Probabilités Numériques (G. Pagès & V. Lemaire)
M2 Probabilités & Finance
M2 Probabilités & Modèles Aléatoires
SU-X
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**3 h**

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**Handouts, course notes, books, mobile phones and smart watches not allowed**

**Exercise (Quasi-Monte Carlo).** Let $(\xi_n)_{n\geq 1}$ be a $(0,1)$-valued sequence supposed uniformly distributed over $[0,1]$ and $(\varepsilon_n)_{n\geq 1}$ a non-increasing $(0,1)$-valued sequence such that $\varepsilon_n \to 0$ as $n \to +\infty$.

PRELIMINARY QUESTIONS. **a.** Recall the condition of uniform distribution over $[0,1]$ for $(\xi_n)_{n\geq 1}$, the definitions of the discrepancy $D_n^*(\xi)$ at the origin (a.k.a. *star*-discrepancy) and the uniform discrepancy $D_n^\infty(\xi)$ (a.k.a. *extreme*-discrepancy).

**b.** Show that, for every real $x \in (0,1)$,

$$D_n^\infty(\xi) \leq 2D_n^*(\xi).$$

**c.** Show that if $\varphi : [0,1] \to \mathbb{R}$ is right continuous and left limited and satisfies $\varphi(0)-\varphi(1)=0$, then

$$\sup_{0\leq x\leq y\leq 1} |\varphi(y)-\varphi(x)| = \sup_{0\leq x\leq y\leq 1} |\varphi(y)-\varphi(x-)|.$$

Let $\{u\}$ denote the fractional part of a real number $u \in \mathbb{R}$ defined such that $u = \lfloor u \rfloor + \{u\}$. In the sequel we consider the disturbed sequence $(\tilde{\xi}_n)_{n\geq 1}$ defined by $\tilde{\xi}_n = \{\xi_n + \varepsilon_n\}$.

**1.** Prove with Weyl's criterion that the sequence $(\tilde{\xi}_n)_{n\geq 1}$ is $[0,1)$-valued and uniformly distributed.

**2.a.** Show that, for every real number $x \in (0,1)$,

$$\left|\frac{1}{n}\sum_{k=1}^{n}\left(\mathbf{1}_{\{\tilde{\xi}_k\leq x\}} - \mathbf{1}_{\{\xi_k\leq x\}}\right)\right| \leq \frac{1}{n}\sum_{k=1}^{n}\left(\mathbf{1}_{\{x-\varepsilon_k<\xi_k\leq x\}} + \mathbf{1}_{\{1-\varepsilon_k<\xi_k\leq 1\}}\right).$$

**2.b.** Deduce that, for every $x \in (0,1)$ and every integer $\ell \in \{1,\ldots,n\}$,

$$\left|\frac{1}{n}\sum_{k=1}^{n}\left(\mathbf{1}_{\{\tilde{\xi}_k\leq x\}} - \mathbf{1}_{\{\xi_k\leq x\}}\right)\right| \leq \frac{2(\ell-1)}{n} + \frac{1}{n}\sum_{k=1}^{n}\left(\mathbf{1}_{\{x-\varepsilon_\ell<\xi_k\leq x\}} + \mathbf{1}_{\{1-\varepsilon_\ell<\xi_k\leq 1\}}\right)$$

**2.c.** Show that, pour tout $x \in (0,1)$ and every integer $\ell \in \{1,\ldots,n\}$,

$$\left|\frac{1}{n}\sum_{k=1}^{n}\left(\mathbf{1}_{\{\tilde{\xi}_k\le x\}}-\mathbf{1}_{\{\xi_k\le x\}}\right)\right|\le\frac{2(\ell-1)}{n}+2(D_n^\infty(\xi)+\varepsilon_\ell).$$

Then deduce that

$$D_n^*(\tilde{\xi})\le D_n^*(\xi)+\frac{2(\ell-1)}{n}+2(D_n^\infty(\xi)+\varepsilon_\ell).$$

[Hint: one could rely on preliminary questions especially the 3rd one]

**3.a.** Let $\ell_n=\lfloor nD_n^*(\xi)\rfloor\vee 1$ for $n\ge 1$. Prove that $\ell_n\in\{1,\dots,n\}$ and that $\frac{\ell_n-1}{n}\le D_n^*(\xi)$. Deduce that

$$\forall n\ge 1,\quad D_n^*(\tilde{\xi})\le C_1D_n^*(\xi)+2\,\varepsilon_{\ell_n},$$

for some real constant to be specified.

**3.b.** Retrieve the fact the $(\tilde{\xi}_n)_{n\ge 1}$ is uniformly distributed over $[0,1]$ and give an as sharp as you can sufficient condition/criterion on the sequence $(\varepsilon_n)_{n\ge 1}$ to ensure that

$$D_n^*(\tilde{\xi})\le CD_n^*(\xi)$$

for some positive real constant $C>0$.

