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Convexity, Optimization and Stochastic Control
Master 2 Probabilités et Finance
Session IV

We denote throughout ∂_t the partial gradient with respect to t. Subscripts indicate partial gradients with respect to space variables, e.g. $f_x := \frac{\partial f}{\partial x}$, $f_{xx} := \frac{\partial^2 f}{\partial x^2}$, $f_{xy} := \frac{\partial^2 f}{\partial x \partial y}$, etc...

EXERCISE

Let W be a scalar Brownian motion, and $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the corresponding completed filtration. Answer the following questions with a formal justification.

√ 1. Provide the dynamic programming equation of the stochastic control problem

$$V(t,x,y) := \inf_{\nu \in \mathcal{U}} \mathbb{E}_{t,x,y}\big[e^{-\rho(T-t)} g(X_1^\nu)\big], \;\; t \leq T, x, y \in \mathbb{R}$$

where $\rho \in \mathbb{R}$, $\mathcal{U}$ is the collection of all $\mathbb{F}-$progressively measurable processes with values in $[-1,1]$, and the controlled process is defined by :

$$dX_s^\nu = Y_s^\nu dW_s, \qquad dY_s^\nu = X_s^\nu ds + \nu_s dW_s, \quad s \in [t,1].$$

√ 2. What is the dynamic programming equation corresponding to the control problem

$$V(t,x) := \sup_{\nu \in \mathcal{U}} \mathbb{E}_{t,x}\Big[\int_t^T \frac{1}{2}\nu_s^2 \, ds\Big],$$

where $\mathcal{U}$ is the collection of all $\mathbb{F}-$progressively measurable processes with values in $\mathbb{R}$, and the controlled process is defined by :

$$dX_s^\nu \;\; = \;\; \nu_s dW_s, \quad s \in [t,T].$$

√ 3. Which problem induces the following nonlinear PDE as the corresponding dynamic programming equation :

$$\min\big\{-\partial_t v - 8v_{xx} + \rho v, v - g\big\} = 0, \quad \text{on } [0,T) \times \mathbb{R}, \quad v\big|_{t=T} = g,$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a given bounded continuous function.

√ 4. Which stochastic control problem induces the following nonlinear PDE as the corresponding dynamic programming equation :

$$-\partial_t v - v_x^+ + v_x^- - 2v_{xx} + 3v - x^2 = 0, \quad \text{on } [0,T) \times \mathbb{R}, \quad v\big|_{t=T} = 0,$$

where $y^+ := \max\{y, 0\}$, $y^- := (-y)^+$, $y \in \mathbb{R}$.

PROBLEM : Optimal mean-variance stopping

Let W be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Given two fixed parameters $\mu \in \mathbb{R}$, $\sigma > 0$, and an initial value $x > 0$, we consider the geometric Brownian motion X_t^x defined by

$$X_0^x = x \quad \text{and} \quad dX_t^x = X_t^x\big(\mu\, dt + \sigma\, dW_t\big), \quad t \geq 0.$$

and we set $X_\infty^x := \limsup_{t\to\infty} X_t^x$.

We denote by $\mathcal{T}$ the collection of all stopping times τ with values in $[0,\infty]$, satisfying the square integrability $\mathbb{E}\big[(X_\tau^1)^2\big] < \infty$. Our objective is to solve the the mean-variance optimal stopping problem

$$V(x) := \sup_{\tau\in\mathcal{T}} J(x,\tau) \quad \text{where} \quad J(x,\tau) := \mathbb{E}\big[X_\tau^x\big] - \frac{1}{2}\mathbb{V}\text{ar}\big[X_\tau^x\big], \quad x > 0.$$

1. (a) Explain why this problem can not be solved by the standard dynamic programming approach.
   (b) Assume $\mu \leq 0$. Verify that the process X^x is a supermartingale, and deduce that $V(x) = x$ with corresponding optimal stopping policy $\hat{\tau} = 0$.
   (c) We next assume that $\mu \geq \frac{\sigma^2}{2}$. For all $n \geq 1$, denote $\tau_n := \inf\{t \geq 0 : X_t^x \geq n\}$. Justify that $\limsup_{t\to\infty} X_t^x = \infty$, a.s. and that $\tau_n \in \mathcal{T}$. Deduce that $V(x) = \infty$, and discuss whether the corresponding optimal stopping policy is in $\mathcal{T}$.

In the rest of this problem we consider the remaining case

$$0 < \mu < \frac{\sigma^2}{2}, \quad \text{and we denote} \quad \gamma := \frac{2\mu}{\sigma^2} \in (0,1). \tag{1}$$

2. For $M \geq 0$, we introduce the subset of stopping times $\mathbb{T}_M := \big\{\tau \in \mathcal{T} : \mathbb{E}\big[X_\tau^x\big] = M\big\}$.
   (a) Verify that
   $$V(x) = \sup_{M\geq 0}\Big\{M + \frac{M^2}{2} - V_M(x)\Big\}, \quad \text{where} \quad V_M(x) := \inf_{\tau\in\mathbb{T}_M} \frac{1}{2}\mathbb{E}\big[(X_\tau^x)^2\big].$$
   (b) By defining $g_\lambda(y) := \frac{1}{2}y^2 - \lambda y$, $y \in \mathbb{R}$, verify that
   $$V_M(x) \geq \sup_{\lambda\geq 0}\big\{\lambda M + G_\lambda(x)\big\}, \quad \text{where} \quad G_\lambda(x) := \inf_{\tau\in\mathcal{T}} \mathbb{E}\big[g_\lambda(X_\tau^x)\big].$$
   (c) Let $\hat{\lambda} \geq 0$ and $\hat{\tau} \in \mathcal{T}$ be such that
   $$G_{\hat{\lambda}}(x) = \mathbb{E}\big[g_{\hat{\lambda}}(X_{\hat{\tau}}^x)\big] \quad \text{and} \quad \mathbb{E}\big[X_{\hat{\tau}}^x\big] = M.$$
   Show that $\hat{\tau}$ is an optimal stopping policy for the problem V_M.

