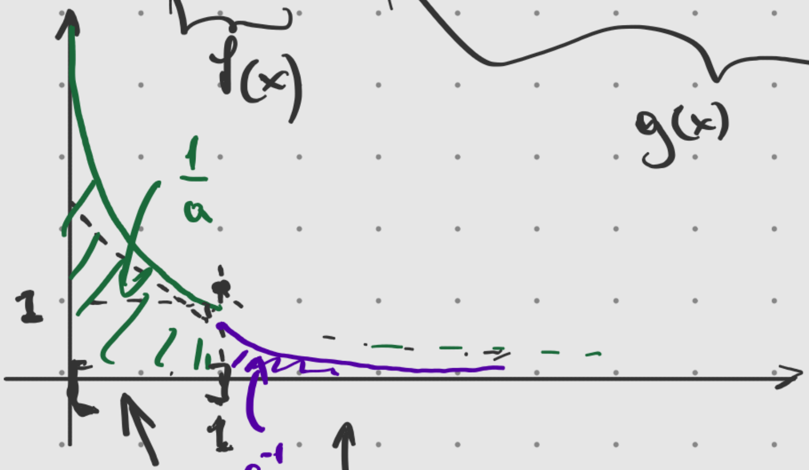
③ Simuler méthode du rejet $\gamma(a) = \dfrac{e^{-x}x^{a-1}}{\Gamma(a)} \mathbb{1}_{x>0}$ $0<a<1$

$a \geq 1$ $\quad a = \lfloor a \rfloor + \{a\}$

$\lfloor a \rfloor$: somme de $\mathcal{E}(1)$

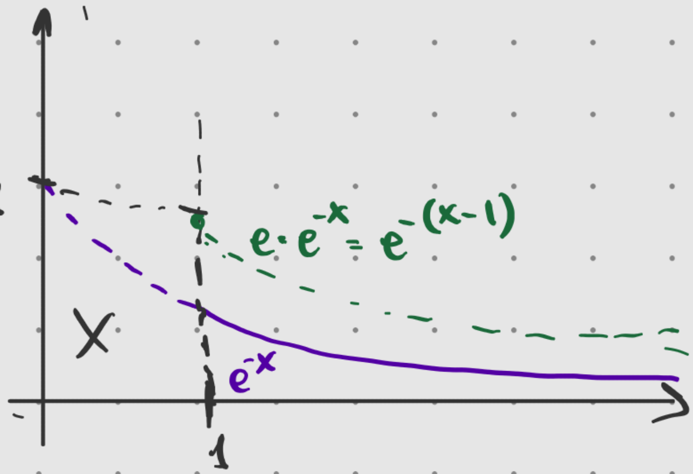
__Indication__ $\underbrace{\dfrac{e^{-x}x^{a-1}}{\Gamma(a)}}_{f(x)} \leq \underbrace{\dfrac{1}{\Gamma(a)} \begin{cases} x^{a-1} & \text{si } x \in]0,1[\ - \text{Pareto} \\ e^{-x}, & x > 1 \ - \mathcal{E}(1) \text{ transl.} \end{cases}}_{g(x)}$



$(Y+1)$, $Y \sim \mathcal{E}(1)$

$\dfrac{1}{x^{1-a}}$, $x \in]0,1]$ $\qquad \displaystyle\int_1^{\infty} e^{-x}dx = e^{-1}$

$$\boxed{e^{-(x-1)} \mathbb{1}_{x \geq 1}}$$
$$= \underbrace{e \cdot e^{-x} \mathbb{1}_{x \geq 1}}$$



$p(x) = x^{a-1} \mathbb{1}_{]0,1]} \cdot a$

$F(x) = \displaystyle\int_0^x y^{a-1}dy = \dfrac{x^a}{a}$ $\qquad F(1) = \dfrac{1}{a} \overset{\text{renorm}}{\rightsquigarrow} F(x) = x^a = u$

$x \in]0,1]$ $\qquad x = (u)^{1/a}$

$e^{-x} = u \quad x = -\log u$

1) Simuler $U_i \sim [0,1]$, $V_i \sim [0,1]$. Si $U_i < \frac{1/a}{1/a + e^{-1}}$, $Y_i = (V_i)^{1/a}$

sinon, $Y_i = 1 - \log V_i$

2) On accepte Y_i avec la proba $\frac{f(V_i)}{g(V_i)}$

Lemme $Y : \Omega \to \mathbb{R}^{d_Y}$, $Z : \Omega \to \mathbb{R}^{d_Z}$ $\mathcal{B} \subset \mathcal{A}$ -sous-tribu

Y - $\mathcal{B}$-mesurable et $Z \perp\!\!\!\perp \mathcal{B}$. Alors $\varphi : \mathbb{R}^{d_Y} \times \mathbb{R}^{d_Z} \to \mathbb{R}$ t.q. $\varphi(Y,Z) \in L^1(\mathbb{R})$

$$\mathbb{E}[\varphi(Y,Z) \mid \mathcal{B}] = [\mathbb{E}\varphi(y,Z)]\big|_{y=Y}$$

D $h(y) = \mathbb{E}\varphi(y,Z) = \mathbb{E}[\varphi(y,Z) \mid \mathcal{B}]$

$\mathbb{E}[\varphi(Y,Z) \mid \mathcal{B}] \overset{?}{=} h(Y)$

1) $h(Y)$ est $\mathcal{B}$-mesurable

2) $\forall B \in \mathcal{B}$ $\mathbb{E}[\mathbb{1}_B h(Y)] \overset{?}{=} \mathbb{E}\varphi(Y,Z)\mathbb{1}_B(\omega)$

$X \in \mathcal{B}$ $\mathbb{E}[X\varphi(Y,Z)] =$

$= \mathbb{E}[X(\text{truc } \mathcal{B}\text{-mes.})] =$

$= \int x\varphi(y,z)\, \mathbb{P}_{X,Y,Z} =$

$= \{\mathbb{P}_{X,Y,Z} = \mathbb{P}_{X,Y} \otimes \mathbb{P}_Z\} = \iint x\varphi(y,z)\, \mathbb{P}_{(X,Y)}(dx,dy)\, \mathbb{P}_Z(dz)$

