TD - MEDAF

Exercice 1

a). D'après le MEDAF : $r_p = r_f + \beta_p (r_M - r_f)$

$$\Rightarrow \beta_p = \frac{r_p - r_f}{r_M - r_f} = \frac{0{,}2 - 0{,}05}{0{,}15 - 0{,}05} = 1{,}5$$

Or $\beta_p = \frac{\sigma_{pM}}{\sigma_M} \Rightarrow \sigma_{pM} = \beta_p \sigma_M^2 = 1{,}5 \times 0{,}2^2 = 0{,}06$

• P efficient $\Rightarrow P \in CML \Rightarrow r_p = r_f + \frac{r_M - r_f}{\sigma_M} \sigma_p$

$$0{,}2 = 0{,}05 + \frac{0{,}15 - 0{,}05}{0{,}2} \times \sigma_p \Rightarrow \sigma_p = 0{,}3$$

b) On sait que $\sigma_k^2 = \underbrace{\beta_k^2 \sigma_M^2}_{\text{R. systématique}} + \underbrace{\sigma_{\varepsilon k}^2}_{\text{R. spécifique}}$

D'après le MEDAF. $\beta_k = \frac{r_k - r_f}{r_M - r_f} = \frac{0{,}25 - 0{,}05}{0{,}15 - 0{,}05} = 2$

$\Rightarrow$ risque systématique $= 2^2 \times 0{,}2^2 = 16\%$

$\Rightarrow$ risque spécifique $= \sigma_k^2 - \beta_k^2 \sigma_M^2 = 52\% - 16\% = 36\%$

Exercice 2

a) • Méthode 1 $\qquad \mu_P = \frac{\mu_A + \mu_B + \mu_C}{3}$

D'après le MEDAF : $\mu_A = r_f + \beta_A(\mu_M - r_f)$

$$= 0{,}1632$$

$$\mu_B = 0{,}2082$$

$$\mu_C = 0{,}2208$$

$$\Rightarrow \mu_P = \frac{0{,}1632 + 0{,}2082 + 0{,}2208}{3} = 0{,}1974$$

• Méthode 2 : Avec le MEDAF : $\mu_P = r_f + \underbrace{\beta_P}_{?}(\mu_M - r_f)$

Or β_P est une mesure linéaire $\quad \beta_P = \frac{\beta_A + \beta_B + \beta_C}{3} = 0{,}86$

$$\Rightarrow \mu_P = 0{,}1974$$

b) $$\underbrace{\sigma_P^2}_{\text{r total}} = \underbrace{\beta_P^2 \, \sigma_M^2}_{\text{risque systématique}} + \underbrace{\sigma_{\varepsilon_P}^2}_{\text{risque spécifique}}$$

risque systématique $= 0{,}86^2 \times 0{,}24^2 = 0{,}0426$

risque spécifique $= Var\left(\frac{\varepsilon_A + \varepsilon_B + \varepsilon_C}{3}\right)$

Explication : D'après le modèle diagonal + MEDAF

$$\tilde{R}_i = \underbrace{r_f + \beta_i(\tilde{R}_M - r_f) + \varepsilon_i}_{= cte + \beta_i \tilde{R}_M + \varepsilon_i} \quad \text{avec} \quad \tilde{R}_M \perp\!\!\!\perp \varepsilon_i$$

$$cte = r_f(1-\beta_i)$$

$$\tilde{R}_p = \frac{\tilde{R}_A + \tilde{R}_B + \tilde{R}_C}{3}$$

$$\sigma_p^2 = Var(\tilde{R}_p) = \frac{1}{9} Var(\tilde{R}_A + \tilde{R}_B + \tilde{R}_C)$$

$$= \frac{1}{9} Var(\underbrace{cte + \beta_A \tilde{R}_M + \varepsilon_A}_{= \tilde{R}_A} + \underbrace{cte_B + \beta_B \tilde{R}_M + \varepsilon_B}_{\tilde{R}_B} + \underbrace{cte_C + \beta_C \tilde{R}_M + \varepsilon_C}_{= \tilde{R}_C})$$

$$= \frac{1}{9} Var(cte + \tilde{R}_M(\beta_A + \beta_B + \beta_C) + \varepsilon_A + \varepsilon_B + \varepsilon_C)$$

