3 h

Polycopié, notes de cours, livres, téléphones mobiles et montres connectées non autorisés
Handouts, course notes, books, mobile phones and smart watches not allowed

Exercise I (Correlation, regression and et variance reduction). Let X and X' be two square integrable random variables defined on a probbaility space $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying

$$\mathbb{E}\,X = \mathbb{E}\,X' = m \text{ (quantity of interest)} \quad \text{and} \quad \mathrm{Var}(X - X') > 0.$$

PRELIMINARY QUESTIONS. **a.** For every $\lambda \in \mathbb{R}$, we set $X_\lambda = (1-\lambda)X + \lambda X' = X - \lambda(X - X')$. Determine λ such that $\mathrm{Var}(X_\lambda)$ is minimal and explain briefly how exploit $\lambda_{\min}$ to reduce the variance of the computation by simulation of the quantity m. One practical method to be implemented should be described.

We assume in what follows that we have access to an accurate enough estimate for $\lambda_{\min}$.

b. Prove that if $\kappa_{(X,X')} \simeq \kappa_X$ (where κ_Y denotes the complexity of the simulation of a random variable Y) then such a method always outperforms any other to estimate m with a prescribed budget B for the simulation. Give a classical example classique in Finance of derivative products where such an assumption is satisfied.

c. Assume $\mathrm{Var}(X) = \mathrm{Var}(X')$. Prove that $\lambda_{\min} = \frac{1}{2}$. Deuce that if $\kappa_X \simeq \kappa_{X'}$ and $\kappa_{(X,X')} \simeq \kappa_X + \kappa_{X'}$, the method can improve the global performance of the simulation only if $\mathrm{Cov}(X, X') < 0$. Give a classical example of such a situation.

d. Let $Z : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ be a random variable with distribution $\mathcal{N}(0, 1)$. Prove that for any continuously differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that $\lim_{|z| \to +\infty} e^{-\frac{z^2}{2}} g(z) = 0$

$$\mathbb{E}\,g'(Z) = \mathbb{E}\,[g(Z)Z].$$

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function with (at most) exponential growth in the sense that

$$\exists C = C_f \text{ such that } \forall x \in \mathbb{R}, \quad |f(x)| \le C e^{C|x|}$$

and let $(Z, \tilde{Z}) : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^2$ be a random vector with distribution $\mathcal{N}(0, I_2)$.

We set, for every $\rho \in [-1, 1]$, $\tilde{\rho} = \sqrt{1 - \rho^2}$ and

$$X_{\lambda,\rho} = (1-\lambda) f(Z) + \lambda f(\rho Z + \tilde{\rho}\tilde{Z}) \quad \text{and} \quad \varphi(\rho) = \mathbb{E}\big[f(Z) f(\rho Z + \tilde{\rho}\tilde{Z})\big].$$

Prove that ρ being fixed, $\mathbb{E}\,[(1-\lambda)f(Z) + \lambda f(\rho Z + \tilde{\rho}\tilde{Z})] = \mathbb{E}\,f(Z)$ and that

$$\begin{aligned} V(\rho) := \min_{\lambda \in \mathbb{R}} \mathrm{Var}(X_{\lambda,\rho}) = \mathrm{Var}(X_{1/2,\rho}) &= \mathrm{Var}\Big(\frac{f(Z) + f(\rho Z + \tilde{\rho}\tilde{Z})}{2}\Big) \\ &= \frac{1}{2}\Big(\mathrm{Var}(f(Z)) + \mathrm{Cov}\big(f(Z), f(\rho Z + \tilde{\rho}\tilde{Z})\big)\Big) \\ &= \frac{1}{2}\Big(\mathrm{Var}(f(Z)) + \varphi(\rho) - (\mathbb{E}\,f(Z))^2\Big). \end{aligned}$$

2.a. Prove that, if f is differentiable and f' has an exponential growth (see above), then

$$\forall \rho \in]1, 1[, \quad \varphi'(\rho) = \mathbb{E}\big[f(Z) f'(\rho Z + \tilde{\rho}\tilde{Z}) Z\big] - \frac{\rho}{\tilde{\rho}} \mathbb{E}\big[f(Z) f'(\rho Z + \tilde{\rho}\tilde{Z})\tilde{Z}\big].$$

2.b. Assume that f is twice (continuously) differentiable with f' et f'' having an exponential growth. Prove using an integration by parts on the state space $\mathbb{R}$ that, for all $\tilde{a} \in \mathbb{R}$,

$$\mathbb{E}\big[f(Z) f'(\rho Z + \tilde{a}) Z\big] = \mathbb{E}\big[f'(Z) f'(\rho Z + \tilde{a})\big] + \rho\,\mathbb{E}\big[f(Z) f''(\rho Z + \tilde{a})\big].$$

