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Master 2 – Probabilités et Finance
Sorbonne Université et Ecole Polytechnique

Convexity, Optimization and Stochastic Control

Exercise 1 : Maximum Principle

Let $T > 0$ and $A \subset \mathbb{R}$. We give ourselves the maps $[0,T]\times\mathbb{R}\times A \ni (t,x,a) \mapsto (b,f)(t,x,a) \in \mathbb{R}\times\mathbb{R}$ and $\mathbb{R} \ni x \mapsto g(x) \in \mathbb{R}$ Lipschitz in (x,a) and there exists $C > 0$ s.t. for each (t,x,a)

$$|b(t,x,a)| + |f(t,x,a)| + |g(x)| \leq C(1+|x|).$$

We denote by $\mathcal{A}$ the set of all predictable process $(\alpha_t)_{t\in[0,T]}$ satisfying $\int_0^T \alpha_t^2 \mathrm{d}t < \infty$. Wet set $x \in \mathbb{R}$. For any $\alpha \in \mathcal{A}$, let X^α be a process satisfying:

$$X_t^\alpha = x + \int_0^t b(r, X_r^\alpha, \alpha_r)\,\mathrm{d}r, \quad \text{for each } t \leq T.$$

The goal of this exercise is to solve the following optimization problem

$$R := \sup_{\alpha\in\mathcal{A}} J(\alpha) \;\text{ with }\; J(\alpha) := \int_0^T f(t, X_t^\alpha, \alpha_t)\mathrm{d}t + g(X_T^\alpha).$$

We introduce for any α and $\beta \in \mathcal{A}$, the process $V^{\alpha,\beta} := V$ verifying:

$$V_t = \int_0^t V_r \partial_x b(r, X_r^\alpha, \alpha_r) + \beta_r \partial_a b(r, X_r^\alpha, \alpha_r)\,\mathrm{d}r, \quad \text{for each } t \leq T.$$

1. Let $\alpha, \beta \in \mathcal{A}$. We assume that for any sufficiently small $\varepsilon > 0$, $\alpha^\varepsilon := \alpha + \varepsilon\beta \in \mathcal{A}$. Show that

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \left| \frac{X_t^{\alpha^\varepsilon} - X_t^\alpha}{\varepsilon} - V_t^{\alpha,\beta} \right| = 0.$$

2. Deduce that

$$\lim_{\varepsilon\to 0} \frac{J(\alpha+\varepsilon\beta) - J(\alpha)}{\varepsilon} = \int_0^T V_t^{\alpha,\beta} \partial_x f(t, X_t^\alpha, \alpha_t) + \beta_t \partial_a f(t, X_t^\alpha, \alpha_t)\,\mathrm{d}t$$

We introduce the map $H : [0,T]\times\mathbb{R}\times\mathbb{R}\times A \to \mathbb{R}$ defined by

$$H(t,x,y,a) := b(t,x,a)\,y + f(t,x,a).$$

The map H is called the Hamiltonian. For $\alpha \in \mathcal{A}$, we denote Y^α the process verifying

$$Y_t = \partial_x g(X_T^\alpha) - \int_t^T \partial_x H(r, X_r^\alpha, Y_r^\alpha, \alpha_r)\,\mathrm{d}r, \quad \text{for all } t \leq T.$$

3. Show that
$$Y_T^\alpha V_T^{\alpha,\beta} = \int_0^T Y_t^\alpha \partial_a b(t, X_t^\alpha, \alpha_t)\beta_t - \partial_x f(t, X_t^\alpha, \alpha_t) V_t^{\alpha,\beta} \,\mathrm{d}t.$$
4. Deduce that
$$\lim_{\varepsilon \to 0} \frac{J(\alpha+\varepsilon\beta) - J(\alpha)}{\varepsilon} = \int_0^T \partial_a H(t, X_t^\alpha, Y_t^\alpha, \alpha_t)\beta_t \,\mathrm{d}t.$$
5. We assume that H is convex in a, show that if α is an optimal control then for each $a \in A$,
$$H(t, X_t^\alpha, Y_t^\alpha, a) \leq H(t, X_t^\alpha, Y_t^\alpha, \alpha_t).$$
6. Let us assume that g is convex, H is convex in (x, a) and $\alpha \in \mathcal{A}$ s.t.
$$H(t, X^\alpha, Y_t^\alpha, \alpha_t) = \sup_{a \in A} H(t, X_t^\alpha, Y_t^\alpha, a).$$
Show that $\alpha \in \mathcal{A}$ is an optimal control i.e. $R = J(\alpha)$.

