Quasi One-Dimensional Nozzle Flow
Fluid Mechanics

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Contents

1 Description of the Problem

1.1 History and Background

A **de Laval nozzle** (or **convergent-divergent nozzle**, **CD nozzle** or **con-di nozzle**) is a tube that is pinched in the middle, making a carefully balanced, asymmetric hourglass-shape. It is used to accelerate a hot, pressurized gas passing through it to a supersonic speed, and upon expansion, to shape the exhaust flow so that the heat energy propelling the flow is maximally converted into directed kinetic energy. Because of this, the nozzle is widely used in some types of steam turbines, and is used as a rocket engine nozzle. It also sees use in supersonic jet engines.

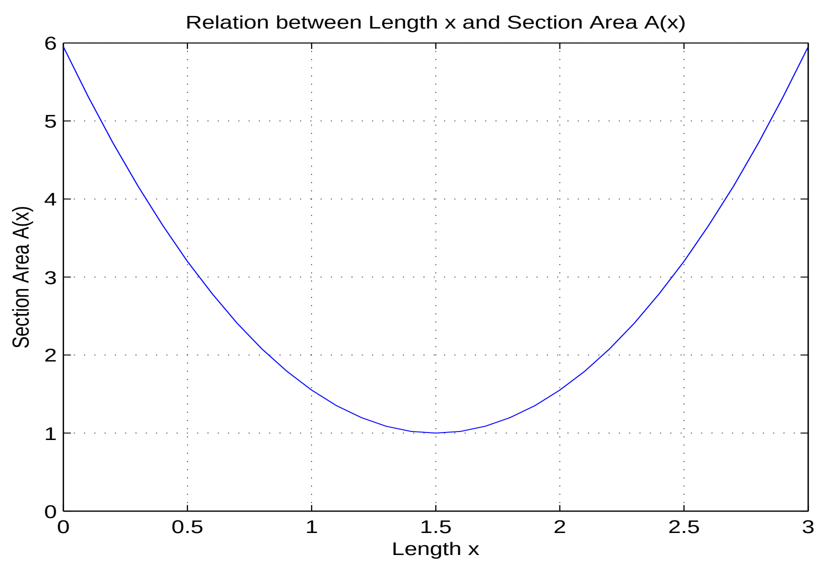
Similar flow properties have been applied to jet streams within astrophysics.The nozzle was developed by Swedish inventor Gustaf de Laval in 1888 for use on a steam turbine.

This principle was first used in a rocket engine by Robert Goddard. Very nearly all modern rocket engines that employ hot gas combustion use de Laval nozzles.

1.2 Description

In this problem, the shape of Laval nozzle is given, whose section area A submits the quadratic function of variable x:

$$A(x) = 2.2(x - 1.5)^2 + 1 \quad (0 \le x \le 3) \tag{1.1}$$



We can see from this function that the section area submits the **convergent-divergent model**. When gases flow throw the Laval nozzle, the speed of gases will be changed by the variation of the shape of the nozzle.

Its operation relies on the different properties of gases flowing at subsonic and supersonic speeds. The speed of a subsonic flow of gas will increase if the pipe carrying it narrows because the mass flow rate is constant.At subsonic flow the gas is compressible; sound, a small pressure wave, will propagate through it. At the "**throat**", where the cross sectional area is a minimum, the gas velocity locally becomes sonic (Mach number = 1.0), a condition called choked flow. As the nozzle cross sectional area increases the gas begins to expand and the gas flow increases to supersonic velocities where a sound wave will not propagate backwards through the gas as viewed in the frame of reference of the nozzle (Mach number > 1.0).

1.3 Problems Given

- Analytical solutions should be solved by the Matlab, while all parameters should be coherent with the knowledge in class.
- In this project, our main object is to use the **finite difference method** in simulating and calculating the result of the Mach number, gas pressure, temperature, speed which are related to the position in the nozzle.
- To simplify and well analyze the problem, we should make the physical variables dimensionless. The dimension depends the choice of fundamental variables, so we need to describe the physical problem deeply without dimensional effects. By this method, we can facilitate equations as well.
- The flow is considered as one-dimensional.

2 Analysis of Analytical Solutions

2.1 Definition of Variables

x	position at axis x
A	section area
ρ	density of gas
u	speed of gas
p	pressure
e	internal energy
T	temperature
V	volume

2.2 Fundamental Equations in One Dimension

- Initial Model
- Equation of Continuity
- Equation of Momentum
- Equation of Energy
- Equation of Perfect Gas

Initial Model

p, T, u, ρ, A are used to represent the pressure, temperature, speed, density and section area respectively. The control volume is chosen between the length x and $x + dx$.

Equation of Continuity

$$\rho u A = (\rho + \mathrm{d}\rho)(u + \mathrm{d}u)(A + dA)$$

Neglect the high order terms.

$$\rho u \mathrm{d}A + uA\mathrm{d}\rho + \rho A \mathrm{d}u = 0$$

$$\frac{\mathrm{d}A}{A} + \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}u}{u} = 0 \tag{2.1}$$

$$\rho u A = const$$

Equation of Momentum

$$\rho u A(u + \mathrm{d}u) - \rho u A u = pA - (p + \mathrm{d}p)(A + \mathrm{d}A) + (p + \frac{\mathrm{d}p}{2})\mathrm{d}A$$

In this equation, $\rho u A(u + \mathrm{d}u)$ represents the momentum of leaving while $\rho u A u$ represent the momentum of entering.
Neglect the high order terms.

$$\rho u \mathrm{d}u = -\mathrm{d}p \tag{2.2}$$

Equation of Energy

$$h + \frac{u^2}{2} = const \tag{2.3}$$

Equation of Perfect Gas

$$\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}T}{T} \tag{2.4}$$

We consider the **isentropic** process.

$$\mathrm{d}S = 0$$

$$\frac{p}{\rho^\gamma} = const \qquad \gamma = \frac{c_\mathrm{p}}{c_\mathrm{v}}$$

γ : adiabatic constant

Question 1: Relation between Mach number Ma and section area A.

Firstly, we admit that the process is isentropic,

$$\mathrm{d}S = 0$$

$$\frac{\mathrm{d}p}{p} = \gamma\frac{p}{\rho}$$

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \gamma\frac{p}{\rho} = c^2$$

c : speed of sound

From the equation of momentum: $\rho u \mathrm{d}u = -\mathrm{d}p$, we have:

$$\frac{\rho u \mathrm{d}u}{\mathrm{d}\rho} = -\frac{\mathrm{d}p}{\mathrm{d}\rho} = -c^2$$

Put this equation into the equation of continuity:

$$\frac{\mathrm{d}A}{A} + \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}u}{u} = 0$$

$$\frac{\mathrm{d}A}{A} - \frac{u\mathrm{d}u}{c^2} + \frac{\mathrm{d}u}{u} = 0$$

$$\begin{aligned}\frac{\mathrm{d}A}{A} &= \frac{u\mathrm{d}u}{c^2} - \frac{\mathrm{d}u}{u} \\ &= \frac{u^2\mathrm{d}u}{c^2 u} - \frac{\mathrm{d}u}{u}\end{aligned}$$

$$\frac{\mathrm{d}A}{A} = (Ma^2 - 1)\frac{\mathrm{d}u}{u} \tag{2.5}$$

Now, there are 2 different conditions that we should discuss:

1. $Ma > 1$
   $\mathrm{d}A > 0$, $\mathrm{d}u > 0$ Acceleration
   $\mathrm{d}A < 0$, $\mathrm{d}u < 0$ Deceleration

2. $Ma < 1$
   $\mathrm{d}A > 0$, $\mathrm{d}u > 0$ Deceleration
   $\mathrm{d}A < 0$, $\mathrm{d}u < 0$ Acceleration

We can see from the above result that the speed changes differently when considering the subsonic and supersonic conditions. When gas speed is supersonic ($Ma > 1$), the speed is in positive correlation with the section area changes, while in the subsonic condition ($Ma < 1$), the speed is in negative correlation with the section area changes.

Question 2: Relation between mass flow $\dot{m}$ Mach number Ma.

