Online teaching - Lecture notes
"Financial derivatives and stochastic processes"
M2 Probability et Finance

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Week 5

4 Hedging risk with several assets in the same currency market

4.1 Modelling the volatility

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ modelling the true (historical) evolution of d risky assets. The filtration is generated by k independent Brownian motions $(\widehat{W}^j : 1 \leq j \leq k)$, completed by the $\mathbb{P}$-null sets: therefore, the filtration satisfies the usual conditions in the sense that it is right-continuous and complete. The Brownian motions $\widehat{W} = (\widehat{W}^j : 1 \leq j \leq k)$ model the noise in the markets.

We also consider a risk-free asset (the money-market asset) which return at time t is equal to r_t, this is a predictable process. The $d+1$ assets evolve as

$$\begin{aligned} \mathrm{d}S_t^i &= S_t^i \left(b_t^i \mathrm{d}t + \sum_{j=1}^{k} \sigma_j^i(t) \mathrm{d}\widehat{W}_t^j \right) \quad \text{for } i \in \{1, 2, \dots, d\}, \\ \mathrm{d}S_t^0 &= S_t^0 r_t \mathrm{d}t. \end{aligned} \tag{4.1.1}$$

Here, to simplify the analysis, we assume that the assets do not pay dividend. All prices are expressed according to the same currency, say € .

Terminology.

- b_t^i is the drift of the asset i.
- $\sigma_t^i := (\sigma_j^i(t) : 1 \leq j \leq k)$ is the vector of volatility of the asset i at time t.

- $\sigma_t = \begin{pmatrix} \vdots \\ \sigma_t^i \\ \vdots \end{pmatrix}$ is the matrix of volatility of the risky assets at time t: its size is $d \times k$.
- $|\sigma_t^i|$ is the spot (instantaneous) volatility at time t of asset i.

All these coefficients are predictable (measurable with respect to the predictable σ-algebra $\mathcal{P}$ on $\mathbb{R}^+ \times \Omega$, which is the smallest σ-algebra that makes measurable any $\mathcal{F}$-adapted cadlag[1] process) [Pro04, Chapter III].

The link between vector volatility and spot volatility is as follows.

Proposition 4.1. *One has*

$$\mathrm{d}S_t^i = S_t^i \left(b_t^i \mathrm{d}t + |\sigma_t^i| \mathrm{d}B_t^i \right) \text{ for } i \in \{1, 2, \ldots, d\},$$

where the B^i's are Brownian motions with instantaneous correlation

$$\mathrm{d}\langle B^i, B^j \rangle_t = \rho_t^{i,j} \mathrm{d}t, \qquad \rho_t^{i,j} := \frac{\sigma_t^i \cdot \sigma_t^j}{|\sigma_t^i| \, |\sigma_t^j|}.$$

4.2 Self-financing portfolio and no arbitrage

Theorem 4.2. *In a market without friction and without arbitrage, there is a k-dimensional process λ (called risk premium) such that*

$$b_t - r_t \mathbf{1}_d = \sigma_t \lambda_t, \qquad \mathrm{d}t \otimes \mathrm{d}\mathbb{P} \text{ a.e.} \tag{4.2.1}$$

Here $\mathbf{1}_d$ is the vector of size d with value 1 on all the components.

See slides and NEK lectures notes.

Consider a portfolio invested in $\delta^i(t)$ asset i at time t: under the self-financing condition and without friction, its value

$$V_t = \delta^0(t) S_t^0 + \sum_{i=1}^d \delta^i(t) S_t^i \tag{4.2.2}$$

must evolve like

$$\mathrm{d}V_t = \delta^0(t) \mathrm{d}S_t^0 + \sum_{i=1}^d \delta^i(t) \mathrm{d}S_t^i$$

