A STOCHASTIC COMPETITION MODEL WITH HETEROGENEOUS FIRMS AND ENVIRONMENT EXTERNALITIES

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The impacts of anthropogenic global warming is one of the greatest concern of our time. To address it in an economic model is however a difficult task. Causes and impact are very heterogeneous. They vary in the geographic location and in the economic and social positions. Two different firms will not necessarily be affected in the same manner: depending on their sector of production, consequences range from dramatical to mostly harmless in the short run. The primary sector is particularly touched, while the industry appears to be less perturbed, while industry is responsible for a large part of the emission of C02 too for example.

Disentangling these logics at work may be hard, and appreciating them in a model even harder. To complicate even more the task, global warming and pollution issues are inherently dynamic. Take C02 emissions again, if firms emits a lot producing now, it will not have a direct impact. We may need to wait to assess the future cost.

Finally these concerns cannot really be individualized, as any warming would be the consequence of emissions of a large number of heterogeneous agent. They fall into the realm of public good, or rather public bad. Each polluter has the incentive to shirk, especially as the emissions of house-warming gas is a byproduct of production, which has no direct impact on immediate profit. For one individual firm, reducing its emissions is not rational if it is costly, since the impact would be nil if the others maintain their emission policies unchanged. In that sense, the emissions game is really a mean-field game, as no player alone has an impact on the macroeconomic quantities (given that there is a large number of polluters/players), but all together they have.

This is therefore the approach we will adopt in our attempt to tackle firms competition in the presence of environmental externalities. We are particularly interested in the evolution on total quantities and prices for a given demand. Hence we would like to understand how firm adapt to being imposed constraints on the environmental externalities induced by their activity. Does it tend to reduce production and increase prices? Do firms invest more in a mitigating technology?

The paper is organized in the following way. The first section introduces the model and derives the analytic optimality condition. Unfortunately no closed form solution can be derive when agents are heterogeneous. So the second section presents the numerical scheme to solve the model and discuss the code. Finally the third section analyzes the results and the last section concludes.

1 Model

In what follows, we fix a filtered probability space $(\Omega, \mathbb{F} = \{\mathcal{F}_s\}_{s \geq t_0}, \mathbb{P})$. All random quantities considered are assumed to be adapted to $\mathbb{F}$, i.e., measurable with respect to the sigma algebra generated by $\{\mathcal{F}_f\}_{s \geq t_0}$.

We can now construct our model. We choose to focus on firm as the only polluter, and will mostly ignore individuals whether they are workers or consumers. They will only appear through the demand to close the model. Note that the model was inspired by the model in [3], so some part are similar. This is the case for example of the demand structure.

Microeconomic agents Hence, here, all the microeconomic agents are firms. At each instant t, a firm (she) is characterized by her past history of emissions (e_t) and her technology (k_t).

Each firm controls her production policy (q_t) and investment policy (i_t). It produces an industrial good and emits CO2 during the process. However, she is able mitigate the emissions by investing in a mitigating technology, and she is only permitted to emit a given amount. After that we may assume that she is penalized by a government or consumers who would refuse to buy.

Microeconomic dynamics The evolution of the state variables is governed by the following two equations (a stochastic differential equation and an ordinary differential equation):

$$de_t = [(K_{max} - k_t)\, q_t]\, dt + \sigma_e\, (e_t - E_{max})\, dB_t$$

$$dk_t = i_t\, dt$$

where B_t is and adapted Brownian motion and we assume that distinct firms are affected by independent Brownian motions. σ_e is a measure of the intensity of the noise. We also assume that the noise is proportional to the remaining capacity to pollute. This could be justified by the fact that firms are more careful when they approach the penalization barrier. These equations, together with the initial conditions $e_{t_0} = e_0$ and $k_{t_0} = k_0$, fully specify the evolution of the firms' mitigation technologies and stock of emissions, where emissions are a byproduct of industrial activity.

Macroeconomic agent and quantities The only macroeconomic variable we consider is the price of industrial good. We assume that it is deterministic (non-random).

Furthermore we also assume the presence of a macroeconomic agent, the government, who is in charge in our story to deliver the permit to pollute, i.e., at time $t = t_0$, the government fix the maximal quantity of emissions allowed per firm. Note that these quantities can vary across firms. At first we assume the government to be non-strategic, and we will treat the initial distribution of E_{max} as exogenous.

