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Résumé

Le calcul extérieur discret a été populaire au cours des deux dernières décennies pour une variété d'applications de traitement de la géométrie. Ces dernières années ont également vu une utilisation accrue des connexions et de leurs courbures, bien que principalement sur des surfaces de dimension 2. Dans cette thèse, nous formulons une définition de dérivée extérieure covariante de connexion discrète sur un complexe cellulaire arbitraire. Nous validons son évaluation basée sur l'holonomie par le biais d'une nouvelle construction à base de propagation parallèle de repères orthonormés, et nous soulignons comment cette notion discrète de courbure respecte l'identité différentielle de Bianchi dans le domaine discret. Dans les chapitres suivants, nous expliquons comment la notion de dérivée extérieure covariante discrète peut être étendue à des formes générales évaluées dans une fibre vectorielle qui satisfont les identités, différentielle et algébrique, de Bianchi.

Abstract

Discrete exterior calculus has been popular over the past two decades in a variety of geometry processing applications. Recent years have also seen an increased use of connections and their curvatures, although mostly over two-dimensional surfaces. In this thesis, we formulate the covariant exterior derivative of a discrete connection one-form over collapsible two-chains on an arbitrary cell complex. We validate its holonomy-based evaluation via a novel construction of parallel-propagated frames, and highlight how this discrete notion of curvature enforces the differential Bianchi identity in the discrete realm. In the following chapters we explain how the notion of discrete exterior derivative can be extended to general vector bundle valued forms. We illustrate how a wedge product between the curvature form and a bundle-valued form can be defined on a tetrahedron such that for this new covariant exterior derivative, the differential and algebraic Bianchi identities are satisfied in a discrete sense.

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1. Introduction

Over the past two decades, discrete exterior calculus (DEC) has been widely used in geometry processing [CdGDS13, CWW17] parallel to its development in computational physics [TK01, STDM15, Arn18]. At the core of this line of work is the discretization of (scalar-valued) *differential forms* through cochains and of the exterior derivative operator through the boundary operator [DKT08] to offer structure preservation. More recently, the use of discrete connections (a discretization of Lie algebra-valued one-forms [CDS10]) has become prevalent for, e.g., the treatment of tangent vectors or frame fields [LTGD16]: connections induce a discrete notion of parallel transport, and hence, of covariant derivatives and other differential operators acting on tangent vectors and other bundles. However, both discretizations remain mostly used separately; for instance, a recent paper proposing a general approach to computations over polygonal meshes [DGBD20] presented a form-based structure-preserving approach alongside a connection-based approach. Yet, the properties of the curvature of a connection have long been recognized as crucial for topics varying from Yang-Mills theory to general relativity since curvature is an invariant measure (it does not depend on the arbitrary choice of a frame field); and in fact, geometry processing is starting to use connection curvatures on 3-manifolds for the design of frame fields in hexahedral meshing [CC19], while one could derive a covariant discretization of elasticity if covector-valued differential one-forms could be properly handled numerically [KAT$^+$07].

In this thesis, we analyze a notion of discrete connection curvature on arbitrary cell complexes, which applies a discrete exterior covariant derivative $\mathbf{d}^\nabla$ (i.e., an extension of the notion of exterior derivative to the setting of vector or principal bundles with connections) to the connection one-form. We detail how a novel multi-prong evaluation of discrete bundle-valued forms over chains is, in fact, an exact evaluation of the continuous curvature definition, and highlight its structure-preserving nature by discussing how it trivially satisfies the second Bianchi identity. In the following chapters we show in what way we can define a discrete wedge product between the curvature form and the vector-valued forms on a generalized dual complex. Finally, we show how a covariant exterior derivative can be defined on the primal complex which still satisfies the differential Bianchi identity, but at the same time satisfies the algebraic Bianchi identity.

1.1 Motivation

Given a cell complex M embedded in $\mathbb{R}^n$, k-chains are linear combinations of k-cells, forming a linear space $C_k(M)$ which conveniently provides a discretization of k-dimensional domains for which the notions of union and boundary (and thus, homology) are well defined. Conversely, a k-cochain maps a chain to $\mathbb{R}$, making their linear space $C^k(X) = \mathrm{Hom}(C_k(M), \mathbb{R})$ a convenient discretization of k-forms: the *pairing* $\langle w, c\rangle$ of a cochain w and a chain c becomes a discrete analog of the *integration* of a continuous form ω over a domain σ [DHLM05a]. A discrete notion of the exterior derivative d acting on forms is then derived so as to satisfy Stokes' theorem

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega; \tag{1.1.1}$$

i.e., by defining the discrete exterior derivative $\mathbf{d}$ as the adjoint of the boundary operator, we enforce:

$$\langle \mathbf{d}w, c\rangle :- \langle w, \partial c\rangle. \tag{1.1.2}$$

Ensues a simple definition of the discrete (de Rham) cohomology respecting the structures of its continuous counterpart, since $\partial\,\partial = 0$.

However, this approach only applies to *scalar-valued* forms. Lie-algebra-valued (or more generally, bundle-valued) forms do not seamlessly admit such a simple discretization: the pointwise expression of a bundle-valued form requires a pointwise frame (or more generally, a fiber), making it difficult to properly define a discrete counterpart to the covariant exterior derivative where the sum over various cells can only be achieved in a common fiber. Yet, this more general framework of forms would allow a number of covariant computations of vector fields or displacements on meshes as it allows for the computation of connection curvatures. It is also a first step towards a more general discrete approach to Riemannian geometry.

1.2 Previous Work

A first extension from scalar-based differential forms to *infinitesimal-rotation-based differential forms* was proposed in graphics in [CDS10] for discrete 2-manifolds. These *connections*, corresponding to a notion of parallel-transport of tangent orthogonal frames (or alternatively, complex tangent planes [KP17]), reduces in this case to a local angle-valued form (since 2D rotations are commutative) once a frame field is chosen, for which DEC tools are readily available. This allows, for instance, to define a notion of holonomy measuring the extent to which parallel transport around closed loops fails to preserve the geometric object being transported. For 3-manifolds, discrete connections must be stored as rotation-matrix-valued discrete forms since rotations are no longer commutative in general. Manipulating these non-scalar forms thus requires a new methodology. In [CC19], symmetric frames are handled based on a discrete structure equation, mimicking Cartan's second structural identity, which can be seen as an approximation based on a second-order expansion of the local monodromy. Since structure preservation was not particularly key in their hex meshing application, no effort was made towards formulating a structure-preserving discrete covariant exterior derivative.

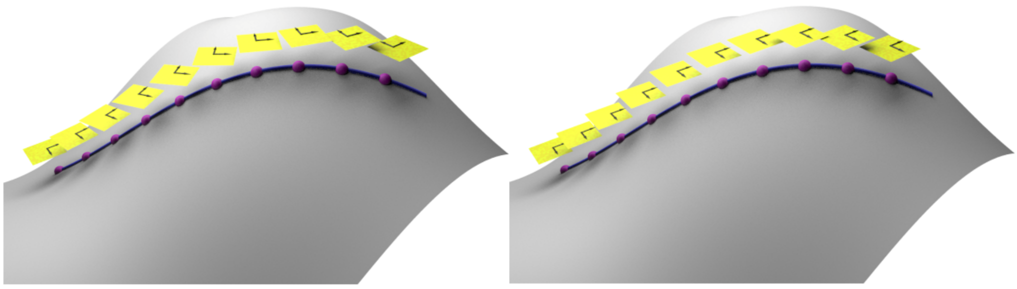


Figure 1.1: **Connections.** A connection is a map between neighboring fibers. Two different connections encoded on the same frame bundle lead to different notions of parallel transport, and different curvatures.

Building upon the theory of synthetic differential geometry [Koc97, Koc06], recent works [Sch18, BEHS21] have offered a discrete covariant exterior derivative operator for *simplicial meshes*. In particular, a formal way to define a curvature tensor as a Lie-algebra-valued differential 2-form was formulated, and a discrete version of the differential Bianchi identity (obtained in differential geometry by applying the exterior derivative to Cartan's second structural equation) is shown to hold. While their work presents a general definition of the discrete covariant exterior derivative operator, we will show their operator expression may suffer various inadequacies, both in terms of accuracy and in terms of theoretical justification. Moreover, they do not provide a treatment of arbitrary cell complexes.

1.3 Contributions

The goal of this thesis is to handle the computation of curvature of connections over cell complexes in a manner compatible with DEC. We adopt a purposely-biased exposition which favors geometric and algorithmic definitions rather than formal mathematical statements. However, for completeness and to cater to a general audience, we provide a summary of the differential geometric notions we discretize in Chap. 2. We also establish how our formulation is structure-preserving as it leads to a simple and well-defined discrete version of the differential Bianchi identity, a staple of Einstein's field equations which informs natural symmetries of the curvature tensor in Riemannian geometry [Ein16]. In particular, we show that this differential Bianchi identity becomes a tautology in the discrete setting, just like Stokes' theorem was for discrete scalar-valued forms. Finally, we discuss how our new formulation helps towards the creation of general discrete differential operators on bundle-valued forms that satisfy a discrete algebraic Bianchi-Identity as well.

2. Smooth Theory of Connections

Let M be an $n-$dimensional manifold and E a vector bundle of rank k over M with projection map $\pi\colon E \to M$. The frame bundle $F(M, E)$ over M at a point $p \in M$ is then defined as the space of all frames for E_p. In this case for the projection maps $\pi\colon F(M) \to M$, the fiber $\pi^{-1}(p)$ consists of invertible maps $\Phi_p\colon E_p \to \mathbb{R}^k$ that can be seen as coordinate maps for the fiber space. Choosing a (local) section $\varphi \in \Gamma F(M, E)$ defines a coordinate representation for every fiber. In the common case of geometry processing, we also have a metric $\langle \cdot, \cdot \rangle \in \Gamma\mathrm{Sym}^2(M, E)$ on the fiber space, so we often consider orthonormal frames, i.e., we enforce $\varphi_p : E_p \to \mathbb{R}^k$ to be an isometry.

2.1 Connection and Covariant Derivative

Given a local frame on a vector bundle E, we can define a differential operator d acting on the components of a section. It should be noted that the d operator is dependent of the frame. Any affine connection will differ from it by an endomorphism-valued 1-form $\omega \in \Omega^1(M, \mathrm{End}(E))$ [CCL99], i.e., we can write any connection as

$$\nabla X = e_i \otimes (dX^i + \omega^i_j X^j), \tag{2.1.1}$$

where e_i is the i-th basis vector in T_pM associated with φ_p, i.e., $\varphi^i_p(e_j) = \delta^i_j$. For a chosen *orthonormal* frame field, the components of a metric-preserving connection can be expressed as an antisymmetric matrix (i.e., $\omega^i_j \in \mathfrak{so}(n)$), with:

$$\omega^i_j = e_i \cdot \nabla e_j.$$

A change in the local frame field by a rotation $\tilde{\varphi} = R\varphi$, where $R \in SO(n)$ induces a *gauge transformation* in the connection ω as:

$$\begin{aligned}\tilde{\omega} = \tilde{e} \cdot \nabla \tilde{e} &= (eR^{-1})^T \nabla (eR^{-1}) = Re^T e(d(R^{-1}) + \omega R^{-1}) \\ &= R(-R^{-1}dRR^{-1} + \omega R^{-1}) = \mathrm{Ad}_R(\omega - R^{-1}dR).\end{aligned} \tag{2.1.2}$$

Given a connection, we can parallel-transport a vector $v(x_0)$ within the vector bundle along a curves $\gamma(t)$ with $\gamma(t_0) = x_0$ by solving the differential equation $\nabla_{\dot{\gamma}(t)} v(\gamma(t)) = 0$, see [LTGD16].

Generalizing to the covariant derivative of any rank-(m,n) tensor T (sections of $E \otimes ... \otimes E \otimes E^* \otimes ... \otimes E^*$), through enforcing the product rule, we have

$$\nabla T^{\alpha...}_{\beta...} = dT^{\alpha...}_{\beta...} + \omega^{\alpha}_{\gamma} T^{\gamma...}_{\beta...} - \omega^{\gamma}_{\beta} T^{\alpha...}_{\gamma...},$$

which is a rank-$(m, n+1)$ tensor.

Once we are given a connection 1-form we can define a connection-dependent integral for a bundle valued 1-form.

