Exo 8. Modèle HJM gaussien

$$\frac{dB(t,T)}{B(t,T)} = r_t dt - \Gamma(t,T) dW_t \quad \text{sous } \mathbb{Q}$$

$$\Gamma(t,T) = \int_t^T \gamma(t,s) ds, \quad \gamma(\cdot,T) \text{ déterministe, bornée}$$

(1) $$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} := \frac{B(t,T)}{B(0,T)} \cdot \frac{1}{B_t} = \mathcal{E}\left(-\int_0^{\cdot} \Gamma(s,T) dW_s\right)_t$$

c'est $\mathcal{E}(\cdots)$ et la seul partie stoch est $-\int_0^t \Gamma(s,T) dW_s$

(2) $\frac{B(t,S)}{B(t,T)}$ est une $\mathbb{Q}^T$-mart $\Leftrightarrow$ $\frac{B(t,S)}{B_t}$ est une $\mathbb{Q}$-mart. C'est le cas grâce à la condition d'AOA. $d\left(\frac{B(t,S)}{B(t,T)}\right) \Big/ \frac{B(t,S)}{B(t,T)} = (\Gamma(t,S) - \Gamma(t,T)) dW_t^T$

(3) $\Psi_T = (B(T,T+\delta) - K)^+$ en T

$$V_t = B(t,T)\, \mathbb{E}\left[\left(\frac{B(T,T+\delta)}{B(T,T)} - K\right)^+\right] \overset{BS}{=} B(t,T+\delta)\mathcal{N}(d_+) - K B(t,T)\mathcal{N}(d_-)$$

où $d_\pm = \dfrac{\ln\left(\frac{B(t,T+\delta)}{K B(t,T)}\right)}{\Sigma} \pm \frac{1}{2}\Sigma$ où $\Sigma^2 = \int_t^T (\Gamma(s,T+\delta) - \Gamma(s,T))^2 ds$

(4) Caplet d'échéance $T+\delta$ de strike K sur $L(T,\delta) = \frac{1}{\delta}\left(\frac{1}{B(T,T+\delta)} - 1\right)$

$$\Psi_{T+\delta} = \delta(L(T,\delta) - K)^+ = \Big(\frac{1}{B(T,T+\delta)} - \underbrace{(1+K\delta)}_{=\tilde{K}^{-1}}\Big)^+ \text{ en } T+\delta \Rightarrow$$

$\sim$ en T $\Psi_T = B(T,T+\delta)\Psi_{T+\delta} = \tilde{K}^{-1}(\tilde{K} - B(T,T+\delta))^+ \Rightarrow$ caplet = put sur ZC

Par (3) et BS pour l'option put,

$$\text{Caplet}_t = \tilde{K}^{-1}(\tilde{K}B(t,T)\mathcal{N}(-d_-) - B(t,T+\delta)\mathcal{N}(-d_+)) = B(t,T)\mathcal{N}(-d_-) - \tilde{K}^{-1}B(t,T+\delta)\mathcal{N}(-d_+)$$

(5) $0 < T_1 < \cdots < T_{n+1}$, $\bar{\delta} = T_{k+1} - T_k$

Taux swap $$S(t,T_1,T_{n+1}) = \frac{B(t,T_1) - B(t,T_{n+1})}{\underbrace{\bar{\delta}\sum_{i=2}^{n+1} B(t,T_i)}_{= A_t}}$$

(a) A_t est un pf. autofinançant (long 1 $B(t,T_i)$ $i=2,\dots,n+1$) $\Rightarrow$ $\frac{A_t}{B_t}$ est une $\mathbb{Q}$-mart $\Rightarrow$ $\Rightarrow$ $\frac{A_t}{B(t,T)}$ est une $\mathbb{Q}^T$-mart.

(6) $\frac{d\mathbb{Q}^{swap}}{d\mathbb{Q}^{T_1}} = \frac{A_{T_1}}{B(T_1,T_1)} \cdot \frac{B(0,T_1)}{A_0}$ $\Rightarrow$ A_t est un numéraire sous $\mathbb{Q}^{swap}$ $\Rightarrow$

$\Rightarrow$ $S(t,T_1,T_{n+1}) = \dfrac{\bar{\delta}^{-1}(B(t,T_1) - B(t,T_{n+1}))}{A_t}$ ← pf. autofinançant est une $\mathbb{Q}^{swap}$-martingale.

A_t ← numéraire

(c) La dynamique de $S(\cdot, T_1, T_{n+1})$ sous $\mathbb{Q}^{swap}$:

$$dS(t,T_1,T_{n+1}) = \sigma_t^S \cdot S(t,T_1,T_{n+1})\, dW_t^{swap}$$

Exo 7 Modèles affines de taux court

$$dr_t = \mu(t,r_t)\,dt + \sigma(t,r_t)\,dW_t, \quad r_0 > 0$$

$$\mu(t,r) = \alpha(t) + \beta(t) r$$
$$\sigma(t,r)^2 = \gamma(t) + \delta(t) r$$

ppté de Markov pour l'EDS

r_t mesurable → fct. mesurable de r_t

(1) $B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du} \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\Phi((r_s), s \in [t,T]) \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du} \middle| r_t\right] = F(t,r_t)$

EDP est donnée par la formule de Feynman-Kac:

$$\begin{cases} \mathcal{L}F(t,r) = rF(t,r) \\ F(T,r) = 1 \end{cases} \quad \text{où } \mathcal{L}F(t,r) = \partial_t F + \mu \cdot \partial_r F + \frac{1}{2}\sigma^2 \cdot \partial_{rr}^2 F$$

Et donc
$$\begin{cases} \partial_t F + (\alpha(t) + \beta(t) r)\partial_r F + \frac{1}{2}(\gamma(t) + \delta(t) r)\partial_{rr}^2 F = rF \\ F|_{t=T} = 1 \end{cases}$$