However to take into account this specificities, we shall rewrite the two above equations to facilitate our analysis using the following changes of variable $k_t = K_{max} - k_t$ and $e_t = E_{max} - e_t$ with a slight abuse of notation. Then the state dynamics becomes

$$
\begin{aligned}
de_t &= -k_t q_t \, dt + \sigma_e e_t dB'_t \\
dk_t &= -i_t \, dt
\end{aligned}
$$

where $B'_t = -B_t$ is a Brownian motion.

Preferences and firms problem Firms are willing to maximize their inter-temporal profit, so they make production choice and investment choice to maximize

$$
\mathbb{E}_{e,k} \int_{t_0}^{+\infty} \exp(-r(t - t_0)) \pi(p_t, q_t, e_t, i_t, k_t) \, dt
$$

subject to $q_t \geq 0$, $i_t \geq 0$, $e_t \geq 0$, and $k_t \geq 0$, where $\mathbb{E}_{e,k}$ is the conditional expectation operator given $e_t = e$ and $k_t = k$. r is the interest rate.

Then each firm maximization problem can be rewritten as an optimal control problem:

$$V(e,k,t) = \sup_{\{q_t,i_t\}_{t\geq t_0}} \mathbb{E}_{e,k} \int_{t_0}^{+\infty} \exp(-r(t-t_0))\pi(p_t,q_t,e_t,i_t,k_t)\,dt \tag{1}$$

subject to

$$de_t = -k_t q_t\,dt + \sigma_e e_t dB_t', \quad e_{t_0} = E_{max} - e_0 \tag{2}$$

$$dk_t = -i_t\,dt, \quad k_{t_0} = K_{max} - k_0 \tag{3}$$

and the non-negativity constraints $q_t \geq 0$, $i_t \geq 0$, $e_t \geq 0$, and $k_t \geq 0$ for all $t \geq 0$.

For π regular enough (in particular if π is of class $\mathcal{C}^2$ in all its variables), it can be shown that V is of class $\mathcal{C}^2$ in the state space and $\mathcal{C}^1$ in time. However the proof is not very direct, so is spared here[1]. Using the principle of dynamic programming, we then have that V (under the above regularity conditions) solves the following Hamilton-Jacobi-Bellman equation

$$\partial_t V - rV + H(e_,k_t,\partial_e V,\partial_k V) + \frac{\sigma_e^2}{2}e_t^2\partial_{ee}^2 V = 0 \tag{4}$$

where $\partial_j V$ stands for the partial derivative of V with respect to variable j, and H is the Hamiltonian:

$$H(e,k,\lambda_e,\lambda_k) = \sup_{\{q,i\}} [\pi - kq\lambda_e - i\lambda_k]$$

To move forward, we know need to define the profit criterion of firms. Assuming a quadratic cost of production, a fix tax on the emission level, and a quadratic cost of investment, we have:

$$\pi(p_t,q_t,e_t,i_t,k_t) = p_t q_t - aq_t - \frac{b}{2}q_t^2 + Te_t - \frac{c}{2}i_t^2$$

A few thing should be noted about this profit function. First the only technology purpose of technology is mitigation: it does not affect the firm productivity and therefore her profit at all. This may be a quite unrealistic assumption, since investing in technology should probably also have an impact on production. However we choose not to take it into account to focus on our question[2]: in this framework, the only reason to invest is to pollute less. Secondly, tax on emission stock T is positive:

[1]If you are interested in it, I have done an identical proof for a project at the Cired studying dynamic differential games, where the optimal control structure is similar also there is only one state variable.

[2]It could be done quite easily by making a or b depend on k, and most of the analysis would remain the same.

this is because of our change of variable. Finally p_t is the price of the industrial good at time t. It will be fix to equalize demand and supply in equilibrium. We will come back to it later on.