In the rest of the problem, we solve the problem V_M by following the criterion of Question 2c.

3. We first derive an explicit solution for the problem G_λ introduced in Question 2b, $\lambda \geq 0$. Denote

$$H_a^x := \inf\{t \geq 0 : X_t^x = a\}, \quad \text{for all} \quad a \geq 0.$$

   (a) In the present setting (1), verify that $\lim_{t\to\infty} X_t^x = 0$, a.s. and, for $x \geq \lambda$, show that $H_\lambda^x \in \mathcal{T}$.

   (b) By exploring the maximizer of the function g_λ, deduce that, for all $x \geq \lambda$, H_λ^x is an optimal stopping policy for $G_\lambda(x)$, and derive the corresponding value of $G_\lambda(x)$.

   (c) Justify **formally** (i.e. without detailed proof) that the dynamic programming equation corresponding to the problem G_λ, for all $\lambda \geq 0$, is

$$\max\{v - g_\lambda\,,\; -\gamma v' - x v''\} = 0, \quad \text{on} \quad \mathbb{R}_+, \tag{2}$$

   where γ is defined in (1).

   (d) For $\lambda > 0$, find a scalar $b < \lambda$ together with a function $v \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{b\})$ such that

$$v \;\text{ is a solution of (2)}, \quad v(0) = 0, \quad v = G_\lambda \;\text{ on }\; [\lambda, \infty),$$
$$v = g_\lambda \;\text{ on }\; [b, \lambda], \;\text{ and }\; v < g_\lambda \;\text{ on }\; (0, b].$$

   (e) Justify that $\hat\tau_\lambda^x := \mathbf{1}_{\{x \geq \lambda\}} H_\lambda^x + \mathbf{1}_{\{x < b\}} H_b^x \in \mathcal{T}$ for all $x > 0$.

   (f) We recall that Itô's formula holds for the function v, although it is not C^2 at the point b. By using the verification argument for optimal stopping problems, show that $v = G_\lambda$ with corresponding optimal stopping policy $\hat\tau_\lambda^x$.

4. In order to apply the optimality criterion of Question 2c, we now seek for a value $\hat\lambda > 0$ such that $\mathbb{E}\big[X_{\hat\tau_{\hat\lambda}^x}^x\big] = M$.

   (a) Using the identity $\mathbb{P}\big[\sup_{t\geq 0}\{W_t - \alpha t\} \geq \beta\big] = e^{-2\alpha\beta}$ for all $\alpha, \beta > 0$, show that $\mathbb{P}[H_b^x < \infty] = \big(\frac{x}{b}\big)^{1-\gamma}$ for all $x \in (0, b)$.

   (b) Deduce that

$$\mathbb{E}\big[X_{\hat\tau_\lambda^x}^x\big] = \mathbf{1}_{\{\lambda < (1+\frac{1}{\gamma})\frac{x}{2}\}}(\lambda \wedge x) + \mathbf{1}_{\{\lambda \geq (1+\frac{1}{\gamma})\frac{x}{2}\}} \left(\frac{\lambda}{(1+\frac{1}{\gamma})\frac{x}{2}}\right)^{\gamma} x.$$

   (c) Find $\hat\lambda > 0$ such that $\mathbb{E}\big[X_{\hat\tau_\lambda^x}^x\big] = M$, and verify that

$$\hat\tau := \hat\tau_{\hat\lambda}^x = \mathbf{1}_{\{x \geq M\}} H_M^x + \mathbf{1}_{\{x < M\}} H_{M_x}^x \quad \text{with} \quad M_x := x\Big(\frac{M}{x}\Big)^{\frac{1}{\gamma}}.$$

   (d) Deduce from Question 2c an explicit expression for $V_M(x)$ for all $x > 0$.

   (e) Compute $V(x)$ for all $x > 0$ in terms of the function the possible root of the equation $1 + M - \frac{x}{2}\big(1 + \frac{1}{\gamma}\big)\big(\frac{M}{x}\big)^{\frac{1}{\gamma}} = 0$.

Exercise

$(\rho = 0)$

(4) $V(t,x,y) = \inf_{\vartheta \in \mathcal{V}} \mathbb{E}_{t,x,y}\left[e^{-\rho(T-t)} g(X_T^\vartheta)\right]$ $t \leq T$ $x, y \in \mathbb{R}$ 1

$\rho \in \mathbb{R}$, $\mathcal{U}$ ensemble de proc. progressifs avec la valeur

dans $[-1,1]$ $\vartheta_t \in [-1,1]$

$$\begin{cases} dX_s^\vartheta = Y_s^\vartheta dW_s \\ dY_s^\vartheta = X_s^\vartheta ds + \vartheta_s dW_s \end{cases} \quad s \in [t,1]$$

Equation de PD?

$$\begin{cases} \inf_{\vartheta \in [-1,1]} \left[\partial_t v + \partial_x v \cdot 0 + \partial_y v \cdot x + \frac{1}{2}\partial_{xx} v \cdot y^2 + \frac{1}{2}\partial_{yy} v \cdot \vartheta^2 + \partial_{xy}\vartheta \cdot y\vartheta - \rho v\right] = 0 \\ V(T,x,y) = g(x) \end{cases}$$

$$\begin{cases} \partial_t v + x\partial_y v + \frac{1}{2}y^2 \partial_{xx} v + \inf_{\vartheta \in [-1,1]} \left\{\frac{1}{2}\vartheta^2 \partial_{yy} v + \vartheta \cdot y\, \partial_{xy} v\right\} - \rho v = 0 \\ V(T,x,y) = g(x) \end{cases}$$

BS:

PPD si V est mesurable alors

Si $\rho = 0$, $V(t,x,y) = \inf_{\vartheta \in \mathcal{U}} \mathbb{E}\left[V(\theta, X_\theta^{t,x,y}, Y_\theta^{t,x,y})\right]$ $\forall \theta$ t.q.