(2) $B(t,T) = e^{m(t,T) - n(t,T) r_t}$

on cherche une solution dans cette forme: $F(t,r) = e^{m(t) - n(t) r}$

EDP $\rightsquigarrow$ $\begin{cases} (\dot m - \dot n r) - (\alpha + \beta r) n + \frac{1}{2}(\gamma + \delta r) n^2 = r & \forall r > 0 \\ m(T) = 0 \\ n(T) = 0 \end{cases}$ →

→ $\begin{cases} \dot m = \alpha n - \frac{\gamma}{2} n^2 \\ m(T) = 0 \end{cases}$

$\begin{cases} \dot n = \frac{\delta}{2} n^2 - \beta n - 1 \\ n(T) = 0 \end{cases}$ ← Riccati

(3) $$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{B(t,T)}{B_t} \cdot \frac{1}{B(0,T)} = \frac{e^{m(t,T) - n(t,T) r_t}}{e^{\int_0^t r_s ds} \cdot e^{m(0,T) - n(0,T) \cdot r_0}}$$

(4) $\Psi_T = (K - B(T,S))^+, \quad S > T$

$P_0 = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_T}(K - B(T,S)) \mathbb{1}_{\{K \geq B(T,S)\}}\right] = B(0,T)\mathbb{E}^{\mathbb{Q}^T}\left[K \mathbb{1}_{\{K \geq B(T,S)\}}\right] + B(0,S)\mathbb{E}^{\mathbb{Q}^S}\left[\mathbb{1}_{\{K \geq B(T,S)\}}\right] =$

$= \left\{ K \geq B(T,S) \Leftrightarrow \log K \geq m(T,S) - n(T,S) r_T \Leftrightarrow r_T \geq \frac{m(T,S) - \log K}{n(T,S)} = r^* \right\} =$

$= K B(0,T) \mathbb{Q}^T(r_T \geq r^*) - B(0,S)\mathbb{Q}^S(r_T \geq r^*)$

(5) On suppose $n(t,T)>0$. $0 \le T_0 < T_1 < \dots < T_n$, $\delta = T_{k+1} - T_k$.

Coupons c_k, $k=1,\dots,n$. On considère une put de strike K sur l'obligation.
($c_n = 1 +$ coupon)

$$\left(K - \sum_{j=1}^{n} c_j B(T_0,T_j)\right)^+ = \left(K - \sum_j c_j e^{m(T_0,T_j) - n(T_0,T_j) r_{T_0}}\right) \mathbb{1}_{\{K - \sum e^{\cdots} \ge 0\}} =$$

$$= \{K = \sum \underbrace{c_j e^{m(T_0,T_j) - n(T_0,T_j)\tilde r}}_{K_j}\} \ominus$$

fonction monotone de r $\to \exists$ une sol. $\tilde r$

$$\sum \left(K_j - c_j e^{m(T_0,T_j) - n(T_0,T_j) r_{T_0}}\right) \ge 0 \iff$$

$$\sum c_j \left(e^{m(T_0,T_j) - n(T_0,T_j)\tilde r} - e^{m(T_0,T_j) - n(T_0,T_j) r_T}\right) \ge 0 \iff$$

$$\iff \tilde r \le r_T$$

$$\ominus \sum_j \left(K_j - c_j e^{m(T_0,T_j) - n(T_0,T_j) r_{T_0}}\right)^+$$

(6) Swaption: entrer en T_0 dans un swap de taux fixe R.

(a) $$\text{Swaption}_{T_0} = \left(PV_{swap}(T_0)\right)^+ = \Big(\underbrace{B(T_0,T_0) - B(T_0,T_n)}_{\text{jambe variable}} - \sum_{j=1}^{n} R\cdot\delta\, B(T_0,T_j)\Big)^+ =$$

$$= \left(1 - \sum_{j=1}^{n} c_j B(T_0,T_j)\right)^+ \quad \text{où} \quad c_1 = \dots = c_{n-1} = \delta\cdot R, \quad c_n = 1+\delta R$$

put sur l'obligation de strike $K=1$

(b) $$\text{Swaption}_0 = \mathbb{E}^{\oplus}\left[\frac{B_t}{B_T}\left(1 - \sum c_j B(T_0,T_j)\right)^+\right] \overset{(5)}{=} \sum_j \mathbb{E}^{\oplus}\left[\frac{B_1}{B_T}\left(K_j - c_j B(T_0,T_j)\right)^+\right] \overset{(4)}{=}$$

$$= \sum_{j=1}^{n} c_j \left(B(0,T_0)\cdot\frac{K_j}{c_j}\,\oplus^T(r_T \ge r^*) - B(0,T_j)\,\oplus^S(r_T \ge r^*)\right) =$$

$$= \sum_{j=1}^{n} \left[K_j B(0,T_0)\,\oplus^T(r_T \ge r^*) - c_j B(0,T_j)\,\oplus^S(r_T \ge r^*)\right]$$

Exo 1 Remplacements du taux Libor

$$r_t = -\frac{\partial}{\partial T}\log B(t,T)\Big|_{T=t} = f(t,T) = \text{taux RFR}$$

OBS $\quad B_t = \exp\{\int_0^t r_s ds\}$

$$B(t,T) = B(0,T)\exp\left\{\int_0^t (r_u - A(u,T))\,du - \int_0^t \rho(u,T)\,dW_u\right\} \qquad W = (W^1, W^2)$$

détерm. bornées

$$F(t,T,\delta) = F(0,T,\delta)\exp\left\{\int_0^t \alpha(u,T,\delta)\,du + \int_0^t \sigma(u,T,\delta)\,dW_u\right\}$$

