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Convexity, Optimization and Stochastic Control
Master 2 Probabilités et Finance
Session III

Exercise I. Let $T > 0$ and W be a $\mathbb{F}$–Brownian motion. We denote by $\mathcal{A}$ the set of $\mathbb{F}$–predictable processes $(\alpha_t)_{t\in[0,T]}$ satisfying $\|\alpha\|_\infty \leq 1$. For any $\alpha \in \mathcal{A}$, we introduce (Y^α, Z^α) two processes such that

$$Y_t^\alpha = Y_0 + \int_0^t \alpha_s \mathrm{d}W_s \ \text{ and } \ Z_t^\alpha = Z_0 + \int_0^t Y_s^\alpha \mathrm{d}W_s, \ \text{with } t \in [0,T].$$

Let λ be a positive parameter verifying $2\lambda T < 1$. We define the function $[0,T] \times \mathbb{R} \times \mathbb{R} \ni (t,y,z) \mapsto v(t,y,z) \in \mathbb{R}$ by

$$v(t, Y_t^1, Z_t^1) := \mathbb{E}\big[e^{2\lambda Z_T^1} \big| \mathcal{F}_t\big].$$

1. Provide the partial differential equation satisfied by v.
2. Compute explicitly $v(t,y,z)$ for any (t,y,z).
3. Show that v is strictly convex in y and that $y\, v_{yz}(t,y,z) \geq 0$.
4. Show that

$$v(0, Y_0, Z_0) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\big[e^{2\lambda Z_T^\alpha}\big].$$

Exercise II. We consider a continuous time financial market, containing a risk-free asset S^0, with a price normalized to 1, and a risky asset S with a price process defined by the SDE :

$$\frac{\mathrm{d}S_t}{S_t} \;=\; \sigma(Y_t)\,\big(\lambda(Y_t)\mathrm{d}t + \mathrm{d}W_t^1\big)\ , \tag{1}$$

where Y is a state variable whose dynamics is governed by :

$$dY_t \;=\; \eta(Y_t)\,dt + \gamma(Y_t)\left[\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right]. \tag{2}$$

Here, $W = (W^1, W^2)$ denotes a $\mathbb{R}^2$–valued Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, et ρ is a constant in $]-1,1[$. The coefficients $\eta(y)$, $\gamma(y)$, $s\lambda(y)$, $s\sigma(y)$ satisfy the usual conditions for the existence and uniqueness of strong solution of two dimensional SDE (1)-(2). In addition, we assume that the maps $\lambda(.)$, $\sigma(.)$ are bounded, and that $\inf_{y\in\mathbb{R}}\ \sigma(y)^2 + \gamma^2(y) > 0$. We will denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the canonical filtration of W.

A strategy θ is a $\mathbb{F}$ predictable process. For any initial capital x and any strategy θ, one defines the wealth process by :

$$X_t^{x,\theta} = x + \int_0^t \theta_r \frac{\mathrm{d}S_r}{S_r}\,, \qquad t \geq 0\,.$$

We say that θ is an admissible strategy if $X^{0,\theta}$ is well–defined and is uniformly bounded by below by a constant. We use the notation $\mathcal{A}$ for the set of admissible strategies.

Finally, consider a European option on the non-exchangeable variable Y :

$$G \;=\; g(Y_T) \quad \text{with} \quad g\ :\ \mathbb{R} \longrightarrow \mathbb{R} \text{ bounded continuous}$$

and we define the optimal investment problem by :

$$V^g(0, x, Y_0) := \sup_{\theta\in\mathcal{A}} \mathbb{E}\left[U\left(X_T^{x,\theta} - g(Y_T)\right)\right] \quad \text{where} \quad U(x) := -e^{-ax},\ a > 0,\ x \in \mathbb{R}.$$

Our goal is to determine an explicit expression for :

$$p(0,x,y) \;:=\; \inf\left\{\pi\ :\ V^g(0, x+\pi, y)\ \geq\ V^0(0,x,y)\right\}.$$

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In the following, we extend as usual the preceding quantities in the case where the initial time is t by noting $V^g(t,x,y)$ and $p(t,x,y)$. Besides, we suppose that the map V^g is $C^{1,2}\left([0,T),\mathbb{R}\times\mathbb{R}\right)\cap C\left([0,T]\times\mathbb{R}\times\mathbb{R}\right)$, for any function g satisfying the previous conditions