[1] continu à droite avec une limite à gauche

$$
\begin{aligned}
&\underset{(4.1.1)}{=} \delta^0(t)S_t^0 r_t \mathrm{d}t + \sum_{i=1}^{d} \delta^i(t)S_t^i(b_t^i \mathrm{d}t + \sigma_t^i \mathrm{d}\widehat{W}_t) \\
&\underset{(4.2.2)}{=} (V_t - \sum_{i=1}^{d} \delta^i(t)S_t^i) r_t \mathrm{d}t + \sum_{i=1}^{d} \delta^i(t)S_t^i(b_t^i \mathrm{d}t + \sigma_t^i \mathrm{d}\widehat{W}_t) \qquad (4.2.3)\\
&= r_t V_t \mathrm{d}t + \sum_{i=1}^{d} \delta^i(t)S_t^i((b_t^i - r_t)\mathrm{d}t + \sigma_t^i \mathrm{d}\widehat{W}_t) \\
&\underset{(4.2.1)}{=} r_t V_t \mathrm{d}t + \sum_{i=1}^{d} \delta^i(t)S_t^i \sigma_t^i(\mathrm{d}\widehat{W}_t + \lambda_t \mathrm{d}t).
\end{aligned}
$$

Denote by

$$\pi_t^i = \delta^i(t)S_t^i \tag{4.2.4}$$

for the amount invested in asset i. Then the above simply rewrites

$$\mathrm{d}V_t = r_t V_t \mathrm{d}t + \sum_{i=1}^{d} \pi_t^i \sigma_t^i(\mathrm{d}\widehat{W}_t + \lambda_t \mathrm{d}t).$$

For the trader aiming at replicating a payoff Ψ_T at time T, the problem is to find V_0 and $(\pi_t^i, 1 \le i \le d : 0 \le t \le T)$ or equivalently (according to (4.2.4)) V_0 and $(\delta^i(t), 1 \le i \le d : 0 \le t \le T))$ such that

$$V_T = \Psi_T.$$

It can be rewritten as a backward problem

$$
\begin{cases}
\mathrm{d}V_t = r_t V_t \mathrm{d}t + \sum_{i=1}^{d} \pi_t^i \sigma_t^i(\mathrm{d}\widehat{W}_t + \lambda_t \mathrm{d}t), \\
V_T = \Psi_T,
\end{cases}
\tag{4.2.5}
$$

called Backward Stochastic Differential Equation (BSDE), initiated in [PP90] and developed in the context of finance in [EPQ97]. The outputs of this problem are the portfolio value V and the hedging strategy π (or equivalently δ).

For connecting the above with martingale measures and risk-neutral valuation rules, see Slides and NEK lectures notes.

Besides, the above equation (4.2.5) can be put in a more general form which opens the way for important extensions and applications.

Definition 4.3. *A BSDE with terminal condition ξ and driver (or generator) f is an equation with unknown processes (Y, Z) of the form*

$$-\mathrm{d}Y_t = f(t, Y_t, Z_t)\mathrm{d}t - Z_t \mathrm{d}\widehat{W}_t, \qquad Y_T = \xi. \tag{4.2.6}$$

Here Y is a process in dimension 1, Z is a process in dimension k written as a row. Both ξ and f can be stochastic (see conditions later).

We use the convention of writing $-\mathrm{d}Y_t$ because the terminal condition ξ is fixed and not the initial one Y_0. Because of the sign convention, the equation (4.2.6) rewrites in an integral form as

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}\widehat{W}_s.$$

Proposition 4.4. *In the case of a market without friction, the pricing/hedging problem (4.2.5) is equivalent to solving a BSDE with terminal condition $\xi = \Psi_T$ and driver*

$$f(t, y, z) := -r_t y - z\lambda_t.$$

Since f is linear in (y, z), the BSDE is called linear BSDE.

