Probabilités Numériques (G. Pagès & V. Lemaire)
M2 Probabilités & Finance
M2 Probabilités & Modèles Aléatoires
SU-X
3 janvier 2023

3 h

Polycopié, notes de cours, livres, téléphones mobiles et montres connectées non autorisés

Handouts, course notes, books, mobile phones and smart watches not allowed

Exercise (Quasi-Monte Carlo). Let $n \in \mathbb{N}$ and $\xi_1, \ldots, \xi_n \in]0,1[$ such that $\xi_1 < \cdots < \xi_n$.

1.a. Show that the function $\varphi_n : x \mapsto \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{\{\xi_k \le x\}} - x$ is right continuous with left limits everywhere it has a sense on the unit interval $[0,1]$ and $\varphi_n(0) = \varphi_n(1) = \cancel{1}0$.

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1.b. Prove that

$$\sup_{x\in[0,1]} \varphi_n(x) = \max_{1\le k\le n} \varphi_n(\xi_k) \quad \text{and} \quad \inf_{x\in[0,1]} \varphi_n(x) = \overset{\min}{\cancel{\max}}_{1\le k\le n} \varphi_n(\xi_k-)$$

where $\varphi_n(u-)$ denotes the left limit of φ_n at $u \in]0,1]$.

1.c. Deduce (with the notations from the course) that

$$D_n^*(\xi_1, \ldots, \xi_n) = \max_{1\le k\le n}\left(\left|\xi_k - \frac{k}{n}\right|, \left|\xi_k - \frac{k-1}{n}\right|\right) = \frac{1}{2n} + \max_{1\le k\le n}\left|\xi_k - \frac{2k-1}{2n}\right|.$$

2. Let $f : [0,1] \to \mathbb{R}$ be an α-Hölder-continuous function, $\alpha \in]0,1]$ (i.e. $[f]_\alpha := \sup_{u\neq v, u,v\in[0,1]} \frac{|f(v)-f(u)|}{|v-u|^\alpha} < +\infty$).

2.a. Prove that

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(\xi_k) - \int_0^1 f(u)du\right| \le \frac{[f]_\alpha}{n} \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |\xi_k - x|^\alpha dx.$$

2.b. Deduce that

$$\left|\frac{1}{n}\sum_{l=1}^{n} f(\xi_k) - \int_0^1 f(u)du\right| \le [f]_\alpha D_n^*(\xi_1, \ldots, \xi_n)^\alpha.$$

Problem I (Multi-asset (pseudo-)risk measure). We introduce the function $V : \mathbb{R}^d \to \mathbb{R}$ ($d \ge 1$) – a (pseudo-)risk measure – associated to an integrable d-dimensional random vector $X = (X^1, \ldots, X^d)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a confidence/risk level $\alpha \in (0,1)$ defined by

$$\forall \xi = (\xi^1, \ldots, \xi^d) \in \mathbb{R}^d, \quad V(\xi) = \sqrt{1+|\xi|^2} - 1 + \frac{1}{1-\alpha}\sum_{i=1}^{d} \mathbb{E}(X^i - \xi^i)^+$$

where $u^+ = \max(u, 0)$. The variable X^i is representative of the loss induced by a traded asset i in a portfolio made up from d assets $1, \ldots, d$, in the sense that $X^i \ge 0$ means a loss of X^i euros. We assume for simplicity that the distribution of the vector X has no atom.

1.a. Prove that V is non-negative, differentiable on $\mathbb{R}^d$, compute its gradient at every $\xi \in \mathbb{R}^d$.

1.b. Prove that $\lim_{|\xi|\to+\infty} V(\xi) = +\infty$ and that V is strictly convex.

1.c. Deduce that $\mathrm{argmin}_{\mathbb{R}^d} = \{\xi^*\}$ where ξ^* is solution to a system of non-linear equations to be specified.

1.d. Briefly interpret the equation in terms of risk of loss.