↳ taux forward à terme sur $[T,T+\delta]$

(1) Condition de drift : $\dfrac{B(t,T)}{B_t}$ doit être martingale

$$\frac{B(t,T)}{B_t} = B(0,T)\exp\left\{-\int_0^t A(u,T)\,du - \int_0^t \Gamma(u,T)\,dW_u\right\}$$

une mart ssi $A(u,T) = \frac{1}{2}|\Gamma(u,T)|^2$

(2) $$B(t,T) = \frac{B(t,T)}{B(t,t)} = \frac{B(0,T)}{B(0,t)}\exp\left\{-\int_0^t (A(s,T)-A(s,t))\,ds - \int_0^t (\Gamma(s,T)-\Gamma(s,t))\,dW_s\right\},\ t\le T$$

(3) $$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{B(t,T)}{B(0,T)}\frac{1}{B_t} = \exp\left\{-\int_0^t \overset{\Gamma(u,T)^2/2}{A(u,T)}\,du - \int_0^t \Gamma(u,T)\,dW_u\right\} = \mathcal{E}\left(-\int_0^{\cdot} \Gamma(u,T)\,dW_u\right)_t$$

(4) Sous $\mathbb{Q}^{T+\delta}$: $W_t^{T+\delta} = W_t + \int_0^t \Gamma(u,T+\delta)\,du$

$$F(t,T,\delta) = F(0,T,\delta)\exp\left\{\int_0^t [\alpha(u,T,\delta) - \Gamma(u,T+\delta)\sigma(u,T,\delta)]\,du + \int_0^t \sigma(u,T,\delta)\,dW_u^{T+\delta}\right\}$$

une $\mathbb{Q}^{T+\delta}$-martingale ssi $\alpha(u,T,\delta) - \Gamma(u,T+\delta)\sigma(u,T,\delta) = -\dfrac{|\sigma(u,T,\delta)|^2}{2}$

(5) $\Psi_{T+\delta} = (F(T,T,\delta)-K)^+\delta$ une $\mathbb{Q}^{T+\delta}$-mart de vol. $\sigma(u,T,\delta)$

$$V_t = \delta B(t,T+\delta)\,\mathbb{E}_t^{T+\delta}\left[(F(T,T,\delta)-K)^+\right] = \delta B(t,T,\delta)\big(F(t,T,\delta)\mathcal{N}(d_+) - K\mathcal{N}(d_-)\big)$$

où $d_\pm = \dfrac{\log\frac{F(t,T,\delta)}{K}}{\Sigma} \pm \frac{1}{2}\Sigma$, $\Sigma^2 = \int_t^T \sigma(u,T,\delta)^2\,du$

(6) $$F(t,T,\delta) = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

(a) Dynamique de $F(\cdot,T,\delta)$: c'est une $\mathbb{Q}^{T+\delta}$-martingale car $\dfrac{B(t,T)}{B(t,T+\delta)}$ est $\mathbb{Q}^{T+\delta}$-mart dont la vol est $\Gamma(t,T)-\Gamma(t,T+\delta)$

Donc $$dF(t,T,\delta) = \frac{-1}{\delta}\underbrace{\frac{B(t,T)}{B(t,T+\delta)}}_{1+\delta F(t,T,\delta)}(\Gamma(t,T)-\Gamma(t,T+\delta))\,dW_t^{T+\delta} = F(t,T,\delta)\underbrace{\frac{1+\delta F(t,T,\delta)}{F(t,T,\delta)}\begin{pmatrix}-\Gamma(t,T)+\\ +\Gamma(t,T+\delta)\end{pmatrix}}_{\text{vol de } F.\ \text{Pas déterministe si } \Gamma \text{ est déterministe}}dW_t^{T+\delta}$$

(b) Prix de caplet : payoff en T

$$\Psi_T = \Big(\frac{B(T,T)}{B(T,T+\delta)}\underbrace{-1-\delta K}_{=-\tilde{K}^{-1}}\Big)^+ \cdot B(T,T+\delta) = \tilde{K}^{-1}\big(\tilde{K} - B(T,T+\delta)\big)^+$$

Donc $$V_t = B(t,T)\tilde{K}^{-1}\,\mathbb{E}_t^T\left[\left(\tilde{K} - \frac{B(T,T+\delta)}{B(T,T)}\right)^+\right] = B(t,T)\tilde{K}^{-1}\left(\tilde{K}\mathcal{N}(-d_-) - \frac{B(t,T+\delta)}{B(t,T)}\mathcal{N}(-d_+)\right) =$$

$$= B(t,T)\,\mathcal{N}(d_-) - \tilde{k}^{-1} B(t,T+\delta)\,\mathcal{N}(-d_+)$$

où
$$d_\pm = \frac{\log \dfrac{B(t,T+\delta)}{\tilde{k}\,B(t,T)}}{\Sigma} \pm \frac{1}{2}\Sigma \,, \quad \Sigma^2 = \int_t^T |\Gamma(s,T+\delta) - \Gamma(s,T)|^2\, ds$$

$$V_t = B(t,\tilde{T})\,\mathbb{E}_t^{\tilde{T}}\left[\frac{\Psi_T}{B(T,\tilde{T})}\right]$$

$$\frac{B_T}{B_{T+\delta}} = e^{\int_T^{T+\delta} r_u\, du} = B(T,T+\delta)$$

(7)
$$\pi_0^{swn} = B(0,T_0)\,\bar{\delta}\,\mathbb{E}^{T_0}\left[\left(\sum_{k=1}^n B(T_0,T_k)F(T_0,T_{k-1},\delta) - R\sum_{k=1}^n B(T_0,T_k)\right)^+\right]$$

$$\Gamma(t,S) - \Gamma(t,T) = \begin{pmatrix} \lambda(t)\Phi(T,S) \\ 0 \end{pmatrix}$$ tout les fcts déterministes bornées

$$\sigma(t,T,\delta) = \begin{pmatrix} \lambda(t)\Phi(T,\delta) \\ \bar{\lambda}(t)\bar{\Phi}(T,\delta) \end{pmatrix}$$