Proof. Set $Z_t := \sum_{i=1}^{d} \pi_t^i \sigma_t^i = (\pi_t^1 \cdots \pi_t^d)\sigma_t = \pi_t^\top \sigma_t$ and $Y_t := V_t$. □

Definition 4.5. *In all the sequel, ξ and f satisfy standard assumptions (we will talk about standard BSDE) if*

- *the terminal condition ξ is square integrable: $\mathbb{E}\left(\xi^2\right) < +\infty$,*
- *$f : (t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^k \mapsto f(t, \omega, y, z) =: f(t, y, z)$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^k)$-measurable map,*
- *$f(., 0, 0)$ is such that $\mathbb{E}\left(\int_0^T f^2(t, 0, 0)\mathrm{d}t\right) < +\infty$,*
- *f is globally Lipschitz in the sense that for some finite constant $C_f > 0$, we have*

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C_f(|y_1 - y_2| + |z_1 - z_2|), \qquad \forall y_1, z_1, y_2, z_2.$$

4.3 Markets with friction and non-linear valuation

4.3.1 Examples

As a difference with Proposition 4.4 there are other situations in finance where f is non-linear. Assume that σ is invertible (requiring $k = d$), so that λ defined in (4.2.1) by

$$\lambda_t := \sigma_t^{-1}(b_t - r_t \mathbf{1}_d)$$

is unique.

Pricing/hedging with two interest rates [Ber95]. Assuming that borrowing is made at rate R_t and lending at rate r_t: of course $r_t \leq R_t$. Both rates can be stochastic (assumed predictable). Rewriting the self-financing condition with this gives that the equation (4.2.3) is now

$$\mathrm{d}V_t = (V_t - \sum_{i=1}^d \delta^i(t) S_t^i)_+ r_t \mathrm{d}t - (V_t - \sum_{i=1}^d \delta^i(t) S_t^i)_- R_t \mathrm{d}t + \sum_{i=1}^d \delta^i(t) S_t^i (b_t^i \mathrm{d}t + \sigma_t^i \mathrm{d}\widehat{W}_t).$$

Owing to the notation (4.2.4) and the definition of λ, we get

$$\mathrm{d}V_t = (V_t - \sum_{i=1}^d \pi_t^i)_+ r_t \mathrm{d}t - (V_t - \sum_{i=1}^d \pi_t^i)_- R_t \mathrm{d}t + \sum_{i=1}^d \pi_t^i (r_t + \sigma_t^i \lambda_t \mathrm{d}t + \sigma_t^i \mathrm{d}\widehat{W}_t).$$

Using now $Z_t := \sum_{i=1}^d \pi_t^i \sigma_t^i = \pi_t^\top \sigma_t$, i.e. $\pi_t^\top = Z_t \sigma_t^{-1}$, and $Y_t := V_t$, we get

$$\begin{aligned}\mathrm{d}Y_t &= (Y_t - Z_t \sigma_t^{-1} \mathbf{1}_d)_+ r_t \mathrm{d}t - (Y_t - Z_t \sigma_t^{-1} \mathbf{1}_d)_- R_t \mathrm{d}t + Z_t \sigma_t^{-1} \mathbf{1}_d r_t \mathrm{d}t + Z_t \lambda_t \mathrm{d}t + Z_t \mathrm{d}\widehat{W}_t \\ &= r_t Y_t \mathrm{d}t - (Y_t - Z_t \sigma_t^{-1} \mathbf{1}_d)_- (R_t - r_t) \mathrm{d}t + Z_t \lambda_t \mathrm{d}t + Z_t \mathrm{d}\widehat{W}_t \qquad (4.3.1)\end{aligned}$$

using $x_+ = x + x_-$. We just obtain the same equation as (4.2.5) with an additional cost $(Y_t - Z_t \sigma_t^{-1} \mathbf{1}_d)_- (R_t - r_t)$ corresponding to the extra cost of borrowing cash (with the amount $Y_t - Z_t \sigma_t^{-1} \mathbf{1}_d = V_t - \sum_{i=1}^d \pi_t^i$) at rate R_t instead of r_t.

This gives for (4.3.1) a non-linear equation (in the class of non-linear BSDE).