From the above specification of π, the Hamiltonian is given by

$$\begin{aligned} H(e,k,\partial_e V,\partial_k V) &= \sup_{\{q,i\}} \left[\pi - kq\partial_e V - i\partial_k V\right] \\ &= \sup_{\{q,i\}} \left[pq - aq - \frac{b}{2}q^2 + Te - \frac{c}{2}i^2 - kq\partial_e V - i\partial_k V\right] \end{aligned}$$

Solving for the maximizers q^* and i^*, we get

$$\begin{aligned} q^*(t,e,k) &= \left(\frac{p-a-k\partial_e V}{b}\right)^+ \\ i^*(t,e,k) &= \left(-\frac{\partial_k V}{c}\right)^+ \end{aligned}$$

and H becomes

$$H(e,k,\partial_e V,\partial_k V) = Te_t + \frac{1}{2\beta}\left[\left(\frac{p-a-k\partial_e V}{b}\right)^+\right] + \frac{1}{2c}\left[\left(-\frac{\partial_k V}{c}\right)^+\right]$$

Therefore, combining the above expression and (4) yields

$$\partial_t V - rV + \frac{\sigma_e^2}{2}e_t^2\partial_{ee}^2 V + Te_t + \frac{1}{2\beta}\left[\left(\frac{p-a-k\partial_e V}{b}\right)^+\right] + \frac{1}{2c}\left[\left(-\frac{\partial_k V}{c}\right)^+\right] = 0 \tag{5}$$

Evolution of the firms population So far, we have considered only the problem of an individual firm. We consider now the population dynamics implied by the firms individual choices and investigate the evolution of the firms' distribution in the state space. Assume that the initial distribution of firms is given by $m(e,k,0) = m_0$.

Note that the optimal controls defined above are Markovian. Then we may fully summarize the population dynamics by the infinitesimal generator G (given the optimal controls)[3]:

$$Gf = -kq\partial_e f - i\partial_k f + \frac{\sigma_e}{2}e^2\partial_{ee}^2 f$$

[3] This is a linear operator defined on the space of smooth function with compact support, $\mathcal{C}_c^\infty$. Defining it on this "simple" functional space is almost without loss of generality since $\mathcal{C}_c^\infty$ is dense in any $\mathcal{L}^p$-space and avoid complications.

for any $f \in \mathcal{C}_c^\infty$. Then the evolution of the density is given by the Fokker-Planck equation

$$\partial_t m(e,k,t) - G^* m(e,k,t) = 0$$

where G^* is the adjoint of G in $\mathcal{L}^2$. Therefore the population dynamics are given by

$$\partial_t m + \partial_e \left(-k_t q_t m\right) + \partial_k \left(-i_t m\right) = \frac{\sigma_e}{2} \partial_{ee}^2 \left(e^2 m\right)$$

and under the assumption of rationality (which is the crucial assumption to close the mean-field game), it becomes

$$\partial_t m + \partial_e \left(-k_t q_t^* m\right) + \partial_k \left(-i_t^* m\right) = \frac{\sigma_e}{2} \partial_{ee}^2 \left(e^2 m\right) \tag{6}$$

Macroeconomic closure Equations (5) and (6) are the classical equations of a mean-field game. They are coupled trough the price, which is determined at the macroeconomic level by the equalization of demand and supply. We only focused on the supply side so far. However we still need to define demand to get the price. Assuming CES utility function for the consumers, demand is given by

$$D(p_t, t) = \frac{W \exp(\rho(t - t_0))}{p_t^\epsilon}$$

where W is the wealth of the economy, ρ is the growth rate, and ϵ is the elasticity of substitution. Then, at every instant $t \geq t_0$, the price solves

$$\begin{aligned} & \mathit{Supply} = \mathit{Demand} \\ \Leftrightarrow \int q^*(e,k,t) m(e,k,t)\, de\, dk = & \frac{W \exp(\rho(t - t_0))}{p_t^\epsilon} \end{aligned}$$

Then the equilibrium of the model is describe by the next proposition.

Proposition 1.1. *The equilibrium of the model is triplet (V, m, p), where the value function V, the* *distribution of firms m, and the price p solves the following equations:*

$$\partial_t V - rV + \frac{\sigma_e^2}{2} e_t^2 \partial_{ee}^2 V + Te_t + \frac{1}{2\beta}\left[\left(\frac{p-a-k\partial_e V}{b}\right)^+\right] + \frac{1}{2c}\left[\left(-\frac{\partial_k V}{c}\right)^+\right] = 0$$

$$\partial_t m + \partial_e\left(-k_t q_t^* m\right) + \partial_k\left(-i_t^* m\right) = \frac{\sigma_e}{2}\partial_{ee}^2\left(e^2 m\right)$$

$$\int q^*(e,k,t) m(e,k,t)\, de\, dk = \frac{W\exp(\rho(t-t_0))}{p_t^\epsilon}$$

Proof. The proposition results from the above derivation. However to be fully rigorous, we still need to show that any triplet that solves these equation is indeed an equilibrium. This is not guaranteed yet as the principle of dynamic programming yields necessary conditions only. However, we will show that they are sufficient too.