2. Devise a recursive stochastic algorithm based on the simulation of an i.i.d. sequence $(X_n)_{n\ge1}$ of random vectors with the distribution of X that converges to ξ^* $\mathbb{P}$-$a.s.$ We denote by $(\xi_n)_{n\ge0}$ this recursive stochastic procedure which reads

$$\xi_{n+1} = \xi_n - \gamma_{n+1} H(\xi_n, X_{n+1}),$$

supposed to be initialized at some deterministic $\mathbb{R}^d$-valued vector $\xi_0 \in \mathbb{R}^d$, where $H : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}_+$ is a Borel function to be specified and $(\gamma_n)_{n\ge1}$ is a step sequence satisfying a condition to be specified as well.

3. We assume that $X \in L^2(\mathbb{P})$. We define the stochastic recursive procedure $(V_n)_{n\ge0}$ by

$$V_{n+1} = V_n - \frac{1}{n+1}\big(V_n - v(\xi_n, X_{n+1})\big), \quad n \ge 0$$

where $V(\xi) = \mathbb{E}\, v(\xi, x)$ in the above definition and $V_0 = 0$.

3.a. Prove that for every $n \ge 1$,

$$V_n = \frac{1}{n}\sum_{k=1}^{n} v(\xi_{k-1}, X_k) = \frac{1}{n}\sum_{k=1}^{n} V(\xi_{k-1}) + \frac{S_n}{n}$$

where

$$S_n := \sum_{k=1}^{n} v(\xi_{k-1}, X_k) - V(\xi_{k-1}), \; n \ge 1,$$

is a square integrable martingale with respect to the filtration $\mathcal{F}_n^X = \sigma(X_1, \ldots, X_n)$, $n \ge 0$.

3.b. Show that for every $k \ge 1$,

$$\big|v(\xi_{k-1}, X_k) - V(\xi_{k-1})\big| \le \frac{1}{1-\alpha}\sum_{i=1}^{d} \int |X_k^i - x^i| \mu_i(dx^i)$$

where μ_i denotes the distribution of the marginal X^i of X, $i = 1, \ldots, d$.

3.c. Show that

$$\mathbb{E}\Big|\sum_{k=1}^{n} v(\xi_{k-1}, X_k) - V(\xi_{k-1})\Big|^2 \le \frac{2dn}{(1-\alpha)^2}\sum_{i=1}^{d}\mathrm{Var}(X^i)$$

and deduce that $V_n \to V(\xi^*)$ in probability.

BONUS QUESTION. Now we consider

$$\widetilde{S}_n = \sum_{k=1}^{n}\frac{v(\xi_{k-1}, X_k) - V(\xi_{k-1})}{k}, \qquad n \ge 1.$$

Show that $(\widetilde{S}_n)_{n\ge 1}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_n^X)_{n\ge 1}$ and that its bracket process $\langle \widetilde{S}\rangle_n$ satisfies

$$\mathbb{E}\,\langle \widetilde{S}\rangle_\infty < +\infty$$

Deduce that $V_n \to V(\xi^*)$ $\mathbb{P}$-*a.s.*

Problem II (Flow of an SDE and applications). We consider a Stochastic Differential Equation (*SDE*)

$$X_t = X_0 + \int_0^t b(X_t)dt + \int_0^t \sigma(X_t)dW_t,$$

where b, $\sigma : \mathbb{R} \to \mathbb{R}$ are two Lipschitz continuous functions with respective Lipschitz coefficientss $[b]_{\rm Lip}$ and $[\sigma]_{\rm Lip}$ and $(W_t)_{t\ge 0}$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and X_0 defined on the same probability space, independent of W. We denote par $(X_t)_{t\ge 0}$ the unique $(\mathcal{F}_t^{X_0,W})_{t\ge 0}$-adapted solution to the above *SDE* where $\mathcal{F}_t^{X_0,W} = \sigma(\mathcal{N}_{\mathbb{P}}, X_0, W_s, 0 \le s \le t)$, $t \ge 0$. All the notations are those used during the course.

