b) $F_X(x;y) = \int_0^{+\infty} \mathbb{P}(X_1 + X_3 \leq y \cap X_1 \leq x \mid X_1 = x_1) f_{X_1}(x_1)\,dx_1$

$$= \int_0^{+\infty} \mathbb{P}(X_3 \leq y - x_1 \cap x_1 \leq x)\, f_{X_1}(x_1)\,dx_1$$

$$= \int_0^{\min(x;y)} \mathbb{P}(X_3 \leq y - x_1)\, f_{X_1}(x_1)\,dx_1$$

$$= \int_0^{\min(x;y)} (1 - e^{-2(y-x_1)})e^{-x_1}\,dx_1$$

$$= \int_0^{\min(x;y)} e^{-x_1} - e^{-2y}e^{x_1}\,dx_1$$

$$= 1 - e^{-\min(x;y)} - e^{-2y}[e^{\min(x;y)} - 1]$$

$$= 1 - e^{-\min(x;y)} - e^{-2y+\min(x;y)} + e^{-2y}.$$

On peut verifier ce resultat en calculant F_{X_2} :

$F_{X_2}(y) = \lim\limits_{x \to +\infty} F_X(x;y)$

$$= 1 - e^{-y} - e^{-y} + e^{-2y}$$

$$= 1 - 2e^{-y} + e^{-2y} = (1 - e^{-y})^2.$$

c) $F_2(y) = (1 - e^{-y})^2 = v \iff -\ln(1 - \sqrt{v}) = y$

$F_1(x) = 1 - e^{-x} = u \iff -\ln(1 - u) = x$

$\min(-\ln(1-u), -\ln(1-\sqrt{v})) = -\ln(1 - \min(u;v))$ $\quad C(u;v) = C(F(F_1^{-1}(u), F_2^{-1}(v)))$

$$= (1 + e^{2\ln(1-\sqrt{v})}) - (1 - \min(u;\sqrt{v})) + \frac{(1-\sqrt{v})^2}{1 - \min(u;\sqrt{v})}$$

$$= (1 - \sqrt{v})^2 + \min(u;\sqrt{v}) + \frac{(1-\sqrt{v})^2}{1 - \min(u;\sqrt{v})}$$

ERM

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21/10/22

TD 2 : Copules

Exercice n°3 (suite)

$(X_1 ; X_2) \perp\!\!\!\perp (X_1' ; X_2')$

2) b) i) $\tau = \mathbb{P}[\overbrace{(X_1 - X_1')(X_2 - X_2')}^{A} > 0] - \mathbb{P}[(X_1 - X_1')(X_2 - X_2') < 0]$

$= \mathbb{P}[A > 0] - (1 - \mathbb{P}[A > 0]) = 2\mathbb{P}[A > 0] - 1$.

Or $\mathbb{P}[A > 0] = 2\mathbb{P}[\overbrace{X_1 < X_1' ; X_2 < X_2'}^{B}]$

car $A > 0 : \begin{cases} X_1 > X_1' \text{ et } X_2 > X_2' \\ X_1 < X_1' \text{ et } X_2 < X_2' \end{cases}$ et $(X_1 ; X_2) ; (X_1' ; X_2')$ iid

$B = \mathbb{P}[X_1 < X_1' ; X_1 + X_3 < X_1' + X_3']$

$= \int_0^{\infty} \int_0^{x_1'} \mathbb{P}[X_1 + X_3 < X_1' + X_3' \mid X_1 = x_1, X_1' = x_1'] f(x_1) f(x_1') \, dx_1 \, dx_1'$

$= \int_0^{\infty} \int_0^{x_1'} \int_{x_1 - x_1'}^{+\infty} \mathbb{P}[X_3 < x_3' - (x_1 - x_1')] f(x_1) f(x_1') f(x_3') \, dx_3' \, dx_1 \, dx_3'$

$= \int_0^{+\infty} \int_0^{x_1'} \int_{x_1 - x_1'}^{+\infty} (1 - e^{-2(x_3' - (x_1 - x_1'))}) e^{-x_1 - x_1' - 2x_3'} \, dx_3' \, dx_1 \, dx_3'$

$= 5/12 \quad \Rightarrow \quad \tau = 2/3$.

ii)