Slide 15 :

$$dX_t = X_t(r\,dt + \sigma\,dW_t)$$

$$h(t,x,y) = x\underbrace{e^{(r-\frac{\sigma^2}{2})(T-t)-\sigma y}}_{E(t,y)}$$

$$h(t,X_t,W_t) = h(0,X_0,W_0) - (r-\tfrac{\sigma^2}{2})\int_0^t X_s E(s,W_s)\,ds + \int_0^t E(s,W_s)\,dX_s - \sigma\int_0^t X_s E(s,W_s)\,dW_s + \sigma^2\int_0^t X_s E(s,W_s)\,ds - \frac{\sigma^2}{2}\int_0^t X_s E(s,W_s)\,ds.$$

$$= h(0,X_0,W_0) - (r-\tfrac{\sigma^2}{2})\int_0^t X_s E(s,W_s)\,ds + \int_0^t E(s,W_s)X_s r\,ds + \int_0^t E(s,W_s)X_s\sigma\,dW_s - \sigma\int_0^t E(s,W_s)X_s\,dW_s + \sigma^2\int_0^t E(s,W_s)X_s\,ds - \frac{\sigma^2}{2}\int_0^t E(s,W_s)X_s\,ds$$

Donc $h(T,X_T,W_T) = h(0,X_0,W_0)$

$$X_T e^{-\sigma W_T} = X_0 e^{(r-\frac{\sigma^2}{2})T}$$

$$\rightarrow X_T = X_0 e^{(r-\frac{\sigma^2}{2})T+\sigma W_T}$$

Slide 21 :

Soit $X_0 = \tilde{X}_0$

$$dX_t = b(t,X_t)\,dt + \sigma\,dW_t$$

Fixons $T>0$

On définit une suite $(X_t^{(k)})_{t\in[0,T]}$

avec $X_t^{(0)} = \tilde{X}_0 \quad \forall t\in[0,T]$

$$\begin{cases} X_0^{(1)} = \tilde{X}_0 \\ dX_t^{(1)} = b(t,X_t^{(0)})\,dt + \sigma\,dW_t \end{cases}$$

Plus généralement, $\forall k\geq 1$

$$\begin{cases} X_0^{(k)} = \tilde{X}_0 \\ dX_t^{(k)} = b(t,X_t^{(k-1)})\,dt + \sigma\,dW_t \end{cases}$$

À la fin, si $X_t^{(k)} \longrightarrow X_t^{(\infty)} \quad \forall t\in[0,T]$

On aura $X_0^{\infty} = \tilde{X}_0$

$dX_t^{\infty} = b(t,X_t^{\infty})\,dt + \sigma\,dW_t$ Solut° de l'EDS

On va travailler dans $\mathcal{C}([0,T], \|\cdot\|_\infty)$

On introduit, pour $(Y_t)_{t\in[0,T]} \in \mathcal{C}([0,T],\|\cdot\|_\infty)$

$$|||Y_\cdot||| = \sup_{t\in[0,T]} e^{-ct}|Y_t| \quad \text{avec } c>0 \text{ à choisir}$$

On a $e^{-cT}\|Y\|_\infty \leq |||Y||| \leq \|Y\|_\infty$

Donc $\mathcal{C}([0,T], |||\cdot|||)$ est un espace de Banach.

Banach (espace complet
Suite de Cauchy cvte)

Soit $t \in [0,T]$, $k \geq 1$

On calcule $|X_t^{(k+1)} - X_t^{(k)}| = \tilde{X}_0 + \int_0^t b(s, X_s^{(k)})\,ds + \sigma W_t - \tilde{X}_0 - \int_0^t b(s, X_s^{(k-1)})\,ds - \sigma W_t$

$$\leq \int_0^t |b(s, X_s^{(k)}) - b(s, X_s^{(k-1)})|\,ds$$

b est Lipschitz donc $\exists L > 0$ tq $\forall t \geq 0$ $|b(t,x) - b(t,y)| \leq L|x-y|$

D'où $|X_t^{(k+1)} - X_t^{(k)}| \leq L \int_0^t e^{cs} e^{-cs} |X_s^{(k)} - X_s^{(k-1)}|\,ds$

$$\leq L \, |||X_\cdot^{(k)} - X_\cdot^{(k-1)}||| \, \frac{e^{ct}}{c}$$

$$e^{-ct} |X_t^{(k+1)} - X_t^{(k)}| \leq \frac{L}{c} |||X_\cdot^{(k)} - X_\cdot^{(k-1)}|||$$