(a) Vérifier que
$$B(T_0,T_k) = e^{\varsigma(T_0,T_k) + \vartheta(T_0,T_k) Z^1_{T_0}}$$
$$F(T_0,T_{k-1},\bar{\delta}) = e^{\bar{\varsigma}(T_0,T_{k-1},\bar{\delta}) + \bar{\vartheta}_1(T_0,T_{k-1},\bar{\delta}) Z^1_{T_0} + \bar{\vartheta}_2(T_0,T_{k-1},\bar{\delta}) Z^2_{T_0}}$$

$$Z^1_{T_0} = \int_0^{T_0} \lambda(t)\, dW^1_t \,, \quad Z^2_{T_0} = \int_0^{T_0} \bar{\lambda}(t)\, dW^2_t$$

D'après (2),
$$B(T_0,T_k) = \frac{B(0,T_k)}{B(0,T_0)} \exp\left\{-\int_0^{T_0}\left(\frac{\Gamma(s,T_k)^2}{2} - \frac{\Gamma(s,T_0)^2}{2}\right)ds - \underbrace{\int_0^{T_0}(\Gamma(s,T_k) - \Gamma(s,T_0))\cdot dW_s}_{-\int_0^{T_0}\lambda(s)\Phi(T_0,T_k)\,dW^1_s}\right\} =$$
$$\longrightarrow \varsigma(T_0,T_k)$$

$$= \exp\{\varsigma(T_0,T_k) \underbrace{- \Phi(T_0,T_k)}_{\vartheta(T_0,T_k)} Z^1_{T_0}\}$$

$$F(T_0,T_{k-1},\bar{\delta}) \overset{(4)}{=} \exp\Big\{\underbrace{\log F(0,T_{k-1},\bar{\delta}) + \int_0^{T_0}\Big(\underbrace{\Gamma(u,T_{k-1}+\bar{\delta})\sigma(u,T_{k-1},\delta)}_{\Gamma_1\lambda\Phi + \Gamma_2\bar{\lambda}\bar{\Phi}} - \underbrace{\frac{|\sigma(u,T_{k-1},\bar{\delta})|^2}{2}}_{\frac{(\lambda\Phi)^2 + (\bar{\lambda}\bar{\Phi})^2}{2}}\Big)du}_{\bar{\varsigma}(T_0,T_{k-1},\bar{\delta})} + \int_0^{T_0}\sigma(u,T_{k-1},\bar{\delta})\,dW_u\Big\}$$

$$\int_0^{T_0}\sigma(u,T_{k-1},\bar{\delta})\,dW_u = \underbrace{\Phi(T_{k-1},\bar{\delta})}_{\bar{\vartheta}_1}\underbrace{\int_0^{T_0}\lambda(u)\,dW^1_u}_{Z^1_{T_0}} + \underbrace{\bar{\Phi}(T_{k-1},\bar{\delta})}_{\bar{\vartheta}_2}\underbrace{\int_0^{T_0}\bar{\lambda}(u)\,dW^2_u}_{Z^2_{T_0}}$$

(b)
$$\pi_0^{swn} = B(0,T_0)\,\bar{\delta}\,\mathbb{E}^{T_0}\left[\left(\sum_{k=1}^n B(T_0,T_k)F(T_0,T_{k-1},\delta) - R\sum_{k=1}^n B(T_0,T_k)\right)^+\right] =$$

$$= B(0,T_0)\,\bar{\delta}\,\mathbb{E}^{T_0}\left[\left(\sum_{k=1}^n e^{\varsigma(T_0,T_k) + \vartheta(T_0,T_k)Z^1_{T_0}}\left(e^{\bar{\varsigma}(T_0,T_{k-1},\bar{\delta}) + \bar{\vartheta}_1(T_0,T_{k-1},\bar{\delta})Z^1_{T_0} + \bar{\vartheta}_2(T_0,T_{k-1},\bar{\delta})Z^2_{T_0}} - R\right)\right)^+\right] =$$

$$= \left\{R = \sum_{k=1}^n e^{\bar{\varsigma}(T_0,T_{k-1},\bar{\delta}) + \bar{\vartheta}_1(T_0,T_{k-1},\bar{\delta})Z^1_{T_0} + \bar{\vartheta}_2(T_0,T_{k-1},\bar{\delta})\bar{z}_2(Z^1_{T_0})}\right\} =$$
(Jamshidian)

$$= B(0,T_0)\,\bar{\delta}\,\mathbb{E}^{T_0}\left[\sum_{k=1}^n e^{\underbrace{\varsigma(T_0,T_k) + \bar{\varsigma}(T_0,T_{k-1},\bar{\delta})}_{\eta} + \underbrace{(\vartheta(T_0,T_k) + \bar{\vartheta}_1(T_0,T_{k-1},\bar{\delta}))}_{\xi_1} Z^1_{T_0}}\left(e^{\underbrace{\bar{\vartheta}_2(T_0,T_{k-1},\bar{\delta})}_{\xi_2} Z^2_{T_0}} - \underbrace{e^{\bar{\vartheta}_2(T_0,T_{k-1},\bar{\delta})\bar{z}_2}}_{K(Z^1_{T_0})}\right)^+\right] =$$

$Z^i_{T_0} \sim f_i(z_i)$

$$= B(0,T_0)\,\delta \sum_{k=1}^{n} e^{\eta(T_0,T_{k-1},\delta)} \int_{\mathbb{R}} e^{\xi_1(T_0,T_{k-1},\delta) z_1} \left(\int_{\mathbb{R}} \left(e^{\xi_2(T_0,T_{k-1},\delta) z_2} - K(z_1) \right)^+ f_2(z_2)\, dz_2 \right) f(z_1)\, dz_1$$

