UNIVERSITY COLLEGE, OXFORD

S1: FUNCTIONS OF A COMPLEX VARIABLE

SUGGESTED SOLUTIONS TO SELECTED QUESTIONS

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S1 2019

1. Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$, where $u(x,y)$ and $v(x,y)$ are real functions. If $f(z)$ is analytic, derive the Cauchy-Riemann conditions which u and v must satisfy, and show that the curves of constant $u(x,y)$ are orthogonal to curves of constant $v(x,y)$.

The Cauchy-Riemann conditions are

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}.$$

Show that if $f(z)$ is analytic, then $u(x,y)$ and $v(x,y)$ are harmonic functions.

For $u(x,y)$ and $v(x,y)$ to be harmonic over the entire plane, we require $\nabla^2 u = \nabla^2 v = 0$. Since $f(z)$ is analytic, in addition to the Cauchy-Riemann conditions, Clairaut's theorem must apply to $u(x,y)$ and $v(x,y)$, i.e. their partial derivatives are interchangeable. Evaluating their derivatives and applying the Cauchy-Riemann conditions, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0},$$
$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \boxed{\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}.$$

Therefore, $\boxed{u(x,y) \text{ and } v(x,y) \text{ are harmonic functions}}$.

Prove that $u(x,y) = \mathrm{e}^{-2xy}\sin(x^2 - y^2)$ is a harmonic function, and find the corresponding $v(x,y)$ that makes $f(z)$ analytic. Find an expression for $f(z)$ in terms of $z = x + \mathrm{i}y$.

Differentiating $u(x,y)$ with respect to x, we have

$$\frac{\partial u}{\partial x} = -2y\mathrm{e}^{-2xy}\sin(x^2 - y^2) + 2x\mathrm{e}^{-2xy}\cos(x^2 - y^2),$$
$$\frac{\partial^2 u}{\partial x^2} = -2(2x^2 - 2y^2 - 1)\mathrm{e}^{-2xy}\sin(x^2 - y^2) - 8xy\mathrm{e}^{-2xy}\cos(x^2 - y^2).$$

Next, differentiating $u(x,y)$ with respect to y, we have

$$\frac{\partial u}{\partial y} = -2x\mathrm{e}^{-2xy}\sin(x^2 - y^2) - 2y\mathrm{e}^{-2xy}\cos(x^2 - y^2),$$
$$\frac{\partial^2 u}{\partial y^2} = 2(2x^2 - 2y^2 - 1)\mathrm{e}^{-2xy}\sin(x^2 - y^2) + 8xy\mathrm{e}^{-2xy}\cos(x^2 - y^2).$$

Clearly, $\boxed{\nabla^2 u = 0}$. By inspection, notice that

$$\frac{\partial}{\partial x}\mathrm{e}^{-2xy}\cos(x^2 - y^2) = -2x\mathrm{e}^{-2xy}\cos(x^2 - y^2) + 2y\mathrm{e}^{-2xy}\sin(x^2 - y^2),$$
$$\frac{\partial}{\partial y}\mathrm{e}^{-2xy}\cos(x^2 - y^2) = -2y\mathrm{e}^{-2xy}\cos(x^2 - y^2) - 2x\mathrm{e}^{-2xy}\sin(x^2 - y^2).$$

Thus, from the Cauchy-Riemann conditions, we have

$$v(x,y) = \int -2y\mathrm{e}^{-2xy}\sin(x^2 - y^2) + 2x\mathrm{e}^{-2xy}\cos(x^2 - y^2)\,\mathrm{d}y$$
$$= \boxed{-\mathrm{e}^{-2xy}\cos(x^2 - y^2)}$$

where we have taken the arbitrary integration constant to be zero. Our expression for $f(z)$ is therefore

$$f(z) = \boxed{\mathrm{e}^{-2xy}\sin(x^2 - y^2) - \mathrm{i}\mathrm{e}^{-2xy}\cos(x^2 - y^2)}.$$

For $z = re^{i\theta}$, show that the Cauchy-Riemann conditions can be rewritten as

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Define $x = r\cos\theta$ and $y = r\sin\theta$. To obtain the first Cauchy-Riemann condition, notice that

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\
&= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
&= \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta \\
&= \frac{1}{r}\left(\frac{\partial v}{\partial x} r\sin\theta + \frac{\partial v}{\partial y} r\cos\theta\right) \\
&= \frac{1}{r}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}\right) \\
&= \frac{1}{r}\frac{\partial v}{\partial \theta}.
\end{aligned}$$

As for the other condition, we have

$$\begin{aligned}
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\
&= -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta \\
&= -\frac{\partial v}{\partial y} r\sin\theta - \frac{\partial v}{\partial x} r\cos\theta \\
&= -\frac{\partial v}{\partial x} r\cos\theta - \frac{\partial v}{\partial y} r\sin\theta \\
&= -r\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right) \\
&= -r\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right) \\
&= -r\frac{\partial v}{\partial r}.
\end{aligned}$$

Thus, the Cauchy-Riemann conditions in polar coordinates are

$$\boxed{r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}.$$

Define $g(r,\theta)$ as

$$g(r,\theta) = e^{-\theta}\cos\ln r + ie^{-\theta}\sin\ln r.$$

Show that $g(r,\theta)$ is analytic, and find dg/dz.

Although the intention of the examiner was to make candidates substitute $g(r,\theta)$ into the Cauchy-Riemann conditions in polar coordinates, it is perhaps easier to show that $g(r,\theta) \equiv g(z,\bar{z})$ is independent of $\bar{z}$. Since $\cos\ln r + \sin\ln r = e^{i\ln r}$, therefore $g(r,\theta) = e^{-\theta + i\ln r}$. Furthermore, since $\ln z = \ln r + i\theta$, therefore $i\ln z = -\theta + i\ln r$ and $g(z) = e^{i\ln z}$, $g(z,\bar{z}) \equiv g(z)$ is an entire function is therefore $\boxed{\text{analytic}}$. We then obtain

$$\frac{dg}{dz} = \boxed{\frac{ie^{i\ln z}}{z}}.$$

S1 2019

2. Consider the functions

$$f(z) = (ze^{\frac{1}{z}})^n = z^n \mathrm{e}^{\frac{n}{z}} \quad \text{and} \quad g(z) = f(z) - f(-z) = z^n[\mathrm{e}^{\frac{n}{z}} + (-1)^{n+1}\mathrm{e}^{-\frac{n}{z}}],$$

where $n \in \mathbb{Z}^+$.

Find and classify the singularities of $f(z)$.

Recall that an essential singularity at $z = z_0$ is a singularity at which $f(z)(z-z_0)^k$ is not differentiable $\forall\, k \in \mathbb{Z}$. These singularities are neither poles nor removable singularities. $\boxed{z = 0}$ is an $\boxed{\text{essential singularity}}$. We will show this in the next part of the question.

On the other hand, consider the substitution $\xi = 1/z$, such that $f(z)$ becomes

$$f\left(\frac{1}{\xi}\right) = \frac{\mathrm{e}^{n\xi}}{\xi^n}.$$

Taking the Laurent expansion about $\xi = 0$, we have

$$\begin{aligned} f\left(\frac{1}{\xi}\right) &= \frac{1}{\xi^n}\left(1 + n\xi + ... + \frac{n^{n-1}}{(n-1)!}\xi^{n-1} + \frac{n^n}{n!}\xi^n + ...\right) \\ &= \frac{1}{\xi^n} + \frac{n}{\xi^{n-1}} + ... + \frac{n^n}{n!} + \frac{n^{n+1}}{(n+1)!}\xi + ... \end{aligned}$$

Therefore, $\boxed{z = \infty}$ is an $\boxed{n^{\text{th}}\text{-order pole}}$.

Find the Laurent expansion of $f(z)$ about $z = 0$.

The Laurent expansion of $f(z)$ about the point $z = z_0$ is

$$f(z) = \sum_{k=-N}^{\infty} a_k (z - z_0)^k, \quad a_{-N} \neq 0.$$

In this case, we have

$$\begin{aligned} f(z) &= z^n\left(1 + \frac{n}{z} + \frac{n^2}{2z^2} + ... + \frac{n^n}{n!z^n} + \frac{n^{n+1}}{(n+1)!z^{n+1}} + ...\right) \\ &= \boxed{z^n + nz^{n-1} + \frac{n^2}{2}z^{n-2} + ... + \frac{n^n}{n!} + \frac{n^{n+1}}{(n+1)!z} + ...}. \end{aligned}$$

As expected there are infinitely many negative powers of z in the expansion, proving that $z = 0$ is indeed an essential singularity.

Evaluate the integral of the function $f(z)$ around the circle of centre $z = 0$ and arbitrary radius.

Denote the contour as C_∞, and expand this contour to infinity. Since

$$f(z)\,\mathrm{d}z = -\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right)\mathrm{d}\xi\,,$$

we can obtain the residue at infinity as the coefficient of $1/\xi$ from the expansion

$$\begin{aligned} -\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right) &= -\left(\frac{1}{\xi^{n+2}} + \frac{n}{\xi^{n+1}} + ... + \frac{n^n}{n!\xi^2} + \frac{n^{n+1}}{(n+1)!\xi} + ...\right) \\ \Rightarrow \operatorname*{Res}_{z=\infty} f(z) &= -\frac{n^{n+1}}{(n+1)!}. \end{aligned}$$

Since the interior of a curve $-C_\infty$ is everything outside of C_∞, by Cauchy's residue theorem, we can solve for our contour integral to be

$$\operatorname*{Res}_{z=\infty} f(z) = -\sum_j \operatorname{Res}_j f(z)$$
$$\Rightarrow \oint_{C_\infty} f(z)\,\mathrm{d}z = -2\pi\mathrm{i} \operatorname*{Res}_{z=\infty} f(z) = \boxed{\frac{2\pi\mathrm{i}n^{n+1}}{(n+1)!}}.$$

Alternatively, we can use the Laurent expansion about $z=0$ obtained earlier. The residue at $z=0$ is

$$\operatorname*{Res}_{z=0} f(z) = a_{-1} = \frac{n^{n+1}}{(n+1)!}$$

By Cauchy's residue theorem,

$$\oint_{C_\infty} f(z)\,\mathrm{d}z = \boxed{\frac{2\pi\mathrm{i}n^{n+1}}{(n+1)!}}.$$

Evaluate the integral of the function $g(z)$ around the circle of centre $z=1$ and radius 2.

The positions of the singularities do not change, and the circle still encloses $z=0$. The Laurent expansion of $f(-z)$ about $z=0$ is

$$\begin{aligned} f(-z) &= (-1)^{n+1}z^n\left(1-\frac{n}{z}+\frac{n^2}{2z^2}+...+(-1)^n\frac{n^n}{n!z^n}+(-1)^{n+1}\frac{n^{n+1}}{(n+1)!z^{n+1}}+...\right) \\ &= (-1)^{n+1}z^n+(-1)^{n+2}nz^{n-1}+...+(-1)^{2n+1}\frac{n^n}{n!}+(-1)^{2n+2}\frac{n^{n+1}}{(n+1)!z}+... \\ &= (-1)^{n+1}z^n+(-1)^{n+2}nz^{n-1}+...-\frac{n^n}{n!}+\frac{n^{n+1}}{(n+1)!z}+... \end{aligned}$$

and hence, the residue of $g(z)$ at $z=0$ is

$$\operatorname*{Res}_{z=0} g(z) = \frac{2n^{n+1}}{(n+1)!}.$$

By Cauchy's residue theorem,

$$\oint_{|z-1|=2} g(z)\,\mathrm{d}z = \boxed{\frac{4\pi\mathrm{i}n^{n+1}}{(n+1)!}}.$$

S1 2019

3. State Cauchy's theorem.

Cauchy's theorem states that if $f(z)$ is holomorphic in the simply connected region R enclosed by a closed, non-intersecting, rectifiable contour Γ, then the contour integral of $f(z)$ along Γ when traversed in an anticlockwise manner is

$$\boxed{\oint_\Gamma f(z)\,\mathrm{d}z = 0}.$$

Use contour integration to evaluate

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\,\mathrm{d}\theta\,.$$

Note that the numerator can be expressed as $1 + \cos 6\theta$. Substituting $z = \mathrm{e}^{\mathrm{i}\theta}$, we have

$$\cos 2\theta = \frac{z^2 + z^{-2}}{2} \text{ and } \cos 6\theta = \frac{z^6 + z^{-6}}{2}.$$

The differential of z is $\mathrm{d}z = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}z\,\mathrm{d}\theta$, so $\mathrm{d}\theta = \mathrm{d}z\,/\mathrm{i}z$. We first convert the given integral into a contour integral. Making the above substitution and powering through some laboriously lengthy algebra, we arrive at

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\,\mathrm{d}\theta = \frac{\mathrm{i}}{4}\oint_\Gamma \frac{z^{12} + 2z^6 + 1}{z^5(2z^4 - 5z^2 + 2)}\,\mathrm{d}z\,.$$

Additionally, we notice that the denominator vanishes when $z = 0$, or when $2z^4 - 5z^2 + 2 = 0$. After even more algebra, we obtain the roots of the latter, which are $z = \pm\sqrt{2}$ and $z = \pm 1/\sqrt{2}$. The only poles enclosed by the unit circle are the simple poles at $z = \pm 1/\sqrt{2}$ and the fifth-order pole at $z = 0$. Let

$$f(z) = \frac{z^{12} + 2z^6 + 1}{z^5(2z^4 - 5z^2 + 2)} = \frac{z^{12} + 2z^6 + 1}{2z^5(z + \sqrt{2})\,(z - \sqrt{2})\,\left(z + \frac{1}{\sqrt{2}}\right)(z - \frac{1}{\sqrt{2}})}.$$

The residues at $z = \pm 1/\sqrt{2}$ are

$$\operatorname*{Res}_{z=\frac{1}{\sqrt{2}}} f(z) = \left(z - \frac{1}{\sqrt{2}}\right) f(z)\Big|_{z=\frac{1}{\sqrt{2}}} = -\frac{27}{16}$$

and

$$\operatorname*{Res}_{z=-\frac{1}{\sqrt{2}}} f(z) = \left(z + \frac{1}{\sqrt{2}}\right) f(z)\Big|_{z=-\frac{1}{\sqrt{2}}} = -\frac{27}{16}$$

respectively. As for the residue at $z = 0$, it is best obtained by expanding the denominator of $f(z)$ about $z = 0$, which gives

$$\begin{aligned} f(z) &= \frac{z^{12} + 2z^6 + 1}{2z^5}\left[1 + \left(z^4 - \frac{5}{2}z^2\right)\right]^{-1} \\ &= \frac{z^{12} + 2z^6 + 1}{2z^5}\left[1 + \frac{5}{2}z^2 - z^4 + \frac{25}{4}z^4 + \mathcal{O}(z^6)\right] \\ &= \frac{1}{2z^5} + \frac{5}{4z^3} + \frac{21}{8z} + \mathcal{O}(z) \end{aligned}$$

Thus,

$$\operatorname*{Res}_{z=0} f(z) = \frac{21}{8}.$$

By Cauchy's residue theorem, we thus obtain

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\,\mathrm{d}\theta = \frac{\mathrm{i}}{4}\oint_\Gamma f(z)\,\mathrm{d}z = \frac{\mathrm{i}}{4}(2\pi\mathrm{i})\left(\frac{21}{8} - \frac{27}{16} - \frac{27}{16}\right) = \boxed{\frac{3\pi}{8}}.$$

Find the singularities of

$$f(z) = \frac{z^4}{\sqrt{z^2-1}},$$

and find two possible sets of branch cuts.

There are $\boxed{\text{first-order branch points}}$ located at $\boxed{z = \pm 1}$. The possible branch cuts are $\boxed{[-1, 1]}$ and $\boxed{(-\infty, -1] \cup [1, \infty)}$.

Evaluate the integral

$$\oint_{C_\infty} f(z)\,\mathrm{d}z\,,$$

where C_∞ is a circle of infinite radius.

Substitute $z = 1/\xi$. We perform an expansion in the neighbourhood of $\xi = 0$ to obtain

$$\begin{aligned} f\left(\frac{1}{\xi}\right) &= \frac{1}{\xi^3}(1-\xi^2)^{-\frac{1}{2}} \\ &= \frac{1}{\xi^3} - \frac{1}{2\xi} + \frac{3}{8}\xi + \mathcal{O}(\xi^3) \end{aligned}$$

Since

$$f(z)\,\mathrm{d}z = -\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right)\mathrm{d}\xi\,,$$

we can obtain the residue at infinity as the coefficient of $1/\xi$ from the expansion

$$-\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right) = -\frac{1}{\xi^5} + \frac{1}{2\xi^3} - \frac{3}{8\xi} + \mathcal{O}(1) \Rightarrow \operatorname*{Res}_{z=\infty} f(z) = -\frac{3}{8}.$$

Since the interior of a curve $-\Gamma$ is everything outside of Γ, by Cauchy's residue theorem,

$$\operatorname*{Res}_{z=\infty} f(z) = -\sum_j \mathrm{Res}_j\, f(z)$$

$$\Rightarrow \oint_{C_\infty} f(z)\,\mathrm{d}z = -2\pi\mathrm{i} \operatorname*{Res}_{z=\infty} f(z) = \boxed{\frac{3\pi\mathrm{i}}{4}}.$$

Alternatively, by noting that the result along a contour stretched to infinity is controlled by the large z behavior of the integrand, we can perform an expansion as if there was a pole at origin to obtain

$$f(z) = z^3\left[1 + \frac{1}{2z^2} + \frac{3}{8z^4} + \mathcal{O}\left(\frac{1}{|z|^6}\right)\right] \sim z^3 + \frac{z}{2} + \frac{3}{8z}.$$

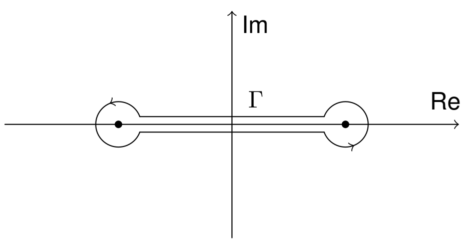
By Cauchy's residue theorem,

$$\oint_{C_\infty} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\left(\frac{3}{8}\right) = \boxed{\frac{3\pi\mathrm{i}}{4}}.$$

Choose a contour that encloses a branch cut to evaluate

$$\int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}}\,\mathrm{d}x\,.$$

We collapse C_∞ into the dogbone contour as shown below, which we call Γ.



To determine the behaviour of $f(z)$ above and below the branch cut, we introduce the substitutions $z-1 = r\mathrm{e}^{\mathrm{i}\theta}$ (vector with head at z and tail at 1) and $z+1 = R\mathrm{e}^{\mathrm{i}\phi}$ (vector with head at z and tail at -1) such that $\sqrt{z^2-1}$ can be expressed in polar form as

$$\sqrt{z^2-1} = \left[rR\mathrm{e}^{\mathrm{i}(\theta+\phi)}\right]^{\frac{1}{2}} = \sqrt{rR}\mathrm{e}^{\mathrm{i}\frac{\theta+\phi}{2}}.$$

We assert that when $z = x + \mathrm{i}y$ lies on the real axis with $x > 1$, $\theta = \phi = 0$. Directly above the branch cut (where $y = 0^+$), $r = |z-1| = 1-x$, $R = |z+1| = 1+x$, $\theta = \arg(z-1) = \pi$ and $\phi = \arg(z+1) = 0$. Thus,

$$\sqrt{z^2-1} = \sqrt{rR}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}\sqrt{1-x^2} \Rightarrow f_{\text{above}} = \frac{x^4}{\mathrm{i}\sqrt{1-x^2}} = -\frac{\mathrm{i}x^4}{\sqrt{1-x^2}}.$$

On the other hand, directly below the branch cut (where $y = 0^-$), $\theta = \pi$ while $\phi = 2\pi$. Thus,

$$\sqrt{z^2-1} = \sqrt{rR}\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}} = -\mathrm{i}\sqrt{1-x^2} \Rightarrow f_{\text{below}} = -\frac{x^4}{\mathrm{i}\sqrt{1-x^2}} = \frac{\mathrm{i}x^4}{\sqrt{1-x^2}}.$$

We shall assume without proof that the integrals along the small deformations tend to zero. Thus, integrating in an anticlockwise fashion, we have

$$\begin{aligned}\oint_\Gamma f(z)\,\mathrm{d}z &= \mathrm{i}\int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}}\,\mathrm{d}x - \mathrm{i}\int_{1}^{-1} \frac{x^4}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &= 2\mathrm{i}\int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{3\pi\mathrm{i}}{4}.\end{aligned}$$

The requested integral is therefore

$$\int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}}\,\mathrm{d}x = \boxed{\frac{3\pi}{8}}.$$

S1 2018

1. Define the concepts of a branch point and a branch cut, illustrating your answer using the function $\sqrt{z}$.

Let z_0 be a point on the complex plane, and define $z - z_0 = r\mathrm{e}^{\mathrm{i}\theta}$. If $z = z_0$ is a branch point of a function $f(z)$, then $f(z_0 + r\mathrm{e}^{\mathrm{i}\theta}) \neq f\left(z_0 + r\mathrm{e}^{\mathrm{i}(\theta+2\pi)}\right)$ i.e. the function is multivalued if that point is encircled. In the case of $f(z) = \sqrt{z}$, notice that $f(r\mathrm{e}^{\mathrm{i}\theta}) = \sqrt{r}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}} \neq f\left(r\mathrm{e}^{\mathrm{i}(\theta+2\pi)}\right) = \sqrt{r}\mathrm{e}^{\mathrm{i}(\frac{\theta}{2}+\pi)} = -\sqrt{r}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}$. This shows the multivalued behaviour of $\sqrt{z}$ around $z = 0$, and hence $\boxed{z = 0 \text{ is a branch point}}$.