$= \int x\, \mathbb{E}\varphi(y,Z)\, d\mathbb{P}_{X,Y}(dx,dy) =$

$= \mathbb{E}[X \underbrace{(\mathbb{E}\varphi(y,Z))\big|_{y=Y}}_{\mathbb{E}[\varphi(Y,Z)\mid\mathcal{B}]}]$

$$\iint \varphi(y,z)\mathbb{1}_B\, d\mu_Y\, d\mu_Z$$

$$\int_B \varphi(Y,Z)\, d\mathbb{P} = \int_{(\mu_Y \otimes \mu_Z)^{-1}(B)} \varphi(y,z)$$

$$\int \mathbb{1}_B \underbrace{\int \varphi(y,z)\, d\mu_Z}_{h(y)}\, d\mu_Y = \mathbb{E}\mathbb{1}_B h(Y)$$

$\mathbb{E}[\varphi(Y,Z) \mid Y] = h(Y)$

$\int \mathbb{1}_B \varphi(y,z)$

Indép. de sous-tribu : $Z \perp\!\!\!\perp \mathcal{B}$: $\forall \varphi$ - $\mathcal{B}$-mes $\mathbb{E} Z\varphi(\omega) = \mathbb{E}Z \cdot \mathbb{E}\varphi(\omega)$

$$Y = \sum_{k\in\mathbb{N}} y_k \mathbb{1}_{B_k} \qquad B_k \in \mathcal{B}$$

$$\mathbb{E}[\varphi(Y,Z)\mathbb{1}_B] = \sum_k \mathbb{E}\,\varphi(y_k,Z)\mathbb{1}_{B\cap B_k} = \sum_k \overbrace{\mathbb{E}\varphi(y_k,Z)}^{h(y_k)}\cdot \mathbb{P}(B_k\cap B) =$$

$$= \mathbb{E}\Big[\Big(\overbrace{\sum_k h(y_k)\mathbb{1}_{B_k}}^{h(Y)}\Big)\cdot\mathbb{1}_B\Big] = \mathbb{E}[h(Y)\mathbb{1}_B].$$

TD

N3 La méthode de Marsaglia

$$(U^1,U^2)\sim\mathcal{U}(B(0,1)) \rightsquigarrow (X_1,X_2) = \left(\frac{U^1}{R},\frac{U^2}{R}\right)\sqrt{-2\log R^2}$$

$$R^2 = (U^1)^2+(U^2)^2$$

$$\mathbb{E}\,\varphi(X_1,X_2) = \iint_{R^2=u_1^2+u_2^2\le 1} \varphi\left(\frac{u_1}{R}\sqrt{-2\log R^2},\ \frac{u_2}{R}\sqrt{-2\log R^2}\right) du_1\, du_2$$

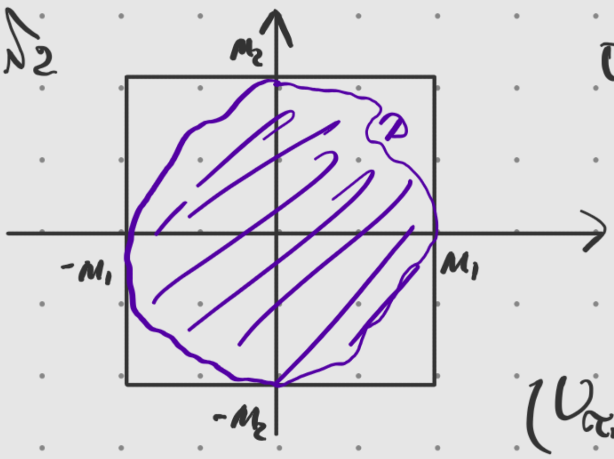
$$P(-2\log R^2 \ge x) = P(R^2 \le e^{-x/2}) = \frac{\pi e^{-x/2}}{\pi\cdot 1} = e^{-x/2} \Rightarrow -2\log R^2 \sim \mathcal{E}xp(1/2)$$

$\theta = \arctan\frac{U^2}{U^1}$ indép. de R^2.

$$\mathbb{P}(\theta\in[\theta_1,\theta_2],\ R\in[R_1,R_2]) =$$  $$= \frac{|\theta_2-\theta_1|}{2\pi}(R_2^2-R_1^2) =$$

$= \mathbb{P}(\theta\in[\theta_1,\theta_2])\,\mathbb{P}(R\in[R_1,R_2]) \Rightarrow \theta$ est indép. de R.

N2



$$U = (U^1,U^2) \sim \mathcal{U}([-M_1,M_1]\times[-M_2,M_2])$$

$(U_n)_{n\ge1}$ i.i.d $\overset{\mathcal{L}}{\sim} U$

$$\tau_n = \min\{k>\tau_{n-1} : U_k\in D\}$$

$(U_{\tau_n})_{n\ge1}$ i.i.d. $\overset{d}{\sim} \mathcal{U}(D) = \dfrac{\mathbb{1}_D(u)\,du}{\lambda(D)}$

$$\mathbb{P}(U_{\tau_1}\in A)=\sum_{\ell\geq 1}\mathbb{P}(\tau_1=\ell,\ U_\ell\in A)=\sum_{\ell\geq 1}\mathbb{P}(U_1\notin D,\dots,U_{\ell-1}\notin D,\ U_\ell\in A\cap D)=$$

$$=\sum_{\ell\geq 1}\mathbb{P}(U_1\notin D)^{\ell-1}\mathbb{P}(U_1\in A\cap D)=\frac{\mathbb{P}(U_1\in A\cap D)}{\mathbb{P}(U_1\in D)}=\frac{\lambda_2(A\cap D)}{\lambda_2(D)}=\int_A\frac{\mathbb{1}_D(u)\,du}{\lambda_2(D)}$$

$$\Rightarrow U_{\tau_1}\overset{\mathcal{L}}{\sim}\mathcal{U}(D)$$

<u>Exo</u> $\mathcal{L}(U_{\tau_1},U_{\tau_2})=\mathcal{L}(U_{\tau_1})^{\otimes 2}$

