Exercise

Bao Lam

Background: (The following is taken from "Elementary" by Hoffman. Consider a bar whose temperature is $T(x,t)$ at the point x at time t. The law of heat conduction then asserts that

$$\frac{\partial T}{\partial t}(x,t) = k\frac{\partial^2 T}{\partial x^2}(x,t), \tag{1}$$

where k is a constant determined by the conductivity of the material. This is called the *Heat Equation.*

Theorem: If f is square integrable, then, for each $t > 0$,

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t/l^2} \cos\frac{n\pi x}{l} \tag{2}$$

converges uniformly, is differentiable, and satisfies the heat equation and boundary conditions.

Exercise. Let $T(x,t)$ be a solution to the heat equation. Show that for all t, the mean temperature $(1/l)\int_0^l T(x,t)dx$ approaches $a_0/2$.

Proof.
Fix $l > 0$. We wish to show that

$$g(t) = \frac{1}{l}\int_0^l T(x,t)dx \to \frac{a_0}{2}$$

as $t \to \infty$. First, note that

$$g_n(t) := \frac{a_n \cos(\frac{n\pi x}{l})}{\exp(\frac{n^2\pi^2 t}{l^2})} \to 0$$

as $t \to \infty$, in part because $a_n \cos(\frac{n\pi x}{l})$ does not depend on t, for any n. We now introduce the following theorem and proof, which are both modified versions of Theorem 7.11 (and its proof) in Rudin's "Principles of Mathematical Analysis".

Lemma. Suppose $f_n \to f$ uniformly on a set E in the extended real line and ∞ a limit point of E, and suppose that

$$\lim_{t\to x} f_n(t) = A_n. \tag{3}$$

Then $\{A_n\}$ converges, and

$$\lim_{t\to\infty}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to\infty} f_n(t). \tag{4}$$

Proof. Fix $\epsilon > 0$. By uniform convergence, there exists an N s.t. for $n, m \geq N$ and $t \in \mathbb{R}$ we have that

$$|f_n(t) - f_m(t)| \leq \epsilon. \tag{5}$$

By letting $t \to \infty$, we have that

$$|A_n - A_m| \leq \epsilon \tag{6}$$

for $n, m \geq N$. Thus $\{A_k\}$ is Cauchy and converges to some A. Now choose n s.t.

$$|f(t) - f_n(t)| \leq \epsilon/3 \tag{7}$$

for all $t \in \mathbb{R}$ and

$$|A_n - A| \leq \epsilon/3 \tag{8}$$

The former is possible by uniform convergence whereas the latter by the convergence of $\{A_k\} \to A$. For such an n, choose a neighborhood $\{x \,|\, x > a\}$ of ∞ for which

$$|f_n(t) - A_n| \leq \epsilon/3. \tag{9}$$

Such a neighborhood exists by the hypothesis. Thus, we have that

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \leq \epsilon, \tag{10}$$

as required. $\square$

Now note that because T converges uniformly by Theorem 10.7.2 of Marsden, we must have that g_n converges uniformly as well. Then applying the above lemma, we can write

$$\lim_{t\to\infty}\sum_{n=1}^{\infty} g_n(t) = \sum_{n=1}^{\infty}\lim_{t\to\infty} g_n(t) = 0.$$

Now

$$\lim_{t\to\infty}\frac{1}{l}\int_0^l\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} g_n(t)\right)dx = \frac{a_0}{2} + \frac{1}{l}\lim_{t\to\infty}\int_0^l\sum_{n=1}^{\infty} g_n(t). \tag{11}$$

By Lebesgue's Dominated Convergence Theorem, we can interchange the integral and summation sign and write the above as

$$\frac{a_0}{2} + \frac{1}{l}\lim_{t\to\infty}\int_0^l\sum_{n=1}^{\infty} g_n(t) = \frac{a_0}{2} + \frac{1}{l}\int_0^l\lim_{t\to\infty}\sum_{n=1}^{\infty} g_n(t) = \frac{a_0}{2} + \frac{1}{l}\sum_{n=1}^{\infty} 0 = \frac{a_0}{2}, \tag{12}$$

as required.