Exercise 2 : Fenchel–Rockafellar Theorem and application

Part I : Fenchel–Rockafellar Theorem

The goal here is to prove that: for two proper convex functions $(f, g) : \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}$, for any $x_0 \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$, we have
$$(f+g)^\star(x_0) = \inf_{z \in \mathbb{R}^d} \{f^\star(z) + g^\star(x_0 - z)\} = (f^* \circ g^*)(x) \tag{1}$$

1. Verify that for any $x \in \mathbb{R}^d$ we have $(f+g)^\star(x) \leq \inf_{z \in \mathbb{R}^d}\{f^\star(z) + g^\star(x - z)\}$.
2. Justify why the map $h(u) := \inf_{x \in E} f(x) + g(x + u)$ is convex ?
3. Let $x_0 = 0 \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$.
   (a) Show that there exists $\overline{x} \in \mathbb{R}^d$ s.t.
   $$h(0) + \langle \overline{x}, u \rangle \leq h(u), \quad \text{for any } u \in \mathbb{R}^d.$$
   (b) Verify that for any x and u,
   $$f(x) + g(x+u) - \langle \overline{x}, u \rangle \geq h(0).$$
   (c) Deduce that (1) is true for $x_0 = 0$.
4. Let $z \in \mathbb{R}^d$ and we set $\ell(u) := g(u) - \langle z, u \rangle$.
   (a) Verify that $\ell^\star(x) = g^\star(x + z)$ and $(f + \ell)^\star(0) = (f + g)^\star(z)$.
   (b) Deduce (1) for the general case.

Exo 1 Principe du maximum

$T>0$ $A\subset\mathbb{R}$. g différentiables en (x,a) aux dérivées bornées

$\exists C>0$: $|b(t,x,a)|+|f(t,x,a)|+|g(x)| \leq C(1+|x|)$

$$\mathcal{A}=\left\{(\alpha_t)_{t\in[0,T]} : \text{prévisible } \int_0^T \alpha_t^2 dt<\infty\right\}$$

$$X_t^\alpha = x+\int_0^t b(r,X_r^\alpha,\alpha_r)dr$$

$$R:=\inf_{\alpha\in\mathcal{A}}\left\{\int_0^T f(t,X_t^\alpha,\alpha_t)dt+g(X_T^\alpha)\right\}$$

On introduit $V^{\alpha,\beta}=V$: $V_t=\int_0^t V_r\cdot\partial_x b(r,X_r^\alpha,\alpha_r)+\beta_r\partial_a b(r,X_r^\alpha,\alpha_r)dr$
$\alpha,\beta\in\mathcal{A}$

(1) $\alpha,\beta\in\mathcal{A}$ $\alpha^\varepsilon=\alpha+\varepsilon\beta\in\mathcal{A}$. M.q. $\lim_{\varepsilon\to 0}\sup_{[0,T]}\left|\frac{X_t^{\alpha^\varepsilon}-X_t^\alpha}{\varepsilon}-V_t^{\alpha,\beta}\right|=0$

$$\left|\frac{X_t^{\alpha^\varepsilon}-X_t^\alpha}{\varepsilon}\right|=\left|\frac{1}{\varepsilon}\int_0^t b(r,X_r^{\alpha^\varepsilon},\alpha_r^\varepsilon)-b(r,X_r^\alpha,\alpha_r)dr\right|\leq$$

$$\leq\varepsilon^{-1}\int_0^t |b'(r,\xi,\alpha_r+\tilde{\varepsilon}\beta_r)||X_r^{\alpha^\varepsilon}-X_r^\alpha|+|b'(r,\eta,\alpha_r+\tilde{\tilde{\varepsilon}}\beta_r)||\varepsilon\beta_r|dr\leq$$

$$\leq\int_0^t\|b'\|_\infty\left|\frac{X_r^{\alpha^\varepsilon}-X_r^\alpha}{\varepsilon}\right|dr+\|b'\|\underbrace{\int_0^T|\beta_r|dr}_{\leq C\int\beta_r^2dr<\infty}\underset{\text{Gronwall}}{\leq} Ce^{\|b'\|_\infty T}=:\tilde{C}$$

Taylor pour $b(r,X_r^{\alpha^\varepsilon},\alpha_r^\varepsilon)$

$$\left|\frac{X_t^{\alpha^\varepsilon}-X_t^\alpha}{\varepsilon}-V_t^{\alpha,\beta}\right|=\left|\int_0^t b_x(r,X_r^\alpha,\alpha_r)\left(\frac{X_r^{\alpha^\varepsilon}-X_r^\alpha}{\varepsilon}-V_r^{\alpha,\beta}\right)+\underbrace{o\Big(\overbrace{\frac{|X_r^{\alpha^\varepsilon}-X_r^\alpha|}{\varepsilon}}^{\leq\tilde{C}}\vee 1\Big)}_{m_r}dr\right|\leq$$

$$\leq\|b'\|_\infty\int_0^t\left|\frac{X_r^{\alpha^\varepsilon}-X_r^\alpha}{\varepsilon}-V_r^{\alpha,\beta}\right|dr+\underbrace{\int_0^t m_r dt}_{\to 0}\xrightarrow{\text{Gronwall}}\sup|\cdot\cdot|\leq\int_0^t m_r dr\, e^{\|b'\|_\infty T}\xrightarrow[\varepsilon\to 0]{}0$$