We denote by $T > 0$ a terminal time (or maturity). For every integer $n \ge 1$, we define step $h = h_n = \frac{T}{n}$. We set $t_k = t_k^n = \frac{kT}{n}$, $k = 0, \ldots, n$.

1.a. Recall the definitions of the three Euler schemes with step $h = \frac{T}{n}$: the discrete time, the stepwise constant and the "genuine" (continuous) Euler schemes, denoted $(\bar{X}^n_{t_k})_{k=0,\ldots,n}$, $(\widetilde{X}^n_t)_{t\in[0,T]}$ and $(\bar{X}^n_t)_{t\in[0,T]}$ respectively.

1.b. Let $p \ge 1$. Recall the uniform L^p-moment control results for both the diffusion $(X_t^x)_{t\in[0,T]}$ and the above Euler scheme(s). [No proof requested here.]

1.c. State the L^p-convergence theorems for the above three Euler schemes in an as synthetic way as possible. [No proof requested here.]

The result of 1.b. and 1.c. may be used without proof in what follows.

2. In what follows we denote by $(X_t^x)_{t\in[0,T]}$ the unique solution of the above SDE starting from $X_0 = x$ and by $(\bar{X}^{n,x}_{t_k})_{k=0,\ldots,n}$, etc, the related Euler schemes.

2.a. Prove that, for every x, $y \in \mathbb{R}$,

$$\sup_{0\le s\le t}|\bar{X}_s^{n,x} - \bar{X}_s^{n,y}| \le |x-y| + \int_0^t |b(\bar{X}^{n,x}_{\underline{u}}) - b(\bar{X}^{n,y}_{\underline{u}})|du + \sup_{s\in[0,t]}\Big|\int_0^s (\sigma(\bar{X}^{n,x}_{\underline{u}}) - \sigma(\bar{X}^{n,y}_{\underline{u}}))dW_u\Big|.$$

2.b. We set, for every $t \in [0,T]$, $g(t) = \mathbb{E}\sup_{s\in[0,t]}|\bar{X}_s^{n,x} - \bar{X}_s^{n,y}|^2$. Prove that ($g$ is non-decreasing and) $g(T) < +\infty$.

2.c. Prove that for every a, b, $c \ge 0$, $(a+b+c)^2 \le 3(a^2+b^2+c^2)$.

2.d. Deduce that, for every $t \in [0,T]$,

$$g(t) \le 3\Big(|x-y|^2 + t[b]^2_{\rm Lip}\int_0^t \mathbb{E}|\bar{X}^{n,x}_{\underline{u}} - \bar{X}^{n,y}_{\underline{u}}|^2 du + \mathbb{E}\sup_{s\in[0,t]}\Big|\int_0^s (\sigma(\bar{X}^{n,x}_{\underline{u}}) - \sigma(\bar{X}^{n,y}_{\underline{u}}))dW_u\Big|^2\Big).$$

3.a. Prove that, for every $t \in [0,T]$,

$$g(t) \le 3\Big(|x-y|^2 + (T[b]^2_{\rm Lip} + 4[\sigma]^2_{\rm Lip})\int_0^t g(s)ds\Big).$$

3.b. Conclude that, for every $n \ge 1$,

$$\Big\|\sup_{t\in[0,T]}|\bar{X}_t^{n,x} - \bar{X}_t^{n,y}|\Big\|_2 \le \sqrt{3}|x-y|e^{C_{b,\sigma,T}T}$$

where $C_{b,\sigma,T}$ is a positive real constant to be specified and that

$$\Big\|\sup_{t\in[0,T]}|X_t^x - X_t^y|\Big\|_2 \le \sqrt{3}|x-y|e^{C_{b,\sigma,T}T}.$$