Show that

$$f(z) = \sqrt{\frac{z-1}{z+1}}$$

is holomorphic on $\mathbb{C} \setminus [-1, 1]$.

$z = \pm 1$ are first-order branch points. Cauchy-Riemann. Note: $z = \infty$ is not a branch point. To prove this, we can substitute $w = 1/z$ and check if $w = 0$ is a branch point, which it is not.

Find the first two terms in the expansion of $f(z)$ around $z = \infty$ and hence calculate

$$I_\Gamma = \oint_\Gamma f(z)\,\mathrm{d}z\,,$$

where the contour Γ encloses the real line segment $[-1, 1]$. By collapsing Γ onto the cut, use this result to show that

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x = \pi.$$

Substitute $z = 1/\xi$. We perform an expansion in the neighbourhood of $\xi = 0$ to obtain

$$\begin{aligned} f\left(\frac{1}{\xi}\right) &= \sqrt{\frac{1-\xi}{1+\xi}} = (1-\xi)^{\frac{1}{2}}(1+\xi)^{-\frac{1}{2}} \\ &= 1 - \xi + \mathcal{O}(\xi^2) \end{aligned}$$

Since

$$f(z)\,\mathrm{d}z = -\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right)\mathrm{d}\xi\,,$$

we can obtain the residue at infinity as the coefficient of $1/\xi$ from the expansion

$$-\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right) = -\frac{1}{\xi^2} + \frac{1}{\xi} + \mathcal{O}(1) \Rightarrow \boxed{\operatorname*{Res}_{z=\infty} f(z) = 1}.$$

Since the interior of a curve $-\Gamma$ is everything outside of Γ, by Cauchy's residue theorem,

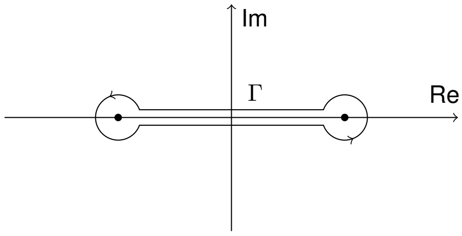
$$\begin{aligned} \operatorname*{Res}_{z=\infty} f(z) &= -\sum_j \mathrm{Res}_j\, f(z) \\ \Rightarrow \oint_\Gamma f(z)\,\mathrm{d}z &= -2\pi\mathrm{i} \operatorname*{Res}_{z=\infty} f(z) = -2\pi\mathrm{i}. \end{aligned}$$

As a sanity check, since $f(z)$ is holomorphic in $\mathbb{C} \setminus [-1, 1]$, the contour integral of $f(z)$ along any contour encircling $[-1, 1]$ gives the same answer. By noting that the result along a contour stretched to infinity is controlled by the large z behavior of the integrand, we can perform an expansion as if there was a pole at the origin to obtain

$$f(z) = \sqrt{1 - \frac{2}{z+1}} \sim \sqrt{1 - \frac{2}{z}} = 1 - \frac{1}{z} + \mathcal{O}\left(\frac{1}{|z|^2}\right) \sim 1 - \frac{1}{z},$$

which gives us the same result.

Next, we collapse Γ into the dogbone contour as shown below.



To determine the behaviour of $f(z)$ above and below the branch cut, we introduce the substitutions $z-1 = re^{i\theta}$ (vector with head at z and tail at 1) and $z+1 = Re^{i\phi}$ (vector with head at z and tail at -1) such that

$$f(z) = \left[\frac{r}{R}e^{i(\theta-\phi)}\right]^{\frac{1}{2}} = \sqrt{\frac{r}{R}}e^{i\frac{\theta-\phi}{2}}.$$

We assert that when $z = x + iy$ lies on the real axis with $x > 1$, $\theta = \phi = 0$. Directly above the branch cut (where $y = 0^+$), $r = |z-1| = 1-x$, $R = |z+1| = 1+x$, $\theta = \arg(z-1) = \pi$ and $\phi = \arg(z+1) = 0$. Thus,

$$f_{\text{above}} = \sqrt{\frac{r}{R}}e^{i\frac{\pi}{2}} = i\sqrt{\frac{1-x}{1+x}}.$$

On the other hand, directly below the branch cut (where $y = 0^-$), $\theta = \pi$ while $\phi = 2\pi$. Thus,

$$f_{\text{below}} = \sqrt{\frac{r}{R}}e^{i\frac{3\pi}{2}} = -i\sqrt{\frac{1-x}{1+x}}.$$

We shall assume without proof that the integrals along the small deformations tend to zero. Thus, integrating in an anticlockwise fashion, we have

$$\begin{aligned}\oint_\Gamma f(z)\,dz &= -i\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,dx + i\int_{1}^{-1}\sqrt{\frac{1-x}{1+x}}\,dx \\ &= -2i\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,dx = -2\pi i.\end{aligned}$$

The requested integral is therefore

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,dx = \boxed{\pi}.$$

By choosing a suitable generalisation of $f(z)$, use the same method to find

$$\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{\frac{1}{n}}dx\,,$$

where $n \in \mathbb{Z}^+$, $n \geq 2$.

Consider

$$f(z) = \left(\frac{z-1}{z+1}\right)^{\frac{1}{n}}.$$

Once again, if we substitute $z = 1/\xi$ and perform an expansion in the neighbourhood of $\xi = 0$, we obtain

$$\begin{aligned}f\left(\frac{1}{\xi}\right) &= (1-\xi)^{\frac{1}{n}}(1+\xi)^{-\frac{1}{n}} \\ &= 1 - \frac{2\xi}{n} + \mathcal{O}(\xi^2) \\ \therefore -\frac{1}{\xi^2}f\left(\frac{1}{\xi}\right) &= -\frac{1}{\xi^2} + \frac{2\xi}{n} + \mathcal{O}(1)\end{aligned}$$

$$\Rightarrow \oint_\Gamma f(z)\,\mathrm{d}z = -2\pi\mathrm{i}\,\operatorname*{Res}_{z=\infty} f(z) = -\frac{4\pi\mathrm{i}}{n}.$$

Likewise for the other method, we have

$$f(z) = \left(1 - \frac{2}{z+1}\right)^{\frac{1}{n}} \sim \left(1 - \frac{2}{z}\right)^{\frac{1}{n}} = 1 - \frac{2}{nz} + \mathcal{O}\left(\frac{1}{|z|^2}\right) \sim 1 - \frac{2}{nz},$$

which gives us the same result. The behaviour of the generalised $f(z)$ around the branch cut and the way we perform the contour integral is the same as before. Therefore,

$$\oint_\Gamma f(z)\,\mathrm{d}z = -2\mathrm{i}\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{\frac{1}{n}}\mathrm{d}x = -\frac{4\pi\mathrm{i}}{n},$$

and we obtain the desired integral

$$\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{\frac{1}{n}}\mathrm{d}x = \boxed{\frac{2\pi}{n}}.$$

S1 2018

2. Using the methods of contour integration, compute

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x$$

and

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{\cos x}{x}\,\mathrm{d}x$$

where $\mathcal{P}$ denotes the principal value.

Recall Jordan's lemma, which states that if $f(z)$ is analytic in the upper half-plane (apart from a finite number of isolated singularities), C_R is the upper semicircular contour of radius R, and

$$\lim_{R\to\infty} \sup_{|z|=R} |f(z)| = 0,$$

then

$$\lim_{R\to\infty} \int_{C_R} f(z)\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}z = 0,$$

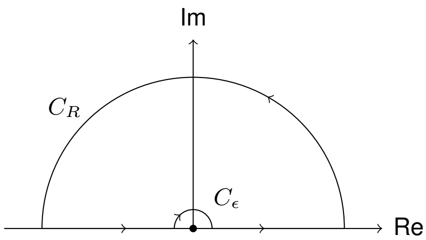
if $k > 0$. Notice that

$$f(z) = \frac{1}{z}$$

satisfies the required conditions to invoke Jordan's lemma. We consider the contour integral

$$\oint_\Gamma \frac{\mathrm{e}^{\mathrm{i}kz}}{z}\,\mathrm{d}z\,.$$

The integrand has a singularity at the origin. Let the contour which we integrate over be $\Gamma = C_R \cup [-R,-\epsilon] \cup C_\epsilon \cup [\epsilon, R]$ as shown below, where C_R is the upper semicircular contour of radius R and C_ϵ is an upper semicircular deformation away from the origin of radius ϵ.



Our choice of Γ allows us to avoid the singularity at the origin. The integral can be split into

$$\oint_\Gamma \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z = \int_{-R}^{-\epsilon} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x + \int_{\epsilon}^{R} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x + \int_{C_R} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z + \int_{C_\epsilon} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z\,.$$

In the limit where $R \to \infty$ and $\epsilon \to 0$, the first two integrals are what we need, while the third integral goes to zero via Jordan's lemma. For the last integral, we prove and invoke the following lemma to help us evaluate the contribution along C_ϵ.

Lemma: Let $f(z)$ be holomorphic on the open disc $\mathbb{D}(z_0, R) = \{z : |z - z_0| < R\} \setminus \{z_0\}$ with a simple pole at $z = z_0$. For $0 < \epsilon < R$, let $C_\epsilon : [\theta_1, \theta_2] \to \mathbb{C}$ be the path $\theta \mapsto z_0 + \epsilon \mathrm{e}^{\mathrm{i}\theta}$ (i.e. the circular arc of radius ϵ centred on $z = z_0$ subtending θ_1 to θ_2). Then,

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z)\,\mathrm{d}z = \mathrm{i}(\theta_2 - \theta_1) \operatorname*{Res}_{z=z_0} f(z).$$

Proof: $f(z)$ can be expressed as

$$f(z) = \frac{\operatorname*{Res}_{z=z_0} f(z)}{z - z_0} + g(z),$$

where $g(z)$ is holomorphic on $\mathbb{D}(z_0, R)$. Then, by the estimation lemma, which states that

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \leq \int_\Gamma |f(z)||\,\mathrm{d}z\,| \leq \sup_{z\in\Gamma}|f(z)| \int_\Gamma |\,\mathrm{d}z\,|,$$

we have

$$\int_{C_\epsilon} f(z)\,\mathrm{d}z \leq \epsilon(\theta_2-\theta_1) \sup_{z\in C_\epsilon} |g(z)|\,.$$

Since $g(z)$ is bounded on $\mathbb{D}(z_0, R)$, thus

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} f(z)\,\mathrm{d}z = 0,$$

and thus

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{\operatorname*{Res}_{z=z_0} f(z)}{z-z_0}\,\mathrm{d}z &= \operatorname*{Res}_{z=z_0} f(z)\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{\mathrm{d}z}{z-z_0}\\ &= \operatorname*{Res}_{z=z_0} f(z)\lim_{\epsilon\to 0}\int_{\theta_1}^{\theta_2}\frac{\mathrm{i}\epsilon\mathrm{e}^{\mathrm{i}\theta}}{\epsilon\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{d}\theta\\ &= \mathrm{i}(\theta_2-\theta_1)\operatorname*{Res}_{z=z_0} f(z).\end{aligned}$$

In this case, $\theta_1 = \pi$ and $\theta_2 = 0$, so

$$\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z = -\mathrm{i}\pi\operatorname*{Res}_{z=0}\frac{\mathrm{e}^{\mathrm{i}kz}}{z}.$$

The origin is a simple pole, so

$$\operatorname*{Res}_{z=0}\frac{\mathrm{e}^{\mathrm{i}kz}}{z} = \lim_{z\to 0}\mathrm{e}^{\mathrm{i}kz} = 1.$$

By Cauchy's residue theorem, as no poles are enclosed, the contour integral must be equal to zero. Therefore, in the limits $R\to\infty$ and $\epsilon\to 0$, we have

$$\oint_\Gamma \frac{\mathrm{e}^{\mathrm{i}kz}}{z}\,\mathrm{d}z = \int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kx}}{x}\,\mathrm{d}x - \mathrm{i}\pi = 0,$$

and

$$\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kx}}{x}\,\mathrm{d}x = \mathrm{i}\pi.$$

Therefore,

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,\mathrm{d}x = \mathrm{Im}\left\{\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kx}}{x}\,\mathrm{d}x\right\} = \boxed{\pi},$$

and likewise

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{\cos x}{x}\,\mathrm{d}x = \mathrm{Re}\left\{\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kx}}{x}\,\mathrm{d}x\right\} = \boxed{0}.$$

Compute

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{13+12\sin\theta}.$$

Consider

$$f(z) = \left(\frac{z-1}{z+1}\right)^{\frac{1}{n}}.$$

Once again, if we substitute $z = 1/\xi$ and perform an expansion in the neighbourhood of $\xi = 0$, we obtain

$$\begin{aligned}f\left(\frac{1}{\xi}\right) &= (1-\xi)^{\frac{1}{n}}(1+\xi)^{-\frac{1}{n}}\\ &= 1-\frac{2\xi}{n}+\mathcal{O}(\xi^2)\end{aligned}$$

$$\begin{aligned} \therefore -\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right) &= -\frac{1}{\xi^2} + \frac{2\xi}{n} + \mathcal{O}(1) \\ \Rightarrow \oint_\Gamma f(z)\,\mathrm{d}z &= -2\pi\mathrm{i} \operatorname*{Res}_{z=\infty} f(z) = -\frac{4\pi\mathrm{i}}{n}. \end{aligned}$$

Likewise for the other method, we have

$$f(z) = \left(1 - \frac{2}{z+1}\right)^{\frac{1}{n}} \sim \left(1 - \frac{2}{z}\right)^{\frac{1}{n}} = 1 - \frac{2}{nz} + \mathcal{O}\left(\frac{1}{|z|^2}\right) \sim 1 - \frac{2}{nz},$$

which gives us the same result. The behaviour of the generalised $f(z)$ around the branch cut and the way we perform the contour integral is the same as before. Therefore,

$$\oint_\Gamma f(z)\,\mathrm{d}z = -2\mathrm{i} \int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^{\frac{1}{n}} \mathrm{d}x = -\frac{4\pi\mathrm{i}}{n},$$

and we obtain the desired integral

$$\int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^{\frac{1}{n}} \mathrm{d}x = \boxed{\frac{2\pi}{n}}.$$

S1 2018

3. Show that the map $f : z \mapsto w$, where $f(z)$ is holomorphic, is locally angle preserving (conformal) except at those points for which $f'(z) = 0$. Find how the map affects angles at z_0 if $f'(z_0) = 0$ but $f''(z_0) \neq 0$.

Consider

$$\delta w = \frac{\mathrm{d}w}{\mathrm{d}z}\delta z = \left(1 - \frac{1}{z^2}\right)\delta z.$$

The condition for f to be a conformal map is therefore

$$\boxed{\arg\frac{\delta w_1}{\delta w_2} = \arg\frac{\delta z_1}{\delta z_2}}.$$

f is conformal in the region of holomorphy precisely where its derivative is non-zero. $f'(z) = 0$ at $z = \pm 1$, and therefore f conformal everywhere in $\mathbb{C}\backslash\{0 \cup \infty\}$ except at $\boxed{z = \pm 1}$. As for a point z_0 wjere $f'(z_0) = 0$ but $f''(z_0) \neq 0$, we have to consider higher order expansions of δw. To $\mathcal{O}(\delta z^2)$,

$$\begin{aligned}\delta w &= \frac{\mathrm{d}w}{\mathrm{d}z}\delta z + \frac{1}{2!}\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\delta z^2 \\ &= \left(1 - \frac{1}{z^2}\right)\delta z + \frac{1}{z^3}\delta z^2.\end{aligned}$$

At $z = 1$, $\delta w = \delta z^2$. Defining $\theta = \arg\,(\delta z_1/\delta z_2)$ and $\phi = \arg\,(\delta w_1/\delta w_2)$, we have

$$\frac{\delta w_1}{\delta w_2} = \frac{\delta z_1^2}{\delta z_2^2} \Rightarrow \arg\frac{\delta w_1}{\delta w_2} = \arg\frac{\delta z_1^2}{\delta z_2^2} = 2\,\arg\frac{\delta z_1}{\delta z_2} \Rightarrow \boxed{\phi = 2\theta}.$$

Show that for

$$f(z) = \sin\frac{\pi z}{2},$$

the semi-infinite strip $-1 \leq \mathrm{Re}\{z\} \leq 1$, $\mathrm{Im}\{z\} \geq 0$ is mapped to the upper half of the w-plane. What is the image in the w-plane of the line segment $-1 \leq \mathrm{Re}\{z\} \leq 1$, $\mathrm{Im}\{z\} = a$, where $a > 0$ is a constant?

Consider

$$f(z) = \left(\frac{z-1}{z+1}\right)^{\frac{1}{n}}.$$

Once again, if we substitute $z = 1/\xi$ and perform an expansion in the neighbourhood of $\xi = 0$, we obtain

$$\begin{aligned} f\left(\frac{1}{\xi}\right) &= (1-\xi)^{\frac{1}{n}}(1+\xi)^{-\frac{1}{n}} \\ &= 1 - \frac{2\xi}{n} + \mathcal{O}(\xi^2) \\ \therefore -\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right) &= -\frac{1}{\xi^2} + \frac{2\xi}{n} + \mathcal{O}(1) \\ \Rightarrow \oint_\Gamma f(z)\,\mathrm{d}z &= -2\pi\mathrm{i}\,\underset{z=\infty}{\mathrm{Res}}\,f(z) = -\frac{4\pi\mathrm{i}}{n}. \end{aligned}$$

Likewise for the other method, we have

$$f(z) = \left(1 - \frac{2}{z+1}\right)^{\frac{1}{n}} \sim \left(1 - \frac{2}{z}\right)^{\frac{1}{n}} = 1 - \frac{2}{nz} + \mathcal{O}\left(\frac{1}{|z|^2}\right) \sim 1 - \frac{2}{nz},$$

which gives us the same result. The behaviour of the generalised $f(z)$ around the branch cut and the way we perform the contour integral is the same as before. Therefore,

$$\oint_\Gamma f(z)\,\mathrm{d}z = -2\mathrm{i}\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{\frac{1}{n}}\mathrm{d}x = -\frac{4\pi\mathrm{i}}{n},$$

and we obtain the desired integral

$$\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{\frac{1}{n}}\mathrm{d}x=\boxed{\frac{2\pi}{n}}.$$

Find the inverse map $f^{-1}(w)$ and show that it is not holomorphic at those points where the original map is not conformal. By considering the behaviour of angles under the map, explain why this is necessarily the case.

Find the inverse map $f^{-1}(w)$ and show that it is not holomorphic at those points where the original map is not conformal. By considering the behaviour of angles under the map, explain why this is necessarily the case.

S1 2017

1. A holomorphic function $f(z)$ may be written in terms of its real and imaginary parts as

$$f(z) = u(x, y) + \mathrm{i}v(x, y).$$

State the Cauchy-Riemann conditions satisfied by $u(x, y)$ and $v(x, y)$ and show that they imply that both $u(x, y)$ and $v(x, y)$ are harmonic functions.