(2) Déduire que

$$\lim_{\varepsilon \downarrow 0} \frac{J(\alpha+\varepsilon\beta)-J(\alpha)}{\varepsilon} = \int_0^T V_t^{\alpha,\beta} f_x(t, X_t^\alpha, \alpha_t) + \beta_t f_a(t, X_t^\alpha, \alpha_t)\, dt + V_T^{\alpha,\beta} g_x(X_T^\alpha)$$

$$\frac{J(\alpha+\varepsilon\beta)-J(\alpha)}{\varepsilon} = \frac{1}{\varepsilon}\Big(\int_0^T \underbrace{\big(f(t, X_t^{\alpha^\varepsilon}, \alpha_t^\varepsilon) - f(t, X_t^\alpha, \alpha_t)\big)}_{f_x(t, X_t^\alpha, \alpha_t)(X_t^{\alpha^\varepsilon} - X_t^\alpha) + f_a(t, X_t^\alpha, \alpha_t)\varepsilon\beta_t} dt + \underbrace{g(X_T^{\alpha^\varepsilon}) - g(X_T^\alpha)}_{g'(\xi)(X_T^{\alpha^\varepsilon} - X_T^\alpha)}\Big) \rightarrow$$

$$g'(\xi) \downarrow g_x(X_T^\alpha), \qquad \underbrace{\frac{X_T^{\alpha^\varepsilon} - X_T^\alpha}{\varepsilon}}_{\to V_T^{\alpha,\beta}}$$

$$\rightarrow \int_0^T f_x(t, X_t^\alpha, \alpha_t) V_t^{\alpha,\beta} + f_a(t, X_t^\alpha, \alpha_t)\beta_t\, dt + g_x(X_T^\alpha) V_T^{\alpha,\beta}$$

$$H(t, x, y, a) := b(t, x, a)\cdot y + f(t, x, a)$$

$\alpha \in \mathcal{A}$ $\quad Y_t^\alpha = g_x(X_T^\alpha) + \int_t^T \partial_x H(r, X_r^\alpha, Y_r^\alpha, \alpha_r)\, dr$ pour $t \leq T$

(3) M.q $\quad Y_T^\alpha V_T^{\alpha,\beta} = \int_0^T Y_t^\alpha b_a(t, X_t^\alpha, \alpha_t)\beta_t - f_x(t, X_t^\alpha, \alpha_t) V_t^{\alpha,\beta} dt$

$$d(Y_t^\alpha V_t^{\alpha,\beta}) = V_t^{\alpha,\beta} \dot{Y}_t^\alpha + Y_t^\alpha \dot{V}_t^{\alpha,\beta} = V_t\big(-\cancel{b_x(t, X_t^\alpha, \alpha_t) Y_t^\alpha} - f_x(t, X_t^\alpha, \alpha_t)\big) +$$

$$+ Y_t^\alpha\big(\cancel{V_t^{\alpha,\beta} b_x(t, X_t^\alpha, \alpha_t)} + \beta_t b_a(t, X_t^\alpha, \alpha_t)\big) = \beta_t Y_t^\alpha b_a(t, X_t^\alpha, \alpha_t) - f_x(t, X_t^\alpha, \alpha_t) V_t^{\alpha,\beta}$$

$Y_0^\alpha V_0^{\alpha,\beta} = 0$ car $V_0^{\alpha,\beta} = 0$ $\longrightarrow$ (3)

(4) Déduire que $\displaystyle \lim_{\varepsilon \downarrow 0} \frac{J(\alpha+\varepsilon\beta)-J(\alpha)}{\varepsilon} = \int_0^T \partial_a H(t, X_t^\alpha, Y_t^\alpha, \alpha_t)\beta_t\, dt$

$$\lim_{\varepsilon \downarrow 0} \frac{J(\alpha+\varepsilon\beta)-J(\alpha)}{\varepsilon} \overset{(2)}{=} \int_0^T (V_t \cdot f_x + \beta_t f_a)\, dt + V_T \underbrace{g_x}_{Y_T^\alpha} =$$

$$\overset{(3)}{=} \int_0^T (\cancel{V_t f_x} + \beta_t f_a)\,dt + \int_0^T (Y_t b_a \beta_t - \cancel{V_t f_x})\,dt = \int_0^T \beta_t \big(\underbrace{b_a \cdot Y_t + f_a}_{\frac{\partial H}{\partial a}}\big)\,dt =$$

$$= \int_0^T \beta_t H_a(t, X_t^\alpha, Y_t^\alpha, \alpha_t)\,dt$$

(5) On suppose que H est convexe en a. M.q. si α est un contrôle optimale $\Rightarrow \forall a \in A \quad H(t, X_t^\alpha, Y_t^\alpha, a) \geq H(t, X_t^\alpha, Y_t^\alpha, \alpha_t)$