Pricing/hedging with short-selling constraints [JK95]. Similarly, if having $\pi_t^i < 0$ (shorting the asset i) gives rise to an additional repo rate q_t^i, then one gets

$$\mathrm{d}V_t = (V_t - \sum_{i=1}^d \pi_t^i) r_t \mathrm{d}t + \sum_{i=1}^d (\pi_t^i)_- q_t^i \mathrm{d}t + \sum_{i=1}^d \pi_t^i (r_t + \sigma_t^i \lambda_t \mathrm{d}t + \sigma_t^i \mathrm{d}\widehat{W}_t). \qquad (4.3.2)$$

Setting again $\pi_t^\top = Z_t \sigma_t^{-1}$ and $Y_t = V_t$, we obtain that it can recasted into a non-linear BSDE.

4.3.2 A priori estimates

We are looking for solutions (Y, Z) to (4.2.6) into the space of processes $\mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^k)$, defined as the predicable processes taking values in $\mathbb{R}$ for Y and in $\mathbb{R}^k$ for Z (as a row) such that

$$\mathbb{E}\left(\int_0^T |Y_t|^2 \mathrm{d}t\right) + \mathbb{E}\left(\int_0^T |Z_t|^2 \mathrm{d}t\right) < +\infty.$$

As a quadratic norm on this space, we set

$$\|U\|_{2,\beta}^2 := \mathbb{E}\left(\int_0^T e^{\beta t} |U_t|^2 \mathrm{d}t\right)$$

for $U = Y, Z$. Equipped with this norm, $(\mathbb{H}_2(\mathbb{R}), \|.\|^2_{2,\beta})$ is now a Banach space for any β. Actually this space does not depend on β since on the interval $[0,T]$, $e^{\beta t}$ is bounded above and away from 0; hence the use of $\beta \neq 0$ seems unnecessary at first sight, in fact it will be useful later for deriving nice a priori estimates, by playing with β large enough.

We now state the following crucial result.

Theorem 4.6. *Let (Y^1, Z^1) and (Y^2, Z^2) two processes in $\mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^k)$ solutions to standard BSDEs associated to two drivers f_1 and f_2, and two terminal conditions ξ_1, ξ_2. Then, denote C the Lipschitz constant for f_1 and set*

$$\delta_2 f_t = f_1(t, Y^2_t, Z^2_t) - f_2(t, Y^2_t, Z^2_t);$$

for any $\mu > 0$ and setting $\beta := C^2 + 3C + \mu^2$, we have

$$\|Y^1 - Y^2\|^2_{2,\beta} \leq T\left[e^{\beta T}\mathbb{E}\left(|\xi_1 - \xi_2|^2\right) + \frac{1}{\mu^2}\|\delta_2 f.\|^2_{2,\beta}\right], \tag{4.3.3}$$

$$\|Z^1 - Z^2\|^2_{2,\beta} \leq (C+1)\left[e^{\beta T}\mathbb{E}\left(|\xi_1 - \xi_2|^2\right) + \frac{1}{\mu^2}\|\delta_2 f.\|^2_{2,\beta}\right]. \tag{4.3.4}$$

Proof. Denote $\delta U := U^1 - U^2$. Observe that

$$\left|f_1(t, Y^1_t, Z^1_t) - f_2(t, Y^2_t, Z^2_t)\right| \leq C|\delta Y_t| + C|\delta Z_t| + |\delta_2 f_t|. \tag{4.3.5}$$

Apply Ito formula to $e^{\beta t}|\delta Y_t|^2$ to get:

$$\begin{aligned} \mathrm{d}(e^{\beta t}|\delta Y_t|^2) = e^{\beta t}(\beta|\delta Y_t|^2 - 2\delta Y_t \times (f_1(t, Y^1_t, Z^1_t) - f_2(t, Y^2_t, Z^2_t)) \\ + |\delta Z_t|^2)\mathrm{d}t - 2e^{\beta t}\delta Y_t \times \delta Z_t \mathrm{d}\widehat{W}_t. \end{aligned} \tag{4.3.6}$$