4. Assume that b and σ are differentiable with bounded derivatives. We admit that $\mathbb{P}(d\omega)$-*a.s*, for every time $t \in [0,T]$, $x \mapsto X_t^x(\omega)$ is differentiable, so that we may define the tangent process by

$$\forall \omega \in \Omega \setminus N_0,\ \forall t \in [0,T],\quad Y_t^{(x)}(\omega) := \frac{d}{dx}X_t^x(\omega)$$

(and $Y_t^{(x)}(\omega) = 0$ if $\omega \in N_0$) where N_0 is a $\mathbb{P}$-negligeable event of the σ-field $\mathcal{A}$.

4.a. Justify *heuristically* why $(Y_t^{(x)})_{t\in[0,T]}$ satisfies an SDE to be specified. Deduce, this time rigorously that, as a solution of this SDE, $Y_t^{(x)}$ satisfies

$$Y_t^{(x)} = \exp\Big(\int_0^t \big(b'(X_s^x) - \tfrac{1}{2}(\sigma')^2(X_s^x)\big)ds + \int_0^t \sigma'(X_s^x)dW_s\Big).$$

4.b. Prove that, $\mathbb{P}(d\omega)$-*a.s*, for every time $t \in [0,T]$, $x \mapsto X_t^x(\omega)$ is non-decreasing.

4.c. Prove that

$$\mathbb{E}\sup_{t\in[0,T]}\big(Y_t^{(x)}\big)^2 < +\infty.$$

5. Let $h : \mathbb{R} \to \mathbb{R}$ be a differentiable Lipschitz function. Prove *rigorously* that the function defined by

$$P(x) = \mathbb{E}\,h(X_T^x).$$

is differentiable on the real line with

$$P'(x) = \mathbb{E}\,h'(X_T^x)Y_T^{(x)}.$$

Exercice QMC :

1.a) $\varphi_m : x \mapsto \frac{1}{m} \sum_{k=1}^{m} \underbrace{\mathbb{1}_{\{\xi_k \le x\}}} - x$

Continue à droite car $\le$
et limite à gauche car $\xi_k \in]0,1[$

$\Rightarrow \varphi_m$ càdlàg

1.b) $\sup_{x\in[0,1]} \varphi_m(x) = \sup_{x\in[0,1]} \left(\frac{1}{m}\sum_{k=1}^{m} \mathbb{1}_{\{\xi_k \le x\}} - x\right)$

Sur $[\xi_i, \xi_{i+1}[$, $\underbrace{\frac{1}{m}\sum_{k=1}^{m} \mathbb{1}_{\{\xi_k\le x\}}}_{=: F_m(x)}$ est cte $\Rightarrow \sup_{x\in[\xi_i,\xi_{i+1}[} (F_m(x) - x) = \underbrace{F_m(\xi_i) - \xi_i}_{= \varphi_m(\xi_i)}$

D'où $\sup_{x\in[0,1]} \varphi_m(x) = \max_{1\le k\le m} \varphi_m(\xi_k)$

D'autre part, $\inf_{x\in[\xi_i,\xi_{i+1}[} \varphi_m(x) = \inf_{x\in[\xi_i,\xi_{i+1}[} (F_m(x) - x) = \lim_{\substack{x\to\xi_{i+1}\\ x<\xi_{i+1}}} \varphi_m(x) = \varphi_m(\xi_{i+1}-)$

Donc $\inf_{x\in[0,1]} \varphi_m(x) = \min_{1\le k\le m} \varphi_m(\xi_k -)$

1.c) D'après le cours, $D_m^*(\xi) = \sup_{x\in[0,1]} |\varphi_m(x)| = \max(|\max(\varphi_m(x))|, |\min(\varphi_m(x))|)$

$\varphi_m(\xi_k) \qquad \varphi_m(\xi_k-)$

$= \max_{1\le k\le m} \left(\left|\frac{k}{m} - \xi_k\right|, \left|\frac{k-1}{m} - \xi_k\right|\right)$