α est optimale $\Rightarrow \forall \beta \in \mathcal{A} \quad \forall \varepsilon > 0 \quad \dfrac{J(\alpha + \varepsilon\beta) - J(\alpha)}{\varepsilon} \geq 0$

En plus, $\forall \beta \in \mathcal{A} \quad \int_0^T \beta_t H_a(t, X_t^\alpha, Y_t^\alpha, \alpha_t)\,dt \geq 0$

On prend $\beta_t^n = n \mathbb{1}_{[t, t+\frac{1}{n}]}$

$$n \int_t^{t+\frac{1}{n}} H_a(t, X_t^\alpha, Y_t^\alpha, \alpha_t)\,dt \geq 0$$

$$\Big\downarrow n \to \infty$$

$$H_a(t, X_t^\alpha, Y_t^\alpha, \alpha_t) \geq 0$$

On suppose que les deux sont admissible. Sinon on a $\alpha_t = \inf A$ où $\alpha_t = \sup A \Rightarrow$ min.
et $H_a \geq 0$ et $H_a \leq 0$

$\Rightarrow H_a(t, X_t^\alpha, Y_t^\alpha, \alpha_t) = 0$

H convexe $\Rightarrow$ minimum

$\beta_t^n = -n \mathbb{1}_{[t, t+\frac{1}{n}]} \Rightarrow H_a(\ldots) \leq 0$

(6) Supposons que g est convexe, H est convexe en (x, a) et $\alpha \in \mathcal{A}$ t.q.

$$H(t, X_t^\alpha, Y_t^\alpha, \alpha_t) = \inf_{a \in \mathcal{A}} H(t, X_t^\alpha, Y_t^\alpha, a)$$

M.q. α est un contrôle optimale.

$h(y) \geq h(x) + \nabla x \cdot (y-x)$

$$J(\tilde{\lambda}) - J(\lambda) = \left[\int_0^T f(t, X_t^{\hat{\lambda}}, \hat{\lambda}_t) - f(t, X_t^{\lambda}, \lambda_t)\, dt + g(X_T^{\tilde{\lambda}}) - g(X_T^{\lambda}) \right] \geq \{\text{convexité } f\} \geq$$

$$\geq \int_0^T f(t, X_t^{\hat{\lambda}}, \hat{\lambda}_t) - f(t, X_t^{\lambda}, \lambda_t)\, dt + \underbrace{g_x(X_T^{\lambda})}_{Y_T^{\lambda}}(X_T^{\tilde{\lambda}} - X_T^{\lambda}) =$$

$$Y_T^{\lambda}(X_T^{\tilde{\lambda}} - X_T^{\lambda}) = \int_0^T \Big(-\partial_x H(t, X_t^{\lambda}, Y_t^{\lambda}, \lambda_t)(X_t^{\tilde{\lambda}} - X_t^{\lambda}) + \underbrace{Y_t^{\lambda}\big(b(t, X_t^{\tilde{\lambda}}, \tilde{\lambda}_t) - b(t, X_t^{\lambda}, \lambda_t)\big)}_{\substack{H(t, X_t^{\tilde{\lambda}}, Y_t^{\lambda}, \tilde{\lambda}_t) - H(t, X_t^{\lambda}, Y_t^{\lambda}, \lambda_t) \\ - f(t, X_t^{\tilde{\lambda}}, \tilde{\lambda}_t) + f(t, X_t^{\lambda}, \lambda_t)}} \Big) dt$$

$H(t, X_t^{\tilde{\lambda}}, Y_t^{\lambda}, \lambda_t)$

$\parallel$

$$= \int_0^T \Big[H(t, X_t^{\tilde{\lambda}}, Y_t^{\lambda}, \tilde{\lambda}_t) - H(t, X_t^{\lambda}, Y_t^{\lambda}, \lambda_t) - H_x(t, X_t^{\lambda}, Y_t^{\lambda}, \lambda_t)(X_t^{\tilde{\lambda}} - X_t^{\lambda}) \Big] dt \geq$$

$$\geq \int_0^T \underbrace{\Big[H(t, X_t^{\tilde{\lambda}}, Y_t^{\lambda}, \lambda_t) - H(t, X_t^{\lambda}, Y_t^{\lambda}, \lambda_t) - H_x(t, X_t^{\lambda}, Y_t^{\lambda}, \lambda_t)(X_t^{\tilde{\lambda}} - X_t^{\lambda}) \Big]}_{\geq 0 \text{ inégalité de convexité.}} dt \geq 0$$