It relies on the concavity of the Hamiltonian and the Arrow's sufficiency theorem or Mangasarian sufficiency theorem.

Let (q^*, i^*) be the optimal controls defined above and let (q, i) be an admissible strategy. Let (e^*, k^*) and (e, k) be the associated trajectories. Then

$$\begin{aligned}
V(q*, i^*) - V(q,i) &= \int_{t_0}^{+\infty} \exp(-r(t-t_0))\left[\pi(p_t, q_t, e_t, i_t, k_t)\right] dt \\
&\geq \int_{t_0}^{+\infty} \exp(-r(t-t_0)) \\
&\quad [H(e^*, k^*, \partial_e V^*, \partial_k V^*) + k^* q^* \partial_e V^* + i^* \partial_k V * - H(e, k, \partial_e V^*, \partial_k V^*) - kq\partial_e V^* - i\partial_k V^*]\, dt \\
&= \int_0^{+\infty} \exp(-rt)\left[H^* - H + \partial_e V^*(k^* q^* - kq) + \partial_k V^*(i^* - i)\right] dt
\end{aligned}$$

where the inequality comes from the definition of the Hamiltonian. Then note that, by concavity of H,

$$H \leq H^* + \partial_e H^*(e - e^*) + +\partial_k H^*(k - k^*)$$

Then, integrating by part,

$$\begin{aligned}
V^* - V &\geq \lim_{t\to+\infty} \exp(-rt)\left[\partial_e V^*(e - e^*) + \partial_k V^*(k - k^*)\right] \\
&\geq 0
\end{aligned}$$

□

Therefore to compute an equilibrium, we need to find a fixed point in (V, p).

2 Numerical solution

This section presents the numerical scheme we use to solve the model and discuss briefly the code.

As it is classic in mean-field game, we take advantage of the dual form of the problem. We first solve for the optimal control faced by the firms and obtained V and the controls q and i. Given the value function and the controls, we update the distribution of the firms in the state space, and then compute the price: given the quantities produced at a point in the state space and the distribution of firms, we can compute the total production of the economy. Then the price equalize demand and supply. We iterate this procedure until the price converges.

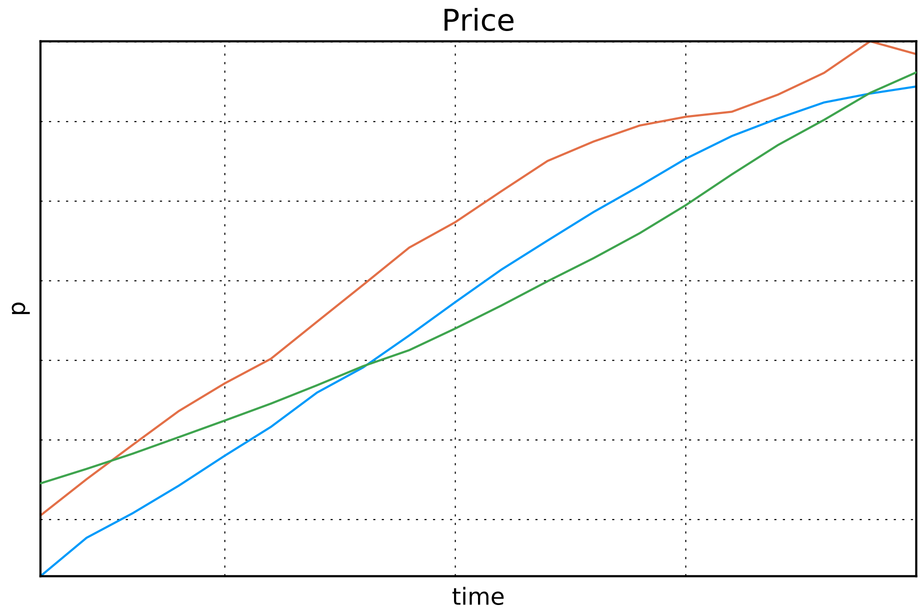
Computation of the value function and the transport equation Following [3], we separate the problem into two parts: a random part and a non-random part. This is the key trick to obtain our solution. Then the non-random part is a pure optimal control problem, and we can solve for it using the classical value function iteration on a discretized grid. This is how we proceed. The random part is approximated using Gaussian convolution.