Exo 2 Modèle CIR

$$dY_t = (2aY_t + \sigma^2)\,dt + 2\sigma\sqrt{Y_t}\,dW_t, \quad Y_0 > 0, \quad a, \sigma > 0$$

$Z_t = \sqrt{Y_t}$. Trouver l'EDS pour Z_t.

$$dZ_t = \frac{1}{2Z_t} dY_t - \frac{1}{8Z_t^3} d\langle Y\rangle_t = \frac{2aZ_t^2 + \cancel{\sigma^2}}{2Z_t} dt + \sigma dW_t - \cancel{\frac{1}{8Z_t^3} 4\sigma^2 Z_t^2 dt} =$$

$$= aZ_t\,dt + \sigma dW_t \text{ - processus d'O.U.}$$

Exo 3 Modèle de Ho et Lee

$$dr_t = \theta(t)\,dt + \sigma dW_t, \quad r_0 > 0, \quad \sigma > 0 \quad \text{sous } \mathbb{Q}$$

θ déterministe, loc bornée

(1) $$r_t = r_0 + \underbrace{\int_0^t \theta(s)\,ds}_{I_t} + \sigma W_t$$

$$\mathbb{E} r_t = r_0 + \int_0^t \theta(s)\,ds \qquad \mathbb{V}ar[r_t] = \sigma^2 t$$

(2) $$X_{t,T} = \int_t^T r_u\,du \qquad \begin{cases} dX_{t,T} = -r_T\,dT & \text{en } T \\ X_{t,t} = 0 \end{cases}$$

$$X_{t,T} = r_0(T-t) + \underbrace{\int_t^T I_s\,ds}_{\text{déterministe}} + \sigma \underbrace{\int_t^T (W_s - W_t)\,ds}_{\perp\!\!\!\perp \mathcal{F}_t} + \underbrace{\sigma(T-t)W_t}$$

$$\mathbb{E}[X_{t,T} | \mathcal{F}_t] = (r_0 + \sigma W_t)(T-t) + \int_t^T \int_0^s \theta(u)\,du\,ds$$

$$\mathbb{V}ar[X_{t,T} | \mathcal{F}_t] = \sigma^2\, \mathbb{V}ar\left[\int_0^{T-t} B_u\,du\right] = \sigma^2 \frac{(T-t)^3}{3}$$

(3) $$B(t,T) = \mathbb{E}\left[\underset{\text{Gaussien sachant } \mathcal{F}_t}{e^{-X_{t,T}}} \middle| \mathcal{F}_t\right] = \exp\left\{ -(r_0 + \sigma W_t)(T-t) - \int_t^T \int_0^s \theta(u)\,du\,ds + \frac{\sigma^2}{6}(T-t)^3 \right\} =$$

$$= \exp\left\{ -\underbrace{(T-t)}_{n(t,T)} r_t + \underbrace{(T-t)\int_0^t \theta(s)\,ds - \int_t^T \int_0^s \theta(u)\,du\,ds + \frac{\sigma^2}{6}(T-t)^3}_{m(t,T)} \right\}$$

(4) On a montré dans (3) que $B(t,T) = B(t,T,r_t) = F(t,r_t)$

Comme toujours, par FK on obtient l'EDP :

$$\begin{cases} \mathcal{L}F(t,r) = rF(t,r) & \text{où } \mathcal{L}F(t,r) = \partial_t F + \theta \cdot \partial_r F + \frac{1}{2}\sigma^2 \cdot \partial^2_{rr} F \\ F(T,r) = 1 \end{cases}$$

$$\begin{cases} \partial_t F + \theta(t)\partial_r F + \frac{1}{2}\sigma^2 \partial^2_{rr} F = rF \\ F|_{t=T} = 1 \end{cases} \leftarrow F(t,r_t) \text{ trouvée dans (3) est une solution de cette EDP}$$

(5) Dans (3), $B(t,T) = e^{-n(t,T)r_t + m(t,T)} \rightarrow F(t,r) = e^{m-nr}$

$\partial_t F = F(\dot m - \dot n r) \quad \partial_r F = -nF \quad \partial^2_{rr} F = n^2 F \rightsquigarrow EDP$

$$\dot m - \dot n r - \theta n + \frac{1}{2}\sigma^2 n^2 = r$$

$\begin{cases} \dot n = -1 \\ n(T) = 0 \end{cases}$ - c'est le cas $n(t) = (T-t)$

$\begin{cases} \dot m = \theta n - \frac{1}{2}\sigma^2 n^2 \\ m(T) = 0 \end{cases}$ $\quad m(t,T) = (T-t)\int_0^t \theta(s)ds - \int_0^t\int_0^s \theta(u)du\,ds + \frac{\sigma^2}{6}(T-t)^3$

$m(T,T) = 0$

$\dot m(t,T) = \theta(t)\underbrace{(T-t)}_{n} - \cancel{\int_0^t \theta(s)ds} + \cancel{\int_0^t \theta(u)du} - \frac{\sigma^2}{2}\underbrace{(T-t)^2}_{n^2} = \theta n - \frac{1}{2}\sigma^2 n^2$

Alors F est une solution de l'EDP

(6) $\theta(t) = \frac{\partial}{\partial t}\hat f(0,t) + \sigma^2 t$

On veut calibrer la courbe initiale $\hat B(0,T) = e^{-\int_0^T \hat f(0,u)du} \overset{(4)}{=} e^{-n(0,T)r_0 + m(0,T)}$

$$-\int_0^T \hat f(0,u)du = -Tr_0 - \int_0^T\int_0^s \theta(u)du\,ds + \frac{\sigma^2}{6}T^3 \quad \forall T$$