**Problem (Yet another Euler scheme).** We consider a (purely one-dimensional) Stochastic Differential Equation ($SDE$)

$$X_t=X_0+\int_0^t b(s,X_s)ds+\int_0^t\sigma(s,X_s)dW_s,\quad t\in[0,T].$$

where $T>0$, $b,\sigma:[0,T]\times\mathbb{R}\to\mathbb{R}$ are two continuous functions, Lipschitz continuous in $x$ Lipschitz coefficients $[b]_{\rm Lip}$ and $[\sigma]_{\rm Lip}$, uniformly in $t\in[0,T]$, such that there exists $\beta\in(0,1]$ and a real constant $C=C_{b,\sigma,T}$ satisfying

$$\forall x\in\mathbb{R},\ \forall s,t\in[0,T],\quad |b(t,x)-b(s,x)|+|\sigma(t,x)-\sigma(s,x)|\le C_{b,\sigma}(1+|x|)|t-s|^\beta.$$

$(W_t)_{t\ge 0}$ is a standard Brownian motion defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ and $X_0$ is defined on the same probability space, independent of $W$. We denote par $(X_t)_{t\ge 0}$ the unique $(\mathcal{F}_t^{X_0,W})_{t\ge 0}$-adapted solution to the above $SDE$ where $\mathcal{F}_t^{X_0,W}=\sigma(\mathcal{N}_\mathbb{P},X_0,W_s,0\le s\le t)$, $t\ge 0$ where $\mathcal{N}_\mathbb{P}$ denotes the $\mathbb{P}$-negligible sets. All the notations are those used during the course.

We denote by $T>0$ a terminal time (or maturity). For every integer $n\ge 1$, we define step $h=h_n=\frac{T}{n}$. We set $t_k=t_k^n=\frac{kT}{n}$, $k=0,\dots,n$ and

$$\underline{t}=t_k\ \text{ if }\ t\in[t_k,t_{k+1}[,\quad \underline{t}=T.$$

**1.a.** Recall the definitions of the three Euler schemes with step $h=\frac{T}{n}$: the discrete time, the stepwise constant and the "genuine" (continuous) Euler schemes, denoted $(\bar{X}_{t_k}^n)_{k=0,\dots,n}$, $(\tilde{X}_t^n)_{t\in[0,T]}$ and $(\bar{X}_t^n)_{t\in[0,T]}$ respectively.

**1.b.** Let $p\ge 1$. Recall the uniform $L^p$-moment control results for both the diffusion $(X_t)_{t\in[0,T]}$ and the above Euler scheme(s). [No proof requested here.]

**1.c.** State the $L^p$-convergence theorems for the above three Euler schemes in an as synthetic way as possible. [No proof requested here.]

**The aim of this problem is to study the $L^2$-convergence of a fourth Euler scheme that we will denote $(\bar{\xi}^n_t)_{t\in[0,T]}$.**

This scheme is defined by the following equation

$$\forall t \in [0,T], \quad \bar{\xi}^n_t = X_0 + \int_0^t b(s,\bar{\xi}^n_{\underline{s}})ds + \int_0^t \sigma(s,\bar{\xi}^n_{\underline{s}})dW_s. \tag{0.1}$$

**Warning!** We temporarily assume that this fourth Euler scheme exists and satisfies the following moment control property: there exists a positive real constant $C_{b,\sigma,T,p}$ such that

$$\forall n \geq 1, \quad \Big\| \max_{k=0,\ldots,n} |\bar{\xi}^n_{t_k}| \Big\|_2 \leq C_{b,\sigma,T}(1+\|X_0\|_2) < +\infty. \tag{0.2}$$

**2.** Prove that the scheme is mathematically well-defined (or well-posed) by showing that, for every $k \in \{0,\ldots,n\}$,

$$\bar{\xi}_{t_{k+1}} = \bar{\xi}_{t_k} + \int_{t_k}^{t_{k+1}} b(s,\bar{\xi}_{t_k})ds + \int_{t_k}^{t_{k+1}} \sigma(s,\bar{\xi}_{t_k})dW_s, \quad \bar{\xi}_0 = X_0. \tag{0.3}$$

and that, for every $t \in [t_k, t_{k+1}[$,

$$\bar{\xi}_t = \bar{\xi}_{t_k} + \int_{t_k}^{t} b(s,\bar{\xi}_{t_k})ds + \int_{t_k}^{t} \sigma(s,\bar{\xi}_{t_k})dW_s. \tag{0.4}$$

**3.a.** Prove that, for every $x, y \in \mathbb{R}$,

$$\sup_{0\leq s\leq t} |X_s - \bar{\xi}^n_s| \leq \int_0^t \big|b(u,X_u) - b(u,\bar{\xi}^n_{\underline{u}})\big|du + \sup_{s\in[0,t]} \Big| \int_0^s \big(\sigma(u,X_u) - \sigma(u,\bar{\xi}^n_{\underline{u}})\big)dW_u \Big|.$$

**3.b.** We set, for every $t \in [0,T]$, $g(t) = \mathbb{E} \sup_{s\in[0,t]} |X_s - \bar{\xi}^n_s|^2$. Prove that ($g$ is non-decreasing and) $g(T) < +\infty$.

