ALGORITHM DESIGN PROJECT

Dănciulescu Theodora - ioana

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Web: Programming Techniques
Link to problem: ORDER

1 Introduction

The goal of my assignment is to develop a software for experimenting with algorithms. The project is focused on the development of skills for good programming of basic algorithms, covering coding, design, documentation and presentation of results. At the end of my project I must produce a set of deliverables including: a technical report, source code and experimental data. I will present the usage of data structures and algorithms as RMQ Segment Trees and Binary Indexed Trees or Fenwick trees.

2 Problem statements

There are N children (numbered from 1 to N) placed in a circle formation in trigonometrical sense. We will see that this trigonometrical sense doesnt even matter.
At each step i (initially 1), the current child (initially 1) starts counting i-children in this trigonometrical sense and the child it reaches is excluded from the circle. The child after the eliminated one will continue the count. This means that the circle will eventually be empty.
Our task is to print the order in which the children are excluded from the circle.

Example Explanation:

- Input: 6
- Output: 2 4 1 3 5 6
- Explanation: The 1st child starts counting (i = 1), so we reach the child 1+1=2 which is eliminated. The next child is 3. children[] = 1, 3, 4, 5, 6
- We are at 2nd step, so we count 2 children starting from 3 (inclusive). We reach child 4. Eliminated child 4 will pass the counting responsibility to child 5. children[] = 1, 3, 5, 6
- Step 3, count 3 children from 5 (inclusive): 5, 6, 1. We eliminate 1, next child is 3. children[] = 3, 5, 6
- Step 4, count 4 children from 3 (inclusive): 3, 5, 6, 3. We eliminate 3, next child is 5. children[] = 5, 6
- Step 5, count 5 children from 5 (inclusive): 5, 6, 5, 6, 5. We eliminate next child is 6. children[] = 6
- Step 6 and last step, child 6 is finally removed.
- So the erase order is: 2, 4, 1, 3, 5, 6.

3 Algorithms

3.1 SEGMENT TREES

In computer science, a segment tree is a tree data structure for storing intervals, or segments. It allows querying which of the stored segments contain a given point. It can be implemented as a dynamic structure.

3.1.1 How to build a segment tree

- There might be more ways to represent a Segment Tree, but most commonly in competitive programming, well see this method of representation:
- A 1D array 4 times larger in dimension than the number of elements. Something like segtree[4*N].
- Then we have:
- segtree[1] - the information of the root.
- segtree[node] information in our current node
- segtree[node * 2] information in the left child of the current node
- segtree[node * 2 + 1] information in the right child of the current node
- A node in a Segment Tree doesnt hold information about a single element of the input array (except in case of leaves), but rather information about an interval. In our case/problem, the information is the maximum on that interval.
- The root holds information about the whole input array, so interval [1, N]. We split this interval in 2 halves:

  [1, N/2] this left half is the responsibility of the left child so for segtree[2*1] = segtree[2]

  [N/2+1, N] this right half is the responsibility of the right child so for segtree[2*1+1] = segtree[3]

- Applying the same logic recursively will then build our whole Segment Tree!

3.1.2 More info about Node Construction

We know the maximum of a single element already, which is that element. That means we know the information for the leaves! Any interval [x, x] has:

- max([x, x]) = max(arr[x]) = arr[x].

Suppose we are in the node with interval [x, x+1]. Now we know the information of its left child and right child. So we use both information to compute the information for our node. In our case, we use the children to compute the maximum for our node:

- max([x, x+1]) = max(max([x,x]), max([x+1,x+1]))=max(arr[x], arr[x+1])
  .

Now we can safely say that the information of a node is a combination/computation of the information of its children.
In our case: segtree[node] = max(segtree[2*node], segtree[2*node+1])
In the general case: segtree[node] = combine(segtree[2*node], segtree[2*node+1]), where combine() uses that information however it needs to be used.

3.1.3 Update function

When we want to modify a position pos, we actually want to modify the information (maximum) in the interval [pos, pos] (leaf node).
We will reach that leaf by splitting intervals starting from the root, that means we will use recurrence to get to the leaf node. We modify the information in the leaf node and also update the parent nodes when returning to them from the recurrence.
By updating the parent nodes I mean reapplying combine() for the children.

3.1.4 Query function

Answering a Query now means combining information from the biggest intervals possible from our Segment Tree to get our interval.
Lets assume we have the Query for interval [2, 5].
The highest intervals we can get from our tree are: [2, 2], [3, 3], [4, 5].
Note that, even though [1, 2] and [1, 3] are bigger than [2, 2] and [3, 3], using their information would mean using information about [1, 1] too, which is not included in our query interval [2, 5].
How do we get the biggest intervals possible? We start from the root and start splitting intervals. At each step we verify if the interval we are at [left, right] is included in the query interval [a, b]: a ¡= left and right ¡= b.