En prenant $\sup_{t \in [0,T]}$

$$|||X_\cdot^{(k+1)} - X_\cdot^{(k)}||| \leq \frac{L}{c} |||X_\cdot^{(k)} - X_\cdot^{(k-1)}|||$$

et pour $c > L$ on a une contraction $\Rightarrow X_\cdot^{(k)}$ CV vers $X_\cdot^{(\infty)}$

Rappel : si f est continue
$\frac{1}{h}\int_t^{t+h} f(s)\,ds \xrightarrow{h \to 0} f(t)$ ⚠

F est tq $F' = f$
$\frac{F(t+h) - F(t)}{h} \longrightarrow F'(t) = f(t)$

$$\phi(X_{t+h}) = \phi(X_t) + \int_t^{t+h} \partial_x \phi(X_s)\,dX_s + \frac{1}{2}\int_t^{t+h} \partial^2_{xx}\phi(X_s)\,d\langle X\rangle_s$$

$$\rightarrow \frac{\phi(X_{t+h}) - \phi(X_t)}{h} = \frac{1}{h}\int_t^{t+h} b(s,X_s)\,\partial_x\phi(X_s)\,ds + \underbrace{\frac{1}{h}\int_t^{t+h}\sigma(s,X_s)\,\partial_x\phi(X_s)\,dW_s}_{\text{Martingale.}} + \frac{1}{h}\int_t^{t+h}\frac{\sigma^2(s,X_s)}{2}\partial^2_{xx}\phi(X_s)\,ds$$

Donc, en prenant l'espérance

$$\mathbb{E}\left[\frac{\phi(X_{t+h}) - \phi(X_t)}{h} \,\middle|\, X_t = x\right] = \mathbb{E}\left[\frac{1}{h}\int_t^{t+h}\partial_x\phi(X_s)\,b(s,X_s)\,ds + \frac{1}{h}\int_t^{t+h}\partial^2_{xx}\phi(X_s)\frac{\sigma^2}{2}(s,X_s)\,ds \,\middle|\, X_t = x\right]$$

$$\xrightarrow[h \to 0]{⚠} \partial_x\phi(t,x)\,b(t,x) + \frac{1}{2}\sigma^2(t,x)\,\partial^2_{xx}\phi(t,x).$$

Slide 22 : Preuve de Feynman-Kac

$Y_t = u(t, X_t)$

$$dY_t = \partial_t u(t,X_t)\,dt + \partial_x u(t,X_t)\,b(t,X_t)\,dt + \partial_x u(t,X_t)\,\sigma(t,X_t)\,dW_t + \frac{1}{2}\partial^2_{xx}u(t,X_t)\underbrace{\sigma^2(t,X_t)\,dt}_{d\langle X\rangle_t}$$

$$= \underbrace{dt\left[\left(\partial_t u + \partial_x u\, b + \frac{1}{2}\sigma^2\partial^2_{xx}u\right)(t,X_t)\right]}_{r\,u(t,X_t)\,dt} + \partial_x u(t,X_t)\,\sigma(t,X_t)\,dW_t$$

Donc $dY_t = \partial_x u(t,X_t)\,\sigma(t,X_t)\,dW_t + r\,Y_t\,dt$

Soit $Z_t = e^{-rt}\,Y_t$

Alors $dZ_t = -r Z_t\,dt + r Z_t\,dt + e^{-rt}\partial_x u(t,X_t)\,\sigma(t,X_t)\,dW_t$

$$= \underbrace{e^{-rt}\partial_x u(t,X_t)\,\sigma(t,X_t)\,dW_t}_{\text{martingale}}$$

$$e^{-rt}u(t,x) = \mathbb{E}\left[e^{-rt}u(t,X_t) \mid X_t = x\right]$$

$$= \mathbb{E}\left[Z_t \mid X_t = x\right]$$

$= \mathbb{E}[Z_T | X_t = x]$ car $(Z_t)_t$ martingale

$= \mathbb{E}[e^{-rT} u(T, X_T) | X_t = x]$

$= \mathbb{E}[e^{-rT} f(X_T) | X_t = x]$

D'où $u(t,x) = \mathbb{E}[e^{-r(T-t)} f(X_T) | X_T = x]$

$\square$.

Slide 23:

f est à croissance au plus polynomiale :

$\exists C > 0, \exists \alpha > 0$ tq $|f(x)| \leq C(1+|x|)$ $\forall x \in \mathbb{R}$.