$X_1^{\theta, X_\theta^{\dots}, Y_\theta^{\dots}}$ $1 \geq \theta$

$$\mathbb{E}\left[e^{-\rho(T-t)} g(X_1^{t,x,y})\right] = \mathbb{E}\left[e^{-\rho(\theta-t)} \underbrace{\mathbb{E}\left[e^{-\rho(T-\theta)} g(X_s^{t,x,y}) \middle| \mathcal{F}_\theta\right]}_{\geq\, =\, V(\theta, X_\theta^{t,x,y}, Y_\theta^{t,x,y})}\right]$$

$$\Rightarrow V(t,x,y) = \inf_{\vartheta \in \mathcal{V}} \mathbb{E}\left[e^{-\rho(\theta-t)} V(\theta, X_\theta^{t,x,y}, Y_\theta^{t,x,y})\right]$$

Soit $\vartheta \in \mathcal{U}$. Si $X \in C^{1,2}$, on applique Itô sur $[t,1]$ pour

$$d\left(e^{-\rho s} V(s, X_s^{t,x,y}, Y_s^{t,x,y})\right) = \left(-\rho e^{-\rho s} V(s, X_s^{t,x,y}, Y_s^{t,x,y}) + e^{-\rho s}\left(\partial_t v + \mathcal{L} v\right)\right) dt +$$

dépend de ϑ

$+\ dM_t$ où M martingale locale.

$$e^{-\rho\theta} V(\theta, X_\theta^{t,x,y}, Y_\theta^{t,x,y}) = e^{-\rho t} V(t,x,y) + \int_t^\theta \Big(-\rho e^{-\rho s} V(s, X_s^{t,x,y}, Y_s^{t,x,y}) + e^{-\rho s} \partial_t v(s, X_s^{t,x,y}, Y_s^{t,x,y})$$

$$+ e^{-\rho s} \mathcal{L} V(s, X_s^{t,x,y}, Y_s^{t,x,y}, \vartheta_s)\Big) ds + (M_\theta - M_t)$$

$$\overset{\mathbb{E}}{\Rightarrow} e^{-\rho t} \cancel{V(t,x,y)} \le \mathbb{E}\Big[e^{-\rho t} \cancel{V(t,x,y)} + \int_t^\theta -\rho e^{-\rho s} V(s, X_s^{t,x,y}, Y_s^{t,x,y}) +$$

$$+ e^{-\rho s} \partial_t V(s, X_s^{t,x,y}, Y_s^{t,x,y}) + e^{-\rho s} \mathcal{L} v(s, X_s^{t,x,y,\vartheta}, Y_s^{t,x,y,\vartheta}, \vartheta_s) ds + \underbrace{M_\theta - M_t}\Big]$$

on localise et on annule l'espérance

$\bullet\ \frac{1}{\theta - t}$, $\theta \downarrow t \Rightarrow$

$$0 \le -e^{-\rho t} \rho V(t,x,y) + e^{-\rho t} \partial_t V(t,x,y) + e^{-\rho t} \mathcal{L} v(t,x,y,u)$$

$$\Downarrow$$

$$0 \le \inf_{u \in [-1,1]} \left\{-\rho v + \partial_t v + \mathcal{L} v\right\} \quad \forall u \in \mathcal{U}$$

(2) L'équation de PD associé au problème

$$V(t,x) = \sup_{\vartheta \in \mathcal{U}} \mathbb{E}\left[\int_t^T \frac{1}{2} \vartheta_s^2 ds\right]$$

$\mathcal{U}$ mesurable aux valeurs dans $\mathbb{R}$

$$dX_s^\vartheta = \vartheta_s dW_s, \quad s \in [t,T], \quad X_t^{t,x,\vartheta} = x$$

$\partial_t V + \mathcal{L} v + f$

$$\begin{cases} \sup_{\vartheta \in \mathbb{R}} \left\{\partial_t V(t,x) + 0 \cdot \partial_x V(t,x) + \frac{1}{2}\vartheta^2 \partial_{xx} V(t,x) + 0 \cdot V(t,x) + \frac{1}{2} u^2\right\} = 0 \\ V(T,x) = 0 \end{cases}$$

$$\begin{cases} \partial_t V(t,x) + \sup\limits_{\vartheta\in\mathbb{R}} \left\{ \frac{1}{2}\vartheta^2 \left(1+\partial_{xx}V(t,x)\right)\right\} = 0 \\ V(T,x) = 0 \end{cases}$$

PPD V mesurable, $V(t,x) = \sup\limits_{\vartheta\in\mathcal{U}} \mathbb{E}\left[\int_t^\theta \frac{1}{2}\vartheta_s^2 ds + V(\theta, X_\theta^{t,x,\vartheta}) \right]$

(4) $\begin{cases} +\partial_t v + \underbrace{v_x^+}_{\text{non-linéarité}} + 2v_{xx} - 3v + x^2 = 0 & \text{sur } [0,T)\times\mathbb{R} \\ v|_{t=T} = 0 \end{cases}$

Quel problème de contrôle stoch?

$$v_x^+ = \sup_{u\in[0,1]} \{u v_x\}$$

$$V(t,x) = \sup_{\substack{\vartheta : \text{ progr.} \\ \text{à val} \\ \text{dans } [0,1]}} \mathbb{E}\left[\int_t^T (X_s^{t,x,\vartheta})^2 e^{-3(s-t)} ds \right]$$

$$\begin{cases} dX_s^{t,x,\vartheta} = \vartheta_s dt + 2dW_t, & s\geq t \\ X_t^{t,x,\vartheta} = x \end{cases}$$

(3) $\begin{cases} 3 - \min\{-\partial_t v - 8\partial_{xx} v + \rho v, \ v - g\} = 0 \\ v(T,x) = g(x) \end{cases}$

$$\begin{cases} v(t,x) = \sup\limits_{\tau\in\mathcal{T}_t} \mathbb{E}\left[e^{-\rho(T-\tau)} g(X_\tau^{t,x}) \right] \\ dX_s^{t,x} = 4dW_s \quad X_t^{t,x} = x \end{cases}$$

où $\mathcal{T}_t$ est l'ens des $\mathbb{F}$-t.a. à valeurs dans $[t,T]$.