The Cauchy-Riemann conditions are

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}.$$

For $u(x, y)$ and $v(x, y)$ to be harmonic over the entire plane, we require $\nabla^2 u = \nabla^2 v = 0$. Clairaut's theorem must apply to $u(x, y)$ and $v(x, y)$, i.e. their partial derivatives are interchangeable. Evaluating their derivatives, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0},$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \boxed{\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}.$$

Therefore, $\boxed{u(x, y) \text{ and } v(x, y) \text{ are harmonic functions}}$.

Show explicitly that

$$u(x, y) = \frac{1}{2} x \ln(x^2 + y^2) - y \tan^{-1} \frac{y}{x}$$

is harmonic.

Differentiating $u(x, y)$ with respect to x, we have

$$\frac{\partial u}{\partial x} = \frac{1}{2} \ln(x^2 + y^2) + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} = 1 + \frac{1}{2} \ln(x^2 + y^2).$$

Next, differentiating $u(x, y)$ with respect to y, we have

$$\frac{\partial u}{\partial y} = \frac{xy}{x^2 + y^2} - \tan^{-1} \frac{y}{x} - \frac{xy}{x^2 + y^2} = -\tan^{-1} \frac{y}{x}.$$

We will need these in later parts. Differentiating the above once more, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{x}{x^2 + y^2},$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{x}{x^2 + y^2}.$$

Clearly, $\boxed{\nabla^2 u = 0}$.

Find a holomorphic function $f(z)$ such that $u = \mathrm{Re}\{f\}$ and determine $v = \mathrm{Im}\{f\}$.

From the first Cauchy-Riemann condition, we have

$$\begin{aligned} v(x, y) &= \int \frac{\partial u}{\partial x} \, \mathrm{d}y \\ &= \int 1 + \frac{1}{2} \ln(x^2 + y^2) \, \mathrm{d}y \\ &= y + \frac{1}{2} y \ln(x^2 + y^2) - \int \frac{y^2}{x^2 + y^2} \, \mathrm{d}y \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}y\ln(x^2+y^2) + \int \frac{x^2}{x^2+y^2}\,\mathrm{d}y \\ &= \frac{1}{2}y\ln(x^2+y^2) + x\tan^{-1}\frac{y}{x} + g(x) \end{aligned}$$

where $g(x)$ is purely a function of x. On the other hand, differentiating $-v(x,y)$ with respect to x and comparing it with $\partial u/\partial y$ the second Cauchy-Riemann condition, we have

$$-\frac{\partial v}{\partial x} = -\left(\frac{xy}{x^2+y^2} + \tan^{-1}\frac{y}{x} - \frac{xy}{x^2+y^2} + \frac{\mathrm{d}g}{\mathrm{d}x}\right) = -\tan^{-1}\frac{y}{x} - \frac{\mathrm{d}g}{\mathrm{d}x}.$$

Thus, $g(x) = c$, where c is a constant which we can take to be zero. Putting everything together,

$$\begin{aligned} v(x,y) &= \boxed{\frac{1}{2}y\ln(x^2+y^2) + x\tan^{-1}\frac{y}{x}}, \\ f(z) &= u(x,y) + \mathrm{i}v(x,y) \\ &= \frac{1}{2}(x+\mathrm{i}y)\ln(x^2+y^2) + \mathrm{i}(x+\mathrm{i}y)\tan^{-1}\frac{y}{x} \\ &= \boxed{z\ln|z|^2 + \mathrm{i}z\arg z}. \end{aligned}$$

Show that the potential

$$V(x,y) = \frac{V_0}{\pi}\left(\frac{1}{2}y\ln(x^2+y^2) + x\tan^{-1}\frac{y}{x}\right)$$

solves the equations for electrostatics in the absence of free charges and with boundary conditions

$$V(x,0) = \begin{cases} V_0 x & x \le 0, \\ 0 & x > 0. \end{cases}$$

We know from Maxwell's equations that $-\nabla V = \mathbf{E}$ and $\nabla^2 V = 0$ in the absence of free charges. Notice that $V(x,y) = (V_0/\pi)v(x,y)$. Since $v(x,y)$ is harmonic as shown in the first part, therefore $V(x,y)$ must satisfy $\boxed{\nabla^2 V = 0}$. Next, recall that $\tan k\pi = 0$ for $k \in \mathbb{N}$. Thus, in the $y = 0$ plane,

$$V(x,0) = \frac{V_0}{\pi}(k\pi x) = kV_0 x, \ \ k \in \mathbb{N}.$$

Choosing $\boxed{k = 1 \text{ for } x \le 0}$ and $\boxed{k = 0 \text{ for } x > 0}$ satisfies the desired boundary conditions.

Show that the gradient of an equipotential line $y(x)$ satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\theta}{1+\ln r},$$

where $x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$. For what ranges of (r,θ) is the gradient respectively positive, infinite, and negative? Sketch the equipotential lines.

The equipotential lines are defined by $V(x,y) =$ constant, which is equivalent to $v(x,y) =$ constant. The tangent vector $\hat{\mathbf{t}}_v$ to these curves are

$$\hat{\mathbf{t}}_v = \frac{1}{\mathrm{d}s}\begin{pmatrix}\mathrm{d}x \\ \mathrm{d}y\end{pmatrix} = \begin{pmatrix}\partial v/\partial y \\ -\partial v/\partial x\end{pmatrix} = \begin{pmatrix}\partial u/\partial x \\ \partial u/\partial y\end{pmatrix} = \hat{\mathbf{n}}_u$$

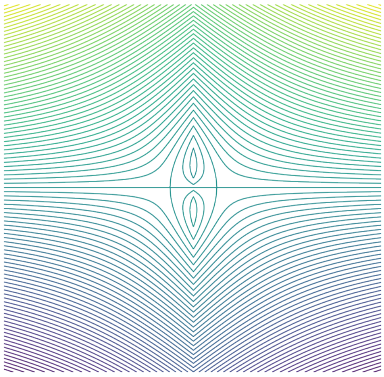
where s is the curvilinear coordinate along the curve. The gradient of these lines therefore satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\tan^{-1}(y/x)}{1+\ln(x^2+y^2)}$$

which in polar coordinates is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \boxed{\frac{-\theta}{1+\ln r}}.$$

Note that $1 + \ln r$ is positive for $r > \mathrm{e}^{-1}$. and The equipotential lines are shown below.



S1 2017

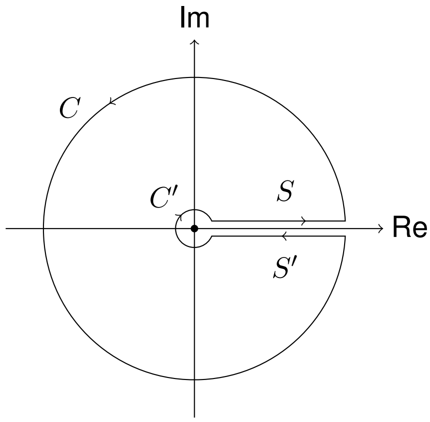
2. What is meant by a branch point and a branch cut? Illustrate your answer using the function $\ln z$.

Let z_0 be a point on the complex plane, and define $z - z_0 = re^{i\theta}$. If z_0 is a branch point of a function $f(z)$, then $f(z_0 + re^{i\theta}) \neq f(z_0 + re^{i(\theta+2\pi)})$ i.e. the function is multivalued if that point is encircled. In the case of $f(z) = \ln z$, notice that $f(re^{i\theta}) = \ln r + i\theta \neq f\left(re^{i(\theta+2\pi)}\right) = \ln r + i(\theta + 2\pi)$. This shows the multivalued behaviour of $\sqrt{z}$ around $z = 0$, and hence $\boxed{z = 0 \text{ is a branch point}}$.

Show that

$$\oint_\Gamma \frac{\ln^n z}{z^2+1}\,\mathrm{d}z = \pi^{n+1}\left(\frac{i}{2}\right)^n (1 - 3^n) \qquad (\star)$$

where $n \in \mathbb{Z}^+$ and the keyhole contour $\Gamma = S \cup C \cup S' \cup C'$ in the plane is cut along the positive real axis as shown in the figure.



We define $f(z)$ to be the integrand of equation $(\star)$ and restrict the argument of z to $\arg(z) \in (0, 2\pi)$. $f(z)$ has simple poles at $z = \pm i$ enclosed within Γ (as well as ∞-order branch points at $z = 0$ and $z = \infty$). The residues at $z = \pm i$ are

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i}(z - i)f(z) = \frac{\ln^n i}{2i} = \frac{\pi^n i^{n-1}}{2^{n+1}},$$

$$\operatorname*{Res}_{z=-i} f(z) = \lim_{z\to -i}(z + i)f(z) = -\frac{\ln^n(-i)}{2i} = -\frac{(3\pi)^n i^{n-1}}{2^{n+1}}.$$

By Cauchy's residue theorem,

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_j \operatorname{Res}_j f(z) = 2\pi i \left[\frac{\pi^n i^{n-1}}{2^{n+1}} - \frac{(3\pi)^n i^{n-1}}{2^{n+1}}\right]$$

$$= \boxed{\pi^{n+1}\left(\frac{i}{2}\right)^n (1 - 3^n)}.$$

Defining

$$I_n = \int_0^\infty \frac{\ln^n x}{x^2+1}\,\mathrm{d}x\,,$$

use $(\star)$ with $n = 2$ to show that

$$I_0 = \frac{\pi}{2} \quad \text{and} \quad I_1 = 0.$$

To determine the behaviour of $f(z)$ above and below the branch cut, we introduce the substitution $z = re^{i\theta}$ (vector with head at z and tail at 0), where $r = |z|$ and $\theta = \arg(z)$. We also assert that when $z = x + \mathrm{i}y$ lies on the real axis with $x > 0$, $\theta = 0$. Directly above the branch cut (where $y = 0^+$), $r = |z| = x$ and $\theta = 0$, so

$$f_{\text{above}} = \frac{\ln^n r}{z^2+1} = \frac{\ln^n x}{x^2+1}.$$

Likewise, directly below the branch cut (where $y = 0^-$), $r = |z| = x$ and $\theta = 2\pi$, so

$$f_{\text{below}} = \frac{\ln^n re^{2\mathrm{i}\pi}}{z^2+1} = \frac{(\ln x + 2\pi\mathrm{i})^n}{x^2+1}.$$

We shall assume without proof that the contributions due to C and C' tend to zero as $R \to \infty$ and $\epsilon \to \infty$. For $n = 2$, the contour integral becomes

$$\begin{aligned}\oint_\Gamma \frac{\ln^2 z}{z^2+1}\,\mathrm{d}z &= \int_0^\infty \frac{\ln^2 x}{x^2+1}\,\mathrm{d}x + \int_\infty^0 \frac{(\ln x + 2\pi\mathrm{i})^2}{x^2+1}\,\mathrm{d}x \\ &= \int_0^\infty \frac{\ln^2 x}{x^2+1}\,\mathrm{d}x - \int_0^\infty \frac{\ln^2 x}{x^2+1}\,\mathrm{d}x - 4\pi\mathrm{i}\int_0^\infty \frac{\ln x}{x^2+1}\,\mathrm{d}x + 4\pi^2\int_0^\infty \frac{1}{x^2+1}\,\mathrm{d}x \\ &= -4\pi\mathrm{i}\int_0^\infty \frac{\ln x}{x^2+1}\,\mathrm{d}x + 4\pi^2\int_0^\infty \frac{1}{x^2+1}\,\mathrm{d}x = 2\pi^3.\end{aligned}$$

Equating the real and imaginary parts, we can immediately conclude that $I_0 = \boxed{\pi/2}$ and $I_1 = \boxed{0}$.

Find I_2 and I_4.

To obtain I_2, we must evaluate the contour integral for $n = 3$.

$$\oint_\Gamma \frac{\ln^3 z}{z^2+1}\,\mathrm{d}z = -6\pi\mathrm{i}I_2 + 12\pi^2 I_1 + 8\pi^3\mathrm{i}I_0 = \frac{13\pi^4\mathrm{i}}{4}$$

$$\Rightarrow I_2 = \frac{1}{6}\left[8\pi^2\left(\frac{\pi}{2}\right) - \frac{13\pi^3}{4}\right] = \boxed{\frac{\pi^3}{8}}.$$

As for I_4, we must evaluate the contour integral for $n = 5$.

$$\oint_\Gamma \frac{\ln^5 z}{z^2+1}\,\mathrm{d}z = -10\pi\mathrm{i}I_4 + 40\pi^2 I_3 + 80\pi^3\mathrm{i}I_2 - 80\pi^4 I_1 - 32\pi^5\mathrm{i}I_0 = -\frac{121\pi^6\mathrm{i}}{16}.$$

The integrals are all real, so we do not need the value of I_3 to obtain I_4. Equating imaginary parts, we have

$$I_4 = \frac{1}{10}\left[80\pi^2\left(\frac{\pi^3}{8}\right) - 32\pi^4\left(\frac{\pi}{2}\right) + \frac{121\pi^5}{16}\right] = \boxed{\frac{5\pi^5}{32}}$$

S1 2017

3. State Cauchy's residue theorem, and explain how it can be used to calculate some definite integrals of the form

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta\,,$$

where $f(\theta) \equiv f(\theta + 2\pi)$. For what forms of $f(\theta)$ is this method useful?

Let $\mathcal{D} \in \mathbb{C}$ be a simply connected domain with $\{z_1, ..., z_n\} \subseteq \mathcal{D}$ such that $f : \mathcal{D} \setminus \{z_1, ..., z_n\} \mapsto \mathbb{C}$ is holomorphic, and Γ be a closed, non-intersecting, rectifiable contour Γ encircling $z_1, ..., z_n$. Cauchy's residue theorem states that the then the contour integral of $f(z)$ along Γ, when traversed in an anticlockwise manner, is

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_j \mathrm{Res}_j\, f(z),$$

where $1 \le j \le n$ runs over the enclosed poles.

Evaluate

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + 2\tan\theta + 3\sec\theta}.$$

Since $z = 0$ is a pole of order two, we have...

Use the method of contour integration to evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x - \mathrm{i}a}\,\mathrm{d}x\,,$$

where $a \in \mathbb{R}^+$, and hence show that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

S1 2016

1. Find the location and residues of the simple poles of the function

$$f(z) = \frac{1}{z^2 \sin \pi(z+\alpha)}$$

where $0 < \alpha < 1$.

Since $0 < \alpha < 1$, therefore $\alpha \notin \mathbb{Z}$. $\sin \pi(z+\alpha) = 0$ has roots $\pi(z+\alpha) = m\pi$, where $m \in \mathbb{Z}$. Thus, the locations of the simple poles are $z = \boxed{m - \alpha,\ m \in \mathbb{Z}}$. L'Hôpital's rule is a local one, i.e. we only concern ourselves with the behavior in the neighbourhood of the singularities, and ignore the global issues such as multivaluedness. We can therefore use it to calculate the residues of the simple poles as

$$\begin{aligned}
\operatorname*{Res}_{z=m-\alpha} f(z) &= \lim_{z \to m-\alpha} (z - m + \alpha) f(z), \\
&= \lim_{z \to m-\alpha} \frac{z - m + \alpha}{z^2 \sin \pi(z+\alpha)} \\
&= \lim_{z \to m-\alpha} \frac{1}{2z \sin \pi(z+\alpha) + z^2 \pi \cos \pi(z+\alpha)} \\
&= \frac{1}{(m-a)^2 \pi \cos m\pi} \\
&= \boxed{\frac{(-1)^m}{(m-a)^2 \pi}}.
\end{aligned}$$

Note that $z = 0$ is a second-order pole, however we will eventually need the residue there, so we will calculate it here. Recall that an n^{th}-order pole at $z = c$ has residue

$$\operatorname*{Res}_{z=c} f(z) = \frac{1}{(n-1)!} \lim_{z \to c} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} (z-c)^n f(z).$$

Since $z = 0$ is a pole of order two, we have

$$\begin{aligned}
\operatorname*{Res}_{z=0} f(z) &= \lim_{z \to 0} \frac{\mathrm{d}}{\mathrm{d}z} z^2 f(z), \\
&= \lim_{z \to 0} \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{\sin \pi(z+\alpha)} \\
&= \lim_{z \to 0} \frac{-\pi \cos(z+\alpha)}{\sin^2 \pi(z+\alpha)} \\
&= -\pi \operatorname{cosec} \pi\alpha \cot \pi\alpha.
\end{aligned}$$

Let Γ_N be the square contour with vertices at $N(1+\mathrm{i})$, $N(-1+\mathrm{i})$, $N(1-\mathrm{i})$, and $N(-1-\mathrm{i})$ where $N \in \mathbb{Z}^+$. Explain why

$$\oint_{\Gamma_N} f(z)\, \mathrm{d}z \to 0$$

as $N \to \infty$ and hence show that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n-\alpha)^2} = \pi^2 \operatorname{cosec} \pi\alpha \cot \pi\alpha.$$

First, note that our choice of Γ_N ensures that Γ_N does not pass through any singularities. Next, recall the estimation lemma, which states that

$$\left| \int_{\Gamma_N} f(z)\, \mathrm{d}z \right| \le \int_{\Gamma_N} |f(z)| |\, \mathrm{d}z\, | \le \sup_{z \in \Gamma_N} |f(z)| \int_{\Gamma_N} |\, \mathrm{d}z\, |.$$

The perimeter of Γ_N is $8N$, so

$$\left|\int_{\Gamma_N} f(z)\,\mathrm{d}z\right| \leq 8N \sup_{z\in\Gamma_N}\left|\frac{1}{z^2\sin\pi(z+\alpha)}\right|$$
$$\leq \frac{8}{N}\sup_{z\in\Gamma_N}|\operatorname{cosec}\pi(z+\alpha)|.$$

All that remains is to show that $|\operatorname{cosec}\pi(z+\alpha)|$ is bounded on Γ_N. Along the vertical sides, we have $z = N \pm \mathrm{i}y$, so

$$\begin{aligned}
|\operatorname{cosec}\pi(z+\alpha)|^2 &= |\operatorname{cosec}\pi(N+\alpha+\mathrm{i}y)|^2 \\
&\equiv |\operatorname{cosec}\pi(\alpha+\mathrm{i}y)|^2 \\
&= \left|\frac{1}{\sin\pi\alpha\cos\mathrm{i}\pi y \pm \sin\mathrm{i}\pi y\cos\pi\alpha}\right|^2 \\
&= \left|\frac{1}{\sin\pi\alpha\cosh\pi y \pm \mathrm{i}\sinh\pi y\cos\pi\alpha}\right|^2 \\
&= \frac{1}{\sin^2\pi\alpha\cosh^2\pi y + \sinh^2\pi y\cos^2\pi\alpha} \\
&\leq \frac{1}{\sinh^2\pi y\cos^2\pi\alpha} \leq \frac{1}{\sinh^2\pi y}.
\end{aligned}$$

Therefore, as $N \to \infty$, the integrands approach zero, so

$$\boxed{\oint_{\Gamma_N} f(z)\,\mathrm{d}z \to 0}.$$

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(n-\alpha)^2} &= \pi\sum_{m\in\{\mathbb{Z}\setminus 0\}}\operatorname*{Res}_{z=m-\alpha} f(z) \\
&= \pi\left(\frac{1}{2\pi i}\oint_{\Gamma_N} f(z)\,\mathrm{d}z - \operatorname*{Res}_{z=0} f(z)\right) \\
&= -\pi\operatorname*{Res}_{z=0} f(z) \\
&= \boxed{\pi^2\operatorname{cosec}\pi\alpha\cot\pi\alpha}.
\end{aligned}$$

By expanding about $\alpha = 0$, show that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

and find

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}.$$

[*You may assume that*

$$\pi^2\operatorname{cosec}\pi\alpha\cot\pi\alpha = \frac{1}{\alpha^2} - \frac{\pi^2}{6} - \frac{7\pi^4\alpha^2}{120} + \mathcal{O}(\alpha^4).$$
]

Due to even function symmetry,

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(n-\alpha)^2} = \frac{(-1)^0}{(0-\alpha)^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-\alpha)^2}$$

$$= \frac{1}{\alpha^2} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-\alpha)^2}.$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} &= \lim_{\alpha\to 0} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-\alpha)^2} \\
&= \lim_{\alpha\to 0} \frac{1}{2}\left[\left(\frac{1}{\alpha^2} - \frac{\pi^2}{6} - \frac{7\pi^4\alpha^2}{120} + \mathcal{O}(\alpha^4)\right) - \frac{1}{\alpha^2}\right] \\
&= \lim_{\alpha\to 0} \frac{1}{2}\left(-\frac{\pi^2}{6} - \frac{7\pi^4\alpha^2}{120} + \mathcal{O}(\alpha^4)\right) \\
&= \boxed{-\frac{\pi^2}{12}}.
\end{aligned}$$

Furthermore, notice that

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-\alpha)^2} = 6\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-\alpha)^4}.$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} &= \lim_{\alpha\to 0} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-\alpha)^4} \\
&= \lim_{\alpha\to 0} \frac{1}{12}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\left[\left(\frac{1}{\alpha^2} - \frac{\pi^2}{6} - \frac{7\pi^4\alpha^2}{120} + \mathcal{O}(\alpha^4)\right) - \frac{1}{\alpha^2}\right] \\
&= \lim_{\alpha\to 0} \frac{1}{12}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\left(-\frac{\pi^2}{6} - \frac{7\pi^4\alpha^2}{120} + \mathcal{O}(\alpha^4)\right) \\
&= \lim_{\alpha\to 0} \frac{1}{12}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\left(-\frac{7\pi^4}{60} + \mathcal{O}(\alpha^2)\right) \\
&= \boxed{-\frac{7\pi^4}{720}}.
\end{aligned}$$

S1 2016

2. What is meant by a conformal map? Does a conformal map conserve shapes? Find the condition that the map $f : z \mapsto w$ is conformal.

