N2 Practical et theoretical challenges of

Practical:

- In general, non-Markovian -> difficult to simulate

Theorical

- Non-semimart -> cannot apply Itô formula & stochastic calculus.

N3. FBM = centered Gaussian $\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$

2. Problem

$$dP_t = P_t \sqrt{V_t}\, dB_t$$
$$V_t = V_0 + \int_0^t K_H(t-s)\theta\, ds + \int_0^t K_H(t-s)\xi \sqrt{V_s}\, dW_s \qquad V_0, \theta, \xi \geq 0$$
$$K_H(t) = t^{H-1/2} \qquad B = \rho W + \sqrt{1-\rho^2}\, W^\perp$$

1) $\rho < 0$

2) Why $H > 0$?

If $H \leq 0$, we lose integrability of $\int_0^T K_H(t)^2 dT < \infty$ the guarantee the existence and all the theoretical results we had obtained

3) Rough Heston in practice

- Standard Heston → Characteristic function → Fourier pricing
- Volterra Heston: combines possibility to find CF with rough models.

2.1. Reversionary Heston

4) $\varepsilon > 0$ $\quad K_{H,\varepsilon}(t) = (t+\varepsilon)^{H-\frac{1}{2}}, \; t > 0$

$$dP_t^\varepsilon = P_t^\varepsilon \sqrt{V_t^\varepsilon}\, dB_t$$
$$V_t^\varepsilon = V_0 + \int_0^t K_{H,\varepsilon}(t-s)\theta\, ds + \int_0^t K_{H,\varepsilon}(t-s)\xi \sqrt{V_s^\varepsilon}\, dW_s$$

$$U_t^\varepsilon = U_0 + \theta \cdot (K_{H,\varepsilon} * 1)(t) + \xi (K_{H,\varepsilon} * \sqrt{U_\cdot^\varepsilon} dW_\cdot)(t)$$

<u>Prop</u> $K_{H,\varepsilon} \in C^1([0,T]) \Rightarrow U_t^\varepsilon$ est une semimartingale

$$K_{H,\varepsilon}(t) = K_{H,\varepsilon}(0) + (1 * K'_{H,\varepsilon})(t) \qquad K_{H,\varepsilon}(0) = \varepsilon^{H-\frac{1}{2}}$$

$$U_t^\varepsilon = U_0 + \theta K_{H,\varepsilon}(0)(1 * dt)(t) + \theta \cdot 1 * (K'_{H,\varepsilon} * dt)(t) + \xi \cdot K(0) \cdot (1 * \sqrt{U_\cdot^\varepsilon}\, dW_\cdot)(t) +$$
$$+ \xi \cdot 1 * (\underset{\in C \to \in \mathcal{L}^2}{\underbrace{K'}} * \sqrt{U_\cdot^\varepsilon}\, dW_\cdot)(t)$$

$$dU_t^\varepsilon = \theta K_{H,\varepsilon}(0) dt + \xi \cdot \left(\int_0^t K'_{H,\varepsilon}(t-s)\sqrt{U_s^\varepsilon}\, dW_s\right) dt + \theta \left(\int_0^t K'_{H,\varepsilon}(t-s)\, ds\right) dt + \xi K(0) \sqrt{U_t^\varepsilon}\, dW_t$$

We need $K_{H,\varepsilon}(0) < \infty$ V   We need $K'_{H,\varepsilon} \in \mathcal{L}^2$ V

$$K'_{H,\varepsilon} = \left(H - \tfrac{1}{2}\right)(t+\varepsilon)^{H-\frac{3}{2}} \in \mathcal{L}^2$$

5) Assume that L_ε resolvent of the first kind. $K_{H,\varepsilon} * L_\varepsilon = 1$

$$L_\varepsilon(dt) = \frac{\delta_0(dt)}{K_{H,\varepsilon}(0)} + \ell_\varepsilon(t)\, dt$$

locally integrable such that: $t \mapsto (K'_{H,\varepsilon} * L_\varepsilon)(t) \in C^1$