PPD v est mesurable

$t \leq \theta \leq \tau$

θ t.a., $\quad v(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\mathbb{1}_{\{\tau > \theta\}} e^{-\rho(\tau - t)} g(X_\tau) + \mathbb{1}_{\{\tau \leq \theta\}} e^{-\rho(\theta-\tau)} v(\theta, X_\theta^{t,x}) \right]$

$$\mathbb{E}\left[e^{-\rho(\tau-t)} g(X_\tau^{t,x})\right] \underset{\theta \text{ t.a.}}{=} \mathbb{E}\Big[\underbrace{\mathbb{1}_{\theta < \tau} e^{-\rho(\tau-t)} g(X_\tau)}_{e^{-\rho(\theta-t)} \mathbb{E}\left[e^{-\rho(\tau-\theta)} g(X_\tau) \mid \mathcal{F}_\theta\right]} + \mathbb{1}_{\theta \geq \tau} e^{-\rho(\tau-t)} g(X_\tau)\Big] =$$

$$= \mathbb{E}\left[\mathbb{1}_{\{\theta \geq \tau\}} e^{-\rho(\tau-t)} g(X_\tau) + \mathbb{1}_{\{\theta < \tau\}} e^{-\rho(\theta-t)} V(\theta, X_\theta)\right]$$

EDP En appliquant Itô à $e^{-\rho s} v(s, X_s^{t,x}) \rightarrow$

$$e^{-\rho s} v(s, X_s^{t,x}) = e^{-\rho t} v(t,x) + \int_t^s \left(\partial_t v + \mathcal{L} \partial_{xx} v - \rho v\right)(r, X_r^{t,x}) dr + M_s$$

$$e^{-\rho t} v(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\mathbb{1}_{\{\tau \leq \theta\}} e^{-\rho(\tau - t)} g(X_\tau^{t,x}) + \mathbb{1}_{\{\tau > \theta\}} e^{-\rho(\theta-t)} V(\theta, X_\theta^{t,x})\right]$$

$$\downarrow$$

$$e^{-\rho t} v(t,x) \geq \mathbb{E}\left[\mathbb{1}_{\{\hat\tau \leq \theta\}} e^{-\rho\hat\tau} g(X_{\hat\tau}^{t,x}) + \mathbb{1}_{\{\hat\tau > \theta\}}\left(e^{-\rho t} v(t,x) + \int_t^\theta (\partial_t v + \mathcal{L}\partial_{xx} v - \rho v) dr + M_\theta\right)\right]$$

$$- \mathbb{1}_{\{\hat\tau \leq \theta\}} v(t,x) e^{-\rho t}$$

$$0 \geq \mathbb{E}\left[\mathbb{1}_{..} e^{-\rho t}(g - v) + \mathbb{1}_{..} (\text{---})\right]$$

Problem: Optimal mean-variance stopping

$\mu \in \mathbb{R}$, $\sigma > 0$ MBG $\quad \begin{cases} dX_t^x = X_t^x(\mu dt + \sigma dW_t), & t \geq 0 \\ X_0^x = x \end{cases}$

$$X_\infty^x = \limsup_{t \to \infty} X_t^x$$

$\mathcal{T} = \{\tau \text{ t.a. à valeurs ds } [0, \infty],\ \mathbb{E}[(X_\tau^1)^2] < \infty\}$

But $\quad V(x) = \sup\limits_{\tau \in T} \Big\{ \overbrace{\mathbb{E}[X_\tau^x] - \frac{1}{2} \underbrace{\mathbb{V}\mathrm{ar}[X_\tau^x]}_{\mathbb{E}[(X_\tau^x)^2 - \mathbb{E}[X_\tau^x]^2]}}^{J(x,\tau)} \Big\}, \quad x > 0$

(1) $J(x,\tau)$ n'est pas une fonction linéaire de $\mathbb{E}\, g(X_\tau^x)$ à cause du terme $\mathbb{E}[X_\tau^x]^2$

(2) $\mu \leq 0$. Vérifier que X^x est sur-martingale et déduire que $V(x) = x$, $\hat{\tau} = 0$

$$X_t^x = x \exp\Big\{\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big\} = e^{\mu t} \cdot \overbrace{S_t}^{\text{martingale}}$$

$$\forall t \geq s \quad \mathbb{E}[X_t^x \mid \mathcal{F}_s] = e^{\mu t} S_s = \underbrace{e^{\mu(t-s)}}_{\leq 1} X_s^x \; \text{-> } X \text{ est surmartingale}$$

$$\text{->}\; \mathbb{E}[X_\tau^x] \leq \mathbb{E}[X_0^x] = x \quad \forall \tau \in T \;\text{->}\; \forall \tau \in T \quad \mathbb{E} X_\tau^x - \frac{1}{2}\mathbb{V}\mathrm{ar}[X_\tau^x] \leq x$$

Si $\tau = 0$ $\;\mathcal{C}(x,\tau) = x$ -> contrôle optimale.