Hence we first fix p. We discretize the state space and compute at each point of the state space the optimal Markovian control given the value function and then update the firm distribution and the value function. The procedure is classic. The main difficulties are to update the distribution and to account for the randomness. However the discretization makes everything clearer. A value function V taking into account the randomness is obtained by convoluting the non-random function and a normal kernel. Given this new value function, optimal policies and the new value of the value function are computed. Then the distribution is updated accordingly: non-random evolution is fully describe by the optimal controls. Then randomness is incorporated in the distribution by taking again the convolution product of m and a normal kernel.

We iterate this procedure until V converges.

Finding p Once a fixed point for V is found, we update p equalizing supply and demand, and start again until a fixed point for p is found too.

Figure 1: Prices evolution



Remark In the code, demand is modified as suggested in [3] to speed the algorithm. We can also note a end of the time effect in the results. Therefore, results are truncated before the end of time.

3 Results

Some results are presented in this section. Price are globally increasing, but this is not surprising as demand is slowly increasing and firms are constrained in their emissions and therefore their production, and so the total production is decreasing, until we approach the very end, where firms starts produce more. This is a feature of the tax and finite time. Hence to match a decreasing production with an increasing demand, price has to increase over time.

The investment behavior is more interesting: to avoid being constrained in the future and to save on the tax, firms tend to invest more in a mitigation technology when the constraint on emission is tighter. This allows them to produce at level almost identical to the level when the constraint are slacker. But this happens at the expense of the firm. In particular, if the cost of investing in a clean technology is large, it reduces firms profitability.

Figure 2: Quantities evolution

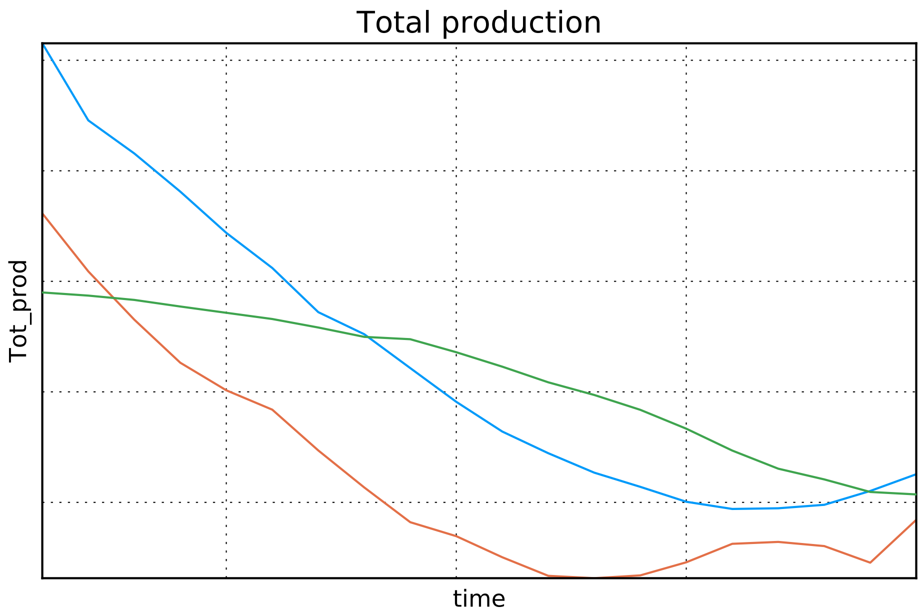
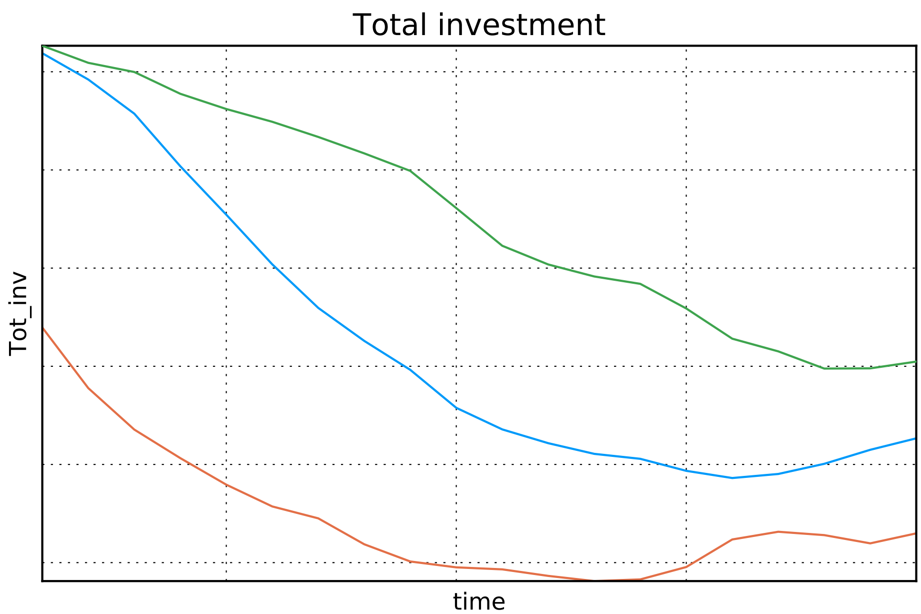