We use the subscript t to represent *total*, along the streamline, so we have:

$$T_t = T + \frac{u^2}{2c_p} \qquad p_t = p(\frac{T_t}{T})^{\frac{\gamma}{\gamma-1}} \qquad \rho_t = \rho(\frac{p_t}{p})^{\frac{1}{\gamma}}$$

$$\dot{m} = \rho u A \qquad \rho = \frac{p\gamma}{c^2} \qquad p = \frac{\rho c^2}{\gamma} = \rho RT$$

$$
\begin{aligned}
\dot{m} &= \frac{p\gamma}{c^2}uA \\
&= \frac{p\gamma A}{c}\frac{u}{c} \\
&= \frac{\gamma p A}{\sqrt{\gamma RT}}Ma \\
&= \sqrt{\frac{\gamma}{R}}\frac{p_t}{\sqrt{T_t}}AMa\left(\frac{1}{1+\frac{\gamma-1}{2}Ma^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}
\end{aligned}
$$

There is no shaft work.

$$T_{total} = const \qquad \mathrm{d}S = 0 \qquad p_{total} = const$$

Using the parameters in stationary point T_t, p_t:

$$T = \frac{T_t}{1+\frac{\gamma-1}{2}Ma^2} \tag{2.6}$$

$$p = \frac{p_t}{1+\frac{\gamma-1}{2}Ma^2} \tag{2.7}$$

So, we obtain the relation between $\dot{m}$ and Ma:

$$\dot{m} = \sqrt{\frac{\gamma}{R}}\frac{p_t}{\sqrt{T_t}}AMa\left(\frac{1}{1+\frac{\gamma-1}{2}Ma^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{2.8}$$

Now, we prove the position of $Ma = 1$ is throat.

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R}}\frac{p_t}{\sqrt{T_t}}Ma\left(\frac{1}{1+\frac{\gamma-1}{2}Ma^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \qquad K = \sqrt{\frac{\gamma}{R}}\frac{p_t}{\sqrt{T_t}} = const$$

$$\frac{\dot{m}}{A} = KMa\left(\frac{1}{1+\frac{\gamma-1}{2}Ma^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

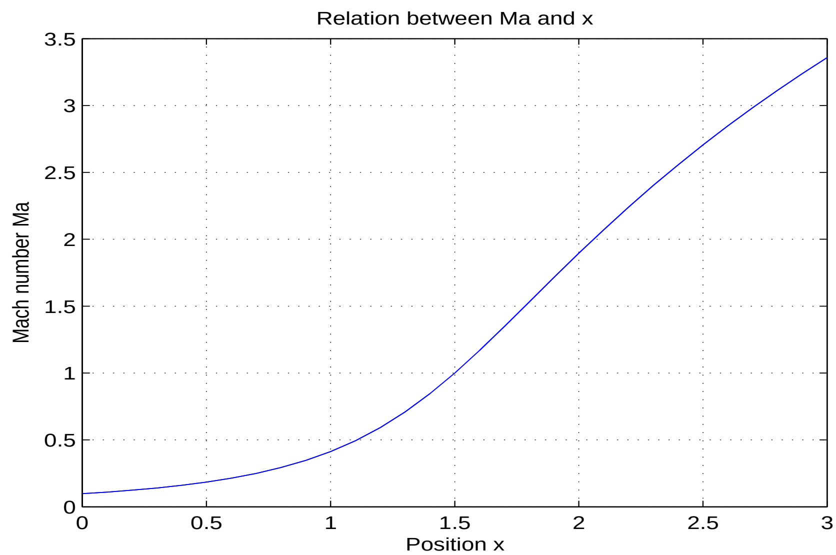
At throat: the section area A reaches minimum and $\frac{\dot{m}}{A}$ reaches maximum.

$$\mathrm{d}(\frac{\dot{m}}{A}) = \mathrm{d}\left(KMa\left(\frac{1}{1+\frac{\gamma-1}{2}Ma^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\right) = 0$$

After calculating:

$$Ma = 1$$

The following graph is about Mach number Ma and position x.



So, the position of $Ma = 1$ is throat.

Question 3: Relation between parameters of stationary point and static parameters.

$A^*, Ma^*, T^*, p^*, \rho^*$ represent the parameters of stationary point, $Ma^* = 1$.
From (2.8), We can conclude that:

$$\frac{A}{A^*} = \frac{1}{Ma}\left(\frac{1+\frac{\gamma-1}{2}Ma^2}{\frac{\gamma+1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{2.9}$$

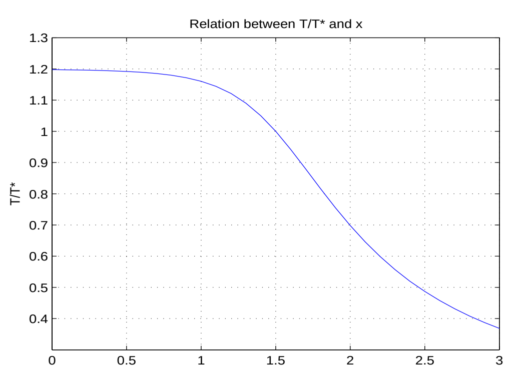
$$\begin{aligned} T_t &= T\left(1+\frac{\gamma-1}{2}Ma^2\right) \\ &= T^*\left(1+\frac{\gamma-1}{2}Ma^{*2}\right) \end{aligned}$$

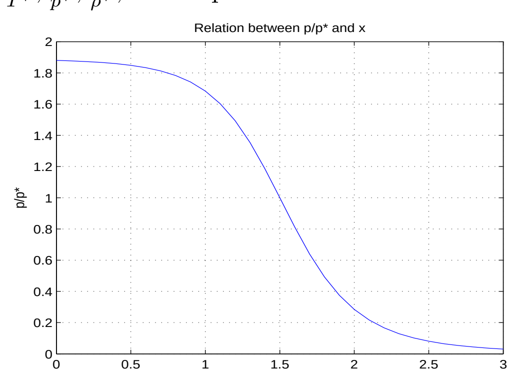
$$\frac{T}{T^*} = \frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}Ma^2} \tag{2.10}$$

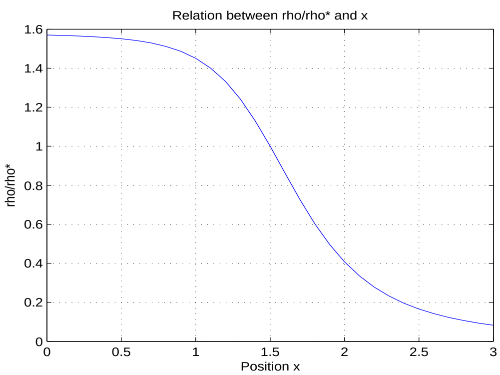
$$\frac{p}{p^*} = \left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}Ma^2}\right)^{\frac{\gamma}{\gamma-1}} \tag{2.11}$$

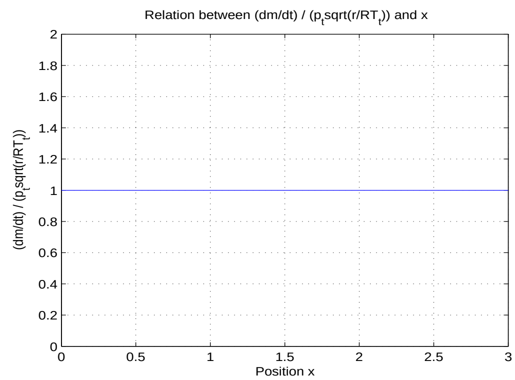
$$\frac{\rho}{\rho^*} = \left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}Ma^2}\right)^{\frac{1}{\gamma-1}} \tag{2.12}$$

The following graph is about the relations between $\frac{T}{T^*}$, $\frac{p}{p^*}$, $\frac{\rho}{\rho^*}$, $\dot{m}$ and position x.









2.3 Analytical Solutions

2.3.1 Equation of Continuity

$$\rho v_n \mathrm{d}S = \rho\vec{v}\mathrm{d}\vec{S}$$

$$\iint_S \rho\vec{v}\mathrm{d}\vec{S} = -\frac{\partial}{\partial t}\iiint_V \rho \mathrm{d}V$$

$$\iint_S \rho\vec{v}\mathrm{d}\vec{S} + \frac{\partial}{\partial t}\iiint_V \rho \mathrm{d}V = 0 \tag{2.13}$$

$$\iint_S \rho\vec{v}\mathrm{d}\vec{S} = -\rho v A + (\rho + \mathrm{d}\rho)(v + \mathrm{d}v)(A + \mathrm{d}A) = \mathrm{d}(\rho v A)$$

$$\frac{\partial}{\partial t}\iiint_V \rho \mathrm{d}V = \frac{\partial}{\partial t}(\rho A \mathrm{d}x)$$

$$\frac{\partial}{\partial t}(\rho A \mathrm{d}x) + \mathrm{d}(\rho v A) = 0 \tag{2.14}$$

$$\frac{\partial}{\partial t}(\rho A) + \frac{\mathrm{d}(\rho v A)}{\mathrm{d}x} = 0 \tag{2.15}$$