Deduce that, for every $\rho \in [-1, 1]$,

$$\mathbb{E}\big[f(Z) f'(\rho Z + \tilde{\rho}\tilde{Z}) Z\big] = \mathbb{E}\big[f'(Z) f'(\rho Z + \tilde{\rho}\tilde{Z})\big] + \rho\,\mathbb{E}\big[f(Z) f''(\rho Z + \tilde{\rho}\tilde{Z})\big].$$

3.a. The function f being still like in **2.b.** show using one of the preliminary questions that, for every $z \in \mathbb{R}$,

$$\forall \rho \in]-1, 1[, \quad \mathbb{E} f''(\rho z + \tilde{\rho}\tilde{Z}) = \frac{1}{\tilde{\rho}} \mathbb{E}\big[f'(\rho z + \tilde{\rho}\tilde{Z})\tilde{Z}\big]$$

and then that

$$\forall \rho \in]-1, 1[, \quad \mathbb{E}\,f(Z) f''(\rho Z + \tilde{\rho}\tilde{Z}) = \frac{1}{\tilde{\rho}} \mathbb{E}\big[f(Z) f'(\rho Z + \tilde{\rho}\tilde{Z})\tilde{Z}\big].$$

3.b. Conclude that the following identity holds true (treat the case $\rho = \pm 1$ apart)

$$\forall \rho \in [-1, 1], \quad \varphi'(\rho) = \mathbb{E}\big[f'(Z) f'(\rho Z + \tilde{\rho}\tilde{Z})\big].$$

We will admit that the above formula is still valid as soon as f is continuously differentiable, with an at most exponential growth as well as f'.

4. Show that if f is (strictly) monotonic

$$V(-1) = \frac{1}{2}\Big(\mathrm{Var}(f(Z)) + \mathrm{Cov}\big(f(Z), f(-Z)\big)\Big) = \inf_{\rho \in [-1,1]} V(\rho) < V(1) = \mathrm{Var}(f(Z)).$$

Which condition do we retrieve in this framework concerning the interest of the variance reduction method associated to the computation of m if we assume $\kappa_{(f(Z), f(-Z))} \simeq 2\kappa_{f(Z)}$?

5.a. We assume that f is even (symmetric) and such that $\mathbb{E}\,f'(Z)^2 > 0$. Prove that $\varphi'(-1) < 0$. What can we deduce ? Show that, in fact under the hypothesis, φ is even as well and minimal at 0. What does this correspond to in terms of simulation?

5.b. Propose (at least heuristically) a condition on f ensuring the strict convexity of V. With which consequences ?

5.c. Propose in that case a method based on simulation (i.e. stochastic) to determine (at least roughly) $\mathrm{argmin}_{[-1,1]} V = \mathrm{argmin}_{[-1,1]} \varphi$. [No proof of convergence is asked.]

$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t$

$\bar{X}_{t_k}^{n,x}, \tilde{X}_t^{n,x}, \bar{X}_t^{n,x}$

$(1+h)(b(X_t)-b(X_{t_k}))$

Exercise II (One step Euler scheme) We consider a Stochastic Differential Equation (SDE)

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t, \quad X_0 = x \in \mathbb{R},$$

where $b, \sigma : \mathbb{R} \to \mathbb{R}$ are two Lipschitz continuous functions with respective Lipschitz ratios $[b]_{\mathrm{Lip}}$ and $[\sigma]_{\mathrm{Lip}}$ and $(W_t)_{t\geq 0}$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote par $(X_t^x)_{t\geq 0}$ the unique $(\mathcal{F}_t^W)_{t\geq 0}$-adapted solution to the above SDE. The notations are those of the "polycopié".

We denote by $T > 0$ a terminal time (or maturity). For every integer $n \geq 1$, we define step $h = h_n = \frac{T}{n}$. We set $t_k = t_k^n = \frac{kT}{n}$, $k = 0, \ldots, n$.

1.a. Recall the definitions of the three Euler schemes with step $h = \frac{T}{n}$: the discrete time, the stepwise constant and the genuine (continuous) Euler schemes, denoted $(\bar{X}_{t_k}^{n,x})_{k=0,\ldots,n}$, $(\tilde{X}_t^{n,x})_{t\in[0,T]}$ and $(\bar{X}_t^{n,x})_{t\in[0,T]}$ respectively.

1.b. Recall the uniform L^p-moment control results for both the diffusion $(X_t^x)_{t\in[0,T]}$ and the Euler above schemes in this setting (deterministic starting value). [No proof requested here.]

1.c. State the L^p-convergence theorems for the above three Euler schemes in an as synthetic way as possible.

The result of 1.b. may be used without proof in what follows, but the aim of the problem being to investigate the strong and weak convergence performances of the Euler scheme on one step, 1.c. is definitely useless for that purpose.