(c) $\mu \geq \frac{\sigma^2}{2}$ $\quad \forall n \geq 1 \quad \tau_n = \inf\{t \geq 0 : X_t^x \geq n\}$

Justifier que $\limsup\limits_{t \to \infty} X_t^x = \infty$ a.s. et $\tau_n \in T$

En déduire que $V(x) = \infty$. Est-ce que t.a. optimale $\tau^* \in T$?

$$X_t^x = x \exp\Big\{\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big\}$$

Par la LLI $\quad \limsup\limits_{t \to \infty} \dfrac{W_t}{\sqrt{t \log\log t}} = 1 \;\Rightarrow\; \limsup W_t = +\infty$

Si $\mu > \frac{\sigma^2}{2}$ $\quad \Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W_t \xrightarrow[t \to \infty]{} +\infty$

Si $\mu = \frac{\sigma^2}{2} \to \limsup W_t = +\infty \to \limsup X_t^x = +\infty$

$\limsup X_t^x = +\infty \to \tau_n < +\infty$ p.s. $\forall n \in \mathbb{N}$, $X_{\tau_n}^x = n$

$\mathbb{E}[X_{\tau_n}^x] = n$, $\mathbb{V}ar[X_{\tau_n}^x] = 0 \to J(x, \tau_n) = n \underset{n\to\infty}{\longrightarrow} +\infty \to V(x) = +\infty$

$\tau_n \to +\infty \to \tau^* = +\infty \quad \mathbb{E}(X_\infty^x)^2 = +\infty$

Désormais, on considère le cas $0 < \mu < \frac{\sigma^2}{2}$ et on note $\gamma = \frac{2\mu}{\sigma^2} \in (0,1)$

(2) $M \geq 0 \quad \mathbb{T}_M := \{\tau \in \mathcal{T} : \mathbb{E}[X_\tau^x] = M\}$

(a) Vérifier que $V(x) = \sup\limits_{M \geq 0} \{M + \frac{M^2}{2} - V_M(x)\}$ où $V_M(x) = \inf\limits_{\tau \in \mathbb{T}_M} \frac{1}{2}\mathbb{E}[(X_\tau^x)^2]$

$$\mathcal{T} = \bigsqcup_{M \geq 0} \mathbb{T}_M \to V(x) = \sup_{M \geq 0} \sup_{\tau \in \mathbb{T}_M} \{\mathbb{E}[X_\tau^x] - \frac{1}{2}(\mathbb{E}[(X_\tau^x)^2] - \mathbb{E}[X_\tau^x]^2)\} =$$

$$= \sup_{M \geq 0} \{M + \frac{M^2}{2} - \inf_{\tau \in \mathbb{T}_M} \{\frac{1}{2}\mathbb{E}(X_\tau^x)^2\}\}$$

(b) $g_\lambda(y) = \frac{1}{2}y^2 - \lambda y$, $y \in \mathbb{R}$

Vérifier que $V_M(x) \geq \sup\limits_{\lambda \geq 0} \{\lambda M + \mathcal{C}_\lambda(x)\}$ où $\mathcal{C}_\lambda(x) = \inf\limits_{\tau \in \mathcal{T}} \mathbb{E}[g_\lambda(X_\tau^x)]$

$$\lambda M + \mathcal{C}_\lambda(x) = \lambda M + \inf_{\tau \in \mathcal{T}} \mathbb{E}[\frac{1}{2}(X_\tau^x)^2 - \lambda X_\tau^x] \leq \inf_{\tau \in \mathbb{T}_M} \frac{1}{2}\mathbb{E}[(X_\tau^x)^2]$$

$$\lambda M + \mathcal{C}_\lambda(x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}[\frac{1}{2}(X_\tau^x)^2 - \lambda(X_\tau^x - M)] \leq \inf_{\tau \in \mathbb{T}_M} \mathbb{E}[\ldots] = \inf_{\tau \in \mathbb{T}_M} \mathbb{E}[\frac{1}{2}(X_\tau^x)^2] \quad \forall \lambda \geq 0$$

$$\to \sup_{\lambda \geq 0} \{\lambda M + \mathcal{C}_\lambda(x)\} \leq V_M$$

(c) Soit $\hat{\lambda} \geq 0$ et $\hat{\tau} \in \mathcal{T}$ t.q. $\mathcal{C}_{\hat{\lambda}}(x) = \mathbb{E}[g_{\hat{\lambda}}(X_{\hat{\tau}}^x)]$ et $\mathbb{E}[X_{\hat{\tau}}^x] = M$

M.q. $\hat{\tau}$ est un t.a. optimale pour V_M

$$\mathbb{E}[X^x_{\hat\tau}] = m \Rightarrow \hat\tau \in \mathbb{T}_m$$

$$\hat\lambda m + \mathcal{C}_{\hat\lambda}(x) = \cancel{\hat\lambda m} + \mathbb{E}\left[\frac{1}{2}(X^x_{\hat\tau})^2 - \cancel{\hat\lambda X^x_{\hat\tau}}\right] \le \sup_{\lambda \ge 0}\{\dots\} \le V_m$$

Donc $\frac{1}{2}\mathbb{E}(X^x_{\hat\tau})^2 \le \inf_{\tau \in \mathbb{T}_m} \mathbb{E}\left[\frac{1}{2}(X^x_\tau)^2\right]$ et $\hat\tau \in \mathbb{T}_m \Rightarrow \hat\tau$ est optimale

On va chercher $\hat\tau$: $\hat\tau \in \mathbb{T}_m$ est $\mathcal{C}_{\hat\lambda}(x) = \mathbb{E}[g_{\hat\lambda}(X^x_{\hat\tau})]$

(3) Solution explicit pour $\mathcal{C}_\lambda(x)$, $\lambda \ge 0$.

$$H^x_a := \inf\{t \ge 0 : X^x_t = a\} \quad \forall a \ge 0$$

(a) Vérifier que $\lim_{t \to \infty} X^x_t = 0$. Pour $x \ge \lambda$ m.q. $H^x_\lambda \in \mathcal{T}$

$$\log X^x_t = \underbrace{\left(\mu - \frac{\sigma^2}{2}\right)}_{<0} t + \sigma W_t \to -\infty \text{ par LLI}$$

$$\left(\limsup \underbrace{\left(\mu - \frac{\sigma^2}{2}\right)}_{<0} \underbrace{\frac{t}{\sqrt{t \log\log t}}}_{\to \infty} + \overset{\sigma}{\overset{\shortparallel}{\frac{\sigma W_t}{\sqrt{t \log\log t}}}} = -\infty\right)$$

$\Rightarrow X^x_t \to 0$ p.s.