The stochastic integral $M_s := \int_0^s e^{\beta t}\delta Y_t \delta Z_t \mathrm{d}\widehat{W}_t$ has zero expectation: indeed, its root-square bracket is bounded by

$$\sqrt{\int_0^s e^{2\beta t}|\delta Y_t|^2|\delta Z_t|^2 \mathrm{d}t} \leq e^{|\beta| T} \sup_{0\leq t\leq T}|\delta Y_t| \sqrt{\int_0^T |\delta Z_t|^2 \mathrm{d}t}$$

which is in integrable since Z^i $(i = 1, 2)$ is in $\mathbb{H}_2(\mathbb{R}^k)$, and $\sup_{0\leq t\leq T}|Y^i_t| \in \mathbb{L}_2$, $i = 1, 2$ in view of (4.2.6) (because $(Y^i, Z^i) \in \mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^k)$). By Burkholder-David-Gundy inequalities, $\mathbb{E}\left(\sup_{0\leq t\leq T}|M_t|\right) \leq C_{\mathrm{BDG}}\mathbb{E}\left(\langle M\rangle_t^{1/2}\right) < +\infty$ and therefore $\mathbb{E}\left(M_t\right) = 0$ for any t.

Taking expectation in (4.3.6), we get

$$\begin{aligned} &\mathbb{E}\left(e^{\beta s}|\delta Y_s|^2 + \int_s^T e^{\beta t}\beta|\delta Y_t|^2 \mathrm{d}t + \int_s^T e^{\beta t}|\delta Z_t|^2 \mathrm{d}t\right) \\ \leq &\mathbb{E}\left(e^{\beta T}|\delta Y_T|^2 + 2\int_s^T e^{\beta t}|\delta Y_t|\ |f_1(t, Y^1_t, Z^1_t) - f_2(t, Y^2_t, Z^2_t)|\mathrm{d}t\right). \end{aligned}$$

Using repeatedly that $2|ab| \leq \frac{a^2}{\lambda} + \lambda b^2$ for any $\lambda > 0$ and any a, b in combination with (4.3.5) and with the value of β in the Theorem statement, we get

$$\begin{aligned}
&\mathbb{E}\left(e^{\beta s}|\delta Y_s|^2 + \int_s^T e^{\beta t}(C^2+3C+\mu^2)|\delta Y_t|^2 \mathrm{d}t + \int_s^T e^{\beta t}|\delta Z_t|^2 \mathrm{d}t\right)\\
\leq &\mathbb{E}\left(e^{\beta T}|\delta\xi|^2 + 2C\int_s^T e^{\beta t}|\delta Y_t|^2\mathrm{d}t\right)\\
&+\mathbb{E}\left(C\int_s^T e^{\beta t}(|\delta Y_t|^2(C+1) + |\delta Z_t|^2/(C+1))\mathrm{d}t\right)\\
&+\mathbb{E}\left(\int_s^T e^{\beta t}(|\delta Y_t|^2\mu^2 + |\delta_2 f_t|^2/\mu^2)\mathrm{d}t\right).
\end{aligned}$$

Simplifying the above gives

$$\mathbb{E}\left(e^{\beta s}|\delta Y_s|^2 + \int_s^T e^{\beta t}\frac{1}{C+1}|\delta Z_t|^2\mathrm{d}t\right) \leq \mathbb{E}\left(e^{\beta T}|\delta\xi|^2\right) + \mathbb{E}\left(\int_s^T e^{\beta t}\frac{|\delta_2 f_t|^2}{\mu^2}\mathrm{d}t\right).$$

Then, (4.3.3) is obtained by integrating the above in $s \in [0, T]$, while (4.3.4) is directly proved. □

4.3.3 Existence/uniqueness to BSDE

Theorem 4.7 ([EPQ97]). *Assume standard conditions on ξ and f, then there is a unique solution $(Y, Z) \in \mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^k)$ to (4.2.6). In addition the solution Y is continuous.*

For (4.2.5)-(4.3.1)-(4.3.2), it is easy to check that the conditions of Theorem 4.7 on the driver are met when $r_t, R_t, \lambda_t, \sigma_t, \sigma_t^{-1}$ are bounded processes.