$dX_t = rX_t\,dt + \sigma X_t\,dW_t$

Slide 23:

EDP à résoudre

$$\begin{cases} \partial_t u(t,x) + rx\,\partial_x u(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} u(t,x) = r u(t,x) \\ u(T,x) = f(x) \end{cases}$$

Slide 30:

$dY_t = (r - \frac{1}{2}\sigma^2(t, X_t))dt + \sigma(t, X_t)\,dW_t$ $\quad Y_0 = y_0$

$X_t = e^{Y_t} = h(Y_t)$ $\quad h(y) = e^y$ $\quad \partial_y h(y) = e^y$ $\quad \partial^2_{yy} h(y) = e^y$

Par Itô, $e^{Y_t} = e^{Y_t} dY_t + \frac{1}{2} e^{Y_t} d\langle Y \rangle_t$

$= e^{Y_t}(r - \frac{1}{2}\sigma^2(t,X_t))dt + e^{Y_t}\sigma(t, e^{Y_t})dW_t + \frac{1}{2}\sigma^2(t, e^{Y_t})e^{Y_t}dt$

$dX_t = rX_t\,dt + \sigma(t, X_t)X_t\,dW_t$

$u(t,x) = v(T-t, \log(x))$ $\qquad \partial_y v(\tau, y) = e^y \partial_x u(T-\tau, e^y)$

$v(\tau, y) = u(T-\tau, \log(y))$ $\qquad \partial^2_{yy} v(\tau, y) = e^y \partial_x u(T-\tau, e^y) + e^{2y}\partial^2_{xx} u(T-\tau, e^y)$

$\partial_\tau v(\tau, y) = -\partial_t u(T-\tau, e^y)$

$= re^y \partial_x u(T-\tau, e^y) + \frac{1}{2}\sigma^2 (e^y)^2 \partial^2_{xx} u(T-t, y) - r u(T-\tau, e^y)$

$= r\,\partial_y v(\tau, y) + \frac{1}{2}\sigma^2(\partial^2_{yy} v(\tau,y) - \partial_y v(\tau, y)) - r v(\tau, y)$

$= r\,\partial_y v(\tau, y) - \frac{1}{2}\sigma^2 \partial_y v(\tau, y) + \frac{1}{2}\sigma^2 \partial^2_{yy} v(\tau, y) - r v(\tau, y)$

$u(T, x) = f(x)$

$v(0, y) = u(T, e^y) = f(e^y)$

Slide 35 :

Taylor-Lagrange : $f : I \to \mathbb{R}$ f est C^m sur I avec $m \in \mathbb{N}^*$
Alors $\forall a<b$ avec $a,b \in I$, $\exists c \in]a,b[$ tq

$$f(b) - \sum_{k=0}^{m} \frac{f^{(k)}(a)(b-a)^k}{k!} = \frac{f^{(m+1)}(c)(b-a)^{m+1}}{(m+1)!}$$

Preuve : ① et ③ à faire en exo

② Par TL_3, $\exists \xi_+ \in]x, x+h[$ et $\exists \xi_- \in]x-h, x[$ tq

L_1 $u(x+h) = u(x) + u'(x)h + u''(x)\frac{h^2}{2} + u^{(3)}(\xi_+)\frac{h^3}{6}$

ξ

L_2 $u(x-h) = u(x) - u'(x)h + u''(x)\frac{h^2}{2} - u^{(3)}(\xi_-)\frac{h^3}{6}$

$$\left(\frac{L_1 - L_2}{2h}\right) \Longrightarrow \frac{u(x+h)-u(x-h)}{2h} - u'(x) = u^{(3)}(\xi_+)\frac{h^2}{12} + u^{(3)}(\xi_-)\frac{h^2}{12}$$
$$\leq \frac{h^2}{6} \sup_{y \in]a,b[} |u^{(3)}(y)|$$

De même, $u'(x) - \frac{u(x+h)-u(x-h)}{2h} = -u^{(3)}(\xi_+)\frac{h^2}{12} - u^{(3)}(\xi_-)\frac{h^2}{12} \leq \frac{h^2}{6} \sup_{y \in]a,b[} |u^{(3)}(y)|$

Schéma = façon d'estimer les dérivés.

Slide 3 (Problématique) du chap 3 :

$$\mathbb{E}[f(X_T)] \approx \frac{1}{m} \sum f(\tilde{X}_T^i)$$

Slide 4 :

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$
$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s$$

Slide 11 :

$X \sim p$ $\quad$ p densité sur $\mathbb{R}$

Sampling : $\mathbb{E}[f(X)] = \int_{\mathbb{R}} f(x)p(x)\,dx$

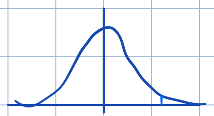
Soit q densité sur $\mathbb{R}$ et $\{x : q(x) = 0\} \subseteq \{x : f(x)p(x) = 0\}$