$x \ge \lambda$, $X^x_0 = x$ $\quad X^x_t \xrightarrow[t \to \infty]{} 0 \Rightarrow H^x_\lambda < \infty$ p.s., $\mathbb{E}[(X^x_{H^x_\lambda})^2] = \lambda^2 < \infty \Rightarrow H^x_\lambda \in \mathcal{T}$

(b) Déduire que $\forall x \ge \lambda$ H^x_λ est t.a. optimal pour $\mathcal{C}_\lambda(x)$. Dériver la valeur de $\mathcal{C}_\lambda(x)$.

$g(\lambda) = -\frac{\lambda^2}{2}$

$$\mathcal{C}_\lambda(x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}[g_\lambda(X^x_\tau)] \Rightarrow \mathcal{C}_\lambda(x) \ge -\frac{\lambda^2}{2}$$

$$g_\lambda : x \mapsto \frac{1}{2}x^2 - \lambda x \quad \forall x \quad g_\lambda(x) \ge g(\lambda) = -\frac{\lambda^2}{2}$$

Pour $\tau = H^x_\lambda$ $\quad \mathbb{E}[g_\lambda(X^x_{H^x_\lambda})] = -\frac{\lambda^2}{2} \Rightarrow H^x_\lambda$ t.a. optimale, $\mathcal{C}_\lambda(x) = -\frac{\lambda^2}{2}$

(c) Justifier formalement l'équation $\max\{u - g_\lambda, -\gamma v' - x v''\} = 0$ (pour G_λ)

où $\gamma = \frac{2\mu}{\sigma^2}$

$$v(x) = \inf_{\tau \in T} \mathbb{E}\Big[\underbrace{\tfrac{1}{2}(X_\tau^x)^2 - \lambda X_\tau^x}_{g_\lambda(X_\tau^x)}\Big]$$

Pour θ t.q. $v(x) = \inf_{\tau \in T} \mathbb{E}\left[1\!\!1_{\{\tau \leq \theta\}} g_\lambda(X_\tau^x) + 1\!\!1_{\{\tau > \theta\}} v(X_\theta^x) \right]$

$$v(X_\theta^x) = v(\underset{\overset{\|}{x}}{X_0^x}) + \int_0^\theta \left(v'(X_s^x)\mu X_s^x + \tfrac{1}{2} v''(X_s^x)\sigma^2(X_s^x)\right) ds + \underbrace{\int_0^\theta v'(X_s^x)\sigma X_s^x \, dW_s}_{M_\theta}$$

$$v(x) = \inf_{\tau \in T} \mathbb{E}\Big[1\!\!1_{\{\tau \leq \theta\}} g_\lambda(X_\tau^x) + 1\!\!1_{\{\tau > \theta\}} v(x) + 1\!\!1_{\{\tau > \theta\}} \cdot \int_0^\theta \left(v'(X_s^x)\mu X_s^x + \tfrac{1}{2} v''(X_s^x)\sigma^2(X_s^x)^2\right) ds$$

$$+ 1\!\!1_{\{\tau > \theta\}} M_\theta \Big]$$

$$\inf_{\tau \in T} \mathbb{E}\left\{ 1\!\!1_{\{\tau \leq \theta\}}\left(g_\lambda(X_\tau^x) - v(x)\right) + 1\!\!1_{\{\tau > \theta\}}\left(\int_0^\theta \cdots ds + M_\theta\right)\right\} = 0$$

Si $\tau = 0 \Rightarrow g_\lambda(x) - v(x) \geq 0$

Si $\tau > 0$ on prend $\frac{1}{\theta}\mathbb{E}\left(\int_0^\theta \cdots ds + M_\theta\right) \xrightarrow[\theta \downarrow 0]{} \mu x v'(x) + \frac{1}{2} v''(x)\sigma^2 x^2 \geq 0 \;\Big|\; \cdot \frac{1}{\sigma^2/2} \cdot \frac{1}{x}$

$\Rightarrow \gamma v'(x) + x \cdot v''(x) \geq 0$

Donc $\inf\{g_\lambda - v, \gamma v' + x v''\} = 0 \quad \forall x \geq 0 \quad (2)$

(d) Pour $\lambda > 0$ trouver $b < \lambda$ et $v \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{b\})$ t.q.

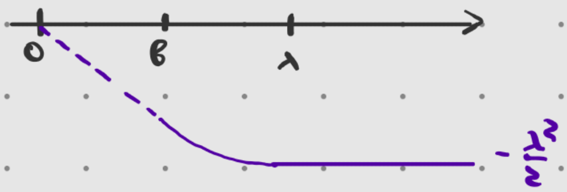
$\min\{g_\lambda - v, \gamma v' + x v''\} = 0$

$v(0) = 0$

$v = G_\lambda$ sur $[\lambda, \infty) = -\frac{\lambda^2}{2}$ $\quad v(\lambda+) = -\frac{\lambda^2}{2}$ $\quad v'(\lambda+) = v''(\lambda+) = 0$ $\quad C^1$ en λ

$v = g_\lambda$ sur $[b, \lambda] = \frac{1}{2}x^2 - \lambda x$ $\quad v(\lambda-) = -\frac{\lambda^2}{2}$ $\quad v'(\lambda-) = \lambda - \lambda = 0$ $\quad v''(\lambda-) = 1$

$v < g_\lambda$ sur $(0, b]$



Sur $(0, b]$ $v < g_\lambda \Rightarrow \gamma v' + x v'' = 0$

$w = v'$

$$\frac{w'}{w} = -\frac{\gamma}{x} \qquad \ln|w| = c - \gamma \ln x$$
$$w = c x^{-\gamma}$$