The a priori estimates give for free estimates on the above solution: take $(Y^2, Z^2, \xi_2, f_2) \equiv 0$ and $\mu = 1$ in Theorem 4.6, then

$$\begin{aligned}
\|Y\|_{2,\beta}^2 &\leq T\left[e^{\beta T}\mathbb{E}\left(|\xi|^2\right) + \|f(.,0,0)\|_{2,\beta}^2\right],\\
\|Z\|_{2,\beta}^2 &\leq (C_f+1)\left[e^{\beta T}\mathbb{E}\left(|\xi|^2\right) + \|f(.,0,0)\|_{2,\beta}^2\right],\\
\beta &:= C_f^2 + 3C_f + 1.
\end{aligned}$$

Proof. Given a pair of processes $(y, z) \in \mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^k)$, then consider (Y, Z) solution to

$$Y_t = \xi + \int_t^T f(s, y_s, z_s)\mathrm{d}s - \int_s^T Z_s \mathrm{d}\widehat{W}_s. \tag{4.3.7}$$

The solution is obtained by observing that $M_t := \mathbb{E}\left(\xi + \int_0^T f(s, y_s, z_s)\mathrm{d}s \mid \mathcal{F}_t\right)$ is a continuous square integrable martingale (M is a conditional expectation of a square integrable quantity, and Brownian martingales are all continuous). Therefore it can be written as $M_t = M_0 +$

$\int_0^t Z_s \mathrm{d}\widehat{W}_s$ for some $Z \in \mathbb{H}_2(\mathbb{R}^k)$ (apply the predictable representation theorem of martingales in the Brownian filtration). From (4.3.7) we can write $Y_t + \int_0^t f(s, y_s, z_s)\mathrm{d}s = M_T - \int_s^T Z_s \mathrm{d}\widehat{W}_s = M_t$, and thus by setting $Y_t = M_t - \int_0^t f(s, y_s, z_s)\mathrm{d}s$ (in $\mathbb{H}_2(\mathbb{R})$), we get the existence of $(Y, Z) \in \mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^k)$.

All in all, the mapping

$$(y, z) \in \mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^k) \mapsto (Y, Z) \in \mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^k)$$

as defined in (4.3.7) defines a mapping from $\mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^k)$ onto itself. Such a mapping is a contraction for some $\|.\|_{2,\beta}$ norm provided that β is large enough: indeed, for two inputs (y^1, z^1, y^2, z^2) and two outputs (Y^1, Z^1, Y^2, Z^2), the a priori estimates from Theorem 4.6 (with $C = 0$ and $\beta = \mu^2$) yield

$$\begin{aligned}
\delta\xi &= 0, \\
\delta_2 f_t &= f(t, y_t^2, z_t^2) - f(t, y_t^2, z_t^2), \\
|\delta_2 f_t| &\le C_f|\delta y_t| + C_f|\delta z_t|, \\
\|\delta Y\|_{2,\beta}^2 + \|\delta Z\|_{2,\beta}^2 &\le \frac{T+1}{\mu^2}\|\delta_2 f.\|_{2,\beta}^2 \le \frac{2(T+1)C_f^2}{\mu^2}(\|\delta y\|_{2,\beta}^2 + \|\delta z\|_{2,\beta}^2).
\end{aligned}$$

This proves the contraction property as soon as $\mu^2 > 2(T+1)C_f^2$ hence the choice of $\beta = \mu^2$: then the Picard fixed point theorem in Banach spaces can apply and we are done. □

4.3.4 Explicit solution in the linear case

Theorem 4.8. *Assume that*

$$f(t, y, z) = \varphi_t + \beta_t y + \gamma_t \cdot z$$

with predictable coefficients s.t. $\varphi \in \mathbb{H}_2(\mathbb{R})$ and β, γ are bounded. Then for any square integrable ξ, the BSDE solution to the data (ξ, f) is explicitly given by

$$Y_t = \mathbb{E}\left(\xi\frac{H_T}{H_t} + \int_t^T \varphi_s \frac{H_s}{H_t}\mathrm{d}s \mid \mathcal{F}_t\right), \tag{4.3.8}$$

where

$$H_t := \exp\left(\int_0^t (\beta_u - \frac{1}{2}|\gamma_u|^2 \mathrm{d}u) + \int_0^t \gamma_u \mathrm{d}\widehat{W}_u\right).$$

