Chapter 9

Viscosity solutions for the dynamic programming equation

9.1 Viscosity solutions for parabolic PDEs

Let F be an operator from $[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ into $\mathbb{R}$ where $\mathbb{S}^n$ stands for the set of n-dimensional symmetric matrices. In this chapter, we will be mostly interested by the case

$$F(t,x,u,q,p,M) \quad = \quad -\sup_{a\in A}\big\{q+\mathcal{L}^a[t,x,u,p,M]+f(x,a)\big\}$$

where

$$\mathcal{L}^a[t,x,u,p,M] \quad = \quad b(x,a).p+\frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^\top(x,a)M\big)$$

for $(t,x,u,q,p,M)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ where the coefficients b,σ and f and the set A are those of the previous chapters and satisfy the related assumptions. This case is called the HJB case.

In the sequel, we assume that F is an elliptic operator, that is, F is nonincreasing in the variable M. We are interested in the PDE

$$F\big(t,x,u(t,x),\partial_t u(t,x),\nabla u(t,x),\nabla^2 u(t,x)\big) \quad = \quad 0 \tag{9.1.1}$$

for $(t,x)\in[0,T)\times\mathbb{R}^n$, together with the terminal condition

$$u(T,x) \quad = \quad g(x)\ , \quad x\in\mathbb{R}^n\ . \tag{9.1.2}$$

Our objective is to give a notion of weak solution to such a PDE.

Definition 9.1.2 *(i) A continuous function* $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *is called a viscosity subsolution to* (9.1.1) *if for any function* $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^n)$ *and* $(t, x) \in [0, T) \times \mathbb{R}^n$ *such that*

$$0 \quad = \quad (u - \varphi)(t, x) \quad = \quad \max_{[0,T]\times\mathbb{R}^n} (u - \varphi)$$

we have

$$F(t, x, \varphi(t, x), \partial_t\varphi(t, x), \nabla\varphi(t, x), \nabla^2\varphi(t, x)) \quad \leq \quad 0 \, .$$

(ii) A continuous function $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *is called a viscosity supersolution to* (9.1.1) *if for any function* $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^n)$ *and* $(t, x) \in [0, T) \times \mathbb{R}^n$ *such that*

$$0 \quad = \quad (u - \varphi)(t, x) \quad = \quad \min_{[0,T]\times\mathbb{R}^n} (u - \varphi)$$

we have

$$F(t, x, \varphi(t, x), \partial_t\varphi(t, x), \nabla\varphi(t, x), \nabla^2\varphi(t, x)) \quad \geq \quad 0 \, .$$

(iii) A function $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *is called a viscosity solution to* (9.1.1) *if it is a viscosity subsolution to* (9.1.1) *and a viscosity supersolution to* (9.1.1).

9.2 Viscosity properties of the value function

We consider the HJB case for PDE (9.1.1) and the value function v defined by (7.1.2).

Theorem 9.2.7 *The value function* v *is a viscosity solution to* (9.1.1) *and satisfies* (9.1.2).

Proof. The fact that v satisfies (9.1.2) comes from its definition. We turn to the viscosity properties w.r.t. equation to (9.1.1).

Step 1. Subsolution property. Let $(\hat{t},\hat{x}) \in [0,T) \times \mathbb{R}^n$ and $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^n)$ be such that

$$0 \;=\; (v-\varphi)(\hat{t},\hat{x}) \;=\; \max_{[0,T]\times\mathbb{R}^n}(v-\varphi)\,. \tag{9.2.3}$$

We proceed by contradiction and assume that

$$-\sup_{a\in A}\big\{(\partial_t+\mathcal{L}^a)\varphi(\hat{t},\hat{x})+f(\hat{x},a)\big\} \;>\; 0\,.$$

By continuity of the coefficients, we can find $r \in (T-\hat{t},T)$ such that

$$-\sup_{a\in A}\big\{(\partial_t+\mathcal{L}^a)\varphi(t,x)+f(x,a)\big\} \;>\; 0 \tag{9.2.4}$$

for $(t,x) \in B := B(\hat{t},r)\times B(\hat{x},r)$. Without loss of generality, we can assume that the minimum in (9.2.5), is strict and that

$$\max_{\partial B}(v-\varphi) \;\le\; -\eta\,.$$

for some $\eta > 0$. Let ν be the first exit time of $(s, X_s^{t,x,\hat{\alpha}})_{s\ge t}$ from B. From the DPP, there exists a control $\hat{\alpha} \in \mathcal{A}_{\hat{t}}$ such that

$$v(\hat{t},\hat{x}) \;\le\; \mathbb{E}\Big[v(\theta, X_\theta^{t,x,\hat{\alpha}}) + \int_t^\theta f\big(X_s^{t,x,\hat{\alpha}},\hat{\alpha}_s\big)ds\Big] + \frac{\eta}{2}\,.$$