A conformal map is a transformation that $\boxed{f \text{ locally preserves angles}}$. Consider

$$\delta w = \frac{\mathrm{d}w}{\mathrm{d}z}\delta z = \left(1 - \frac{1}{z^2}\right)\delta z.$$

The condition for f to be a conformal map is

$$\boxed{\arg\frac{\delta w_1}{\delta w_2} = \arg\frac{\delta z_1}{\delta z_2}}.$$

f is conformal in the region of holomorphy precisely where its derivative is non-zero, i.e. $\boxed{f'(z) = 0}$.

Let

$$f(z) = \frac{z-1}{z+1}.$$

To what region of the w-plane is the real axis of the z-plane mapped?

We can rewrite $f(z)$ as

$$f(z) = 1 - \frac{2}{z+1}.$$

This shows that...

Show that a circle of radius r and centre $z = -1$ maps to a circle in the w-plane and find its centre and radius. Describe what happens to the interior of the circle under the map.

A...

Show that the unit circle in the z-plane is mapped to the imaginary axis in the w-plane. Which region of the z-plane maps to the unit circle in the w-plane?

A...

S1 2016

3. Using the method of contour integration, evaluate the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{3 + 2\cos\theta + \sin\theta}.$$

Let $z = \mathrm{e}^{\mathrm{i}\theta} \Rightarrow \mathrm{d}z = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}z\,\mathrm{d}\theta \Rightarrow \mathrm{d}\theta = \mathrm{d}z\,/\mathrm{i}z$. We first convert the given integral into a contour integral. Making the above substitution, we obtain

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{3 + 2\cos\theta + \sin\theta} = -2\mathrm{i}\oint_\Gamma \frac{\mathrm{d}z}{(2-\mathrm{i})\,z^2 + 6z + (2-\mathrm{i})}.$$

Next, we need to find the poles. The denominator vanishes when

$$\begin{aligned} z &= \frac{-6 \pm \sqrt{6^2 - 4(2+\mathrm{i})(2-\mathrm{i})}}{2(2-\mathrm{i})} = \frac{-6 \pm \sqrt{36-20}}{2(2-\mathrm{i})} \\ &= \frac{-3 \pm 2}{2-\mathrm{i}} = -2-\mathrm{i} \text{ or } -\frac{2}{5} - \frac{\mathrm{i}}{5}. \end{aligned}$$

The denominator can thus be expressed as

$$(2-\mathrm{i})(z+2+\mathrm{i})\left(z + \frac{2}{5} + \frac{\mathrm{i}}{5}\right).$$

The only pole which remains within the region bounded by the contour is $z = -\frac{2}{5} - \frac{\mathrm{i}}{5}$. The residue is

$$\begin{aligned} \operatorname*{Res}_{z=-\frac{2}{5}-\frac{\mathrm{i}}{5}} \frac{1}{(2-\mathrm{i})(z+2+\mathrm{i})\left(z+\frac{2}{5}+\frac{\mathrm{i}}{5}\right)} &= \lim_{z\to-\frac{1}{5}+\frac{2\mathrm{i}}{5}} \frac{1}{(2-\mathrm{i})(z+2+\mathrm{i})} \\ &= \frac{1}{(2-\mathrm{i})\left(-\frac{2}{5}-\frac{\mathrm{i}}{5}+2+\mathrm{i}\right)} \\ &= \frac{1}{(2-\mathrm{i})\left(\frac{8}{5}+\frac{4\mathrm{i}}{5}\right)} = \frac{1}{4}. \end{aligned}$$

By Cauchy's residue theorem, we thus obtain

$$I = -2\mathrm{i}\oint_\Gamma \frac{\mathrm{d}z}{(2-\mathrm{i})\,z^2 + 6z + (2-\mathrm{i})} = -2\mathrm{i}(2\pi\mathrm{i})\frac{1}{4} = \boxed{\pi}.$$

State Cauchy's theorem and derive the criterion satisfied by the contours Γ and Γ' for the function $f(z)$ if

$$\oint_\Gamma f(z)\,\mathrm{d}z = \oint_{\Gamma'} f(z)\,\mathrm{d}z\,.$$

Cauchy's theorem states that if $f(z)$ is holomorphic in the simply connected region R enclosed by a closed, non-intersecting, rectifiable contour Γ, then the contour integral of $f(z)$ along Γ when traversed in an anticlockwise manner is

$$\boxed{\oint_\Gamma f(z)\,\mathrm{d}z = 0}.$$

Additionally, for

$$\oint_\Gamma f(z)\,\mathrm{d}z = \oint_{\Gamma'} f(z)\,\mathrm{d}z\,,$$

then $\boxed{f(z) \text{ must be holomorphic in region between } \Gamma \text{ and } \Gamma'}$ (denoted as the shaded region Σ), i.e. Γ and Γ' must $\boxed{\text{enclose the same poles}}$.

Give the location and type of the singular points of the function $f(z) = (z^2 - 1)^{\frac{1}{2}}$.

Since $z^2 - 1 = (z-1)(z+1)$, there are $\boxed{\text{first-order branch points}}$ located at $\boxed{z = \pm 1}$. $z = \infty$ is not a branch point, but we can define a residue at $z = \infty$. Recall that

$$f(z)\,\mathrm{d}z = -\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right) \mathrm{d}\xi\,,$$

where we have substituted $z = 1/\xi$. We then perform an expansion in the neighbourhood of $\xi = 0$ to obtain the residue as the coefficient of $1/\xi$.

$$\begin{aligned}
-\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right) &= -\frac{1}{\xi^2}\left(\frac{1}{\xi^2} - 1\right)^{\frac{1}{2}} \\
&= -\frac{1}{\xi^3}(1-\xi^2)^{\frac{1}{2}} \\
&= -\frac{1}{\xi^3}\left(1 - \frac{1}{2}\xi^2 + \mathcal{O}(\xi^4)\right) \\
&= -\frac{1}{\xi^3} + \frac{1}{2\xi} + \mathcal{O}(\xi) \\
\Rightarrow \operatorname*{Res}_{z=\infty} f(z) &= \boxed{\frac{1}{2}}.
\end{aligned}$$

The contour Γ encircles the segment $[-1, 1]$ of the real axis. By considering

$$\oint_\Gamma f(z)\,\mathrm{d}z\,,$$

evaluate

$$\int_{-1}^{1} \sqrt{1 - x^2}\,\mathrm{d}x\,.$$

We choose our branch cut to be $[-1, 1]$, and as stated by the question let Γ contain the branch cut. Since $f(z)$ is holomorphic on the slit plane $\mathbb{C} \setminus [-1, 1]$, any Γ enclosing the branch cut will return the same answer to our contour integral. Let Γ extend to infinity. Recall that the interior of a curve $-\Gamma$ is everything outside of Γ. Thus, by Cauchy's residue theorem,

$$\operatorname*{Res}_{z=\infty} f(z) = -\sum_j \mathrm{Res}_j\, f(z)$$

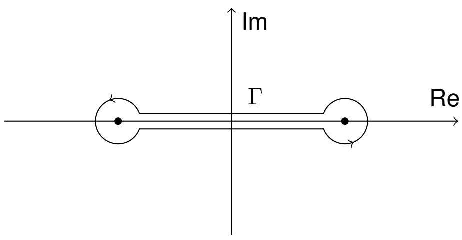
$$\Rightarrow \oint_\Gamma f(z)\,\mathrm{d}z = -2\pi\mathrm{i} \operatorname*{Res}_{z=\infty} f(z) = -\pi\mathrm{i}.$$

Alternatively, even if we did not calculate the residue at infinity, by noting that the result along a contour stretched to infinity is controlled by the large z behavior of the integrand, we can perform an expansion as if there was a pole at origin to obtain

$$f(z) = z\left[1 - \frac{1}{2z^2} + \mathcal{O}\left(\frac{1}{|z|^4}\right)\right] \sim z - \frac{1}{2z}.$$

The coefficient of $1/z$ is $-1/2$, which gives us the same result.

Next, we collapse Γ into the dogbone contour as shown below.



To determine the behaviour of $f(z)$ above and below the branch cut, we introduce the substitutions $z-1 = r\mathrm{e}^{\mathrm{i}\theta}$ (vector with head at z and tail at 1) and $z+1 = R\mathrm{e}^{\mathrm{i}\phi}$ (vector with head at z and tail at -1) such that

$$f(z) = \left[rR\mathrm{e}^{\mathrm{i}(\theta+\phi)}\right]^{\frac{1}{2}} = \sqrt{rR}\mathrm{e}^{\mathrm{i}\frac{\theta+\phi}{2}}.$$

We assert that when $z = x + \mathrm{i}y$ lies on the real axis with $x > 1$, $\theta = \phi = 0$. Directly above the branch cut (where $y = 0^+$), $r = |z-1| = 1-x$, $R = |z+1| = 1+x$, $\theta = \arg(z-1) = \pi$ and $\phi = \arg(z+1) = 0$. Thus,

$$f_{\text{above}} = \sqrt{rR}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}\sqrt{1-x^2}.$$

On the other hand, directly below the branch cut (where $y = 0^-$), $\theta = \pi$ while $\phi = 2\pi$. Thus,

$$f_{\text{below}} = \sqrt{rR}\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}} = -\mathrm{i}\sqrt{1-x^2}.$$

We shall assume without proof that the integrals along the small deformations tend to zero. Thus, integrating in an anticlockwise fashion, we have

$$\begin{aligned}\oint_\Gamma f(z)\,\mathrm{d}z &= -\mathrm{i}\int_{-1}^{1}\sqrt{1-x^2}\,\mathrm{d}x + \mathrm{i}\int_{1}^{-1}\sqrt{1-x^2}\,\mathrm{d}x \\ &= -2\mathrm{i}\int_{-1}^{1}\sqrt{1-x^2}\,\mathrm{d}x = -\pi\mathrm{i}.\end{aligned}$$

Rearranging, we see that

$$\int_{-1}^{1}\sqrt{1-x^2}\,\mathrm{d}x = \boxed{\frac{\pi}{2}}.$$

S1 2015

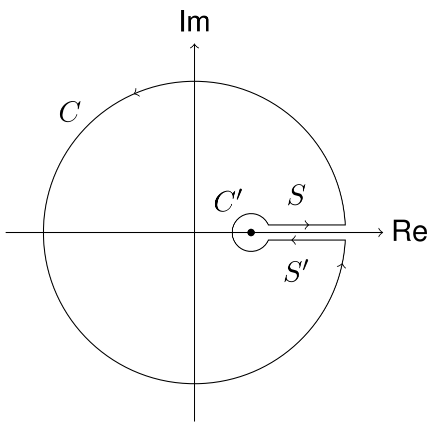
1. Classify the singularities of the function

$$f(z) = \frac{\ln^2(1-z)}{z^3},$$

and identify the region of the complex plane in which it is holomorphic.

$z = 0$ is a third-order pole. $z = 1$ and $z = \infty$ are ∞-order branch points. Choosing our branch cut to be $[1, \infty)$, $f(z)$ is holomorphic on the punctured slit plane $\mathbb{C} \setminus \{0 \cup [1, \infty)\}$.

The contour $\Gamma = S \cup C \cup S' \cup C'$ is defined as shown in the figure below.



Evaluate

$$\oint_\Gamma f(z)\,\mathrm{d}z$$

in the case that the radius of $C' \to 0$ and the radius of $C \to \infty$, and hence show that

$$\int_1^\infty \frac{\ln(x-1)}{x^3}\,\mathrm{d}x = -\frac{1}{2}.$$

Since $z = 0$ is not a simple pole, it is easier to obtain the residue there via an expansion in the neighbourhood of $z = 0$ and taking the coefficient of $1/z$. Doing so, we obtain

$$f(z) = \frac{1}{z^3}\left[-z + \mathcal{O}(|z|^2)\right]^2 = \frac{1}{z^3}\left[z^2 + \mathcal{O}(|z|^3)\right] \sim \frac{1}{z},$$

and hence, by Cauchy's residue theorem,

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}.$$

We introduce the substitution $1 - z = r\mathrm{e}^{\mathrm{i}\theta}$ (vector with head at 1 and tail at z), where $r = |1-z|$ and $\theta = \arg(1-z)$. We also assert that when $z = x + \mathrm{i}y$ lies on the real axis with $x < 1$, $\theta = 0$. Directly above the branch cut (where $y = 0^+$), $r = |1-z| = x - 1$ and $\theta = -\pi$, so

$$f_{\text{above}} = \frac{\ln^2 r\mathrm{e}^{-\mathrm{i}\pi}}{x^3} = \frac{\ln^2(x-1)}{x^3} - \frac{2\pi\mathrm{i}\ln(x-1)}{x^3} - \frac{\pi^2}{x^3}.$$

Likewise, directly below the branch cut (where $y = 0^-$), $r = |1-z| = x - 1$ and $\theta = \pi$, so

$$f_{\text{below}} = \frac{\ln^2 r\mathrm{e}^{\mathrm{i}\pi}}{x^3} = \frac{\ln^2(x-1)}{x^3} + \frac{2\pi\mathrm{i}\ln(x-1)}{x^3} - \frac{\pi^2}{x^3}.$$

We shall assume without proof that the contributions due to C and C' tend to zero as $R \to \infty$ and $\epsilon \to \infty$. The contour integral becomes

$$\oint_\Gamma f(z)\,\mathrm{d}z = \int_1^\infty \frac{\ln^2(x-1)}{x^3} - \frac{2\pi\mathrm{i}\ln(x-1)}{x^3} - \frac{4\pi^2}{x^3}\,\mathrm{d}x + \int_\infty^1 \frac{\ln^2(x-1)}{x^3} + \frac{2\pi\mathrm{i}\ln(x-1)}{x^3} - \frac{4\pi^2}{x^3}\,\mathrm{d}x$$
$$= -4\pi\mathrm{i}\int_1^\infty \frac{\ln(x-1)}{x^3}\,\mathrm{d}x\,.$$

Hence, as requested, we obtain

$$\int_1^\infty \frac{\ln(x-1)}{x^3}\,\mathrm{d}x = \boxed{-\frac{1}{2}}.$$

By considering

$$\oint_\Gamma g(z)\,\mathrm{d}z\,,$$

where $g(z)$ is an appropriately chosen function, show that

$$\int_1^\infty \frac{\ln^2(x-1)}{x^3}\,\mathrm{d}x = \frac{\pi^2}{6}.$$

We choose $g(z)$ to be

$$g(z) = \frac{\ln^3(1-z)}{z^3}.$$

Like before, $z = 0$ is a third-order pole. To obtain the residue at $z = 0$, we perform an expansion again. Since the coefficient of $1/z$ is 0, the residue is 0 as well. Thus, by Cauchy's residue theorem,

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

Using the same substitutions as before, above the branch cut we have

$$f_{\text{above}} = \frac{\ln^3(x-1)}{x^3} - \frac{3\pi\mathrm{i}\ln^2(x-1)}{x^3} - \frac{3\pi^2\ln(x-1)}{x^3} + \frac{\pi^3\mathrm{i}}{x^3},$$

and below the branch cut we have

$$f_{\text{below}} = \frac{\ln^3(x-1)}{x^3} + \frac{3\pi\mathrm{i}\ln^2(x-1)}{x^3} - \frac{3\pi^2\ln(x-1)}{x^3} - \frac{\pi^3\mathrm{i}}{x^3}.$$

Hence,

$$\oint_\Gamma f(z)\,\mathrm{d}z = -6\pi\mathrm{i}\int_1^\infty \frac{\ln^2(x-1)}{x^3}\,\mathrm{d}x + 2\pi^3\mathrm{i}\int_1^\infty \frac{1}{x^3}\,\mathrm{d}x$$
$$= -6\pi\mathrm{i}\int_1^\infty \frac{\ln^2(x-1)}{x^3}\,\mathrm{d}x + \pi^3\mathrm{i} = 0.$$

Putting it all together, we have

$$\int_1^\infty \frac{\ln^2(x-1)}{x^3}\,\mathrm{d}x = \boxed{\frac{\pi^2}{6}}.$$

S1 2015

2. Explain what is meant by a conformal map.

Consider the mapping from the complex z-plane to the complex w-plane $f : z \mapsto w = \mathrm{e}^z$. Find the region in the z-plane that maps to the entire w-plane.

Find the image in the w-plane of the map f applied to the square in the z-plane with vertices at 0, a, $a(1+\mathrm{i})$ and $a\mathrm{i}$, where $0 < a < \pi/2$ is real. Show that the internal angles at the vertices of the square are unchanged by the mapping.

Consider the map f^{-1} applied to the square in the w-plane with vertices at 0, a, $a(1+\mathrm{i})$, and $a\mathrm{i}$, where $a \in \mathbb{R}^+$. At which of the vertices of the square are the interior angles unchanged by the mapping? Sketch the image of the square in the z-plane.

S1 2015

3. Using the method of contour integration:

a) Show that

$$h(k) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx}}{1+x^2}\,\mathrm{d}x$$

is given by $\pi \mathrm{e}^{-k}$ when k is real and positive, and find $h(k)$ when k is real and negative.

Recall Jordan's lemma, which states that if $f(z)$ is analytic in the upper half-plane (apart from a finite number of isolated singularities), C_R is the upper semicircular contour of radius R, and

$$\lim_{R\to\infty} \sup_{|z|=R} |f(z)| = 0,$$

then

$$\lim_{R\to\infty} \int_{C_R} f(z)\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}z = 0,$$

if $k > 0$. Notice that

$$f(z) = \frac{1}{1+z^2}$$

satisfies the required conditions to invoke Jordan's lemma. The integrand has simple poles at $z = \pm\mathrm{i}$. If we let the contour we integrate over be $\Gamma = C_R \cup [-R, R]$, then we only require the residue at $z = \mathrm{i}$, which is

$$\operatorname*{Res}_{z=\mathrm{i}} f(z)\mathrm{e}^{\mathrm{i}kz} = (z-\mathrm{i})f(z)\mathrm{e}^{\mathrm{i}kz}\Big|_{z=\mathrm{i}} = \frac{\mathrm{e}^{-k}}{2\mathrm{i}}.$$

The integral can be split into

$$\oint_\Gamma \frac{\mathrm{e}^{\mathrm{i}kz}}{1+z^2}\,\mathrm{d}z = \int_{-R}^{R} \frac{\mathrm{e}^{\mathrm{i}kx}}{1+x^2}\,\mathrm{d}x + \int_{C_R} \frac{\mathrm{e}^{\mathrm{i}kz}}{1+z^2}\,\mathrm{d}z\,.$$

Using Cauchy's residue theorem and invoking Jordan's lemma as $R \to \infty$, we have

$$\begin{aligned}
\oint_\Gamma \frac{\mathrm{e}^{\mathrm{i}kz}}{1+z^2}\,\mathrm{d}z &= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx}}{1+x^2}\,\mathrm{d}x + \lim_{R\to\infty}\int_{C_R} \frac{\mathrm{e}^{\mathrm{i}kz}}{1+z^2}\,\mathrm{d}z \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx}}{1+x^2}\,\mathrm{d}x + 0 \\
&= \pi e^{-k} \\
\Rightarrow \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx}}{1+x^2}\,\mathrm{d}x &= \boxed{\pi\mathrm{e}^{-k}}.
\end{aligned}$$

For real and negative k, we consider the semicircular contour in the lower half-plane instead.