Show that $dU_t^\varepsilon = \left(\varepsilon^{H-\frac{1}{2}}\theta - \left(\frac{1}{2} - H\right)\varepsilon^{-1}(U_t^\varepsilon - U_0) + \int_0^t (K'_{H,\varepsilon} * L_\varepsilon)'(t-s)(U_s^\varepsilon - U_0)\, ds\right) dt +$
$$+ \underbrace{\varepsilon^{H-\frac{1}{2}}}_{K_{H,\varepsilon}(0)} \xi \sqrt{U_t^\varepsilon}\, dW_t$$

By (2), we should show that

$$\xi \int_0^t K'_{H,\varepsilon}(t-s)\sqrt{U_s^\varepsilon}\, dW_s + \theta \int_0^t K'_{H,\varepsilon}(t-s)\, ds \overset{?}{=} -\left(\tfrac{1}{2} - H\right)\varepsilon^{-1}(U_t^\varepsilon - U_0) + \int_0^t (K'_{H,\varepsilon} * L_\varepsilon)'(t-s)(U_s^\varepsilon - U_0)\, ds$$

$$K'_{H,\varepsilon} * L = K'_{H,\varepsilon} * L(0) + 1 * (K'_{H,\varepsilon} * L)' \quad | * K_{H,\varepsilon}$$

$$K'_{H,\varepsilon} * 1 = K'_{H,\varepsilon} * L(0)(1 * K_{H,\varepsilon}) + 1 * (K'_{H,\varepsilon} * L)' * K_{H,\varepsilon}$$

$$K'_{H,\varepsilon} = (K'_{H,\varepsilon} * L)(0) \cdot K_{H,\varepsilon} + (K'_{H,\varepsilon} * L)' * K_{H,\varepsilon}$$

$$\Updownarrow$$

$$K'_{H,\varepsilon} * \theta ds + K'_{H,\varepsilon} * \xi\sqrt{U^\varepsilon}\, dW = (K'_{H,\varepsilon} * L)(0)\underbrace{(K_{H,\varepsilon} * \theta ds + K_{H,\varepsilon} * \xi\sqrt{U^\varepsilon}\, dW)}_{U_t^\varepsilon - U_0} + (K'_{H,\varepsilon} * L)' * (U_t^\varepsilon - U_0)$$

$$(K'_{H,\varepsilon} * L)(0) = \int_0^0 K'_{H,\varepsilon}(-s) L_\varepsilon(ds) = K'_{H,\varepsilon}(0) / K_{H,\varepsilon}(0) = \frac{(H - \frac{1}{2})\varepsilon^{H-\frac{3}{2}}}{\varepsilon^{H-\frac{1}{2}}} = \left(H - \tfrac{1}{2}\right)\varepsilon^{-1}$$

(6)

$$dV_t^\varepsilon = \Big(\underbrace{\varepsilon^{H-\frac{1}{2}}\theta}_{\text{Bro}} - \underbrace{\left(\tfrac{1}{2}-H\right)\varepsilon^{-1}(V_t^\varepsilon - V_0)}_{\text{Bro}} + \underbrace{\int_0^t (K'_{H,\varepsilon} * L_\varepsilon)'(t-s)(V_s^\varepsilon - V_0)\,ds}_{\text{Not Bro: path-dependent}}\Big)dt + \underbrace{\varepsilon^{H-\frac{1}{2}}\xi\sqrt{V_t^\varepsilon}}_{\text{Bro}}\,dW_t$$

scaling

Not Bro: path-dependent

Drop him!

$$\Downarrow$$

$$\begin{cases} dV_t^\varepsilon = \left(\varepsilon^{H-\frac{1}{2}}\cdot\theta - \varepsilon^{-1}(V_t^\varepsilon - V_0)\right)dt + \varepsilon^{H-\frac{1}{2}}\xi\sqrt{V_t^\varepsilon}\,dW_t \\ dS_t^\varepsilon = S_t^\varepsilon\sqrt{V_t^\varepsilon}\,dB_t \end{cases}$$

Bro, cool Markovian model approximating Volterra semimartingale with shifted power-law kernel. $\varepsilon \downarrow 0$ (kernel $\to$ power law) $\to$ $\to \varepsilon^{-1}, \varepsilon^{H-\frac{1}{2}} \to \infty$ mean-reversion and vol-of-vol explose