Proof. The existence of (Y, Z) is ensured by Theorem 4.7. The Ito formula applied to $U_t := Y_t H_t + \int_0^t \varphi_s H_s \mathrm{d}s$ yields

$$\mathrm{d}U_t = Y_t \mathrm{d}H_t + H_t \mathrm{d}Y_t + \mathrm{d}\langle Y, H\rangle_t + \varphi_t H_t \mathrm{d}t$$

$$= Y_t H_t(\beta_t \mathrm{d}t + \gamma_t \mathrm{d}\widehat{W}_t) - H_t(\varphi_t + \beta_t Y_t + \gamma_t \cdot Z_t)\mathrm{d}t + H_t Z_t \mathrm{d}\widehat{W}_t + H_t\gamma_t \cdot Z_t \mathrm{d}t + \varphi_t H_t \mathrm{d}t$$
$$= (Y_t H_t \gamma_t + H_t Z_t)\mathrm{d}\widehat{W}_t.$$

Thus, U is a local martingale. But

$$\mathbb{E}\left(\langle U\rangle_t^{1/2}\right) = \mathbb{E}\left(\sqrt{\int_0^T |Y_t H_t \gamma_t + H_t Z_t|^2 \mathrm{d}t}\right) \leq C\mathbb{E}\left(\sup_{t\leq T}|H_t|\left(\sqrt{\int_0^T |Y_t|^2\mathrm{d}t} + \sqrt{\int_0^T |Z_t|^2 \mathrm{d}t}\right)\right)$$

Because β, γ are bounded, $\sup_{t\leq T}|H_t|$ is square integrable, and (Y, Z) being solution in $\mathbf{L}_2$ spaces, we get $\mathbb{E}\left(\langle U\rangle_t^{1/2}\right) < +\infty$ yielding that U is true (integrable) martingale. Therefore,

$$Y_t H_t + \int_0^t \varphi_s H_s = U_t = \mathbb{E}\left(U_T \mid \mathcal{F}_t\right) = \mathbb{E}\left(\xi H_T + \int_0^T \varphi_s H_s \mathrm{d}s \mid \mathcal{F}_t\right)$$

and by rearranging terms, we obtain the announced formula (4.3.8). □

In the linear pricing rule case of Proposition 4.4 (where $\varphi_t \equiv 0$, $\beta_t = -r_t$, $\gamma_t = -\lambda_t^\top$), we obtain

$$Y_t = \mathbb{E}\left(e^{-\int_t^T r_s \mathrm{d}s}\xi e^{-\int_t^T \lambda_u^\top \mathrm{d}\widehat{W}_u - \frac{1}{2}\int_t^T |\lambda_u|^2 \mathrm{d}u} \mid \mathcal{F}_t\right).$$

If we define the new probability measure $\mathbb{Q}^\lambda$ by

$$\mathbb{Q}^\lambda|_{\mathcal{F}_T} = e^{-\int_0^T \lambda_u^\top \mathrm{d}\widehat{W}_u - \frac{1}{2}\int_0^T |\lambda_u|^2\mathrm{d}u}\mathbb{P}|_{\mathcal{F}_T},$$

the Bayes rule for change of measure (Theorem 4.11) gives

$$Y_t = \mathbb{E}_{\mathbb{Q}^\lambda}\left(e^{-\int_t^T r_s \mathrm{d}s}\xi \mid \mathcal{F}_t\right)\ !$$

We retrieve the risk-neutral valuation using the risk-neutral measure $\mathbb{Q}^\lambda$. But the BSDE tools provide the price but also the hedging strategy through the Z-process.

4.3.5 Comparison rule

Another important tool for comparing solution of BSDEs is the following.