1. Give a financial interpretation of the map p. Let $\hat{\pi}\in\mathbb{R}$ and $\hat{\theta}\in\mathcal{A}$ such that $X_T^{\hat{\pi},\hat{\theta}}\geq G$ a.s. and $X_T^{\hat{\pi},\hat{\theta}}$ is bounded. Show that $p(0,x,y)\leq\hat{\pi}$, and interpret this result.
2. Show that there exists a non–negative function $F^g(t,y)$ satisfying :
$$V^g(t,x,y)=-e^{-ax}F^g(t,y)\quad\text{for all}\quad t\leq T,\ (x,y)\in\mathbb{R}^2\,,$$
and deduce that
$$p(t,x,y)=p(t,y)=\frac{1}{a}\ln\left(\frac{F^g(t,y)}{F^0(t,y)}\right),\qquad (t,x,y)\in[0,T]\times\mathbb{R}^2.$$
3. We assume that the map $V^g(t,x,y)$ is a classic solution of the equation
$$\inf_{\theta\in\mathbb{R}}-\mathcal{L}^\theta V^g(t,x,y)\ =\ 0\,,\qquad V^g(T,x,y)\ =\ U(x-g(y))$$
where
$$\mathcal{L}^\theta V^g\quad:=\quad V_t^g+\eta V_y^g+\frac{1}{2}\gamma^2V_{yy}^g+\lambda\sigma\theta V_x^g+\frac{1}{2}\sigma^2\theta^2V_{xx}^g+\theta\rho\sigma\gamma V_{xy}^g$$
Show that $F^g(t,y)$ verifies :
$$F_t^g+\eta F_y^g+\frac{1}{2}\gamma^2F_{yy}^g-\frac{\left(\lambda F^g+\rho\gamma F_y^g\right)^2}{2F^g}\ =\ 0,\qquad F^g(T,y)\ =\ e^{ag(y)}\,.$$
4. Find a parameter δ such that the function $f^g(t,y):=\left[F^g(t,y)\right]^{1/\delta}$ is the solution of a **linear** partial differential equation.
5. Deduce that
$$f^g(t,y)\quad=\quad\mathbb{E}\left[e^{(1-\rho^2)\left(ag(\widehat{Y}_T)-\frac{1}{2}\int_t^T\lambda^2(\widehat{Y}_u)\mathrm{d}u\right)}\right]$$
where $\widehat{Y}$ is the solution of an SDE to be specified.
6. Give a probability measure $\widehat{\mathbb{Q}}$ equivalent to $\mathbb{P}$ so that p is written as
$$p(t,y)\quad=\quad\frac{1}{a(1-\rho^2)}\ln\left(\mathbb{E}^{\widehat{\mathbb{Q}}}\left[e^{a(1-\rho^2)g(\widehat{Y}_T)}\right]\right).$$
7. Find a random variable $\mathcal{F}_T$–measurable H s.t. for any function with payoff g :
$$\frac{\partial p}{\partial y}(t,y)\quad=\quad\frac{1}{a(1-\rho^2)}\,\frac{\mathbb{E}^{\widehat{\mathbb{Q}}}\left[He^{a(1-\rho^2)g(\widehat{Y}_T)}\right]}{\mathbb{E}^{\widehat{\mathbb{Q}}}\left[e^{a(1-\rho^2)g(\widehat{Y}_T)}\right]}\,.$$

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Exo 1

(1) $dY_t^\lambda = \lambda_t \, dW_t$

$dZ_t^\lambda = Y_t^\lambda \, dW_t$

$Z' = Z^{\hat{\lambda}}$ où $\lambda_t \equiv 1$

$v(t, Y_t^1, Z_t^1) = \mathbb{E}\left[e^{2\lambda Z_T^1} \middle| \mathcal{F}_t\right]$

$\partial_t v + \mathcal{L} v = 0$

$$\begin{cases} \partial_t v + \frac{1}{2}\partial_{yy} v + \frac{1}{2}\partial_{zz} v \cdot y^2 + \partial_{zy} v \cdot y = 0 \\ v(T, y, z) = e^{2\lambda z} \end{cases}$$

