Examen *Processus stochastiques et produits dérivés*
Master 2 *Probabilités et Finance* SU-X

Sans document, ni calculatrice, ni téléphone.

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The notations are those of the course. The two exercises are independent. It is requested to justify the answers carefully and clearly ; the grading will take this into account. The 20-points scale is indicative.

We recall the Black-Scholes formula for a Call, with risk-free rate r, dividend rate q and volatility σ :

$$\text{Call}^{\text{BS}}(t,S,T,K,\sigma,r) = Se^{-q(T-t)}\mathcal{N}\left(d^+\right) - Ke^{-r(T-t)}\mathcal{N}\left(d^-\right),$$
$$\text{with } d^+ = \frac{1}{\sigma\sqrt{T-t}}\ln\left(\frac{Se^{-q(T-t)}}{Ke^{-r(T-t)}}\right) + \frac{1}{2}\sigma\sqrt{T-t}, \qquad d^- = d^+ - \sigma\sqrt{T-t},$$

where $\mathcal{N}$ is the c.d.f. of the normal distribution, with zero mean and unit variance.

Exercice 1 - Zero-coupon bond in Vasicek model (5 points)

Consider the Vasicek model for interest rate. Under the risk-neutral measure $\mathbb{Q}$, the evolution of short rate r_t is given by an Ornstein-Uhlenbeck process :

$$\mathrm{d}r_t = a(b - r_t)\mathrm{d}t - \sigma \mathrm{d}W_t, \quad r_0 \text{ given.}$$

Part 1 : pricing of the zero-coupon bond.

1. Prove that
$$r_t = r_0 e^{-at} + b(1-e^{-at}) - \sigma\int_0^t e^{-a(t-s)}\mathrm{d}W_s,$$
then deduce that r is a Gaussian process with explicit mean and covariance parameters (give these parameters).

2. We now define $I_{t,T} := \int_t^T r_s \mathrm{d}s$. Prove that
$$aI_{t,T} = ab(T-t) - (r_T - r_t) - \sigma\int_t^T \mathrm{d}W_s$$
and deduce the explicit form of $I_{t,T}$ with respect to r_t.

3. Compute the mean and variance of $I_{t,T}$, conditionally on r_t, and deduce the price of the zero-coupon bond $B(t,T)$

$$B(t,T) = \exp\left[-b(T-t) + (b - r_t)\frac{1-e^{-a(T-t)}}{a} - \frac{\sigma^2}{4a^3}\left(1-e^{-a(T-t)}\right)^2 + \frac{\sigma^2}{2a^2}\left(T - t - \frac{1-e^{-a(T-t)}}{a}\right)\right].$$

Part 2 : pricing of a call option on a forward contract with Heath-Jarrow-Morton model. Now consider the zero-coupon as asset under the risk neutral probability $\mathbb{Q}$, i.e.

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \Gamma(t,T)dW_t$$

where $\Gamma(t,T)$ is the volatility of the zero-coupon bond.

4. Give the definition of $\mathbb{Q}^T$, the *forward-neutral probability* with maturity T, in terms of $B(\cdot,T)$ and $\mathbb{Q}$.
5. What is the value of $\Gamma(t,T)$? What is the SDE of the zero-coupon bond under $\mathbb{Q}$, and what is it under $\mathbb{Q}^T$? Find the solution to these SDEs.
6. For any $\delta \geq 0$, show that the forward price at time t for a payoff $B(T,T+\delta)^{-1}$ at maturity T is $X_t := \mathbb{E}_{\mathbb{Q}^T}[B(T,T+\delta)^{-1}|\mathcal{F}_t]$ and compute it.
7. Deduce that $(X_t)_{t\geq 0}$ is a stochastic exponential under $\mathbb{Q}^T$ and compute its volatility.
8. Deduce the value of a call option on the forward contract at maturity T with strike K, i.e. the payoff is $(X_T - K)_+$.

Exercice 2 - Call-Put symmetry (3 points)

The dynamic of the price of an asset paying continuous dividend satisfies the SDE

$$\frac{\mathrm{d}S_t}{S_t} = (r-q)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

where W is a 1-dimensional Brownian Motion under the risk neutral measure $\mathbb{Q}$, σ is a strictly positive constant, r is the constant interest rate, and q is the dividend rate. The aim is to prove the Call-Put symmetry

$$\mathrm{Call}(0, S_0 e^{-\nu T}; T, K) = \mathrm{Put}(0, K e^{-\nu T}; T, S_0), \qquad \nu = r - q,$$

where $\mathrm{Call}(0,x,T,k)$ (respectively $\mathrm{Put}(0,x,T,k)$) is the price at $t=0$ of a call (respectively put) at time 0 with maturity T, spot x and strike k, with an asset paying dividends at rate q and with an interest rate r.

1. Justify that the spot price and the forward price for receiving S_T at T are respectively

$$C_t(S_T, T) = S_t e^{-q(T-t)}, \quad F_t(S_T, T) = S_t e^{\nu(T-t)}.$$

2. Show that

$$X_t = C_t(S_T, T)$$

can be used as a numéraire and that the price of $\mathrm{Call}(0, x; T, k)$ is equal to

$$\mathbb{E}_{\mathbb{Q}^X}[e^{-qT}(x - \frac{kx}{S_T})_+]$$

where $\mathbb{E}_{\mathbb{Q}^X}$ is the expectation under the measure related to the numéraire X. Here $x = S_0$.

3. Let $W_t^X := W_t - \sigma t$. Write down the formula for $\frac{kx}{S_T}$ as an exponential of W_T^X.
4. Using the fact that W^X is a Brownian motion under $\mathbb{Q}^X$, prove the Call-Put symmetry.
5. **Applications.** Consider a Down-In Call (DIC) with Strike K and lower barrier D which payoff is $(S_T - K)_+ \mathbf{1}_{\inf_{t\le T} S_t \le D}$ at maturity T. We assume that this barrier option is regular, i.e. $K \ge D$. When the spot S_0 is larger than the barrier D, prove that the price of the DIC equals $\frac{K}{D}$ Puts with strike $\frac{D^2}{K}$ and maturity T. Provide a semi-static hedging strategy.

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