2.3.2 Equation of Momentum

$$F_x = m a_x$$

$$m \ : \ \rho \mathrm{d}x\mathrm{d}y\mathrm{d}z = \rho A \mathrm{d}x$$

$$a_x \ : \ \frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + \vec{v}\cdot\nabla u$$

$$\frac{\partial}{\partial t}\iiint_V \rho u \mathrm{d}V + \iint_S \rho u\vec{v}\mathrm{d}\vec{S} = -\iint_S (p\mathrm{d}\vec{S})_x \tag{2.16}$$

$$\frac{\partial}{\partial t}\iiint_V \rho u \mathrm{d}V = \frac{\partial}{\partial t}(\rho u A)\mathrm{d}x \tag{2.17}$$

$$\iint_S (\rho u\vec{v})\mathrm{d}\vec{S} = -\rho v^2 A + (\rho + \mathrm{d}\rho)(v + \mathrm{d}v)^2(A + \mathrm{d}A) \tag{2.18}$$

$$\iint_S (p\mathrm{d}\vec{S})_x = pA - (p + \mathrm{d}p)(A + \mathrm{d}A) + 2p\cdot\frac{\mathrm{d}A}{2} \tag{2.19}$$

Put (2.17)(2.18)(2.19) into (2.16):

$$\frac{\partial}{\partial t}(\rho u A)\mathrm{d}x - \rho v^2 A + (\rho + \mathrm{d}\rho)(v + \mathrm{d}v)^2(A + \mathrm{d}A) = pA - (p + \mathrm{d}p)(A + \mathrm{d}A) + 2p\cdot\frac{\mathrm{d}A}{2}$$

$$\frac{\partial}{\partial t}(\rho v A \mathrm{d}x) + \mathrm{d}(\rho v^2 A) = -A\mathrm{d}p$$

$$\frac{\partial}{\partial t}(\rho v A) + \frac{\partial(\rho v^2 A)}{\partial x} = -A\frac{\partial p}{\partial x} \tag{2.20}$$

The equation (2.20) is called **Equation of Momentum in Conservation Form**.
Using $v\cdot$(2.15):

$$v\frac{\partial}{\partial t}(\rho A) + v\frac{\partial(\rho v A)}{\partial x} = 0 \tag{2.21}$$

(2.20)-(2.21), we get:

$$\frac{\partial}{\partial t}(\rho v A) - v\frac{\partial}{\partial t}(\rho A) + \frac{\partial(\rho v^2 A)}{\partial x} - v\frac{\partial(\rho v A)}{\partial x} = -A\frac{\partial p}{\partial x}$$

$$\rho A\frac{\partial v}{\partial t} + \rho A v\frac{\partial v}{\partial x} = -A\frac{\partial p}{\partial x}$$

$$\rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial x} \tag{2.22}$$

The equation (2.22) is called **Equation of Momentum in Non-Conservation Form**.

2.3.3 Equation of Energy

In this section, the volumed force is not considered. And the fluid is adiabatic and non-viscosity.

$$\frac{\partial}{\partial t}\iiint_V \rho(e+\frac{v^2}{2})\mathrm{d}V + \iint_S \rho(e+\frac{v^2}{2})\vec{v}\mathrm{d}\vec{S} = -\iint_S (p\vec{v})\mathrm{d}\vec{S}$$

$$\iint_S \rho(e+\frac{v^2}{2})\vec{v}\mathrm{d}\vec{S} = (\rho+\mathrm{d}\rho)(A+\mathrm{d}A)(v+\mathrm{d}v)\left(e+\mathrm{d}e+\frac{(v+\mathrm{d}v)^2}{2}\right) - \rho A v(e+\frac{v^2}{2})$$

$$\iint_S (p\vec{v})\mathrm{d}S = (p+\mathrm{d}p)(A+\mathrm{d}A)(v+\mathrm{d}v) - pAv - 2\cdot\frac{\mathrm{d}A}{2}pv$$

$$\frac{\partial}{\partial t}\iiint_V \rho(e+\frac{v^2}{2})\mathrm{d}V = \frac{\partial}{\partial t}\left(\rho(e+\frac{v^2}{2})A\mathrm{d}x\right) \tag{2.23}$$

$$\frac{\partial}{\partial t}\left(\rho(e+\frac{v^2}{2})A\mathrm{d}x\right) + \mathrm{d}(\rho e v A) + \frac{\mathrm{d}(\rho v^3 A)}{2} = -\mathrm{d}(pAv) \tag{2.24}$$

Take (2.23) into (2.24):

$$\frac{\partial\left(\rho(e+\frac{v^2}{2}A)\right)}{\partial t} + \frac{\partial\left(\rho(e+\frac{v^2}{2}vA)\right)}{\partial x} = -\frac{\partial(pAv)}{\partial x} \tag{2.25}$$

$(e+\frac{v^2}{2})$ is used to represent the equation in conservation form.
Using $v\cdot$(2.20):

$$v\frac{\partial(\rho v A)}{\partial t} + v\frac{\partial(\rho v^2 A)}{\partial x} = -Av\frac{\partial p}{\partial x} \tag{2.26}$$

Using $Av\cdot$(2.22):

$$\rho v A\frac{\partial v}{\partial t} + \rho v^2 A\frac{\partial v}{\partial x} = -Av\frac{\partial p}{\partial x} \tag{2.27}$$

Calculating the arithmetic mean:

$$\frac{1}{2}v\frac{\partial(\rho A v)}{\partial t} + \frac{1}{2}\rho v A\frac{\partial v}{\partial t} + \frac{1}{2}v\frac{\partial(\rho v^2 A)}{\partial x} + \frac{1}{2}\rho v^2 A\frac{\partial v}{\partial t} = -Av\frac{\partial p}{\partial x}$$

$$\frac{\partial(\rho\frac{v^2}{2}A)}{\partial t} + \frac{\partial(\rho\frac{v^3}{2}A)}{\partial x} = -Av\frac{\partial p}{\partial x} \tag{2.28}$$

(2.25)-(2.28)

$$\frac{\partial(\rho e A)}{\partial t} + \frac{\partial(\rho e v A)}{\partial x} = -p\frac{\partial(Av)}{\partial x} \tag{2.29}$$

e·(2.15)

$$e\frac{\partial(\rho A)}{\partial t} + e\frac{\rho v A}{\partial x} = 0 \tag{2.30}$$

(2.29)-(2.30)

$$\begin{aligned}
\rho A\frac{\partial e}{\partial t} + \rho A v\frac{\partial e}{\partial t} &= -p\frac{\partial(Av)}{\partial x} \\
\rho A\frac{\partial e}{\partial t} + \rho A v\frac{\partial e}{\partial t} &= -pA\frac{\partial v}{\partial x} - pv\frac{\partial A}{\partial x} \\
\rho\frac{\partial e}{\partial t} + \rho v\frac{\partial e}{\partial t} &= -p\frac{\partial v}{\partial x} - p\frac{v}{A}\frac{\partial A}{\partial x} \\
\rho\frac{\partial e}{\partial t} + \rho v\frac{\partial e}{\partial t} &= -p\frac{\partial v}{\partial x} - pv\frac{\partial(\ln A)}{\partial x}
\end{aligned} \tag{2.31}$$

2.3.4 Equation of Perfect Gas

In this section, we consider the perfect gas $e = c_{\mathrm{v}}T$.

$$\begin{cases}
\rho c_{\mathrm{v}}\frac{\partial T}{\partial t} + \rho v c_{\mathrm{v}}\frac{\partial T}{\partial x} = -p\frac{\partial v}{\partial x} - pv\frac{\partial(\ln A)}{\partial x} \\
\rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial x} \\
\frac{\partial(\rho A)}{\partial t} + \frac{(\rho A v)}{\partial x} = 0
\end{cases}$$

$$p = \rho R T \qquad \frac{\partial p}{\partial x} = R\left(\rho\frac{\partial T}{\partial x} + T\frac{\partial \rho}{\partial x}\right)$$

$$\frac{\partial(\rho A)}{\partial t} + \rho A\frac{\partial v}{\partial x} + \rho v\frac{\partial A}{\partial x} + vA\frac{\partial \rho}{\partial x} = 0 \tag{2.32}$$

$$\rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial x} = -R\left(\rho\frac{\partial T}{\partial x} + T\frac{\partial \rho}{\partial x}\right) \tag{2.33}$$

$$\rho c_{\mathrm{v}}\frac{\partial T}{\partial t} + \rho v c_{\mathrm{v}}\frac{\partial T}{\partial x} = -\rho R T\left(\frac{\partial v}{\partial x} + v\frac{\partial(\ln A)}{\partial x}\right) \tag{2.34}$$

2.3.5 Dimensionless Method

$$T' = \frac{T}{T_0} \qquad \rho' = \frac{\rho}{\rho_0} \qquad x' = \frac{x}{L}$$

$$v' = \frac{v}{a_0} \qquad t' = \frac{t}{\frac{L}{a_0}} = \frac{a_0}{L}t \qquad A' = \frac{A}{A^*}$$

The subscript 0 represents the parameters of coming flow, L represents the length of nozzle, $a_0 = \sqrt{\gamma R T_0}$ represents the local speed of sound, A^* represents the section area at throat.

$$\frac{\partial \rho'}{\partial t} = -v'\frac{\partial \rho'}{\partial x'} - \rho'\frac{\partial v'}{\partial x'} - \rho' v'\frac{\partial(\ln A')}{\partial x'} \tag{2.35}$$

$$\frac{\partial v'}{\partial t'} = -v\frac{\partial v'}{\partial x'} - \frac{1}{\gamma}\left(\frac{\partial T'}{\partial x'} + \frac{T'}{\rho'}\frac{\partial \rho'}{\partial x'}\right) \tag{2.36}$$

$$\frac{\partial T'}{\partial t} = -v'\frac{\partial T'}{\partial x'} - (\gamma - 1)T'\left(\frac{\partial v'}{\partial x'} + v'\frac{(\ln A')}{\partial x'}\right) \tag{2.37}$$

3 Diagram of Programming

3.1 Definition of Variables

x	position at axis x
A	section area
ρ	density of gas
u	speed of gas
p	pressure
e	internal energy
T	temperature
V	volume
Ma	Mach number

3.2 Mac-Cormack Finite Difference Method

In mathematics, finite-difference methods (FDM) are numerical methods for approximating the solutions to differential equations using finite difference equations to approximate derivatives.
In this problem, we use the form of Mac-Cormacle.