Compute

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{1+8\sin^2\theta}.$$

AAAAA.

S1 2014

1. Explain why the condition

$$\frac{\partial f(x,y)}{\partial(x-\mathrm{i}y)}=0$$

is equivalent to the statement $f(x,y)\equiv F(z)$ where $z=x+\mathrm{i}y$. You may assume that all first derivatives exist.

AAAAA.

Letting $f(z)=u(x,y)+\mathrm{i}v(x,y)$, where $u(x,y)$ and $v(x,y)$ are real functions, show that the condition above implies the Cauchy-Riemann conditions for an analytic function

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Show also that if $f(z)$ is an analytic function, then $u(x,y)$ and $v(x,y)$ satisfy Laplace's equation and the families of curves $u(x,y)=$ constant and $v(x,y)=$ constant are orthogonal.

AAAAA.

In a two-dimensional electrostatic system, a thin electrode lying along the real axis between $z=\pm1$ is held at potential $V=0$. At large distance from the origin, $V\simeq A\ln|z|$, where A is a real positive constant. Show that the potential

$$V=\mathrm{Re}\left\{A\ln\left(z+\sqrt{z^2-1}\right)\right\}$$

satisfies the boundary conditions. You may assume that the electrostatic potential is unique.

AAAAA.

In another system, the segment $[0,1]$ of the real axis and the segment $[0,\mathrm{i}]$ of the imaginary axis are held at potential $V=0$ as shown in the figure below.

AAAAA.

By considering the mapping between this configuration and the configuration in the previous part, find V at a general point in the first quadrant.

Under the transfromation of the conformal map $f:z\mapsto w=z^2$, we have the same scenario as before. The complex potential in the w-plane is then

$$\Phi(w)=A\ln\left(w+\sqrt{w^2-1}\right).$$

Transforming back to the z-plane we have

$$\Phi(z)=A\ln\left(z^2+\sqrt{z^4-1}\right).$$

The electrostatic potential in the first quadrant is therefore

$$V=\boxed{\mathrm{Re}\left\{A\ln\left(z^2+\sqrt{z^4-1}\right)\right\}},\quad \text{where } \boxed{0\le\arg z\le\frac{\pi}{2}}.$$

S1 2014

2. Classify the singularities of the function

$$f(z) = \frac{1}{(z^2+a^2)\sin \pi z}$$

where a is real and positive. Find the residues at all the simple poles.

For this part, it is worth remembering that L'Hôpital's rule is a local one, i.e. we only concern ourselves with the behavior in the neighbourhood of the singularities, and so we can ignore the global issues such as multivaluedness. It is worth noting that

$$\mathrm{i}\sin \mathrm{i}z = -\sinh z.$$

The simple poles occur at $z = \pm a\mathrm{i}$ and $z = n$, $n \in \mathbb{N}$. The residues at $z = \pm a\mathrm{i}$ are

$$\operatorname*{Res}_{z=a\mathrm{i}} f(z) = \lim_{z\to a\mathrm{i}} (z-a\mathrm{i})f(z) = \lim_{z\to a\mathrm{i}} \frac{1}{(z+a\mathrm{i})\sin \pi z} = \frac{1}{2a\mathrm{i}\sin \mathrm{i}\pi a} = \boxed{-\frac{1}{2a\sinh \pi a}},$$

$$\operatorname*{Res}_{z=-a\mathrm{i}} f(z) = \lim_{z\to -a\mathrm{i}} (z+a\mathrm{i})f(z) = \lim_{z\to -a\mathrm{i}} \frac{1}{(z-a\mathrm{i})\sin \pi z} = \frac{1}{-2a\mathrm{i}\sin(-\mathrm{i}\pi a)} = \boxed{-\frac{1}{2a\sinh \pi a}}.$$

As for the residues at $z = n$, we make use of L'Hôpital's rule to obtain

$$\begin{aligned}\operatorname*{Res}_{z=n} f(z) = \lim_{z\to n}(z-n)f(z) &= \lim_{z\to n} \frac{z-n}{(z^2+a^2)\sin \pi z} \\ &= \lim_{z\to n} \frac{1}{2z\sin \pi z + \pi(z^2+a^2)\cos \pi z} \\ &= \frac{1}{\pi(n^2+a^2)\cos n\pi} = \boxed{\frac{(-1)^n}{\pi(n^2+a^2)}}.\end{aligned}$$

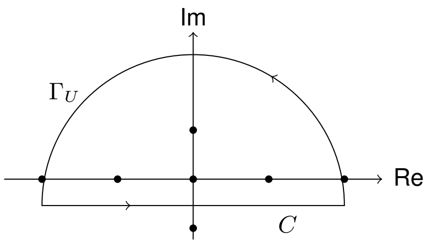
Consider the integral

$$I = \int_C f(z)\,\mathrm{d}z$$

where C is the straight line contour running from $-\infty - \mathrm{i}a/2$ to $\infty - \mathrm{i}a/2$. By closing the contour in both the upper and lower half planes find two distinct expressions for I, and hence show that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{a\sinh a\pi}.$$

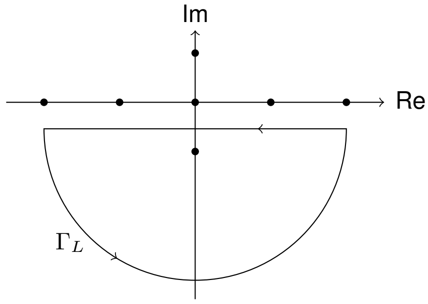
We first close C via an upper semicircular contour Γ_U as shown below.



As the radius of Γ_U tends to infinity, the poles enclosed by $C \cup \Gamma_U$ in this case are $z = a\mathrm{i}$ and $z = n$, $n \in \mathbb{N}$. By Cauchy's residue theorem,

$$\oint_{C\cup\Gamma_U} f(z)\,\mathrm{d}z = I + I_U = 2\pi\mathrm{i}\left[-\frac{1}{2a\sinh \pi a} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\pi(n^2+a^2)}\right].$$

Next, we close C via a lower semicircular contour Γ_L as shown below.



The only pole enclosed by C in this case is $z = -a\mathrm{i}$. By Cauchy's residue theorem,

$$\oint_{C\cup\Gamma_L} f(z)\,\mathrm{d}z = -I + I_L = 2\pi\mathrm{i}\left(-\frac{1}{2a\sinh\pi a}\right).$$

The contributions from the semicircular contours in both cases are zero. To see this, recall that if $f(z)$ is defined along a piecewise continuously differentiable curve Γ, then by the estimation lemma,

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \le \int_\Gamma |f(z)||\,\mathrm{d}z\,| \le \sup_{z\in\Gamma}|f(z)|\int_\Gamma|\,\mathrm{d}z\,|.$$

The perimeter of Γ_U is πR, so

$$\begin{aligned}\left|\int_{\Gamma_U}\frac{\mathrm{d}z}{(z^2+a^2)\sin\pi z}\right| &\le \pi R\sup_{z\in\Gamma_U}\left|\frac{1}{(z^2+a^2)\sin\pi z}\right| \\ &\le \frac{\pi R}{R^2+a^2}\sup_{z\in\Gamma_U}|\operatorname{cosec}\pi z|.\end{aligned}$$

All that remains is to show that $|\operatorname{cosec}\pi z|$ is bounded on Γ. Along the vertical sides, we have $z = N \pm \mathrm{i}y$, so

$$\begin{aligned}\left|\int_{\Gamma_U}\frac{\mathrm{d}z}{(z^2+a^2)\sin\pi z}\right| &\le \sup_{z\in\Gamma_U}\pi R\left|\frac{1}{(z^2+a^2)\sin\pi z}\right| \\ &\sim \sup_{z\in\Gamma_U} R\cdot\mathcal{O}\left(\frac{1}{R^2}\right)|\operatorname{cosec}\pi z| \\ &\sim \sup_{z\in\Gamma_U}\mathcal{O}\left(\frac{1}{R}\right)|\operatorname{cosec}\pi z|.\end{aligned}$$

A similar proof exists for the integral along Γ_L. Altogether, we have

$$\frac{1}{2a\sinh\pi a} = -\frac{1}{2a\sinh\pi a} + \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\pi(n^2+a^2)},$$

and after some rearranging, we obtain

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+a^2} = \boxed{\frac{\pi}{a\sinh a\pi}}.$$

By considering the limit $a \to 0$, show that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

Due to even function symmetry,

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+a^2} = \frac{(-1)^0}{0^2+a^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}$$

$$= \frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2}.$$

The expansion of $\pi/a \sinh a\pi$ about $a = 0$ is

$$\begin{aligned} \frac{\pi}{a \sinh a\pi} &= \frac{\pi}{a[a\pi + (a\pi)^3/3! + \mathcal{O}(a^5)]} \\ &= \frac{1}{a^2[1 + (a\pi)^2/3! + \mathcal{O}(a^4)]} \\ &= \frac{1}{a^2}\left[1 - \frac{(a\pi)^2}{3!} + \mathcal{O}(a^4)\right] \\ &= \frac{1}{a^2} - \frac{\pi^2}{6} + \mathcal{O}(a^2). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} &= \lim_{a\to 0} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-\alpha)^2} \\ &= \lim_{a\to 0} \frac{1}{2}\left[\left(\frac{1}{a^2} - \frac{\pi^2}{6} + \mathcal{O}(a^2)\right) - \frac{1}{\alpha^2}\right] \\ &= \lim_{a\to 0} \frac{1}{2}\left(-\frac{\pi^2}{6} + \mathcal{O}(a^2)\right) \\ &= \boxed{-\frac{\pi^2}{12}}. \end{aligned}$$

S1 2013

1. Give the location and type of the branch points of the function $f(z) = (z^2 - 1)^{-\frac{1}{2}}$. The contour Γ encircles the segment $[-1, 1]$ of the real axis. By considering

$$\oint_\Gamma f(z)\,\mathrm{d}z\,,$$

evaluate

$$\int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}}.$$

There are $\boxed{\text{first-order branch points}}$ located at $\boxed{z = \pm 1}$. $z = \infty$ is not a branch point, but we can define a residue at $z = \infty$. Recall that

$$f(z)\,\mathrm{d}z = -\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right)\mathrm{d}\xi\,,$$

where we have substituted $z = 1/\xi$. We then perform an expansion in the neighbourhood of $\xi = 0$ to obtain the residue as the coefficient of $1/\xi$.

$$\begin{aligned}
-\frac{1}{\xi^2} f\left(\frac{1}{\xi}\right) &= -\frac{1}{\xi^2}\left(\frac{1}{\xi^2} - 1\right)^{-\frac{1}{2}} \\
&= -\frac{1}{\xi}(1-\xi^2)^{-\frac{1}{2}} \\
&= -\frac{1}{\xi}\left[1 + \mathcal{O}(\xi^2)\right] \\
&= -\frac{1}{\xi} + \mathcal{O}(\xi) \\
\Rightarrow \operatorname*{Res}_{z=\infty} f(z) &= -1.
\end{aligned}$$

As stated by the question, we let Γ contain the branch cut $[-1, 1]$. Since $f(z)$ is holomorphic on the slit plane $\mathbb{C} \setminus [-1, 1]$, any Γ enclosing the branch cut will return the same answer to our contour integral. Let Γ extend to infinity. Recall that the interior of a curve $-\Gamma$ is everything outside of Γ. Thus, by Cauchy's residue theorem,

$$\operatorname*{Res}_{z=\infty} f(z) = -\sum_j \mathrm{Res}_j\, f(z)$$

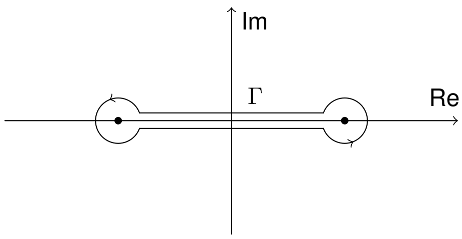
$$\Rightarrow \oint_\Gamma f(z)\,\mathrm{d}z = -2\pi\mathrm{i} \operatorname*{Res}_{z=\infty} f(z) = 2\pi\mathrm{i}.$$

Alternatively, even if we did not calculate the residue at infinity, by noting that the result along a contour stretched to infinity is controlled by the large z behavior of the integrand, we can perform an expansion as if there was a pole at origin to obtain

$$f(z) = \frac{1}{z}\left[1 + \mathcal{O}\left(\frac{1}{|z|^2}\right)\right] \sim \frac{1}{z}.$$

The coefficient of $1/z$ is 1, which gives us the same result.

Next, we collapse Γ into the dogbone contour as shown below.



To determine the behaviour of $f(z)$ above and below the branch cut, we introduce the substitutions $z-1 = re^{i\theta}$ (vector with head at z and tail at 1) and $z+1 = R\mathrm{e}^{\mathrm{i}\phi}$ (vector with head at z and tail at -1) such that

$$f(z) = \left[rR\mathrm{e}^{\mathrm{i}(\theta+\phi)}\right]^{-\frac{1}{2}} = \frac{1}{\sqrt{rR}}\mathrm{e}^{-\mathrm{i}\frac{\theta+\phi}{2}}.$$

We assert that when $z = x + \mathrm{i}y$ lies on the real axis with $x > 1$, $\theta = \phi = 0$. Directly above the branch cut (where $y = 0^+$), $r = |z-1| = 1-x$, $R = |z+1| = 1+x$, $\theta = \arg(z-1) = \pi$ and $\phi = \arg(z+1) = 0$. Thus,

$$f_{\text{above}} = \frac{1}{\sqrt{rR}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} = -\mathrm{i}\frac{1}{\sqrt{1-x^2}}.$$

On the other hand, directly below the branch cut (where $y = 0^-$), $\theta = \pi$ while $\phi = 2\pi$. Thus,

$$f_{\text{below}} = \frac{1}{\sqrt{rR}}\mathrm{e}^{-\mathrm{i}\frac{3\pi}{2}} = \mathrm{i}\frac{1}{\sqrt{1-x^2}}.$$

We shall assume without proof that the integrals along the small deformations tend to zero. Thus, integrating in an anticlockwise fashion, we have

$$\begin{aligned}\oint_\Gamma f(z)\,\mathrm{d}z &= \mathrm{i}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x - \mathrm{i}\int_{1}^{-1}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &= 2\mathrm{i}\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^2}} = 2\pi\mathrm{i}.\end{aligned}$$

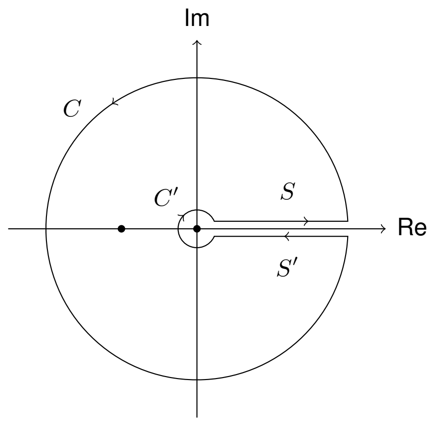
Rearranging, we see that

$$\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \boxed{\pi}.$$

Give the location and type of the singularities of the function $g(z) = z^{-\frac{1}{4}}(z+1)^{-1}$. By considering the plane cut along the positive real axis and a suitable contour, show that

$$\int_0^\infty \frac{x^{-\frac{1}{4}}}{1+x}\,\mathrm{d}x = \sqrt{2}\pi.$$

There is a $\boxed{\text{fourth-order branch point}}$ located at $\boxed{z=0}$ and a $\boxed{\text{simple pole}}$ located at $\boxed{z=-1}$. A suitable branch cut is $[0,\infty)$, and our contour of choice is the keyhole contour shown below.



The residue at $z=-1$ is

$$\operatorname*{Res}_{z=-1}\, g(z) = \lim_{z\to-1}(z+1)g(z) = (-1)^{-\frac{1}{4}} = \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}.$$

By Cauchy's residue theorem,

$$\oint_\Gamma g(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_j \mathrm{Res}_j\, g(z) = 2\pi\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}.$$

To determine the behaviour of $g(z)$ above and below the branch cut, we introduce the substitution $z = re^{i\theta}$ (vector with head at z and tail at the origin). We assert that when $z = x + iy$ lies on the real axis with $x > 0$, $\theta = 0$. Directly above the branch cut (where $y = 0^+$), $r = |z| = x$ and $\theta = 0$. Thus,

$$g_{\text{above}} = \frac{x^{-\frac{1}{4}}}{1+x}.$$

On the other hand, directly below the branch cut (where $y = 0^-$), $\theta = 2\pi$. Thus,

$$g_{\text{below}} = \frac{x^{-\frac{1}{4}}e^{-i\frac{\pi}{2}}}{1+x} = -\frac{ix^{-\frac{1}{4}}}{1+x}.$$

We shall assume without proof that the integrals along C and C' tend to zero. Thus, integrating in an anticlockwise fashion, we have

$$\begin{aligned}\oint_\Gamma f(z)\,dz &= \int_0^\infty \frac{x^{-\frac{1}{4}}}{1+x}\,dx - i\int_\infty^0 \frac{x^{-\frac{1}{4}}}{1+x}\,dx \\ &= (1+i)\int_0^\infty \frac{x^{-\frac{1}{4}}}{1+x}\,dx \\ &= \sqrt{2}e^{-i\frac{\pi}{4}}\int_0^\infty \frac{x^{-\frac{1}{4}}}{1+x}\,dx = 2\pi i e^{-i\frac{\pi}{4}}.\end{aligned}$$

Rearranging, we see that

$$\int_0^\infty \frac{x^{-\frac{1}{4}}}{1+x}\,dx = \boxed{\sqrt{2}\pi}.$$

Calculate the real integral

$$\int_0^\infty \frac{x^{-\frac{1}{2}}}{(1+x)^n}\,dx\,,$$

where $n \in \mathbb{Z}^+$.

Consider the function

$$h(z) = \frac{z^{-\frac{1}{2}}}{(z+1)^n}.$$

$z = 0$ remains a branch point (although this time it is second-order), and $z = -1$ is now a pole of order n. Recall that an n^{th}-order pole at $z = c$ has residue

$$\operatorname*{Res}_{z=c} h(z) = \frac{1}{(n-1)!}\lim_{z\to c}\frac{d^{n-1}}{dz^{n-1}}(z-c)^n h(z).$$

Thus,

$$\begin{aligned}\operatorname*{Res}_{z=-1} h(z) &= \frac{1}{(n-1)!}\lim_{z\to -1}\frac{d^{n-1}}{dz^{n-1}}(z+1)^n h(z) \\ &= \frac{1}{(n-1)!}\lim_{z\to -1}\frac{d^{n-1}}{dz^{n-1}}z^{-\frac{1}{2}} \\ &= \frac{1}{(n-1)!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left[-\frac{2(n-1)-1}{2}\right]\lim_{z\to-1} z^{-n+\frac{1}{2}} \\ &= \frac{(2n-3)!}{2^{2n-3}(n-1)!(n-2)!}(-1)^{-\frac{1}{2}} \\ &= -\frac{(2n-3)!\,i}{2^{2n-3}(n-1)!(n-2)!}.\end{aligned}$$

We use the same contour as in the previous part. By Cauchy's residue theorem,

$$\oint_\Gamma h(z)\,dz = 2\pi i\sum_j \operatorname{Res}_j h(z) = 2\left[\frac{(2n-3)!\,\pi}{2^{2n-3}(n-1)!(n-2)!}\right].$$

Once again, we introduce the substitution $z = r\mathrm{e}^{\mathrm{i}\theta}$, and assert that when $z = x + \mathrm{i}y$ lies on the real axis with $x > 0$, $\theta = 0$. Directly above the branch cut (where $y = 0^+$), $r = |z| = x$ and $\theta = 0$. Thus,

$$h_{\text{above}} = \frac{x^{-\frac{1}{2}}}{(1+x)^n}.$$

On the other hand, directly below the branch cut (where $y = 0^-$), $\theta = 2\pi$. Thus,

$$h_{\text{below}} = \frac{x^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\pi}}{(1+x)^n} = -\frac{x^{-\frac{1}{2}}}{(1+x)^n}.$$

We shall assume without proof that the integrals along C and C' tend to zero. Thus, integrating in an anticlockwise fashion, we have

$$\begin{aligned}\oint_\Gamma h(z)\,\mathrm{d}z &= \int_0^\infty \frac{x^{-\frac{1}{2}}}{(1+x)^n}\,\mathrm{d}x - \int_\infty^0 \frac{x^{-\frac{1}{2}}}{(1+x)^n}\,\mathrm{d}x \\ &= 2\int_0^\infty \frac{x^{-\frac{1}{2}}}{(1+x)^n}\,\mathrm{d}x \\ &= 2\left[\frac{(2n-3)!\,\pi}{2^{2n-3}(n-1)!(n-2)!}\right].\end{aligned}$$

Rearranging, we see that

$$\int_0^\infty \frac{x^{-\frac{1}{2}}}{(1+x)^n}\,\mathrm{d}x = \boxed{\frac{(2n-3)!\,\pi}{2^{2n-3}(n-1)!(n-2)!}}.$$

S1 2013

2. Let a, b, c, d be complex numbers such that $ad - bc \neq 0$, and consider the mappings from the complex z-plane to the complex w-plane specified by the functions

$$\begin{aligned} f_1 &: z \mapsto w = z + \frac{d}{c}, \\ f_2 &: z \mapsto w = z^{-1}, \\ f_3 &: z \mapsto w = -\frac{ad-bc}{c^2} z, \\ f_4 &: z \mapsto w = z + \frac{a}{c}. \end{aligned}$$

Describe geometrically the action of each of the mappings f_1, f_2 and f_3.