Alors $\mathbb{E}[f(X)] = \int_{\mathbb{R}} \frac{f(x)p(x)}{q(x)} q(x)\,dx$

$= \mathbb{E}\left[\frac{f(Y)p(Y)}{q(Y)}\right]$ où $Y \sim q$

Estimateur $\hat{F}_m = \frac{1}{m} \sum_{i=1}^{m} \frac{f(y_i)p(y_i)}{q(y_i)}$ où $Y_i \overset{iid}{\sim} q$

$$\frac{|f(x)|\,p(x)}{q(x)} \approx C$$



$$p(x) = \frac{e^{-\frac{|x|^2}{2}}}{\sqrt{2\pi}} \qquad q(x) = e^{-x+5}\,\mathbb{1}_{x>5}$$

$$\frac{p(x)}{q(x)} = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{|x|^2}{2}}\, e^{x-5}\,\mathbb{1}_{x>5}$$

Variable antithétique

<u>Ex</u> : Estimation de $\mathbb{E}[e^{-rT}(S_T-K)_+] = \mathbb{E}[h(x)]$ avec $X \sim \mathcal{N}(0,1)$ Estimateur

$$\hat{F}_m = \frac{1}{m}\sum_{i=1}^{m} \frac{h(X_i)+h(-X_i)}{2}$$

<u>Variable de contrôle :</u>

Soit $h_0 : \mathbb{R} \to \mathbb{R}$ tq $\mathbb{E}[h_0(X)] = \alpha$ avec $\alpha \in \mathbb{R}$ connu

Alors $\forall b \in \mathbb{R}$, $\mathbb{E}[h(x)] = \mathbb{E}[h(x) + b[h_0(X) - \alpha]]$

Donc (LFGN)

$$\hat{F}_m = \frac{1}{m}\sum_{i=1}^{m} \{h(X_i) + b[h_0(X_i) - \alpha]\} \xrightarrow[m\to\infty]{p.s.} \mathbb{E}[h(X)]$$

$$\mathbb{E}[h(X) + b(h_0(X)-\alpha)] = \mathbb{E}[h(X)] + b(\mathbb{E}[h_0(X)] - \alpha) \quad \text{(Linéarité)}$$
$$= \mathbb{E}[h(X)] + b\times 0 = \mathbb{E}[h(X)]$$

$$Var(\hat{F}_m) = Cov\left\{\frac{1}{m}\sum_i [h(X_i) + b(h_0(X_i)-\alpha)], \frac{1}{m}[h(X_j) + b(h_0(X_j)-\alpha)]\right\}$$

$$\underset{\text{Linéarité}}{=} \frac{1}{m^2}\sum_i\sum_j Cov(h(X_i) + b(h_0(X_i)-\alpha), h(X_j) + b(h_0(X_j)-\alpha))$$

$$= \frac{1}{m^2}\sum_i Cov(h(X_i) + b(h_0(X_i)-\alpha), h(X_i) + b(h_0(X_i)-\alpha))$$

$$\underset{X_i \text{ iid}}{=} \frac{1}{m} Cov(h(X_1) + b\times h_0(X_1) - \cancel{b\alpha}, h(X_1) + b h_0(X_1) - \cancel{b\alpha})$$

$$= \underbrace{\frac{1}{m} Var(h(X_1))}_{} + \frac{2}{m} b\, Cov(h(X_1), h_0(X_1)) + \frac{b^2}{m} Var(h_0(X_1))$$

$Var(\hat{E}_m) = Var\left(\frac{1}{m}\sum_{i=1}^{m} h(X_i)\right)$

$= \frac{1}{m} Var(h(X_1))$

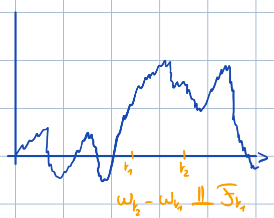
Réduct° de variance si $2b\, Cov(h(X_1), h_0(X_1)) + b^2\, Var(h_0(X_1)) <$

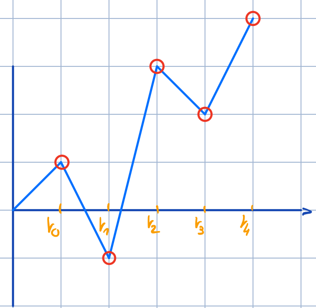
Exercice Cholesky :

$$K = \begin{pmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{pmatrix}$$

$$L = \begin{pmatrix} \sqrt{k_{1,1}} & 0 \\ \frac{k_{1,2}}{\sqrt{k_{1,1}}} & \sqrt{k_{2,2} - \frac{k_{1,2}^2}{k_{1,1}}} \end{pmatrix}$$