(2) calculer $v(t, y, z)$

$$Z_s^{t,y,z} = z + \int_t^s (y + \underbrace{W_u - W_t}_{W_u^t})\, dW_u = z + y W_s^t + \underbrace{\int_t^s W_u^t \, dW_u}_{=\int_t^s W_u^t dW_u^t} = z + y W_s^t + \frac{(W_s^t)^2}{2} - \frac{(s-t)}{2}$$

$$e^{2\lambda Z_T^{t,y,z}} = \exp\left\{ 2\lambda z + 2\lambda y W_T^t + \lambda\left((W_T^t)^2 - (T-t)\right) \right\}$$

$$\mathbb{E}[\ \ ] = e^{2\lambda z} \int e^{2\lambda y \sqrt{T-t}\, \xi + \lambda((T-t)\xi^2 - (T-t))} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi =$$

il faut que ça soit positif $(2\lambda T < 1)$

$\bullet (1 - 2\lambda(T-t))$

$$= e^{2\lambda z - \lambda(T-t)} \frac{1}{\sqrt{2\pi}} \int \exp\left\{ -\frac{1}{2}\left(\xi^2(1 - 2\lambda(T-t)) - \xi(4\lambda y\sqrt{T-t}) + \mu^2 \right) \right\} d\xi \cdot e^{\frac{1}{2}\mu^2} =$$

$$= e^{2\lambda z - \lambda(T-t)} e^{\frac{1}{2}\mu^2(1-2\lambda(T-t))} \underbrace{\frac{1}{\sqrt{2\pi}} \int \exp\left\{ -\frac{1}{2} \frac{(\xi - \mu)^2}{\underbrace{(1 - 2\lambda(T-t))^{-1}}_{\sigma^2}} \right\} d\xi}_{=\sigma} =$$

$$= \sigma\, e^{2\lambda z - \lambda(T-t) + \frac{1}{2}\mu^2(1 - 2\lambda(T-t))} \quad \text{où} \quad \mu = \frac{4\lambda y\sqrt{T-t}}{1 - 2\lambda(T-t)}$$

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$$= \frac{1}{\sqrt{1-2\lambda(T-t)}} e^{2\lambda z - \lambda(T-t) + \frac{8\lambda^2 y^2 (T-t)}{1-2\lambda(T-t)}}$$

(3) v est str^t convexe en y et $y\partial_{yz} v > 0$

$$c = \frac{8\lambda^2(T-t)}{1-2\lambda(T-t)} > 0 \rightarrow cy^2 \text{ str}^t \text{ convexe} \rightarrow e^{cy^2} \text{ aussi}$$

$$y\partial_{yz} v = y \cdot \frac{1}{\sqrt{\cdots}} \cdot 2\lambda \cdot 2cy\, e^{\cdots} = \underbrace{4c\lambda y^2}_{>0} \cdot \underbrace{v}_{>0} > 0$$

(4) Montrer que

$$v(0, Y_0, Z_0) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[e^{2\lambda Z_T^\alpha}]$$

Par (2), on voit que $v \in C^{1,2}$ donc v vérifie bien l'EDP de (1)

On va m.q. $v(0, Y_0, Z_0) \geq \mathbb{E}[e^{2\lambda Z_T^\alpha}]$

$$\boxed{HJB = FK + \sup_{a \in A}}$$

$$HJB: \quad \sup_{a \in [-1,1]} \left\{ \partial_t v + \frac{\alpha^2}{2} \partial_{yy} v + \frac{1}{2}\partial_{zz} v \cdot y^2 + \partial_{zy} v \cdot y\, \alpha \right\} = 0$$

$$\alpha^2 \underbrace{\frac{\partial_{yy} v}{2}}_{>0} + \alpha \big(\underbrace{\partial_{zy} v \cdot y}_{>0} \big) \xrightarrow[\alpha \in [-1,1]]{} \max \rightarrow \alpha^* = 1$$

Par le thm. de vérification, $\hat{\alpha}_t \equiv 1$ est un contrôle optimale $\rightarrow$

$$\rightarrow v(0, Y_0, Z_0) = \mathbb{E}[e^{2\lambda Z_T^{\hat{\alpha}}}] \geq \mathbb{E}[e^{2\lambda Z_T^\alpha}] \quad \forall \alpha \in \mathcal{A}.$$

(on peut aussi calculer directement
$\mathbb{E} e^{2\lambda Z_T^2} = \mathbb{E} e^{2\lambda Z_0^2} + \underbrace{\mathbb{E} \mathcal{L}^a v}_{=0} + \underbrace{\mathbb{E} \int \cdots dW}_{0}$ — mart)

