Quasi One-Dimensional Nozzle Flow

Fluid Mechanics & Finite Difference Method

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Outline

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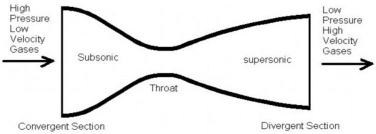
Why we discuss the Laval nozzle

Principle of Laval Nozzle

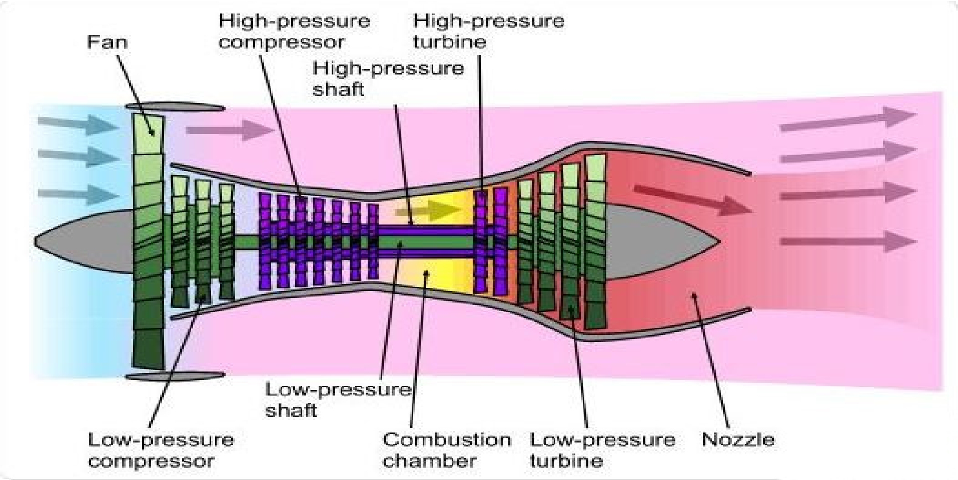
- Composition
  - Convergent Tube
  - Divergent Tube



Principle

- Acceleration & Thrust Augmentation
  - Subsonic Speed (before throat)
  - Transonic Speed (at throat $Ma = 1$)
  - Supersonic Speed (after throat)

Application Field



Two Different method

Analytical Method

- Knowledge of Fluid Mechanics
  - Ma / x relations
  - $\frac{\rho}{\rho^*}, \frac{p}{p^*}, \frac{T}{T^*}$ / x relations
  - $\dot{m}$ / x relations

Finite Difference Method

- Programming
  - Mac-Cormack Method
  - Boundary and Initial Conditions
  - Mesh Δx

Analytical Relations

Definition

- x : position at axis x
- A : section area
- ρ : density of gas
- u : speed of gas
- p : pressure
- e : internal energy
- T : temperature
- V : volume
- $A = 2.2(x - 1.5)^2 + 1$

Analytical Relations

Theorem

$$\frac{\mathrm{d}A}{A} = (Ma^2 - 1)\frac{\mathrm{d}u}{u}$$

Conclusion

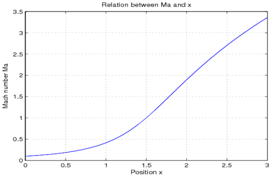
- $Ma > 1$
  $\mathrm{d}A > 0 \ , \ \mathrm{d}u > 0$ *Acceleration*
  $\mathrm{d}A < 0 \ , \ \mathrm{d}u < 0$ *Deceleration*
- $Ma < 1$
  $\mathrm{d}A > 0 \ , \ \mathrm{d}u > 0$ *Deceleration*
  $\mathrm{d}A < 0 \ , \ \mathrm{d}u < 0$ *Acceleration*

Analytical Relations

Theorem

$$\mathrm{d}\left(\frac{\dot{m}}{A}\right) = \mathrm{d}\left(KMa\left(\frac{1}{1+\frac{\gamma-1}{2}Ma^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\right) = 0$$

At throat, the section area A reaches minimum while $\frac{\dot{m}}{A}$ reaches maximum and the Mach number $Ma = 1$.



Analytical Relations

Theorem

Relations between $\frac{A}{A^*}$, $\frac{T}{T^*}$, $\frac{p}{p^*}$, $\frac{\rho}{\rho^*}$ and Ma.