$\frac{k}{m} - \frac{1}{2m} \qquad \frac{2k-1}{m}$

$= \max_{1\le k\le m} \left(\left|\xi_k - \frac{k}{m}\right|, \left|\xi_k - \frac{k-1}{m}\right|\right) = \frac{1}{2m} + \max_{1\le k\le m}\left(\left|\xi_k - \frac{2k-1}{2m}\right|, \left|\xi_k - \frac{2k-1}{2m}\right|\right)$

$= \frac{1}{2m} + \max_{1\le k\le m} \left|\xi_k - \frac{2k-1}{2m}\right|$

2) Soit $f : [0,1] \to \mathbb{R}$ une α-Hölder fct° $\alpha \in]0,1]$ i.e $[f]_\alpha = \sup_{u\ne v} \frac{|f(v)-f(u)|}{|v-u|^\alpha} < \infty$

2.a) $\sum_{k=1}^{m} \frac{1}{m} f(\xi_k) = \sum_{k=1}^{m} \int_{\frac{k-1}{m}}^{k/m} f(\xi_k)\,du$

$\Rightarrow \sum_{k=1}^{m}\frac{1}{m} f(\xi_k) - \int_0^1 f(u)\,du = \sum_{k=1}^{m} \int_{\frac{k-1}{m}}^{\frac{k}{m}} f(\xi_k) - f(u)\,du$

$\Rightarrow \left|\frac{1}{m}\sum_{k=1}^{m} f(\xi_k) - \int_0^1 f(u)\,du\right| \le \sum_{k=1}^{m} \int_{\frac{k-1}{m}}^{k/m} |f(\xi_k) - f(u)|\,du \le \sum_{k=1}^{m} \int_{\frac{k-1}{m}}^{k/m} |\xi_k - u|^\alpha [f]_\alpha\,du$

$\le [f]_\alpha \sum_{k=1}^{m} \int_{\frac{k-1}{m}}^{k/m} |\xi_k - x|^\alpha\,dx$

2.b) $\max_{[\frac{k-1}{m},\frac{k}{m}]} |\xi_k - x|^\alpha = \max\left(\left|\xi_k - \frac{k}{m}\right|^\alpha, \left|\xi_k - \frac{k-1}{m}\right|^\alpha\right)$

$= D_m^*(\xi_1, \dots, \xi_n)^\alpha$

$\Rightarrow \left|\frac{1}{m}\sum_{k=1}^{m} f(\xi_k) - \int_0^1 f(u)\,du\right| \le [f]_\alpha\, D_m^*(\xi_1,\dots,\xi_m)^\alpha$

Problem 1: $V(\xi) = \sqrt{1+|\xi|^2} - 1 + \frac{1}{1-\alpha}\sum_{i=1}^{d} \mathbb{E}[(X^i-\xi^i)^+]$

Différentiable $\longleftrightarrow$ Toutes les dérivée partielles existent et sont continues

1.a) Ok car $\xi \in \mathbb{R}^d \Rightarrow \sqrt{1+|\xi|^2} \geq 1 \Rightarrow \sqrt{1+|\xi|^2} - 1 \geq 0$

et $\alpha \in]0,1[\Rightarrow \frac{1}{1-\alpha} > 0$ et $\mathbb{E}[(X^i-\xi^i)^+] \geq 0 \quad \forall i$

$\Rightarrow V \geq 0$

De plus, $\xi \mapsto \sqrt{1+|\xi|^2}$ est bien dérivable et $\forall \xi,\ 1+|\xi|^2 \geq 1 \Rightarrow \frac{\partial\sqrt{1+|\xi|^2}}{\partial \xi}$ existe $\forall \xi \in \mathbb{R}^d$

Reste à vérifier si $\mathbb{E}[(X^i-\xi^i)^+]$ est dérivable

$\rightarrow$ Thm de dérivation sous l'intégrale ou $\mathbb{E}$

$(E, \mathcal{E}, \mu)$ espace mesuré, I intervalle de $\mathbb{R}$ non trivial ($I \neq \emptyset, \{x\}$)

Soit $\varphi : I \times E \to \mathbb{R} \quad (x,\xi) \mapsto \varphi(x,\xi)$

a) Version locale : $x_0 \in I$.