3.2.1 Mesh

Firstly, determine the number of nodes N.
Then, fix the unit length $\Delta x = \frac{x_{out} - x_{in}}{N}$.
Finally, determine the position, $\forall\ i \in [1, N], x_i = (i-1)\Delta x$.

3.2.2 Global Equation

At time t, all results of node position are known. Find out all results of node position at time $t + \Delta t$.

$$\rho_{i,j}^{t+\Delta t} = \rho_{i,j}^{t} + \left(\frac{\partial \rho}{\partial t}\right)_{av} \cdot \Delta t \tag{3.1}$$

$$u_{i,j}^{t+\Delta t} = u_{i,j}^{t} + \left(\frac{\partial u}{\partial t}\right)_{av} \cdot \Delta t \tag{3.2}$$

$$T_{i,j}^{t+\Delta t} = T_{i,j}^{t} + \left(\frac{\partial T}{\partial t}\right)_{av} \cdot \Delta t \tag{3.3}$$

In these 3 equations, we need to solve these unknown values: $\left(\frac{\partial \rho}{\partial t}\right)_{av}, \left(\frac{\partial u}{\partial t}\right)_{av}, \left(\frac{\partial T}{\partial t}\right)_{av}$ and determine the unit time Δt. To find out these values, we should use the form of Mac-Cormacle which can be conclude in the next step.

3.2.3 Finite Difference

Find out the average values: $\left(\frac{\partial \rho}{\partial t}\right)_{av}, \left(\frac{\partial u}{\partial t}\right)_{av}, \left(\frac{\partial T}{\partial t}\right)_{av}$.

a. Estimated Step, Forward Difference and Replace the Spatial Derivatives.

$$\frac{\partial \rho}{\partial t} = -u\frac{\partial \rho}{\partial x} - \rho\frac{\partial u}{\partial x} - \rho u\frac{\partial(\ln A)}{\partial x}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_i^t = -u_i^t \frac{\rho_{i+1}^t - \rho_i^t}{\Delta x} - \rho_i^t \frac{u_{i+1}^t - u_i^t}{\Delta x} - \rho_i^t u_i^t \frac{\ln A_{i+1} - \ln A_i}{\Delta x} \tag{3.4}$$

$$\left(\frac{\partial u}{\partial t}\right)_i^t = -u_i^t \frac{u_{i+1}^t - u_i^t}{\Delta x} - \frac{1}{\gamma}\left(\frac{T_{i+1}^t - T_i^t}{\Delta x} + \frac{T_i^t}{\rho_i^t} \cdot \frac{\rho_{i+1}^t - \rho_i^t}{\Delta x}\right) \tag{3.5}$$

$$\left(\frac{\partial T}{\partial t}\right)_i^t = -u_i^t \frac{T_{i+1}^t - T_i^t}{\Delta x} - (\gamma - 1) T_i^t \left(\frac{u_{i+1}^t - u_i^t}{\Delta x} + u_i^t \frac{\ln A_{i+1} - \ln A_i}{\Delta x}\right) \tag{3.6}$$

b. Estimated Value of ρ, u, T.

$$\overline{\rho}_i^t = \rho_i^t + \left(\frac{\partial \rho}{\partial t}\right)_i^t \cdot \Delta t \tag{3.7}$$

$$\overline{u}_i^t = u_i^t + \left(\frac{\partial u}{\partial t}\right)_i^t \cdot \Delta t \tag{3.8}$$

$$\overline{T}_i^t = T_i^t + \left(\frac{\partial T}{\partial t}\right)_i^t \cdot \Delta t \tag{3.9}$$

c. Correct Step, Backward Difference.

$$\begin{aligned}\left(\frac{\overline{\partial \rho}}{\partial t}\right)_i^{t+\Delta t} =& -\overline{u}_i^{t+\Delta t} \frac{\overline{\rho}_i^{t+\Delta t} - \overline{\rho}_{i-1}^{t+\Delta t}}{\Delta x} \\ & - \overline{\rho}_i^{t+\Delta t} \frac{\overline{u}_i^{t+\Delta t} - \overline{u}_{i-1}^{t+\Delta t}}{\Delta x} - \overline{\rho}_i^{t+\Delta t} \overline{u}_i^{t+\Delta t} \frac{\ln A_i - \ln A_{i-1}}{\Delta x}\end{aligned} \tag{3.10}$$

$$\begin{aligned}\left(\frac{\overline{\partial u}}{\partial t}\right)_i^{t+\Delta t} =& -\overline{u}_i^{t+\Delta t} \frac{\overline{u}_i^{t+\Delta t} - \overline{u}_{i-1}^{t-1}}{\Delta x} \\ & - \frac{1}{\gamma}\left(\frac{\overline{T}_i^{t+\Delta t} - \overline{T}_{i-1}^{t+\Delta t}}{\Delta x} + \frac{\overline{T}_i^{t+\Delta t}}{\overline{\rho}_i^{t+\Delta t}} \cdot \frac{\overline{\rho}_i^{t+\Delta t} - \overline{\rho}_{i-1}^{t+\Delta t}}{\Delta x}\right)\end{aligned} \tag{3.11}$$

$$\begin{aligned}\left(\frac{\overline{\partial T}}{\partial t}\right)_i^{t+\Delta t} =& -\overline{u}_i^{t+\Delta t} \frac{\overline{T}_i^{t+\Delta t} - \overline{T}_{i-1}^{t+\Delta t}}{\Delta x} \\ & - (\gamma - 1)\overline{T}_i^{t+\Delta t}\left(\frac{\overline{u}_i^{t+\Delta t} - \overline{u}_{i-1}^{t+\Delta t}}{\Delta x} + \overline{u}_i^{t+\Delta t} \frac{\ln A_i - \ln A_{i-1}}{\Delta x}\right)\end{aligned} \tag{3.12}$$

d. Average Derivatives by Time.

$$\left(\frac{\partial \rho}{\partial t}\right)_{av} = \frac{1}{2}\left[\left(\frac{\partial \rho}{\partial t}\right)_i^t + \left(\frac{\overline{\partial \rho}}{\partial t}\right)_i^{t+\Delta t}\right] \tag{3.13}$$

$$\left(\frac{\partial u}{\partial t}\right)_{av} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)_i^t + \left(\frac{\overline{\partial u}}{\partial t}\right)_i^{t+\Delta t}\right] \tag{3.14}$$

$$\left(\frac{\partial T}{\partial t}\right)_{av} = \frac{1}{2}\left[\left(\frac{\partial T}{\partial t}\right)_i^t + \left(\frac{\overline{\partial T}}{\partial t}\right)_i^{t+\Delta t}\right] \tag{3.15}$$

Now, these three average derivatives can be determined when we fix Δt.

3.2.4 Unit Time

Fix the Unit Time Δt.

$$\Delta t = C\frac{\Delta x}{|v| + a} \tag{3.16}$$

$$\Delta t = C\frac{\Delta x}{v + a}$$

$$\Delta t_i^t = C\frac{\Delta x}{v_i^t + a_i^t}$$

$$\Delta t = \min\left[(\Delta t)_i^t\right] \tag{3.17}$$

C : Courant Number

Till now, the form of Mac-Cormacle has been completed. The rests include boundary and initial conditions.