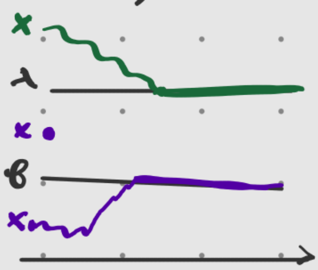
$$v(x) = C + C_2 \cdot b^{\gamma} x^{1-\gamma} \quad \rightsquigarrow \quad v(x) = c x^{1-\gamma}, \; x \in (0,b]$$
$$v(0) = 0$$

$$v(b_+) = \frac{1}{2} b^2 - \lambda b \quad \to \quad c = b^{\gamma}\left(\frac{b}{2} - \lambda\right)$$
$$\shortparallel$$
$$v(b_-) = c\, b^{1-\gamma}$$

$$v(x) = \left(\frac{b}{2} - \lambda\right) b^{\gamma} x^{1-\gamma}, \quad x \in (0,b]$$

$$v'(b_-) = \left(\frac{b-2\lambda}{2}\right)(1-\gamma) = v'(b_+) = b - \lambda$$
$$(b - 2\lambda)(1-\gamma) = 2(b-\lambda)$$
$$b(1+\gamma) = 2\lambda\gamma \Rightarrow \boxed{b = \frac{2\lambda\gamma}{1+\gamma}}$$
$$\Downarrow$$
$$v \in C^1(\mathbb{R}_+)$$

$$v(x) = \begin{cases} \left(\frac{b}{2} - \lambda\right) b^{\gamma} x^{1-\gamma}, & x \in (0,b] \\ \frac{x^2}{2} - \lambda x, & x \in [b,\lambda] \\ -\frac{\lambda^2}{2}, & x > \lambda \end{cases}$$



(e) Justifier que $\hat{\tau}^x_\lambda = \mathbb{1}_{\{x \geq \lambda\}} H^x_\lambda + \mathbb{1}_{\{x < b\}} H^x_b \in \mathcal{T}$ $\quad x > 0$

$$X_{H^x_b} \leq b \to \mathbb{E}(X_{H^x_b})^2 < b^2 < \infty$$
$$x \geq \lambda \Rightarrow H^x_\lambda \in \mathcal{T} \quad \text{par (a)}$$

(f) Formule d'Itô marche pour v qui n'est pas C^2 sur l'ensemble fini.

Vérification : m.q. $v = \mathcal{C}_\lambda$ avec $\hat{\tau}^x_\lambda$ contrôle optimale

$$v(X^x_\theta) = \underset{\substack{\shortparallel \\ x}}{v(X^x_0)} + \int_0^\theta \left(v'(X^x_s)\gamma X^x_s + \frac{1}{2} v''(X^x_s)\sigma^2 (X^x_s)^2 \right) ds + \underbrace{\int_0^\theta v'(X^x_s)\sigma X^x_s \, dW_s}_{M_\theta}$$

$$\mathbb{E} \downarrow 0 \qquad \text{on localise si nécessaire}$$

On veut m.q. $v(x) = \mathcal{C}_\lambda(x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}[g_\lambda(X^x_\tau)]$

$$v(x) = \mathbb{E}\, v(X^x_\tau) - \mathbb{E}\left[\int_0^\tau \overbrace{\frac{\sigma^2}{2} X^x_s \Big(\underbrace{\gamma v'(X^x_s) + X^x_s v''(X^x_s)}_{\geq 0} \Big)}^{\geq 0} ds \right] \leq \mathbb{E}\, g(X^x_\tau)$$

On a $\min\{g - v, \gamma v' + x v''\} = 0$
$\Rightarrow v \leq g_\lambda$

Donc $v(x) \leq \mathcal{C}_\lambda(x)$

Quand on prend $\tau = \hat{\tau}^x_\lambda$ on a

$$x \in [b, \lambda] \to \begin{cases} \hat{\tau}^x_\lambda = 0 \\ v(x) = g_\lambda(x) \end{cases} \Rightarrow v(x) = \mathbb{E}\, g(X_{\hat{\tau}^x_\lambda}) \to v(x) \geq G_\lambda(x) \to \smile$$

$$x > \lambda \to \begin{cases} \hat{\tau}^x_\lambda = H^x_\lambda \\ v = -\frac{\lambda^2}{2} \Rightarrow \gamma v' + x v'' = 0 \end{cases} \Rightarrow v(x) = \mathbb{E}\, v(X_{H^x_\lambda}) = v(\lambda) = g_\lambda(\lambda) = \mathbb{E}[g(X_{H^x_\lambda})] \to v(x) = G_\lambda(x)$$

soit 0, soit b

$v(0) = g_\lambda(0) \quad v(b) = g_\lambda(b)$

$$x < b \Rightarrow \begin{cases} \hat{\tau}^x_\lambda = H^x_b \\ \gamma v' + x v'' = 0 \quad x < b \end{cases} \Rightarrow v(x) = \mathbb{E}[v(X_{H^x_b})] = \mathbb{E}[g_\lambda(X_{H^x_b})] \to v(x) = G_\lambda(x)$$

(4) Pour appliquer 2c on cherche $\hat{\lambda} > 0 : \mathbb{E}[X_{\hat{\tau}^x_{\hat{\lambda}}}] = \mathcal{M}$

(a) En utilisant $\mathbb{P}(\sup_{t \geq 0} \{W_t - \alpha t\} \geq \beta) = e^{-2\alpha\beta} \quad \forall \alpha, \beta > 0$

m.q. $\mathbb{P}(H^x_b < \infty) = \left(\frac{x}{b}\right)^{1-\gamma} \quad \forall x \in (0, b)$

$$\mathbb{P}(H^x_b < \infty) = \mathbb{P}\left(\sup_{t \geq 0} \left\{ x e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \right\} \geq b\right) = \mathbb{P}\left(\sup_{t \geq 0} \left\{ W_t - \underbrace{\frac{(\frac{\sigma^2}{2} - \mu)}{\sigma}}_{\alpha} t \right\} \geq \underbrace{\frac{1}{\sigma} \log \frac{b}{x}}_{\beta}\right) =$$

$$= e^{-2\alpha\beta} = \left(\frac{x}{b}\right)^{\frac{2}{\sigma^2}(\frac{\sigma^2}{2} - \mu)} = \left(\frac{x}{b}\right)^{(1-\gamma)}$$