3.1.5 Time Complexity

Segment Trees offers us a time complexity O(logN) for both Update and Query operation, and that is a very good complexity on any case.

3.2 BINARY INDEXED TREES

A Fenwick tree or binary indexed tree is a data structure that can efficiently update elements and calculate prefix sums in a table of numbers. This structure was proposed by Peter Fenwick in 1994 to improve the efficiency of arithmetic coding compression algorithms.

3.2.1 Time complexity and memory

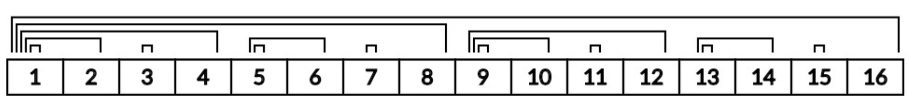
When compared with a flat array of numbers, the Fenwick tree achieves a much better balance between two operations: element update and prefix sum calculation. In a flat array of n numbers, you can either store the elements, or the prefix sums. In the first case, computing prefix sums requires linear time; in the second case, updating the array elements requires linear time (in both cases, the other operation can be performed in constant time).
Fenwick trees allow both operations to be performed in O(log n) time. This is achieved by representing the numbers as a tree, where the value of each node is the sum of the numbers in that subtree. The tree structure allows operations to be performed using only O(log n) node accesses.

3.2.2 Constructing the intervals of a Binary Indexed Tree / Fenwick Tree

If all intervals have the same length for instance, the best balance is when we use [$\sqrt{n}$] as the length of our intervals. This is a common technique when you know one of your operations takes B steps, and the other one N/B steps, we would pick B = [$\sqrt{n}$], to minimize the worst case.

But we can get even better results if we consider the length of the interval ending at i to be the largest power of 2 that's a divisor of i.

Below you can see a graphical representation of the intervals, for N=16:



These intervals may seem a bit chaotic at a first glance.

Let's say F[i] is the length of the interval ending at index i. Basically are dealing with intervals of the form [i-F[i]+1, i]. For the example above F=[1,2,1,4,1,2,1,8,1,2,1,4,1,2,1,16].

We can show how to build F incrementally over lengths that are powers of 2:

- if N = 1, it only makes sense to maintain an interval of length 1, so F=[1]
- The next step is for N=2. We will keep track of the first element and the sum of the first two, F=[1,2]:
- To find F for the next power of 2, we take the previous F, append a copy of it, and change the last element to be equal to N. So in the case of N=4,

appending a copy to the previous F means we get [1, 2, 1, 2], but then we change the last element, so the new F=[1, 2, 1, 4]

- For N=8, we have F=[1,2,1,4,1,2,1,8]
- And for N=16, we get our result F=[1, 2, 1, 4, 1, 2, 1, 8, 1, 2, 1, 4, 1, 2, 1, 16]

3.2.3 Finding all intervals that contain a certain index

Since we've seen that intervals are either disjoint or fully contained one inside another, this part is equivalent with repeatedly just finding the next smallest interval that contains our own. Notice the father of an interval is always identified by a higher index, so we can say we need to find the difference between i and its father.

If i is a power of 2, the difference is equal to i, because the father is the next power of 2. Otherwise, we can again make use of the inductive way the Fenwick tree is build, and conclude that the difference for i is equal to the difference for i-p (p is the largest power of 2, smaller than i).

You can already guess where this is going: the difference between an interval i and its father is equal to the interval's length which can be computed really fast with bitwise and operation between pos and -pos.

When we change to updating the interval ending at pos, its father's father, and so on.

In the function below we consider the first parameter to be the updated position, and the second one the difference between the new value and the old value being updated:

```
void update(int pos, int val) {
    while (pos <= N) {
        fenwick[pos] += val;
        pos += lsb(pos);
    }
}

```

As for the query, we start at the index being queried, add the sum of its interval, and move to the left. This is where knowing the length of the intervals comes in handy:

```
int query(int pos) {
    int sum = 0;
    while (pos > 0) {
        sum += fenwick[pos];
        pos -= lsb(pos);
    }
    return sum;
}

```

4 Pseudocode Algorithms

4.1 General logic aspects of the problem

Since the array is supposed to be circular (array[] = 2, 3, 4, after 4 follows 2), we will use modulo. Suppose we are at i-th child and we have to count k children (inclusive). Since its inclusive, we can subtract 1 from k. Suppose we go out of boundaries i+k-1 ¿ remaining_children. We can use modulo to get out dirt and provide the correct position: (i+k-1) % remaining_children. That was easy. But what if the result is 0? That just means i+k-1 == remaining_children, so we set our result to remaining_children.
But wait, if we start counting from the first child, he is not included. Thats fine, we will just start using the formula from the second child, not from the first.