$\Big\Downarrow \partial_T$

$$-\hat f(0,T) = -r_0 - \int_0^T \theta(u)du + \frac{\sigma^2 T^2}{2}$$

$\Big\Downarrow \partial_T$

$$-\partial_T \hat f(0,T) = -\theta(T) + \sigma^2 T$$

Alors $\theta(T) = \partial_T \hat f(0,T) + \sigma^2 T$

Exo 4 Modèle de Vasiček exponentiel

$y_t = \log r_t \qquad r_t = e^{y_t}$

$dy_t = (\theta - a y_t)dt + \sigma dW_t$ - O.U., $\theta, a, \sigma > 0$

(1) $dr_t = r_t dy_t + \frac{1}{2} r_t d\langle y\rangle_t = r_t\big((\theta - a\underbrace{y_t}_{\log r_t})dt + \sigma dW_t + \frac{1}{2}\sigma^2 dt\big)$

Alors l'EDS pour r_t: $\frac{dr_t}{r_t} = (\theta + \frac{\sigma^2}{2} - a\log r_t)dt + \sigma dW_t$

(2) La forme explicite de r_t?

$$d(e^{at}y_t) = e^{at}\theta dt + \sigma e^{at}dW_t$$

$$e^{at}y_t = y_0 + \theta\frac{e^{at}-1}{a} + \sigma\int_0^t e^{as}dW_s$$

$$y_t = e^{-at} y_0 + \frac{\theta}{a}(1-e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dW_s$$

$$r_t = \exp\left\{ e^{-at} y_0 + (1-e^{-at})\frac{\theta}{a} + \underbrace{\sigma \int_0^t e^{-a(t-s)} dW_s}_{\mathcal{N}\left(0,\, \sigma^2 \frac{1-e^{-2at}}{2a}\right)} \right\}$$

$$\mathbb{E}[r_t] = \exp\left\{ \underbrace{e^{-at} y_0 + (1-e^{-at})\frac{\theta}{a}}_{\mu(t)} + \underbrace{\frac{\sigma^2}{2} \cdot \frac{(1-e^{-2at})}{2a}}_{\Sigma(t)^2/2} \right\}$$

$$r_t^2 = \exp\left\{ 2\mu(t) + 2\sigma \int_0^t e^{-a(t-s)} dW_s \right\}$$

$$\mathbb{E}[r_t^2] = \exp\left\{ 2\mu(t) + 4\frac{\Sigma(t)^2}{2} \right\} = \exp\left\{ 2\mu(t) + 2\Sigma(t)^2 \right\}$$

$$\mathbb{V}\mathrm{ar}[r_t] = \exp\left\{ 2\mu(t) + \Sigma(t)^2 \right\} \left(e^{\Sigma(t)^2} - 1 \right)$$

(4) $0 \leq s \leq t \quad y_t = e^{-a(t-s)} y_s + (1-e^{-a(t-s)})\frac{\theta}{a} + \sigma \int_s^t e^{-a(t-u)} dW_u$

$$r_t = \exp\left\{ \underbrace{e^{-a(t-s)} y_s}_{\mathcal{F}_s^{\theta}\text{-mes}} + \underbrace{(1-e^{-a(t-s)})\frac{\theta}{a}}_{\text{déterm.}} + \underbrace{\sigma \int_s^t e^{-a(t-u)} dW_u}_{\perp\!\!\!\perp \mathcal{F}_s^{\theta},\ \sim \mathcal{N}\left(0,\, \sigma^2 \frac{1-e^{-2a(t-s)}}{2a}\right)} \right\}$$

$$\mathbb{E}[r_t | \mathcal{F}_s^{\theta}] = \exp\left\{ \underbrace{e^{-a(t-s)} y_s + (1-e^{-a(t-s)})\frac{\theta}{a}}_{\mu_{s,t}} \right\} \cdot \exp\left\{ \underbrace{\frac{\sigma^2}{2} \frac{1-e^{-2a(t-s)}}{2a}}_{\Sigma_{s,t}^2} \right\}$$

Comme dans (3), on trouve

$$\mathbb{V}\mathrm{ar}[r_t | \mathcal{F}_s^{\theta}] = \exp\left\{ 2\mu_{s,t} + \Sigma_{s,t}^2 \right\} \left(e^{\Sigma_{s,t}^2} - 1 \right)$$

Quand $t \to +\infty$ $\quad \mu_{s,t} \to \frac{\theta}{a} \quad \Sigma_{s,t}^2 \to \frac{\sigma^2}{4a}$ et donc

$$\lim_{t\to\infty} \mathbb{E}^{\theta}[r_t | \mathcal{F}_s^{\theta}] = \exp\left\{ \frac{\theta}{a} + \frac{\sigma^2}{4a} \right\}$$

$$\lim_{t\to\infty} \mathbb{V}\mathrm{ar}[r_t | \mathcal{F}_s^{\theta}] = \exp\left\{ 2\frac{\theta}{a} + \frac{\sigma^2}{4a} \right\} \left(e^{\sigma^2/4a} - 1 \right)$$