Exo 2

1) $(f+g)^*(x) \leq \inf_z \{f^*(z)+g^*(x-z)\}$

$\|$

$\sup_y\{\langle x,y\rangle - f(y) - g(y)\} = \{x = z + (x-z)\} =$

$= \sup_y \{\langle z,y\rangle - f(y) + \langle x-z,y\rangle - g(y)\} \leq$

$\leq \sup_y\{\langle z,y\rangle - f(y)\} + \sup_y\{\langle x-z,y\rangle - g(y)\} = f^*(z)+g^*(x-z)$

Donc $\forall z$ $(f+g)^*(x) \leq f^*(z)+g^*(x-z)$

$\Downarrow$

$(f+g)^*(x)\leq \inf_z\{f^*(z)+g^*(x-z)\}$

2) Justifier que $h(u) = \inf_x\{f(x)+g(x+u)\}$ est convexe

$h(\lambda u+(1-\lambda)v) = \inf_x\{f(x)+g(x+\lambda u+(1-\lambda)v)\} \leq$

$\leq \{g(x+\cdot)$ est cvxe$\} \leq \inf_x\{\underbrace{f(x)}_{\lambda f(x)+(1-\lambda)f(x)}+\lambda g(x+u)+(1-\lambda)g(x+v)\} \leq$

$\overset{?}{\cancel{\leq}}\ \lambda \inf_x\{f(x)+g(x+u)\}+(1-\lambda)\inf_x\{f(x)+g(x+u)\}$

$\inf\{f(x)+g(x+\lambda u+(1-\lambda)v)\} \overset{?}{\leq} \lambda\inf_x\{f(x)+g(x+u)\}+(1-\lambda)\inf_x\{f(x)+g(x+v)\}$

$\vee|$

$\uparrow \varepsilon\to 0$ $\qquad \lambda(f(x^u)+g(x^u+u))+(1-\lambda)(f(x^v)+g(x^v+v))-\varepsilon$

$\vee|$

$\inf_x\{f(x)+g(x+\lambda u+(1-\lambda)v)\}-\varepsilon \leq f(\underbrace{\lambda x^u+(1-\lambda)x^v}) + g(\underbrace{\lambda x^u+(1-\lambda)x^v}+\lambda u+(1-\lambda)v)$ $-\varepsilon$

Autre solution: $f(x)+g(x+\cdot)$ x^o convexe → inf de fct x^o cvxe est cvxe.

3) $x_0 = 0 \in \mathrm{Dom}\, f\cap \mathrm{Dom}\, g$ ↑ ça ne marche pas pour inf:

≈ n'est pas convexe

(a) Mq $\exists\bar x$: $h(0)+\langle\bar x,u\rangle\leq h(u)$ $\forall u$

(b) Vérifier que $\forall x,u$ $f(x)+g(x+u)-\langle\bar x,u\rangle\geq h(0)$

(c) Dériver $(f+g)^*(x_0) = \inf_z \{f^*(z) + g^*(x_0 - z)\}$ pour $x_0 = 0$

(a) h est convexe

$h'_d(0, v)$ - dérivée directionnelle

$h'_d(0,v) \leq h(v) - h(0)$ $\forall v$ grâce à convexité

$v \mapsto h'_d(0,v)$ est convexe est positivement 1-homogène

i.e. $h'_d(0, \lambda v) = \lambda h'_d(0,v)$, $\lambda \geq 0$

On a $h'_d(0,v) = \sup\{\langle \tilde{x}, v\rangle + \alpha : \langle \cdot, v\rangle + \alpha \leq h'_d(0,v)\}$

$h(\lambda v) \leq \lambda h(v) + (1-\lambda) h(0)$, $\lambda > 0$

$$\underbrace{\frac{h(\lambda v) - h(0)}{\lambda}}_{\downarrow\ h'_d(0,v)} \leq h(v) - h(0)$$

$$h'_d(0,v) = \lim_{\lambda \to 0} \frac{h(\lambda \xi v) - h(0)}{\xi \lambda} = \frac{1}{\xi} \underbrace{\lim \frac{h(\lambda \cdot \xi v) - h(0)}{\lambda}}_{h'_d(0, \xi v)}$$

$\exists \tilde{x}, \alpha :$ $\langle \tilde{x}, v\rangle + \alpha \leq h'_d(0,v)$