En dimension d, si $m \in \mathbb{R}^d$, $K \in \mathcal{M}_{d,d}^{sym}$
et $LL^T = K$ $\quad \tilde{X} \sim \mathcal{N}(0, I_d)$
$X = m + L\tilde{X} \sim \mathcal{N}(m, K)$





Chapitre 3 :

Slide 2 :

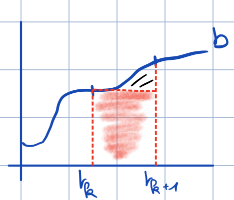
Problème : Calculer $\mathbb{E}[f(X_T)]$ où typiquement $f(x) = (x-K)_+$ et X_T satisfait une EDS par exemple
$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$

Méthode 1 : Interpréter $\mathbb{E}[f(X_T)]$ comme la solution d'une EDP
$\longrightarrow$ méthodes numériques pour les EDP

Méthode 2 : Si on sait simuler X_T, on peut simuler une suite iid $(X_T^\ell)_{\ell=1,\dots,N}$ et estimer par Monte-Carlo $\quad \frac{1}{N}\sum_{\ell=1}^{N} f(X_T^\ell) \xrightarrow[N\to\infty]{p.s} \mathbb{E}[f(X_T)]$

Quest° : Comment simuler une solut° d'EDS ?

$X_0 = x$
$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$
$X_t = x + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s$ $\quad \longrightarrow$ méthode des rectangles



$$X_{t_{k+1}} = X_{t_k} + \underbrace{\int_{t_k}^{t_{k+1}} b(s, X_s)\,ds}_{\simeq (t_{k+1} - t_k)\, b(t_k, X_{t_k})} + \underbrace{\int_{t_k}^{t_{k+1}} \sigma(s, X_s)\,dW_s}_{\simeq \sigma(t_k, X_{t_k})(W_{t_{k+1}} - W_{t_k})}$$

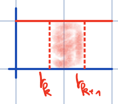
Soit $m \geq 1$ le pas de discrétisat°
$\Delta_m = \frac{T}{m}$, $\forall k \in \{0, \dots, N\}$, $t_k = k\Delta_m$

Schéma d'Euler à m étapes

$X_0^m = x$

$X_{t_{k+1}}^m = X_{t_k}^m + b(t_k, X_{t_k}^m)\Delta_m + \sigma(t_k, X_{t_k}^m)(W_{t_{k+1}} - W_{t_k})$

Si b constant



$dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t$

$\mathbb{E}\left[\left(\int_{t_k}^{t_{k+1}} b(s,X_s)\,ds\right)^2\right] \le (t_{k+1}-t_k)\,\mathbb{E}\left[\int_{t_k}^{t_{k+1}} b(s,X_s)^2\,ds\right]$

b borné $\le c(t_{k+1}-t_k)^2 = c(\Delta_m)^2$

$\mathbb{E}\left[\left(\int_{t_k}^{t_{k+1}} \sigma(s,X_s)\,dW_s\right)^2\right] \underset{\text{Isométrie Itô}}{=} \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \sigma(s,X_s)^2\,ds\right]$ $\qquad c \in \mathbb{R}$.

σ borné $\le c(t_{k+1}-t_k) = c\,\Delta_m$

$X_{t_{k+1}} = X_{t_k} + \int_{t_k}^{t_{k+1}} b(s,X_s)\,ds + \int_{t_k}^{t_{k+1}} \sigma(s,X_s)\,dW_s$

Si σ indépendant du temps, $\sigma(X_s) \approx \sigma(X_{t_k}) + \partial_x \sigma(X_{t_k})(X_s - X_{t_k})$

$\int_{t_k}^{t_{k+1}} \sigma(X_s)\,dW_s = \sigma(X_{t_k})(W_{t_{k+1}} - W_{t_k}) + \partial_x\sigma(X_{t_k})\,\mathbb{E}\left[\int_{t_k}^{t_{k+1}} (X_s - X_{t_k})\,dW_s\right]$ (2)

$(X_s - X_{t_k}) = \underbrace{\int_{t_k}^{s} b(u,X_u)\,du}_{\text{petit p/r à}} + \int_{t_k}^{s} \sigma(X_u)\,dW_u$

$\approx \sigma(X_{t_k})(W_s - W_{t_k})$ (1)

En injectant (1) dans (2),

$$\boxed{\int_{t_k}^{t_{k+1}} \sigma(X_s)\,dW_s = \sigma(X_{t_k})(W_{t_{k+1}} - W_{t_k}) + \partial_x\sigma(X_{t_k})\,\sigma(X_{t_k})\int_{t_k}^{t_{k+1}} (W_s - W_{t_k})\,dW_s}$$

$\to \int_{t_k}^{t_{k+1}} (W_s - W_{t_k})\,dW_s = \frac{1}{2}\left((W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)\right)$

IPP stochastique.