Lien avec la méthode du rejet

$$f(x)=\mathbb{1}_D(x)\leq 4M_1M_2\underbrace{\left(\frac{1}{4M_1M_2}\mathbb{1}_{[-M_1,M_1]\times[-M_2,M_2]}\right)}_{\mathcal{U}([-M_1,M_1]\times[-M_2,M_2])}$$

Projet basket

$$G\to -G\ \Rightarrow\ \tilde{G}\to -G$$

$$\tilde{G}\sim\mathcal{N}(0,\Sigma)\qquad G\sim\mathcal{N}(0,I_d)\qquad \tilde{G}=LG$$

$$S_T^i=S_0^i\exp\left\{\left(r-\frac{\sigma_i^2}{2}\right)T+\sigma_i\sqrt{T}\,\tilde{G}_i\right\}\qquad \mathbb{E}S_T^i=S_0^ie^{rT}$$

$$\mathbb{E}\left[\frac{1}{d}\sum_1^d S_T^i\right]=\frac{1}{d}e^{rT}\sum_i S_0^i=\bar{S}_0e^{rT}$$

Variable de contrôle $Y=(\bar{S}_0e^{Z}-K)_+\qquad \bar{S}_0=\frac{1}{d}\sum S_0^i$

$$Z=\sum_i a_0^i\left(\mu_iT+\sigma_i\sqrt{T}\,\tilde{G}_i\right)$$

$$\mathbb{E}e^Z=e^{\mathbb{E}Z+\frac{\operatorname{Var}Z}{2}}$$

$$\mathbb{E}Z=T\langle a_0,\mu\rangle$$

$$\operatorname{Var}[Z]=\operatorname{Var}\left[\sum_i a_0^i\sigma_i\sqrt{T}\tilde{G}_i\right]=\sum_i\sum_j a_0^ia_0^jT\Sigma_{ij}=$$

$$=a_0^T\Sigma a_0T$$

$$\mathbb{E}Z+\frac{\operatorname{Var}Z}{2}=T\langle a_0,\mu\rangle+\frac{T}{2}a_0^T\Sigma a_0=rT$$

$$\boxed{\langle a_0, \mu \rangle + \frac{1}{2} a_0^T \Sigma a_0 = r}$$

$$a_0^i = \frac{1}{d} \qquad \underset{\shortparallel}{\bar{\mu}} + \frac{1}{2d^2} \sum_{i,j} \Sigma_{ij}$$

$$\frac{1}{d} \sum_i \mu^i$$

$$\mu^i + \frac{1}{2d} \sum_j \Sigma_{ij} = r$$

$$\boxed{\begin{array}{l} \mu^i = r - \frac{1}{2} \cdot \frac{1}{d} \sum_j (\Sigma_{ij}) \\ a_0^i = \frac{1}{d} \end{array}}$$

Asiatique

$$W^n = \{W_{t_k}\}_{k=0,\dots,n} \qquad \int_0^T W_t \, dt$$

$$\int_0^T W_t \, dt = \sum_{k=0}^n \alpha_k W_{t_k} + \gamma \cdot \varepsilon, \quad \varepsilon \perp\!\!\!\perp W^n$$

$$\shortparallel$$

$$\sum_{k=0}^n \beta_k \underbrace{(W_{t_k} - W_{t_{k-1}})}_{\Delta W_{t_k}} + \gamma \cdot \varepsilon^{\sim \mathcal{N}(0,1)} \qquad (t_{-1} = 0)$$

$$\mathrm{Cov}\left(\int_0^T W_t dt, \Delta W_{t_k}\right) = \beta_k \cdot (t_k - t_{k-1}) = \beta_k \Delta t_k$$

$$\shortparallel$$

$$\mathbb{E} \int_0^T W_t (W_{t_k} - W_{t_{k-1}}) dt = \int_0^T (t \wedge t_k - t \wedge t_{k-1}) dt = \int_0^{t_{k-1}} (t - t) dt +$$

$$+ \int_{t_{k-1}}^{t_k} (t - t_{k-1}) dt + \int_{t_k}^T \Delta t_k \, dt = \int_0^{\Delta t_k} t \, dt + \Delta t_k (T - t_k) = \frac{\Delta t_k^2}{2} + \Delta t_k (T - t_k)$$

$$\beta_k = \frac{\Delta t_k}{2} + (T - t_k)$$

$$\mathbb{E}\left[\int_0^T W_t dt \,\middle|\, \underset{\uparrow}{W^n}\right] = \sum_{k=0}^n \beta_k \Delta W_{t_k}$$

$$(W_{t_1}, \dots, W_{t_n})$$

$$\mathbb{V}\mathrm{ar}\left[\int_0^T W_t dt\right] = \frac{T^3}{3} = \sum \beta_k^2 \cdot \Delta t_k + \gamma^2 \Rightarrow \gamma^2 = \frac{T^3}{3} - \sum \beta_k^2 \cdot \Delta t_k$$

$$\int_0^T W_t dt \,\Big|\, W^n \sim \mathcal{N}\left(\sum \beta_k \Delta W_{t_k}, \ \frac{T^3}{3} - \sum \beta_k^2 \cdot \Delta t_k\right)$$

$$\int_0^T W_t dt = \sum \beta_k \Delta W_{t_k} + \gamma \cdot \varepsilon$$

$$\mathbb{E}\left[\int_0^T W_t dt \cdot \overbrace{(W_{t_k} - W_{t_{k-1}})}^{\Delta W_{t_k}}\right] = \beta_k \Delta t_k$$

$$\parallel$$

$$\underbrace{\int_{t_{k-1}}^{t_k} (t - t_{k-1}) dt}_{\frac{\Delta t_k^2}{2}} + \Delta t_k (T - t_k)$$

$\mathrm{Var}(T)$?