2.2. Jumps at the limit

(7) $u \in \mathbb{R}$. Sketch of proof for the characteristic function

$$\mathbb{E}\left[e^{iu\cdot\log\frac{S_T^\varepsilon}{S_t^\varepsilon}} \,\middle|\, \mathcal{F}_t\right] = \exp\left\{\varphi_\varepsilon(T-t) + \varepsilon^{\frac{1}{2}-H}\cdot\psi_\varepsilon(T-t)V_t^\varepsilon\right\},\quad t \le T \qquad (*)$$

$$\mathbb{E}\left[e^{iu\log S_T^\varepsilon} \,\middle|\, \mathcal{F}_t\right] = \exp\Big\{\underbrace{\varphi_\varepsilon(T-t) + \varepsilon^{\frac{1}{2}-H}\cdot\psi_\varepsilon(T-t)V_t^\varepsilon + iu\cdot\log S_t^\varepsilon}_{Y_t}\Big\} = M_t$$

Terminal condition $M_T = e^{iu\log S_T^\varepsilon}$ $\to \begin{cases}\varphi_\varepsilon(0) = 0 \\ \psi_\varepsilon(0) = 0\end{cases}$

If $(*)$ holds, M_t is a martingale $M_t = e^{Y_t}$

$$\frac{dM_t}{M_t} = dY_t + \frac{1}{2}d\langle Y\rangle_t \qquad \text{"}dt\text{"-part should be } 0.$$

$$dY_t = \Big[-\dot\varphi_\varepsilon(T-t) - \varepsilon^{\frac{1}{2}-H}\dot\psi_\varepsilon(T-t)V_t^\varepsilon + \varepsilon^{\frac{1}{2}-H}\psi_\varepsilon(T-t)\left(\varepsilon^{H-\frac{1}{2}}\cdot\theta - \varepsilon^{-1}(V_t^\varepsilon - V_0)\right) - iu\cdot\tfrac{1}{2}V_t^\varepsilon +$$
$$+ \frac{1}{2}\left[\cancel{\varepsilon^{1-2H}}\cdot\psi_\varepsilon(T-t)^2\cdot\cancel{\varepsilon^{2H-1}}\,\xi^2\cdot V_t^\varepsilon - u^2\cdot V_t^\varepsilon + iu\,\cancel{\varepsilon^{\frac{1}{2}-H}}\cdot\rho\,\psi_\varepsilon(T-t)\cdot\cancel{\varepsilon^{H-\frac{1}{2}}}\cdot\xi V_t^\varepsilon\right]\Big]dt$$
$$+ [\dots]dW_t$$

$$(\text{ODEs})\quad \begin{cases}\dot\varphi_\varepsilon = \psi_\varepsilon(\theta + \varepsilon^{-H-\frac{1}{2}}V_0) \\ \varphi_\varepsilon(0) = 0\end{cases} \qquad \begin{cases}\varepsilon^{\frac{1}{2}-H}\dot\psi_\varepsilon = -\psi_\varepsilon\cdot\varepsilon^{-\frac{1}{2}-H} + \frac{1}{2}\xi^2\cdot\psi_\varepsilon^2 + \frac{1}{2}\rho\, iu\,\xi\cdot\psi_\varepsilon - \frac{1}{2}(u^2 + iu) \\ \psi_\varepsilon(0) = 0\end{cases}$$

We showed that $(*) \to$ (ODEs)

For the inverse resultat, it is suffisient do provide the existence of the solution to (ODEs) and guarantee that M_t is a martingale.

This will imply $(*)$.

(8) We accept that

$$\varphi_\varepsilon(t) = (\varepsilon^{-H-\frac{1}{2}}\cdot\vartheta + \varepsilon^{-1-2H}\cdot V_0)\,\xi^{-2}\left((1-i\rho\varepsilon^{H+\frac{1}{2}}\xi u - d)t - 2\varepsilon\log\left(\frac{1-ge^{-\varepsilon^{-1}td}}{1-g}\right)\right)$$

$$\psi_\varepsilon(t) = \varepsilon^{-H-\frac{1}{2}}\cdot\xi^{-2}(1-i\rho\varepsilon^{H+\frac{1}{2}}\xi u - d)\,\frac{1-e^{-\varepsilon^{-1}td}}{1-ge^{-\varepsilon^{-1}td}}$$

one can show that $\varphi_\varepsilon, \psi_\varepsilon \leq 0$.