Erreur forte du schéma d'Euler

$\mathbb{E}[|g(X_T) - g(X_T^m)|] \le \mathbb{E}[|g(X_T) - g(X_T^m)|^2]^{1/2}$

Regardons déjà,

$\mathbb{E}[|g(X_{t_1}) - g(X_{t_1}^m)|^2] \le C\,\mathbb{E}[|X_{t_1} - X_{t_1}^m|^2]$ $\qquad$ Si g Lipschitz

$\le C\,\mathbb{E}\left[\left|\int_0^{t_1} b(s,X_s)\,dW_s - b(0,x)(t_1) + \int_0^{t_1} \sigma(s,X_s)\,dW_s + \sigma(0,x)(W_{t_1})\right|^2\right]$

$\le C_2\,\mathbb{E}\left[\left(\int_0^{t_1} (b(s,X_s) - b(0,x))\,ds\right)^2\right] + C_2\,\mathbb{E}\left[\left(\int_0^{t_1} (\sigma(s,X_s) - \sigma(0,x))\,dW_s\right)^2\right]$

Car $(a+b)^2 \le 2(a^2+b^2)$

Cauchy-Schwartz pour b. Itô pour σ

Donc $\mathbb{E}[|g(X_{t_1}) - g(X_{t_1}^m)|^2] \le C_2\, t_1\,\mathbb{E}\left[\int_0^{t_1} (b(s,X_s) - b(0,x))^2\,ds\right] + C_2\,\mathbb{E}\left[\int_0^{t_1} (\sigma(s,X_s) - \sigma(0,x))^2\,ds\right]$

Si b, σ bornés $\qquad \le C_2\, t_1^2 + C_2\, t_1 = C_2(\Delta_m)^2 + C_2\,\Delta_m$

De m̂, $\mathbb{E}[|g(X_T) - g(X_T^m)|^2] \le C_2(\Delta_m)^2 + C_2\,\Delta_m$ $\qquad C_1, C_2$ de $\perp\!\!\!\perp$ de m

Donc $\mathbb{E}[|g(X_T) - g(X_T^m)|] \le C_3\sqrt{\Delta_m} \longrightarrow$ CV en $\frac{1}{\sqrt{m}}$

Extrapolation de Romberg

On a $\mathbb{E}[f(X_T^m)] - \mathbb{E}[f(X_T)] = \frac{C_1}{m} + \frac{C_2}{m^2} + o(\frac{1}{m^3})$

$$\longrightarrow 2\mathbb{E}[f(X_T^{2m})] - \mathbb{E}[f(X_T^m)] - \mathbb{E}[f(X_T)] = 2(\mathbb{E}[f(X_T^{2m})] - \mathbb{E}[f(X_T)]) - (\mathbb{E}[f(X_T^m)] - \mathbb{E}[f(X_T)])$$
$$= 2\left(\frac{C_1}{2m} + \frac{C_2}{4m^2} + o(\tfrac{1}{m^3})\right) - \left(\frac{C_1}{m} + \frac{C_2}{m^2} + o(\tfrac{1}{m^3})\right)$$
$$= \frac{C_2}{2m^2} - \frac{C_2}{m^2} + o(\tfrac{1}{m^3})$$
$$= -\frac{C_2}{m^2} + o(\tfrac{1}{m^3}) \quad \text{d'ordre } \frac{1}{m^2} \,!$$

polynômes de Richardson

Chapitre 4 :

Modèle CIR

$X_0 \geq 0$

$dX_t = \theta(b - X_t)dt + \sigma\sqrt{X_t}\, dW_t$

Schéma d'Euler : $X_{t_{k+1}} = X_{t_k} + \theta(b - X_{t_k})(t_{k+1} - t_k) + \sigma\sqrt{X_{t_k}}\,(W_{t_{k+1}} - W_{t_k})$

$\mathbb{E}[f(X_T, I_T)]$

$$I_T = \int_0^T X_t\, dt \qquad dI_t = X_t\, dt$$

Reformulat° en 2 dim

$\bar{X}_t = \begin{pmatrix} X_t \\ I_t \end{pmatrix}$

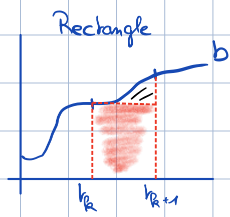
$d\bar{X}_t = \begin{pmatrix} dX_t \\ dI_t \end{pmatrix} = \begin{pmatrix} b(t, X_t) \\ X_t \end{pmatrix} dt + \begin{pmatrix} \sigma(t, X_t) \\ 0 \end{pmatrix} dW_t$