(5) soit $t \geq s$ $\quad \mathbb{E}^{\theta}[r_t r_s] = \mathbb{E}^{\theta}\left[\mathbb{E}^{\theta}\left[\exp\{y_s\} \exp\left\{ \mu_{s,t} + \sigma \int^t e^{-a(t-u)} dW_u \,\middle|\, \mathcal{F}_s^{\theta} \right\} \right] \right] =$

$$= \mathbb{E}^{\theta}\left[e^{\overbrace{(1+e^{-a(t-s)})}^{A(s,t)} y_s + (1-e^{-a(t-s)})\frac{\theta}{a} + \frac{\sigma^2}{2} \frac{1-e^{-2a(t-s)}}{2a}} \right] =$$

$$\mathbb{V}\mathrm{ar}[\dots] = \sigma^2 A^2 \frac{1-e^{-2as}}{2a}$$

$$= \exp\left\{ (1-e^{-a(t-s)})\frac{\theta}{a} + \frac{\sigma^2}{2} \frac{1-e^{-2a(t-s)}}{2a} \right\} \mathbb{E}^{\theta}\left[\exp\left\{ A\left(e^{-as} y_0 + \frac{\theta}{a}(1-e^{-as})\right) + \overbrace{\sigma A \int_0^s e^{-a(s-u)} dW_u}^{} \right\} \right] =$$

$$= \exp\left\{ (1-e^{-a(t-s)})\frac{\theta}{a} + \frac{\sigma^2}{2} \frac{1-e^{-2a(t-s)}}{2a} + (1+e^{-a(t-s)})\left(e^{-as} y_0 + \frac{\theta}{a}(1-e^{-as})\right) + \frac{1}{2}\sigma^2 (1+e^{-a(t-s)})^2 \frac{1-e^{-2as}}{2a} \right\} =$$

$$= \exp\left\{(e^{-as}+e^{-at})y_0 + (2-e^{-as}-e^{-at})\frac{\theta}{a} + \frac{\sigma^2}{2}\frac{1-e^{-2as}}{2a}\left(1+(1+e^{-a(t-s)})^2\right)\right\}$$

(6) $$B(0,T) \approx B_1(0,T) = \mathbb{E}^{\mathbb{Q}}\left[1-\int_0^T r_s ds\right] = 1-\int_0^T \mathbb{E}^{\mathbb{Q}}[r_s]ds =$$

$$= 1-\int_0^T \exp\left\{e^{-at}y_0 + (1-e^{-at})\frac{\theta}{a} + \frac{\sigma^2}{2}\cdot\frac{(1-e^{-2at})}{2a}\right\}dt = \left\{\begin{array}{l} u = e^{-at} \\ du = -a\underbrace{e^{-at}}_{u}dt \end{array}\right\} =$$

$$= 1-\int_{e^{-aT}}^1 \exp\left\{u y_0 + (1-u)\frac{\theta}{a} + \frac{\sigma^2}{2}\frac{(1-u^2)}{2a}\right\}\frac{du}{au}$$

donné par (6)

(7) $$B_2(0,T) = B_1(0,T) + \frac{1}{2}\mathbb{E}\left[\left(\int_0^T r_s ds\right)^2\right]$$

donnée par (5)

$$\frac{1}{2}\mathbb{E}\left[\left(\int_0^T r_s ds\right)^2\right] = \frac{1}{2}\mathbb{E}\left[\int_0^T\int_0^T r_s r_t ds dt\right] = \int_0^T\int_s^T \mathbb{E}[r_t r_s]dtds$$

Exo 5

(1) $\psi_T = (B(T,T+\delta)-K)^+$. $V_0 = ?$

Vasiček-Hull-White étendu:

$$dr_t = (b(t)-ar_t)dt + \sigma dW_t$$
$$d(r_t e^{at}) = e^{at}b(t)dt + \sigma e^{at}dW$$
$$r_t = e^{-at}r_0 + \int_0^t e^{-a(t-s)}b(s)ds + \sigma\int_0^t e^{-a(t-s)}dW_s$$

$$I_t = \int_0^t r_s ds \qquad a\,dI_t = ar_t dt = b(t)dt + \sigma dW_t - dr_t$$
$$a(I_T - I_t) = \int_t^T b(s)ds + \int_t^T \sigma dW_s - (r_T - r_t)$$

$$r_T = e^{-a(T-t)}r_t + \int_t^T e^{-a(T-s)}b(s)ds + \sigma\int_t^T e^{-a(T-s)}dW_s$$

$$r_t - r_T = (1-e^{-a(T-t)})r_t - \int_t^T \ldots ds - \sigma\int_t^T \ldots dW_s$$

$$I_T - I_t = \frac{1-e^{-a(T-t)}}{a}r_t + \int_t^T b(s)\left(\frac{1-e^{-a(T-s)}}{a}\right)ds + \sigma\underbrace{\int_t^T\left(\frac{1-e^{-a(T-s)}}{a}\right)dW_s}_{\mathcal{N}\left(0,\frac{\sigma^2}{a^2}\frac{1-e^{-2a(T-t)}}{2a}\right)}$$

$$B(t,T) = \mathbb{E}\left[e^{-(I_T-I_t)}\middle|\mathcal{F}_t\right] = \exp\left\{-\left(\frac{1-e^{-a(T-t)}}{a}\right)r_t - \int_t^T b(s)\left(\frac{1-e^{-a(T-s)}}{a}\right)ds + \frac{\sigma^2}{2a^2}\left(\frac{1-e^{-2a(T-t)}}{2a}\right)\right\}$$

On va noter $k(x) = \frac{1-e^{-ax}}{a}$, $k_2(x) = \frac{1-e^{-2ax}}{2a}$

$$B(T,T+\delta) = \exp\left\{-k(\delta) r_T - \int_T^{T+\delta} \theta(s) k(T+\delta-s)\,ds + \frac{\sigma^2}{2a^2} k_2(\delta)\right\} =$$

$$= \exp\left\{-k(\delta)\left(e^{-aT} r_0 + \int_0^T e^{-a(T-s)}\theta(s)\,ds + \sigma\int_0^T e^{-a(T-s)}\,dW_s\right) - \int_T^{T+\delta}\theta(s)k(T+\delta-s)\,ds + \frac{\sigma^2}{2a^2}k_2(\delta)\right\} =$$

$$= \exp\Big\{\underbrace{-\sigma k(\delta)\int_0^T e^{-a(T-s)}\,dW_s - \frac{1}{2}\sigma^2 k(\delta)^2 k_2(T)}_{\text{partie martingale}} + \xi(T,\delta)\Big\}$$