(i) $\forall x,\ \varphi(x,\cdot) \in \mathcal{L}^1(\mu)$ $\qquad$ Ici : $(X^i-\xi^i)^+ \in \mathcal{L}^1$ si $X^i \in \mathcal{L}^1$

(ii)$_{loc}$ $\frac{\partial\varphi}{\partial x}(x_0,\xi)$ existe $\mu(d\xi)$ p.p $\qquad$ Ici : $\frac{\partial\varphi}{\partial\xi^i} = -\mathbb{1}_{\{X^i-\xi^i\geq 0\}}$

(iii)$_{loc}$ $\exists\, \Xi : (E,\mathcal{E}) \to \mathbb{R}_+$ tq $\Xi \in \mathcal{L}^1(\mu)$

tq $\forall x \in I,\ |\varphi(x,\xi) - \varphi(x_0,\xi)| \leq \Xi \cdot |x-x_0| \quad \mu(d\xi)$-pp $\qquad$ Ici : $(X^i-\xi^i)^+$ est Lipschitzienne $(\Xi \equiv 1)$

Alors, $\phi : I \to \mathbb{R}$ est bien définie par $\phi(x) = \int_E \varphi(x,\xi)\,\mu(d\xi)$ et $\phi'(x_0) = \int \frac{\partial\varphi}{\partial x_0}(x_0,\xi)\,\mu(d\xi)$ $\quad (*)$

b) Version globale. Si φ vérifie

(i) —

(ii)$_{Glob}$ $\mu(d\xi)$-pp, $x \mapsto \varphi(x,\xi)$ est dérivable sur I.

(iii)$_{Glob}$ $\exists\, \Xi : (E,\mathcal{E}) \to \mathbb{R}_+$, $\Xi \in \mathcal{L}^1(\mu)$ tq $\left|\frac{\partial\varphi}{\partial x}(x,\xi)\right| \leq \Xi(\xi) \quad \forall x \in I \quad \mu(d\xi)$-pp

Alors, $\phi \in D(I)$ et $\phi'(x)$ est donnée par $(*)$ $\forall x \in I$.

Donc par le thm locale, V est bien dérivable $\forall \xi \in \mathbb{R}^d$

et $\nabla V(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}} - \frac{1}{1-\alpha}\begin{pmatrix} \mathbb{P}(X^1 \geq \xi^1) \\ | \\ \mathbb{P}(X^d \geq \xi^d)\end{pmatrix}$ $\leftarrow \sum_{i=1}^{d} \mathbb{E}[-\mathbb{1}_{\{X^i-\xi^i\geq 0\}}]$

1.b) $V(\xi) \geq \sqrt{1+|\xi|^2} - 1 \underset{|\xi|\to\infty}{\sim} |\xi| \Rightarrow V(\xi) \underset{|\xi|\to\infty}{\longrightarrow} \infty$

De plus, $\xi \mapsto \frac{1}{1-\alpha}\sum_{i=1}^{d}\mathbb{E}[(X^i-\xi^i)^+]$ est convexe car $(X^i - \cdot)^+$ est cvx (car linéaire)

Il suffit de mq $\xi \mapsto \sqrt{1+|\xi|^2}$ est stt cvx

$\frac{\partial^2\sqrt{1+|\xi|^2}}{\partial\xi^2} = \frac{1}{(1+|\xi|^2)^{3/2}} > 0$