3.3 Analysis of Entering and Leaving

After the construction of the form of Mac-Cormacle, conditions of entering and leaving should be determined in order to start the calculation by Matlab.
We consider the conditions of subsonic entering and supersonic leaving.

$$Ma = 0.2 \qquad p_{in} = 47892\ N/m \qquad \rho_{in} = 1.2218\ kg/m^3$$

3.3.1 Boundary Conditions

For the first node, the density, temperature, speed are the same with the conditions of entering.

$$\rho_1 = \rho_{in} \qquad T_1 = T_{in} \qquad u_1 = u_{in}$$

For the last node, the density and temperature are defined by the method of **linear interpolation** which is stable in the program of solution.

$$\rho_N = 2\rho_{N-1} - \rho_{N-2} \qquad T_N = 2T_{N-1} - T_{N-2} \qquad u_N = 2u_{N-1} - u_{N-2}$$

3.3.2 Initial Conditions

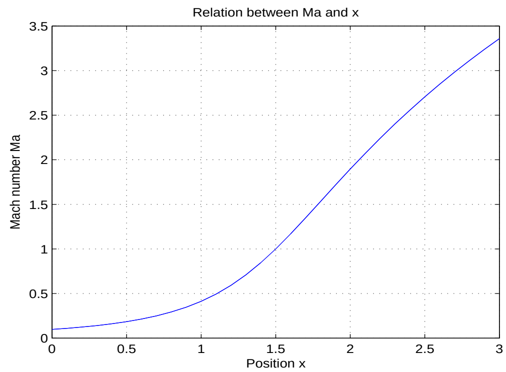
For the initial time, all nodes are in the same conditions which are the entering parameters.

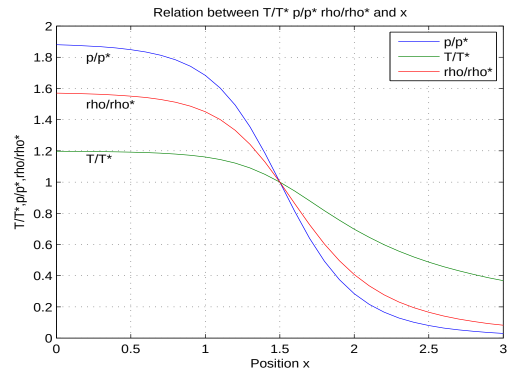
$$At \quad t = 0, \quad \begin{cases} \rho = 1 - 0.314x \\ T = 1 - 0.2314x \\ u = (0.2 + 1.09x)\sqrt{T} \end{cases}$$

4 Calculation and Analysis

4.1 Fundamental Solutions

From the section 2, we obtain the relations between state parameters and position x. Now, we will analyze the solution of these results and graphs.





From the graph, we can conclude that the speed of gas increases rapidly when it pass the throat, and finally reaches $Ma \approx 3.5$ which is a relatively high speed.

In one-dimensional flow, the state parameters $\frac{T}{T^*}$, $\frac{p}{p^*}$, $\frac{\rho}{\rho^*}$ decrease when the gas traverses the nozzle, and reaches the critical condition at throat.

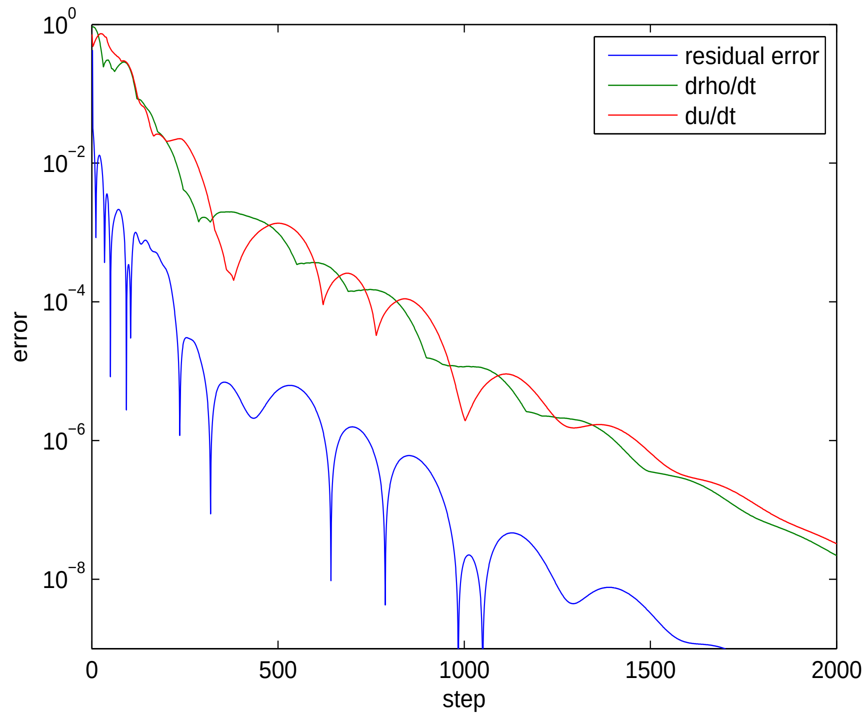
The above flow will not appear directly, because we need give a force difference between the entering and the leaving position. In this problem, the force is given by the pressure. Usually, a high-pressure tank is located at the entering position and a vacuum tank at the leaving position.

4.2 FDM Solutions

In this section, the solution will be mostly presented by the graphs which show the relations among all the parameters.
Before the last two subsections, we use the mesh with $\Delta_x = 0.1$, $step = 2000$ and Courant Number $C = 0.5$.

4.2.1 Figure of Error and Step

Firstly, please look at the figure of residual error. The residual error converge to a small quantity rapidly, which we can conclude that the system is stable and convergent, and the stability of other parameters can be assumed from this conclusion.
The graph also includes another two parameters $(\frac{\partial\rho}{\partial t})_{av}$ and $\left(\frac{\partial u}{\partial t}\right)_{av}$ which can represent the conditions of convergence as well.



We take the time derivatives of speed and density as their functions of time (or the functions of step) to analyze, which can represent the variation of speed and density.

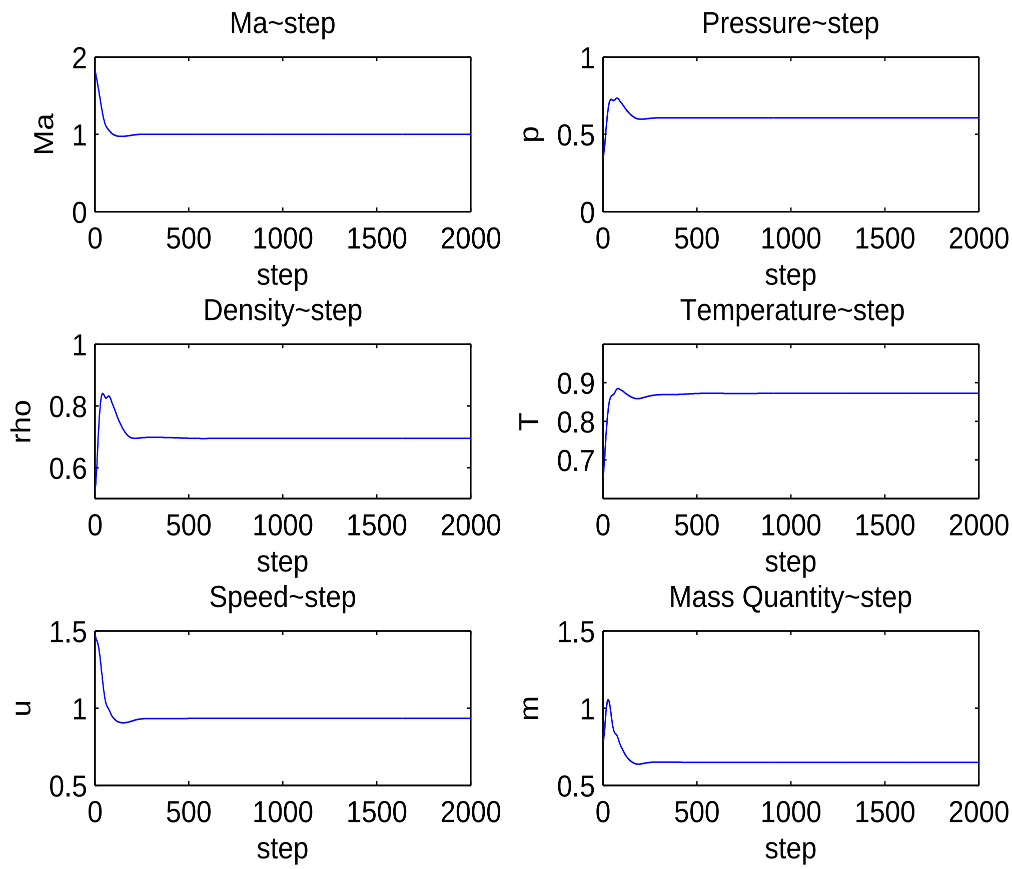
From the above calculation and analysis, we can inform that:

1. At the initial time of calculation, the derivatives are relatively large with oscillation. This oscillation is related with the initial conditions and the unstable waves at initial time.

