are all finite sets) there exists a Bayesian Nash equilibrium, perhaps in mixed strategies. The proof closely parallels the proof of the existence of a mixed-strategy Nash equilibrium in finite games of complete information, and so is omitted here.

3.2 Applications

3.2.A Mixed Strategies Revisited

As we mentioned in Section 1.3.A, Harsanyi (1973) suggested that player j's mixed strategy represents player i's uncertainty about j's choice of a pure strategy, and that j's choice in turn depends on the realization of a small amount of private information. We can now give a more precise statement of this idea: a mixed-strategy Nash equilibrium in a game of complete information can (almost always) be interpreted as a pure-strategy Bayesian Nash equilibrium in a closely related game with a little bit of incomplete information. (We will ignore the rare cases in which such an interpretation is not possible.) Put more evocatively, the crucial feature of a mixed-strategy Nash equilibrium is not that player j chooses a strategy randomly, but rather that player i is uncertain about player j's choice; this uncertainty can arise either because of randomization or (more plausibly) because of a little incomplete information, as in the following example.

Recall that in the Battle of the Sexes there are two pure-strategy Nash equilibria (Opera, Opera) and (Fight, Fight) and a mixed-strategy Nash equilibrium in which Chris plays Opera with probability 2/3 and Pat plays Fight with probability 2/3.

		Pat	
		Opera	Fight
Chris	Opera	$2, 1$	$0, 0$
	Fight	$0, 0$	$1, 2$

The Battle of the Sexes

Now suppose that, although they have known each other for quite some time, Chris and Pat are not quite sure of each other's payoffs. In particular, suppose that: Chris's payoff if both attend the Opera is $2 + t_c$, where t_c is privately known by Chris; Pat's payoff if both attend the Fight is $2 + t_p$, where t_p is privately known by Pat; and t_c and t_p are independent draws from a uniform distribution on $[0, x]$. (The choice of a uniform distribution on $[0, x]$ is not important, but we do have in mind that the values of t_c and t_p only slightly perturb the payoffs in the original game, so think of x as small.) All the other payoffs are the same. In terms of the abstract static Bayesian game in normal form $G = \{A_c, A_p; T_c, T_p; p_c, p_p; u_c, u_p\}$, the action spaces are $A_c = A_p = \{\text{Opera, Fight}\}$, the type spaces are $T_c = T_p = [0, x]$, the beliefs are $p_c(t_p) = p_p(t_c) = 1/x$ for all t_c and t_p, and the payoffs are as follows.

		Pat	
		Opera	Fight
Chris	Opera	$2 + t_c, 1$	$0, 0$
	Fight	$0, 0$	$1, 2 + t_p$

The Battle of the Sexes with Incomplete Information

We will construct a pure-strategy Bayesian Nash equilibrium of this incomplete-information version of the Battle of the Sexes in which Chris plays Opera if t_c exceeds a critical value, c, and plays Fight otherwise and Pat plays Fight if t_p exceeds a critical value, p, and plays Opera otherwise. In such an equilibrium, Chris plays Opera with probability $(x - c)/x$ and Pat plays Fight with probability $(x - p)/x$. We will show that as the incomplete information disappears (i.e., as x approaches zero), the players' behavior in this pure-strategy Bayesian Nash equilibrium approaches their behavior in the mixed-strategy Nash equilibrium in the original game of complete information. That is, both $(x - c)/x$ and $(x - p)/x$ approach 2/3 as x approaches zero.

Suppose Chris and Pat play the strategies just described. For a given value of x, we will determine values of c and p such that these strategies are a Bayesian Nash equilibrium. Given Pat's strategy, Chris's expected payoffs from playing Opera and from playing Fight are

$$\frac{p}{x}(2+t_c) + \left[1-\frac{p}{x}\right]\cdot 0 = \frac{p}{x}(2+t_c)$$

and

$$\frac{p}{x}\cdot 0 + \left[1-\frac{p}{x}\right]\cdot 1 = 1-\frac{p}{x},$$

respectively. Thus playing Opera is optimal if and only if

$$t_c \geq \frac{x}{p} - 3 = c. \tag{3.2.1}$$

Similarly, given Chris's strategy, Pat's expected payoffs from playing Fight and from playing Opera are

