Exercise

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Exercise. Prove that the complex projective n-space $\mathbb{CP}^n$ is compact. Furthermore, prove that $\mathbb{CP}^1$ is diffeomorphic to the 2-sphere S^2.

Proof. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the quotient map associated with $\mathbb{CP}^n$. We want to first show that the restriction $\pi|_{S^{2(n+1)-1}}$ of π to the $(2(n+1)-1)$-sphere is a surjection onto the complex projective plane. Then fix $v = (x^1, x^2, \dots, x^{n+1}) \in \mathbb{R}^{2(n+1)}$ and let $l = \{\lambda v | \lambda \in \mathbb{C}\}$ denote a one-dimensional subspace of $\mathbb{C}^{n+1}$. Let $p \in l$ so that $p = \lambda v$ for some $\lambda = a + bi$ and define

$$C = \lambda^2 \sum_{i=1}^{2(n+1)} (x^i)^2.$$

Divide both sides by C to obtain

$$\frac{\lambda^2}{C} \sum_{i=1}^{2(n+1)} (x^i)^2 = 1.$$

If we let $\lambda_0 = \frac{a}{\sqrt{C}} + \frac{b}{\sqrt{C}} i$ then $\lambda_0 v$ lies on $S^{2(n+1)-1}$ and $\pi(\lambda_0 v) = l$, as required.

Because π is a continuous map and $S^{2(n+1)-1}$ is compact, we must have that $\pi(S^{2(n+1)-1}) = \mathbb{CP}^n$ is compact as well.

The work for the second part of the problem is as follows: For a point $l \in \mathbb{CP}^1$, we can express l in homogenous coordinates as $l = [z, w]$, for some $z, w \in \mathbb{C}$. Scale $l = [1, z']$, where $z' = w/z$, and note that the only other point in $\mathbb{CP}^1$ is $[0, 1]$.

Recall that the stereographic projection to the north pole $s : \mathbb{R}^2 \to S^2 \setminus \{n.p.\}$ where $n.p.$ denotes the north pole of the sphere, is defined by

$$s(x^1, x^2) = \frac{(2x^1, 2x^2, |x|^2 - 1)}{|x|^2 + 1},$$

Of course, s is a bijection. Furthermore, it is smooth because the four mappings

$$\begin{aligned} x &\mapsto 2x, \\ x &\mapsto 2x^2, \\ (x^1, x^2) &\mapsto (x^1)^2 + (x^2)^2 - 1, \\ (x^1, x^2) &\mapsto (x^1)^2 + (x^2)^2 + 1 \end{aligned}$$

are Euclidean smooth. Now define the map $d : \mathbb{CP}^1 \to S^2$ by

$$\begin{aligned} [1, x + iy] &\overset{d}{\mapsto} s(x, y), \\ [0, 1] &\overset{d}{\mapsto} n.p. \end{aligned} \tag{1}$$

Of course, d presents a surjection from $\mathbb{CP}^1$ to S^2 because s maps to every point but the north pole and now we have a point $[0, 1]$ which d can map to that point. Meanwhile, injection of d follows from the injectivity of s. Thus the inverse $d^{-1} : S^2 \to \mathbb{CP}^1$ is given by

$$d^{-1}(x^1, x^2, x^3) = [1, \frac{x^1}{1 - x^3} + \frac{x^2}{1 - x^3} i] \tag{2}$$

for $x \neq (0, 0, 1)$, and $d^{-1}(0, 0, 1) = [0, 1]$.

Our goal now is to find suitable charts for $\mathbb{CP}^1$. Because $\mathbb{CP}^1$ is endowed with the quotient topology, we must have that $U \subset \mathbb{CP}^1$ is open iff $\tilde{U} = \pi^{-1}(U)$ is open in $\mathbb{C}^2 \setminus \{0\}$. Let $\tilde{U_i}$ be an open set in $\mathbb{C}^2 \setminus \{0\}$ s.t. $x^i \neq 0$ and let $p = (u, v) \in \tilde{U_i}$. Then $[u, v] \in \pi(\tilde{U_i}) = U_i$.