Theorem 4.9. *Consider two standard BSDEs such that*

- $\xi_1 \geq \xi_2$ *a.s.*
- $\delta_2 f_t = f_1(t, Y_t^2, Z_t^2) - f_2(t, Y_t^2, Z_t^2) \geq 0, \mathrm{d}t \otimes \mathrm{d}\mathbb{P}$ *a.e.*

Then, $Y_t^1 \geq Y_t^2$ *for any* $t \in [0, T]$ *a.s.*

As a direct consequence, in the case of the linear pricing rule (Proposition 4.4) and non-linear pricing rule, then the above applied to $f_1 \equiv f_2 \equiv f$ gives that two payoffs such that

$$\xi_1 \geq \xi_2$$

generate two prices

$$Y_t^1 \geq Y_t^2;$$

this is consistent with the no-arbitrage rules.

Exercise 4.10. *Assume a Black-Scholes model with drift μ and volatility $\sigma \neq 0$, and two interest rates for borrowing and lending $R > r$. Prove that*

- *the call price is given by the usual Black-Scholes formula with the interest R.*
- *the put price is given by the usual Black-Scholes formula with the interest r.*

Give a financial interpretation.

Proof of Theorem 4.9. Here we assume Z is one-dimensional to simplify notations. Observe that the BSDE difference $(\delta Y, \delta Z)$ solves a new BSDE with terminal condition $\delta\xi \geq 0$, and linear driver described by

$$\begin{aligned}
\delta f_t = f_1(t, Y_t^1, Z_t^1) - f_2(t, Y_t^2, Z_t^2) &= f_1(t, Y_t^2, Z_t^2) - f_2(t, Y_t^2, Z_t^2) \\
&+ \frac{f_1(t, Y_t^1, Z_t^1) - f_1(t, Y_t^2, Z_t^1)}{\delta Y_t} \mathbf{1}_{\delta Y_t \neq 0} \delta Y_t \\
&+ \frac{f_1(t, Y_t^2, Z_t^1) - f_1(t, Y_t^2, Z_t^2)}{\delta Z_t} \mathbf{1}_{\delta Z_t \neq 0} \delta Z_t \\
&=: \delta_2 f_t + \beta_t \delta Y_t + \gamma_t \delta Z_t.
\end{aligned}$$

The newly defined coefficients are such that $\delta_2 f_t \geq 0$, β_t and γ_t are bounded (because f_1 is Lipschitz). The application of Theorem (4.8) ensures that $\delta Y_t = \mathbb{E}\left(\delta\xi \frac{H_T}{H_t} + \int_t^T \delta_2 f_s \frac{H_s}{H_t} \mathrm{d}s \mid \mathcal{F}_t\right) \geq 0$. We are done. □

Appendice: Bayes formula for change of probability measures

Theorem 4.11. *Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ and a positive $\mathbb{Q}$-martingale L starting from $L_0 = 1$, define a new probability measure*

$$\mathbb{Q}^L \big| \mathcal{F}_T := L_T \quad \mathbb{Q}\big|_{\mathcal{F}_T}.$$

Let Z_T be a $\mathcal{F}_T$-mesurable random variable, assumed to be integrable under $\mathbb{Q}$ ($\mathbb{E}_{\mathbb{Q}}(|Z_T|) < +\infty$). Then, the following properties hold:

1. $\mathbb{E}_{\mathbb{Q}^L}\left(\frac{|Z_T|}{L_T}\right) < +\infty$,

2. $\mathbb{E}_{\mathbb{Q}^L}\left(\frac{Z_T}{L_T} \mid \mathcal{F}_t\right) = \frac{1}{L_t}\mathbb{E}_{\mathbb{Q}}\left(Z_T \mid \mathcal{F}_t\right)$.

In addition, M is a $\mathbb{Q}$-martingale if and only if $\frac{M}{L}$ is a $\mathbb{Q}^L$-martingale.

For the proof, see [KS91, Lemma 5.3, Section 3.5].

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