Definition 2.1. *Let M be a manifold and $E \to M$ be a vector bundle with connection $\nabla = d + \omega$. Further let $\alpha \in \Omega^1(M; E)$ be a bundle-valued 1-form. For a curve $\gamma\colon [0,1] \to M$, we have:*

$$\oint\!\!\!\!\!\!\!/\;\;_{\gamma} \alpha = \oint\!\!\!\!\!\!\!/\;\;_{[0,1]} \gamma^* \alpha = \int_0^1 \underbrace{P \exp\left(\int_0^t \omega_{\gamma(\tau)}[\dot{\gamma}(\tau)]\, d\tau \right)}_{R_{0,t}} \alpha_{\gamma(t)}[\dot{\gamma}(t)]\, dt,$$

where P indicates the path ordering of the matrix product.

2.2 Covariant Exterior Derivative

For a given vector bundle E with connection ∇, the covariant exterior derivative d^{∇} of a E-valued k-form $\alpha \in \Omega^k(M; E)$ is defined as:

$$d^{\nabla} \alpha = (d + \omega \wedge) \alpha,$$

which corresponds to the usual covariant derivative for $k=0$. Contrary to the case of scalar-valued differential forms, we will see that $d^{\nabla}\, d^{\nabla}$ is not zero in general, except if the fiber space is flat (as in the case of the trivial bundle $M \times \mathbb{R}^n$): in general, the fiber space induces a curvature which renders the covariant exterior derivative rather different from its scalar-based counterpart, see Equation (2.3.2). Note that this definition changes slightly for Lie-Algebra valued form β to

$$d^{\nabla} \beta = d\beta + [\omega \wedge \beta], \tag{2.2.1}$$

where $[\cdot \wedge \cdot]$ indicates the commutator with the wedge product.

2.3 Curvature Two-form of a Connection

An important geometric invariant of connections, the curvature, quantifies the impossibility of endowing the base manifold with a Euclidean structure. It measures whether a parallel-transported frame around a loop "closes up", that is, matches up with its starting frame: the mismatch expressed as a rotation is called the holonomy of the connection — a key concept used in [CDS10]. The *curvature 2-form* Ω is an infinitesimal version of this concept, measuring an infinitesimal holonomy.

This is also known as Ambrose-Singer theorem [AS53]. It is expressed using the connection 1-form ω as its covariant exterior derivative, i.e.,

$$\Omega^{\nabla} = d^{\nabla}\omega = d\omega + \omega \wedge \omega. \tag{2.3.1}$$

Remark 2.2. Note that although the connection 1-form takes values in the vector bundle of endomorphisms, we treat it as a vector valued form and not as a Lie-Algebra valued form. Therefore we apply the exterior derivative for vector valued forms to ω and not the operator introduced in 2.2.1.

For given vector fields $X, Y \in \Gamma TM$, we can also express it as $\Omega^{\nabla} \in \Omega^2(M; \mathrm{End}(E))$ through

$$\Omega^{\nabla}(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Note that the above d^{∇} applied to ω is not the d^{∇} for a rank-$(1,1)$ tensor-valued 1-form but instead, the covariant derivative of a vector-valued 1-form for each ω_i. The reason is that ω does not transform as a tensor-valued 1-form under a change of frame R, which would have been $R\omega R^{-1}$, but instead undergoes a gauge transform. The curvature Ω^{∇} is, in fact, precisely the measurement of the failure of $d^{\nabla}d^{\nabla}$ to vanish: one can prove that for an arbitrary bundle-valued k-form α, we have

$$d^{\nabla}d^{\nabla}\alpha = \Omega^{\nabla} \wedge \alpha. \tag{2.3.2}$$

Indeed, expanding $d^{\nabla}d^{\nabla}\alpha$ proves this equality:

$$\begin{aligned}(d + \omega\wedge)(d + \omega\wedge)\alpha &= d(\omega \wedge \alpha) + \omega \wedge d\alpha + \omega \wedge \omega \wedge \alpha \\ &= (d\omega + \omega \wedge \omega) \wedge \alpha = \Omega \wedge \alpha.\end{aligned}$$

2.4 Differential Bianchi Identity

There is an important identity related to the curvature form, called the differential (or second) *Bianchi identity*:

$$d^{\nabla}\Omega^{\nabla} = 0. \tag{2.4.1}$$

Following the definition of d^{∇}, this time applied to a rank-(1,1) tensor-valued 2-form (as Ω does transform into $R\Omega R^{-1}$ under a change of frame field R), one indeed can check that:

$$\begin{aligned}d^{\nabla}\Omega^{\nabla} &= d\Omega^{\nabla} + [\omega \wedge \Omega^{\nabla}] = d\Omega^{\nabla} + \omega \wedge \Omega^{\nabla} - \Omega^{\nabla} \wedge \omega \\ &= d\omega \wedge \omega - \omega \wedge d\omega + \omega \wedge (d\omega + \omega \wedge \omega) - (d\omega + \omega \wedge \omega) \wedge \omega \\ &= 0.\end{aligned}$$

This indicates that the covariant exterior derivative applied to the curvature two-forms should be trivially zero, independent of the choice of frame, of connection, or metric.

2.5 Torsion and the Algebraic Bianchi Identity

For a manifold M and the special case of $E = TM$, we can define the tautological 1-form, or *solder form*, through

$$\theta(X) = X, \quad \theta \in \Omega^1(M, TM).$$

For a connection $\nabla = d + \omega$, its exterior derivative is a 2-form called the torsion Θ, i.e.,

$$\Theta = d^\nabla \theta = d\theta + \omega \wedge \theta \ \in \Omega^2(M, TM). \tag{2.5.1}$$

Just as the curvature form, the torsion is not a frame-dependent quantity. If we express the torsion differently via

$$\Theta(X, Y) = d^\nabla \theta(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

it becomes apparent that Θ measures the deviation of the covariant derivative from the Lie bracket. The exterior derivative of the torsion is a special case of Eq. 2.3.2:

$$d^\nabla \Theta = d^\nabla d^\nabla \theta = \Omega \wedge \theta. \tag{2.5.2}$$

This is called the *algebraic Bianchi Identity.* In [Car23] Cartan gives a geometric intuition for this quantity. Given a connection $\nabla = d + \omega$ and a loop
$\gamma \colon \mathbb{S}^1 \cong [0, 1] \to M$, we define for $\gamma_\tau \colon [0, \tau] \to M$ the map

$$\bar{\theta}(\tau) = \int_{\gamma_\tau} \theta, \tag{2.5.3}$$

where the integral is to be understood in the sense of Def. 2.1. The arising vector $\bar{\theta}(1) \in T_{\gamma(1)}M$ will converge under shrinking to the pointwise torsion of the connection.

Remark 2.3. The torsion form is a vector valued 2-form. In order to obtain a vector, we need to feed two tangent vectors to it. In the loop based interpretation these two vectors span a surface surface element and we can use the triangle formed by them to evaluate the loop based integral. For discrete forms, instead of thinking of the 2-form as a multilinear map with two vectors as input, we can think of it in a more geometric sense, i.e., that the 2-form acts on an element of surface.
Given $a, b \in \mathbb{R}^3$ let $n = a \times b \in \mathbb{R}^3$. In this case we have for the basis of the 2-forms

$$\begin{aligned} dx \wedge dy[a, b] &= n_z \\ dz \wedge dx[a, b] &= n_y \\ dy \wedge dz[a, b] &= n_x. \end{aligned}$$

Therefore for a general scalar valued 2-form
$\alpha_p = f(p)\ dy \wedge dz + g(p)\ dz \wedge dx + h(p)\ dx \wedge dy,$ we have

$$\alpha_p[a, b] = f(p)n_x + g(p)n_y + h(p)n_z.$$

Thus, for the evaluation of a 2-form it is sufficient to provide the normal vector of a surface element. For an evaluation point $p \in M$, we have (since we assume that M is always orientable) a well-defined surface normal $n_p \in N_pM$, where NM is the normal bundle of M. Since the components of the torsion form are scalar valued 2-forms, we can evaluate it on an area-weighted surface normal to obtain a vector.

3. The Differential Bianchi Identity on the Primal Mesh

In this work, we consider an arbitrary discrete orientable manifold M represented by a cell complex embedded in $\mathbb{R}^3$. We denote its vertices by $\mathcal{V} = \{v_i\}$, its edges by $\mathcal{E} = \{e_{ij}\}$ (where e_{ij} is the oriented edge between v_i and v_j), its faces by $\mathcal{F} = \{f_{ijk...}\}$ (whose boundaries consists of oriented edges), and its 3D cells by $\mathcal{C} = \{c_{ijkl...}\}$ similarly; we will not make use of higher dimensional cells in our exposition, but our explanations will be general enough to extend easily to higher dimensions.

3.1 From Discrete Frames to Discrete Connection Forms

We first equip our manifold M with a discrete frame field and a definition of local parallel transport.

Frames. When working with tangent vectors or other fiber bundles, one needs to first define a set of local frames in which to express these geometric quantities. We thus define an arbitrary, but fixed local *orthonormal frame* F_i per mesh vertex v_i; a frame is typically given as a set of orthonormal vectors. These frames form a *discrete section of the frame bundle*.

Parallel transport. Given a vector bundle E over a simplicial complex M, the fibers at different vertices are a priori unrelated. As illustrated in Fig. 1.1 we need to define a way to transport vector quantities between fibers. Thus, for each edge e_{ij}, we provide a rotation matrix

$$R_{ij} \in \mathrm{SO}(n) \tag{3.1.1}$$

which indicates how the fibers are *connected*, that is, how the coordinates of a vector at v_j represented in the frame F_j need to be changed in order to return the coordinates in the frame F_i of the parallel-transported vector along the edge from v_j to v_i. Equivalently, this is how the frame F_i can be parallel-transported to a frame $\tilde{F}_j = F_j R_{ij}^{-1}$ at E_j, so that the components of a vector expressed in $\tilde{F}_j$ can be directly used as the components of its parallel transport at E_i. This follows from the fact that any vector $V \in E_j$ expressed in F_j by $X \in \mathbb{R}^n$ would become $\tilde{F}_j = F_j R_{ij}^{-1}\tilde{X} = R_{ij}X$ when expressed in $\tilde{F}_j$, since $\tilde{F}_j = F_j R_{ij}^{-1}\tilde{F}_j\tilde{X} = F_j X$. This local definition of parallel transport along edges has become common in graphics, typically stored on primal edges as in [CC19] — or on dual edges as in [CDS10]. By definition, one has: $R_{ij} = R_{ji}^{-1}$. This discrete notion of parallel transport on the mesh M is thus stored as the set of all these edge rotations: $\boldsymbol{\nabla} = \{R_{ij}\}_{e_{ij}\in\mathcal{E}}$. Note that $\boldsymbol{\nabla}$ depends on the choice of frames: changing the frames would require another set of rotations in order to represent the same notion of parallel transport.

Connection 1-forms. In order to mimic the continuous theory, we finally establish a *discrete connection 1-form w*, the discrete counterpart of the differential 1-form ω in Sec. 2.1. As a discrete 1-form, it consists of a matrix value per edge, which we define as

$$w_{ij} = R_{ij} - \mathrm{Id}. \tag{3.1.2}$$

This oriented-edge based expression w_{ij} encodes the *change to the components* of a vector at v_j once parallel-transported to the fiber of v_i, *expressed in the local frame* F_i. We can then verify that $R_{ij}w_{ji} = -w_{ij}$, indicating that our oriented-edge values are indeed akin to a $\mathfrak{so}(n)$-valued *1-form* integrated along the oriented edge *when evaluated at the same fiber E_i*. The discrete connection w is then given by the set of all edge matrices w_{ij}. It should be noted, that [CC19] define the connection 1-form as $w_{ij} := \log(R_{ij})$. Our definition is only the first order approximation of this power series. If we would use the entire power series, we would lose the property, that the connection 1-form is actually an anti-symmetric form.

Remark 3.1. It should be pointed out that the connection 1-form takes values in the homomorphisms per edge. Nevertheless it is not a discrete Lie-Algebra valued form! We should rather regard it as a quantity per edge that measures how much the frame alignment along the edge differs from the identity.

3.2 Discrete Curvature

Given a discrete parallel transport $\boldsymbol{\nabla}$, or equivalently, the induced connection 1-form w, we should be able to evaluate a notion of *connection curvature*, akin to the continuous definition of Ω in the continuous case (see Sec. 2.3).

