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Optimization and stochastic control – M2 Probabilités et Finance

Exercice I. We consider a process X defined by the stochastic differential equation

$$\mathrm{d}X_t = (a + bX_t)\mathrm{d}t + \sqrt{\sigma + \theta X_t}\mathrm{d}W_t,$$

with W a scalar Brownian motion, and given scalar parameters a, b, σ, θ. The existence and uniqueness of a square integrable solution $X \in \mathbb{H}^2$ of this equation is admitted.

1. Compute $\mathbb{E}[X_t]$.
2. Provide the stochastic differential equation satisfied by the process $(X_t^n, t \geq 0)$ for any positive integer n.
3. Compute $\mathbb{E}[X_t^2]$ (assuming that all local martingales appearing in the calculation are martingales).
4. Justify that, for $t < T$ and $\lambda \in \mathbb{R}$, $\mathbb{E}\big[e^{-\frac{\lambda^2}{2}X_T}\big|\mathcal{F}_t\big] = \psi(t, X_t)$ for some function ψ. Assuming that ψ is $C^{1,2}$, show that the calculation of $\mathbb{E}\big[e^{-\frac{\lambda^2}{2}X_T}\big]$ can be reduced to a partial differential equation.
5. Show that the calculation of $\mathbb{E}\big[e^{-\frac{\lambda^2}{2}X_T - \mu\int_0^T X_s \mathrm{d}s}\big]$ can be reduced to a partial differential equation.

Exercice II. Let S be the price process of a risky security with dynamics $\frac{\mathrm{d}S_t}{S_t} = (r - \delta)\mathrm{d}t + \sigma \mathrm{d}B_t$, where r is a constant interest rate, δ is a constant dividend rate paid by the security, and B is a Brownian motion (under the risk neutral measure). The price of a European option with payoff $g(S_T)$ is given by

$$V(t, S_t) := \mathbb{E}_t\Big[e^{-r(T-t)}g(S_T)\Big],$$

with $\mathbb{E}_t$ denoting the conditional expectation on the information available at time t.

1. Justify that V is a smooth $C^{1,2}$ function.
2. Provide the partial differential equation satisfied by V.

Exercice III. Let S be the price process of a risky security with dynamics $\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma \mathrm{d}B_t$, where r is a constant interest rate, and B is a Brownian motion (under the risk neutral measure). An Asian option is defined by the payoff $(Y_T - K)^+$ with $Y_T := \int_0^T S_u \mathrm{d}u$. Given the spot price S_t and the integral $Y_t := \int_0^t S_u \mathrm{d}u$, the price at time t of the Asian option is given by

$$V(t, S_t, Y_t) := \mathbb{E}_t\Big[e^{-r(T-t)}(Y_T - K)^+\Big],$$

with $\mathbb{E}_t$ denoting the conditional expectation on the information available at time t.

1. Derive the dynamics of the process Y, and justify that the pair process (S, Y) is Markovian.
2. Assuming V is smooth, provide the partial differential equation satisfied by V.
3. We introduce the guess that $V(t, s, y) = s\phi(t, z(s, y))$ with $z(s, y) = \frac{y-k}{s}$. By plugging this form in the PDE satisfied by V, show that ϕ satisfies a PDE in the variables (t, z).
4. Provide a stochastic representation for the function ϕ.

Exercice IV. The price of a $T-$maturity zero-coupon bond is $P(t, r_t) := \mathbb{E}_t\big[e^{-\int_t^T r_u \mathrm{d}u}\big]$, where $\mathbb{E}_t$ denotes the conditional expectation on the information available at time t, and the interest rate process r is defined by an Ornstein-Uhlenbeck process driven by the Brownian motion B :

$$\mathrm{d}r_t = k(b - r_t)\mathrm{d}t + \sigma \mathrm{d}B_t, \text{ with given constants } k, b, \sigma.$$

1. Assuming P is smooth, provide the partial differential equation satisfied by P.
2. We introduce the guess that $P(t, r) = e^{A(t)+B(t)r}$, for some deterministic functions A and B. By plugging this form in the PDE satisfied by P, show that A and B can be characterized as solutions of ordinary differential equations.