$\beta_k =$

$$t_{k^*} \leq T < t_{k^*+1} < \ldots$$

$$\beta_{k^*+1+l} = 0, \ l > 0$$

$$\beta_{k^*+1} = \int_{t_{k^*}}^T (t - t_{k^*}) dt = \frac{(T - t_{k^*})^2}{2 \Delta t_{k^*+1}}$$

$$\beta_k = \frac{\Delta t_k}{2} + (T - t_k) \qquad t_1 = 1 \qquad t_2^* = 2$$

$$\beta_1 = \frac{1}{2} + \frac{1}{2} = 1$$

$$\beta_2^* = \frac{(1/2)^2}{2} = \frac{1}{8}$$

$$\mathbb{E}\left[\int_0^{1.5} W_t dt \,\Big|\, W_1, W_2\right] = W_1 + \frac{1}{8}(W_2 - W_1) = \frac{7}{8} W_1 + \frac{1}{8} W_2$$

$$\frac{T^3}{3} - 1 - \frac{1}{64}$$

$$\tilde{X}_k = X_k - \hat{\lambda}_{-k}\,\Xi$$

$$\hat{\lambda}_{-k} = \frac{\sum_{j\neq k} X_j \Xi_j}{\sum_{j\neq k} \Xi_j^2} = \frac{\overline{X\Xi_N} - \frac{1}{N}X_k\Xi_k}{\overline{\Xi_N^2} - \frac{1}{N}\Xi_k^2}$$

$$\boxed{\hat{\lambda}_{-k} = \frac{\overline{X\Xi_N} - \frac{X_k\Xi_k}{N}}{\overline{\Xi_N^2} - \frac{\Xi_k^2}{N}}}$$

$$\alpha_1 W_{t_1} + \alpha_2 W_{t_2} + \alpha_3 W_{t_3} + \lambda Y$$

$$\parallel$$

$$\beta_1(W_{t_1}-0) + \beta_2(W_{t_2}-W_{t_1}) + \beta_3(W_{t_3}-W_{t_2}) + \lambda Y$$

$$\begin{cases} \beta_1 - \beta_2 = \alpha_1 \\ \beta_2 - \beta_3 = \alpha_2 \\ \beta_3 = \alpha_3 \end{cases} \rightsquigarrow (\beta_1, \beta_2, \beta_3)$$

$$\beta_3 = \alpha_3$$
$$\beta_2 = \alpha_2 + \alpha_3$$
$$\beta_1 = \alpha_1 + \alpha_2 + \alpha_3$$

$$\mathbb{E}\int_0^T W_t\left(W_{(k+1)\frac{T}{n}} - W_{k\frac{T}{n}}\right)dt = \mathbb{E}\int_0^T W_t W_{(k+1)\frac{T}{n}}\,dt - \mathbb{E}\int_0^T W_t W_{k\frac{T}{n}}\,dt$$

$$= \int_0^{(k+1)\frac{T}{n}} t\,dt + \int_{(k+1)\frac{T}{n}}^{T} (k+1)\frac{T}{n}\,dt - \int_0^{k\frac{T}{n}} t\,dt - \int_{k\frac{T}{n}}^{T} k\frac{T}{n}\,dt =$$

$$\underbrace{\int_{k\frac{T}{n}}^{(k+1)\frac{T}{n}} t\,dt}_{\frac{1}{2}\left[\left((k+1)\frac{T}{n}\right)^2 - \left(\frac{kT}{n}\right)^2\right]} + \left((k+1)\frac{T}{n}\right)\left(T - (k+1)\frac{T}{n}\right) - \left(k\frac{T}{n}\right)\left(T - k\frac{T}{n}\right)$$

$$= \underbrace{(k+1)\frac{T}{n}}_{t_{k+1}}\left(T - \frac{1}{2}\underbrace{(k+1)\frac{T}{n}}_{t_{k+1}}\right) - \underbrace{\left(k\frac{T}{n}\right)}_{t_k}\left(T - \frac{1}{2}\underbrace{k\frac{T}{n}}_{t_k}\right)$$

Exercice (mini-projet) Relation de parité Call-Put "asiatique".

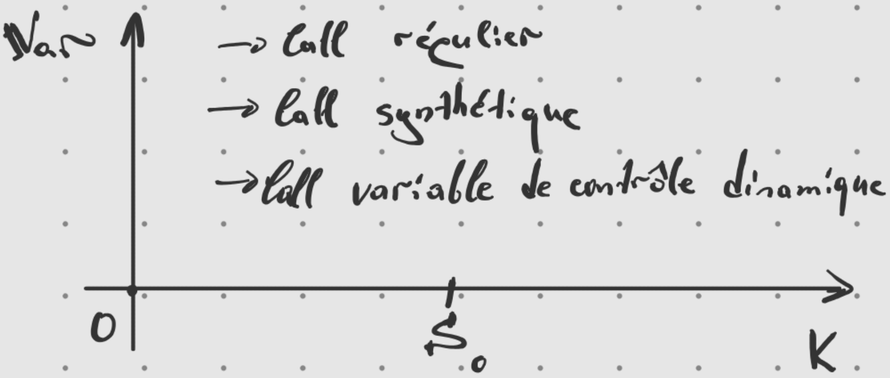
$$0 < T_0 < T \qquad \underbrace{\left(\frac{1}{T-T_0}\int_{T_0}^{T} S_t\,dt - K\right)^+}_{\text{payoff de call asiatique}} \quad \text{et} \quad \underbrace{\left(K - \frac{1}{T-T_0}\int_{T_0}^{T} S_t\,dt\right)^+}_{\text{payoff put asiatique}}$$