Limitations of existing holonomy-based approaches However, the fact that we are dealing with quantities that require a local frame to be evaluated renders the task not quite obvious. For a triangle face f_{ijk}, [CC19] define the curvature over it through

$$\langle \Omega, \sigma \rangle_{v_i} = R_{ij}R_{jk}R_{ki} - \mathrm{Id}. \tag{3.2.1}$$

This definition implicitly makes use of the fact that we use vertex v_i and its fiber E_i to express the integrated 1-forms over a loop around the triangle. This is, indeed, in agreement with the fact that the integrated curvature measures the local holonomy of the connection; it also trivially extends to arbitrary polygonal faces. However, this is not a *constructive* definition: one cannot meaningfully *sum* this definition over two triangles, even if they share the same evaluation vertex v_i. In other words, since this curvature is associated to a loop instead of a simple chain, we cannot use the sum of chains to induce a notion of "curvature integration" over a larger region. We need to leverage the underlying algebraic structure of chains.

Summable curvatures. Based on ideas from synthetic geometry for infinitesimal cells [Koc97] and the derived simplicial mesh case proposed in [Sch18, BEHS21], we propose not to directly use loops to evaluate curvatures, but instead, the difference of two 1-chains forming a loop. This might seem tautological, but it will give us the opportunity to define an addition on 2-chains that share a 1-chain. Concretely, it means that for an oriented face f formed by vertices v_i, v_j, v_k, v_l, and v_m for instance, we can arbitrarily split the boundary at v_l, and define the integrated connection curvature expressed at v_i for cut v_l as:

$$\langle \Omega, f \rangle_{v_i}^{v_l} :- R_{ij} R_{jk} R_{kl} - R_{im} R_{ml}.$$

This can be understood as integrating the form w over the two 1-chains $e_{ij} + e_{jk} + e_{kl}$ and $e_{lm} + e_{mi}$ by each time parallel-transporting the result back to fiber E_i (see Fig. 3.1): for the first 1-chain for instance, we evaluate w_{ij} which is the change of coordinates "seen" from v_i when traveling back along e_{ij}, then we add $R_{ij} w_{jk}$ as this is the change of coordinates when traveling back along e_{jk} but still expressed in E_i, and so on all the way to v_l. We then find that

$$\begin{aligned} & w_{ij} + R_{ij} w_{jk} + R_{ij} R_{jk} w_{kl} \\ & = (R_{ij} - \mathrm{Id}) + R_{ij}(R_{jk} - \mathrm{Id}) + R_{ij} R_{jk}(R_{kl} - \mathrm{Id}) \\ & = R_{ij} R_{jk} R_{kl} - \mathrm{Id}. \end{aligned}$$

Thus, our two-prong evaluation corresponds to a formal integration at v_i of w along the 1-chain $e_{ij} + e_{jk} + e_{kl}$ added to the evaluation $e_{lm} + e_{mi}$ along the 1-chain at v_i too. Note that our evaluation is in fact very much in line with a trivial extension of the traditional way of expressing the curvature on face f which consists in using $R_{ij} R_{jk} R_{kl} R_{lm} R_{mi} - \mathrm{Id}$: the two evaluations only differ by a post-multiplication by $R_{im} R_{ml}$. Similarly, changing the evaluation fiber to another vertex would simply require a pre-multiplication by the accumulated parallel-transport from the old evaluation fiber to the new one. We thus see that our evaluation is, in a sense, a *representative* value of the curvature of the connection on f, which can be trivially evaluated for another choice of the two prongs through pre- and post-multiplication using the connection $\boldsymbol{\nabla}$. The value of this encoding is in its summability: if we now have an oriented face g whose boundary shared with f is the 1-chain $e_{im} + e_{ml}$ and we know its connection curvature at v_i with cut v_l, then we can simply sum these two evaluations to get the connection curvature over the 2-chain $f + g$ evaluated at v_i for cut v_l; i.e.,

$$\langle \Omega, f \rangle_{v_i}^{v_l} + \langle \Omega, g \rangle_{v_i}^{v_l} = \langle \Omega, f + g \rangle_{v_i}^{v_l}. \tag{3.2.2}$$

Indeed, the terms $R_{im} R_{ml}$ will simply cancel out, since their values accumulated from v_i over E_i differ only in sign since f and g are oriented the same way. From this approach, we will derive a procedure to sum curvatures on a cell-by-cell basis *à la DEC* in Sec. 3.6.

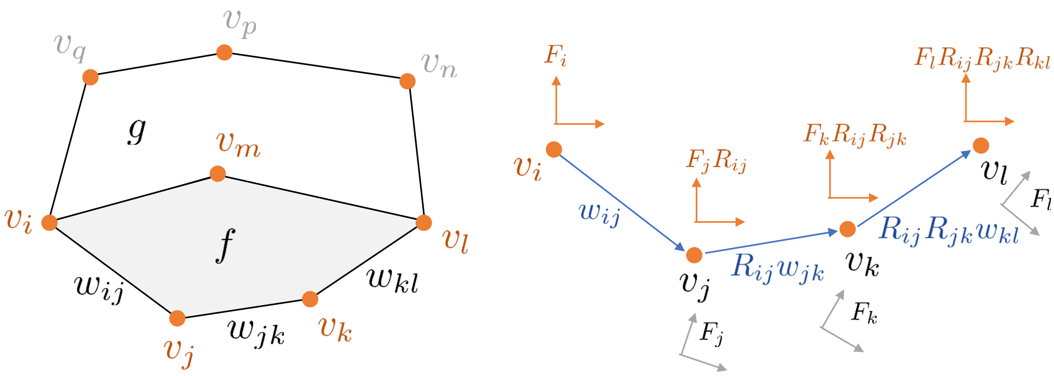


Figure 3.1: **Two-prong evaluation of connection curvature.** For two 2-chains f and g sharing a set of edges, we can evaluate the curvature on the sum of the chains by choosing the extremities of the shared boundary (left) as two prongs for the integration of the connection 1-form.

3.3 Discrete Torsion

In the case of the curvature, we made use of the fact, that the curvature can be expressed as the limit of the vector valued integral of the connection 1-form over a shrinking loop around a chosen point of evaluation $p \in \mathbb{R}^3$. In this case we split the integral with the choice of a cut fiber into two Lie-Algebra valued integrals. In the case of the curvature, this measures how much the integral of the connection is path-dependent. For the torsion we have a similar expression in Eq. 2.5.3. In order to define a discrete notion of torsion we will discretize this integral. Let $T_{x_0}X$ be the evaluation fiber over x_0 and $x_i \in \sigma$ the cut vertex. In this case we define two paths $\gamma_1 = \{x_0, \ldots, x_{i-1}, x_i\}$ and $\gamma_2 = \{x_0, x_n, \ldots, x_{i+1}, x_i\}$. A suggestive definition for the discrete torsion with cut at i and evaluation at 0 through

$$
\begin{aligned}
\langle \Theta, \sigma \rangle_0^i &= \oint_{\gamma_1} \theta - \oint_{\gamma_2} \theta \\
&= (\theta_{01} + R_{01}\theta_{12} + \ldots + R_{01} \cdot \ldots \cdot R_{i-2,i-1}\theta_{i-1,i}) \\
&\quad - (\theta_{0n} + R_{0n}\theta_{n,n-1} + \ldots + R_{0n} \cdot \ldots \cdot R_{i+2,i+1}\theta_{i+1,i})
\end{aligned}
$$

In contrast to the curvature, we only need to specify an evaluation fiber since the torsion is vector valued and not Lie-algebra valued. Nevertheless we realized in the numerical tests, that this suggestive definition does not converge to the smooth notion. The reason is that the frame becomes aligned through the rotation matrices along the edge and the solder form is evaluated in the rotated frame. Since the solder form is also obtained through integrating the identity along the edge, the alignment of frame is taken into account twice.

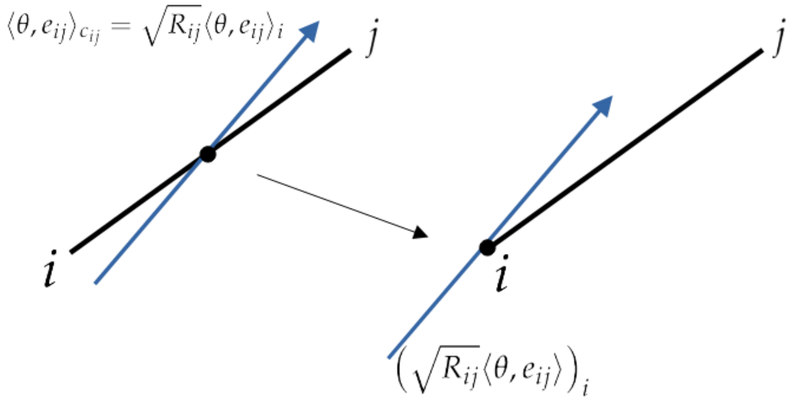


Figure 3.2: The frame is changing along an edge. The solder form is changing in this moving frame. We are storing the transported solder form from the center point of the edge in the starting point of the edge.

We write

$$\sqrt{R_{ij}} = \exp\left(\frac{1}{2}\int_{e_{ij}} \omega\right),$$

to express the "root" respectively the transport map to the center of the corresponding edge. We assume, that the integrated connection can be thought of the constant rotation speed along the edge.
Instead of storing $\langle\theta, e_{ij}\rangle_i = v_j - v_i \in T_iM$, we store the solder form, transported to the center point of the edge, in the starting fiber of the edge, see Fig. 3.2.

Furthermore, we will use different transport maps to carry the solder forms to the corresponding evaluation fibers. We replace R_{ij} with $\sqrt{R_{ij}}$. We denote

$$\tilde{\theta}_{ij} = \sqrt{R_{ij}}\theta_{ij}.$$

With this new definition, we have

$$R_{ji}\tilde{\theta}_{ij} = \sqrt{R_{ji}}\sqrt{R_{ji}}\sqrt{R_{ij}}\,\theta_{ij} = -\sqrt{R_{ji}}\,\theta_{ji} = -\tilde{\theta}_{ji}$$

Note that the second equality holds true, because the matrices in the exponential commute, therefore we can reduce the product of the matrix exponentials to the matrix exponential of their sum. Furthermore, as soon as we apply the rotation to the center point, we can give sense to the flipping of the edge; it just reduces to a sign flip. With the introduced notation we can now give a definition of the discrete torsion.

Definition 3.2. *Let $M = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a discrete orientable manifold with discrete affine connection $\nabla = \{R_{ij}\}_{e_{ij} \in \mathcal{E}}$. Furthermore let $\sigma \in \mathcal{F}$, $\sigma = \{x_0, \ldots, x_i, \ldots, x_n\} \subseteq \mathcal{V}$ be an arbitrary 2-cell. In this case we define the discrete torsion*

$$\begin{aligned}\langle \Theta, \sigma \rangle_0^i &= \fint_{\gamma_1} \theta - \fint_{\gamma_2} \theta \\ &= (\tilde{\theta}_{01} + R_{01}\tilde{\theta}_{12} + \ldots + R_{01} \cdot \ldots \cdot R_{i-2,i-1}\tilde{\theta}_{i-1,i}) \\ &\quad - (\tilde{\theta}_{0n} + R_{0n}\tilde{\theta}_{n,n-1} + \ldots + R_{0n} \cdot \ldots \cdot R_{i+2,i+1}\tilde{\theta}_{i+1,i})\end{aligned}$$

Remark 3.3. Note that this can be interpreted as if we are averaging the solder form over the edge before we apply the exterior derivative to it. The idea that we take the average of the discrete differential form over different orientations before applying the outer derivative will play a major role in Chapter 5.

In Appendix A we will illustrate how this discrete notion of torsion is related to the smooth definition of torsion through the exterior derivative of the solder form to first order.

If we store the solder forms directly in the center points of the edges and we are given connection matrices from the center points of the edges to the center point of the 2-cell, we can define the torsion on a 2-cell σ through

$$\langle \Theta, \sigma \rangle_{c_\sigma} = \sum_{e \in \sigma} R_{c_\sigma, c_e} \langle \theta, c_e \rangle \tag{3.3.1}$$

We will examine this notion of discrete torsion and discrete exterior derivative in Chapter 4 more in detail.