Exercice 1

X s.t.

$$dX_t = (a + bX_t)dt + \sqrt{\sigma + \theta X_t}\, dW_t$$

(a, b, σ, θ) sont t.q. $\exists! \; X \in \mathbb{H}^2$ i.e. $\mathbb{E}\left[\int_0^T X_t^2 dt\right] < \infty$

1) $\mathbb{E}X_t = ?$ a-t-on $\mathbb{E}|X_t| < \infty$?

$$\mathbb{E}|X_t| = \mathbb{E}\Big[\Big|\underbrace{X_0}_{<+\infty} + \underbrace{\int_0^t (a+bX_s)ds}_{①} + \underbrace{\int_0^t \sqrt{\sigma + \theta X_s}\, dW_s}_{②}\Big|\Big]$$

$$① \leq \mathbb{E}\left[\int_0^T |a| + |bX_s|\, ds\right] \leq |a|T + |b|\,\mathbb{E}\left[\int_0^T X_s^2 ds\right]^{1/2} < +\infty \text{ car } X \in \mathbb{H}^2$$

$$② \leq \mathbb{E}\left[\left|\int_0^t \sqrt{\sigma+\theta X_s}\, dW_s\right|^2\right]^{1/2} \leq \mathbb{E}\left[\int_0^t \sigma + \theta X_s\, ds\right]^{1/2} \leq \sqrt{\sigma T} + |\theta|\,\mathbb{E}\left[\int_0^T |X_s|^2 ds\right]^{1/2} < +\infty$$

On a bien que $\mathbb{E}|X_t| < \infty$. Au fait on a $\mathbb{E}\left[\sup_{t \leq T} |X_t|^2\right] < +\infty$

$$\underbrace{\mathbb{E}X_t}_{m_t} = \mathbb{E}\left[X_0 + \int_0^t a + bX_s\, ds + \underbrace{\int_0^t \sqrt{\sigma + \theta X_s}\, dW_s}_{\text{une mart.}}\right] = X_0 + at + b\int_0^t \mathbb{E}X_s\, ds$$

$$\begin{cases} m_0 = X_0 \\ \dot{m}_t = a + b m_t \end{cases}$$

$m_t = c(t)e^{bt}$ $\quad c(0) = X_0$

$\dot{c}(t)e^{bt} = a \Rightarrow \dot{c}(t) = ae^{-bt}$

$c(t) = X_0 + \int_0^t ae^{-bt}dt = X_0 + \frac{a}{b}(1 - e^{-bt})$

$$m_t = X_0 e^{bt} + \frac{a}{b}(e^{bt} - 1) = \left(X_0 + \frac{a}{b}\right)e^{bt} - \frac{a}{b}$$

2) EDS pour $(X_t^n)_{t \in [0,T]}$, $n \in \mathbb{N}^*$

$$dX_t^n = nX_t^{n-1} \underbrace{dX_t}_{(a+bX_t)dt + \sqrt{\sigma+\theta X_t}\, dW_t} + \frac{1}{2}n(n-1)X_t^{n-2} \underbrace{d\langle X\rangle_t}_{(\sigma+\theta X_t)dt} =$$

$$= \left(a n X_t^{n-1} + b_n X_t^n + \frac{\sigma}{2} n(n-1) X_t^{n-2} + \frac{\theta}{2} n(n-1) X_t^{n-1}\right) dt + n X_t^{n-1} \sqrt{\sigma + \theta X_t}\, dW_t$$

3) Calculer $\mathbb{E} X_t^2$

$$dX_t^2 = \left(\sigma + (2a+\theta) X_t + 2b X_t^2\right) dt + 2 X_t \sqrt{\sigma + \theta X_t}\, dW_t$$

pas de problème car positive (peut-être infinie) — on admet que c'est une vraie mart.

$$\underbrace{\mathbb{E} X_t^2}_{f(t)} = \int_0^t \left(\sigma + (2a+\theta) \underbrace{\mathbb{E} X_s}_{m(s)} + 2b \underbrace{\mathbb{E} X_s^2}_{f(s)}\right) ds + X_0^2 \quad 0$$

Pourquoi elle est finie?

$$\begin{cases} \dot f(t) = \sigma + (2a+\theta) m(t) + 2b f(t) \\ f(0) = X_0^2 \end{cases}$$

$$f(t) = c(t) e^{2bt}$$

$$\begin{cases} \dot c\, e^{2bt} = \sigma + (2a+\theta) m(t) \\ c(0) = X_0^2 \end{cases} \Rightarrow c(t) = X_0^2 + \int_0^t e^{-2bs} \left(\sigma + (2a+\theta) m(s)\right) ds$$

$$f(t) = \left(X_0^2 + \int_0^t e^{-2bs} \left(\sigma + (2a+\theta) m(s)\right) ds\right) e^{2bt}$$

cf. pt (1)

$$f(t) = X_0^2 e^{2bt} + \int_0^t e^{2b(t-s)} \left(\sigma + (2a+\theta) m(s)\right) ds$$