2. At the later period, the derivatives become small rapidly and finally $residual\ error \leq 10^{-6}$, which is our destination. In this condition, we can regard the system as a stable condition (only in infinity, the derivatives could be 0). Actually, after 1000 steps, the residual error is considerable small which could be seen as no variations.

The precision of MC method is in second order so that the high order will cause more vibrations of the residual error and the next parameters. When we try to compare some methods who is the best, we usually use the residual error as the evaluating parameters. Those who can make the residual error converges rapidly to 0 can be the best method.

4.2.2 Figure of $(Ma, \rho, T, u, p, \dot{m})$ and Step at one point (throat)



This 6 pictures represent the conditions of the parameters **at throat**, with the steps going.

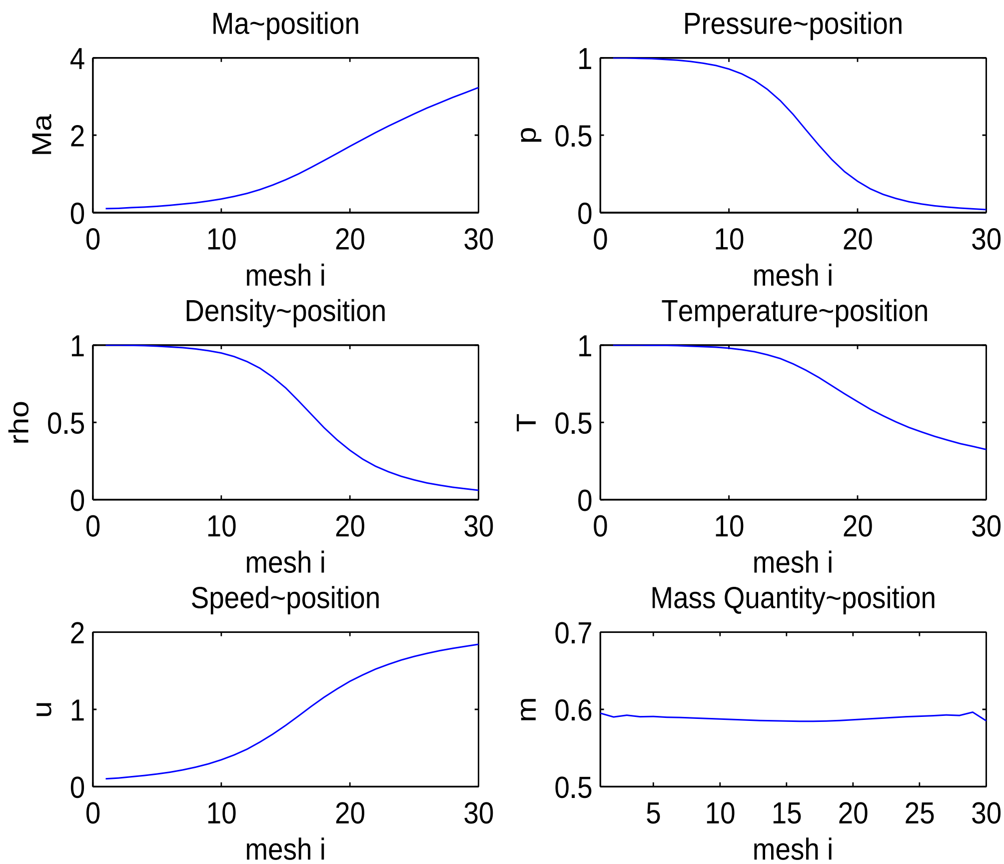
All those parameters turn into stable conditions, which is coherent with the former result of the residual error.

1. From 0 to 100 steps, these parameters have large variations. Actually, it can be easily explained by the former residual error of $(\frac{\partial \rho}{\partial t})_{av}$ and $\left(\frac{\partial u}{\partial t}\right)_{av}$. They all have a great potential power to make them change rapidly.

2. After 500 steps, all these parameters are likely a straight line which is considered as the final exact solutions. From the pictures, we can conclude that these parameters converge to the stable solution.

Finally, we should notice that the viscosity is not considered in this calculation, because there is no need to do that now, and we neglect it.

4.2.3 Figure of $(Ma, \rho, T, u, p, \dot{m})$ and position x (1900 step)

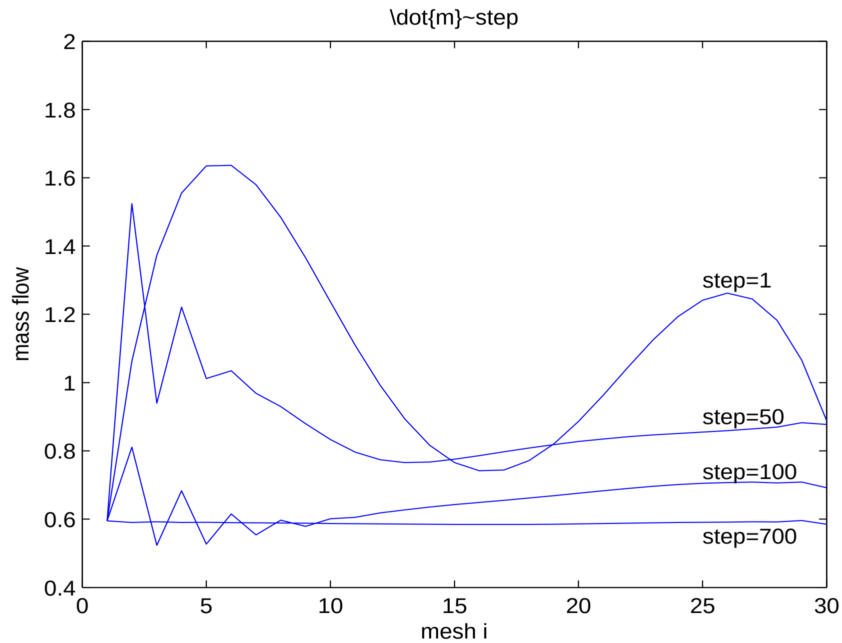


From the graphs above, compared with the analytical solutions, we can conclude that the variation tendency of FDM solutions is coherent with the analytical solutions.
However, some errors exist. We can see some small vibrations at the end of the curves of Ma, p, ρ, u, these vibrations could represent the leaving conditions which is not stable. In this problem, we do not consider the phenomenon of shock wave and expansion wave, which may cause the above vibrations. On the other hand, the mesh is chosen with $\Delta x = 0.1$ which means that the mesh is not dense enough. When we make the mesh denser and more detailed, the vibration area will become small but it will never disappear. In next subsection, the result will be proved by the graphs.

Usually, we do not mention the initial values effects. By the analytical solution, there is not any problem. However in the FDM calculation, we need good and reasonable initial values to make the calculation faster and with less errors. In fact, a good initial condition can reduce much meaningless calculation. If the initial values are far away from the analytical results, the initial derivatives will be large enough. By the experience of experts of CFD, too large gradient will make the calculation unstable and finally obtain a false result.

4.2.4 Figure of Mass Flow $\dot{m}$ and Step

From the above analysis and graphs, we can conclude that all parameters turn to a stable condition after a long calculation of iteration. So, it is evident that the mass flow will tend to stable. The following graph shows that the variation of mass flow and different step.



In this graph, we obtain that the mass flow tends to be stable with the augmentation of step. However, at the first several grids, the mass flow has a large vibration. This error refers to the initial values. Because of the large gradient of first several step of calculation, some sawtooth-like lines appear to the first instability. In fact, it will not affect the following calculation because they will converge into a smooth curve after several iteration. When the step is long enough (after 700 steps), the mass flow tends to a straight line which represent the mass flow is converge to the stable condition.

Compared to the analytical solutions, the FDM method can have a correct answer if the initial values and boundary conditions are suitable.

4.2.5 Effect of Mesh

It is obvious that the density of mesh will affect the speed of calculation, and the detailed mesh can represent much more exactly the parameters of the system. In this problem, the mesh is not very complicate. However, we only consider the flow in the Laval nozzle is one-dimensional and the mesh is one-dimensional as well. In fact, the real flow is 3 dimensional, many conditions are not considered. So, when we choose quite a detailed mesh, the convergent speed will be slower and may cause some points that do not converge.

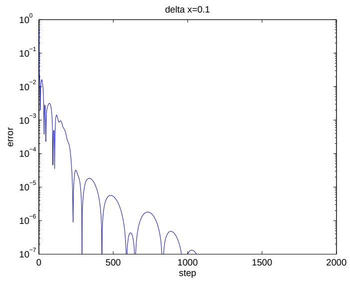
Actually, the density of mesh will affect the precision of the result, but very little. We use the data at throat to prove this assumption.