1) Relation Call-Put

2) Faire de simulation

$S_0 = 100$, $\sigma = 0{,}3$

$r = 0.05$, $T = 1$



$$\left(\frac{1}{T-T_0}\int_{T_0}^{T} S_t\,dt - K\right)^+ - \left(K - \frac{1}{T-T_0}\int_{T_0}^{T} S_t\,dt\right)^+ = \frac{1}{T-T_0}\int_{T_0}^{T} S_t\,dt - K$$

$$\Big\downarrow \; e^{-rT}\mathbb{E}[\ldots]$$

$$Call_0^{Asian}(T,K;T_0) - Put_0^{Asian}(T,K;T_0) = \frac{1}{T-T_0}\int_{T_0}^{T} e^{-r(T-t)}\,dt\,\mathbb{E}\,e^{-rt}S_t - e^{-rT}K$$

$$= S_0 \cdot \frac{1-e^{-r(T-T_0)}}{r(T-T_0)} - e^{-rT}K$$

$$\text{Call régulier} = \left(\frac{1}{T-T_0}\int_{T_0}^{T} S_t\,dt - K\right)^+$$

$$\text{Call synthétique} = S_0 \cdot \frac{1-e^{-r(T-T_0)}}{r(T-T_0)} - e^{-rT}K + \text{Put régulier}$$

Call + variable de contrôle : Call régulière $- \lambda_{adopt} \cdot$ (var. de contrôle)

$$x\int_{T_0}^{T}\exp\{\mu t+\sigma W_t\}\frac{dt}{T-T_0} \geq x\,e^{\frac{1}{T-T_0}\int_{T_0}^{T}\mu t+\sigma W_t\,dt}$$

$$\downarrow$$

$$\varphi(\dots) \geq \varphi(\dots)$$

$$k_T = \varphi\left(x\exp\left\{\frac{1}{T-T_0}\int_{T_0}^{T}(\mu t+\sigma W_t)dt\right\}\right)$$

$$\frac{1}{T-T_0}\int_{T_0}^{T}(\mu t+\sigma W_t)dt = \mu\frac{T+T_0}{2} + \sigma\cdot\frac{1}{T-T_0}\int_{T_0}^{T}W_t\,dt$$

QMC

Exo $d=1$ $(\xi_1,\dots,\xi_n)\in[0,1]$. Reordrement croissant $\xi_1^{(n)}\leq\dots\leq\xi_n^{(n)}$

t.q. $\{\xi_i,\ i=1:n\} = \{\xi_i^{(n)},\ i=1:n\}$

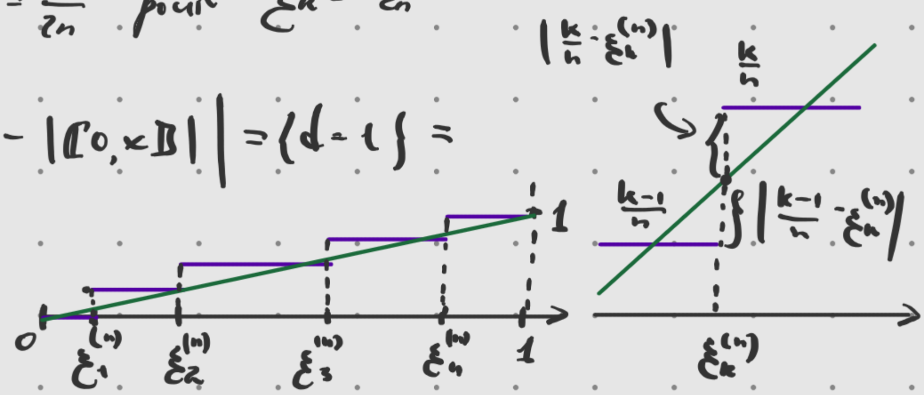
Alors $D_n^*(\xi_{1:n}) = \max\limits_{k=1:n}\left(\left|\xi_k^{(n)}-\frac{k}{n}\right| \vee \left|\xi_k^{(n)}-\frac{k-1}{n}\right|\right)$

$$D_n^*(\xi_{1:n}) = \frac{1}{2n} + \max_{k=1:n}\left|\xi_k^{(n)}-\frac{2k-1}{2n}\right|$$

Discrepance minimale $=\frac{1}{2n}$ pour $\xi_k = \frac{2k-1}{2n}$

$$D_n^* = \sup_x\left\{\left|\frac{1}{n}\sum \mathbb{1}^{(x)}_{\xi_k\in[0,x]} - |[0,x]|\right|\right\} = \{d=1\} =$$

$$= \sup\left\{\left|\frac{1}{n}\sum\mathbb{1}_{\{\xi_k\leq x\}} - x\right|\right\}$$



Le sup est atteint sur $\{\xi_j^{(n)}\}_{j=1}^{n}$. En plus, $\frac{1}{n}\sum\limits_{j=1}^{k}\mathbb{1}_{\{\xi_j^{(n)}\leq x\}} = \frac{1}{n}\mathrm{card}\{k : \xi_k^{(n)}\leq x\}$

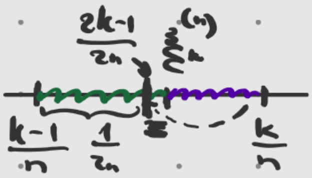
Donc $D_n^* = \max\limits_{x=\xi_j^{(n)},\ j=1,\dots,n}\left\{\frac{\mathrm{card}\{k:\xi_k^{(n)}\leq x\}}{n} - x\right\} = \max\limits_{k}\left\{\left|\frac{k}{n}-\xi_k^{(n)}\right| \vee \left|\frac{k-1}{n}-\xi_k^{(n)}\right|\right\}$

$\max_k \left| \xi_k^{(n)} - \frac{2k-1}{2n} \right| = 0$ si $\xi_k^{(n)} = \frac{2k-1}{2n}$

$\frac{2k-1}{2n} = -\frac{1}{2n} + \frac{k}{n}$

$\qquad\; \|$

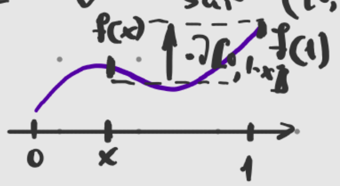
$\frac{k-1}{n} + \frac{1}{2n}$



Def $f : [0,1]^d \to \mathbb{R}$ est à variation finie au sens de la mesure

s'il existe une mesure signée ν sur $([0,1]^d, \mathcal{B}([0,1]^d))$ t.q.

(i) $\nu(\{0\}) = 0$ pour $d=1$



(ii) $\forall x \in [0,1]^d \;\; f(x) = f(1) + \nu(⟦0, 1-x⟧) = f(0) - \nu(⟦0,1-x⟧^c)$

① $d=1$ f v.f au sens de la mesure si f est à v.f.