$$= \underbrace{\frac{(\beta_A + \beta_B + \beta_C)^2}{9}}_{= \beta_p^2} Var(\tilde{R}_M) + \frac{1}{9} Var(\varepsilon_A + \varepsilon_B + \varepsilon_C)$$

$$\sigma_p^2 = \beta_p^2 \sigma_M^2 + \underbrace{\frac{1}{9} Var(\varepsilon_A + \varepsilon_B + \varepsilon_C)}_{= \sigma^2_{\varepsilon_p} \text{ risque spécifique}}$$

risque spécifique $= \frac{1}{9} Var(\varepsilon_A + \varepsilon_B + \varepsilon_C)$

$$= \frac{1}{9}\Big[\underbrace{\sigma^2_{\varepsilon_A} + \sigma^2_{\varepsilon_B} + \sigma^2_{\varepsilon_C}}_{3^e \text{ colonne}} + \underbrace{2cov(\varepsilon_A, \varepsilon_B) + 2cov(\varepsilon_A, \varepsilon_C) + 2cov(\varepsilon_B, \varepsilon_C)}_{\text{dernière colonne du tbl.}}\Big]$$

Donc $\sigma_{\varepsilon_p}^2 = \frac{1}{9}\left[0,35+0,21+0,3-2\times0,0171-2\times0,026-2\times0,04322\right]$

$= 7,6\ \%$

$\Rightarrow \sigma_p^2 = 4,26\% + 7,6\% = 11,86\ \%$

Exercice 3

a) $Sh_i = \dfrac{\mu_i - rf}{\sigma_i}$

b) $(\omega \quad 1-\omega)'$

$Sh_p = \dfrac{\mu_p - rf}{\sigma_p}$ où $\mu_p = \omega\mu_1 + (1-\omega)\mu_2$

$\sigma_p^2 = \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\rho\sigma_1\sigma_2$

$$Sh_p = \frac{\omega\mu_1 + (1-\omega)\mu_2 - rf}{\sqrt{\omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\rho\sigma_1\sigma_2}}$$

c) $\mu_2 = rf$ et $\sigma_2 = 0$

$$Sh_p = \frac{\omega\mu_1 + (1-\omega)rf - rf}{\sqrt{\omega^2\sigma_1^2}} = \frac{\omega\mu_1 - \omega rf}{\sqrt{\omega^2\sigma_1^2}} = \frac{\omega(\mu_1 - rf)}{\sigma_1|\omega|}$$

car les ventes à découvertes peuvent être autorisées

$$Sh_p = \frac{\mu_1 - rf}{\sigma_1} \times \underbrace{\frac{\omega}{|\omega|}}_{=\,\mathrm{sgn}(\omega)} = \begin{cases} -Sh_1 & \text{si } \omega < 0 \\ +Sh_1 & \text{si } \omega > 0 \end{cases}$$

Exercice 4

$$P = \left(\frac{1}{n} \quad \frac{1}{n} \quad \dots \quad \frac{1}{n}\right)'$$

a) $Sh_p = \dfrac{\mu_p - r_f}{\sigma_p}$ où $\mu_p = \dfrac{1}{n} \sum_{i=1}^{n} \mu_i$

$$\sigma_p^2 = Var(\tilde{R}_p) = \frac{1}{n^2} Var(\tilde{R}_1 + \tilde{R}_2 + \dots + \tilde{R}_n)$$

$$\overset{\tilde{R}_i \perp\!\!\!\perp}{=} \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2$$

$$\Rightarrow Sh_p = \frac{\frac{1}{n}\left(\sum_{i=1}^{n} \mu_i\right) - r_f}{\frac{1}{n}\sqrt{\sum_{i=1}^{n} \sigma_i^2}} = \frac{\cancel{\frac{1}{n}} \sum_{i=1}^{n} (\mu_i - r_f)}{\cancel{\frac{1}{n}} \sqrt{\sum_{i=1}^{n} \sigma_i^2}}$$

b) $Sh_p = \sum_{i=1}^{n} \underbrace{\dfrac{\mu_i - r_f}{\sigma_i}}_{= Sh_i} \times \dfrac{\sigma_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}}$

$$= \sum_{i=1}^{n} Sh_i \times \underbrace{\frac{\sigma_i}{\sum_{i=1}^{n} \sigma_i^2}}_{\text{Notation } = p_i}$$

c) $p_i > 0$ car $\sigma_i > 0$

$\sigma_i^2 < \sum \sigma_j^2 \Rightarrow \sigma_i < \sqrt{\sum_{j=1}^{n} \sigma_j^2} \Rightarrow p_i < 1$