3.4 Numerical Verification of the Discrete Torsion and Curvature

In the following we will examine the new notion of discrete torsion and curvature. In order to do this we will define a smooth connection, calculate its curvature and torsion. Next we derive from it a notion of discrete connection, which we will use to check if our new definitions match under shrinking the smooth values for curvature and torsion. We define a connection 1-form ω through

$$\omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} (x\, dz) + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (-y\, dx - dz)$$

In this case a connection can be defined through $\nabla = d + \omega$. The solder form is given by

$$\theta = e_1 \otimes dx + e_2 \otimes dy + e_3 \otimes dz = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$

The torsion is then computed through

$$\Theta = d^{\nabla}\theta = \underbrace{d\theta}_{=0} + \omega \wedge \theta,$$

where the wedge product in this case basically means to take the matrix product between the forms, but to use as component wise multiplication the wedge product for scalar valued forms. This yields

$$\Theta = \begin{pmatrix} y\, dx \wedge dy - dy \wedge dz \\ -dz \wedge dx \\ -x\, dz \wedge dx \end{pmatrix} \tag{3.4.1}$$

We obtain the curvature through $\Omega = d^{\nabla}\omega = d\omega + \omega \wedge \omega$.

$$d\omega = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right)(-dz \wedge dx) + \left(\begin{smallmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) dx \wedge dy = \begin{pmatrix} 0 & -dx \wedge dy & -dz \wedge dx \\ dx \wedge dy & 0 & 0 \\ dz \wedge dx & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \omega \wedge \omega = & \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} (x\, dz) + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (-y\, dx - dz) \right\} \\ & \wedge \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} (x\, dz) + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (-y\, dx - dz) \right\} \\ = & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & xy\, dz \wedge dx \\ 0 & -xy\, dz \wedge dx & 0 \end{pmatrix} \end{aligned}$$

This yields

$$\Omega^{\nabla} = \begin{pmatrix} 0 & -dx \wedge dy & -dz \wedge dx \\ dx \wedge dy & 0 & xy\, dz \wedge dx \\ dz \wedge dx & -xy\, dz \wedge dx & 0 \end{pmatrix}.$$

Remark 3.4. In Equation 3.1.1 we made the assumption to use rotation matrices per edge to align the frames. Therefore, ω needs to be a 1-form returning a skew-symmetric matrix since the exponential of a matrix $A \in \mathbb{R}^{n \times n}$ satisfying $A^T = -A$ is a rotation, i.e., $e^A \in \mathrm{SO}(n)$,

Let $a, b \in \mathbb{R}^3$ be two points. We will now give a closed form expression for the integrated connection form along the line segment joining them. This segment is defined as the set of all convex combinations of these two points, i.e

$$\gamma_{(ab)}\colon [0,1] \to \mathbb{R}^3\colon t \mapsto (1-t)a + tb = (1-t)\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + t\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}.$$

In our particular case, this yields

$$\begin{aligned}
\int_{\gamma_{ab}} \omega &= \int_{[0,1]} \gamma^*\omega = \int_{[0,1]} \omega(\gamma(t))[\dot{\gamma}] \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \int_{\gamma_{ab}} (x\,dz) + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \int_{\gamma_{ab}} (-y\,dx - dz) \\
&= \left(\begin{smallmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \left(\frac{-(b_1+b_2)(a_2-a_1)}{2} - (c_2 - c_1) \right) + \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right) \left(\frac{(a_1+a_2)(c_2-c_1)}{2} \right)
\end{aligned}$$

This expression can be regarded as $\omega_m(b-a)$, the evaluation of the 1-form at the midpoint $m = \frac{1}{2}(a+b)$. With this notion, we can obtain a one-point quadrature discrete connection 1-form. To obtain the Lie-group element from it, we apply the matrix exponential to it. We denote in the following for on edge e_{ij} joining two vertices i and j, the matrix

$$R_{ij} = \exp\left(\int_{e_{ij}} \omega \right).$$

Note that the above is only a one-point quadrature approximation. One may verify that, when $[\omega_i, \omega_j] \neq 0$, $R_{ij} \neq R_{im}R_{mj}$, where R_{im} (resp., R_{mj}) is the one-point quadrature rotation from vertex i to the midpoint (resp., from the midpoint to vertex j). The true integral of the one-form ω along the edge e_{ij} yields

$$R_{ij} = \mathcal{T}[\exp\left(\int_{e_{ij}} \omega \right)].$$

The second order approximation for a linear ω along edge e_{ij} can be evaluated as

$$\begin{aligned}
\mathcal{T}\left[\left(\int_{e_{ij}} \omega \right) \right] &= \mathcal{T}\left[I + \int_{e_{ij}} \omega + \frac{1}{2!} \left(\int_0^L \omega(\tau_1) d\tau_1 \right) \left(\int_0^L \omega(\tau_2) d\tau_2 \right) \right] + O(L^3) \\
&= I + \bar{\omega}L + \int_0^L \left[\omega(\tau_1) \int_{\tau_1}^L \omega(\tau_2) d\tau_2 + \left(\int_0^{\tau_1} \omega(\tau_2) d\tau_2 \right) \omega(\tau_1) \right] d\tau_1 + O(L^3) \\
&= I + \bar{\omega}L + \frac{L^2}{2}(\bar{\omega}^2 + \frac{1}{6}[\omega_i, \omega_j]) + O(L^3),
\end{aligned}$$

where $\bar{\omega}$ is the average value of ω, i.e., its value at the edge midpoint.

Convergence To test the validity of the discrete formulas, we implemented the formulas discussed in Sec. 3.3 in a test program. We used the libigl library [JP$^+$18] for it. To create different test cases, we use triangles which we rotate and shift to evaluate the discrete torsion and curvature form for different cases. The rotation is represented by Euler angles $R(\alpha, \beta, \gamma) = R(x, \alpha) \cdot R(y, \beta) \cdot R(z, \gamma)$, where the first parameter of $R(\cdot, \cdot)$ indicates the rotation axis. After the rotation we shift the triangle such that vertex 0 of the triangle, is located at $v = (v_x, v_y, v_z)$. We compare the three notions of discrete torsion in the test. First we store the torsion in the center point of the triangle see Eq. 3.3.1. We also compute the torsion as a loop based integral, i.e., in the sense of Sec. 3.3 we are using the same evaluation and cut fiber. Lastly we use the vertex 2 as cut vertex to evaluate the loop based integral. We then computed the discrete torsion for on the triangle with different methods and in each step we scaled the triangle down by a factor of 0.5.

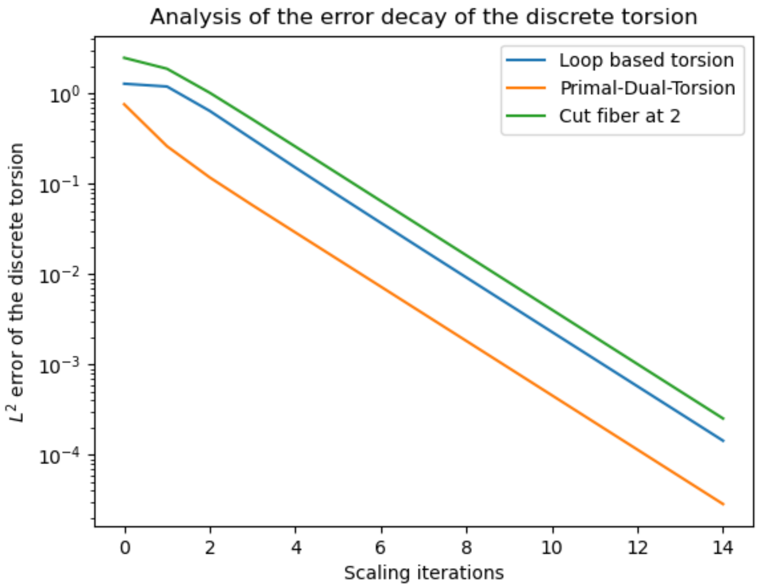


Figure 3.3: Error plot of the discrete torsions, where we use rotation $R(1.3, 1.1, -1.6)$ and displacement $v = (3.3, 1.6, -0.3)$.

We observe that all three notions of torsion converge towards the smooth value. Nevertheless, the notion of torsion where we store the quantity on the center point of the 2-cell seems to converge fastest in all our tested cases—albeit with the same order of convergence. For the convergence of the discrete curvature formulas we also obtain linear convergence for the Frobenius norm of the error of the evaluations of the curvature matrices.

3.5 Covariant Exterior Derivative

While the fact that our discrete notion of curvature seems to agree with the notion of holonomy, it also seems to clash with the continuous definition: indeed, the usual differential geometric expression of the curvature Ω^∇ of a connection is $\Omega^\nabla = d^\nabla \omega := d\omega + \omega \wedge \omega$ (see Sec. 2.3). While the presence of an exterior deriva-

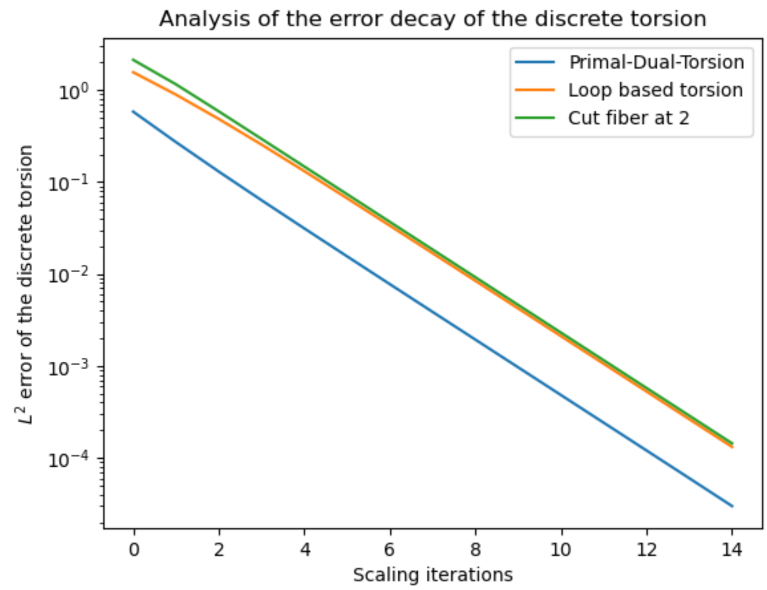


Figure 3.4: Error plot of the discrete torsions, where we use rotation $R(1.7, 2.1, -2.6)$ and displacement $v = (1.3, 1.8, -1.3)$.

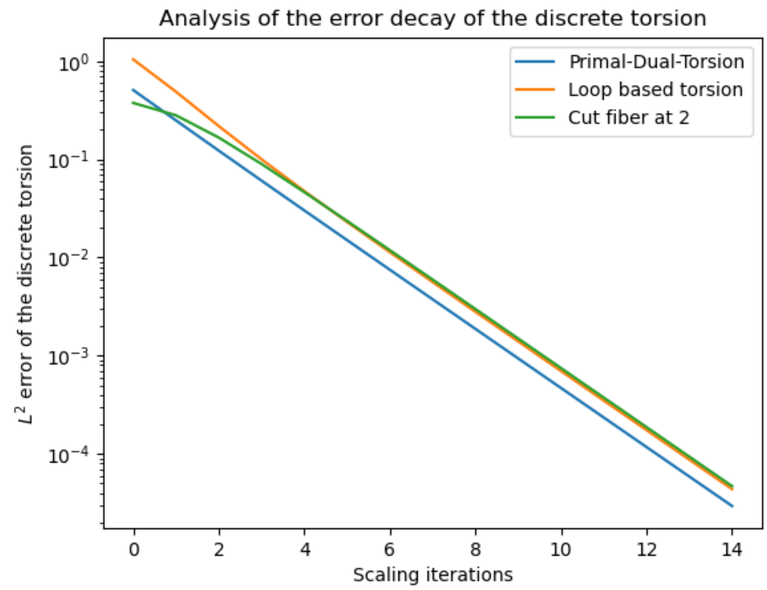


Figure 3.5: Error plot of the discrete torsions, where we use rotation $R(2.3, -1.1, 2)$ and displacement $v = (2.3, -0.6, 1.1)$.

tive d might explain our summation of the connection 1-form along the boundary of a face over which the curvature is evaluated as explained in Sec. 3.2, the extra terms that make up the covariant exterior derivative d^{∇} appear not to factor into

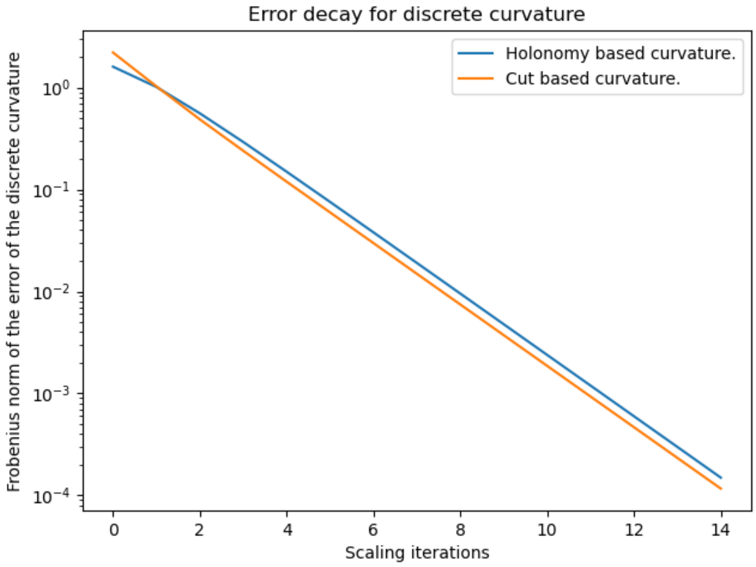


Figure 3.6: Error plot of the discrete curvature notions, where we use rotation $R(2.3, -1.1, 2)$ and displacement $v = (2.3, -0.6, 1.1)$.

our evaluation. In this section, we return to the continuous world for a moment to explain that our parallel-transported frame from the evaluation fiber along the boundary towards the cut is, in fact, accounting for *all* terms, making the evaluation exact given a discrete connection that exactly integrates a continuous connection along edges. Furthermore, we prove that our notion of discrete curvature is structure-preserving.