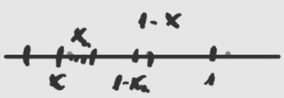
au sens classique et $x \mapsto f(1-x)$ est càdlàg

② f v.f. au sens de la mesure s'il existe ν mesure signée

t.q. $\begin{cases} \text{(i)} \; \nu(\{1\}) = 0 \\ \text{(ii)} \; f(x) = f(1) + \nu(⟦x,1⟧) \end{cases}$

① f au sens de la mesure $\Leftrightarrow$ f à v.f. au sens classique $x \mapsto f(1-x)$ est càdlàg

$\Rightarrow$ $f(x) = f(1) + \nu([0,1-x])$



$f(1-x)$ càd? $\quad f(1-x) = f(1) + \nu([0,x]) \quad x_n \downarrow x \to [0,x] = \bigcap_n [0,x_n] \to \nu([0,x]) = \lim \nu([0,x_n])$ continuité de mesure

f à v.f. au sense classique?

$$\sum_k |f(x_{k+1}) - f(x_k)| = \sum_k |\vartheta([x_k, x_{k+1}])| \leq |\vartheta|([0,1]) < \infty$$

$$\Downarrow$$

$$\lim \leq |\vartheta|([0,1])$$

$\Leftarrow$ On définit $\vartheta([0,x]) = \underbrace{f(1-x)}_{\text{càd}} - f(1)$

Il faut juste m.q. $\vartheta(\{0\}) = 0$

$\{0\} = \bigcap [0, \frac{1}{n}]$ $\quad \vartheta([0,\frac{1}{n}]) = f(1-(0+\frac{1}{n})) \xrightarrow{\text{grâce à càd}} f(1) = \vartheta(\{0\})$

② $\tilde{\vartheta}$ mesure signée

(i) $\tilde{\vartheta}(\{1\}) = 0$

(ii) $f(x) = f(1) + \tilde{\vartheta}([x,1])$

$f(x) = f(1) + \vartheta([0, 1-x])$

$\tilde{\vartheta}([x,1])$ - mesure signée

$\tilde{\vartheta}(\{1\}) = \tilde{\vartheta}([1,1]) = \vartheta([0,0]) = 0$

$\exists \vartheta$: $f(x) = f(1) + \vartheta([0, 1-x])$ $\quad \vartheta(\{0\}) = 0$

$\Longleftrightarrow$

$\exists \vartheta$: $f(x) = f(1) + \vartheta([x, 1])$ $\quad \vartheta(\{1\}) = 0$

Exo $\underbrace{f \text{ Lipschitz}}$ $\to$ à variation finie au sens de mesure

$\Downarrow$

à v.f. au sens classique + continue $\Big\}$ $\to$ à v.f. au sens de la mesure

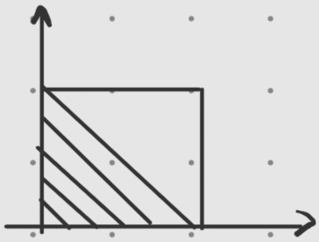
2. Show that the function f defined on $[0,1]^2$ by

$$f(x^1,x^2) := (x^1+x^2)\wedge 1, \quad (x^1,x^2)\in[0,1]^2$$

has finite variation in the measure sense [Hint: consider the distribution of $(U, 1-U)$, $U \overset{d}{=} \mathcal{U}([0,1])$].

$f(x^1,x^2) = (x^1+x^2)\wedge 1$

la mesure?

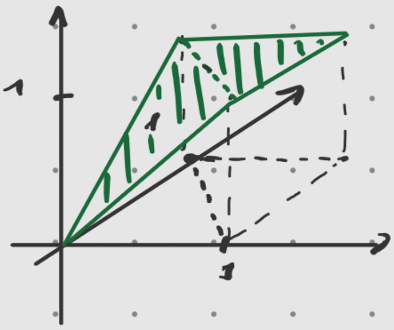


$\nu(]\!]x,1]\!]) = f(x^1,x^2) - \overset{=1}{f(\mathbb{1})} = (x^1+x^2-1)\wedge 0 =$

$= (x^1+x^2-1) - (x^1+x^2-1)^+$

$X = (X_1,X_2) = (U, 1-U)$

$x^1, x^2 \in [0,1]$



$F_X(x^1,x^2) = \mathbb{P}(X^1 \le x^1, X^2 \le x^2) = \mathbb{P}(U\le x^1, 1-U \le x^2)$

$= \mathbb{P}(1-x^2 \le U \le x^1) = (x^1+x^2-1)^+ \Rightarrow$

$\Rightarrow \nu \approx \mathcal{L}((U,1-U))$

Exo $f(x^1,x^2,x^3) = (x^1+x^2+x^3)\wedge 1$ n'est pas à v.f au sens de la mesure

$f(x) - f(\mathbb{1}) = (x^1+x^2+x^3)\wedge 1 \overset{?}{=} \int_{[x,1]} \nu(dy)$

$\nu([0,1-x]) = \int_0^{1-x_3} dy_3 \int d\nu_{y_3}$

$\{0\le y_1 \le 1-x_1,\ 0 \le y_2 \le 1-x_2\}$

$\mathbb{P}(X^i \le 1-x_i,\ i=1,2,3)$

$v(x) = (x^1+x^2+x^3)\wedge 1$

$\partial_{x_1} v = \mathbb{1}_{x^1+x^2+x^3<1}$

$\partial_{x_2}\partial_{x_1} v = \delta_{1-x^1-x^2}(dx_3)$

$\partial_{x_1}\partial_{x_2}\partial_{x_3} v = ??$

mesure

$\nu([0,\underbrace{1-x}_{z}]) = \int_0^{1-x_1} dx_1 \int_0^{1-x_2} dx_2 \int_0^{1-x_3} \nu_{x_1,x_2}(dx_3) =$

$= \int_0^{z_1} dz_1 \int_0^{z_2} dz_2 \int_0^{z_3} \nu_{z_1,z_2}(dz_3) = (x_1+x_2+x_3)\wedge 1$