NB : si les actifs sont corrélés, on doit prendre en compte les cov dans σ_p^2

Exercice 5

a) $(\alpha \quad 1-\alpha)'$ $\quad \tilde{R}_p = \alpha \tilde{R}_A + (1-\alpha)\tilde{R}_B$

On cherche α tq $cov(\tilde{R}_p, \tilde{R}_n) = 0$

$cov(\alpha \tilde{R}_A + (1-\alpha)\tilde{R}_B ; \tilde{R}_n) = 0$

$\Rightarrow cov(\alpha \beta_A \tilde{R}_n + \alpha \Sigma_A + (1-\alpha)\beta_B \tilde{R}_n + (1-\alpha)\Sigma_B ; \tilde{R}_n) = 0$

$\Rightarrow \alpha \beta_A cov(\tilde{R}_n, \tilde{R}_n) + \alpha \underbrace{cov(\Sigma_A, \tilde{R}_n)}_{0} + (1-\alpha)\beta_B cov(\tilde{R}_n, \tilde{R}_n) + (1-\alpha) cov(\Sigma_B, \tilde{R}_n) = 0$

$\Rightarrow \underbrace{\sigma_n^2}_{>0} [\alpha \beta_A + (1-\alpha)\beta_B] = 0 \Rightarrow \alpha \beta_A + (1-\alpha)\beta_B = 0 \Rightarrow \alpha = \dfrac{\beta_B}{\beta_B - \beta_A}$

- 2ème méthode : $\alpha = ?$ tq $cov(\tilde{R}_p, \tilde{R}_n) = 0 \Rightarrow \beta_p = \dfrac{cov(\tilde{R}_p, \tilde{R}_n)}{\sigma_n^2} = 0$

β est une mesure linéaire

$$\beta_p = \alpha \beta_A + (1-\alpha)\beta_B = 0 \quad \Rightarrow \alpha = \frac{\beta_B}{\beta_B - \beta_A}$$

Rmq : $\sigma_p^2 = \underbrace{\beta_p^2 \sigma_n^2}_{=0} + \sigma_{\varepsilon_p}^2$

Le risque total = le risque spécifique

b) On cherche α solution de

$$\min_{\alpha} Var(\alpha \tilde{R}_A + (1-\alpha)\tilde{R}_B)$$

$$= \min_{\alpha} Var(\alpha \beta_A \tilde{R}_m + \alpha \varepsilon_A + (1-\alpha)\beta_B \tilde{R}_m + (1-\alpha)\varepsilon_B)$$

$$= \min_{\alpha} Var((\alpha \beta_A + (1-\alpha)\beta_B)\tilde{R}_m + \alpha \varepsilon_A + (1-\alpha)\varepsilon_B)$$

$$= \min_{\alpha} [\alpha \beta_A + (1-\alpha)\beta_B]^2 \sigma_m^2 + \alpha^2 \sigma^2_{\varepsilon_A} + (1-\alpha)^2 \sigma^2_{\varepsilon_B}$$

$$= \min_{\alpha} \alpha^2 \underbrace{[(\beta_A - \beta_B)^2 \sigma_m^2 + \sigma^2_{\varepsilon_A} + \sigma^2_{\varepsilon_B}]}_{\geq 0 \Rightarrow \text{le min existe}}$$

$$+ 2\alpha[\beta_B \sigma_m (\beta_A - \beta_B) - \sigma^2_{\varepsilon_B}] + \beta_B^2 \sigma_m^2 + \sigma^2_{\varepsilon_B}$$

Rappel: $ax^2 + bx + c$

$a \geq 0 \Rightarrow$ le min existe $x^* = -\frac{b}{2a}$

$$\Rightarrow \alpha^* = \frac{\sigma^2_{\varepsilon_B} - \beta_B \sigma_m (\beta_A - \beta_B)}{(\beta_A - \beta_B)^2 \sigma_m^2 + \sigma^2_{\varepsilon_A} + \sigma^2_{\varepsilon_B}}$$

$(\alpha^* \quad 1-\alpha^*)$ le port. contenant A et B le moins risqué.