From the previous inequalities, we get

$$\varphi(\hat{t},\hat{x}) \;\le\; \mathbb{E}\Big[\varphi(\theta, X_\theta^{t,x,\hat{\alpha}}) - \eta + \int_t^\theta f\big(X_s^{t,x,\hat{\alpha}},\hat{\alpha}_s\big)ds\Big] + \frac{\eta}{2}\,.$$

Applying Itô's Lemma to φ, we get

$$0 \;\le\; -\frac{\eta}{2} + \mathbb{E}\Big[\int_t^\theta \big((\partial_t+\mathcal{L}^{\hat{\alpha}_s})\varphi\big(s,X_s^{t,x,\hat{\alpha}}\big) + f\big(X_s^{t,x,\hat{\alpha}},\hat{\alpha}_s\big)\big)ds\Big]\,.$$

Using (9.2.4) we get

$$0 \;\le\; -\frac{\eta}{2}\,.$$

Step 2. Supersolution property. Let $(t,x) \in [0,T)\times\mathbb{R}^n$ and $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$ be such that

$$0 \quad = \quad (v-\varphi)(t,x) \quad = \quad \min_{[0,T]\times\mathbb{R}^n}(v-\varphi)\ . \tag{9.2.5}$$

Fix $a \in A$ and denote by $\hat\alpha$ the control constant equal to a. From the dynamic programming principle we have

$$v(t,x) \quad \geq \quad \mathbb{E}\Big[v\big(\theta, X_\theta^{t,x,\hat\alpha}\big) + \int_t^{t+h} f\big(X_s^{t,x,\hat\alpha},a\big)ds\Big]$$

for $h \in (0, T-\hat t)$. Using (9.2.5), we get

$$\varphi(t,x) \quad \geq \quad \mathbb{E}\Big[\varphi\big(t+h, X_{t+h}^{t,x,\hat\alpha}\big) + \int_t^{t+h} f\big(X_s^{t,x,\hat\alpha},a\big)ds\Big]$$

Applying Itô's formula and diving by $h > 0$, we get

$$0 \quad \geq \quad \frac{1}{h}\mathbb{E}\Big[\int_t^{t+h}\big[(\partial_t + \mathcal{L}^a)\varphi\big(X_s^{t,x,\hat\alpha},a\big) + f\big(X_s^{t,x,\hat\alpha},a\big)\big]ds\Big]$$

Sending h to $0+$, we get by the mean value theorem

$$(\partial_t + \mathcal{L}^a)\varphi\big(t,x\big) + f\big(x,a\big) \quad \leq \quad 0\ .$$

Since this holds for all $a \in A$, we get the supersolution property. □

9.3 Comparison and uniqueness

We first have the following result which allow to compare a supersolution and a subsolution as soon as we can compare their values at the terminal time T.

Theorem 9.3.8 *Let U (resp. V) $\in C^{1,2}([0,T)\times\mathbb{R}^n)\cap C^0([0,T]\times\mathbb{R}^n)$ be a subsolution (resp. supersolution) with polynomial growth to* (9.1.1) *in the HJB case. If $U(T,.) \leq V(T,.)$ on $\mathbb{R}^n$, then $U \leq V$ on $[0,T]\times\mathbb{R}^n$.*

Proof. We present the proof in the regular case. The proof for viscosity solutions uses an additional result called Ishii's lemma which gives a similar condition to second order optimality condition. We proceed in 3 steps.
Step 1. Let $\tilde{U}(t,x) = e^{\lambda t}U(t,x)$ and $\tilde{V}(t,x) = e^{\lambda t}V(t,x)$. Then a straightforward calculation shows that $\tilde{U}$ (resp. $\tilde{V}$) is a subsolution (resp. supersolution) to

$$\tilde{F}\big(t,x,w,\partial_t w(t,x),\nabla w(t,x),\nabla^2 w(t,x)\big) \;=\; 0 \qquad (9.3.6)$$

with

$$\begin{aligned}
F\big(t,x,u,q,p,M\big) &= -\sup_{a\in A}\big\{q+\tilde{\mathcal{L}}^a[t,x,u,p,M]+\tilde{f}(t,x,a)\big\}\ ,\\
\tilde{\mathcal{L}}^a[t,x,u,p,M] &= b(x,a).p+\frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^\top(x,a)M\big)-\lambda u\ ,\\
\tilde{f}(t,x,a) &= e^{\lambda t}f(x,a)
\end{aligned}$$

for $(t,x,u,q,p,M)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ and $a\in A$.