<u>Exo 2</u>

$$\frac{dS_t}{S_t} = \sigma(Y_t)\left[\lambda(Y_t)\,dt + dW_t^1\right]$$

$$dY_t = \eta(Y_t)\,dt + \gamma(Y_t)\left[\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right]$$

λ, σ bornées, $\inf_y \{\sigma^2(y) + \gamma^2(y)\} > 0$

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$$X_t^{x,\vartheta} = x + \int_0^t \vartheta_u \frac{dS_u}{S_u}$$

capital initial — stratégie admissible $(\vartheta \in \mathcal{A})$ — bornée à l'inf. par const

$G = g(Y_T)$ $\quad V^g(0,x,Y_0) = \sup_{\vartheta\in\mathcal{A}} \mathbb{E}\left[U\left(X_T^{x,\vartheta} - g(Y_T)\right)\right]$ où $U(x) = -e^{-ax}$, $a>0$, $x\in\mathbb{R}$

$U \nearrow$

$p(0,x,y) = \inf\{\pi : V^g(0, x+\pi, y) \geq V^0(0,x,y)\}$ ← prix d'option g

On suppose que $V^g \in C^{1,2}([0,T)\times\mathbb{R}\times\mathbb{R}) \cap C([0,T]\times\mathbb{R}\times\mathbb{R})$

(1) Interpretation de p? Superhedging: l'utilité moyenne avec le prime π doit être plus grand que si on n'a ni prime ni option à couvrir.

Soit $\hat\pi \in \mathbb{R}$, $\hat\theta \in \mathcal{A}$ t.q. $X_T^{\hat\pi,\hat\theta} \geq G$ et $X_T^{\hat\pi,\hat\theta}$ est bornée

M.q. $p(0,x,y) \leq \hat\pi$

Il faut m.q. $V^g(0, x+\hat\pi, y) \geq V^0(0,x,y)$

$$V^0(0,x,y) = \sup_{\vartheta\in\mathcal{A}} \mathbb{E}\left[U(X_T^{x,\vartheta})\right]$$

$$\tilde\theta = \theta + \hat\theta \Rightarrow X_T^{x+\hat\pi,\, \theta+\hat\theta} = X_T^{x,\theta} + X_T^{\hat\pi,\hat\theta}$$

$U \nearrow$

$$U(X_T^{x+\hat\pi,\vartheta+\hat\vartheta} - G) = U(X_T^{x,\vartheta} + \underbrace{X_T^{\hat\pi,\hat\theta} - G}_{\geq 0}) \geq U(X_T^{x,\vartheta})$$

$\sup_{\vartheta\in\mathcal{A}}$ $\quad$ ($\hat\theta$ bornée $\Rightarrow \mathcal{A} - \hat\theta = \mathcal{A}$)

$$V^g(0, x+\hat\pi, y) \geq V^0(0,x,y)$$

On peut m.q. si $\hat\theta$: $X_T^{\hat\pi,\hat\theta} = G \Rightarrow p(0,x,y) = \hat\pi$

(2) M.q. $\exists\, F^g(t,y) \geq 0$: $\quad V^g(t,x,y) = -e^{-ax} F^g(t,y)$

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En déduire que $p(t,x,y) = p(t,y) = \frac{1}{a} \ln\left(\frac{F^g(t,y)}{F^0(t,y)}\right)$

$$V^g(t,x,y) = \sup_{\theta \in \mathcal{A}} \mathbb{E}\left[U(x + X_T^{0,\theta} - G)\right] = -e^{-ax} \underbrace{\sup_{\theta \in \mathcal{A}} \mathbb{E}\left[U(X_T^{0,\theta} - G)\right]}_{=: F^g(t,y)}$$

$$U(x+y) = -e^{-ax} U(y)$$

$$p(t,x,y) = \inf\{\pi > 0 : V^g(t, x+\pi, y) \geq V^0(t,x,y)\} = \inf\{\pi > 0 : e^{-a\pi} F^g(t,y) \leq F^0(t,y)\}$$

$$-e^{-a(x+\pi)} F^g(t,y) \geq -e^{-ax} F^0(t,y)$$

$$\inf\left\{\pi > 0 : -a\pi \leq \ln \frac{F^0(t,y)}{F^g(t,y)}\right\} = \frac{1}{a} \ln\left(\frac{F^g(t,y)}{F^0(t,y)}\right)$$