$$\left[1-\frac{c}{x}\right]\cdot 0 + \frac{c}{x}(2+t_p) = \frac{c}{x}(2+t_p)$$

and

$$\left[1-\frac{c}{x}\right]\cdot 1 + \frac{c}{x}\cdot 0 = 1-\frac{c}{x},$$

respectively. Thus, playing Fight is optimal if and only if

$$t_p \geq \frac{x}{c} - 3 = p. \tag{3.2.2}$$

Solving (3.2.1) and (3.2.2) simultaneously yields $p = c$ and $p^2 + 3p - x = 0$. Solving the quadratic then shows that the probability that Chris plays Opera, namely $(x-c)/x$, and the probability that Pat plays Fight, namely $(x-p)/x$, both equal

$$1 - \frac{-3+\sqrt{9+4x}}{2x},$$

which approaches 2/3 as x approaches zero. Thus, as the incomplete information disappears, the players' behavior in this pure-strategy Bayesian Nash equilibrium of the incomplete-information game approaches their behavior in the mixed-strategy Nash equilibrium in the original game of complete information.

3.2.B An Auction

Consider the following first-price, sealed-bid auction. There are two bidders, labeled $i = 1, 2$. Bidder i has a valuation v_i for the good—that is, if bidder i gets the good and pays the price p, then i's payoff is $v_i - p$. The two bidders' valuations are independently and uniformly distributed on $[0,1]$. Bids are constrained to be nonnegative. The bidders simultaneously submit their bids. The higher bidder wins the good and pays the price she bid; the other bidder gets and pays nothing. In case of a tie, the winner is determined by a flip of a coin. The bidders are risk-neutral. All of this is common knowledge.

In order to formulate this problem as a static Bayesian game, we must identify the action spaces, the type spaces, the beliefs, and the payoff functions. Player i's action is to submit a (nonnegative) bid, b_i, and her type is her valuation, v_i. (In terms of the abstract game $G = \{A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2\}$, the action space is $A_i = [0, \infty)$ and the type space is $T_i = [0,1]$.) Because the valuations are independent, player i believes that v_j is uniformly distributed on $[0,1]$, no matter what the value of v_i. Finally, player i's payoff function is

$$u_i(b_1, b_2; v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j, \\ (v_i - b_i)/2 & \text{if } b_i = b_j, \\ 0 & \text{if } b_i < b_j. \end{cases}$$

To derive a Bayesian Nash equilibrium of this game, we begin by constructing the players' strategy spaces. Recall that in a static Bayesian game, a strategy is a function from types to actions. Thus, a strategy for player i is a function $b_i(v_i)$ specifying the bid that each of i's types (i.e., valuations) would choose. In a Bayesian Nash equilibrium, player 1's strategy $b_1(v_1)$ is a best response to player 2's strategy $b_2(v_2)$, and vice versa. Formally, the pair of strategies $(b(v_1), b(v_2))$ is a Bayesian Nash equilibrium if for each v_i in $[0,1]$, $b_i(v_i)$ solves

$$\max_{b_i} \; (v_i - b_i)\text{Prob}\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i)\text{Prob}\{b_i = b_j(v_j)\}.$$

We simplify the exposition by looking for a linear equilibrium: $b_1(v_1) = a_1 + c_1 v_1$ and $b_2(v_2) = a_2 + c_2 v_2$. Note well that we are *not* restricting the players' strategy spaces to include only linear strategies. Rather, we allow the players to choose arbitrary strategies but ask whether there is an equilibrium that is linear. It turns out that because the players' valuations are uniformly distributed, a linear equilibrium not only exists but is unique (in a sense to be made precise). We will find that $b_i(v_i) = v_i/2$. That is, each player submits a bid equal to half her valuation. Such a bid reflects the fundamental trade-off a bidder faces in an auction: the higher the bid, the more likely the bidder is to win; the lower the bid, the larger the gain if the bidder does win.

Suppose that player j adopts the strategy $b_j(v_j) = a_j + c_j v_j$. For a given value of v_i, player i's best response solves

$$\max_{b_i} \, (v_i - b_i)\text{Prob}\{b_i > a_j + c_j v_j\},$$

where we have used the fact that $\text{Prob}\{b_i = b_j(v_j)\} = 0$ (because $b_j(v_j) = a_j + c_j v_j$ and v_j is uniformly distributed, so b_j is uniformly distributed). Since it is pointless for player i to bid below player j's minimum bid and stupid for i to bid above j's maximum bid, we have $a_j \leq b_i \leq a_j + c_j$, so

$$\text{Prob}\{b_i > a_j + c_j v_j\} = \text{Prob}\left\{v_j < \frac{b_i - a_j}{c_j}\right\} = \frac{b_i - a_j}{c_j}.$$