$\mathbf{d}^\nabla$ acting on connection 1-forms. Let us assume that we are given a *contractible two-dimensional region* C of a manifold M equipped with a section of the frame bundle (i.e., a frame F at each point) and a connection 1-form ω on this frame field. Furthermore, suppose we select two points, v_{top} and v_{cut}, on the boundary ∂C of C. The frame F_{top} of point v_{top} corresponds to the frame of the evaluation fiber discussed above, while v_{cut} corresponds to what we described as the cut vertex. For the sake of simplicity, we will think of this region as a disk (technically, the region C can always be mapped onto a disk through the Riemann mapping theorem). The construction we described in Sec. 3.5 amounts to parallel-transporting the frame F_{top} along the two boundary paths between v_{top} and v_{cut}, where v_{cut} is thought of as a *puncture* (the parallel-transported frames on each side of v_{cut} may not match when expressed in fiber E_{cut}). Because we are dealing with a contractible region (integrating a connection over a non-contractible region would not even make sense, given its local nature), we can now parallel transport these boundary frames towards the inside through *retraction*: in effect, we parallel-transport the frames inwards by shrinking the boundary continuously while keeping v_{cut} in place. More precisely, a retraction induces a 2D (x, y) parameterization where $\mathbf{e}_x$ is always aligned with the retracted boundary and $\mathbf{e}_y$ is in the shrinking direction (see Fig. 3.7); we thus parallel-transport the frames already present on the boundary towards the inside along the y direction. By construction, the connection $\tilde{\omega}$ coming from the same notion of parallel transport but expressed on this *new* frame field is thus satisfying $\tilde{\omega}(\mathbf{e}_y) = R(\omega - R^{-1}\, dR)(\mathbf{e}_y)R^{-1} = 0$ in the interior of C, where R denotes the local rotation between the input frame and our parallel-propagated frame F_{top} (see Eq. Equation (2.1.2)). Consequently, $\tilde{\omega} \wedge \tilde{\omega}$ is identically null everywhere in the punctured region C since $\tilde{\omega} \wedge \tilde{\omega}(\mathbf{e}_x, \mathbf{e}_y) = \tilde{\omega}(\mathbf{e}_x)\tilde{\omega}(\mathbf{e}_y) - \tilde{\omega}(\mathbf{e}_y)\tilde{\omega}(\mathbf{e}_x) = 0$ — except at the singularity v_{cut}. The consequence is that the integral of $d^\nabla \tilde{\omega}$ over C is reduced to $\int_C d\tilde{\omega} = \int_{\partial C} \tilde{\omega}$. Moreover, as $\tilde{\omega}$ is also 0 on the cut boundary due to our use of parallel transport, the curvature has been entirely "pushed" to v_{cut} where the mismatch between the parallel transported frames $F_{top}^{+/-}$ seen from the left and the right of v_{cut} and expressed in the same frame F_{cut} is thus the total integrated curvature over the region. In other words, the fact that we parallel-propagated the frame F_{top} along the boundary to evaluate our two-prong curvature form is indeed offering an *exact* evaluation of the integrated curvature.

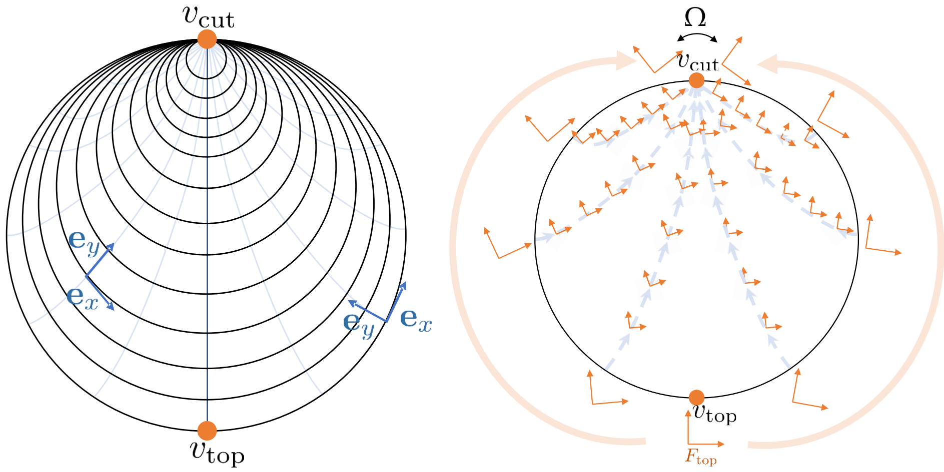


Figure 3.7: **Retraction-based parameterization.** (Left): a retraction of the disk defines a (x, y) parameterization of the (punctured) disk, where the x direction is along the boundary, while the y direction corresponds to the deflation direction. (Right): Starting from a parallel-transported frame field F_{top} from v_{top} along the boundary, we can parallel-propagate this boundary frame field towards the interior of the punctured disk along the curves corresponding to the y direction of the retraction-based parameterization. With this new choice of frames, the curvature Ω reduces to the mismatch of the frame witnessed at v_{top}, expressed in F_{cut}.

d^∇ **of the curvature.** The curvature Ω^∇ is a Lie-algebra valued two-form, independent of the input frame field (it is an invariant of the connection by conjugation of rotation). This is a major difference with the case above: while ω is also endomorphism valued, it is a one-form on the bundle, i.e., it depends entirely on the choice of the bundle section (with the additional term $-R^{-1}dR$ when a rotation field R is applied to the representation). Nevertheless, this only changes the expression of d^∇ slightly as one gets:

$$d^\nabla \Omega^\nabla = d\Omega^\nabla + [\omega \wedge \Omega] = d\Omega^\nabla + \omega \wedge \Omega^\nabla - \Omega^\nabla \wedge \omega$$

as explained in Sec. 2.3. This expression shows that the integral of $d^\nabla \Omega$ (a 3-form) over a three-dimensional ball B turns into a boundary integral over the sphere surface: indeed, the term $d\Omega$ trivially satisfies Stokes' theorem, and so do the left-over terms since:

$$\int_B \omega \wedge \Omega^\nabla - \Omega^\nabla \wedge \omega = \int_B \omega \wedge d\omega - d\omega \wedge \omega = \int_B -d(\omega \wedge \omega).$$

Moreover, it turns out that our aforementioned idea of parallel-propagating frames along the boundary still applies. Better yet, because the curvature was proven to be expressible purely from boundary terms itself (after proper parallel transport of the evaluation frame), we can show that the covariant exterior derivative of the curvature is null over any contractible ball — a celebrated property named the *second Bianchi identity* (see Sec. 2.4). To understand this, we can perform the same parallel-propagation of a reference frame as follows. From the ball B (again, assumed to be sphere-shaped only to simplify our explanations), pick a point of the boundary ∂B we will call v_{top}: its frame F_{top} will encode the evaluation fiber in which to express the result. Then, make a cut going through v_{top}; in three dimension, a cut of the ball corresponds to a surface membrane, which we will think of as a plane for simplicity. The cut C is thus shaped as a disk containing v_{top}, bringing us back to the previous case. We can then make a cut of the boundary ∂C of C by selecting another point v_{cut}. We are now ready to parallel propagate our reference frame F_{top} in a way similar to the 2D case: first, we reproduce exactly what we did earlier by parallel-transporting the evaluation frame from v_{top} along the two branches of ∂C towards v_{cut}. Once this is done, we can parallel-propagate the resulting frames through 2D retraction to each of the two half-spheres since they are, themselves, two-dimensional domains as in the previous construction. Now that we have equipped the whole surface of the ball with frames, we can parallel-propagate the boundary frames to the inside using this time a 3D retraction, which can be thought of as continuously deflating the ball while keeping v_{cut} in place, see Fig. 3.8. In this manner, we have parallel-propagated our evaluation frame through the (punctured) ball. Now because the terms of $d^\nabla \Omega^\nabla$ all turn into a boundary integral over ∂B, we evaluate this boundary integral independently over the two half-sphere separated by ∂C. Because of the consistent choice of our propagated frame field, each independent surface integral equals the frame mismatch at F_{cut} coming from the two branches of ∂C; and since the orientation of ∂C is opposite for each identically-oriented half-sphere, these mismatches exactly cancel each other — showing that the curvature is such that its covariant exterior derivative over an arbitrary collapsible ball is just zero.

3.6 Discrete Curvature Evaluation

We now return to the discrete realm and discuss an algorithmic scheme to evaluate the integrated DEC-compatible curvature of a discrete connection on an arbitrary

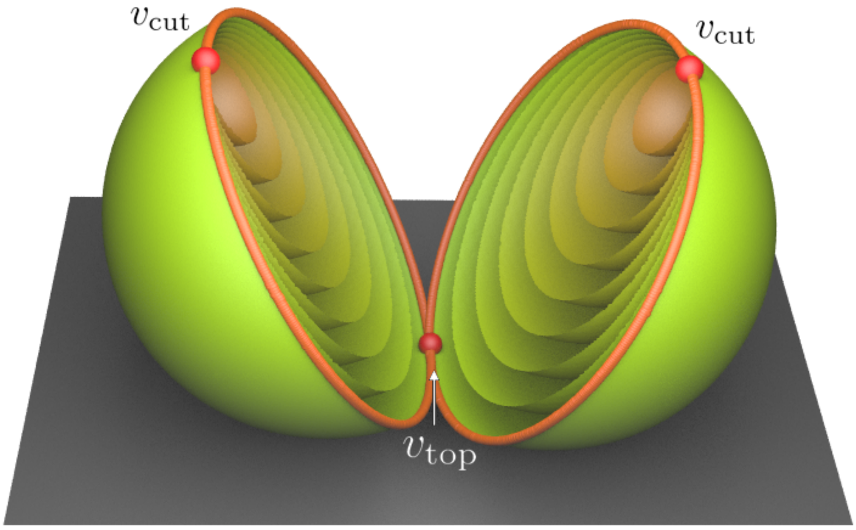


Figure 3.8: **3D retraction.** The parameterization from the boundary of the (punctured) disk defined in Fig. 3.7 extends naturally to the (punctured) ball through retraction onto v_{cut} (exploded view illustration),

contractible 2-chain by leveraging the arguments laid out in the above discussion.

Given the set of adjacent faces over which the curvature must be summed along with an evaluation vertex and a cut vertex, we begin by building a dual spanning tree of the domain whose root is a face containing the cut vertex. We then traverse the tree in depth-first fashion, which provides an ordering in which the sum of each face curvature will be treated. For each new dual node visited, the region $\mathcal{R}$ over which the curvature has been integrated thus far is grown by the corresponding primal face which gets added through a shared edge with $\mathcal{R}$; choosing the extremities of this shared edge as respectively the evaluation vertex and the cut vertex, we then compute the discrete curvature of the new face with these two prongs; we then sum the resulting matrix to the prior curvature result (Equation (3.2.2)) once pre- and post-multiplications of rotation (parallel transport) matrices from $\partial\mathcal{R}$ so as to re-express the previous integrated curvature using the same shared prong. We then handle the next dual node in the spanning tree and repeat, until the tree has been entirely traversed.