Step 2. From the polynomial growth of U and V, we may choose an integer $p\geq 1$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n}\frac{|\tilde{U}(t,x)|+|\tilde{V}(t,x)|}{1+|x|^p} \;<\; +\infty\ ,$$

and we consider the function $\phi(t,x)=e^{-\lambda t}(1+|x|^{2p}) =: e^{-\lambda t}\psi(x)$. From the linear growth condition on b and σ, a straightforward computation shows that there exists some constant $c>0$ s.t.

$$\begin{aligned}
&-\partial_t\phi(t,x)+\lambda\phi(t,x)-\sup_{a\in A}b(x,a).\nabla\phi(t,x)+\frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(x,a)\nabla^2\phi(t,x))\\
\geq\;& e^{-\lambda t}\psi(x)(\lambda-c)\ .
\end{aligned}$$

By taking $\lambda\geq c$, the function $\tilde{V}_\varepsilon := \tilde{V}+\varepsilon\phi$ is a supersolution to (9.1.1) for any $\varepsilon>0$. Moreover, from the growth conditions on U,V and ϕ, we have

$$\lim_{|x|\to+\infty}\sup_{t\in[0,T]}(\tilde{U}-\tilde{V}_\varepsilon)(t,x) \;=\; -\infty\ . \qquad (9.3.7)$$

for all $\varepsilon>0$.

Step 3. We finally argue by contradiction to show that $\tilde{U} - \tilde{V}_\varepsilon \leq 0$ on $[0,T] \times \mathbb{R}^n$ for all $\varepsilon > 0$, which gives the required result by sending ε to 0. On the contrary, by continuity of $\tilde{U} - \tilde{V}_\varepsilon$, and from (9.3.7), there exists $\varepsilon > 0$, $(t,x) \in [0,T] \times \mathbb{R}^n$ such that

$$\sup_{[0,T]\times\mathbb{R}^n} (\tilde{U} - \tilde{V}_\varepsilon) \;=\; (\tilde{U} - \tilde{V}_\varepsilon)(t,x) \;>\; 0\,. \tag{9.3.8}$$

Since $(\tilde{U} - \tilde{V}_\varepsilon)(T,.) \leq (\tilde{U} - \tilde{V})(T,.) \leq 0$ on $\mathbb{R}^n$, we have $t < T$. Therefore, the first and second order optimality conditions (9.3.8) give

$$\begin{aligned}
\partial_t(\tilde{U} - \tilde{V}_\varepsilon)(t,x) &\leq 0\,,\\
\nabla(\tilde{U} - \tilde{V}_\varepsilon)(t,x) &= 0\,,\\
\nabla^2(\tilde{U} - \tilde{V}_\varepsilon)(t,x) &\leq 0\,.
\end{aligned}$$

By writing that $\tilde{U}$ (resp. $\tilde{V}_\varepsilon$) is a subsolution (resp. supersolution) to (9.3.6), and recalling that $\tilde{F}$ is nondecreasing in its last argument, we then deduce that

$$\begin{aligned}
& \lambda(\tilde{U} - \tilde{V}_\varepsilon)(t,x)\\
\leq\; & \tilde{F}(t,x,\partial_t\tilde{U}(t,x),\nabla\tilde{U}(t,x),\nabla^2\tilde{U}(t,x))\\
& -\tilde{F}(t,x,\partial_t\tilde{V}_\varepsilon(t,x),\nabla\tilde{V}_\varepsilon(t,x),\nabla^2\tilde{V}_\varepsilon(t,x))\\
\leq\; & 0
\end{aligned}$$

which contradicts (9.3.8). □

Corollary 9.3.1 *The value function v is the unique viscosity solution to* (9.1.1) *in the HJB case satisfying* (9.1.1) *and having polynomial growth.*

Proof. We simply need to prove that v has polynomial growth. It actually follows from the polynomial growth assumption on f and g and Proposition (7.1.3). □