} T & \frac{dT}{dt} \end{bmatrix} = \kappa_{\pm}\frac{ds}{dt}.$$

□

## Problem 2.1.9

Show that

$$q(s) = \left(\int_{s_0}^{s} \cos(\phi(u))\, du,\ \int_{s_0}^{s} \sin(\phi(u))\, du\right),$$

is a unit speed curve with $\kappa_{\pm} = \frac{d\phi}{ds}$.

*Proof.* By Proposition 2.1.4, the signed curvature can be calculated using the formula

$$\kappa_{\pm} = \frac{\det\begin{bmatrix} v & a \end{bmatrix}}{|v|^3}.$$

We first calculate $v(s) = \frac{dq}{ds}(s) = \dot{q}(s)$ using the Fundamental Theorem of Calculus:

$$\begin{aligned} v(s) &= \left( \frac{d}{ds} \int_{s_0}^{s} \cos\left(\phi\left(u\right)\right)\, du,\ \frac{d}{ds} \int_{s_0}^{s} \sin\left(\phi\left(u\right)\right)\, du \right) \\ &= \left(\cos\left(\phi\left(s\right)\right),\ \sin\left(\phi\left(s\right)\right)\right). \end{aligned}$$

Now, we calculate $a(s) = \frac{d^2q}{dt^2}(s) = \ddot{q}(s)$:

$$a(s) = \left( -\sin\left(\phi\left(s\right)\right) \frac{d\phi}{ds},\ \cos\left(\phi\left(s\right)\right) \frac{d\phi}{ds} \right)$$

Now, we compute $\det \begin{bmatrix} v & a \end{bmatrix}$:

$$\begin{aligned} \det \begin{bmatrix} v & a \end{bmatrix} &= \begin{bmatrix} \cos\left(\phi\left(s\right)\right) & -\frac{d\phi}{ds} \sin\left(\phi\left(s\right)\right) \\ \sin\left(\phi\left(s\right)\right) & \frac{d\phi}{ds} \cos\left(\phi\left(s\right)\right) \end{bmatrix} \\ &= \frac{d\phi}{ds} \cos^2\left(\phi\left(s\right)\right) + \frac{d\phi}{ds} \sin^2\left(\phi\left(s\right)\right) \\ &= \frac{d\phi}{ds} \end{aligned}$$

Lastly, we compute $|v|^3$. Since $|v| = 1$, $|v|^3 = 1$. Thus, putting everything together, we have:

$$\kappa_{\pm} = \frac{\det \begin{bmatrix} v & a \end{bmatrix}}{|v|^3} = \frac{\frac{d\phi}{ds}}{1} = \frac{d\phi}{ds}.$$

□

# Problem 2.1.11

For a planar unit speed curve $q(s)$ consider the parallel curve

$$q_{\epsilon} = q + \epsilon N_{\pm}$$

for some fixed $\epsilon$.

(a) Show that this curve is regular as long as $\epsilon \kappa_{\pm} \neq 1$.

(b) Show that this curvature is

$$\frac{\kappa_{\pm}}{|1 - \epsilon \kappa_{\pm}|}.$$

*Proof.* (a) We begin by calculating $\frac{dq_{\epsilon}}{ds}$:

$$\frac{dq_{\epsilon}}{ds} = \frac{dq}{ds} + \frac{d}{ds}\left(\epsilon N_{\pm}\right)$$

By Theorem 2.1.5, $\frac{dq}{ds} = T$ and $\frac{dN_\pm}{ds} = -\kappa_\pm T$. Thus, making the substitutions and simplifying, we get:

$$\begin{aligned}\frac{dq_\epsilon}{ds} &= T + \epsilon \frac{dN_\pm}{ds} \\ &= T + \epsilon\left(-\kappa_\pm\right)T \\ &= T - T\epsilon\kappa_\pm\end{aligned}$$

If $q_\epsilon$ is *not* regular, then $\frac{dq_\epsilon}{ds} = T - T\epsilon\kappa_\pm = T\left(1 - \epsilon\kappa_\pm\right) = 0$. That is, when $\epsilon\kappa_\pm = 1$, $q_\epsilon$ is *not* regular. Therefore, $q_\epsilon$ **is regular** as long as $\epsilon\kappa_\pm \neq 1$. □

(b) We will calculate the curvature using the formula from Proposition 2.1.4:

$$\kappa_\pm = \frac{\det\begin{bmatrix} v & a \end{bmatrix}}{|v|^3}$$

From part (a), we have shown that $v(s) = \frac{dq_\epsilon}{ds} = T - T\epsilon\kappa_\pm = T\left(1 - \epsilon\kappa_\pm\right)$. Next, we calculate $a(s) = \frac{d^2q_\epsilon}{ds}$:

$$\begin{aligned}a(s) = \frac{d^2q_\epsilon}{ds} &= \frac{dT}{ds} - \frac{d}{ds}\left(T\epsilon\kappa_\pm\right) \\ &= \frac{dT}{ds} - \epsilon\kappa_\pm\frac{dT}{ds}\end{aligned}$$

By Theorem 2.1.5, since $\frac{dT}{ds} = \kappa_\pm N_\pm$, we have then:

$$a(s) = \kappa_\pm N_\pm - \epsilon\kappa_\pm^2 N_\pm = \kappa_\pm N_\pm\left(1 - \epsilon\kappa_\pm\right).$$

Now, we will calculate $|v\left(s\right)|^3$. Since

$$|v\left(s\right)|^2 = \left(1 - \epsilon\kappa_\pm\right)^2, \quad (T^2 = 1 \text{ because } T \text{ is a unit vector}),$$

$$|v\left(s\right)| = |1 - \epsilon\kappa_\pm|.$$

Thus,

$$|v\left(s\right)|^3 = |1 - \epsilon\kappa_\pm|^3.$$

Now, we calculate $\det\begin{bmatrix} v & a \end{bmatrix}$, using determinant properties and that $\det\begin{bmatrix} T & N_\pm \end{bmatrix} = 1$ :

$$\begin{aligned}\det\begin{bmatrix} v & a \end{bmatrix} &= \begin{bmatrix} T\left(1 - \epsilon\kappa_\pm\right) & \kappa_\pm N_\pm\left(1 - \epsilon\kappa_\pm\right) \end{bmatrix} \\ &= \kappa_\pm\left(1 - \epsilon\kappa_\pm\right)^2 \det\begin{bmatrix} T & N_\pm \end{bmatrix} \\ &= \kappa_\pm\left(1 - \epsilon\kappa_\pm\right)^2 \\ &= \kappa_\pm|v\left(s\right)|^2\end{aligned}$$

Thus, putting everything together:

$$\begin{aligned}\kappa_{\pm} = \frac{\det \begin{bmatrix} v & a \end{bmatrix}}{|v|^3} &= \frac{\kappa_{\pm} |v|^2}{|v|^3} \\ &= \frac{\kappa_{\pm}}{|v|} \\ &= \frac{\kappa_{\pm}}{|1 - \epsilon \kappa_{\pm}|}.\end{aligned}$$

□