4.2 Segment Trees Method

Suppose we know the position of the child wed like to remove in the children[] list from the Example Explanation. We know that at each step, a child is removed from that list.
What if we build a ST in which a nodes information represents how many children from that nodes interval are still in the circle? Thats the correct idea, by the way, but lets see why this works.

We build the ST before any counting, that means the circle has all N children. That means that we already know the information for every interval.
How many children are in an interval [a, b]? b-a+1, of course. So that gets rid of the building part easily.
We can also consider knowing information only about a leaf, just for the sake of it. So in a leaf there is only 1 child. Then we have to think about the combine() function I was talking about in the Segment Trees General Aspect section.
We have 2 children nodes of the [a, b] interval and know their information:

- [a, mid] is interval A
- [mid+1, b] is interval B

where mid=(a+b)/2
Then we know that for interval [a, b] we have A+B children in it.
So the combine() method just takes both values of the children and sums them up.

Pseudocode for Building the Segment Tree

When do we need to UPDATE? When we remove a child from the circle. So we go to the leaf which corresponds to that child and subtract 1. But a leaf has 1 child anyway, so we can just set it to 0.
Now here comes the tricky part. We know the number of the child we should remove. How do find out what index that child actually has (so we can go to the correct leaf)?

```
build_tree( node, left, right){
        if left = right then
                aint[ node ] <- 1
                return
        mid <- (left + right) / 2
        build_tree ( left_child, left, mid)
        build_tree( right_child, mid + 1, right)
        aint[ nod ] = aint[ left_child ] + aint[ right_child ]
}
```

Lets take a concrete example: we have 5 children out of 8 remaining.
The root of the ST has interval [1, 8]. Doing updates correctly, we should have segtree[root]=5.
Suppose the left_child of the root has segtree[left_child] = 3. That can only mean segtree[right_child] = 5-3 = 2, otherwise the tree cant be correct, because segtree[left_child]+segtree[right_child] = segtree[parent].

We are searching the leaf of the 2nd child. If we have 2 children in the left_child, that means the 2nd child must be somewhere in that interval, so we go to the left_child. But what if we looked for the 3rd child? There are only 2 children in the left_child, so the 3rd must be somewhere in the right_child. But if we go to the right_child, are we still looking for the 3rd child of that interval?

We know that in the left_child there are 2 children, so that means 2 children are already excluded from our search. Left_child and right_child are basically 2 continuous halves, uniting them would create a proper big interval. Knowing that our child is in the right_child means we know that in this big interval the child were looking for is somewhere in the right half. That means already half of the big interval is excluded from the search. So in the right half we are not looking for the 3rd child anymore, but rather for the (3-2)=1st child.

We reach the following condition:

- if (searched_child <= segtree[left_child]) go search for searched_child in left_child
- else go search for the searched_child − segtree[left_child] in the right_child

In other words: if there are enough children in the left half, go look in that half, otherwise, go look in the remaining interval of the right half.

Querying here actually means finding the index of the child were looking for. And that is basically searching in the same way as when updating, but instead of modifying some value, we just return the index of that leaf.
As you can see, since the updateand query methods do the same thing until they reach the leaf, we can make a single method that does both updateand query when it reaches that leaf.

Pseudocode for Query and Update function:

Pseudocode for Order processing function:

```
function(remaining_children, n, child)
    for i = 1, n do
        child <- (child + i - 1) modulo remaining_children
        if child = 0 then
            child <- remaining_children

        remaining_children <- remaining_children - 1
        left_pos <- 0
        accmlate <- 0
        for j = 15 , 0 do
            if left_pos + 1 << j < n then
                if accmlate + aib[left_pos + (1 << j)] < child then
                    accmlate <- accmlate + aib[left_pos + (1 << j)]
                    left_pos <- left_pos + (1 << j)

        left_pos <- left_pos + 1
        update(left_pos, -1, n)
```

```
query_and_update_tree(nod, left, right, nr)
        if left = right then
                aint[nod] <- 0
                return left

        mid <- (left + right) / 2
        pos <- 0

        if nr <= aint[left_child] then
                pos <- query_and_update_tree(left_child, left, mid, nr)
        else
                pos <- query_and_update_tree(right_child, mid + 1, right, nr - aint[left_child])

        aint[nod] <- aint[left_child] + aint[right_child]

        return pos
```

4.3 Binary Indexed Tree Method

In this method we will use the same formula described in the 4.1 General aspects of the problem section.