$X_{\hat{\tau}^x_\lambda}$ si $x > b$

$\parallel$

(b) Déduire que $\mathbb{E}[X_{\hat{\tau}^x_\lambda}] = \mathbb{1}_{\{\lambda < (1 + \frac{1}{\gamma}) \frac{x}{2}\}} (\lambda \wedge x) + \mathbb{1}_{\{\lambda \geq (1 + \frac{1}{\gamma}) \frac{x}{2}\}} \left(\frac{\lambda}{(1 + \frac{1}{\gamma}) \frac{x}{2}}\right)^\gamma x$

$b = \frac{2\lambda\gamma}{1+\gamma} = 2\lambda \left(1 + \frac{1}{\gamma}\right)^{-1}$ $\qquad \overset{\parallel}{\{b < x\}} \qquad \overset{\parallel}{\{b \geq x\}}$

Si $x < b \quad \mathbb{E}[X_{\hat{\tau}^x_\lambda}] = \mathbb{E}[X_{H^x_b}] = \mathbb{P}(H^x_b < \infty) b + 0 \cdot \mathbb{P}(H^x_b = \infty) = \left(\frac{b}{x}\right)^{\gamma - 1} \cdot b = \left(\frac{b}{x}\right)^\gamma \cdot x$

(c) Trouver $\hat{\lambda} > 0$ t.q. $\mathbb{E}[X_{\hat{\tau}^x_{\hat{\lambda}}}] = \mathcal{M}$ et vérifier que

$$\hat{\tau} := \hat{\tau}^x_{\hat{\lambda}} = \mathbb{1}_{\{x \geq \mathcal{M}\}} H^x_{\mathcal{M}} + \mathbb{1}_{\{x < \mathcal{M}\}} H^x_{\mathcal{M}_x} \quad \text{où} \quad \mathcal{M}_x := x \left(\frac{\mathcal{M}}{x}\right)^{\frac{1}{\gamma}}$$

Si $\mu \le x \to \hat{\lambda} = \mu \to \tau^x_{\hat{\lambda}} = H^x_{\mu}$

Si $\mu > x \to \mathbb{E}\left[X^x_{\tau^x_{\hat{\lambda}}}\right] = \left(\frac{\lambda}{(1+\frac{1}{\gamma})\frac{x}{2}}\right)^{\gamma} x = \mu \to \hat{\lambda} = \left(\frac{\mu}{x}\right)^{\frac{1}{\gamma}} \cdot \frac{x}{2}\left(1+\frac{1}{\gamma}\right)$

$$b = \frac{2\hat{\lambda}}{1+\frac{1}{\gamma}} = \underbrace{x\left(\frac{\mu}{x}\right)^{\frac{1}{\gamma}}}_{=\mu_x} > x \xrightarrow{(e)} \tau^x_{\hat{\lambda}} = H^x_{\mu_x}$$

(d) La formule explicite pour $V_\mu(x)$

Par (2c), $V_\mu = \frac{1}{2}\mathbb{E}\left[\left(X^x_{\tau^x_{\hat{\lambda}}}\right)^2\right] = \begin{cases} \frac{1}{2}(\mu \wedge x)^2 & \text{si } \mu \le x \\ \frac{1}{2}\left(\frac{\mu_x}{x}\right)^{\gamma-1} \cdot \mu_x^2 & \text{si } \mu > x \end{cases} = \begin{cases} \frac{1}{2}\mu^2 & \text{si } \mu \le x \\ \frac{x^2}{2}\left(\frac{\mu}{x}\right)^{\frac{\gamma+1}{\gamma}} & \text{si } \mu > x \end{cases}$

$$\frac{1}{2}\left(\frac{\mu_x}{x}\right)^{\gamma-1}\mu_x^2 = \frac{1}{2}\left(\frac{\mu}{x}\right)^{1-\frac{1}{\gamma}} \cdot x^2\left(\frac{\mu}{x}\right)^{\frac{2}{\gamma}} = \frac{x^2}{2}\left(\frac{\mu}{x}\right)^{1+\frac{1}{\gamma}}$$

(e) Calculer $V(x) > 0$ en termes de racines de $1+\mu-\frac{x}{2}\left(1+\frac{1}{\gamma}\right)\left(\frac{\mu}{x}\right)^{\frac{1}{\gamma}} = 0$

Par (2a) $V(x) = \sup_{\mu \ge 0}\left\{\mu + \frac{\mu^2}{2} - V_\mu(x)\right\}$

si $\mu \le x \to V(x) = \mu$

si $\mu > x \to V(x) = \sup_{\mu \ge 0}\left\{\mu + \underbrace{\frac{\mu^2}{2} - \frac{x^2}{2}\left(\frac{\mu}{x}\right)^{\overbrace{1+\frac{1}{\gamma}}^{>2}}}_{\xrightarrow[\mu\to\infty]{} -\infty}\right\} = \sup_{\mu \ge 0} \mathcal{J}(\mu)$

$\mathcal{J}(\mu)$ est concave, $\mathcal{J}(\mu) \xrightarrow[\mu\to\infty]{} -\infty \to \exists!$ min en $\mu^*(x)$:

$$\mathcal{J}'(\mu^*) = 1+\mu^* - \frac{x}{2}\left(1+\frac{1}{\gamma}\right)\left(\frac{\mu^*}{x}\right)^{\frac{1}{\gamma}} = 0 \to V(x) = \mathcal{J}(\mu^*(x)).$$