$*$ $$\mathbb{E}\left[\int_0^t |2X_s(a+bX_s) + \sigma + \theta X_s|\, ds\right] \leq \underbrace{\mathbb{E}\left[\int_0^t 2|X_s|\,|a+bX_s|\, ds\right]}_{<} + \overbrace{\mathbb{E}\left[\int_0^t |\sigma + \theta X_s|\, ds\right]}^{<+\infty}$$

$$\left[\mathbb{E} \int_0^t (2a|X_s|)^2 ds\right]^{1/2} + |b|\, \mathbb{E} \int_0^t |X_s|^2 ds < +\infty$$

$*$ $$\mathbb{E}\left[\left|\int_0^t X_s \sqrt{\sigma + \theta X_s}\, dW_s\right|\right] \leq \left(\mathbb{E}\left|\int_0^t X_s \sqrt{\sigma + \theta X_s}\, dW_s\right|^2\right)^{1/2} = \mathbb{E}\left[\int_0^t X_s^2 (\sigma + \theta X_s) ds\right]^{1/2}$$

$$\int_0^t X_s^2 (\sigma + \theta X_s) ds < +\infty$$ à verifier!

$$\mathbb{E} \sup_{s \leq t} X_t^4 \leq C\left(\mathbb{E} X_0^4 + \mathbb{E}\left(\int_0^t |a + b X_s| ds\right)^4 + \mathbb{E}\left[\left(\int_0^t \sqrt{\sigma + \theta X_s}\, dW_s\right)^4\right]\right) \leq$$

$$\leq \tilde{C}\left(c' + |b|^4\, \mathbb{E} \int_0^t X_s^4 ds + \mathbb{E}\left[\int_0^t |\sigma + \theta X_s| ds\right]^2\right) \to \text{Gronwall} \to f(t) \leq f(0) e^{\tilde{C}} < \infty$$

$\sup |X_s|^4$

// On peut aussi utiliser (BDG) $\mathbb{E} \sup_{t \leq T} |M_t|^p \leq C\, \mathbb{E}[\langle M \rangle_t]^{p/2} < \infty$

(3) $\lambda \in \mathbb{R}$ $\quad \mathbb{E}\left[e^{-\frac{\lambda}{2} X_T} \middle| \mathcal{F}_t\right] = \Psi(t, X_t)$ grâce à la propriété de Markov

$$\left(= \mathbb{E}\left[e^{-\frac{\lambda}{2} X_T^{t, X_t^{0,x}}} \middle| \mathcal{F}_t\right] = \mathbb{E}\left[e^{-\frac{1}{2} F(t, X_t^{0,x_0}, (W_r - W_t))} \middle| \mathcal{F}_t\right] = \Psi(t, X_t^{0,x_0})\right)$$

$X_t^{0,x_0} \in \mathcal{F}_t$, $(W_r - W_t)_{t \leq r \leq T}$ $\perp \mathcal{F}_t$

$X_s^{t,x} = F(t, x, (W_r - W_t)_{t \leq r \leq s})$

EDP: $\begin{cases} \mathcal{L}\Psi = 0 \\ \Psi(T,x) = e^{-\frac{\lambda}{2} x} \end{cases}$

$$\begin{cases} \partial_t \Psi + \partial_x \Psi \cdot (a + bx) + \frac{1}{2} \partial^2_{xx} \Psi (\sigma + \theta x) = 0 \\ \Psi(T,x) = e^{-\frac{\lambda}{2} x} \end{cases}$$

On cherche la solution dans la forme $\Psi(t,x) = e^{m(t) + n(t) x}$

$n(T) = -\frac{\lambda}{2}$, $m(T) = 0$

$$(\dot{m} + \dot{n} x)\Psi + n\Psi(a + bx) + \frac{1}{2} n^2 \Psi(\sigma + \theta x) = 0$$

$$\begin{cases} \dot{m} + na + \frac{1}{2} n^2 \sigma = 0 \\ \dot{n} + bn + \frac{1}{2} \theta n^2 = 0 \end{cases} \qquad m(t) = -\int_t^T a\, n(s) + \frac{\sigma}{2} n(s)^2\, ds$$