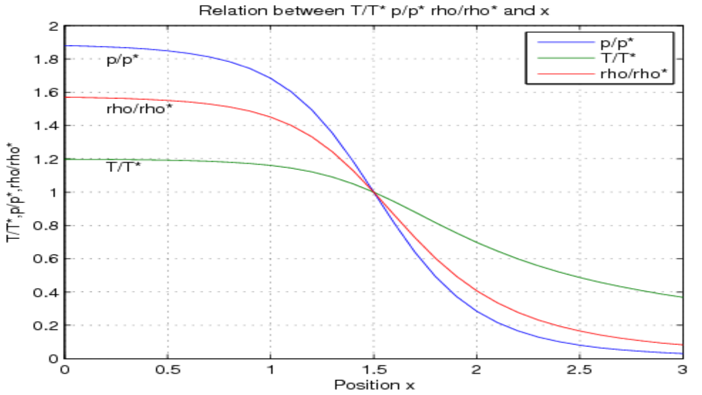
$$\frac{A}{A^*} = \frac{1}{Ma}\left(\frac{1+\frac{\gamma-1}{2}Ma^2}{\frac{\gamma+1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \qquad \frac{T}{T^*} = \frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}Ma^2}$$

$$\frac{p}{p^*} = \left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}Ma^2}\right)^{\frac{\gamma}{\gamma-1}} \qquad \frac{\rho}{\rho^*} = \left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}Ma^2}\right)^{\frac{1}{\gamma-1}}$$

The script $*$ represent the parameters at throat.

Analytical Relations

Conclusion



Mesh

Definition

Number of nodes N.
Unit length $\Delta x = \frac{x_{out} - x_{in}}{N}$.
Position, $\forall\ i \in [1, N], x_i = (i - 1)\Delta x$.

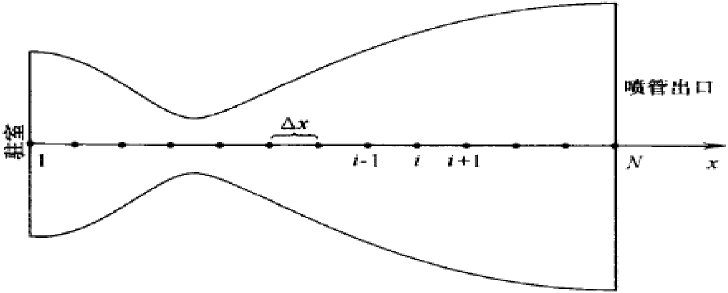


图 7-5　沿喷管的网格点分布示意图

Global Equation

Definition

We suppose to know the parameters at time t, the parameters at $t + \Delta t$ are defined in the following equations:

$$\rho_{i,j}^{t+\Delta t} = \rho_{i,j}^{t} + \left(\frac{\partial \rho}{\partial t}\right)_{av} \cdot \Delta t$$

$$u_{i,j}^{t+\Delta t} = u_{i,j}^{t} + \left(\frac{\partial u}{\partial t}\right)_{av} \cdot \Delta t$$

$$T_{i,j}^{t+\Delta t} = T_{i,j}^{t} + \left(\frac{\partial T}{\partial t}\right)_{av} \cdot \Delta t$$

- We need to confirm the $(\frac{\partial \rho}{\partial t})_{av}, (\frac{\partial u}{\partial t})_{av}, (\frac{\partial T}{\partial t})_{av}$ and determine the unit time Δt.

Estimated Step, Estimated Value, Correct Step.

Definition

Estimated Step $(\frac{\partial\rho}{\partial t})_i^t, (\frac{\partial u}{\partial t})_i^t, (\frac{\partial T}{\partial t})_i^t$.

Definition

Estimated Value $\overline{\rho}_i^t, \overline{u}_i^t, \overline{T}_i^t$.

Definition

Correct Step $\left(\overline{\frac{\partial\rho}{\partial t}}\right)_i^{t+\Delta t}, \left(\overline{\frac{\partial u}{\partial t}}\right)_i^{t+\Delta t}, \left(\overline{\frac{\partial T}{\partial t}}\right)_i^{t+\Delta t}$.

- To know the detailed equations and calculations, please consult the report directly.

Average Derivatives by Time and Unit Time.

Definition

Average Derivatives by Time $(\frac{\partial \rho}{\partial t})_{av}, (\frac{\partial u}{\partial t})_{av}, (\frac{\partial T}{\partial t})_{av}$.