Positivement homogène, minorée par la fct affine $\Rightarrow \alpha = 0$ (linéaire)

(Éxo)

$\rightarrow$ $\langle \tilde{x}, v\rangle \leq h'_d(0,v) \leq h(v) - h(0)$

(b) $$\inf_x \{\underbrace{f(x) + g(x+u)}_{\forall x \ \wedge\ f(x)+g(x+u)}\} \geq \underbrace{\inf\{f(x) + g(x)\}}_{h(0)} + \langle \bar{x}, u\rangle$$

Donc $h(0) \leq f(x) + g(x+u) - \langle \bar{x}, u\rangle$

(c) $(f+g)^*(0) \underset{(1)}{\leq} \inf_z \{ f^*(z) + g^*(-z) \}$

$$h(0) = \inf\{f(x)+g(x)\} = -\sup\{-(f+g)(x)\} = -(f+g)^*(0)$$

$$\overset{(b)}{\Rightarrow} \quad -(f+g)^*(0) + \langle \bar{x}, u\rangle \leq f(x) + g(x+u) \quad \forall x, u$$

$$(f+g)^*(0) \geq \langle \bar{x}, u\rangle - f(x) - g(x+u) \quad \forall x, u \ \overset{u=0}{②}$$

$$f^*(z) = \sup_y \{\langle z, y\rangle - f(y)\} \qquad f^*(z) + f(y) \geq \langle z, y\rangle$$

$$g^*(-z) = \sup_y \{-\langle z, y\rangle - g(y)\} \qquad g^*(-z) + g(y) \geq -\langle z, y\rangle$$

$$\left.\right\} f(y) + g(y) + f^*(z) + g^*(-z) \geq 0$$

$$\inf_z \{ f^*(z) + g^*(-z) \} \leq f^*(z) + g^*(-z)$$

$$② \ -f(x) + \langle \bar{x}, x\rangle - \langle \bar{x}, x+u\rangle - g(x+u) + \langle \bar{x}, u\rangle \overbrace{- \langle \bar{x}, x\rangle + \langle \bar{x}, x+u\rangle} \rightsquigarrow \sup_x, \sup_v$$

$$u = -x + v$$

4. $\ell(u) = g(u) - \langle z, u\rangle$

(a) Vérifier que $\ell^*(x) = g^*(x+z)$ $\quad (f+\ell)^*(0) = (f+g)^*(z)$

(b) Déduire (1) dans le cas général

$$\ell^*(x) = \sup_y \{\langle x, y\rangle - \ell(y)\} = \sup_y \{\langle x+z, y\rangle - g(y)\} = g^*(x+z)$$

$$(f+\ell)^*(0) = \sup_y \{-(f+\ell)(y)\} = \sup_y \{-(f+g-\langle z, \cdot\rangle)(y)\} =$$

$$= \sup_y \{\langle y, z\rangle - (f+g)(y)\} = (f+g)^*(z)$$

$$(f^* \square g^*)(x_0) \overset{?}{\geq} (f+g)^*(x_0)$$

On connait ça pour $x_0 = 0$

$$(f^* \square \ell^*)(0) = \underbrace{\inf\{f^*(y) + \underbrace{\ell^*(-y)}_{g^*(z-y)}\}}_{(f^* \square g^*)(z)} \overset{3.}{=} (f+\ell)^*(0) = (f+g)^*(z)$$

Alors $(f+g)^*(z) = (f^* \square g^*)(z)$

Part II – Application: Monge Kantorovich duality

Let $d, \ell \geq 1$. We give ourselves two probability distributions $\nu \in \mathcal{P}(\{1, \cdots, d\})$ and $\mu \in \mathcal{P}(\{1, \cdots, \ell\})$. We identify ν and μ as vectors of $\mathbb{R}^d$ and $\mathbb{R}^\ell$ respectively. We denote by $\Gamma(d, \ell) \subset \mathbb{R}^d \times \mathbb{R}^\ell$ the set $(\gamma(i,j))_{1 \leq i \leq d, 1 \leq j \leq \ell}$ s.t. $\gamma \geq 0$, $\sum_{j=1}^{\ell} \gamma(i,j) = \nu(j)$ and $\sum_{i=1}^{d} \gamma(i,j) = \mu(i)$. We consider $(c(i,j))_{1 \leq i \leq d, 1 \leq j \leq \ell} \in \mathbb{R}^d \times \mathbb{R}^\ell$. We want to prove that

$$\begin{aligned}
&\inf_{\gamma \in \Gamma(d,\ell)} \sum_{i=1}^{d} \sum_{j=1}^{\ell} \gamma(i,j) c(i,j) \\
&= \sup \Big\{ \sum_{i=1}^{d} \phi(i)\nu(i) + \sum_{j=1}^{\ell} \psi(j)\mu(j) : \quad \phi(i) + \psi(j) \leq c(i,j), \;\; \text{for any } (i,j) \Big\}.
\end{aligned}$$