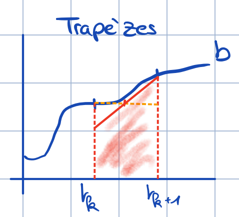
Schéma d'Euler

$X_0^1 = X_0 \quad I_0 = 0$

$X_{t_{k+1}}^m = X_{t_k}^m + b(t_k, X_{t_k})\Delta_m + \sigma(t_k, X_{t_k})(W_{t_{k+1}} - W_{t_k})$

$I_{t_{k+1}}^m = I_{t_k}^m + X_{t_k}^m \Delta_m$





$$I_{t_{k+1}}^m = I_{t_k}^m + \frac{X_{t_{k+1}}^m + X_{t_k}^m}{2}\Delta_m$$

V - "Greeks"

Référence : J. Hull

$$\underset{\nearrow}{V}(x) = \mathbb{E}_x\left[e^{-rT} f(X_T)\right]$$

Valeur actualisée

$$\Delta = \frac{\partial V}{\partial x} \qquad \Gamma = \frac{\partial^2 V}{\partial x^2}$$
$$\rho = \frac{\partial V}{\partial r} \qquad Vega = \frac{\partial V}{\partial \sigma}$$

"Δ - Hedging"

Exemple :

Portefeuille au temps 0 :

$$\Pi^0 = \Delta_x - \mathbb{E}_x[f(X_T)] \qquad\qquad \delta_x = X_h - x$$

Au temps $h \ll 1$

$$\Pi^h = \Delta X_h - \mathbb{E}_{X_h}[f(X_T)]$$
$$= \Delta(X_h - x) + \Delta x - (\mathbb{E}_{X_h}[f(X_T)] - \mathbb{E}_x[f(X_T)]) - \mathbb{E}_x[f(X_T)]$$
$$= \Delta\delta_x + \Delta x - \left(\frac{\partial V}{\partial x}(x)(X_h - x)\right) - \mathbb{E}_x[f(X_T)] + o(h)$$
$$= \Delta\delta_x + \Delta x - (\Delta\delta x) - \mathbb{E}_x[f(X_T)] + o(h)$$

$$\Rightarrow \Pi^h = \Pi^0 + o(h)$$

Modèle de B-S

$$P(S_0, K, r, T, \sigma) = -S_0\mathcal{N}(-d_1) + Ke^{-rT}\mathcal{N}(-d_2)$$
$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right)$$
$$d_2 = d_1 - \sigma\sqrt{T}$$
$$\mathcal{N}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy$$

Calculer Δ et ρ ici

$(1) = (2)$

$$\Delta = \frac{\partial P}{\partial S_0} = -\mathcal{N}(-d_1) - S_0\mathcal{N}'(-d_1)\overset{(1)}{\boxed{\frac{\partial d_1}{\partial S_0}}} + Ke^{-rT}\mathcal{N}'(-d_1 + \sigma\sqrt{T})\overset{(2)}{\boxed{\frac{\partial(d_1 - \sigma\sqrt{T})}{\partial S_0}}}$$
$$= -\mathcal{N}(-d_1) + \frac{\partial d_1}{\partial S_0}\left[-S_0\mathcal{N}'(-d_1) + Ke^{-rT}\mathcal{N}'(-d_1 + \sigma\sqrt{T})\right] \quad (1)$$

$$\mathcal{N}'(-d_1 + \sigma\sqrt{T}) = \frac{e^{-\frac{(d_1 - \sigma\sqrt{T})^2}{2}}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_1)^2}{2}} e^{\frac{2d_1\sigma\sqrt{T}}{2}} e^{-\frac{\sigma^2 T}{2}}$$
$$= \mathcal{N}'(-d_1)\, e^{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}T\right)} e^{-\frac{\sigma^2}{2}T}$$
$$= \mathcal{N}'(-d_1)\frac{S_0}{K}e^{-rT} \quad (2)$$

(2) dans (1)

$$\Rightarrow \frac{\partial P}{\partial S_0} = -\mathcal{N}(-d_1) + \frac{\partial d_1}{\partial S_0}\Big[\underbrace{-S_0\mathcal{N}'(d_1) + Ke^{-rT}\frac{S_0}{K}e^{rT}\mathcal{N}'(-d_1)}_{=0}\Big]$$
$$= -\mathcal{N}(-d_1)$$

$$\rho = \frac{\partial P}{\partial r} = -S_0 N'(d_1)\frac{\partial d_1}{\partial r} + Ke^{-rT}N'(d_1+\sigma\sqrt{T})\frac{\partial(d_1-\sigma\sqrt{T})}{\partial r} - KTe^{-rT}N(-d_2)$$
$$= \frac{\partial d_1}{\partial r}[-S_0 N'(-d_1) + Ke^{-rT}N(-d_1+\sigma\sqrt{T}) - KTe^{-rT}N(-d_2)$$
$$= -KTe^{-rT}N(-d_2)$$

