Annale 2019

Exo 1 :

on suppose $\rho = 1$

a) $P : (\alpha \quad 1-\alpha)'$

Les ventes à découverts sont interdites $\Rightarrow \alpha \in [0,1]$

$$\mu_P = \alpha \mu_1 + (1-\alpha)\mu_2$$

$$\sigma_P^2 = Var(\alpha A_1 + (1-\alpha)A_2)$$
$$= \alpha^2 \sigma_1^2 + (1-\alpha)^2 \sigma_2^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2$$
$$= (\alpha\sigma_1 + (1-\alpha)\sigma_2)^2$$

b) $$\min_{\alpha \in [0,1]} Var(\alpha A_1 + (1-\alpha)A_2) = \min_{\alpha \in [0,1]} \alpha^2\sigma_1^2 + \sigma_2^2 - 2\alpha\sigma_2^2 + \alpha^2\sigma_2^2 + 2\alpha\sigma_1\sigma_2 - 2\alpha^2\sigma_1\sigma_2$$

$$= \min_{\alpha \in [0,1]} \alpha^2 \underbrace{(\sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2)}_{= (\sigma_1 - \sigma_2)^2 > 0} + 2\alpha(\sigma_1\sigma_2 - \sigma_2^2)$$

$$= \min_{\alpha \in [0,1]} a\alpha^2 + b\alpha \qquad \text{avec } a = (\sigma_1 - \sigma_2)^2 > 0$$
$$\text{et } b = 2(\sigma_1\sigma_2 - \sigma_2^2)$$

$\Rightarrow$ le min existe et est atteint en $\alpha^* = \frac{-b}{2a} = \frac{\sigma_2^2 - \sigma_1\sigma_2}{(\sigma_1 - \sigma_2)^2}$

Donc le port. le moins risqué sur ce marché serait $\left(\frac{\sigma_2^2 - \sigma_1\sigma_2}{(\sigma_1-\sigma_2)^2} \quad \frac{\sigma_1^2 - \sigma_1\sigma_2}{(\sigma_1-\sigma_2)^2} \right)$

Exo 2 :

a) $\mu_P = w'\mu$ $\qquad w'1_{\mathbb{R}^n} = 1$

$$\sigma_P^2 = w'\Sigma w$$

b) $$\max_{w \in \mathbb{R}^n} w'\mu - \frac{k}{2} w'\Sigma w$$

c) $$\mathcal{L}(w) = w'\mu - \frac{k}{2} w'\Sigma w$$

$$\nabla_w \mathcal{L}(w) = \mu - k\Sigma w = 0 \Rightarrow w = \frac{1}{k}\Sigma^{-1}\mu$$

De + $\nabla_w(\nabla_w \mathcal{L}(w)) = -k\Sigma < 0 \Rightarrow w \mapsto \mathcal{L}(w)$ est concave

donc le max existe bien

Le port. optimal est donc $w = \frac{1}{k}\Sigma^{-1}\mu$.

$$\Rightarrow \mu_p = \omega'\mu = \frac{1}{k}\mu'\Sigma^{-1}\mu.$$

Exo 3:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1/2 & 0 & -1/2 & 1 & 0 & 0 \end{pmatrix} \qquad q = \begin{pmatrix} 2\% \\ 5\% \\ 3\% \end{pmatrix}$$

$$\Omega = \begin{pmatrix} 7.5\% & & \\ & 0 & \\ & & 1.8\% \end{pmatrix}$$

Exo 4:

Hyp du MEDAF : $\mu_i = r_f + \beta_i(\mu_M - r_f)$

$$\beta_A = \frac{\mu_A - r_f}{\mu_M - r_f} = \frac{0{,}1 - 0{,}1}{0{,}2 - 0{,}1} = 0$$

$$\mu_B = r_f + \beta_B(\mu_M - r_f)$$
$$= 0{,}1 + 1{,}5(0{,}3 - 0{,}1)$$
$$= 0{,}4 = 40\%$$

$$\mu_M^C = \frac{\mu_C - r_f}{\beta_C} + r_f = \frac{0{,}2 - 0{,}1}{2} + 0{,}1$$
$$= 0{,}15$$
$$= 15\%$$

Exo 5:

$$\tilde{R}_A = \beta_{AF} F + \varepsilon_A$$
$$\tilde{R}_B = \beta_{BF} F + \varepsilon_B$$

$$\beta_{iF} = \frac{Cov(\tilde{R}_i; F)}{\sigma_F^2}$$

a) $P : (\alpha \quad 1-\alpha)' \qquad \alpha \in [0,1]$

On cherche α tq $Cov(\alpha \tilde{R}_A + (1-\alpha)\tilde{R}_B ; F) = 0$

$$\Rightarrow \alpha\, Cov(\tilde{R}_A; F) + (1-\alpha) Cov(\tilde{R}_B; F) = 0$$
$$\Rightarrow \alpha\, \beta_{AF}\sigma_F^2 + (1-\alpha)\beta_{BF}\sigma_F^2 = 0$$

$$\Rightarrow \alpha(\beta_{AF} - \beta_{BF}) + \beta_{BF} = 0$$

$$\Rightarrow \alpha = \frac{\beta_{BF}}{\beta_{BF} - \beta_{AF}} \qquad \Rightarrow 1 - \alpha = \frac{-\beta_{AF}}{\beta_{BF} - \beta_{AF}}$$

b) $\min_{\alpha \in [0,1]} Var(\alpha \tilde{R}_A + (1-\alpha)\tilde{R}_B)$

$$= \min_{\alpha \in [0,1]} Var(\alpha(\beta_{AF} F + \varepsilon_A) + (1-\alpha)(\beta_{BF} F + \varepsilon_B))$$

$$= \min_{\alpha \in [0,1]} \alpha^2 \beta_{AF}^2 \sigma_F^2 + \alpha^2 \sigma_{\varepsilon_A}^2 + (1-\alpha)^2 \beta_{BF}^2 \sigma_F^2 + (1-\alpha)^2 \sigma_{\varepsilon_B}^2$$

$$= \min \alpha^2 \big(\underbrace{\beta_{AF}^2 \sigma_F^2 + \sigma_{\varepsilon_A}^2 + \beta_{BF}^2 \sigma_F^2 + \sigma_{\varepsilon_B}^2}_{a > 0} \big) - 2\alpha \left(\beta_{BF}^2 \sigma_F^2 + \sigma_{\varepsilon_B}^2 \right) + \beta_{BF}^2 \sigma_F^2 + \sigma_{\varepsilon_B}^2$$

$$= \min_{\alpha \in [0,1]} a\alpha^2 + b\alpha + c$$

avec $a = \beta_{AF}^2 \sigma_F^2 + \sigma_{\varepsilon_A}^2 + \beta_{BF}^2 \sigma_F^2 + \sigma_{\varepsilon_B}^2 > 0$

$b = -2\left(\beta_{BF}^2 \sigma_F^2 + \sigma_{\varepsilon_B}^2\right)$

$c = \beta_{BF}^2 \sigma_F^2 + \sigma_{\varepsilon_B}^2$

$\Rightarrow$ le min existe et est atteint pour $\alpha^* = \frac{-b}{2a} = \frac{\beta_{BF}^2 \sigma_F^2 + \sigma_{\varepsilon_B}^2}{\beta_{AF}^2 \sigma_F^2 + \sigma_{\varepsilon_A}^2 + \beta_{BF}^2 \sigma_F^2 + \sigma_{\varepsilon_B}^2}$