$\boxed{f_1}$ is a $\boxed{\text{translation by } d/c}$.

Under the action of f_2, $z = re^{i\theta}$ is mapped to $w = r^{-1}e^{-i\theta}$. Thus, $\boxed{f_2}$ is an $\boxed{\text{inversion about the unit circle}}$ followed by a $\boxed{\text{reflection about the real axis}}$.

$\boxed{f_3}$ is a $\boxed{\text{rescaling by } |(ad-bc)/c^2|}$ combined with a $\boxed{\text{rotation through } \arg\left[-(ad-bc)/c^2\right]}$, both of which are $\boxed{\text{global}}$. The order of the two operations does not matter.

Show that the Möbius mapping $f_M = f_4 \circ f_3 \circ f_2 \circ f_1$ acts as

$$f_M : z \mapsto w = \frac{az+b}{cz+d},$$

and the combination of two Möbius mappings is itself a Möbius mapping. Find f_M^{-1}.

Applying each map step-by-step, we have

$$\begin{aligned} f_2 \circ f_1 : z \mapsto w &= \frac{1}{z + d/c} \\ &= \frac{c}{cz+d}, \\ f_3 \circ f_2 \circ f_1 : z \mapsto w &= -\frac{ad-bc}{c^2}\frac{c}{cz+d} \\ &= -\frac{ad-bc}{c^2 z + cd}, \\ f_4 \circ f_3 \circ f_2 \circ f_1 : z \mapsto w &= -\frac{ad-bc}{c^2 z + cd} + \frac{a}{c} \\ &= \frac{acz + ad - ad + bc}{c^2 z + cd} \\ &= \frac{c(az+b)}{c(cz+d)} \\ &= \boxed{\frac{az+b}{cz+d}}. \end{aligned}$$

The combination of Möbius maps $f_{M'} \circ f_M$

$$f_{M'} \circ f_M : z \mapsto w = \frac{a'\left(\dfrac{az+b}{cz+d}\right) + b'}{c'\left(\dfrac{az+b}{cz+d}\right) + d'} = \boxed{\frac{(a'a + b'c)z + (a'b + b'd)}{(c'a + d'c)z + (c'b + d'd)}}$$

is a Möbius map provided

$$(a'a+b'c)(c'b+d'd)-(a'b+b'd)(c'a+a'c)=\boxed{(a'd'-b'c')(ad-bc)\neq 0}.$$

$f_M^{-1}=f_1^{-1}\circ f_2^{-1}\circ f_3^{-1}\circ f_4^{-1}$ is the corresponding inverse map. Applying each inverse map individually, we obtain

$$f_M^{-1}:w\mapsto z=\left[-\frac{c^2}{ad-bc}\left(w-\frac{a}{c}\right)\right]^{-1}-\frac{d}{c}=\boxed{\frac{-dw+b}{cw-a}}.$$

Show that there is always an f_M that maps the set of complex numbers $\{z_1,z_2,z_3\}$ to $\{0,1,\infty\}$. Hence, explain why the invariance of the cross-ratio

$$C=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$$

under any Möbius mapping can be demonstrated by considering the special initial values $\{z_1=0,z_2=1,z_3=\infty,z_4\}$. Demonstrate the Möbius invariance of C.

Notice that

$$\begin{aligned}
z_1&\mapsto 0\Rightarrow az_1+b=0,\\
z_2&\mapsto 1\Rightarrow az_2+b=cz_2+d,\\
z_3&\mapsto \infty\Rightarrow cz_3+d=0.
\end{aligned}$$

Solving these equations simultaneously, we obtain

$$\frac{c}{a}=\frac{z_2-z_1}{z_2-z_3},\quad \frac{d}{a}=-\frac{z_3(z_2-z_1)}{z_2-z_3}.$$

S1 2013

3. Using the method of contour integration evaluate the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\cos\theta}.$$

Let $z = \mathrm{e}^{\mathrm{i}\theta} \Rightarrow \mathrm{d}z = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}z\,\mathrm{d}\theta \Rightarrow \mathrm{d}\theta = \mathrm{d}z\,/\mathrm{i}z$. We first convert the given integral into a contour integral. Making the above substitution, we obtain

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\cos\theta} = -\mathrm{i}\oint_\Gamma \frac{\mathrm{d}z}{2z^2+5z+2}.$$

Next, we need to find the poles. The denominator vanishes when

$$z = \frac{-5\pm\sqrt{25-16}}{4} = -2 \text{ or } -\frac{1}{2}.$$

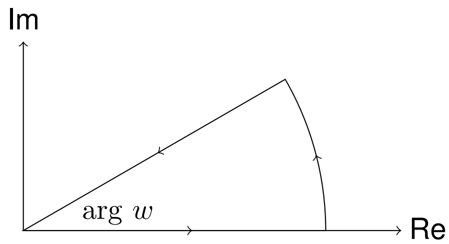
The only pole which remains within the region bounded by the contour is $z = -\frac{1}{2}$. The residue is

$$\operatorname*{Res}_{z=-\frac{1}{2}} \frac{1}{2(z+2)\left(z+\frac{1}{2}\right)} = \lim_{z\to-\frac{1}{2}} \frac{1}{2(z+2)} = \frac{1}{3}.$$

By Cauchy's residue theorem, we thus obtain

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\cos\theta} = -\mathrm{i}(2\pi\mathrm{i})\frac{1}{3} = \boxed{\frac{2\pi}{3}}.$$

Consider the contour Γ shown below.



Show that

$$\int_0^\infty \mathrm{e}^{-w^2x^2}\,\mathrm{d}x = \frac{1}{w}\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x\,, \quad 0 \le \arg\,w < \frac{\pi}{4}. \qquad (\star)$$

Consider the function

$$f(z) = \mathrm{e}^{-z^2}.$$

$f(z)$ has no singularities enclosed within Γ, so

$$\oint_\Gamma \mathrm{e}^{-z^2}\,\mathrm{d}z = \int_0^R \mathrm{e}^{-x^2}\,\mathrm{d}x + \int_0^{\arg\,w} \mathrm{i}R\mathrm{e}^{-R^2(\cos 2\theta+\mathrm{i}\sin 2\theta)}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta - \int_0^R w\mathrm{e}^{-w^2x^2}\,\mathrm{d}x = 0.$$

If $0 \le \arg\,w < \frac{\pi}{4}$, then $\mathrm{Re}\{\cos 2\theta + \mathrm{i}\sin 2\theta\} > 0$, so the contribution of the second term vanishes as $R\to\infty$, and

$$\boxed{\int_0^\infty \mathrm{e}^{-w^2x^2}\,\mathrm{d}x = \frac{1}{w}\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x}.$$

Show that

$$\lim_{R\to\infty}\left| R\int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2\mathrm{e}^{2\mathrm{i}\theta}}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta\right| = 0.$$

You may assume that $\sin\phi > \phi\cos\epsilon$ for $0 \le \phi \le \epsilon$ where $\epsilon = 0^+$. Hence extend the result ($\star$) to the case $\arg w = \pi/4$ and evaluate

$$\int_0^\infty \cos x^2 \, \mathrm{d}x \, .$$

To prove this, we can use the following inequalities

$$\begin{aligned}
\left| R \int_0^{\frac{\pi}{4}} e^{-R^2 \mathrm{e}^{2\mathrm{i}\theta}} \mathrm{e}^{\mathrm{i}\theta} \, \mathrm{d}\theta \right| &< \int_0^{\frac{\pi}{4}} R\mathrm{e}^{-R^2 \cos 2\theta} \, \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} R\mathrm{e}^{-R^2 \sin 2\theta} \, \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{R}{2} e^{-R^2 \sin 2\phi} \, \mathrm{d}\phi \\
&= \frac{R}{2} \left(\int_0^{\epsilon} \mathrm{e}^{-R^2 \sin 2\phi} \, \mathrm{d}\phi + \int_\epsilon^{\frac{\pi}{2}} \frac{R}{2} \mathrm{e}^{-R^2 \sin 2\phi} \, \mathrm{d}\phi \right) \\
&< \frac{R}{2} \left(\int_0^{\epsilon} \mathrm{e}^{-R^2 \phi \cos\epsilon} \, \mathrm{d}\phi + \frac{\pi}{2} \mathrm{e}^{-R^2 \sin\epsilon} \right) \\
&< \frac{1}{2R\cos\epsilon} + \frac{\pi}{4R} \mathrm{e}^{-R^2 \sin\epsilon} .
\end{aligned}$$

Taking the limit $R \to \infty$, we can immediately see that

$$\boxed{\lim_{R\to\infty} \left| R \int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2 e^{2\mathrm{i}\theta}} \mathrm{e}^{\mathrm{i}\theta} \, \mathrm{d}\theta \right| = 0} .$$

Next, we let $w = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$ such that $w^2 = \mathrm{i}$ and $\arg w = \pi/4$. Then,

$$\int_0^\infty \mathrm{e}^{\mathrm{i}x^2} \, \mathrm{d}x = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \int_0^\infty \mathrm{e}^{-x^2} \, \mathrm{d}x \, .$$

Taking the real part, we see that

$$\int_0^\infty \cos x^2 \, \mathrm{d}x = \boxed{\frac{1}{2}\sqrt{\frac{\pi}{2}}} .$$

S1 2012

2. Consider the mapping specified by the function

$$f : z \mapsto w = z + \frac{1}{z}$$

from the complex z-plane to the complex w-plane.

Give the subset of the complex z-plane in which f is holomorphic, and determine whether the mapping is conformal in the region of holomorphy.

---

f is holomorphic in the punctured plane $\mathbb{C} \setminus \{0 \cup \infty\}$. For the mapping to be conformal, we require that $\boxed{f \text{ locally preserves angles}}$. Consider

$$\delta w = \frac{\mathrm{d}w}{\mathrm{d}z}\delta z = \left(1 - \frac{1}{z^2}\right)\delta z.$$

The condition for f to be a conformal map is therefore

$$\boxed{\arg \frac{\delta w_1}{\delta w_2} = \arg \frac{\delta z_1}{\delta z_2}}.$$

f is conformal in the region of holomorphy precisely where its derivative is non-zero. $f'(z) = 0$ at $z = \pm 1$, and therefore f conformal everywhere in $\mathbb{C} \setminus \{0 \cup \infty\}$ except at $\boxed{z = \pm 1}$.

Determine how the mapping transforms angles at the point $z = 1$.

To check this, we have to consider higher order expansions of δw. To $\mathcal{O}(\delta z^2)$,

$$\begin{aligned}\delta w &= \frac{\mathrm{d}w}{\mathrm{d}z}\delta z + \frac{1}{2!}\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\delta z^2 \\ &= \left(1 - \frac{1}{z^2}\right)\delta z + \frac{1}{z^3}\delta z^2.\end{aligned}$$

At $z = 1$, $\delta w = \delta z^2$. Defining $\theta = \arg\,(\delta z_1/\delta z_2)$ and $\phi = \arg\,(\delta w_1/\delta w_2)$, we have

$$\frac{\delta w_1}{\delta w_2} = \frac{\delta z_1^2}{\delta z_2^2} \Rightarrow \arg \frac{\delta w_1}{\delta w_2} = \arg \frac{\delta z_1^2}{\delta z_2^2} = 2 \arg \frac{\delta z_1}{\delta z_2} \Rightarrow \boxed{\phi = 2\theta}.$$

Determine the image through f of the upper and lower halves of the unit circle $|z| = 1$.

Along the unit circle, $z = \mathrm{e}^{\mathrm{i}\theta}$, and so $w = \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} = 2\cos\theta$. Thus, f conformally maps the upper and lower halves of the unit circle $|z| = 1$ onto the $\boxed{\text{finite straight slit } w : |\mathrm{Re}\{w\}| \leq 2, \mathrm{Im}\{w\} = 0}$. Note that the argument doubles at $z = \pm 1$ as shown from the previous part.

Determine the image through f of a circle $|z| = \rho$ with $\rho \neq 1$.

Along this circle, $z = \rho \mathrm{e}^{\mathrm{i}\theta}$, and so

$$\begin{aligned}w &= \rho \mathrm{e}^{\mathrm{i}\theta} + \rho^{-1}\mathrm{e}^{-\mathrm{i}\theta} \\ &= \rho(\cos\theta + \mathrm{i}\sin\theta) + \rho^{-1}(\cos\theta - \mathrm{i}\sin\theta) \\ &= (\rho + \rho^{-1})\cos\theta + \mathrm{i}(\rho - \rho^{-1})\sin\theta.\end{aligned}$$

Notice that $x = \mathrm{Re}\{w\} = (\rho + \rho^{-1})\cos\theta$ and $y = \mathrm{Im}\{w\} = (\rho - \rho^{-1})\sin\theta$. Eliminating θ, we have

$$\left(\frac{x}{\rho + \rho^{-1}}\right)^2 + \left(\frac{y}{\rho - \rho^{-1}}\right)^2 = 1.$$

Thus, f conformally maps the circle $|z| = \rho$ onto an $\boxed{\text{ellipse}}$ with equation as described above.

S1 2011

1. Solve the equation

$$\sin z = 3$$

for z in $\mathbb{C}$.

The equation can be rewritten as

$$\mathrm{e}^{2\mathrm{i}z} - 6\mathrm{i}\mathrm{e}^{\mathrm{i}z} - 1 = 0.$$

Defining $w = \mathrm{e}^{\mathrm{i}z}$, we have the quadratic equation $w^2 - 6\mathrm{i}w - 1 = 0$. Solving for w, we obtain

$$w = \frac{6\mathrm{i} \pm \sqrt{-36+4}}{2} = \frac{6\mathrm{i} \pm \sqrt{32}\mathrm{i}}{2} = (3 \pm 2\sqrt{2})\mathrm{i}.$$

Consider the Laurent series expansion about $z = 0$ of the function

$$f(z) = \frac{\sin z}{z^4}.$$

Determine the principal part of the series and the first term of the analytic part of the series.

The Laurent expansion of $f(z)$ about the point $z = z_0$ is

$$f(z) = \sum_{k=-N}^{\infty} a_k (z - z_0)^k, \quad a_{-N} \neq 0.$$

In this case, we have

$$\begin{aligned} f(z) &= \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...\right) \\ &= \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{120} + ... \end{aligned}$$

The principal part is thus $\boxed{z^{-3} - (z^{-1}/6)}$ and the first term of the analytic part is $\boxed{z/120}$.

Evaluate the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2 + \cos\theta}$$

using complex contour integration methods.

S1 2011

2. Consider the function

$$u(x,y) = \mathrm{e}^{-x}\cos y + xy.$$

Show that $u(x,y)$ is harmonic over the whole xy-plane.

For $u(x,y)$ to be harmonic over the entire plane, we require $\nabla^2 u = 0$. Evaluating the derivatives of $u(x,y)$, we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= -\mathrm{e}^{-x}\cos y + y \Rightarrow \frac{\partial^2 u}{\partial x^2} = \mathrm{e}^{-x}\cos y, \\
\frac{\partial u}{\partial y} &= -\mathrm{e}^{-x}\sin y + x \Rightarrow \frac{\partial^2 u}{\partial y^2} = -\mathrm{e}^{-x}\cos y.
\end{aligned}$$

Therefore,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \mathrm{e}^{-x}\cos y - \mathrm{e}^{-x}\cos y = 0 \Rightarrow \boxed{u(x,y) \text{ is harmonic in the entire } xy\text{-plane}}.$$

Find a holomorphic function $f(z)$ whose real part is given by $u(x,y)$ and which is real-valued at $z=0$, where $z = x + \mathrm{i}y$.

Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$. From the Cauchy-Riemann conditions,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

From the first condition,

$$\frac{\partial v}{\partial y} = -\mathrm{e}^{-x}\cos y + y \Rightarrow v(x,y) = -\mathrm{e}^{-x}\sin y + \frac{y^2}{2} + g(x)$$

where $g(x)$ is purely a function of x. As for the other condition,

$$-\frac{\partial v}{\partial x} = -\mathrm{e}^{-x}\sin y + x \Rightarrow v(x,y) = -\mathrm{e}^{-x}\sin y - \frac{x^2}{2} + \frac{y^2}{2} + c$$

where c is a constant, which we can take to be zero. Note that $z^2 = (x+\mathrm{i}y)^2 = x^2 - y^2 + 2\mathrm{i}y$. Altogether, we have

$$\begin{aligned}
f(z) &= u(x,y) + \mathrm{i}v(x,y) \\
&= \mathrm{e}^{-x}\cos y + xy - \mathrm{i}\mathrm{e}^{-x}\sin y - \frac{\mathrm{i}x^2}{2} + \frac{\mathrm{i}y^2}{2} \\
&= \mathrm{e}^{-x}(\cos y - \mathrm{i}\sin y) + xy - \frac{\mathrm{i}}{2}(x^2 - y^2) \\
&= \mathrm{e}^{-x}\mathrm{e}^{-\mathrm{i}y} - \frac{\mathrm{i}}{2}(x^2 - y^2 + 2ixy) \\
&= \mathrm{e}^{-(x+\mathrm{i}y)} - \frac{\mathrm{i}}{2}(x+\mathrm{i}y)^2 \\
&= \boxed{\mathrm{e}^{-z} - \frac{\mathrm{i}z^2}{2}}.
\end{aligned}$$

Write down the family of curves in the xy-plane which are orthogonal to the curves $\mathrm{e}^{-x}\cos y + xy =$ constant.

For the curve $u(x,y) =$ constant, $\hat{\mathbf{n}}_u = \nabla u = (\,\partial u/\partial x\,,\, \partial u/\partial y\,)$ represents the normal vector to the curve. The tangent vector to the curve $\hat{\mathbf{t}}_u$ satisfies $\hat{\mathbf{t}}_u \cdot \hat{\mathbf{n}}_u = 0$, so $\hat{\mathbf{t}}_u = (\,\partial u/\partial y\,, -\,\partial u/\partial x\,)$. We want the family of curves $v(x,y) =$ constant with tangent vector $\hat{\mathbf{t}}_v$ perpendicular to $\hat{\mathbf{t}}_u$. This is achieved by letting $\hat{\mathbf{t}}_v = \hat{\mathbf{n}}_u$, or $(\,\partial v/\partial y\,, -\,\partial v/\partial x\,) = (\,\partial u/\partial x\,,\, \partial u/\partial y\,)$. These are the Cauchy-Riemann conditions as discussed in the previous part. Thus, the family of curves orthogonal to the curves $\mathrm{e}^{-x}\cos y + xy =$ constant is $\boxed{-\mathrm{e}^{-x}\sin y - \frac{x^2}{2} + \frac{y^2}{2} = \text{constant}}$.

Evaluate the integral of $u(x, y)$ round the circle in the xy-plane with centre at the origin and radius 1.