**3.c.** Prove that for every $a, b \geq 0$, $(a+b)^2 \leq 2(a^2+b^2)$.

**3.d.** Deduce that, for every $t \in [0,T]$,

$$g(t) \leq 2\Big(t\,[b]^2_{\rm Lip} \int_0^t \mathbb{E}\,|X_u - \bar{\xi}^n_{\underline{u}}|^2 du + \mathbb{E} \sup_{s\in[0,t]} \Big| \int_0^s \big(\sigma(u,X_u) - \sigma(u,\bar{\xi}^n_{\underline{u}})\big)dW_u \Big|^2\Big).$$

**3.e.** Prove in detail that, for every $t \in [0,T]$,

$$g(t) \leq 2\big(T[b]^2_{\rm Lip} + 4[\sigma]^2_{\rm Lip}\big) \int_0^t \mathbb{E}\,|X_s - \bar{\xi}^n_{\underline{s}}|^2 ds.$$

**4.a.** Prove that

$$\mathbb{E}\,|X_s - \bar{\xi}^n_{\underline{s}}|^2 \leq 2\big(g(s) + \mathbb{E}\,|\bar{\xi}^n_s - \bar{\xi}^n_{\underline{s}}|^2\big)$$

and deduce that

$$\Big\| \sup_{t\in[0,T]} |X_t - \bar{\xi}^n_t| \Big\|_2 \le \sqrt{K} e^{KT/2}\Big(\int_0^T \mathbb{E}\,|\bar{\xi}^n_s - \bar{\xi}^n_{\underline{s}}|^2 ds\Big)^{1/2}.$$

where $K = K_{b,\sigma,T}$ is a positive real constant to be specified.

**4.b.** Establish the inequality

$$\mathbb{E}\,|\bar{\xi}^n_s - \bar{\xi}^n_{\underline{s}}|^2 \le 2\Big((t-\underline{t})\int_{\underline{t}}^t \mathbb{E}\, b^2(s,\bar{\xi}^n_{\underline{s}})ds + \int_{\underline{t}}^t \mathbb{E}\,\sigma^2(s,\bar{\xi}^n_{\underline{s}})ds\Big).$$

**4.c.** Prove, using that $t \mapsto b(t,0)$ and $t\mapsto \sigma(t,0)$ are continuous on $[0,T]$ that there exists a real constant $C'' = C_{b,\sigma,T}$ such that

$$\forall s\in[0,T],\ \forall x\in\mathbb{R},\quad |b(s,x)|^2 \vee |\sigma(s,x)^2| \le C''(1+|x|^2).$$

Deduce the existence of positive real constants $C'''$ and $C''''$ such that, for every $s\in[0,T]$,

$$\mathbb{E}\,|\bar{\xi}^n_s - \bar{\xi}^n_{\underline{s}}|^2 \le 2C'''\Big(\Big(\frac{T}{n}\Big)^2 + \Big(\frac{T}{n}\Big)\Big)(1+\mathbb{E}\,X_0^2) \le C''''\Big(\frac{T}{n}\Big)(1+\mathbb{E}\,X_0^2).$$

What is the order of convergence of this "semi-integrated" scheme?

**4.d.** What is the $L^2$-rate of convergence of this Euler scheme that the above computations prove ?

**5.a.** What is the main theoretical difference between the *genuine* Euler scheme and this semi-integrated scheme (a.k.a. as *super genuine*)? What is the main em practical difference between the usual stepwise constant Euler scheme $\tilde{X}_t := X_{\underline{t}}$ and the stepwise constant version $\tilde{\xi}_t := \xi_{\underline{t}}$ of the scheme $(\xi_t)_{t\in[0,T]}$ (a.k.a. as *super genuine*)?

**5.b.** A posteriori what are the minimal assumptions needed on $b$ and $\sigma$ to obtain the rate of convergence of this scheme (beyond Borel measurability of course).

**5.c.** Give an explicit example where the genuine Euler scheme and the semi-integrated scheme have different orders of convergence.

Bonus questions (prequel). Le us define the random variables $\xi_{t_k}$ by the recursion from question **2.**

($a$) Prove by induction using (0.3) from question **2.** that if $X_0\in L^p(\mathbb{P})$, $\xi_{t_k}\in L^p(\mathbb{P})$ for every $k=0,\ldots,n$.

($b$) Prove there exists a real constant $\kappa = \kappa_{b,\sigma}$ (not depending on $n$) such that

$$\mathbb{E}\,|\bar{\xi}^n_{t_{k+1}}|^2 \le \mathbb{E}|\bar{\xi}^n_{t_k}|^2\big(1+\kappa\tfrac{T}{n}\big)(1+\mathbb{E}\,X_0^2)$$

and deduce the existence of a real constant $\kappa_{b,\sigma,T}$ such that

$$\max_{k=0,\ldots,n} \mathbb{E}\,|\bar{\xi}^n_{t_k}|^2 \le \kappa_{b,\sigma,T}(1+\mathbb{E}\,X_0^2) < +\infty.$$

($c$) Prove that if we *define* $\bar{\xi}^n_t$ by (0.4) then $(\bar{\xi}^n_t)_{t\in[0,T]}$ satisfies Equation (0.1). Finally prove (0.2).