(3) On suppose que $V^g(t,x,y)$ est une solution classique de

$$\begin{cases} \inf_{\theta \in \mathbb{R}} \{-\mathcal{L}^\theta V^g(t,x,y)\} = 0 \\ V^g(T,x,y) = U(x - g(y)) \end{cases}$$

où $\mathcal{L}^\theta V^g = \partial_t V^g + \eta \partial_y V^g + \frac{1}{2}\gamma^2 \partial_{yy} V^g + \lambda \sigma \theta V_x^g + \frac{1}{2}\sigma^2\theta^2 \partial_{xx} V^g + \theta\rho\sigma\gamma\, \partial_{xy} V^g$

M.q. $F^g(t,y)$ vérifie

$$\begin{cases} \partial_t F^g + \eta \partial_y F^g + \frac{1}{2}\gamma^2 \partial_{yy} F^G - \dfrac{(\lambda F^g + \rho\gamma \partial_y F^g)^2}{2F^g} = 0 \\ F^g(T,y) = e^{ag(y)} \end{cases}$$

$$V^g(t,x,y) = \sup_{\theta \in \mathcal{A}} \mathbb{E}\left[U(X_T^{x,\theta} - g(Y_T^y))\right] \overset{HJB}{\Rightarrow} \sup_{\theta \in \mathbb{R}} \{\mathcal{L}^\theta V^g\} = 0 \Rightarrow \inf_{\theta \in \mathbb{R}} \{-\mathcal{L}^\theta V^g\} = 0$$

$$V^g(t,x,y) = -e^{-ax} F^g(t,y)$$

$$\mathcal{L}^\theta V^g = -e^{-ax}\left[\partial_t F^g + \eta \partial_y F^g + \frac{1}{2}\gamma^2 \partial_{yy} F^g\right] + \max_\theta [\,--\,]$$

$$\theta^2\left(\frac{1}{2}\sigma^2\,\partial_{xx}V^\delta\right) + \theta\left(\rho\sigma\delta\,\partial_{xy}V^\delta + \lambda\sigma\,\partial_x V^\delta\right) \to \max_{\theta\in\mathbb{R}}$$

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$$\partial_{xx}V^\delta < 0 \text{ (j'espère) } \to \theta^* = -\frac{\rho\,\delta\,\partial_{xy}V^\delta + \lambda\,\partial_x V^\delta}{\sigma\,\partial_{xx}V^\delta} =$$

$$\left\{\begin{aligned} \partial_x V^\delta &= a e^{-ax} F^\delta \\ \partial_{xx} V^\delta &= -a^2 e^{ax} F^\delta < 0 \\ \partial_{xy} V^\delta &= a e^{-ax} \partial_y F^\delta \end{aligned}\right\} \quad = \frac{\rho\delta\,\partial_y F^\delta + \lambda F^\delta}{\sigma a F^\delta}$$

$a<0$

$$a\theta^2 + b\theta \to \max \to \theta^* = -\frac{b}{2a}, \quad a\theta^{*2} + b\theta^* = \frac{b^2}{4a} - \frac{b^2}{2a} = -\frac{b^2}{4a} = \frac{b}{2}\theta^*$$

Donc, $\mathcal{L}^\theta V^\delta = -\cancel{e^{-ax}}\left[\partial_t F^\delta + \eta\,\partial_y F^\delta + \frac{1}{2}\delta^2\partial_{yy}F^\delta\right] + \dfrac{\cancel{\sigma a}\cancel{e^{-ax}}\left(\rho\delta\,\partial_y F^\delta + \lambda F^\delta\right)^2}{2\cancel{\sigma a}F^\delta} = 0$

$$\partial_t F^\delta + \eta\,\partial_y F^\delta + \frac{1}{2}\delta^2\partial_{yy}F^\delta - \frac{\left(\lambda F^\delta + \rho\delta\,\partial_y F^\delta\right)^2}{2F^\delta} = 0$$

$$F^\delta(T,y) = -e^{ax}\underbrace{V^\delta(T,x,y)}_{U(x-g(y))} = e^{ax}e^{-ax+ag(y)} = e^{ag(y)}$$

(4) Trouver le paramètre ς t.q. $f^\delta(t,y) := \left[F^\delta(t,y)\right]^{1/\varsigma}$ est une sol de l'EDP linéaire.