$$\underbrace{e^{\varphi_\varepsilon + \varepsilon^{\frac{1}{2}-H}\psi_\varepsilon\cdot V_0^\varepsilon}}_{\leq 1}\cdot\underbrace{e^{iu\log S_1^\varepsilon}}_{|\cdot|=1}$$

$g := \dots$, $d := \dots$ $\quad \Re(d) > 0$

For $H = -\frac{1}{2}$

$$\underbrace{\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\exp\{iu\log\frac{S_T^\varepsilon}{S_0^\varepsilon}\}\right]}_{=e^{\varphi_\varepsilon(T)+\varepsilon^{\frac{1}{2}-H}\psi_\varepsilon(T)V_0}} = \exp\{\eta(u)\cdot T\}$$

$$d = \sqrt{(1-i\rho\xi u)^2 + 2\xi^2\left(\frac{u^2+iu}{2}\right)}$$

$$e^{\frac{1}{2}-H}\psi_\varepsilon(T) = \underbrace{\varepsilon^{-2H}}_{\varepsilon}\xi^{-2}(1-i\rho \quad - d)\,\frac{1-\overbrace{e^{-\varepsilon^{-1}td}}^{\to 0}}{1-g\underbrace{e^{-\varepsilon^{-1}td}}_{\to 0}} \to 0 \qquad g = \frac{1-d}{1+d}$$

$$\varphi_\varepsilon(T)\Big|_{H=-\frac{1}{2}} = (\vartheta+V_0)\,\xi^{-2}\left((1-i\rho\xi u - d)T - \underbrace{2\varepsilon}_{\to 0}\log\left(\frac{1-g\overbrace{e^{-\varepsilon^{-1}Td}}^{\to 0}}{1-g}\right)\right) \to$$

$$\to \underbrace{(\vartheta+V_0)\,\xi^{-2}(1-i\rho\xi u - d)}_{\eta(u)}T$$

Where $\eta(u) = (\vartheta+V_0)\,\xi^{-2}\left(1-i\rho\xi u - \sqrt{(1-i\rho\xi u)^2 + 2\xi^2\left(\frac{u^2+iu}{2}\right)}\right)$

we can identify $\lambda, \beta, \delta, \varsigma$, but we do not want...

(9) (X_t) a mysterious process if $X_0 = 0$ and it has independent stationary increments s.t. $\mathbb{E}[e^{iuX_1}] = e^{\eta(u)}$

We have shown that $\mathbb{E}\left[e^{iu\log\frac{S_T^\varepsilon}{S_0^\varepsilon}}\right] \xrightarrow{\varepsilon\downarrow\frac{1}{2}} e^{\eta(u)}$ for $H = -\frac{1}{2}$ $\to$

$\to$ by continuity theorem $\log S_1^\varepsilon - \log S_0^\varepsilon \xrightarrow{\mathcal{L}} X_1$

$$\mathbb{E}\left[e^{iu\log\frac{S_T^\varepsilon}{S_0^\varepsilon}}\right] \to e^{\eta(u)T}$$

It's enough to show that $\mathbb{E}[e^{iuX_T}] = e^{\eta(u)\cdot T}$

For $T \in \mathbb{N}$ $\quad X_T = (X_T - X_{T-1}) + (X_{T-1} - X_{T-2}) + \dots + (X_1 - X_0)$ $\to \mathbb{E}[e^{iuX_T}] = e^{\eta(u)T}$

indep & stationary

Similar argument for $T = \frac{1}{m}$: $X_1 - X_0 = (X_1 - X_{1-T}) + (X_{1-T} - X_{1-2T}) + \dots + (X_T - X_0)$

So, it's true for $T \in \mathbb{Q}$. By continuity of X and dominated convergence it's true $\forall T \in \mathbb{R}$.

(10) ATM skew is exploding (power law?) and there are jumps in the limiting mysterious process. So, it's a Levy process with jumps (Normal inverse Gaussian-inverse Gaussian proc.)

(11) Main messages: after an approximation we get a model that can describe both the rough behavior and approximate the Volterra process for $H>0$, but also can converge to the jump processes for $H<0$.