$\partial_{z_1}\partial_{z_2}$

$\int_0^{z_3} \nu_{z_1,z_2}(dz_3) = \delta_{x_1+x_3-1}(dx_2)$

$\Rightarrow$ il n'existe pas de dérivée au sens de la mesure

Exo $\quad p=2 \quad \xi_{2n}=\xi_n/2, \quad \xi_{2n+1}=\dfrac{\xi_n+1}{2}$

pqcq $\quad \xi_{pn}=\dfrac{\xi_n}{p}, \quad \xi_{pn+r}=\dfrac{\xi_n+r}{p}, \quad r=0,\dots,p-1$

D

$$n=\sum_{k=0}^{r} a_k 2^k \rightsquigarrow \xi_n=\sum_{k=0}^{r} a_k 2^{-k-1}$$

$$2n=\sum_{k=0}^{r} a_k 2^{k+1} \rightsquigarrow \xi_{2n}=\sum_{k=0}^{r+1} a_{k-1} 2^{-k-1}=\frac{1}{2}\xi_n$$

$$2n+1=\overset{\tilde a_0}{1\cdot}+\overset{\tilde a_1}{a_0\cdot 2}+\overset{\tilde a_2}{a_1 2^2}\dots \rightarrow \xi_{2n+1}=\xi_{2n}+\frac{1}{2}=\frac{\xi_n+1}{2} \qquad \square$$

Exercice $\quad VdC(p) \quad \xi_{pn+r}=\dfrac{\xi_n+r}{p}$ où $r\in\{0,\dots,p-1\}$

En déduire $\displaystyle \frac{1}{n}\sum_{k=1}^{n}\frac{1}{p}\sum_{r=0}^{p-1}\delta_{\xi_{pk+r}} \xrightarrow{w} ?$

$$n=a_0+a_1p+\dots+a_\ell p^\ell \rightsquigarrow \xi_n=a_0p^{-1}+a_1p^{-2}+\dots+a_\ell p^{-\ell-1}$$

$$p\cdot n+r=r+a_0p+a_1p^2+\dots a_{\ell-1}p^\ell+a_\ell p^{\ell+1} \rightsquigarrow \xi_{pn+r}=\underbrace{rp^{-1}}_{r/p}+\underbrace{a_0p^{-2}+\dots+a_\ell p^{-\ell-2}}_{\xi_n/p}=\frac{\xi_n+r}{p}$$

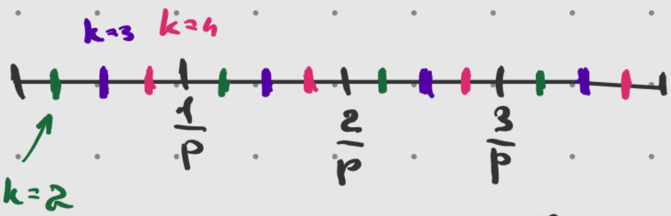
$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{p}\sum_{r=0}^{p-1}\delta_{\xi_{pk+r}} \overset{\int f\,d\mu_n}{\rightsquigarrow} \underbrace{\frac{1}{n}\sum_{k=1}^{n}\frac{1}{p}\sum_{r=0}^{p-1} f\left(\frac{\xi_k+r}{p}\right)}$$

$\xi_1=\dfrac{1}{p} \qquad \xi_p=\dfrac{1}{p^2}$

$\xi_2=\dfrac{2}{p}$

$\vdots$

$\xi_{p-1}=\dfrac{p-1}{p}$



Pour $n=m\cdot p$ on obtient la grille uniforme $\rightsquigarrow$ sous-suite $\rightsquigarrow$ sommes de Riemann $\rightsquigarrow \int f(x)\,dx$

Hammersley procedure

<u>Exo</u> (a) Mq la discrépance $\mathcal{D}_n^*\left(\left(\frac{k}{n}, \xi_k\right)_{k=1:n}\right) \leq C_d \max_{k=1:n} \frac{k\mathcal{D}_k^*(\xi_{1:k})}{n}$

$\downarrow [0,1]^{d-1}$ ↑ $\leq$ de Hammersley

(b) Utiliser (a) pour montrer (b)

$$\xi_k = (\xi_k^{(1)}, \dots, \xi_k^{(d-1)}) \rightsquigarrow \eta_k = \left\{\left(\frac{k}{n}, \xi_k\right)\right\}_{k=1:n}$$

$$\mathcal{D}_n^*(\eta_{1:n}) = \sup_{\substack{x \in [0,1]^{d-1} \\ y \in [0,1]}} \left\{ \left| \frac{1}{n}\sum_{k=1}^n \mathbb{1}_{\{\xi_k \leq x\}} \cdot \mathbb{1}_{\{\frac{k}{n} \leq y\}} - \left(\prod_{i=1}^{d-1} x_i\right) \cdot y \right| \right\} =$$

$$= \max_{k=1:n} \sup_{x \in [0,1]^{d-1}} \left\{ \left| \frac{1}{n}\sum_{i=1}^k \mathbb{1}_{\{\xi_i < x\}} - \frac{k}{n}\prod_{i=1}^{d-1} x_i \right| \right\} \vee \left\{ \frac{1}{n}\sum_{i=1}^{k-1} \mathbb{1}_{\{\xi_i < x\}} - \frac{k}{n}\prod_{i=1}^{d-1} x_i \right\}$$

$\uparrow y = \frac{k}{n}$

$$= \max_{k=1:n} \left(\frac{k}{n} \underbrace{\sup_{x \in [0,1]^{d-1}} \left\{ \frac{1}{k}\sum \mathbb{1}_{\{\xi_k \leq x\}} - \prod_{i=1}^{d-1} x_i \right\}}_{\mathcal{D}_k^*(\xi)} \vee \frac{k-1}{n} \overbrace{\sup_x \left| \frac{1}{k-1}\sum_{i=1}^{k-1} \mathbb{1}_{\{\xi_i < x\}} - \prod_{i=1}^{d-1} x_i + \frac{1}{k-1}\underbrace{\prod_{i=1}^{d-1} x_i}_{\leq 1} \right|}^{\mathcal{D}_{k-1}^*(\xi) + \frac{1}{k-1}} \right) \leq$$

