Exercise

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Exercise. Apply Parseval's identity to $\sin(x/2)$ on $[-\pi, \pi]$ to compute an exact value for the infinite series $\sum_{n=1}^{\infty} n^2/(4n^2-1)^2$.

Proof.
Parseval's Relation states that

$$||f||^2 := \int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2,$$

where

$$c_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-inx} dx.$$

Letting $f(x) = \sin(x/2)$, we then get that

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(x/2) e^{-inx} dx = \frac{4in\cos(\pi n)}{\pi(4n^2-1)}$$

and

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 4,$$

so that

$$\sum_{n=-\infty}^{\infty} \left| \frac{4in\cos(\pi n)}{\pi(4n^2-1)} \right|^2 = \frac{2}{\pi}.$$

Because $|g(n)|^2 = g^2(n)$ for any function g, note that the above equation can be rewritten as

$$\sum_{n=-\infty}^{\infty} \frac{(4in\cos(\pi n))^2}{(\pi(4n^2-1)^2} = \sum_{n=-\infty}^{\infty} \frac{-4^2 n^2 \cos^2(\pi n)}{\pi^2(4n^2-1)^2} = \sum_{n=-\infty}^{\infty} \frac{-16\cos^2(\pi n)}{\pi^2} \cdot \frac{n^2}{(4n^2-1)^2} = \frac{2}{\pi}.$$

As $\cos^2(\pi n) = 1$ for every n, rewrite the above as

$$\sum_{n=-\infty}^{\infty} \frac{n^2}{(4n^2-1)^2} = \frac{\pi}{8},$$

as desired.