Player i's best response is therefore

$$b_i(v_i) = \begin{cases} (v_i + a_j)/2 & \text{if } v_i \geq a_j, \\ a_j & \text{if } v_i < a_j. \end{cases}$$

If $0 < a_j < 1$ then there are some values of v_i such that $v_i < a_j$, in which case $b_i(v_i)$ is not linear; rather, it is flat at first and positively sloped later. Since we are looking for a linear equilibrium, we therefore rule out $0 < a_j < 1$, focusing instead on $a_j \geq 1$ and $a_j \leq 0$. But the former cannot occur in equilibrium: since it is optimal for a higher type to bid at least as much as a lower type's optimal bid, we have $c_j \geq 0$, but then $a_j \geq 1$ would imply that $b_j(v_j) \geq v_j$, which cannot be optimal. Thus, if $b_i(v_i)$ is to be linear, then we must have $a_j \leq 0$, in which case $b_i(v_i) = (v_i + a_j)/2$, so $a_i = a_j/2$ and $c_i = 1/2$.

We can repeat the same analysis for player j under the assumption that player i adopts the strategy $b_i(v_i) = a_i + c_i v_i$. This yields $a_i \leq 0$, $a_j = a_i/2$, and $c_j = 1/2$. Combining these two sets of results then yields $a_i = a_j = 0$ and $c_i = c_j = 1/2$. That is, $b_i(v_i) = v_i/2$, as claimed earlier.

One might wonder whether there are other Bayesian Nash equilibria of this game, and also how equilibrium bidding changes as the distribution of the bidders' valuations changes. Neither of these questions can be answered using the technique just applied (of positing linear strategies and then deriving the coefficients that make the strategies an equilibrium): it is fruitless to try to guess all the functional forms other equilibria of this game might have, and a linear equilibrium does not exist for any other distribution of valuations. In the Appendix, we derive a symmetric Bayesian Nash equilibrium,[3] again for the case of uniformly distributed valuations. Under the assumption that the players' strategies are strictly increasing and differentiable, we show that the unique symmetric Bayesian Nash equilibrium is the linear equilibrium already derived. The technique we use can easily be extended to a broad class of valuation distributions, as well as the case of n bidders.[4]

Appendix 3.2.B

Suppose player j adopts the strategy $b(\cdot)$, and assume that $b(\cdot)$ is strictly increasing and differentiable. Then for a given value of v_i, player i's optimal bid solves

$$\max_{b_i} \, (v_i - b_i)\text{Prob}\{b_i > b(v_j)\}.$$

Let $b^{-1}(b_j)$ denote the valuation that bidder j must have in order to bid b_j. That is, $b^{-1}(b_j) = v_j$ if $b_j = b(v_j)$. Since v_j is uniformly distributed on $[0, 1]$, $\text{Prob}\{b_i > b(v_j)\} = \text{Prob}\{b^{-1}(b_i) > v_j\} = b^{-1}(b_i)$. The first-order condition for player i's optimization problem is therefore

$$-b^{-1}(b_i) + (v_i - b_i)\frac{d}{db_i}b^{-1}(b_i) = 0.$$

[3]A Bayesian Nash equilibrium is called symmetric if the players' strategies are identical. That is, in a symmetric Bayesian Nash equilibrium, there is a single function $b(v_i)$ such that player 1's strategy $b_1(v_1)$ is $b(v_1)$ and player 2's strategy $b_2(v_2)$ is $b(v_2)$, and this single strategy is a best response to itself. Of course, since the players' valuations typically will be different, their bids typically will be different, even if both use the same strategy.

[4]Skipping this appendix will not hamper one's understanding of what follows.

This first-order condition is an implicit equation for bidder i's best response to the strategy $b(\cdot)$ played by bidder j, given that bidder i's valuation is v_i. If the strategy $b(\cdot)$ is to be a symmetric Bayesian Nash equilibrium, we require that the solution to the first-order condition be $b(v_i)$: that is, for each of bidder i's possible valuations, bidder i does not wish to deviate from the strategy $b(\cdot)$, given that bidder j plays this strategy. To impose this requirement, we substitute $b_i = b(v_i)$ into the first-order condition, yielding:

$$-b^{-1}(b(v_i)) + (v_i - b(v_i))\frac{d}{db_i}b^{-1}(b(v_i)) = 0.$$

Of course, $b^{-1}(b(v_i))$ is simply v_i. Furthermore, $d\{b^{-1}(b(v_i))\}/db_i = 1/b'(v_i)$. That is, $d\{b^{-1}(b_i)\}/db_i$ measures how much bidder i's valuation must change to produce a unit change in the bid, whereas $b'(v_i)$ measures how much the bid changes in response to a unit change in the valuation. Thus, $b(\cdot)$ must satisfy the first-order differential equation

$$-v_i + (v_i - b(v_i))\frac{1}{b'(v_i)} = 0,$$

which is more conveniently expressed as $b'(v_i)v_i + b(v_i) = v_i$. The left-hand side of this differential equation is precisely $d\{b(v_i)v_i\}/dv_i$. Integrating both sides of the equation therefore yields

$$b(v_i)v_i = \frac{1}{2}v_i^2 + k,$$

where k is a constant of integration. To eliminate k, we need a boundary condition. Fortunately, simple economic reasoning provides one: no player should bid more than his or her valuation. Thus, we require $b(v_i) \leq v_i$ for every v_i. In particular, we require $b(0) \leq 0$. Since bids are constrained to be nonnegative, this implies that $b(0) = 0$, so $k = 0$ and $b(v_i) = v_i/2$, as claimed.

3.2.C A Double Auction

We next consider the case in which a buyer and a seller each have private information about their valuations, as in Chatterjee and Samuelson (1983). (In Hall and Lazear [1984], the buyer is a firm and the seller is a worker. The firm knows the worker's marginal

product and the worker knows his or her outside opportunity. See Problem 3.8.) We analyze a trading game called a double auction. The seller names an asking price, p_s, and the buyer simultaneously names an offer price, p_b. If $p_b \geq p_s$, then trade occurs at price $p = (p_b + p_s)/2$; if $p_b < p_s$, then no trade occurs.

The buyer's valuation for the seller's good is v_b, the seller's is v_s. These valuations are private information and are drawn from independent uniform distributions on [0, 1]. If the buyer gets the good for price p, then the buyer's utility is $v_b - p$; if there is no trade, then the buyer's utility is zero. If the seller sells the good for price p, then the seller's utility is $p - v_s$; if there is no trade, then the seller's utility is zero. (Each of these utility functions measures the change in the party's utility. If there is no trade, then there is no change in utility. It would make no difference to define, say, the seller's utility to be p if there is trade at price p and v_s if there is no trade.)

In this static Bayesian game, a strategy for the buyer is a function $p_b(v_b)$ specifying the price the buyer will offer for each of the buyer's possible valuations. Likewise, a strategy for the seller is a function $p_s(v_s)$ specifying the price the seller will demand for each of the seller's valuations. A pair of strategies $\{p_b(v_b), p_s(v_s)\}$ is a Bayesian Nash equilibrium if the following two conditions hold. For each v_b in $[0, 1]$, $p_b(v_b)$ solves

$$\max_{p_b} \left[v_b - \frac{p_b + E[p_s(v_s) \mid p_b \geq p_s(v_s)]}{2}\right] \text{Prob}\{p_b \geq p_s(v_s)\}, \quad (3.2.3)$$

where $E[p_s(v_s) \mid p_b \geq p_s(v_s)]$ is the expected price the seller will demand, conditional on the demand being less than the buyer's offer of p_b. For each v_s in $[0, 1]$, $p_s(v_s)$ solves

$$\max_{p_s} \left[\frac{p_s + E[p_b(v_b) \mid p_b(v_b) \geq p_s]}{2} - v_s\right] \text{Prob}\{p_b(v_b) \geq p_s\}, \quad (3.2.4)$$

where $E[p_b(v_b) \mid p_b(v_b) \geq p_s]$ is the expected price the buyer will offer, conditional on the offer being greater than the seller's demand of p_s.