We introduce

$$f(\gamma) = \begin{cases} \sum_{i=1}^{d} \sum_{j=1}^{\ell} \gamma(i,j) c(i,j), & \text{if } \gamma \geq 0 \\ \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$g(\gamma) = \begin{cases} 0, \;\; \text{if } \;\; \sum_{j=1}^{\ell} \gamma(i,j) = \nu(i) \;\; \text{and} \;\; \sum_{i=1}^{d} \gamma(i,j) = \mu(j) \\ \\ +\infty, \;\; \text{otherwise} \end{cases}$$

1. Show that $f^\star(\pi) = 0$ if $\pi(i,j) \leq c(i,j)$ and $f^\star(\pi) = +\infty$ otherwise.
2. Compute that $g^\star(\pi)$ for any $\pi \in \mathbb{R}^d \times \mathbb{R}^\ell$.
3. Let $i_0 \neq i_1$ s.t. $\exists j_0 \neq j_1$, $\pi(i_0, j_0) - \pi(i_1, j_0) \neq \pi(i_0, j_1) - \pi(i_1, j_1)$. We suppose that $\pi(i_0, j_0) - \pi(i_1, j_0) > \pi(i_0, j_1) - \pi(i_1, j_1)$
   (a) Let $\eta \in \mathbb{R}^{d+\ell}$ with $\eta(i_0, j_0) = \eta(i_1, j_1) = 1$ and $\eta(i_0, j_1) = \eta(i_1, j_0) = -1$. Show that, for all t, $\sum_{j=1}^{\ell} (\gamma + t\eta)(i,j) = \nu(i)$ and $\sum_{i=1}^{d} (\gamma + t\eta)(i,j) = \mu(j)$ and
   $$\begin{aligned} &\sum_{i,j} \pi(i,j)(\gamma + t\eta)(i,j) \\ &= \sum_{i,j} \pi(i,j)\gamma(i,j) + t(\pi(i_0, j_0) + \pi(i_1, j_1) - \pi(i_1, j_0) - \pi(i_0, j_1)) \end{aligned}$$
   (b) Deduce that $g^\star(\pi) = \sum_i \phi(i)\nu(i) + \sum_j \psi(j)\mu(j)$ for $\pi(i,j) = \phi(i) + \psi(j)$ and $g^\star(\pi) = +\infty$ otherwise.
   (c) Apply the Fenchel–Rockafellar Theorem and deduce the result

$$f(\gamma) = \begin{cases} \sum_{i,j} \gamma(i,j)\, c(i,j) & \text{si } \gamma \geq 0 \\ +\infty & \text{sinon} \end{cases}$$

$$g(\gamma) = \begin{cases} 0 & \text{si } \sum_{j} \gamma(i,j) = \nu(i) \text{ et } \sum_{i} \gamma(i,j) = \mu(j) \\ +\infty & \text{sinon} \end{cases}$$

1. $$f^*(\pi) = \sup_{\gamma} \{\langle \pi, \gamma \rangle - f(\gamma)\} = \sup_{\gamma \geq 0} \{\langle \pi, \gamma\rangle - f(\gamma)\} =$$

$$= \sup_{\gamma \geq 0} \{\langle \pi - c, \gamma \rangle\} = \begin{cases} 0 & \text{si } \pi \leq c \\ +\infty & \text{sinon} \end{cases}$$

si $\pi \leq c \to \langle \pi - c, \gamma \rangle \leq 0 \quad \forall \gamma \geq 0$

sinon $\exists (i,j) \quad \pi(i,j) - c(i,j) > 0$ On prend $\gamma(i',j') = 0 \quad (i',j') \neq (i,j)$

$\gamma(i,j) \to +\infty$

2) $$g^*(\pi) = \sup_{\gamma} \{\langle \pi, \gamma \rangle - g(\gamma)\} = \sup_{\substack{\gamma : \\ \pi_x \gamma = \mu \\ \pi_y \gamma = \nu}} \{\langle \pi, \gamma \rangle\} = \sup_{\gamma \in \Pi(\mu, \nu)} \langle \pi, \gamma \rangle$$

3) Si $\pi(i,j)$ n'est pas de la forme $\pi(i,j) = \varphi(i) + \psi(j)$ alors $g^*(\pi) = +\infty$

$\exists \; i_0 \neq i_1$ t.q. $\pi(i_0,j_0) - \pi(i_1,j_0) \neq \pi(i_0,j_1) - \pi(i_1,j_1)$