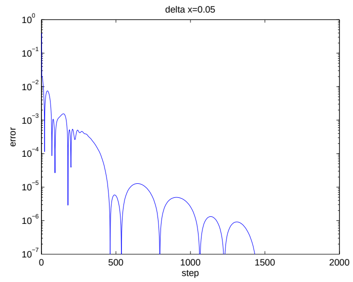
	$\Delta x = 0.1$	$\Delta x = 0.05$	analytical solution
Ma	0.9994	0.9998	1.000
ρ	0.6393	0.6386	0.634
p	0.5349	0.5338	0.528
T	0.8368	0.8359	0.833

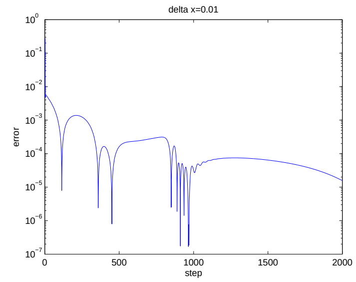
The above table shows that the different density of mesh will not have a great effect to the result of calculation. Although there exist a little differences to the analytical solutions, we can consider they are very close.

So, if we can accept the error of loose mesh, the loose mesh will accelerate the speed of calculation and save much more time. To different problems, we should choose the reasonable mesh.If we want to obtain a more precise result, the mesh should be carefully considered. For example, using

the 2-dimensional or 3-dimensional model and mesh to simulate which can represent better the flow closed to the boundary.
The next graphs represent the residual error in different mesh $\Delta x = 0.1, \Delta x = 0.05, \Delta x = 0.01$.





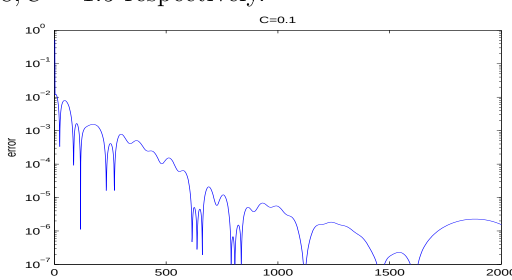


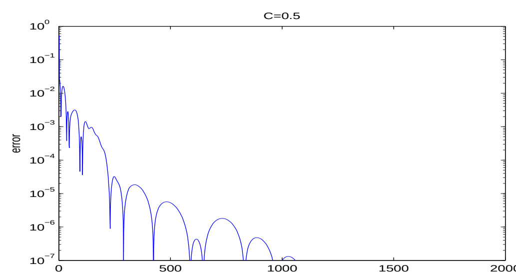
From these two graphs, it is clear that the speed of convergence is related to the density of mesh.

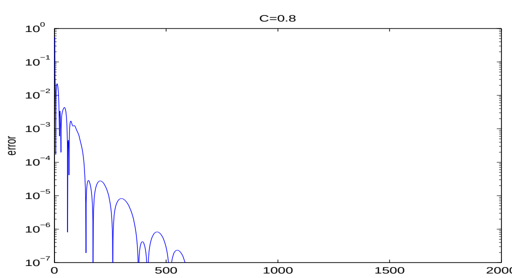
4.2.6 Effect of Courant Number

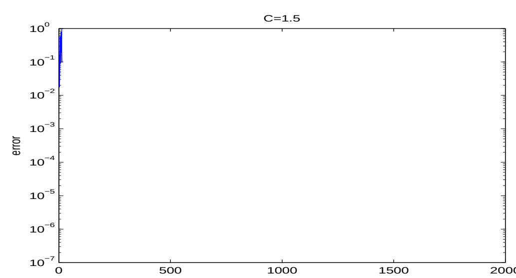
The Courant number is used to control the stability and convergence of calculation. Usually, with the augmentation of Courant number, the speed of convergence will turn faster, but the stability will become weaker. So, for the different problem, when doing the calculation, it is better to start the Courant number from a small initial value. We can have a global view of the convergent conditions of residual error. If the speed of convergence is slow and stable, we could increase the Courant number reasonable. To the definite problem, we need to obtain an appropriate Courant number to make the convergent speed fast enough and keep the stability of the whole system.

The next graphs represent the residual error in different Courant number $C = 0.1, C = 0.5, C = 0.8, C = 1.5$ respectively.









We can conclude from the above graphs that the conditions are coherent with our assumption. When $C < 1$, the speed of convergent turns faster with the increasing of Courant number. Then after $C > 1$, the calculation do not converge and the system cannot be solved. In conclusion, the Courant number reflect the relation between the time step and spatial step. It represent how many grid parameters change within the unit time step.

In this problem, due to the model of **non-conservation form**, we simplify the calculation of derivatives because all derivatives are with one parameters only. However the MC method have to restrict the $C < 1$ to keep the calculation converge.

4.3 Conclusion

- Solutions of the finite difference method are mostly coherent with the analytical solutions. However, some errors exist because of the character of numerical calculation. In fact, the numerical method use the discrete points to simulate the continuous progress.
- If we have time, we should use the advanced softwares such as Fluent and ANSYS to solve the problem directly, which could help us understand whether we obtain a correct result.
- This is a basic and simple project in modeling and simulation which is good for undergraduate students to have an essential view of the numerical methods. It will be beneficial to our following study such as Analysis of Finite Elements etc.

5 Appendix

5.1 Supplements

1. Why we use the method of **Linear Interpolation** to define the boundary conditions?

The method of linear interpolation is wildly used in mathematics and computer calculations. It can be easily considered the variation of parameters refers to be linear. So, the linear interpolation method is well utilized and can be accepted by most kind of numerical calculation. If the variation of parameters is large, the linear interpolation will be less accurate. However, after the plenty of experiments, we can conclude that the boundary should be in a stable condition which is suitable to the usage of linear interpolation method.

2. If initial values can cause great difference in the numeric calculation?

In fact, after obtaining the analytical solutions, we can make a reasonable initial values which are closed to the analytical solutions. In this case, the speed of convergence will be much faster than the unreasonable initiatives and it will save much time and keep a right result in calculation.

3. Is the section area A affect the convergence?

In the first class, we use the section area $A = 1 + 2(x - 4)^2 \quad (0 \leq x \leq 10)$. In fact, due to the dimensionless method, the section area is much larger than 10 when we reach the boundaries $(x = 0,\ x = 10)$. So, the variation of section area is so great that the other parameters' variation is in a small effect. During the initial coding of the former function, the residual error cannot converge, which means that the section area of nozzle have a great effect to the numeric iteration.

5.2 Code

5.2.1 Analytical Solutions

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Section Area A(x) and x
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [ A ] = A_x( x )
x=0:0.1:3;
A=1+2.2*(x-1.5).^2;
plot(x,A);
hold on;
grid on;
title('Relation between Length x and Section Area A(x)');
xlabel('Length x');
ylabel('Section Area A(x)');
axis([0,3,0,6]);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Mach number Ma and x
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc
clear
r=1.4;
for i=1:16
Ma(i)=fzero(@(Ma)((1+0.5*(r-1).*Ma.*Ma)/(0.5*(r+1))).^(0.5*(r+1)/(r-1))...
    ./(1+2.2*((i-1)/10-1.5).^2)-Ma,1);
end
for i=17:31
Ma(i)=fzero(@(Ma)((1+0.5*(r-1).*Ma.*Ma)/(0.5*(r+1))).^(0.5*(r+1)/(r-1))...
    ./(1+2.2*((i-1)/10-1.5).^2)-Ma,3);
end
x=0:0.1:3;
plot(x,Ma);
hold on;
grid on;
title('Relation between Ma and x');
xlabel('Position x');
ylabel('Mach number Ma');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  Temperature T,Pressure p, Density rho and x
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc
clear
r=1.4;
for i=1:16
Ma(i)=fzero(@(Ma)((1+0.5*(r-1).*Ma.*Ma)/(0.5*(r+1))).^(0.5*(r+1)/(r-1))...
    ./(1+2.2*((i-1)/10-1.5).^2)-Ma,1);
end
for i=17:31
Ma(i)=fzero(@(Ma)((1+0.5*(r-1).*Ma.*Ma)/(0.5*(r+1))).^(0.5*(r+1)/(r-1))...
    ./(1+2.2*((i-1)/10-1.5).^2)-Ma,3);
end
x=0:0.1:3;
for i=1:31
```

