Exercise

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Exercise. Let M be an oriented n-manifold with Riemannian metric g and let dvol_g denote the volume form of g. Prove that in local coordinates, $\mathrm{dvol}_g = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n$.

Proof. We want to show that $\det(e^j(\partial_{x^i})) = \sqrt{\det(g_{ij})}$. Note that $\partial_{x^i} = e^j(\partial_{x^i})e_j$, which implies

$$\begin{aligned} g_{ij} &= \langle \partial_{x^i}, \partial_{x^j} \rangle \\ &= \left\langle e^k(\partial_{x^i})e_k, e^s(\partial_{x^j})e_s \right\rangle \\ &= e^k(\partial_{x^i})e^s(\partial_{x^j}) \langle e_k, e_s \rangle \\ &= e^k(\partial_{x^i})e^s(\partial_{x^j})\delta_{ks} \\ &= \sum_{s=1}^{n} e^s(\partial_{x^i})e^s(\partial_{x^j}), \end{aligned} \tag{1}$$

which is the (i, j)-component of the matrix product $(e^k(\partial_{x^s}))(e^k(\partial_{x^s}))^T$. Thus we have that

$$\begin{aligned} \sqrt{\det(g_{ij})} &= \sqrt{\det(e^k(\partial_{x^s}))\det(e^k(\partial_{x^s}))^T} \\ &= \sqrt{\det^2(e^k(\partial_{x^s}))} \\ &= \det(e^k(\partial_{x^s})), \end{aligned} \tag{2}$$

as required.