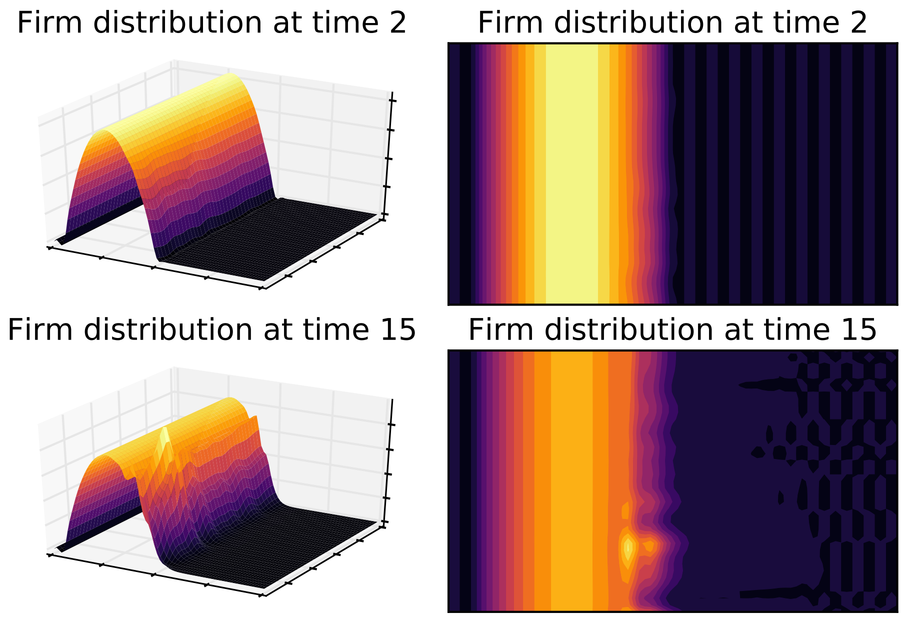
Figure 3: Investment evolution



4 Conclusion

This paper is a first attempt at modeling the behavior of firms in an economy where the environment is not expansible at infinity. In our model, this is the case because of the presence of binding permit

Figure 4: Distribution evolution



to emit. It could be model in another way or at least be made more precise. In particular, the state should be a rational macro player, solving an optimal control problem subject to the solution of the Mean Field Game.

It could also be interesting to look at investment more carefully, i.e., investment here has only an impact on emissions, and not on production. This should probably be integrated in a more complex model.

Finally, environmental issues are international, and assuming the presence of an international regulator (the government here) is quite unrealistic. For this reason another possible extension could be to model parallel economies competing. We would expect regulators to favor their domestic production by imposing slacker environmental constraint.

References

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