There are many, many Bayesian Nash equilibria of this game. Consider the following one-price equilibrium, for example, in which trade occurs at a single price if it occurs at all. For any value x in $[0, 1]$, let the buyer's strategy be to offer x if $v_b \geq x$ and

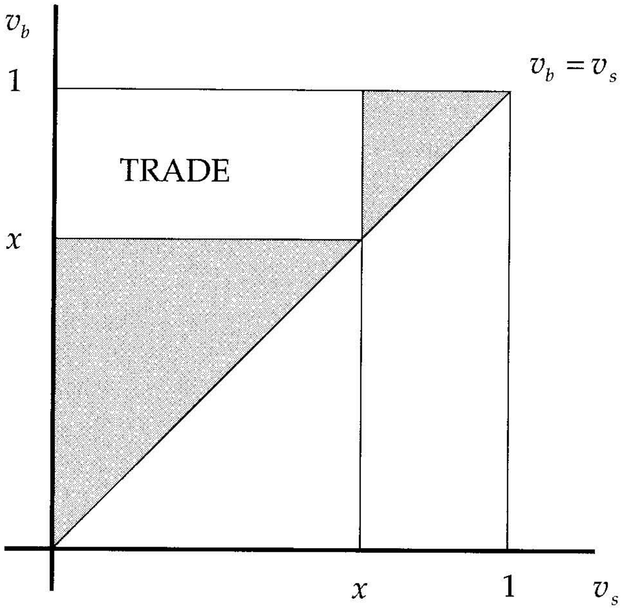

Figure 3.2.1.

to offer zero otherwise, and let the seller's strategy be to demand x if $v_s \leq x$ and to demand one otherwise. Given the buyer's strategy, the seller's choices amount to trading at x or not trading, so the seller's strategy is a best response to the buyer's because the seller-types who prefer trading at x to not trading do so, and vice versa. The analogous argument shows that the buyer's strategy is a best response to the seller's, so these strategies are indeed a Bayesian Nash equilibrium. In this equilibrium, trade occurs for the (v_s, v_b) pairs indicated in Figure 3.2.1; trade would be efficient for all (v_s, v_b) pairs such that $v_b \geq v_s$, but does not occur in the two shaded regions of the figure.

We now derive a linear Bayesian Nash equilibrium of the double auction. As in the previous section, we are *not* restricting the players' strategy spaces to include only linear strategies. Rather, we allow the players to choose arbitrary strategies but ask whether there is an equilibrium that is linear. Many other equilibria exist besides the one-price equilibria and the linear equilibrium, but the

linear equilibrium has interesting efficiency properties, which we describe later.

Suppose the seller's strategy is $p_s(v_s) = a_s + c_s v_s$. Then p_s is uniformly distributed on $[a_s, a_s + c_s]$, so (3.2.3) becomes

$$\max_{p_b} \left[v_b - \frac{1}{2}\left\{ p_b + \frac{a_s + p_b}{2} \right\} \right] \frac{p_b - a_s}{c_s},$$

the first-order condition for which yields

$$p_b = \frac{2}{3}v_b + \frac{1}{3}a_s. \tag{3.2.5}$$

Thus, if the seller plays a linear strategy, then the buyer's best response is also linear. Analogously, suppose the buyer's strategy is $p_b(v_b) = a_b + c_b v_b$. Then p_b is uniformly distributed on $[a_b, a_b + c_b]$, so (3.2.4) becomes

$$\max_{p_s} \left[\frac{1}{2}\left\{ p_s + \frac{p_s + a_b + c_b}{2} \right\} - v_s \right] \frac{a_b + c_b - p_s}{c_b},$$

the first-order condition for which yields

$$p_s = \frac{2}{3}v_s + \frac{1}{3}(a_b + c_b). \tag{3.2.6}$$

Thus, if the buyer plays a linear strategy, then the seller's best response is also linear. If the players' linear strategies are to be best responses to each other, (3.2.5) implies that $c_b = 2/3$ and $a_b = a_s/3$, and (3.2.6) implies that $c_s = 2/3$ and $a_s = (a_b + c_b)/3$. Therefore, the linear equilibrium strategies are

$$p_b(v_b) = \frac{2}{3}v_b + \frac{1}{12} \tag{3.2.7}$$

and

$$p_s(v_s) = \frac{2}{3}v_s + \frac{1}{4}, \tag{3.2.8}$$

as shown in Figure 3.2.2.