This is simply the real part of the integral of $f(z)$ around the unit circle $|z| = 1$ in the complex plane. Using Cauchy's residue theorem,

$$\begin{aligned}
\int_0^{2\pi} u(\theta)\,\mathrm{d}\theta &= \mathrm{Re}\left\{\frac{1}{\mathrm{i}}\oint_{|z|=1}\frac{f(z)}{z}\,\mathrm{d}z\right\} \\
&= \mathrm{Re}\left\{2\pi\sum_j \mathrm{Res}_j\left(\frac{\mathrm{e}^{-z}}{z} - \frac{\mathrm{i}z}{2}\right)\right\} \\
&= \mathrm{Re}\left\{2\pi\operatorname*{Res}_{z=0}\left(\frac{\mathrm{e}^{-z}}{z} - \frac{\mathrm{i}z}{2}\right)\right\} \\
&= \boxed{2\pi}.
\end{aligned}$$

S1 2008

1. Define the terms i) analytic function; and ii) conformal map.

An analytic function $f(z)$ is one where the $\boxed{\text{derviative with respect to } z \text{ exists}}$, i.e. the limit

$$\boxed{\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}}$$

exists $\boxed{\text{independent of the direction of approach}}$. Equivalently, $f(z)$ is $\boxed{\text{independent of } \bar{z}}$, where $\bar{z}$ is the complex conjugate of z, and must satisfy the Cauchy-Riemann conditions

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}.$$

A conformal map $f : z \mapsto w$ is a transformation that $\boxed{\text{locally preserves angles}}$.

Consider the Möbius mapping

$$f_M : z \mapsto w = \frac{az+b}{cz+d},$$

where a, b, c and d are complex constants.

Under what conditions is this mapping conformal?

Consider

$$\delta w = \frac{\mathrm{d}w}{\mathrm{d}z}\delta z = \left(1 - \frac{1}{z^2}\right)\delta z.$$

The condition for f to be a conformal map is therefore

$$\boxed{\arg\frac{\delta w_1}{\delta w_2} = \arg\frac{\delta z_1}{\delta z_2}}.$$

f is conformal in the region of holomorphy where $\boxed{f'(z) \neq 0}$.

Show that the transformation can be written as

$$f_M : z \mapsto w = \frac{a}{c} - \frac{ad-bc}{c^2}\frac{1}{z+d/c}.$$

Hence, show that the mapping may be regarded as a sequence of elementary transformations.

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aaaa

Explain why $|z-a| = r$ represents a circle in the complex plane.

Let $z = x + iy$ and $a = a_R + ia_I$. Then, $|z-a| = |(x-a_R) + i(y-a_I)| = \sqrt{(x-a_R)^2 + (y-a_I)^2} = r$, which corresponds to a $\boxed{\text{circle of radius } r}$ with $\boxed{\text{centre at } z = a}$.

Consider the equation

$$\left|\frac{z-p}{z-q}\right| = k \qquad (\star)$$

Show that i) if $k \neq 1$, $k > 0$, equation ($\star$) describes a circle with centre at $a = (p - k^2 q)/(1-k^2)$ and radius $r = k|p-q|/|1-k^2|$; ii) if $k = 1$, equation ($\star$) describes a straight line; and iii) equation ($\star$) is transformed into another equation of the same type by each member of the sequence of elementary transformations derived from f_M. Hence, deduce that f_M maps circles and lines into circles and lines.

---

aaaa

By viewing

$$w = \frac{z - p}{z - q}$$

as a conformal transformation, deduce that the line $\arg w = \theta$ is mapped into a circle in the z-plane passing through $z = p$ and $z = q$ for any θ. Use the conformal property to show that, for a given choice of p and q, circles of this type are orthogonal to all those given by equation ($\star$).

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aaaa

S1 2008

1. With reference to the analyticity of the function $w = \ln z$, elucidate the terms i) branch point; and ii) Riemann surface.

Let z_0 be a point on the complex plane, and define $z - z_0 = r\mathrm{e}^{\mathrm{i}\theta}$. If z_0 is a branch point of a function $f(z)$, then $f(z_0 + r\mathrm{e}^{\mathrm{i}\theta}) \neq f(z_0 + r\mathrm{e}^{\mathrm{i}(\theta+2\pi)})$ i.e. the function is multivalued if that point is encircled. In the case of $f(z) = \ln z$, notice that $f(r\mathrm{e}^{\mathrm{i}\theta}) = \ln r + \mathrm{i}\theta \neq f\left(r\mathrm{e}^{\mathrm{i}(\theta+2\pi)}\right) = \ln r + \mathrm{i}(\theta + 2\pi)$. This shows the multivalued behaviour of $\sqrt{z}$ around $z = 0$, and hence $\boxed{z = 0 \text{ is a branch point}}$.

Define

$$z^a = \mathrm{e}^{a \ln z}.$$

Evaluate this in the polar representation of the complex z-plane and show that z^a is analytic and single-valued in the cut plane described by $0 < \arg\, z < 2\pi$. Classify the singularities of the function

$$f(z) = \frac{z^a}{z^2 + 1}.$$

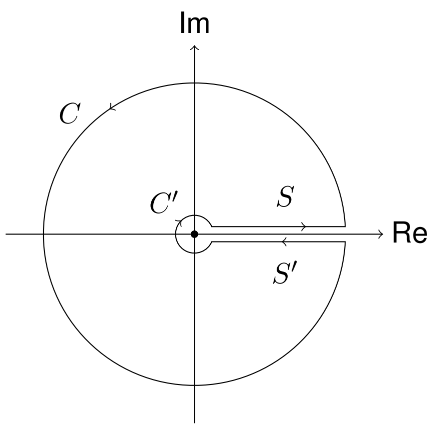
$z = 0$ is a branch point. $z = \pm\mathrm{i}$ are simple poles. Choosing our branch cut to be $[0, \infty)$, $f(z)$ is holomorphic on the punctured slit plane $\mathbb{C} \setminus \{\pm\mathrm{i} \cup [1, \infty)\}$. The residues at $z = \pm\mathrm{i}$ are

$$\operatorname*{Res}_{z=\mathrm{i}} f(z) = \lim_{z\to\mathrm{i}} (z - \mathrm{i}) f(z) = \lim_{z\to\mathrm{i}} \frac{z^a}{z + \mathrm{i}} = \boxed{\frac{\mathrm{i}^a}{2\mathrm{i}}},$$

$$\operatorname*{Res}_{z=-\mathrm{i}} f(z) = \lim_{z\to-\mathrm{i}} (z + \mathrm{i}) f(z) = \lim_{z\to-\mathrm{i}} \frac{z^a}{z - \mathrm{i}} = \boxed{-\frac{(-\mathrm{i})^a}{2\mathrm{i}}},$$

which we have left in their unsimplified forms for convenience later.

Using the residue theorem, evaluate the integral of $f(z)$ on the keyhole contour shown in the figure below.



Hence, by carefully considering the components of the contour integral, deduce that

$$\int_0^\infty \frac{x^a}{x^2 + 1}\,\mathrm{d}x = \frac{\pi}{2\cos(\pi a/2)}$$

provided the parameter a satisfies $-1 < a < 1$, and comment on the cases i) $a = 0$; and ii) $a \to \pm 1$.

Let $\Gamma = S \cup C \cup S' \cup C'$. To determine the behaviour of $f(z)$ above and below the branch cut, we introduce the substitution $z = re^{i\theta}$ (vector with head at z and tail at 0), where $r = |z|$ and $\theta = \arg(z)$. We also assert that when $z = x + iy$ lies on the real axis with $x > 0$, $\theta = 0$. Directly above the branch cut (where $y = 0^+$), $r = |z| = x$ and $\theta = 0$, so

$$f_{\text{above}} = f(z) = \frac{r^a}{z^2+1} = \frac{x^a}{x^2+1}.$$

Likewise, directly below the branch cut (where $y = 0^-$), $r = |z| = x$ and $\theta = 2\pi$, so

$$f_{\text{below}} = \frac{r^a e^{2i\pi a}}{z^2+1} = \frac{x^a e^{2i\pi a}}{x^2+1}.$$

We shall assume without proof that the contributions due to C and C' tend to zero as $R \to \infty$ and $\epsilon \to \infty$. Using Cauchy's residue theorem, the contour integral becomes

$$\begin{aligned}
\oint_\Gamma f(z)\,dz &= \int_0^\infty \frac{x^a}{x^2+1}\,dx + \int_\infty^0 \frac{x^a e^{2i\pi a}}{x^2+1}\,dx \\
&= \left(1 - e^{2i\pi a}\right) \int_0^\infty \frac{x^a}{x^2+1}\,dx \\
&= -2ie^{i\pi a} \sin \pi a \int_0^\infty \frac{x^a}{x^2+1}\,dx \\
&= 2\pi i \left[\frac{i^a}{2i} - \frac{(-i)^a}{2i}\right] \\
&= \pi \left[i^a - (-i)^a\right] \\
&= \pi \left(e^{\frac{i\pi a}{2}} - e^{\frac{3i\pi a}{2}}\right) \\
&= -\pi e^{i\pi a} \left(e^{\frac{i\pi a}{2}} - e^{-\frac{i\pi a}{2}}\right) \\
&= -2i\pi e^{i\pi a} \sin \frac{\pi a}{2}.
\end{aligned}$$

The integral of interest is therefore

$$\int_0^\infty \frac{x^a}{x^2+1}\,dx = \frac{-2i\pi e^{i\pi a} \sin(\pi a/2)}{-2ie^{i\pi a} \sin \pi a} = \boxed{\frac{\pi}{2\cos(\pi a/2)}}.$$

When $a = 0$, $\cos(\pi a/2) = \cos 0 = 1$, and the value of the integral is $\boxed{\pi/2}$. We can confirm this by replacing the numerator of the integrand with $x^0 = 1$, and calculating

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}.$$

On the other hand, as $a \to \pm 1$, $\cos(\pi a/2) \to 0$, so the integral $\boxed{\text{diverges}}$. For $a \to 1$, we see that

$$\int_0^\infty \frac{x}{x^2+1}\,dx = \left[\ln \sqrt{x^2+1}\right]_0^\infty \to \boxed{\infty},$$

whereas for $a \to -1$,

$$\int_0^\infty \frac{1}{x(x^2+1)}\,dx = \int_0^\infty \frac{1}{x} - \frac{x}{x^2+1}\,dx = \left[\ln x - \ln \sqrt{x^2+1}\right]_0^\infty \to \boxed{\infty}.$$

S1 2008

3. Verify the values of the following integrals by contour integration, explaining carefully the methods which you use.

c) $\displaystyle\int_0^\infty \frac{\sinh ax}{\sinh \pi x}\,\mathrm{d}x = \frac{1}{2}\tan\frac{a}{2}$ for real values of a satisfying $|a| < 1$.

We define

$$f(z) = \frac{\sinh az}{\sinh \pi z}.$$

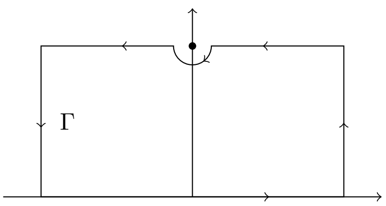
Since $\sinh \pi z = -\mathrm{i}\sin \mathrm{i}\pi z$, we can easily determine the location of the poles, which are the zeros of $\sinh \pi z$ except $z = 0$.

$$\sinh \pi z = -\mathrm{i}\sin \mathrm{i}\pi z = 0 \Rightarrow \sin \mathrm{i}\pi z = 0 \Rightarrow z = \mathrm{i}n, n \in \{\mathbb{N} \setminus 0\}.$$

This means that the poles all lie on the imaginary axis. It is therefore wise to use a rectangular contour. To determine the most appropriate location of the top of the contour, we first find the smallest positive value of y such that $\sinh \pi(x + \mathrm{i}y) = -\sinh \pi x$. Using the trigonometric identity $\sinh(a + \mathrm{i}b) = \sinh a \cos b + \mathrm{i}\cosh a \sin b$, we can deduce that

$$\sinh \pi(x + \mathrm{i}y) = \sinh \pi x \cos \pi y + \mathrm{i}\cosh \pi x \sin \pi y = -\sinh \pi x \Rightarrow y_{\text{min}} = 1.$$

The contour of choice is therefore $\Gamma = [-R, R] \cup [R, R+\mathrm{i}] \cup [\epsilon+\mathrm{i}, R+\mathrm{i}] \cup C \cup [-R+i, -\epsilon+\mathrm{i}] \cup [-R, -R+\mathrm{i}]$ as shown below. C is the semicircular deformation chosen such that none of the poles are bounded by Γ, allowing $f(z)$ to be holomorphic within Γ, such that the contour integral of $f(z)$ along Γ is zero.



The integral can be split into

$$\oint_\Gamma f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_0^1 \mathrm{i}f(R + \mathrm{i}y)\,\mathrm{d}y + \int_R^\epsilon f(x + \mathrm{i})\,\mathrm{d}x$$
$$+ \int_{C_2} f(z)\,\mathrm{d}z + \int_{-\epsilon}^{-R} f(x + \mathrm{i})\,\mathrm{d}x + \int_1^0 \mathrm{i}f(-R + \mathrm{i}y)\,\mathrm{d}y\,.$$

We next consider each integral in the limits $R \to \infty$ and $\epsilon \to 0$. Notice that since $|a| < \pi$, therefore

$$f(R + \mathrm{i}y) = \frac{\sinh a(R + \mathrm{i}y)}{\sinh \pi(R + \mathrm{i}y)} = \frac{\mathrm{e}^{a(R+\mathrm{i}y)} - \mathrm{e}^{-a(R+\mathrm{i}y)}}{\mathrm{e}^{\pi(R+\mathrm{i}y)} - \mathrm{e}^{-\pi(R+\mathrm{i}y)}} \sim \mathrm{e}^{-(\pi-a)R} \to 0,$$

and similarly

$$f(-R + \mathrm{i}y) = \frac{\sinh a(-R + \mathrm{i}y)}{\sinh \pi(-R + \mathrm{i}y)} = \frac{\mathrm{e}^{a(-R+\mathrm{i}y)} - \mathrm{e}^{-a(-R+\mathrm{i}y)}}{\mathrm{e}^{\pi(-R+\mathrm{i}y)} - \mathrm{e}^{-\pi(-R+\mathrm{i}y)}} \sim \mathrm{e}^{-(\pi-a)R} \to 0.$$

The integrals along the vertical legs of the contour can therefore be ignored. Furthermore, the integrals along the horizontal sides in these limits can be combined to obtain

$$\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x} + \frac{\sinh a(x + \mathrm{i})}{\sinh \pi x}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{(1 + \cos a)\sinh ax + \mathrm{i}\sin a \cosh ax}{\sinh \pi x}\,\mathrm{d}x$$
$$= (1 + \cos a)\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\,\mathrm{d}x + \mathrm{i}\sin a\int_{-\infty}^{\infty} \frac{\cosh ax}{\sinh \pi x}\,\mathrm{d}x\,.$$

Next, we prove and invoke the following lemma to help us evaluate the contribution along C.

Lemma: Let $f(z)$ be holomorphic on the open disc $\mathbb{D}(z_0, R) = \{z : |z - z_0| < R\} \setminus \{z_0\}$ with a simple pole at $z = z_0$. For $0 < \epsilon < R$, let $C_\epsilon : [\theta_1, \theta_2] \to \mathbb{C}$ be the path $\theta \mapsto z_0 + \epsilon e^{i\theta}$ (i.e. the circular arc of radius ϵ centred on $z = z_0$ subtending θ_1 to θ_2). Then,

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)\,\mathrm{d}z = \mathrm{i}(\theta_2 - \theta_1) \operatorname*{Res}_{z=z_0} f(z).$$

Proof: $f(z)$ can be expressed as

$$f(z) = \frac{\operatorname*{Res}_{z=z_0} f(z)}{z - z_0} + g(z),$$

where $g(z)$ is holomorphic on $\mathbb{D}(z_0, R)$. Then, by the estimation lemma, which states that

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \leq \int_\Gamma |f(z)||\,\mathrm{d}z\,| \leq \sup_{z\in\Gamma} |f(z)| \int_\Gamma |\,\mathrm{d}z\,|,$$

we have

$$\int_{C_\epsilon} f(z)\,\mathrm{d}z \leq \epsilon(\theta_2 - \theta_1) \sup_{z \in C_\epsilon} |g(z)|\,.$$

Since $g(z)$ is bounded on $\mathbb{D}(z_0, R)$, thus

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)\,\mathrm{d}z = 0,$$

and thus

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{\operatorname*{Res}_{z=z_0} f(z)}{z - z_0}\,\mathrm{d}z &= \operatorname*{Res}_{z=z_0} f(z) \lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{\mathrm{d}z}{z - z_0} \\
&= \operatorname*{Res}_{z=z_0} f(z) \lim_{\epsilon \to 0} \int_{\theta_1}^{\theta_2} \frac{\mathrm{i}\epsilon e^{\mathrm{i}\theta}}{\epsilon e^{\mathrm{i}\theta}}\,\mathrm{d}\theta \\
&= \mathrm{i}(\theta_2 - \theta_1) \operatorname*{Res}_{z=z_0} f(z).
\end{aligned}$$

Therefore, the integral along C is just

$$\int_C f(z)\,\mathrm{d}z = -\mathrm{i}\pi \operatorname*{Res}_{z=\mathrm{i}} f(z),$$

and

$$\begin{aligned}
\operatorname*{Res}_{z=\mathrm{i}} f(z) &= \lim_{z\to \mathrm{i}} \frac{(z - \mathrm{i})\sinh az}{\sinh \pi z} \\
&= \lim_{z \to \mathrm{i}} \frac{\sinh az + a(z - \mathrm{i})\cosh az}{\pi \cosh \pi z} \\
&= \frac{\sinh a\mathrm{i}}{\pi \cosh \pi \mathrm{i}} = \frac{\mathrm{i}\sin a}{\pi \cos \pi} = -\frac{\mathrm{i}\sin a}{\pi}.
\end{aligned}$$

Altogether, this gives

$$-\sin a + (1 + \cos a) \int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\,\mathrm{d}x + \mathrm{i}\sin a \int_{-\infty}^{\infty} \frac{\cosh ax}{\sinh \pi x}\,\mathrm{d}x = 0.$$

Taking the real part, we can easily see that

$$\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\,\mathrm{d}x = \frac{\sin a}{1 + \cos a} = -\frac{2\sin\frac{a}{2}\cos\frac{a}{2}}{2\cos^2\frac{a}{2}} = \tan\frac{a}{2}.$$

Since the integrand is an even function, we can verify that

$$\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\,\mathrm{d}x = \boxed{\frac{1}{2}\tan\frac{a}{2}}.$$

S15 2008

2. The population anomalies of a certain species of caterpillar at the end of the n^{th} breeding season, x_{n+1}, are related to the population anomalies at the end of the previous season by the following discrete quadratic map

$$x_{n+1} = f(x_n) = x_n^2 + c.$$

where c is a parameter. Find and classify the fixed points of this map as a function of c over the range $-1 < c < 1$.

Denote the fixed points as x_*. Then, $x_* = x_*^2 + c$, from which we can solve to obtain

$$x_*^{\pm} = \boxed{\frac{1 \pm \sqrt{1-4c}}{2}}.$$

To determined the stability of the fixed points, we consider a nearby orbit $x_n = x_* + \eta_n$ where η_n is a small parameter. Then,

$$x_{n+1} = x_* + \eta_{n+1} = f(x_*) + \eta_n f'(x_*) + \mathcal{O}(\eta_n^2) \approx x_* + \eta_n f'(x_*).$$

This gives the linearised map $\eta_n f'(x_*) = \eta_{n+1}$, where $f'(x_*)$ is the eigenvalue. In this case,

$$f'(x_*^{\pm}) = 2x_*^{\pm} = 1 \pm \sqrt{1-4c}.$$

- If $\boxed{c > 1/4}$, there are $\boxed{\text{no real-valued fixed points}}$, and $\boxed{\text{all the orbits tend to infinity}}$.
- If $\boxed{c < 1/4}$, there are $\boxed{\text{two fixed points}}$, x_*^+ and x_*^-, as calculated above. $\boxed{|f'(x_*^+)| > 1}$ and is therefore an $\boxed{\text{unstable fixed point}}$, while $\boxed{|f'(x_*^-)| < 1}$ and is therefore a $\boxed{\text{linearly stable fixed point}}$.
- If $\boxed{c = 1/4}$, then there is only $\boxed{\text{one fixed point } x_* = 1/2}$ of multiplicity two.