From this algorithmic perspective, one can note that the use of a spanning tree guarantees that our region $\mathcal{R}$ grows like a reverse retraction. Thus, as in the continuous case, we see that the discrete case *mirrors precisely the continuous case*. Most importantly, the final curvature could have directly been computed from the boundary connection matrices of the entire integration region, seen as just a large face. Our DEC-compatible evaluation of a sum of curvatures over a contractible union of faces is thus entirely consistent with the continuous case.
In this picture the differential Bianchi Identity becomes a tautology. Nevertheless it is not clear at this point how one can define a wedge product between the curvature 2-form and the solder form in order to satisfy the algebraic Bianchi Identity. We will therefore discuss in the following sections how we can nevertheless derive

(at least approximately) the Algebraic Bianchi Identity.

4. The Algebraic Bianchi Identity on the Primal-Dual Mesh

In contrast to scalar valued differential forms where we had $d \circ d = 0$, such an identity does not hold true for d^∇ in the vector bundle valued case. As explained in Sec. 2.3 we have instead

$$d^\nabla d^\nabla \alpha = \Omega^\nabla \wedge \alpha.$$

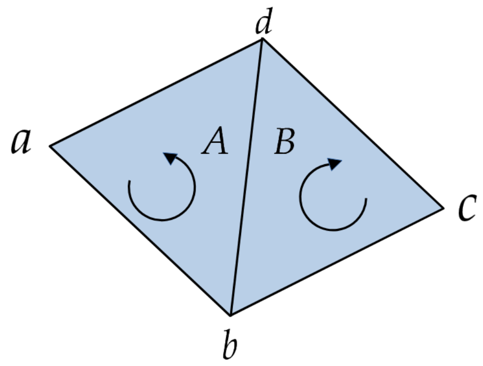
The aim of this section is to understand this identity for discrete bundle valued differential forms. If we stick to the formula of [BEHS21, Sch18] for the calculation of the exterior derivative of a bundle valued 2-form $\alpha \in \Omega^2(M; E)$, we get on a tetrahedron $[abcd]$

$$\langle d^\nabla \alpha, [abcd] \rangle = R_{ab} \langle \alpha, [bcd] \rangle_b + \sum_{i=1}^{3} (-1)^i \langle \alpha, [a, .., \hat{i}, .., d] \rangle_a.$$

In the case of $\Theta = d^\nabla \theta \in \Omega^2(M; TM)$, this yields

$$\begin{aligned} \langle d^\nabla \Theta, [abcd] \rangle_a &= R_{ab}(R_{bc}\theta_{cd} - \theta_{bd} + \theta_{bc}) - (R_{ac}\theta_{cd} - \theta_{ad} + \theta_{ac}) \\ &\quad + (R_{ab}\theta_{bc} - \theta_{ac} + \theta_{ab}) - (R_{ab}\theta_{bc} - \theta_{ac} + \theta_{ab}) \\ &= (R_{ab}R_{bc} - R_{ac})\theta_{cd} = \langle \Omega, [abc] \rangle_a^c \theta_{cd} \end{aligned}$$

The formula reduces to a curvature expression, to which we apply the solder form. But we can verify with numerical tests that this formula does not even converge. Nevertheless if we allow more flexibility in the calculation of the torsion, by using different cut fibers for the computation of the torsion, the expression might change.



In the following, we will use the 1-forms aligned to the fiber over the center of the edge. As above we denote $\tilde\theta_{ij} := \sqrt{R_{ij}}\theta_{ij}$. The torsion of the sum of the chains can be evaluated through

$$\langle \Theta, A+B\rangle_a = \underbrace{\tilde\theta_{ab} + R_{ab}\tilde\theta_{bd} - \tilde\theta_{ad}}_{\Theta_A} + R_{ab}(\underbrace{\tilde\theta_{bc} - \tilde\theta_{bd} + R_{bc}\tilde\theta_{cd}}_{\Theta_B}) = \tilde\theta_{ab} + R_{ab}\tilde\theta_{bc} + R_{ab}R_{bc}\tilde\theta_{cd} - \tilde\theta_{ad}$$

If we had chosen a different prong for the evaluation of the torsion of B, it would have resulted in something of the form

$$\langle \Theta, A+B\rangle_a = \underbrace{\tilde\theta_{ab} + R_{ab}\tilde\theta_{bd} - \tilde\theta_{ad}}_{\Theta_A} + R_{ab}(\underbrace{\tilde\theta_{bc} - \tilde\theta_{bd} - R_{bd}\tilde\theta_{db}}_{\Theta_B}) = \tilde\theta_{ab} + R_{ab}\tilde\theta_{bc} - R_{ab}R_{bd}\tilde\theta_{dc} - \tilde\theta_{ad}.$$

Although these are similar expressions, the paths of transporting are not the same.

If we go back to the formula of [BEHS21], the sum of the forms evaluated on the triangles that contain the evaluation point is such that the contributions along shared edges cancel away. For merging the last triangle, we are faced with the situation that

$$\langle \Theta, [abc] + [acd] + [abd]\rangle_a = R_{ab}\tilde\theta_{bd} - R_{ac}\tilde\theta_{cd} - R_{ab}\tilde\theta_{bc}$$

If we agree to evaluate the torsion of the last triangle in the fiber over vertex b, we have two choices.

$$\langle \Theta, [bcd]\rangle_b = \tilde\theta_{bc} + R_{bc}\tilde\theta_{cd} - \tilde\theta_{bd} \tag{4.0.1}$$
$$\langle \Theta, [bcd]\rangle_b = \tilde\theta_{bc} + R_{bc}\tilde\theta_{cd} + R_{bc}R_{cd}\tilde\theta_{db} = \tilde\theta_{bc} + R_{bc}\tilde\theta_{cd} - R_{bc}R_{cd}R_{db}\tilde\theta_{bd} \tag{4.0.2}$$

Transporting back to the vertex a then yields different curvature expressions.

$$\langle d^\nabla\Theta, [abcd]\rangle_a = \langle \Omega, [abc]\rangle_a^c\tilde\theta_{cd} \tag{4.0.3}$$
$$\langle d^\nabla\Theta, [abcd]\rangle_a = (R_{ab} - R_{ab}R_{bc}R_{cd}R_{db})\tilde\theta_{bd} + (R_{ab}R_{bc} - R_{ac})\tilde\theta_{cd} \tag{4.0.4}$$

If we compute the exterior derivative for the torsion on a cube, there are even more choices to transport the calculated quantities back to the evaluation fiber.

We observe that we obtain expressions that involve curvatures applied to the solder form. But for now it is not clear how to turn this observation into systematic expressions and some notion of a discrete wedge product. Following the idea of Sec. 3.3 we define a notion of bundle valued exterior derivative, that is not stored on one of the primal vertices.

Definition 4.1. *Let X be a simplicial complex of dimension n. In this case let X_ℓ be the subcomplex that consists of all cells $\sigma \in C(X)$ with $\dim(\sigma) < \ell$. We denote the barycentric dual complex of X_ℓ with X_ℓ^*. We define the generalized dual complex of X as $X^* = \bigcup_{\ell<n}(X_\ell^*)$.*

With this definition we want to ensure that for each cell of dimension $k+1$, we have connection matrices from the barycenter of all $k-$cells that form the corresponding $k+1-$cells.

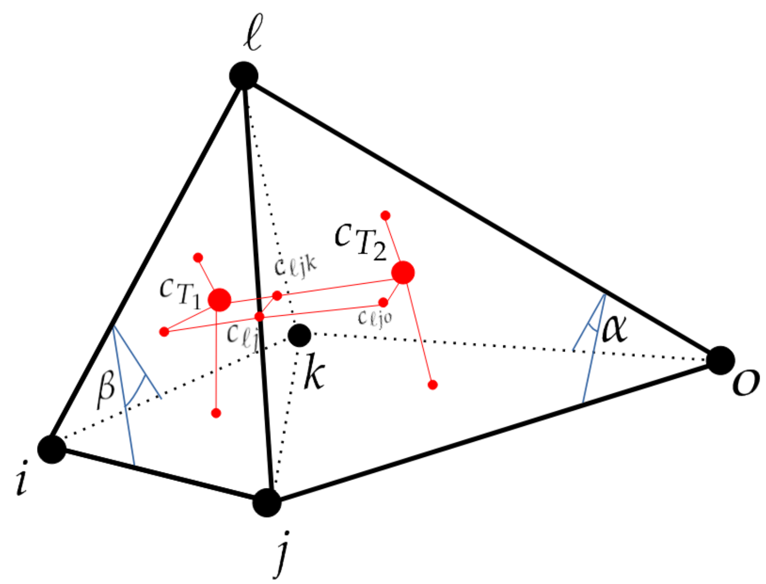


Figure 4.1: Simplicial complex with a subset of the edges of the generalized dual complex.

With this setup we can give an alternative definition of the discrete bundle valued exterior derivative.

Definition 4.2. *Let $f \in C^{k+1}(X)$ be a $(k+1)$-cell with connection 1-form defined on the generalized dual complex. We denote $\mathcal{C}_f$ the set of oriented k-chains forming f. Further let $\alpha \in C_k(X;E)$ be a bundle valued k-form whose value is stored in the center of each k-cell. We denote with a subscript c the barycenter of each cell. In this case we* *define the exterior derivative through*

$$\langle d^\nabla \alpha, f\rangle_{c_f} = \sum_{e\in\mathcal{C}_f} R_{c_f c_e}\langle \alpha, e\rangle_{c_e}.$$

If we use this to take the exterior derivative for $\alpha \in C_k(X;E)$ twice, we obtain

$$\langle d^\nabla d^\nabla \alpha, t\rangle_{c_t} = \sum_{f\in\mathcal{F}_t} R_{c_t c_f} \sum_{e\in\mathcal{E}_f} R_{c_f c_e}\langle \alpha, e\rangle_{c_e} = \sum_{f\in\mathcal{F}_t}\sum_{e\in\mathcal{E}_f} R_{c_t c_f} R_{c_f c_e}\langle \alpha, e\rangle_{c_e}.$$

If the $(k+2)-$cell of evaluation has no boundary, every k-cell will appear exactly twice in the sum, but with opposite orientation. Since we store the values on the center point, a change of orientation of the chain will result in a sign flip in the evaluation of the form. Every oriented k-cell is contained in an unique $k+1$-cell. For a k-cell e we denote the adjacent $k+1$-cell f_e. We denote (e,e') a tuple representing two opposite k-cells, The above sum then reduces to

$$\begin{aligned}\langle d^\nabla d^\nabla \alpha, t\rangle_{c_t} &= \sum_{e\in\mathcal{E}} R_{c_t c_{f_e}} R_{c_{f_e} c_e}\langle \alpha, e\rangle_{c_e} = \sum_{(e,e')} R_{c_t c_{f_e}} R_{c_{f_e} c_e}\langle \alpha, e\rangle_{c_e} + R_{c_t c_{f'_e}} R_{c_{f'_e} c_e}\langle \alpha, e'\rangle_{c_e} \\ &= \sum_{(e,e')} (R_{c_t c_{f_e}} R_{c_{f_e} c_e} - R_{c_t c_{f'_e}} R_{c_{f'_e} c_e})\langle \alpha, e\rangle_{c_e} = \sum_{(e,e')} \langle \Omega, [c_{f_e}, c_e, c_{f_{e'}}, c_t]\rangle_{c_t}^{c_e}\langle \alpha, e\rangle_{c_e}. \\ &=: \langle \Omega^\nabla \wedge \alpha, t\rangle_{c_t}\end{aligned}$$

It should be noted, that in contrast to the primal case where the arising curvature expressions seemed to appear in a random manner in terms of transport paths and cut fiber for the 2-prong, this is not the case in this primal-dual picture.

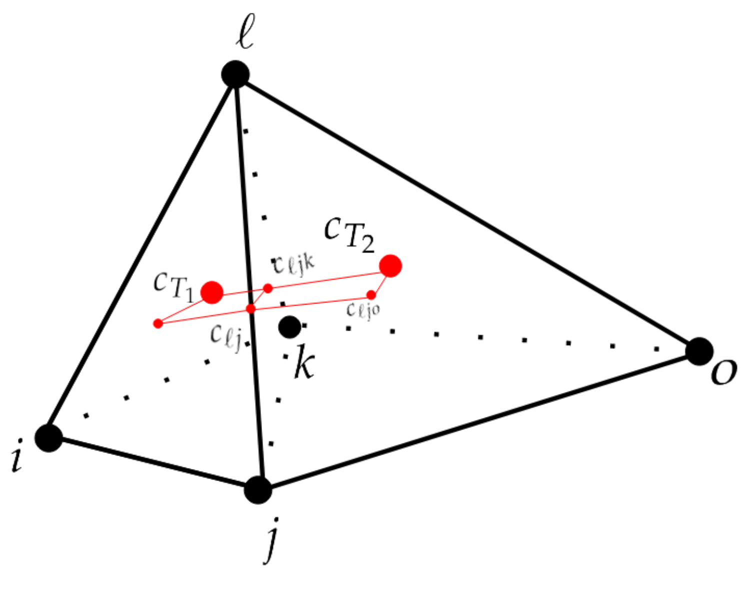


Figure 4.2: Illustration of the domains that are used to calculate the curvature.