Il faut se débarasser de $\dfrac{\left(\lambda F^\delta + \rho\delta\,\partial_y F^\delta\right)^2}{2F^\delta}$

$$F^\delta = \left(f^\delta\right)^\varsigma \qquad \partial_y F^\delta = \varsigma\left(f^\delta\right)^{\varsigma-1}\partial_y f^\delta \qquad \left(\frac{1}{\varsigma} = 1-\rho^2\right)$$

$$\lambda\left[f^\delta\right]^\varsigma + \varsigma\rho\delta\left[f^\delta\right]^{\varsigma-1}\partial_y f^\delta = 0$$

$$\lambda f^\delta + \varsigma\rho\delta\,\partial_y f^\delta = 0$$

$$\partial_t F^\delta + \eta\,\partial_y F^\delta + \frac{1}{2}\delta^2\partial_{yy}F^\delta - \frac{\left(\lambda F^\delta + \rho\delta\,\partial_y F^\delta\right)^2}{2F^\delta} = 0$$

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$$\partial_t F^\delta = \delta\,[f^\delta]^{\delta-1}\,\partial_t f^\delta$$
$$\partial_y F^\delta = \delta\,[f^\delta]^{\delta-1}\,\partial_y f^\delta$$
$$\partial_{yy} F^\delta = \delta(\delta-1)\,[f^\delta]^{\delta-2}(\partial_y f^\delta)^2 + \delta\,[f^\delta]^{\delta-1}\,\partial_{yy} f^\delta$$

$$\frac{(\lambda F^\delta + \rho\gamma\partial_y F^\delta)^2}{2F^\delta} = \frac{\lambda^2}{2}F^\delta + \rho\lambda\gamma\,\partial_y F^\delta + \frac{1}{2}\frac{\rho^2\gamma^2(\partial_y F^\delta)^2}{F^\delta} =$$

$$= \frac{\lambda^2}{2}[f^\delta]^\delta + \rho\lambda\gamma\,\delta\,[f^\delta]^{\delta-1}\partial_y f^\delta + \frac{1}{2}\rho^2\gamma^2\delta^2\,\frac{[f^\delta]^{\overset{\delta-2}{\cancel{2(\delta-1)}}}(\partial_y f^\delta)^2}{[f^\delta]^{\cancel{\delta}}}$$

On choisit δ t.q. $\cancel{\tfrac{1}{2}}\cancel{\gamma^2}\,\cancel{\delta}(\delta-1) - \cancel{\tfrac{1}{2}}\rho^2\cancel{\gamma^2}\,\delta^{\cancel{2}} = 0$

$$\delta - 1 = \delta\rho^2 \;\rightarrow\; \delta = (1-\rho^2)^{-1}$$

L'EDP pour f^δ: (on divise par $\delta[f^\delta]^{\delta-1}$)

$$\partial_t f^\delta + \eta\,\partial_y f^\delta + \frac{1}{2}\gamma^2\partial_{yy} f^\delta - \frac{\lambda^2}{2\delta}f^\delta - \rho\lambda\gamma\,\partial_y f^\delta = 0$$

$$\begin{cases} \partial_t f^\delta + (\eta - \rho\lambda\gamma)\partial_y f^\delta + \frac{1}{2}\gamma^2\partial_{yy} f^\delta = \underbrace{\frac{\lambda(y)^2}{2}(1-\rho^2)}_{r(y)} f^\delta \\ f^\delta|_{t=T} = e^{a(1-\rho^2)g(y)} = \Psi(y) \end{cases}$$

l'EDP linéaire

(5) En déduire que $f^\delta(t,y) = \mathbb{E}\left[e^{(1-\rho^2)\left(a g(\hat{Y}_T) - \frac{1}{2}\int_t^T \lambda^2(\hat{Y}_u)\,du\right)}\right]$

Par la formule de FK, on a

$$f^\delta(t,y) = \mathbb{E}\left[\Psi(\hat{Y}_T)\,e^{-\int_t^T r(\hat{Y}_u)\,du}\right] = \mathbb{E}\left[e^{(1-\rho^2)\left(a g(\hat{Y}_T) - \frac{1}{2}\int_t^T \lambda^2(\hat{Y}_u)\,du\right)}\right]$$

où $\begin{cases} d\hat{Y}_s = [\eta(\hat{Y}_s) - \rho\lambda(\hat{Y}_s)\gamma(\hat{Y}_s)]\,ds + \gamma(\hat{Y}_s)\,dW_s \\ \hat{Y}_t = y \end{cases}$ sous $\mathbb{P}$.