équation de Riccati

(5) $\mathbb{E}\left[e^{\{-\frac{\lambda^2}{2} X_T - \mu \int_0^T X_s ds\}}\right] \rightsquigarrow$ résoudre EDP

$$\mathbb{E}\left[e^{-\frac{\lambda^2}{2} X_T - \mu \int_0^T X_s ds} \middle| \mathcal{F}_t\right] = \Psi(t, \underbrace{(X_s)_{0 \leq s \leq t}}_{\text{:-(}})$$

$$f(t,x) = \mathbb{E}\left[e^{-\mu \int_t^T X_s^{t,x} ds} \cdot e^{-\frac{\lambda^2}{2} X_T^{t,x}} \middle| \mathcal{F}_t\right]$$

$$e^{-\mu \int_0^t X_s ds} f(t,x) = \mathbb{E}\left[e^{-\int_0^T X_s ds} e^{-\frac{\lambda^2}{2} X_T} \middle| \mathcal{F}_t\right]$$ une mart

$$\begin{cases} \mathcal{L}f = g\,x\,f \\ f(T,x) = e^{-\frac{x^2}{2}x} \end{cases}$$

Exercice 2

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma dB_t \qquad g : \mathbb{R} \to \mathbb{R} \text{ mesurable bornée}$$

$$V(t,S_t) = \mathbb{E}\left[e^{-r(T-t)} g(S_T) \mid \mathcal{F}_t\right]$$

1. $V \in C^{1,2}$?

$$S_T^{t,x} = e^{\ln x + (r-\delta-\frac{1}{2}\sigma^2)(T-t) + \sigma(B_T-B_t)} = e^{Z_t}, \quad Z_t \sim \mathcal{N}(\dots)$$

$$V(t,x) = \mathbb{E}\left[e^{-r(T-t)} g\left(x e^{(r-\delta-\frac{\sigma^2}{2})(T-t) + \sigma\sqrt{T-t}\,N}\right)\right] = \qquad N \sim \mathcal{N}(0,1)$$

$$= e^{-r(T-t)} \int g\big(e^{\overbrace{\ln x + \dots + \sigma\sqrt{T-t}\,y}^{s}}\big) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy = e^{-r(T-t)} \frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}} \int g(e^s) e^{-\frac{(s-\dots)^2}{2\sigma^2(T-t)}} ds$$

On remarque $V \in C^{1,2}([0,T) \times \mathbb{R})$ et $V \in C^0([0,T] \times \mathbb{R})$

(2) FK $\rightsquigarrow$
$$\begin{cases} -rv + \partial_t v + \partial_s v \cdot s(r-\delta) + \frac{1}{2}\partial^2_{ss} v\, s^2\sigma^2 = 0 \\ v|_{t=T} = g(x) \end{cases}$$

Exercice 3

$$\frac{dS_t}{S_t} = r\,dt + \sigma dW_t$$

$$\Big(\underbrace{\int_0^T S_t\,dt}_{Y_T} - K\Big)^+ \qquad V(t,S_t,Y_t) = \mathbb{E}\left[e^{-r(T-t)}(Y_T-K)^+ \mid \mathcal{F}_t\right]$$

$$dY_t = S_t\,dt$$

(1) (S_t, Y_t) est-il markovien? Oui, comme la solution de l'EDS 2-dim

$$d\begin{pmatrix}S_t\\Y_t\end{pmatrix} = b\left(r,\begin{pmatrix}Y_t\\S_t\end{pmatrix}\right)dt + \sigma\left(t,\begin{pmatrix}Y_t\\S_t\end{pmatrix}\right)\begin{pmatrix}dW_t\\dB_t\end{pmatrix}$$

$$b\left(r,\begin{pmatrix}y\\s\end{pmatrix}\right) = \begin{pmatrix}rs\\s\end{pmatrix} \qquad \sigma\left(t,\begin{pmatrix}y\\s\end{pmatrix}\right) = \begin{pmatrix}0 & 0\\ \sigma s & 0\end{pmatrix}$$

(2) $V_t = \mathbb{E}_t\left[\left(\frac{1}{T}Y_T - K\right)^+\right] = v(t,S_t,Y_t) \to \{FK\} \Rightarrow$

$$\begin{cases}\mathcal{L}v = \partial_t v + \partial_s v\cdot rs + \partial_y v\cdot s + \frac{1}{2}\sigma^2 s^2\partial_{ss}v = 0\\ v(T,s,y) = (y-K)^+\end{cases}$$

(3) $v(t,s,y) = s\varphi(t,z(s,y))$ où $z(s,y) = \frac{y-k}{s}$ ⇝ EDP sur (t,z)?