The main advantage of this representation for a wedge product is that the evaluation fiber is independent of the vertices of the primal cell. In the previous section, where we discussed the exterior derivative on the primal mesh, it turns out that we had difficulties to even define a wedge product in a systematic way. Furthermore one key property of any wedge product is that it returns an alternating form.

Alternation of the Discrete Wedge-Product. Suppose we are given a $(k+2)-$cell σ with a given orientation. For a simplicial complex, this means that all the $(k+1)$ cells forming σ are consistently oriented, which means on the other hand that all the $k-$cells forming it are consistently oriented. Suppose we flip the orientation of one $k-$cell forming σ. In this case, to have a consistently oriented complex, we need to flip the orientation of every $k-$cell. We call the $(k+2)-$cell with flipped orientation $\tilde{\sigma}$. Flipping the orientation of every $k-$cell forming σ leads to:

$$\begin{aligned}
\langle \Omega^\nabla \wedge \alpha, \tilde{\sigma} \rangle_{c_{\tilde{\sigma}}} &= \sum_{(e,e')} \langle \Omega^\nabla, [c_{f_e}, c_e, c_{f_{e'}}, c_{\tilde{\sigma}}] \rangle_{c_{\tilde{\sigma}}}^{c_e} \langle \alpha, e \rangle_{c_e} \\
&= -\sum_{(e',e)} \langle \Omega^\nabla, [c_{f'_e}, c_e, c_{f_e}, c_{\tilde{\sigma}}] \rangle_{c_{\tilde{\sigma}}}^{c_e} \langle \alpha, e \rangle_{c_e} \\
&= -\sum_{(e,e')} \langle \Omega^\nabla, [c_{f_e}, c_e, c_{f_{e'}}, c_{\sigma}] \rangle_{c_{\sigma}}^{c_e} \langle \alpha, e \rangle_{c_e} = -\langle \Omega^\nabla \wedge \alpha, \sigma \rangle_{c_\sigma}
\end{aligned}$$

Numerical Convergence of the Wedge Product on the Primal-Dual Complex. As before for the torsion, we can verify the numerical validity of this formula. For any connection $\nabla = d + \omega$ it holds for a (smooth) vector valued differential form that

$$d^\nabla d^\nabla \alpha = \Omega^\nabla \wedge \alpha.$$

For the sample connection defined in Sec. 3.4 this means here in particular

$$\Omega^\nabla \wedge \theta = \begin{pmatrix} 0 & -dx \wedge dy & -dz \wedge dx \\ dx \wedge dy & 0 & xy\; dz \wedge dx \\ dz \wedge dx & -xy\; dz \wedge dx & 0 \end{pmatrix} \wedge \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -xy \;\det \end{pmatrix}$$

Similar to the numerical test of the torsion, we place the evaluation fiber of a tetrahedron at a point $p = (x, y, z) \in \mathbb{R}^3$ and compute $d^\nabla \circ d^\nabla \theta$ in the primal dual picture. In this case, we scale the tet in each iteration by a factor of 0.5 and calculate (after dividing by the volume of the tet) $d^\nabla d^\nabla$. We observe in the numerical tests that we have linear convergence to the exact value.

Issues with the Differential Bianchi Identity The notion of storing a vector valued form in the fiber over the barycenter of an evaluation cell has the big advantage that it is independent of the primal vertices. We can consider this way to compute the integral as *delocalizing the cut vertex* onto every edge of the evaluation cell. Nevertheless for a Lie-algebra valued form – especially the curvature form – this delocalization of the cut fiber causes a problem, since we need to specify for the curvature homomorphism an evaluation fiber and a cut fiber. It is unclear

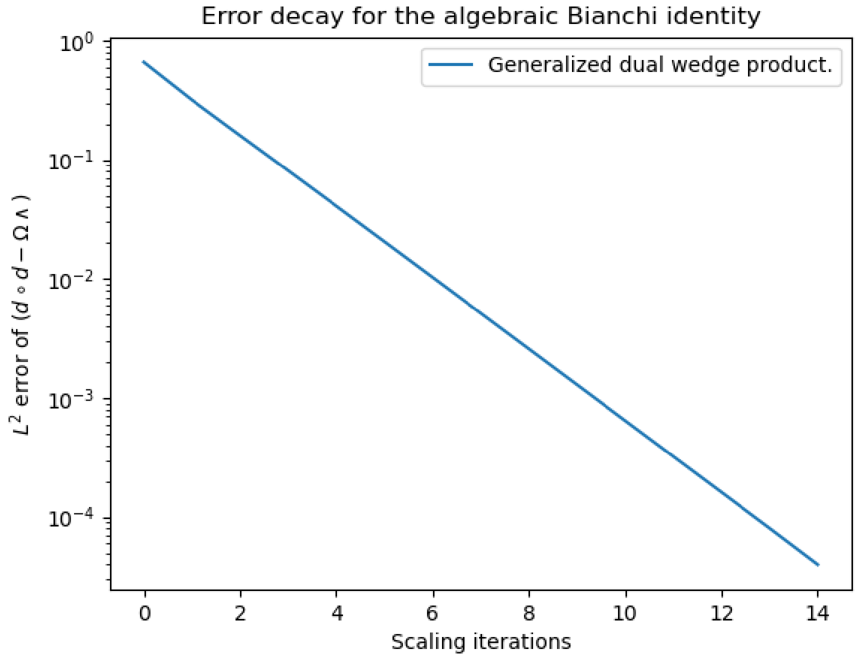


Figure 4.3: Error plot of $d^\nabla d^\nabla - \Omega\wedge$

what the curvature is in this case.
If we use the barycenter as cut and evaluation fiber, it is unclear which path we would choose. Furthermore with this holonomy based approach of different loops will not cancel out in general as in Chapter 3.

5. New Discrete Covariant Exterior Derivative

As discussed in the beginning of Chapter 4 it is a priori not clear how we can define the ordering for merging faces of the covariant exterior derivative. In the previous section, we have seen that for an anti symmetrized expression of the exterior derivative on the generalized dual complex, we can get a notion of a wedge product. In the following we will try to define for a primal tetrahedron a notion of exterior derivative that will turn into a wedge product, but preserves the differential Bianchi-Identity. Following [BEHS21] we can define the unsymmetrized exterior derivative as follows

Definition 5.1. *Let E be a discrete vector bundle with connection over a simplicial complex X. Let $s \in X$ with $s = (x_0, \dots, x_{k+1})$ be a $(k+1)-$simplex.*
For a discrete E-valued $k-$form, we define the unsymmetrized covariant exterior derivative evaluated at the fiber at x_0 with cut at x_{k+1} as

$$\langle \mathfrak{d}^\nabla \alpha, s\rangle_{x_0}^{x_{k+1}} = R_{01}\langle \alpha, [x_1, \dots, x_{k+1}]\rangle_{x_1}^{x_{k+1}} + \sum_{j=1}^{k+1}(-1)^j \langle \alpha, [x_0, \dots, \widehat{x_j}, \dots, x_{k+1}]\rangle_{x_0}^{x_{k+1}}.$$

For a Lie-algebra valued k-form β, we define the unsymmetrized covariant exterior derivative as

$$\begin{aligned}\langle \mathfrak{d}^\nabla \beta, s\rangle_{x_0}^{x_{k+1}} = {} & R_{01}\langle \beta, [x_1, \dots, x_{k+1}]\rangle_{x_1}^{x_k} + \sum_{j=1}^{k+1}(-1)^j \langle \beta, [x_0, \dots, \widehat{x_j}, \dots, x_{k+1}]\rangle_{x_0}^{x_{k+1}} \\ & + (-1)^{k+1}\langle \beta, [x_0, \dots, x_k]\rangle_{x_0}^{x_k} R_{k,k+1}\end{aligned}$$

If we treat the connection 1-form as a vector valued 1-form, we have, as described in Chapter 2.3.1, with the discrete exterior derivative for the curvature

$$\langle \Omega, [abc]\rangle_a^c = \langle \mathfrak{d}^\nabla w, [abc]\rangle_a^c = R_{ab}R_{bc} - R_{ac}.$$

A computation in [BEHS21] shows, that with the Lie-algebra valued curvature form, we have

$$\begin{aligned}\langle \mathfrak{d}^{\nabla}\Omega^{\nabla},[abcd]\rangle_a^d &= R_{ab}(R_{bc}R_{cd}-R_{bd})-(R_{ac}R_{cd}-R_{ad})+(R_{ab}R_{bd}-R_{ad})-(R_{ab}R_{bc}-R_{ac})R_{cd}\\ &=0,\end{aligned}$$

hence we have an exact differential Bianchi identity.
Nevertheless, if we apply the vector valued exterior derivative twice to the solder form for example, we get

$$\langle \mathfrak{d}^{\nabla}\mathfrak{d}^{\nabla}\theta,[abcd]\rangle_a^d = (R_{ab}R_{bc}-R_{ac})\theta_{cd},$$

which is an expression of the form "curvature $\times$ form" but not a wedge product. Furthermore it does not converge to the expected smooth value in numerical tests. Therefore, as in the smooth case, we will average these terms with the goal to get a discrete wedge product notion that also converges to the smooth wedge product under refinement.

For a triangle $[abc]$ we have for the curvature

$$R_{ab}\langle\Omega,[bca]\rangle_b^a R_{ac} = R_{ab}(R_{bc}R_{ca}-R_{ba})R_{ac} = R_{ab}R_{bc}-R_{ac} = \langle\Omega,[abc]\rangle_a^c.$$

In fact we can calculate, that we have

$$R_{a\sigma(a)}\langle\Omega,\sigma([abc])\rangle R_{\sigma(c)c} = \langle\Omega,[abc]\rangle_a^c$$

for all $\sigma\in S_3$ except $\sigma([abc])=[cba]$. In this case we have

$$R_{ac}\langle\Omega,[cba]\rangle_c^a R_{ac} = R_{ac}R_{cb}R_{ba}R_{ac}-R_{ac}.$$

If we perform the averaging in such a way that we fix every vertex as an evaluation fiber, then we consider the triangles with this evaluation fiber and choose as a cut fiber the cut fiber of the triangle with positive orientation.
Now we can sum the curvatures (or more general the Lie-algebra values) by post composing into the same cut fiber. In this case we have

$$\langle\Omega,[cab]\rangle_c^b - \langle\Omega,[cba]\rangle_c^a R_{ab} = 2(R_{ca}R_{ab}-R_{cb}).$$

To average this value with the contribution from the other permutations, we can pre and post multiply the averages of each evaluation fiber by parallel transporting them to a common evaluation and cut fiber. As in the smooth case, we call this the alternation of a linear form and denote it by Alt. We define the exterior derivative in this case through

$$d^{\nabla}\beta = \mathfrak{d}^{\nabla}(\mathrm{Alt}(\beta)).$$

For the case $\beta=\Omega^{\nabla}$ we clearly have, since $\mathrm{Alt}(\Omega^{\nabla})=\Omega^{\nabla}$

$$d^{\nabla}\Omega^{\nabla}=0.$$

In Remark 3.3, we noted that the converging notion of torsion can be obtained by averaging the solder form over each edge before applying it to the exterior derivative $\mathfrak{d}$.

For the case of the vector valued connection 1-form, we basically average the form over edges before we apply the discrete covariant exterior derivative to obtain the curvature 2-form. Nevertheless for the connection 1-form this does not have an averaging effect, as for the curvature over a face.

We generalize this idea to define an exterior derivative in the sense that we average the forms per k-cell and apply to this form the covariant exterior derivative from Definition 5.1.