2.a. Prove that, for every $\varepsilon > 0$,

$$\mathbb{E} \sup_{0\leq t\leq h} |X_t^x - \bar{X}_t^{n,x}|^2 \leq \left((1+\varepsilon)[b]_{\mathrm{Lip}}^2 h + 4(1+1/\varepsilon)[\sigma]_{\mathrm{Lip}}^2\right) \int_0^h \mathbb{E}\,|X_t^x - x|^2 dt.$$

2.b. Deduce from what precedes (or by any other honest mean) that

$$\Big\| \sup_{0\leq t\leq h} |X_t^x - \bar{X}_t^{n,x}| \Big\|_2 \leq \left([b]_{\mathrm{Lip}}\sqrt{h} + 2[\sigma]_{\mathrm{Lip}}\right) \left(\int_0^h \mathbb{E}\,|X_t^x - x|^2 dt\right)^{1/2}.$$

3.a. Prove that there exists a real constant $C_{b,\sigma,T} > 0$ only depending on b, σ and T and related to a constant coming out in **1.b.**, such that, for every $t \geq 0$,

$$\Big\| \sup_{0\leq s\leq t} |X_s^x - x| \Big\|_2 \leq C_{b,\sigma,T}\sqrt{t}(1+|x|).$$

3.b. Deduce the existence of another real constant $C'_{b,\sigma,T} > 0$ such that

$$\Big\| \sup_{0\leq t\leq h} |X_t^x - \bar{X}_t^{n,x}| \Big\|_2 \leq C'_{b,\sigma,T} h(1+|x|).$$

What comment does this result may suggest?

4. Let $Z \sim \mathcal{N}(0;1)$ be a normally distributed random variable defined on $(\Omega, \mathcal{A}, \mathbb{P})$, independent of W (hence of $(X_t^x)_{t\in[0,T]}$). Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Borel function (in all what follows).

4. Show that, for every $\varepsilon > 0$, the function f_ε defined on the real line by

$$f_\varepsilon(x) = \mathbb{E}\, f(x + \sqrt{\varepsilon}Z) = \int_{\mathbb{R}} f(\xi) e^{-\frac{(\xi - x)^2}{2\varepsilon}} \frac{d\xi}{\sqrt{2\pi\varepsilon}}$$

is differentiable with a derivative given by

$$f'_\varepsilon(x) = \mathbb{E}\left[\frac{f(x+\sqrt{\varepsilon}Z)Z}{\sqrt{\varepsilon}}\right].$$

Deduce that f_ε is Lipschitz continuous and give an upper-bound of its Lipschitz ratio.

5. We admit Friedman's Theorem which says that if b is C^4, σ is C^5 and bounded, both with bounded existing derivatives and if moreover σ is uniformly elliptic, that is

$$\forall x \in \mathbb{R}, \quad \sigma(x) > \underline{\sigma} > 0,$$

then, for every $t > 0$, the distribution of X_t^x admits a density $p_t(x, y)$ satisfying

$$\forall x, y \in \mathbb{R}, \quad \left|\frac{\partial}{\partial y} p_t(x,y)\right| \leq \frac{\kappa}{\sqrt{t}} \frac{1}{\sqrt{2\pi v_0 t}} e^{-\frac{(y-x)^2}{2v_0 t}}$$

where $\kappa = \kappa_{b,\sigma} > 0$ and $v_0 = v_{0,b,\sigma} > 0$ denote positive real constants.

The above regularity and uniform ellipticity assumptions on b and σ are in force until the end of the problem.

5.a. Prove that, for every $u \in \mathbb{R}$,

$$\left|\mathbb{E}\, f(X_h^x + u) - \mathbb{E}\, f(X_h^x)\right| \leq \frac{\kappa}{\sqrt{h}} \|f\|_{\sup} |u|$$

and deduce in turn that

$$\left|\mathbb{E}\, f_\varepsilon(X_h^x) - \mathbb{E}\, f(X_h^x)\right| \leq \kappa \|f\|_{\sup} \sqrt{\frac{\varepsilon}{h}}.$$

5.b. Show that there exists an $\eta = \eta(x) > 0$ such that

$$\mathbb{E}\, f(\bar{X}_h^{n,x} + u) = \mathbb{E}\, f_\eta(x + u + hb(x))$$

and derive that

$$\left|\mathbb{E}\, f_\varepsilon(\bar{X}_h^{n,x}) - \mathbb{E}\, f(\bar{X}_h^{n,x})\right| \leq \frac{\|f\|_{\sup}}{\sqrt{h}\,\sigma(x)} \sqrt{\varepsilon}.$$

6.a. Deduce from questions **4.** and **5.** that there exists a real constant $C''_{b,\sigma,T}$

$$\left|\mathbb{E}\, f(X_h^x) - \mathbb{E}\, f(\bar{X}_h^{n,x})\right| \leq C''_{b,\sigma,T}\left(\sqrt{\frac{\varepsilon}{h}} + \frac{h}{\sqrt{\varepsilon}}\right).$$

6.b. Derive a (first) estimate of the one step weak error of the Euler scheme.

Comment. This error estimate can be significantly improved to almost reach $\sqrt{h}$.