Donc $\mathcal{D}_n^*(\eta_{1:n}) \geq \max_k \left\{ \frac{k}{n} \mathcal{D}_k^*(\xi) \right\}$

$$\leq \max_k \left\{ \frac{k}{n}\mathcal{D}_k^*(\xi), \frac{k-1}{n}\mathcal{D}_{k-1}^*(\xi) + \frac{1}{n} \right\} \leq \max_{k=1..n} \frac{k\mathcal{D}_k^*(\xi) + 1}{n}$$

ne dépend pas de n

$$\forall n \ \exists \xi_n \quad \mathcal{D}_n^*(\xi) \leq C_d \frac{1 + (\log n)^{d-1}}{n}$$

Pour $d-1$ ξ_k à discrepence faible $\Rightarrow \mathcal{D}_k(\xi) \leq C \frac{(\log k)^{d-1}}{k}$

Hammersley : $\xi_k = \left(\frac{k}{n}, \xi_k\right)$

$$\mathcal{D}_n^*(\xi) \leq \frac{1}{n}\Big(\max_k \underbrace{k D_k^*(\xi)}_{\leq C_{d-1}(1+(\log k)^{d-1})} + 1\Big) \leq \frac{1}{n}\big(1 + C \quad + C \quad (\log n)^{d-1}\big) \leq \leq C_d \, \frac{1+(\log n)^{d-1}}{n}$$

$$\sigma(t_k, X_{t_k}) \in \mathbb{R}^{d \times q}$$

$$\underset{N\times d}{X_{t_{k+1}}} = \underset{N\times d}{X_{t_k}} + \underset{N\times d\times q}{\sigma(t_k, X_{t_k})} \cdot \underset{N\times q}{\Delta W_{t_{k+1}}}$$

$$X^i_{t_{k+1}} = X^i_{t_k} + \sigma(t_k, X^i_{t_k}) \cdot \Delta W^i_{t_{k+1}}$$

$$dX_t = -\lambda(X_t - m)\,dt + \sigma\, dW_t \qquad \underset{\sim\sim}{m=0}$$

Sinon, $\begin{cases} Y_t = X_t - m \\ Y_0 = X_0 - m \end{cases}$

$$d e^{\lambda t} X_t = e^{\lambda t} \cancel{\lambda m}\, dt + \sigma e^{\lambda t} dW_t$$

$\rightsquigarrow Y_t \rightsquigarrow X_t = Y_t + m$

$$X_t = e^{-\lambda t} X_0 + \sigma \int_0^t e^{-\lambda(t-s)} dW_s$$

$$\mathcal{L}\Big(\underbrace{\int_0^t e^{-\lambda(t-s)} dW_s}_{I_t} \Big| W_t, W_0 = 0\Big)$$

$$\mathrm{Var}\, I_t = \int_0^t e^{-2\lambda(t-s)} ds = \frac{1}{2\lambda}(1 - e^{-2\lambda t})$$

$$I_t = \beta W_t + \overset{(\perp\!\!\!\perp W_t)}{Z_t}$$

$$\mathrm{cov}(I_t, W_t) = \beta t$$

$$\int_0^t e^{-\lambda(t-s)} ds = \frac{1}{\lambda}(1 - e^{-\lambda t}) \qquad \Rightarrow \beta = \frac{1 - e^{-\lambda t}}{\lambda t}$$

$$\mathrm{Var}[I_t] = \beta^2 t + \mathrm{Var}[Z]$$

$$\mathrm{Var}[Z] = \frac{1 - e^{-2\lambda t}}{2\lambda} - \frac{(1 - e^{-\lambda t})^2}{\lambda^2 t}$$

$$I_t = \frac{1 - e^{-\lambda t}}{\lambda t} \cdot W_t + Z_t$$

$$X_{t_{k+1}} = e^{-\lambda(t_{k+1}-t_k)} X_{t_k} + \sigma\left(\frac{1-e^{-\lambda \Delta t_{k+1}}}{\lambda \Delta t_{k+1}} W_{t_{k+1}} + Z_{t_{k+1}}\right)$$

$$dX_t^i = -\lambda X_t^i dt + \sigma dW_t^i$$

$$d(\overbrace{\sum w_i X_t^i}^{Y_t}) = -\lambda \underbrace{\sum w_i X_t^i}_{Y_t} dt + \sigma \sum w_i dW_t^i$$

$$\sum w_i = 1$$

X - la perte

CVaR $\quad VaR_\alpha := \inf\{\xi : \mathbb{P}(X \leq \xi) \geq \alpha\}$ $\quad \alpha \approx 1$

$$CVaR_\alpha = \mathbb{E}[X \mid X \geq VaR_\alpha(X)]$$

X n'a pas d'atomes.

$$CVaR_\alpha(X) = VaR_\alpha(X) + \frac{1}{1-\alpha}\int_{VaR_\alpha(X)}^{+\infty} \mathbb{P}(X > u)\,du$$

$$Y \geq 0 \Rightarrow \mathbb{E}[Y] = \int_0^{+\infty} \mathbb{P}(Y \geq u)\,du$$

$$1-\alpha = \mathbb{P}(X \geq VaR_\alpha(X))$$

$$\mathbb{E}[X \mid X \geq VaR_\alpha(X)] = \int_0^{+\infty} \mathbb{P}(X > u \mid X \geq VaR_\alpha)\,du = \int_0^{VaR_\alpha} \underbrace{\mathbb{P}(X > u \mid X \geq VaR_\alpha)}_{=1}\,du +$$

$$+ \int_{VaR_\alpha}^{+\infty} \frac{\mathbb{P}(X > u)}{\mathbb{P}(X \geq VaR_\alpha)}\,du = VaR_\alpha(X) + \frac{1}{1-\alpha}\int_{VaR_\alpha}^{+\infty} \mathbb{P}(X > u)\,du$$