Definition

Unit Time

C : Courant Number

$$\Delta t = C\frac{\Delta x}{|v| + a} \qquad \Delta t = C\frac{\Delta x}{v + a} \qquad \Delta t = \min\left[(\Delta t)_i^t\right]$$

- To know the detailed equations and calculations, please consult the report directly.

Boundary and Initial Conditions

Definition

Boundary Conditions

$$Ma = 0.2 \qquad p_{in} = 47892\ N/m \qquad \rho_{in} = 1.2218\ kg/m^3$$

linear interpolation

$$\rho_N = 2\rho_{N-1} - \rho_{N-2} \qquad T_N = 2T_{N-1} - T_{N-2} \qquad u_N = 2u_{N-1} - u_{N-2}$$

Initial Conditions

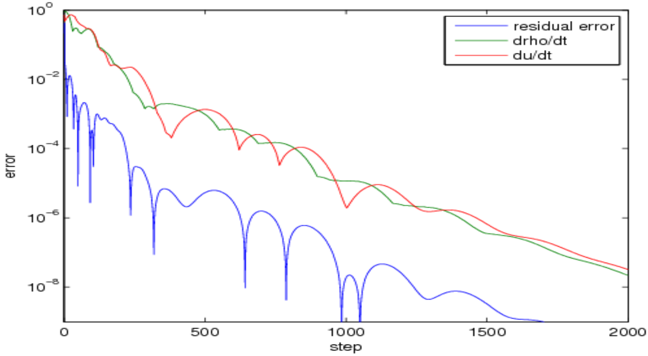
$$At \quad t = 0, \quad \begin{cases} \rho = 1 - 0.314x \\ T = 1 - 0.2314x \\ u = (0.2 + 1.09x)\sqrt{T} \end{cases} \qquad (0 \le x \le 3)$$

Finite Difference Method Solutions

Fixed Conditions:

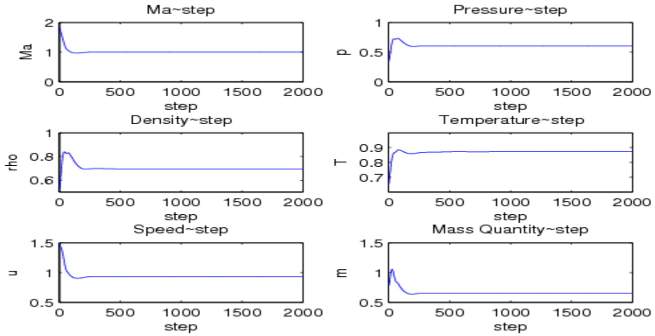
- In this section, the solution will be mostly presented by the graphs which show the relations among all the parameters.
- Before the last two subsections, we use the mesh with $\Delta x = 0.1$, $step = 2000$ and Courant Number $C = 0.5$.

Relations between Residual Error and Step



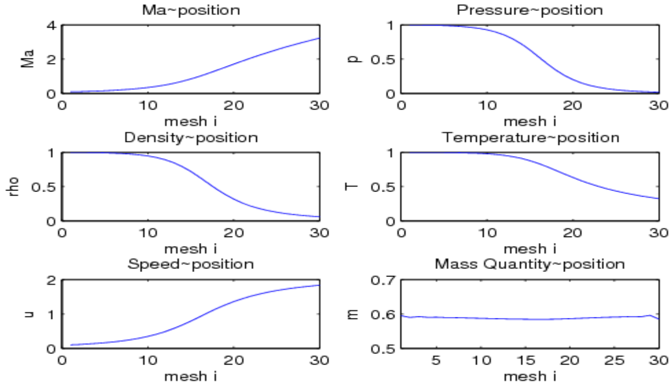
- The graph also includes another two parameters $(\frac{\partial\rho}{\partial t})_{av}$ and $(\frac{\partial u}{\partial t})_{av}$ which can represent the conditions of convergence as well.

Relations between Parameters and Step at Throat



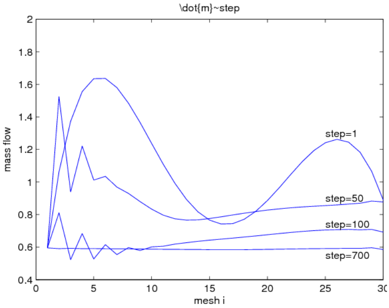
- All those parameters turn into stable conditions, which is coherent with the former result of the residual error.