On veut calculer le prix sous $\mathbb{Q}^T$:

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{B(t,T)}{B_t}\cdot\frac{B_0}{B(0,T)} = \mathcal{E}\left(\int_0^t -\varphi(T-t)\sigma(e^{-a(T-s)}+1)\,dW_s\right)_t$$

$$W_t^T = W_t + \sigma\int_0^t \varphi(T-t)(1+e^{-a(T-s)})\,ds$$

$$B(T,T+\delta) = \exp\Big\{-\sigma k(\delta)\int_0^T e^{-a(T-s)}\,dW_s^T - \frac{1}{2}\sigma^2 k(\delta)^2 k_2(T) + \underbrace{\sigma k(\delta)\int_0^T \varphi(T-t)e^{-a(T-s)}(1-e^{-a(T-s)})\,ds + \xi}_{\tilde{\xi}}\Big\}$$

$$V_0 = B(0,T)\,\mathbb{E}^T\left[\left(e^{\tilde{\xi}}\cdot e^{-\sigma k(\delta)\int_0^T e^{-a(T-s)}\,dW_s^T - \frac{1}{2}\sigma^2 k(\delta)^2 k_2(T)} - K\right)^+\right] =$$

$$= B(0,T)\,e^{\tilde{\xi}}\left(N(d_+) - e^{-\tilde{\xi}} K N(d_-)\right) \qquad d_\pm = \frac{\log\left(\frac{e^{\tilde{\xi}}}{K}\right)}{\Sigma} \pm \frac{1}{2}\Sigma$$

$$\Sigma^2 = \sigma^2 k(\delta)^2 k_2(T)$$

(2) Même question pour Ho-Lee

On a trouvé dans l'exo 3:

$$B(t,T) = \exp\Big\{-\underbrace{(T-t)}_{n(t,T)} r_t + \underbrace{(T-t)\int_0^t \theta(s)\,ds - \int_t^T\int_0^s \theta(u)\,du\,ds + \frac{\sigma^2}{6}(T-t)^3}_{m(t,T)}\Big\}$$

$$r_t = r_0 + \int_0^t \theta(s)\,ds + \sigma W_t$$

$$\int_t^T r_s\,ds = X_{t,T} = r_0(T-t) + \underbrace{\int_t^T I_s\,ds}_{\text{déterministe}} + \sigma\int_t^T \underbrace{(W_s - W_t)}_{\perp\!\!\!\perp \mathcal{F}_t}\,ds + \underbrace{\sigma(T-t)W_t}$$

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{B(t,T)}{B_t}\,\frac{1}{B(0,T)} = \exp\left\{-(T-t)\sigma W_t - \sigma\int_0^t W_s\,ds + \dots\right\} =$$

$$\int_0^t W_s\,ds + \int_0^t s\,dW_s = W_t t$$

$$= \exp\left\{-(T-\cancel{t})\sigma W_t - \cancel{\sigma t W_t} + \sigma\int_0^t s\,dW_s + \dots\right\} = \mathcal{E}\left(-\sigma\int_0^t (T-s)\,dW_s\right)_t$$

$$W_t^T = W_t + \sigma\int_0^t (T-s)\,ds = W_t + \sigma\left(\frac{T^2}{2} - \frac{(T-t)^2}{2}\right) = W_t + \sigma\left(Tt - \frac{t^2}{2}\right)$$

$$B(T,T+\delta) = \exp\left\{-\delta\left(r_0 + \int_0^T \theta(s)ds + \sigma W_T^T - \sigma^2\left(T(T+\delta) - \frac{T^2}{2}\right)\right) + m(T,T+\delta)\right\}$$

$$= \exp\left\{-\delta\sigma W_T^T - \frac{1}{2}\delta^2\sigma^2 T + \tilde{m}(T,T+\delta)\right\}$$

$$V_0 = B(0,T)\,\mathbb{E}^T\left[\left(e^{\tilde{m}} \cdot e^{-\delta\sigma W_T^T - \frac{1}{2}\delta^2\sigma^2 T} - K\right)^+\right] =$$

$$= B(0,T)\,e^{\tilde{m}}\left(N(d_+) - e^{-\tilde{m}} K N(d_-)\right) \qquad d_\pm = \frac{\log\left(\frac{e^{\tilde{m}}}{K}\right)}{\Sigma} \pm \frac{1}{2}\Sigma$$

$$\Sigma^2 = \delta^2\sigma^2 T$$

Exo 6

Nominal K, échéance $T>0$.

K_A montant capitalisé sur $[0,T]$ à partir du montant K en utilisant le taux fixe R. K_B est le montant capitalisé sur $[0,T]$ à partir du montant K en utilisant le taux court shochastique r_t. À la date T:

- $\uparrow K_A - K$ est reçu
- $\downarrow K_B - K$ est payé

- Calculer le valeurs des jambes en $t=0$ et $V(0;K,R,T)$.
- Trouver R^* t.q. $V(0;K,R^*,T)=0$

$$K_A = K e^{RT} \qquad K_B = K e^{\int_0^T r_t dt}$$

$$PV_{fixe} = K(e^{RT}-1)B(0,T)$$

$$PV_{var.} = K\left(\mathbb{E}\left[\frac{1}{B_T} e^{\int_0^T r_t dt} - \frac{1}{B_T}\right]\right) = K(1 - B(0,T))$$

$$V(0;K,R,T) = KB(0,T)e^{RT} - \cancel{KB(0,T)} - K + \cancel{KB(0,T)} = K\left(e^{RT}B(0,T) - 1\right)$$

$$V=0 \text{ si } \quad B(0,T) = e^{-R^*T} \qquad R^* = -\frac{1}{T}\log B(0,T)$$