Méthode des différences finies

$$V(\theta) = \mathbb{E}_x[f(S_T)]$$
$$\frac{\partial V}{\partial V} \simeq \frac{V(\theta+\Delta\theta)-V(\theta)}{\Delta\theta} + o(\Delta\theta)$$
$$\frac{\partial V}{\partial \theta} \simeq \frac{V(\theta+\Delta\theta)-V(\theta-\Delta\theta)}{2\Delta\theta} + o(\Delta\theta^2)$$

Par simulations indépendantes

$$\mathrm{Var}\left(\frac{V(\theta+\Delta\theta)-V(\theta-\Delta\theta)}{2\theta}\right) = \frac{1}{4\theta^2}[\mathrm{Var}(V(\theta+\Delta\theta)) + \mathrm{Var}(V(\theta-\Delta\theta))]$$
$$\simeq \frac{1}{4\theta^2}\left[\frac{\mathrm{Var}[f(S_T)]}{N} + \frac{\mathrm{Var}[f(S_T)]}{N}\right]$$
$$\simeq \frac{1}{2\theta^2}\frac{\mathrm{Var}(f(S_T))}{N}$$

Schéma d'Euler pour $\Delta = \frac{\partial V}{\partial x}$

avec $dX_t = rX_t\,dt + \sigma X_t\,dW_t$

Initialize $r, \sigma, T, N, m, \delta_x$
Let $h = \frac{T}{N}$, $F = 0$, $\tilde{F} = 0$
FOR $i = 1, -, m$
  $X = X_0$, $\tilde{X} = X_0 + \delta_x$
  FOR $j = 1, -, m$
    SIMULATE $Z \sim N(0,1)$, $\tilde{Z} \sim N(0,1)$, $Z \perp\!\!\!\perp \tilde{Z}$
    $X = X(1 + rh + \sigma\sqrt{h}Z)$
    $\tilde{X} = \tilde{X}(1 + rh + \sigma\sqrt{h}\tilde{Z})$
  ENDFOR
  $F = F + f(X)$
  $\tilde{F} = \tilde{F} + f(\tilde{X})$
ENDFOR
$\Delta = \frac{F - \tilde{F}}{2m\delta_x}$

Schéma d'Euler pour $\rho = \frac{\partial V}{\partial r}$

Initialize $r, \sigma, T, N, m, \delta_r$
Let $h = \frac{T}{N}$, $F = 0$, $\tilde{F} = 0$
FOR $i = 1, -, m$
  $X = X_0$, $\tilde{X} = X_0$
  FOR $j = 1, -, m$
    SIMULATE $Z \sim N(0,1)$, $\tilde{Z} \sim N(0,1)$, $Z \perp\!\!\!\perp \tilde{Z}$
    $X = X(1 + (r+\delta_r)h + \sigma\sqrt{h}\tilde{Z})$
    rh $(\sigma+\delta_\sigma)$ pour Vega
    $\tilde{X} = \tilde{X}(1 + (r-\delta_r)h + \sigma\sqrt{h}\tilde{Z})$
  ENDFOR
  $F = F + f(X)$
  $\tilde{F} = \tilde{F} + f(\tilde{X})$
ENDFOR
$\Delta = \frac{F - \tilde{F}}{2m\delta_r}$

Par la même simulation brownienne

$$\mathrm{Var}\left[\frac{V(\theta+\Delta\theta)-V(\theta-\Delta\theta)}{2\Delta\theta}\right] = \frac{\mathrm{Var}(f(S_T(\theta+\Delta\theta)) - f(S_T(\theta-\Delta\theta)))}{2\Delta\theta}$$
$$= \mathrm{Var}\left[f'(S_T(\theta-\Delta\theta))\frac{\partial S_T}{\partial\theta}(\theta-\Delta\theta)\left(\frac{2\Delta\theta}{2\Delta\theta}\right)\right]$$
$$\simeq \frac{1}{N}\mathrm{Var}\left[f'(S_T(\theta))\frac{\partial S_T}{\partial\theta}(\theta)\right]$$

Exemple : Simulation de $\Delta = \frac{\partial V}{\partial x_0}$