$\partial_s z = -\frac{y-k}{s^2}$

$$\partial_t v = s\partial_t\varphi$$
$$\partial_s v = \varphi + s\partial_z\varphi\cdot\left(\frac{k-y}{s^2}\right) = \varphi - z\partial_z\varphi$$
$$\partial_y v = s\partial_z\varphi\cdot\frac{1}{s} = \partial_z\varphi$$

$$\partial_{ss}v = \partial_z\varphi\underbrace{\left(\frac{k-y}{s^2}\right)}_{-\frac{z}{s}} + \underbrace{\frac{y-k}{s^2}}_{z/s}\partial_z\varphi - z\partial_{zz}\varphi\cdot\overbrace{\left(-\frac{y-k}{s^2}\right)}^{-z/s} = \frac{1}{s}\left(-\cancel{z\partial_z\varphi} + \cancel{z\partial_z\varphi} + z^2\partial_{zz}\varphi\right) = \frac{z^2}{s}\partial_{zz}\varphi$$

EDP : $s\partial_t\varphi + rs(\varphi - z\partial_z\varphi) + s\partial_z\varphi + \frac{1}{2}\sigma^2 s z^2\partial_{zz}\varphi = 0$

$\varphi(T,z) = \frac{v(T,s,y)}{s} = \left(\frac{y-K}{s}\right)^+ = \frac{1}{T}\left(\underbrace{\frac{y-k}{s}}_{z}\right)^+$

on pose $k = KT$

$$\begin{cases}\partial_t\varphi + r\varphi + \underbrace{(1-rz)}_{b^z(z)}\partial_z\varphi + \frac{1}{2}\underbrace{\sigma^2 z^2}_{(\sigma^z)^2(z)}\partial_{zz}\varphi = 0\\ \varphi(T,z) = \frac{1}{T}z^+ = h(z)\end{cases}$$

(4) Représentation stochastique de φ?

Par FK, $\varphi(t,z_t) = \mathbb{E}_t\left[e^{r(T-t)}\cdot z_T^+\right]$ où Z_t est la solution de l'EDS $dZ_t = (1 - rZ_t)dt + \sigma Z_t dW_t$

Exercice 4

$$P(t, r_t) = \mathbb{E}_t\left[e^{-\int_t^T r_u du}\right]$$

Vasiček : $dr_t = k(b - r_t)dt + \sigma dB_t$

(1) On suppose que P est régulière. EDP?

$g = 1 \quad f = 0$

Par la formule de FK,

$$\begin{cases} \mathcal{L}P(t,r) = rP(t,r) \\ P(T,r) = 1 \end{cases} \quad \text{où } \mathcal{L}f(t,r) = \left(\partial_t f + \partial_r f \cdot k(b-r) + \frac{1}{2}\partial_{rr} f \cdot \sigma^2\right)(t,r)$$

(2) On cherche $P(t,r) = e^{A(t) + B(t)r}$.

$$\partial_t P = P(\dot{A} + \dot{B}r)$$
$$\partial_r P = B \cdot P \quad \partial_{rr} P = B^2 P$$

$$\text{EDP: } \begin{cases} P\left(\dot{A} + \dot{B}r + k(b-r)\cdot B + \frac{1}{2}B^2\sigma^2\right) = rP \\ A(T) = B(T) = 0 \end{cases}$$

$$\begin{cases} \dot{A} = -k \cdot b\, B(t) - \frac{\sigma^2}{2}B^2(t) \\ A(0) = 0 \end{cases} \qquad \begin{cases} \dot{B} = kB(t) + 1 \\ B(0) = 0 \end{cases}$$

$$B(t) = \frac{e^{kt} - 1}{k}$$
$$A(t) = \int_0^t b(1 - e^{ks})ds - \frac{\sigma^2}{2k^2}\int_0^t (e^{2ks} - 2e^{ks} + 1)ds = bt + \frac{b}{k}(1 - e^{kt}) -$$

$$- \frac{\sigma^2}{2k^2}\left(\frac{e^{2kt}-1}{2k} - 2\frac{e^{kt}-1}{k} + t \right)$$