Donc V est stt cvx

1.c) Par 1.b) on a V stt cvx et $\lim_{|\xi|\to\infty} V(\xi) = \infty$ $\Rightarrow \exists!\, \xi^*$ tq $\nabla V(\xi^*) = 0$ i.e $\frac{\xi}{\sqrt{1-|\xi|^2}} = \frac{1}{1-\alpha}\begin{pmatrix} \mathbb{P}(X^1 \geq \xi^1) \\ | \\ \mathbb{P}(X^d \geq \xi^d)\end{pmatrix}$

1.d) Interprétat° : $d=1$: $\mathbb{P}(X \geq \xi) = (1-\alpha)\frac{\xi}{\sqrt{1+\xi^2}}$ $\leftarrow$ presque VaR

On cherche le ptf tq pour chaque actif la proba de perte + grande que ξ^i soit $\div$ à $(1-\alpha)\xi^i$

On peut considérer cela comme une généralisat° de VaR.

2) On veut résoudre $h(\xi)=0$ où $h(\xi)=(1-\alpha)\frac{\xi}{\sqrt{1+|\xi|^2}}-\mathbb{E}[\mathbb{1}_{\{X^i\geq\xi^i\}}]$

Donc $H(X,\xi)=(1-\alpha)\frac{\xi}{\sqrt{1+|\xi|^2}}-(\mathbb{1}_{\{X_i^i\geq\xi^i\}})_{i=1,\dots,d}$ $X_i\sim X$ iid

$\xi_{m+1}=\xi_m-\gamma_{m+1}H(\xi_m,X_{m+1})\quad\longrightarrow\gamma_m>0,\ \sum_{m\geq1}\gamma_m^2<\infty$ et $\sum_{m\geq1}\gamma_m=\infty$

Pour Robbins-Monro il faut aussi :

$*\ \forall\theta\neq\theta^*,\ \langle h(\xi),\xi-\xi^*\rangle>0$

$*\ \forall\theta\in\mathbb{R}^d,\ \|H(\xi,z)\|_2\leq C(1+|\xi-\xi^*|^2)^{1/2}$

3) $X\in\mathcal{L}^2(\mathbb{P})$

$V_m\geq0,\ V_{m+1}=V_m-\frac{1}{m+1}(V_m-v(\xi_m,X_{m+1}))\quad m\geq0$

avec $V(\xi)=\mathbb{E}[v(\xi,x)]$ et $V_0=0$

$=\mathbb{E}[\underbrace{\sqrt{1+|\xi|^2}-1+\frac{1}{1-\alpha}\sum_{1}^{d}(X^i-\xi^i)^+}_{v(\xi,X)}]$

3.a) Par rec, $V_{m+1}=\frac{m}{m+1}\underbrace{V_m}_{=\frac{1}{m}\sum_{k=1}^{m}v(\xi_{k-1},X_k)}+\frac{1}{m+1}v(\xi_m,X_{m+1})$

$=\frac{1}{m+1}\sum_{1}^{m+1}v(\xi_{k-1},X_k)$

$=\frac{1}{m+1}\sum_{k=1}^{m+1}V(\xi_{k-1})+\frac{1}{m+1}\left(\sum_{k=1}^{m+1}v(\xi_{k-1},X_k)-V(\xi_{k-1})\right)$

$\xi_k\in\mathcal{F}_k^X$

$\mathbb{E}[V_{m+1}|\mathcal{F}_m^X]=\frac{1}{m+1}\sum_{k=1}^{m+1}V(\xi_{k-1})+\frac{1}{m}\mathbb{E}[\underbrace{\sum_{k=1}^{m+1}v(\xi_{k-1},X_k)-V(\xi_{k-1})}_{S_{m+1}}|\mathcal{F}_m^X]$

$\mathbb{E}[S_{m+1}|\mathcal{F}_m^X]=\sum_{1}^{m}(v(\xi_{k-1},X_k)-V(\xi_{k-1}))+\underbrace{\mathbb{E}[v(\xi_{k-1},X)|\mathcal{F}_m^X]}_{=V(\xi_{k-1})}-V(\xi_{k-1})=S_m$