Find the two values of c within this range at which the map undergoes a bifurcation. Classify these bifurcations and sketch cobweb diagrams on either side of each bifurcation to illustrate the nature of the solution.

AAAA.

For $c = -1$, sketch the second-iterate map x_{n+2} against x_n and determine the stability of its fixed points at $x_* = -1$ and $x_* = 0$. Graphically, or otherwise, show that the solution corresponds to a two-cycle in the first-iterate map.

AAAA.

Show that the first-iterate map can be transformed to the Logistic Map,

$$y_{n+1} = ry_n(1 - y_n),$$

by the transformation $y_n = ax_n + b$, and determine the parameters a, b and r in terms of c. This is known as a conjugacy, which is a change of variables that transforms one map into another.

AAAA.

S15 2008

3. A discrete linear stochastic ($\mathrm{AR}(2)$?) process is governed by the equation

$$X_{t+1} = aX_t - 2aX_{t-1} + Z_t,$$

where X_t is real (with zero mean?), a is a real constant, and Z_t is a unit variance uncorrelated noise process with zero mean. Assuming that X_t is stationary, show that the variance of this process is given by

$$c_0 \equiv \mathrm{Var}(X_t) = \mathbb{E}(X_t^2) = \frac{1+2a}{1+2a-5a^2-6a^3}.$$

Note: I think there is a typo in the required expression.

Multiplying the equation by Z_t and taking the expectation, we have

$$\begin{aligned}
\mathbb{E}(X_{t+1}Z_t) &= \mathbb{E}(aX_tZ_t - 2aX_{t-1}Z_t + Z_t^2) \\
&= a\mathbb{E}(X_tZ_t) - 2a\mathbb{E}(X_{t-1}Z_t) + \mathbb{E}(Z_t^2) \\
&= a\mathbb{E}(X_tZ_t) + \mathbb{E}(Z_t^2) \\
&= a\mathbb{E}(X_tZ_t) + 1 = 0.
\end{aligned}$$

This gives $\mathbb{E}(X_tZ_t) = -1/a$. Next, multiplying the equation by X_t and taking the expectation, we have

$$\begin{aligned}
\mathbb{E}(X_{t+1}X_t) &= \mathbb{E}(aX_t^2 - 2aX_{t-1}X_t + X_tZ_t) \\
&= a\mathbb{E}(X_t^2) - 2a\mathbb{E}(X_{t-1}X_t) + \mathbb{E}(X_tZ_t) \\
&= ac_0 - 2ac_1 + \mathbb{E}(X_tZ_t) \\
&= ac_0 - 2ac_1 - \frac{1}{a} = c_1.
\end{aligned}$$

Rearranging, we have $a^2c_0 - a(1+2a)c_1 - 1 = 0$. Lastly, we require require

$$\begin{aligned}
\mathbb{E}(X_{t+1}X_{t+1}) &= \mathbb{E}(aX_{t+1}X_t - 2aX_{t+1}X_{t-1} + X_{t+1}Z_t) \\
&= a\mathbb{E}(X_{t+1}X_t) - 2a\mathbb{E}(X_{t+1}X_{t-1}) + \mathbb{E}(X_{t+1}Z_t) \\
&= ac_1 - 2ac_2 = c_0, \\
\mathbb{E}(X_{t+1}X_{t-1}) &= \mathbb{E}(aX_tX_{t-1} - 2aX_{t-1}^2 + X_{t-1}Z_t) \\
&= a\mathbb{E}(X_tX_{t-1}) - 2a\mathbb{E}(X_{t-1}^2) + \mathbb{E}(X_{t-1}Z_t) \\
&= ac_1 - 2ac_0 = c_2.
\end{aligned}$$

These equations give $a(1+2a)c_1 = (1-4a^2)c_0$. Putting everything together, we have

$$\begin{aligned}
\left[a^2 - \frac{(1-4a^2)(1+2a)}{1-2a}\right]c_0 - 1 &= \left[\frac{a^2(1-2a) - (1-4a^2)(1+2a)}{1-2a}\right]c_0 - 1 \\
&= \frac{-2a^3 + 8a^3 + a^2 + 4a^2 - 2a - 1}{1-2a}c_0 - 1 \\
&= \frac{6a^3 + 5a^2 - 2a - 1}{1-2a}c_0 - 1 = 0.
\end{aligned}$$

Rearranging, we obtain

$$c_0 = \boxed{\frac{2a-1}{1+2a-5a^2-6a^3}}.$$

Given that there is only one value of a in the range $-1 < a < 1$ for which c_0 is a minimum, find this value and the corresponding minimum of c_0.

Solving for a in $\mathrm{d}c_0/\mathrm{d}a = 0$, we have $a = \boxed{-2/3}$ and $c_0 = \boxed{3}$.

Given that the process is non-stationary (has infinite variance) if $a = -1$, find the values of a in the interval $-1 < a < 1$ for which the process is stationary. Hence sketch the process variance (where it is defined) as a function of a over the interval $-1 < a < 1$.

No idea.

A simple model of El Nino represents annual mean sea surface temperatures in the equatorial Pacific as

$$T_{t+1} = 0.125T_t - 0.25T_{t-1} + Z_t,$$

where the units of time are years, and z_t is again a unit variance uncorrelated noise process. Estimate lag-covariances $c_k = \mathbb{E}(T_t T_{t-k})$ for $k = 0$ to $k = 5$, sketch c_k and comment on the implications of your result for the periodicity of El Nino.

Also no idea.

S1 2007

4. Integrate by complex methods

$$I(k)=\int_0^\infty \frac{\sin x}{x(k^2x^2+1)}\,\mathrm{d}x\,,$$

where k is real.

Recall Jordan's lemma, which states that if $g(z)$ is analytic in the upper half-plane (apart from a finite number of isolated singularities), C_R is the upper semicircular contour of radius R, and

$$\lim_{R\to\infty}\sup_{|z|=R}|g(z)|=0,$$

then

$$\lim_{R\to\infty}\int_{C_R} g(z)e^{ikz}\,\mathrm{d}z=0,$$

if $k>0$. Notice that

$$g(z)=\frac{1}{z(k^2z^2+1)}$$

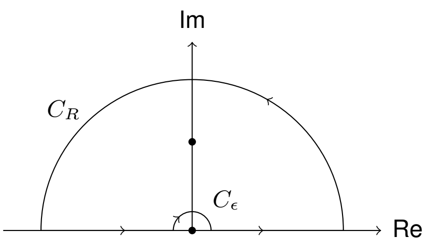
satisfies the required conditions to invoke Jordan's lemma. Since

$$\frac{\sin z}{z(k^2z^2+1)}=\mathrm{Im}\left\{\frac{\mathrm{e}^{\mathrm{i}z}}{z(k^2z^2+1)}\right\},$$

we define

$$f_k(z)=\frac{\mathrm{e}^{\mathrm{i}z}}{z(k^2z^2+1)}.$$

$f_k(z)$ has simple poles at $z=0$ and $z=\pm\mathrm{i}/k$. Let the contour which we integrate over be $\Gamma=C_R\cup[-R,-\epsilon]\cup C_\epsilon\cup[\epsilon,R]$ as shown below, where C_R is the upper semicircular contour of radius R and C_ϵ is an upper semicircular deformation away from the origin of radius ϵ.



The only pole enclosed by Γ is the one at $z=\mathrm{i}/k$, and the residue at that point is

$$\operatorname*{Res}_{z=\frac{\mathrm{i}}{k}} f_k(z)=\left(z-\frac{\mathrm{i}}{k}\right)\frac{\mathrm{e}^{\mathrm{i}z}}{z(k^2z^2+1)}\bigg|_{z=\frac{\mathrm{i}}{k}}=-\frac{\mathrm{e}^{-\frac{1}{k}}}{2}.$$

The integral can be split into

$$\oint_\Gamma \frac{\mathrm{e}^{\mathrm{i}z}}{z(k^2z^2+1)}\,\mathrm{d}z=\int_{-R}^{-\epsilon}\frac{\mathrm{e}^{\mathrm{i}x}}{x(k^2x^2+1)}\,\mathrm{d}x+\int_{\epsilon}^{R}\frac{\mathrm{e}^{\mathrm{i}x}}{x(k^2x^2+1)}\,\mathrm{d}x+\int_{C_R}\frac{\mathrm{e}^{\mathrm{i}z}}{z(k^2z^2+1)}\,\mathrm{d}z+\int_{C_\epsilon}\frac{\mathrm{e}^{\mathrm{i}z}}{z(k^2z^2+1)}\,\mathrm{d}z\,.$$

In the limit where $R\to\infty$ and $\epsilon\to 0$, the first two integrals are what we need, while the third integral goes to zero via Jordan's lemma. For the last integral, we prove and invoke the following lemma to help us evaluate the contribution along C_ϵ.

Lemma: Let $f(z)$ be holomorphic on the open disc $\mathbb{D}(z_0,R)=\{z:|z-z_0|<R\}\setminus\{z_0\}$ with a simple pole at $z=z_0$. For $0<\epsilon<R$, let $C_\epsilon:[\theta_1,\theta_2]\to\mathbb{C}$ be the path $\theta\mapsto z_0+\epsilon\mathrm{e}^{\mathrm{i}\theta}$ (i.e. the circular arc of radius ϵ centred on $z=z_0$ subtending θ_1 to θ_2). Then,

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} f(z)\,\mathrm{d}z=\mathrm{i}(\theta_2-\theta_1)\operatorname*{Res}_{z=z_0} f(z).$$

Proof: $f(z)$ can be expressed as

$$f(z) = \frac{\operatorname*{Res}_{z=z_0} f(z)}{z - z_0} + g(z),$$

where $g(z)$ is holomorphic on $\mathbb{D}(z_0, R)$. Then, by the estimation lemma, which states that

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \leq \int_\Gamma |f(z)||\,\mathrm{d}z\,| \leq \sup_{z\in\Gamma}|f(z)| \int_\Gamma |\,\mathrm{d}z\,|,$$

we have

$$\int_{C_\epsilon} f(z)\,\mathrm{d}z \leq \epsilon(\theta_2 - \theta_1) \sup_{z\in C_\epsilon} |g(z)|\,.$$

Since $g(z)$ is bounded on $\mathbb{D}(z_0, R)$, thus

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z)\,\mathrm{d}z = 0,$$

and thus

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{C_\epsilon} \frac{\operatorname*{Res}_{z=z_0} f(z)}{z - z_0}\,\mathrm{d}z &= \operatorname*{Res}_{z=z_0} f(z) \lim_{\epsilon\to 0}\int_{C_\epsilon} \frac{\mathrm{d}z}{z - z_0} \\
&= \operatorname*{Res}_{z=z_0} f(z) \lim_{\epsilon\to 0}\int_{\theta_1}^{\theta_2} \frac{\mathrm{i}\epsilon\mathrm{e}^{\mathrm{i}\theta}}{\epsilon\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{d}\theta \\
&= \mathrm{i}(\theta_2 - \theta_1) \operatorname*{Res}_{z=z_0} f(z).
\end{aligned}$$

In this case, $\theta_1 = \pi$ and $\theta_2 = 0$, so the integral along C_ϵ is just

$$\int_{C_\epsilon} f(z)\,\mathrm{d}z = -\mathrm{i}\pi \operatorname*{Res}_{z=0} f_k(z).$$

The origin is a simple pole, so

$$\operatorname*{Res}_{z=0} f_k(z) = \lim_{z\to 0} \frac{\mathrm{e}^{\mathrm{i}kz}}{k^2z^2 + 1} = 1.$$

Using Cauchy's residue theorem, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x(k^2x^2 + 1)}\,\mathrm{d}x &= 2\pi\mathrm{i}\left(-\frac{\mathrm{e}^{-\frac{1}{k}}}{2}\right) + \mathrm{i}\pi \\
&= \mathrm{i}\pi\left(1 - \mathrm{e}^{-\frac{1}{k}}\right) \\
\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x(k^2x^2 + 1)} &= \operatorname{Im}\left\{\mathrm{i}\pi\left(1 - \mathrm{e}^{-\frac{1}{k}}\right)\right\} \\
&= \pi\left(1 - \mathrm{e}^{-\frac{1}{k}}\right).
\end{aligned}$$

Since the integrand is an even function, therefore the value of the desired integral is

$$\int_0^\infty \frac{\sin x}{x(k^2x^2 + 1)} = \boxed{\frac{\pi}{2}\left(1 - \mathrm{e}^{-\frac{1}{k}}\right)}.$$

---

b) Discuss the limiting values of $I(k)$ in your answer as k tends to zero and as k tends to infinity. You may assume that

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

We again assume without proof that Lebesgue's dominated convergence theorem applies and that the limit and integral commute. Thus,

$$\lim_{k\to 0} I(k) = \lim_{k\to 0}\int_0^\infty \frac{\sin x}{x(k^2x^2 + 1)}\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^\infty \lim_{k\to 0} \frac{\sin x}{x(k^2x^2+1)}\,\mathrm{d}x \\
&= \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \boxed{\frac{\pi}{2}}.
\end{aligned}
$$

This is in agreement with

$$
\lim_{k\to 0} \frac{\pi}{2}\left(1-\mathrm{e}^{-\frac{1}{k}}\right) = \boxed{\frac{\pi}{2}}.
$$

Likewise,

$$
\begin{aligned}
\lim_{k\to\infty} I(k) &= \lim_{k\to\infty} \int_0^\infty \frac{\sin x}{x(k^2x^2+1)}\,\mathrm{d}x \\
&= \int_0^\infty \lim_{k\to\infty} \frac{\sin x}{x(k^2x^2+1)}\,\mathrm{d}x = \boxed{0},
\end{aligned}
$$

which is also in agreement with

$$
\lim_{k\to\infty} \frac{\pi}{2}\left(1-\mathrm{e}^{-\frac{1}{k}}\right) = \boxed{0}.
$$

S1 2006

1. What is meant by a conformal mapping? Show that, if $w = f(z)$ is analytic in a region R of the complex z-plane, then it may be used to define a conformal mapping from all points in R for which $\mathrm{d}f/\mathrm{d}z = 0$.

---

aaaa

The z-plane is mapped onto the w-plane by the transformation

$$f : z \mapsto w = z + \frac{k(z-a)}{k^2 - az}$$

where k and a are real constants.

Show that when $k \neq a$, the set of points in the w-plane obtained by mapping the curve $|z| = k$ all lie on a circle that is independent of k and a.

---

aaaa

State the region in the w-plane that corresponds to the region $z \leq k$ in the cases i) $k > a$; and ii) $k < a$. What is the mapping in the case $k = a$?

---

aaaa

S1 2006

3. Explain what it means for a function $f(z)$ to have, at $z = z_0$, i) a simple pole; ii) a double pole; iii) an isolated essential singularity; and iv) a branch point.

Consider the Laurent expansion of $f(z)$ about the point $z = z_0$, which is

$$f(z) = \sum_{k=-N}^{\infty} a_k (z - z_0)^k, \quad a_{-N} \neq 0.$$

$f(z)$ has a pole at $z = z_0$ if the lower limit $N > 0$. The order of the pole is N, and the residue is a_{-1}.

S1 2006

4. Evaluate the following real integrals by contour integration.

b) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(x^4+1)^2}$.

S1 2005

2. Evaluate the following by contour integration, explaining carefully your reasoning in each case.

b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)^2}\,\mathrm{d}x\,.$

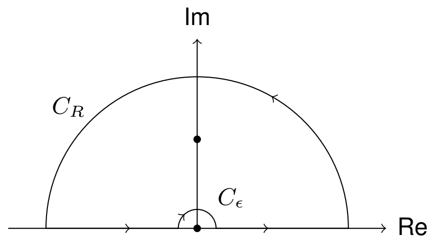
Notice that

$$\frac{\cos z}{(z^2+4)^2} = \mathrm{Re}\left\{\frac{\mathrm{e}^{\mathrm{i}z}}{(z^2+4)^2}\right\}.$$

We define

$$f(z) = \frac{\mathrm{e}^{\mathrm{i}z}}{(z^2+4)^2}.$$

$f(z)$ has second-order poles are located at $z = \pm 2\mathrm{i}$. Let the contour which we integrate over be $\Gamma = C_R \cup [-R, -\epsilon] \cup C_\epsilon \cup [\epsilon, R]$ as shown below, where C_R is the upper semicircular contour of radius R and C_ϵ is an upper semicircular deformation away from the origin of radius ϵ.



The only pole enclosed by Γ is the one at $z = 2\mathrm{i}$, and we want to find the residue at that point. Recall that the expansion of a general complex function $g(z)$ about $z = 2\mathrm{i}$ is

$$g(z) = g(2\mathrm{i}) + (z-2\mathrm{i})g'(2\mathrm{i}) + \mathcal{O}[(z-2\mathrm{i})^2].$$

Thus,

$$\mathrm{e}^{\mathrm{i}z} = \frac{1}{\mathrm{e}^2} + \frac{\mathrm{i}}{\mathrm{e}^2}(z-2\mathrm{i}) + \mathcal{O}[(z-2\mathrm{i})^2].$$

The coefficient of $1/(z-2\mathrm{i})$ in the expansion of $f(z)$ about $z = 2\mathrm{i}$ is the residue at $z = 2\mathrm{i}$. This gives

$$\begin{aligned}
\frac{1}{(z+2\mathrm{i})^2} &= \frac{1}{[(z-2\mathrm{i})-4\mathrm{i}]^2} = \frac{1}{-16[1+\frac{z-2\mathrm{i}}{4\mathrm{i}}]^2} = -\frac{1}{16}\left[1 - \frac{1}{2\mathrm{i}}(z-2\mathrm{i}) + ...\right] \\
\Rightarrow \frac{\mathrm{e}^{\mathrm{i}z}}{(z+2\mathrm{i})^2} &= ... - \frac{3\mathrm{i}}{32\mathrm{e}^2}(z-2\mathrm{i}) + ... \\
\Rightarrow \frac{\mathrm{e}^{\mathrm{i}z}}{(z^2+4)^2} &= ... - \frac{3i}{32\mathrm{e}^2}\frac{1}{z-2\mathrm{i}} + ... \\
\Rightarrow \operatorname*{Res}_{z=2\mathrm{i}} f(z) &= -\frac{3\mathrm{i}}{32\mathrm{e}^2}.
\end{aligned}$$

Consider a semicircular contour in upper half-plane. By Cauchy's residue theorem,

$$\oint_\Gamma \frac{\mathrm{e}^{\mathrm{i}z}}{(z^2+4)^2}\,\mathrm{d}z = 2\pi\mathrm{i}\left(-\frac{3i}{32\mathrm{e}^2}\right) = \frac{3\pi}{16\mathrm{e}^2}.$$

Recall Jordan's lemma, which states that if $h(z)$ is analytic in the upper half-plane (apart from a finite number of isolated singularities), C_R is the upper semicircular contour of radius R, and

$$\lim_{R\to\infty} \sup_{|z|=R} |h(z)| = 0,$$

then

$$\lim_{R\to\infty} \int_{C_R} h(z)e^{ikz}\,\mathrm{d}z = 0,$$

if $k > 0$. Notice that

$$h(z) = \frac{1}{(z^2+4)^2}$$

satisfies the required conditions to invoke Jordan's lemma. Thus,

$$\begin{aligned}
\oint_\Gamma \frac{\mathrm{e}^{\mathrm{i}z}}{(z^2+4)^2}\,\mathrm{d}z &= \int_{C_1} \frac{\mathrm{e}^{\mathrm{i}z}}{(z^2+4)^2}\,\mathrm{d}z + \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{(x^2+4)^2}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{(x^2+4)^2}\,\mathrm{d}x \\
&= \frac{3\pi}{16\mathrm{e}^2}.
\end{aligned}$$

Taking the real part of the integral, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)^2}\,\mathrm{d}x = \boxed{\frac{3\pi}{16\mathrm{e}^2}}.$$

S1 2005

3. Using the method of contour integration, evaluate the integral

S1 2005

4. Explain how each of the following functions $z = x + iy$ may be defined as analytic everywhere in the complex plane except for the portions of the real axis with $x > 1$ and $x < -1$, and as real and positive on the real... where k is real.

aaaa

In each case, what is the difference between the values of the function just above and just below the real axis, for both $x > 1$ and $x < -1$?

aaaa

With reference to either a) or b), explain the concept of a Riemann surface.

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Give an example of a function whose Riemann surface consists of three sheets.

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