Define the mapping $\phi_1 : U_1 \to \mathbb{R}^2$ by $\phi_1[u, v] = v/u$ and similarly the map $\phi_2 : U_2 \to \mathbb{R}^2$ by $\phi_2[u, v] = u/v$, where division in this case corresponds to complex division. Of course, these two coordinate functions are bijective, with inverses $\phi_1^{-1}(x^1, x^2) = [1, (x^1, x^2)]$ and $\phi_2^{-1}(x^1, x^2) = [(x^1, x^2), 1]$:

$$\begin{aligned} \phi_1^{-1} \circ \phi_1[u, v] &= \phi_1^{-1}(v/u) \\ &= \phi_1^{-1}\left(\frac{v^1u^1 + v^2u^2}{|u|^2}, \frac{v^2u^1 - v^1u^2}{|u|^2}\right) \\ &= \left[1, \left(\frac{v^1u^1 + v^2u^2}{|u|^2}, \frac{v^2u^1 - v^1u^2}{|u|^2}\right)\right] \\ &= [u/u, v/u] \\ &= [u, v]; \end{aligned}$$

$$\begin{aligned} \phi_1 \circ \phi_1^{-1}(x^1, x^2) &= \phi_1[1, (x^1, x^2)] \\ &= (x^1, x^2). \end{aligned} \tag{3}$$

For the second coordinate function, we have

$$\begin{aligned}\phi_2^{-1} \circ \phi_2[u,v] &= \phi_2^{-1}(u/v) \\ &= \phi_2^{-1}\left(\frac{v^1u^1 + v^2u^2}{|v|^2}, \frac{v^2u^1 - v^1u^2}{|v|^2}\right) \\ &= \left[\left(\frac{v^1u^1 + v^2u^2}{|v|^2}, \frac{v^2u^1 - v^1u^2}{|v|^2}\right), 1\right] \\ &= [u/v, v/v] \\ &= [u,v];\end{aligned}$$

$$\begin{aligned}\phi_2 \circ \phi_2^{-1}(x^1,x^2) &= \phi_2[(x^1,x^2),1] \\ &= (x^1,x^2).\end{aligned}$$

To show that the charts (U_1, ϕ_1) and (U_2, ϕ_2) are smoothly compatible, we will show that the composition maps $\phi_2 \circ \phi_1^{-1}$ defined on $\phi_1(U_1 \cap U_2)$ and $\phi_1 \circ \phi_2^{-1}$ defined on $\phi_2(U_1 \cap U_2)$ are Euclidean smooth. The map

$$\begin{aligned}\phi_2 \circ \phi_1^{-1}(x^1,x^2) &= \frac{1}{x^1 + x^2 i} \\ &= \left(\frac{x^1}{(x^1)^2 + (x^2)^2}, \frac{-x^2}{(x^1)^2 + (x^2)^2}\right), \end{aligned} \tag{4}$$

is well-defined for $(x^1,x^2) \in \phi_1(U_1 \cap U_2)$ as the denominators in both slots can never equal 0 due to our conditions on U_1 and U_2. Furthermore, the denominators of all of their partials will be in the form $[(x^1)^2 + (x^2)^2]^{2k}$, for some positive integer k, whereas the expressions in the numerators are the usual functions of two variables which lack any discontinuities. The same applies to the other composition of coordinates:

$$\begin{aligned}\phi_1 \circ \phi_2^{-1}(x^1,x^2) &= \frac{1}{x^1 + x^2 i} \\ &= \left(\frac{x^1}{(x^1)^2 + (x^2)^2}, \frac{-x^2}{(x^1)^2 + (x^2)^2}\right),\end{aligned}$$

where $(x^1,x^2) \in \phi_2(U_1 \cap U_2)$.

Finally, to show that d is smooth, we must show that for every $l \in \mathbb{CP}^1$, there exists charts (U, ϕ) on $\mathbb{CP}^1$ and (V, ψ) on S^2 (with $d(l) \in V$) s.t. $\psi \circ d \circ \phi^{-1}$ is Euclidean smooth.

Let $l \in U_1$ so that $\phi_1(l) = (x^1,x^2)$. Then $d(l) \neq [0,1]$ and

$$\begin{aligned}(s^{-1} \circ d \circ \phi_1^{-1})(x^1,x^2) &= s^{-1} \circ d[1,(x^1,x^2)] \\ &= s^{-1} \circ s(x^1,x^2) \\ &= (x^1,x^2),\end{aligned}$$

which is smooth as it is the identity mapping on $\mathbb{R}^2$.

Finally, let $l \in U_2$ so that $\phi_2(l) = (x^1, x^2)$. Then $d(l)$ may $= [0, 1]$ and

$$\begin{aligned}(s^{-1} \circ d \circ \phi_2^{-1})(x^1, x^2) &= s^{-1} \circ [(x^1, x^2), 1] \\ &= (s^{-1} \circ s)\left(\frac{x^1}{|x|^2}, \frac{-x^2}{|x|^2}\right) \\ &= \left(\frac{x^1}{|x|^2}, \frac{-x^2}{|x|^2}\right). \end{aligned} \tag{5}$$