We know that at every step a child is eliminated. We will initialize all the positions of the aib with 1 and when a child is eliminated we do update at that position with -1. So, the aib array for N = 6 will initially look like this :

1	2	1	4	1	2
1	2	3	4	5	6

We will iterate through the aib array making use of as big as possible subsequences of length power of 2. For the above example we will have 2 on position 2, that means we have 2 children in the interval [1,2], on position 4 we have the value 4, which means that in the interval [1,4] we have 4 children. Depending on these values we will make a sort of binary search implemented on binary indexed trees like this: if we need to count more children we will look in the right of the current position, otherwise we will look in the left of the current position.

The aib array is a global variable so the modifications done to it in every function is applied globally.

4.3.1 Update function

In this problem we notice that we only need the update function of binary indexed trees. The function has the classic parameters : the position to update, the value to update with and the length of the aib array.

```
update( position, value, n)
    for( ; position ⩽ n; position <- position + (position & -position))
        aib[position] <- aib[position] + value
```

4.3.2 Build AIB function

We will initialize every position with 1, so we call the update function with the parameter position = 1 for every position from 1...n. The function has only one parameter, the lenght of the aib.

```
build_tree(n)
    for i = 1, n do
         update(i, 1, n)
```

4.3.3 Function generating the elimination of the children

At every step we must eliminate a child, using the formula I know which child to eliminate. We also decrease the variable remaining_children. For the binary search we set the bounds like this: left_pos = 0, and the right bound I called it "accmlate" because it accumulates the number of children I counted and I set it as the right bound and equal to n.

As the problem states that the maximum number of children is 30.000, that means that it is also the maximum length of the aib array. That means that the maximum length is 2 at power 15.

I am iterating in decreasing order with a loop of length n through all the powers of 2 : 15....1; at every position i check if the length power of 2 I am situated is lower than the length of my aib and I also check if accmlate + the value at my iterated position is lower than the number of children to be counted. If the condition is full-filed I add the value from my iterated value to accmlate and actualize the left_pos.

Because I am starting with left_pos from zero, before displaying the index of the eliminated child I increment it with 1, make update with -1 at that position and after that I display the position found.

```
function(remaining_children,n, child)
    for i = 1, n, do
        child <- (child + i - 1) % remaining_children
        if child = 0
            child <- remaining_children
        remaining_children - 1
        left_pos <- 0
        accmlate <- 0
        for j = 15, 0 do
            if left_pos + ( 1 << j) < n then
                if accmlate + aib[left_pos + (1 << j)] < child
                    accmlate <- accmlate + aib[left_pos + (1 << j)]
                    left_pos <- left_pos + (1 << j)

        left_pos + 1
        update(left_pos, -1, n)
        printf left_pos
```

5 Application Design

The high level architectural overview of the application.

5.1 Segment Trees

- use of function to query and update the segment tree accordingly
- function to generate the order of elimination based on the data generated in my segment tree and also print it
- function to build the segment tree
- main function calling the other functions and initializing certain variables

5.2 Binary Indexed Tree

- classic update function for binary indexed trees
- function to initialize the binary indexed tree
- function to compute the final answer using binary search on the data generated in the data structure used
- main function calling the other functions and initializing certain variables

6 Experimental Data

I use a c source to randomly generate my data. The input is represented by a single integer lower than 30000 which represents the number of children who plays the counting game.

I simply genarate integers lower than 30000 and higher or equal than 1 with the standard function rand() from C library.

The time limit per test is 0.05 seconds and both methods pass all the tests successfully on the infoarena platform.

7 Results and Conclusions

A segment tree is a tree data structure for storing intervals or segments. It allows querying which of the stored segments contain a given point. It is, in principle, a static structure; that is, its structure cannot be modified once it is built. A similar data structure is the interval tree.

A segment tree for a set I of n intervals uses O(n log n) storage and can be built in O(n log n) time. Segment trees support searching for all the intervals that contain a query point in O(log n + k), k being the number of retrieved intervals or segments.

Applications of the segment tree are in the areas of computational geometry, and geographic information system.

Fenwick tree or binary indexed tree is a data structure providing efficient methods for calculation and manipulation of the prefix sums of a table of values.Fenwick trees are used to implement the arithmetic coding algorithm. Development of operations it supports were primarily motivated by use in that case.

Using a Fenwick tree it is very easy to generate the cumulative sum table. From this cumulative sum table it is possible to calculate the summation of the frequencies in a certain range

References

[1] `https://csacademy.com/lesson/fenwick_trees` *Fenwick Trees* .

[2] `https://www.infoarena.ro/problema/arbint` *Segment Trees.*