```
p(i)=(0.5*(r+1)/(1+0.5*(r-1)*Ma(i)^2))^(r/(r-1));
t(i)=0.5*(r+1)/(1+0.5*(r-1)*Ma(i)^2);
rho(i)=(0.5*(r+1)/(1+0.5*(r-1)*Ma(i)^2))^(1/(r-1));
end
plot(x,p,x,t,x,rho);
legend('p/p*','T/T*','rho/rho*');
text(0.2,1.8,'p/p*');
text(0.2,1.15,'T/T*');
text(0.2,1.5,'rho/rho*');
hold on;
grid on;
title('Relation between T/T* p/p* rho/rho* and x');
xlabel('Position x');
ylabel('T/T*,p/p*,rho/rho*');
```

5.2.2 FDM of Mac-Cormack

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% MacCormack Method in Solving Quasi One-Dimensional Flow
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear;
delta_x=0.1;
N=ceil(3/delta_x);
r=1.4;
C=0.5;
step=2000;
%% Bounded Conditions
Ma_in=0.2;
P_in=0.47862;
rho_in=1.2218;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Definition of Variables
A=zeros(1,N);       a=zeros(1,N);        T_f=zeros(step,N);
Ma=zeros(step,N);  deltat=zeros(1,N);  u_f=zeros(step,N);
T=zeros(step,N);   m=zeros(step,N);     error_rho=zeros(1,step);
rho=zeros(step,N);p=zeros(step,N);      error_u=zeros(1,step);
u=zeros(step,N);    rho_f=zeros(step,N);error=zeros(1,step);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Initial Values
for i=1:1:N
    A(i)=2.2*((i-1)*delta_x -1.5)^2+1;                  %% section area
    T(1,i)=1-0.2314*(i-1)*delta_x;                      %% temperature
    rho(1,i)=1-0.314*(i-1)*delta_x;                     %% density
    u(1,i)=(0.2+1.09*(i-1)*delta_x)*sqrt(T(1,i));       %% speed
end
for i=1:1:N                                  %% Courant Number delta_t
    a(i)=sqrt(T(1,i));
    deltat(i)=C.*delta_x/(u(1,i)+a(i));
end
delta_t=min(deltat);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for k=1:step
%% Estimated Step
for i=1:1:N-1   %% Forward Difference and Calculate Spatial Derivatives
    rho_t(i)=-u(1,i).*(rho(1,i+1)-rho(1,i))/delta_x...
             -rho(1,i).*(u(1,i+1)-u(1,i))/delta_x...
             -rho(1,i).*u(1,i).*(log(A(i+1))-log(A(i)))/delta_x;

    u_t(i)=-u(1,i).*(u(1,i+1)-u(1,i))/delta_x...
           -1/r.*((T(1,i+1)-T(1,i))/delta_x...
           +T(1,i)./rho(1,i).*(rho(1,i+1)-rho(1,i))/delta_x);

    T_t(i)=-u(1,i).*(T(1,i+1)-T(1,i))/delta_x...
           -(r-1).*T(1,i).*((u(1,i+1)-u(1,i))/delta_x...
           +u(1,i).*(log(A(i+1))-log(A(i)))/delta_x);
end
%% Estimated Value
for i=1:1:N-1
    rho_v(i)=rho(1,i)+rho_t(i).*delta_t;
    u_v(i)=u(1,i)+u_t(i).*delta_t;
    T_v(i)=T(1,i)+T_t(i).*delta_t;
end
%% Correct Step
for i=2:1:N-1   %% Backward Diffenrence and Calculate Spatial Derivatives
    rho_cs(i)=-u_v(i).*(rho_v(i)-rho_v(i-1))/delta_x...
              -rho_v(i).*(u_v(i)-u_v(i-1))/delta_x...
              -rho_v(i).*u_v(i).*(log(A(i))-log(A(i-1)))/delta_x;

    u_cs(i)=-u_v(i).*(u_v(i)-u_v(i-1))/delta_x...
            -1/r.*((T_v(i)-T_v(i-1))/delta_x...
            +T_v(i)/rho_v(i).*(rho_v(i)-rho_v(i-1))/delta_x);

    T_cs(i)=-u_v(i).*(T_v(i)-T_v(i-1))/delta_x...
            -(r-1).*T_v(i).*((u_v(i)-u_v(i-1))/delta_x...
            +u_v(i).*(log(A(i))-log(A(i-1)))/delta_x);
end

%% Average Derivation of Time
for i=2:1:N-1
    rho_av(i)=0.5*(rho_t(i)+rho_cs(i));
    u_av(i)=0.5*(u_t(i)+u_cs(i));
    T_av(i)=0.5*(T_t(i)+T_cs(i));
end
%% Correct Value
for i=2:1:N-1
    rho(2,i)=rho(1,i)+rho_av(i)*delta_t;
    u(2,i)=u(1,i)+u_av(i)*delta_t;
    T(2,i)=T(1,i)+T_av(i)*delta_t;
end
rho(2,1)=2*rho(2,2)-rho(2,3);        %%
u(2,1)=2*u(2,2)-u(2,3);              %%
T(2,1)=2*T(2,2)-T(2,3);              %%
rho(2,N)=2*rho(2,N-1)-rho(2,N-2);    %% Boundary Values
u(2,N)=2*u(2,N-1)-u(2,N-2);          %% Linear Interpolation
T(2,N)=2*T(2,N-1)-T(2,N-2);          %%

for i=1:N
m(k,i)=rho(1,i)*u(1,i)*A(i);         %% mass quantity
Ma(k,i)=u(1,i)/sqrt(T(1,i));         %% Mach number
p(k,i)=rho(1,i).*T(1,i);             %% pressure
rho_f(k,i)=rho(1,i);                 %% density
```

```
T_f(k,i)=T(1,i);                        %% temperature
u_f(k,i)=u(1,i);                        %% speed
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for i=2:1:N
T(1,i)=T(2,i);
rho(1,i)=rho(2,i);
u(1,i)=u(2,i);
end
%% Residual Error
if k>1
e(k)=rho_f(k,i)/(r-1)+0.5*rho_f(k,i)*u_f(k,i)^2;
error(k)=max(abs(e(k)-e(k-1))); %% residual error
end
error_rho(k)=max(abs(rho_av));  %% residual error of drho/dt
error_u(k)=max(abs(u_av));      %% residual error of du/dt

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Graph
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%% Graph of Residual Error
%{
n=1:step;
semilogy(n,error,n,error_rho,n,error_u);axis([0,2000,10e-10,1]);
xlabel('step');
ylabel('error');
legend('residual error','drho/dt','du/dt');
%}
%% Graph of Ma/p/rho/T/u/m and x on one fixed point(throat)
%{
subplot(3,2,1);plot(Ma(:,1.6/delta_x));xlabel('step'),ylabel('Ma');title('Ma~step');
subplot(3,2,2);plot(p(:,1.6/delta_x));xlabel('step');ylabel('p');title('Pressure~step');
subplot(3,2,3);plot(rho_f(:,1.6/delta_x));xlabel('step');ylabel('rho');title('Density~step');
subplot(3,2,4);plot(T_f(:,1.6/delta_x));xlabel('step');ylabel('T');title('Temperature~step');
subplot(3,2,5);plot(u_f(:,1.6/delta_x));xlabel('step');ylabel('u');title('Speed~step');
subplot(3,2,6);plot(m(:,1.6/delta_x));xlabel('step');ylabel('m');title('Mass Quantity~step');
%}
%% Graph of Ma/p/rho/T/u/m and x at one fixed step
%{
s=1900;                          %% choose the step
subplot(3,2,1);plot(Ma(s,:));xlabel('mesh i');ylabel('Ma');title('Ma~position');
subplot(3,2,2);plot(p(s,:));xlabel('mesh i');ylabel('p');title('Pressure~position');
subplot(3,2,3);plot(rho_f(s,:));xlabel('mesh i');ylabel('rho');title('Density~position');
subplot(3,2,4);plot(T_f(s,:));xlabel('mesh i');ylabel('T');title('Temperature~position');
subplot(3,2,5);plot(u_f(s,:));xlabel('mesh i');ylabel('u');title('Speed~position');
subplot(3,2,6);plot(m(s,:));xlabel('mesh i');ylabel('m');title('Mass Quantity~position');axis([1,30,0,1]);
%grid on;
%}
%{
s=1900;
Ma(s,31)
rho_f(s,31)
p(s,31)
T_f(s,31)
%}
%% m and Step
%{
s=1;
plot(m(s,:));text(25,1.3,'step=1');hold on;
s=50;
plot(m(s,:));text(25,0.9,'step=50');hold on;
s=100;
plot(m(s,:));text(25,0.74,'step=100');hold on;
s=700;
plot(m(s,:));text(25,0.55,'step=700');hold on;
title('\dot{m}~step');
xlabel('mesh i');ylabel('mass flow');
%}
%% Mesh Effect
%{
n=1:step;
semilogy(n,error);xlabel('step');ylabel('error');
title('delta x=0.1');axis([0,step,10e-8,1]);
%}
%% Courant Number Effect
%{
n=1:step;
semilogy(n,error);xlabel('step');ylabel('error');
title('C=0.5');axis([0,step,10e-8,1]);
%}
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References

[1] C.J. Clarke and B. Carswell. Principles of Astrophysical Fluid Dynamics (1st ed.). Cambridge University Press. p. 226. ISBN 978-0-521-85331-6. 2007.

[2] John D. Anderson. Computational Fluid Dynamics. McGraw-Hill Education. 1995.

[3] Zhiliang Lu. Aerodynamics. Beihang University press. 2009.