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(6) Donner une proba $\widehat{\mathbb{Q}} \sim \mathbb{P}$ t.q.

$$p(t,y) = \frac{1}{a(1-\rho^2)} \ln\left(\mathbb{E}^{\widehat{\mathbb{Q}}}\left[e^{a(1-\rho^2)g(\widehat{Y}_T)}\right]\right)$$

Par (2), $p(t,y) = \frac{1}{a}\ln\left(\frac{F^g(t,y)}{F^0(t,y)}\right) = \{F^g = [f^g]^\delta\} = \frac{1}{a(1-\rho^2)} \ln\left(\frac{\mathbb{E}\left[e^{(1-\rho^2)\left(ag(\widehat{Y}_T) - \frac{1}{2}\int_t^T \lambda^2 du\right)}\right]}{\mathbb{E}\left[e^{(1-\rho^2)\left(-\frac{1}{2}\int_t^T \lambda^2 du\right)}\right]}\right)$

On veut obtenir $\dfrac{\mathbb{E}_t^{\mathbb{P}}\left[e^{(1-\rho^2)ag(\widehat{Y}_T)} e^{-\frac{1}{2}\int_t^T \lambda^2(\widehat{Y}_u)du}\right]}{\mathbb{E}_t^{\mathbb{P}}\left[e^{-\frac{1}{2}\int \lambda^2 du}\right]} = \mathbb{E}_t^{\widehat{\mathbb{Q}}}\left[e^{(1-\rho^2)ag(\widehat{Y}_T)}\right]$

$$\mathbb{E}_t^{\widehat{\mathbb{Q}}}[\dots] = \frac{\mathbb{E}_t^{\mathbb{P}}\left[\dots \cdot \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right]}{\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right]} \Rightarrow \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} = e^{-\frac{1}{2}\int_0^T \lambda^2(\widehat{Y}_u)du}$$

et $p(t,y) = \dfrac{1}{a(1-\rho^2)} \ln\left(\mathbb{E}^{\widehat{\mathbb{Q}}}\left[e^{(1-\rho^2)ag(\widehat{Y}_T^y)}\right]\right)$

(7) Trouver une v.a. $\mathcal{F}_T^0$-mes. H t.q. $\forall g$

$$\frac{\partial p}{\partial y} = \frac{1}{a(1-\rho^2)} \frac{\mathbb{E}^{\widehat{\mathbb{Q}}}\left[He^{a(1-\rho^2)g(\widehat{Y}_T)}\right]}{\mathbb{E}^{\widehat{\mathbb{Q}}}\left[e^{a(1-\rho^2)g(\widehat{Y}_T)}\right]}$$

$$\frac{\partial p}{\partial y} = \frac{1}{a(1-\rho^2)} \frac{\partial_y \mathbb{E}^{\widehat{\mathbb{Q}}}[\dots]}{\mathbb{E}^{\widehat{\mathbb{Q}}}[\dots]}$$

Donc il suffit m.q. $\partial_y \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\underbrace{e^{(1-\rho^2)ag(\widehat{Y}_T^y)}}_{\Psi_T}\right] = \mathbb{E}^{\widehat{\mathbb{Q}}}[H \cdot \Psi_T]$

On suppose que $\widehat{Y}_T^y$ a une densité $p(\hat{y}, y)$

Dans ce cas, $\partial_y \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\Psi_T(\widehat{Y}_T^y)\right] = \partial_y \int \Psi_T(\hat{y})\, p(\hat{y}, y)\, d\hat{y} = \int \Psi_T(\hat{y})\, \partial_y p(\hat{y}, y)\, d\hat{y} =$

$$= \int \Psi_T(\hat{y})\, \partial_y \ln(p(\hat{y}, y))\, p(\hat{y}, y)\, d\hat{y} = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\Psi_T(\widehat{Y}_T)\, \overbrace{\partial_y \ln(p(\widehat{Y}_T^y, y))}^{H \in \mathcal{F}_T^0}\right]$$