Definition 5.2. *Let E be a discrete vector bundle with connection over a simplicial complex X. Let $s \in X$ with $s = (x_0, \ldots, x_{k+1})$ be a $(k+1)-$simplex. For a discrete vector valued differential $k-$form $\alpha \in C^k(X; E)$ we define the averaged form per $k-$cell via*

$$\mathrm{Alt}(\alpha)[x_0, \ldots, x_k] = \frac{1}{|S_k|} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) R_{0,\sigma(0)} \cdot \alpha[x_{\sigma(0)}, \ldots, x_{\sigma(k)}].$$

We define the anti-symmetrized covariant exterior derivative in this case as

$$\langle d^\nabla \alpha, [x_0, \ldots, x_{k+1}] \rangle = \mathfrak{d}^\nabla(\mathrm{Alt}(\alpha))[x_0, \ldots, x_{k+1}].$$

If we use this definition for the solder form on a tetrahedron, we obtain

$$\begin{aligned}
\langle d^\nabla d^\nabla \theta, [abcd] \rangle_a = {} & \frac{1}{6}\Big((R_{ac}R_{cb} - R_{ad}R_{db})\tilde{\theta}_{ba} - (R_{ab}R_{bc} - R_{ad}R_{dc})\tilde{\theta}_{ca} + (R_{ac} - R_{ab}R_{bd}R_{dc})\tilde{\theta}_{cb} \\
& + (R_{ab}R_{bd} - R_{ac}R_{cd})\tilde{\theta}_{da} - (R_{ad} - R_{ab}R_{bc}R_{cd})\tilde{\theta}_{db} + (R_{ad} - R_{ab}R_{bd})\tilde{\theta}_{dc} \Big) \\
& + \frac{1}{3}\Big((R_{ac} - R_{ab}R_{bc})\tilde{\theta}_{cb} - (R_{ad} - R_{ab}R_{bd})\tilde{\theta}_{db} + (R_{ad} - R_{ab}R_{bd})\tilde{\theta}_{dc} \Big) \\
& + \frac{1}{2}(R_{ab}R_{bc} - R_{ac})\tilde{\theta}_{cd}
\end{aligned}$$

This expression can be interpreted as a weighted sum, that first averages over all edges of the tetrahedron, then only over all contribution of the last face and lastly with the original expression. All the expressions in front of the solder forms are evaluations of the curvature form. We can interpret this sum as a discrete wedge product between the curvature form and the solder form. For the calculation of the anti-symmetrized expressions we make use of the NCAlgebra Mathematica Library [Hel10].

Convergence. With the numerical test setup explained in Sec. 4 we actually get convergence for our sample connections. It turns out that the convergence for the wedge product on the generalized dual complex is faster; nevertheless they converge both linearly. This is similar to the case of the covariant exterior derivative of the solder form, where we also had faster convergence for the version evaluated on the barycenters.
We point out that this new, "averaged" notion has the big advantage that it is compatible with the covariant exterior derivative for Lie-algebra valued forms, and that it also preserves the differential Bianchi identity in the discrete realm.

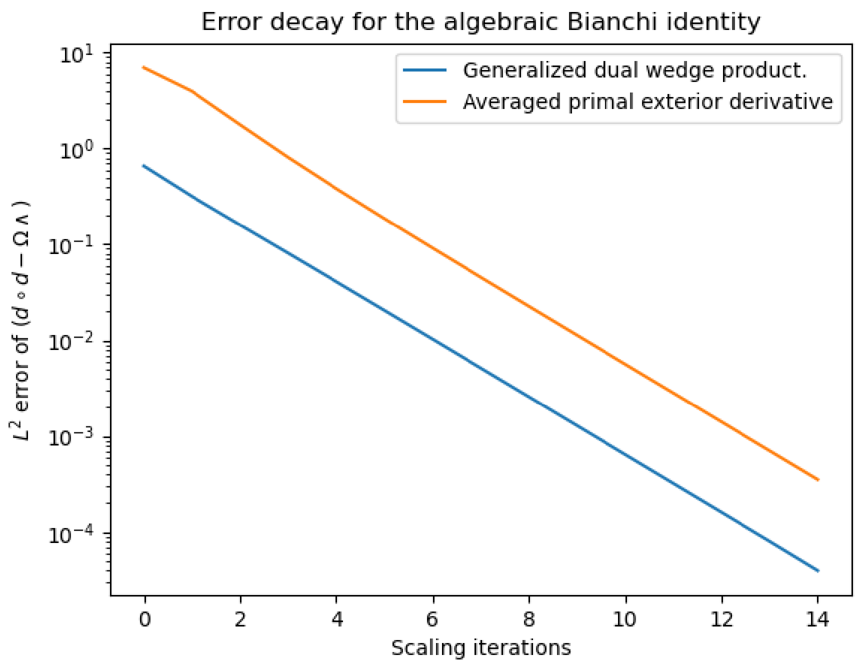


Figure 5.1: Error plot of $d^{\nabla} d^{\nabla} - \Omega^{\nabla}\wedge$ where we use as a rotation $R(-2.2, 2.1, 1.7)$ and displacement $v = (1.1, -1.2, -1.2)$ for the generalized dual and the averaged covariant exterior derivative.

6. Conclusion and Outlook

The presented approach in Chapter 5 yields, for the first time to our knowledge, a notion of discrete covariant exterior derivative that *satisfies* both the differential and the algebraic Bianchi identities — in the sense that a clear and valid notion of wedge product in the discrete realm makes each side of these identities equal. We showed that our notion of covariant exterior derivative converges towards the well known smooth wedge product under refinement. We showed in Chapter 4 that our notion of covariant exterior derivative on a generalized dual complex is close to the notion that we would expect from a smooth wedge product.

If we consider the notion of discrete scalar-valued wedge product for differential forms from[DHLM05b], we observe that their wedge product is basically an averaging over the products of possible stencils for the forms of lower degree. This is precisely what we get in Chap. 5. Further investigations should demonstrate whether this discrete notion is actually already sufficient for a wedge product of bundle-valued forms.

In future work we hope to show how this notion can be extended to more general cellular complexes, which we are unable to do yet.

Another direction of further investigation could be to link the different notions of curvature or torsion, that we found in sec. 3.3. Through different cut and evaluation fibers we get several representations of evaluations of forms over the same cell.

In the case of scalar-valued discrete exterior calculus, discrete differential forms can be obtained through the integration of smooth differential forms over k-cells. In our investigations we were trying to find a similar reasoning for bundle-valued differential forms. The issue with integration of bundle-valued forms over oriented cells is that so far, we only defined an integral for 1-forms, since we need a connection 1-form, see Def. 2.1. As in the scalar-valued case, we can define integrals of higher order forms through the tensor product of 1-dimensional cuts. In the bundle valued case, we are nevertheless faced with the issue that we do not have a bundle valued theorem of Fubini, since the product we are using is non commutative. This has the effect that the order of filling cells of higher order with 1-dimensional line segments actually matters.

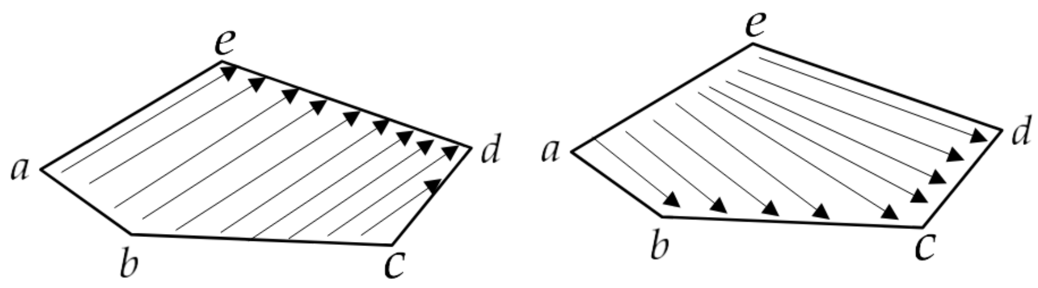


Figure 6.1: Bundle valued integrals of higher dimension can be calculated through different fillings with one dimensional line elements. The integral will depend on this choice.

One hypothesis that we want to understand better in further work is that different patterns of filling of cells for integration of smooth bundle valued forms will result in different representations for the discrete differential forms in terms of cut and evaluation fiber.

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A. Convergence of the Discrete Torsion

The notion of torsion presented in definition 3.2 is based on a discrete evaluation of a loop based integral. Nevertheless with this notion it is a priori not clear how this definition is linked to the smooth torsion defined through the exterior derivative of the solder form as in 2.5.1. In order to show the link to the definition 2.5.1 to first order we need to show, that the discrete notion of torsion can be seen also as a notion of wedge product. To do that, we will only consider the formula on a triangle $[ijk]$ since we can in there easily give sense to a wedge product through the vectors leaving the chosen evaluation fiber.
In the following let $\vartheta \in \Omega^1(\mathbb{R}^3, T\mathbb{R}^3)$ be the smooth solder form. We denote with θ_{ij} the discretization per edge (ij) and with ϑ_{ij} the discretization that is obtained through integrating ϑ along the edge (ij) in the initial frame. On the other hand θ_{ij} corresponds to the integral in the parallel propagated frame as presented in Section 3.5. For a triangle we have following [BEHS21] or definition 3.2

$$\langle \Theta, [ijk] \rangle_i^k = \tilde{\theta}_{ij} + R_{ij}\tilde{\theta}_{jk} - \tilde{\theta}_{ik}.$$

To first order we can approximate the discretized solder form through

$$\tilde{\theta}_{ij} = (I + \underbrace{\omega_{ic_{ij}}}_{\approx \frac{1}{2}\omega_{ij}})\vartheta_{ij} + o(h^2),$$

where h is proportional to the scaling of the triangle. The factor in front of the integral of the solder form corresponds to the first order approximation of the frame alignment. We obtain

$$\begin{aligned}\langle \Theta, [ijk] \rangle_i^k = \tilde{\theta}_{ij} + R_{ij}\tilde{\theta}_{jk} - \tilde{\theta}_{ik} &= \left(I + \frac{\omega_{ij}}{2}\right)\vartheta_{ij} + (I + \omega_{ij})\left(I + \frac{\omega_{jk}}{2}\right)\vartheta_{jk} - \left(I + \frac{\omega_{ik}}{2}\right)\vartheta_{ik} + o(h^2)\\ &= (\vartheta_{ij} + \vartheta_{jk} - \vartheta_{ik}) + \frac{\omega_{ij}\vartheta_{ij}}{2} + \omega_{ij}\vartheta_{jk} + \frac{\omega_{jk}\vartheta_{jk}}{2} - \frac{\omega_{ik}\vartheta_{ik}}{2} + o(h^2)\end{aligned}$$

It should be noted, that the term, that involves two products of ω along different edges can be neglected, since we consider first order approximation only. The terms that involve multiple products are part of the $o(h^2)$. We want to express this derivative entirely through contributions of the edges (ij) and (ik). To do this, note that we have $d\vartheta = 0$, since it is the differential of a constant form. Therefore we have in the discrete sense

$$d\vartheta_{ijk} = \vartheta_{ij} + \vartheta_{jk} - \vartheta_{ik}$$

Therefore

$$\begin{aligned}
\langle \Theta, [ijk] \rangle_i^k &= (d\vartheta)_{ijk} + \frac{\omega_{ij}}{2}\vartheta_{ij} + \omega_{ij}(\vartheta_{ik} - \vartheta_{ij}) + \frac{1}{2}(\omega_{ik} - \omega_{ij})(\vartheta_{ik} - \vartheta_{ij}) - \frac{\omega_{ik}}{2}\vartheta_{ik} \\
&= (d\vartheta)_{ijk} + \frac{\omega_{ij}}{2}\vartheta_{ij} + \omega_{ij}\vartheta_{ik} - \omega_{ij}\vartheta_{ij} + \frac{\omega_{ik}}{2}\vartheta_{ik} + \frac{\omega_{ij}}{2}\vartheta_{ij} - \frac{\omega_{ik}}{2}\vartheta_{ij} - \frac{\omega_{ij}}{2}\vartheta_{ik} - \frac{\omega_{ik}}{2}\vartheta_{ik} \\
&= (d\vartheta)_{ijk} - \frac{\omega_{ik}\vartheta_{ij}}{2} + \frac{\omega_{ij}\vartheta_{ik}}{2} \\
&= (d\vartheta)_{ijk} + \frac{1}{2}(\omega \wedge \vartheta)(e_{ik}, e_{ij})
\end{aligned}$$

Remark A.1. This derivation also shows that it is necessary to use the forms evaluated in the fiber over the center point of the edges.

Remark A.2. This derivation can be done for any arbitrary bundle valued differential form too. We used none of the special properties of the solder form to show the link to the smooth exterior derivative.