$(>)$

	i_0	i_1
j_1	$+$	$-$
j_0	$+$	$-$

$\pi(i_0,j_0) + \pi(i_1,j_1) - \pi(i_1,j_0) - \pi(i_0,j_1) > 0$

a) $\eta(i_0,j_0) = \eta(i_1,j_1) = 1$

$\eta(i_0,j_1) = \eta(i_1,j_0) = -1$ $\quad \eta(i,j) = 0$ sinon

$\gamma + t\eta \in \Pi(\mu, \nu)$

$$\sum_i \pi(i,j)(\gamma + t\eta)(i,j) = \sum \pi(i,j)\gamma(i,j) + t \sum \eta(i,j)\pi(i,j)$$

Si $\pi(i,j) \neq \varphi(i)+\psi(j) \Rightarrow t \to +\infty \Rightarrow g^*(\pi) = +\infty$

Dans ce cas $g^*(\pi) = \sum_i \gamma(i,j)(\varphi(i)+\psi(j)) = \sum_i (\nu(i)\varphi(i) + \mu(j)\psi(j))$

(c) Théorème de Fenchel-Rockafellar

$$(f+g)^*(x) = (f^* \square g^*)(x)$$

$\mathrm{Dom}\, f \cap \mathrm{Dom}\, g \neq \emptyset$ $\quad (\gamma(i,j) = \mu(i)\nu(j))$

$$\inf_{\gamma \in \Gamma(d,c)} \sum\sum_i \gamma(i,j)c(i,j) = -\sup_{\gamma}\{-f(\gamma) - g(\gamma)\} = -(f+g)^*(0) =$$

$$= -(f^* \square g^*)(0) = -\inf_{\pi}\{f^*(\pi) + g^*(-\pi)\} = \sup\{-f^*(\pi) - g^*(-\pi)\} =$$

$$= \sup_{\substack{\pi \leq c \\ \pi(i,j) = \varphi(i)+\psi(j)}}\left\{\sum \varphi(i)\nu(i) + \sum \psi(j)\mu(j)\right\}$$

$\int \varphi(i) + \psi(j) \leq c(i,j)$

Part III – Application: Super replication in complete market

The general version of the Monge Kantorovich duality is

$$\inf_{(X,Y)\in\Theta} \mathbb{E}^{\mathbb{P}}[c(X,Y)]$$
$$= \sup\left\{ \int_{\mathbb{R}^d} \phi(e)\nu(\mathrm{d}e) + \int_{\mathbb{R}^\ell} \psi(u)\mu(\mathrm{d}u) : \quad \phi(e)+\psi(u) \le c(e,u), \ \text{ for any } (e,u)\right\}$$

where $(X,Y)\in\Theta$ if $\mathcal{L}^{\mathbb{P}}(X)=\nu$ and $\mathcal{L}(Y)=\mu$.

Let $(S^1)_{t\le T}$ and $(S^2)_{t\le T}$ be two assets martingale under the risk neutral measures $\mathbb{P}^1$ and $\mathbb{P}^2$ respectively (the interest rate $r=0$). The price p_0 of super replication of an European option with payoff $c(S^1_T, S^2_T)$ is given by

$$p_0 := \inf\left\{ \mathbb{E}^{\mathbb{P}^1}[\lambda^1(S^1_T)] + \mathbb{E}^{\mathbb{P}^2}[\lambda^2(S^2_T)] : \ \ c(x,y) \le \lambda^1(x)+\lambda^2(y) \ \text{ for all } (x,y)\right\}.$$

Apply the Monge Kantorovich duality and comment.

prix de sur-replication = sup $\mathbb{E}^{\mathbb{Q}}$ payoff, $\mathbb{Q}$: S^i_T est martingale ($\mathcal{L}(S^i_T) = \mathbb{P}^i$)

$$p_0 = \sup_{\pi\in\Pi(\mathbb{P}^1,\mathbb{P}^2)} \mathbb{E}^{\pi}[c(S^1_T, S^2_T)] =$$

$$= \inf\Big\{ \underbrace{\int \varphi(s^1_T)\,\mathbb{P}^1(ds^1_T)}_{=\mathbb{E}^{\mathbb{P}^1}[\varphi(S^1_T)]} + \underbrace{\int \psi(s^2_T)\,\mathbb{P}(ds^2_T)}_{=\mathbb{E}^{\mathbb{P}^2}[\psi(S^2_T)]} : \varphi(s^1_T)+\psi(s^2_T) \ge c(s^1_T, s^2_T)\Big\}$$

séparation en deux options européenes aux payoffs φ et ψ : $\varphi+\psi \ge c$ (sur-replication de payoff $c(s_1,s_2)$)