$\Rightarrow S_m$ mart

S_m est intégrable $\Leftrightarrow v$ est de carré intégrable $(X\in\mathcal{L}^2)$

3.b) $v(\xi_{k-1},X_k)-V(\xi_{k-1})=\left(\cancel{\sqrt{1+|\xi_{k-1}|^2}}\cancel{-1}+\frac{1}{1-\alpha}\sum_{i=1}^{d}(X^i-\xi^i)^+\right)-\left(\cancel{\sqrt{1+|\xi_{k-1}|^2}}\cancel{-1}+\frac{1}{1-\alpha}\sum_{i=1}^{d}\mathbb{E}[(X^i-\xi^i)^+]\right)$

$=\frac{1}{1-\alpha}\sum_{i=1}^{d}\left(\underbrace{(X^i-\xi^i)^+}_{=\int(x^i-\xi^i)\mu(dx^i)}-\underbrace{\mathbb{E}[(X^i-\xi^i)^+]}_{=\int(x^i-\xi^i)^+\mu(dx^i)}\right)$

$x\mapsto(x-k)^+$ est Lipschitzienne

$\Rightarrow|v(\xi_{k-1},X_k)-V(\xi_{k-1})|\leq\frac{1}{1-\alpha}\sum_{i=1}^{d}\int|X_k^i-x^i|\mu_i(dx^i)$

3.c) Par 3.b) on a $\forall k,\ |v(\xi_{k-1},X_k)-V(\xi_{k-1})|\leq\frac{1}{1-\alpha}\sum_{i=1}^{d}\int|X_k^i-x^i|\mu_i(dx^i)$

$\mathbb{V}[X^i]=\int|x^i-\mathbb{E}[X^i]|^2\mu_i(dx^i)=\int\left|\int(x^i-\tilde{x}^i)\mu_i(d\tilde{x}^i)\right|^2\mu_i(dx^i)$

$\mathbb{E}[|\sum_{k=1}^{m}v(\xi_{k-1},X_k)-V(\xi_{k-1})|^2]=?$

$\longrightarrow X_k\perp\!\!\!\perp\mathcal{F}_{k-1}^X\Rightarrow\mathbb{E}[(v(\xi_{k-1},X_k)-V(\xi_{k-1}))(v(\xi_{\ell-1},X_\ell)-V(\xi_{\ell-1}))]=\mathbb{E}[\mathbb{E}[\underbrace{\dots|\mathcal{F}_{\ell-1}^X]}_{=0}(v(\xi_{k-1},X_k)-V(\xi_{k-1}))]=0$
$k<\ell$

$\Rightarrow\mathbb{E}[|\sum_{k=1}^{m}v(\xi_{k-1},X_k)-V(\xi_{k-1})|^2]=m\,\mathbb{E}[\frac{1}{1-\alpha}\sum_{i=1}^{d}\int|X^i-x^i|\mu_i(dx^i)]^2$ *

Or $\underbrace{(\sum_{i=1}^{d}a_i)^2}_{\langle a,\mathbb{1}\rangle^2\leq|a|^2\cdot|\mathbb{1}|^2=|a|^2d}\leq d\sum_{i=1}^{d}a_i^2$

Par Robbins - Monro, $\xi_m \xrightarrow{p.s} \xi^*$

$$\Rightarrow \; * \leq \frac{dm}{(1-\alpha)^2} \sum_{i=1}^{d} \underbrace{\mathbb{E}\left[\int |X^i - x^i| \mu_i(dx^i)\right]^2}_{\mathbb{V}[X^i]} = \frac{dm}{(1-\alpha)^2} \sum_{i=1}^{d} \mathbb{V}[X^i] \leq \frac{2dm}{(1-\alpha)^2} \sum_{i=1}^{d} \mathbb{V}[X^i]$$