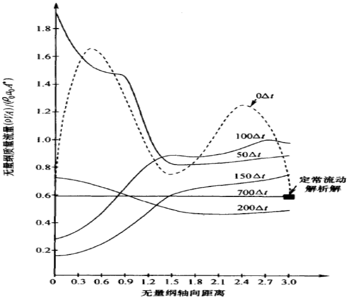
Conditions of Parameters at $step = 1900$



- The conditions of these parameters are coherent with the analytical solutions.

Variations of Mass Flow and Step





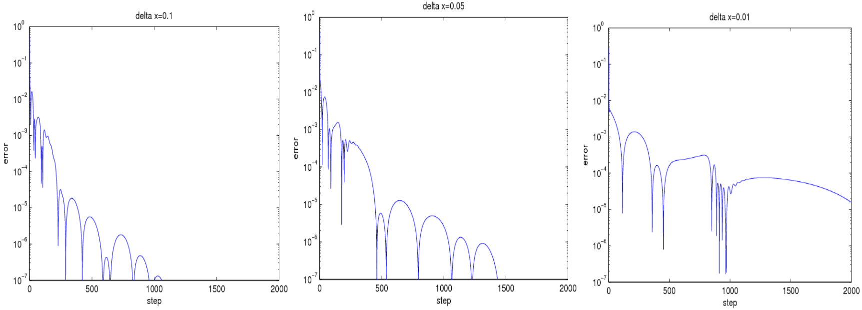
- At first several hundred steps, the mass flow appears a vibration at the entering.
- The vibration tends to become small with the augmentation of steps.

Error Analysis

Vibration exists when the step is small.

- the calculation iteration in first several steps is not stable
- the boundary problem
- the initial values

Effect of Mesh



- A more detailed mesh is chosen, the convergent speed will be slower and may cause some points that do not converge.

Effect of Mesh

	$\Delta x = 0.1$	$\Delta x = 0.05$	analytical solution
Ma	0.9994	0.9998	1.000
ρ	0.6393	0.6386	0.634
p	0.5349	0.5338	0.528
T	0.8368	0.8359	0.833

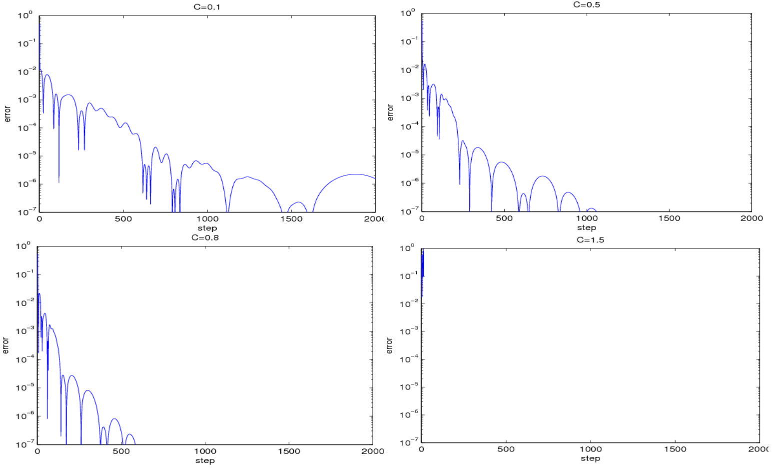
- The above table shows that the different density of mesh increase very little precision to the result of calculation.
- There exist a little differences to the analytical solutions, we can consider they are very close.

Error Analysis

Quasi one-dimensional flow.

- the real flow is three-dimensional
- the mesh should be intensive at the positions with large variations of parameters

Effect of Courant Number



Effect of Courant Number

The Courant number is used to control the stability and convergence of calculation. Usually, with the augmentation of Courant number, the speed of convergence will turn faster, but the stability will become weaker.

- $C < 1$, the speed of convergent turns faster with the increasing of Courant number.
- $C > 1$, the calculation do not converge and the system cannot be solved.

Summary

Conclusion

- *Solutions of the finite difference method are almost coherent with the analytical solutions. However, some errors exist because of the character of numerical calculation. In fact, the numerical method use the discrete points to simulate the continuous progress.*
- *This is a basic and simple project in modeling and simulation which is good for undergraduate students to have an essential view of the numerical methods. It will be beneficial to our following study such as Analysis of Finite Elements etc.*

Outline:

Thanks for your attention!
Q & A