Recall that trade occurs in the double auction if and only if $p_b \geq p_s$. Manipulating (3.2.7) and (3.2.8) shows that trade occurs in the linear equilibrium if and only if $v_b \geq v_s + (1/4)$, as shown in Figure 3.2.3. (Consistent with this, Figure 3.2.2 reveals that seller-types above 3/4 make demands above the buyer's highest offer,

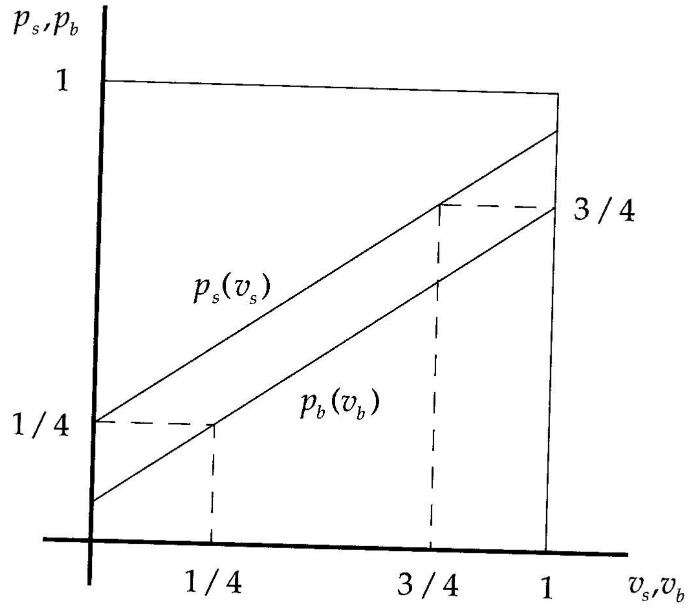

Figure 3.2.2.

$p_b(1) = 3/4$, and buyer-types below 1/4 make offers below the seller's lowest offer, $p_s(0) = 1/4$.)

Compare Figures 3.2.1 and 3.2.3—the depictions of which valuation pairs trade in the one-price and linear equilibria, respectively. In both cases, the most valuable possible trade (namely, $v_s = 0$ and $v_b = 1$) does occur. But the one-price equilibrium misses some valuable trades (such as $v_s = 0$ and $v_b = x - \varepsilon$, where ε is small) and achieves some trades that are worth next to nothing (such as $v_s = x - \varepsilon$ and $v_b = x + \varepsilon$). The linear equilibrium, in contrast, misses all trades worth next to nothing but achieves all trades worth at least 1/4. This suggests that the linear equilibrium may dominate the one-price equilibria, in terms of the expected gains the players receive, but also raises the possibility that the players might do even better in an alternative equilibrium.

Myerson and Satterthwaite (1983) show that, for the uniform valuation distributions considered here, the linear equilibrium yields higher expected gains for the players than any other Bayes-

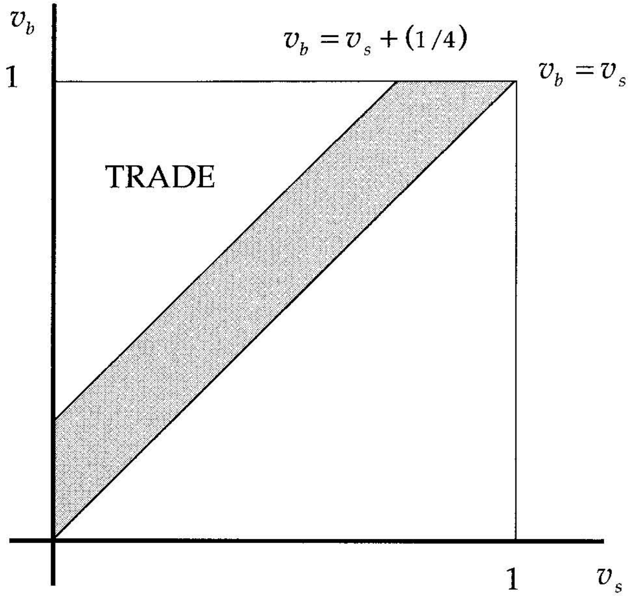

Figure 3.2.3.

ian Nash equilibria of the double auction (including but far from limited to the one-price equilibria). This implies that there is no Bayesian Nash equilibrium of the double auction in which trade occurs if and only if it is efficient (i.e., if and only if $v_b \geq v_s$). They also show that this latter result is very general: if v_b is continuously distributed on $[x_b, y_b]$ and v_s is continuously distributed on $[x_s, y_s]$, where $y_s > x_b$ and $y_b > x_s$, then there is no bargaining game the buyer and seller would willingly play that has a Bayesian Nash equilibrium in which trade occurs if and only if it is efficient. In the next section we sketch how the Revelation Principle can be used to prove this general result. We conclude this section by translating the result into Hall and Lazear's employment model: if the firm has private information about the worker's marginal product (m) and the worker has private information about his or her outside opportunity (v), then there is no bargaining game that the firm and the worker would willingly play that produces employment if and only if it is efficient (i.e., $m \geq v$).