Exercise

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Exercise. Let (M, g) be an oriented and compact Riemannian manifold without boundary. Prove that the volume form on g, dvol_g is not exact, i.e., $\mathrm{dvol}_g \neq d\theta$ for any $\theta \in \Omega^{n-1}(M)$.

Proof. By way of contradiction, we seek a $\theta = \theta_K dx^K \in \Omega^{n-1}(\mathbb{R}^n)$ for which $d\theta = d\mathrm{vol}_g$. From above, we have that

$$d\mathrm{vol}_g = \det(e^j(\partial_{x^i}))dx^1 \wedge \cdots \wedge dx^n. \tag{1}$$

Recall that on $\mathbb{R}^n$, $d\theta$ can be written locally as

$$d\theta = \sum_J{}' \sum_{i=1}^n \frac{\partial \theta_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{n-1}}, \tag{2}$$

where $\sum'$ means that we are summing over every such increasing multi-set (as in Lee). Furthermore, note that because each $\theta_J \in \mathbb{R}^n$ is independent of each other, we can set $J^* = \{2, 3, \ldots, n\}$ and $\partial\theta_J/\partial_{x^i} \equiv 0$ for all $\sigma(n-1) \ni J \neq J^*$ and $i \neq 1$, and constructing the function $\theta_{J^*} = x^1$ so that $\partial\theta_{J^*}/\partial x^1 = \det(g_{ij}) = 1$, hence giving us

$$\begin{aligned} d\theta &= \sum_{J \neq J^*}{}' \sum_{i \neq 1}^n \left(\frac{\partial \theta_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{n-1}} \right) + \frac{\partial \theta_{J^*}}{\partial x^1} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \\ &= dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n. \end{aligned} \tag{3}$$