**Tsinghua University** *2017–2018*

**"Advanced Probability"** (Part III: Brownian motion)

*Exercise sheet #III.1:*

*Construction of Brownian motion*

**Exercice 1.** Let $\xi$ be a Gaussian $\mathscr{N}(0, 1)$ random variable. Let $x > 0$.

**(i)** Prove that $\frac{1}{(2\pi)^{1/2}} \left(\frac{1}{x} - \frac{1}{x^3}\right) \mathrm{e}^{-x^2/2} \le \mathbb{P}(\xi > x) \le \frac{1}{(2\pi)^{1/2}} \frac{1}{x} \mathrm{e}^{-x^2/2}$.

**(ii)** Prove that[1] $\mathbb{P}(\xi > x) \le \mathrm{e}^{-x^2/2}$.

**Solution.** **(i)** We have

$$\mathbb{P}(\xi > x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \mathrm{e}^{-u^2/2}\, \mathrm{d}u \le \frac{1}{\sqrt{2\pi}} \frac{1}{x} \int_x^\infty u\mathrm{e}^{-u^2/2}\, \mathrm{d}u = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \mathrm{e}^{-x^2/2},$$

giving the desired upper bound. For the lower bound, we note that by integration by parts,

$$\mathbb{P}(\xi > x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \mathrm{e}^{-u^2/2}\, \mathrm{d}u = \Big[ - \frac{1}{\sqrt{2\pi}} \frac{1}{u} \mathrm{e}^{-u^2/2} \Big]_x^\infty - \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{1}{u^2} \mathrm{e}^{-u^2/2}\, \mathrm{d}u.$$

This yields the desired lower bound because $\int_x^\infty \frac{1}{u^2} \mathrm{e}^{-u^2/2}\, \mathrm{d}u \le \frac{1}{x^3} \int_x^\infty u\mathrm{e}^{-u^2/2}\, \mathrm{d}u = \frac{1}{x^3}$.

**(ii)** By the Markov inequality, for any $\lambda > 0$,

$$\mathbb{P}(\xi > x) \le \mathrm{e}^{-\lambda x} \mathbb{E}[\mathrm{e}^{\lambda \xi}] = \mathrm{e}^{-\lambda x + \lambda^2/2},$$

which yields the desired inequality by taking $\lambda = x$. $\square$

**Exercice 2.** Let $\xi$ be a Gaussian $\mathscr{N}(0, 1)$ random variable.

**(i)** Compute $\mathbb{E}(\xi^4)$ and $\mathbb{E}(|\xi|)$.

**(ii)** Compute $\mathbb{E}(\mathrm{e}^{a\xi})$, $\mathbb{E}(\xi \mathrm{e}^{a\xi})$ and $\mathbb{E}(\mathrm{e}^{a\xi^2})$, with $a \in \mathbb{R}$.

**(iii)** Let $b \ge 0$. Let $\eta$ be a Gaussian $\mathscr{N}(0, 1)$ random variable, independent of $\xi$. Prove that $\mathbb{E}(\mathrm{e}^{b\xi^2}) = \mathbb{E}(\mathrm{e}^{\lambda \xi \eta})$, where $\lambda := (2b)^{1/2}$.

**Solution.** **(i)** We have $\mathbb{E}(\xi^4) = 3$, $\mathbb{E}(|\xi|) = (\frac{2}{\pi})^{1/2}$.

**(ii)** We have $\mathbb{E}(\mathrm{e}^{a\xi}) = \mathrm{e}^{a^2/2}$, $\mathbb{E}(\xi \mathrm{e}^{a\xi}) = a\mathrm{e}^{a^2/2}$. As for $\mathbb{E}(\mathrm{e}^{a\xi^2})$, it is seen that $\mathbb{E}(\mathrm{e}^{a\xi^2}) = \infty$ if $a \ge \frac{1}{2}$, whereas $\mathbb{E}(\mathrm{e}^{a\xi^2}) = (1-2a)^{-1/2}$ if $a < \frac{1}{2}$.

**(iii)** By conditioning on $\xi$, we habe, by (ii), $\mathbb{E}(\mathrm{e}^{\lambda \xi \eta} \,|\, \xi) = \mathrm{e}^{\lambda^2 \xi^2/2}$, which is nothing else but $\mathrm{e}^{b\xi^2}$. Taking expectation on both sides gives the desired conclusion. $\square$

[1] We will see that $\mathbb{P}(\xi > x) \le \frac{1}{2} \mathrm{e}^{-x^2/2}$.

**Exercice 3.** Let $\xi$, $\xi_1$, $\xi_2$, $\cdots$ be real-valued random variables. Assume that for each $n$, $\xi_n$ is Gaussian $\mathscr{N}(\mu_n, \sigma_n^2)$, with $\mu_n \in \mathbb{R}$ and $\sigma_n \geq 0$, and that $\xi_n \to \xi$ in law. Prove that $\xi$ is Gaussian.

**Solution.** For any random variable $\xi$, we denote its characteristic function by $\varphi_\xi$. By assumption, $\varphi_{\xi_n}(t) = \exp(i\mu_n t - \frac{\sigma_n^2}{2}t^2)$ converges pointwise to $\varphi_\xi(t)$. So $\exp(-\frac{\sigma_n^2}{2}t^2) \to |\varphi_\xi(t)|$ for any $t \in \mathbb{R}$. As a consequence, $\sigma_n^2 \to \sigma^2 \geq 0$ (the possibility that $\sigma_n^2 \to \infty$ is excluded as $\mathbf{1}_{\{t=0\}}$ is not a characteristic function, being discontinuous at point 0).

Suppose that $(\mu_n)$ is unbounded. Then there exists a subsequence $(\mu_{n_k})$ tending to $+\infty$ (or to $-\infty$, but the argument will be identical). Let $a \in \mathbb{R}$. The distribution function $F_\xi$ of $\xi$ being non-decreasing, we can find $b \geq a$ which is a point of continuity of $F_\xi$. Hence

$$F_\xi(a) \leq F_\xi(b) = \lim_{k\to\infty} \mathbb{P}(\xi_{n_k} \leq b) \leq \frac{1}{2},$$

as for large $k$, $\mathbb{P}(\xi_{n_k} \leq b) \leq \mathbb{P}(\xi_{n_k} \leq \mu_{n_k}) = \frac{1}{2}$. So $F_\xi(a) \leq \frac{1}{2}$ for all $a \in \mathbb{R}$, which is absurd because $F_\xi$ is a distribution function and its limit at $+\infty$ is 1.

The sequence $(\mu_n)$ is thus bounded. Let $\mu \in \mathbb{R}$ and $\nu \in \mathbb{R}$ be limits along subsequences, then $\mathrm{e}^{i\mu t} = \mathrm{e}^{i\nu t}$ for all $t \in \mathbb{R}$, which is possible only if $\mu = \nu$. So the sequence $(\mu_n)$ converges, to a limit, denoted by $\mu \in \mathbb{R}$. Since $\sigma_n \to \sigma$, we have $\varphi_\xi(t) = \exp(i\mu t - \frac{\sigma^2}{2}t^2)$. In other words, $\xi$ is Gaussian $\mathscr{N}(\mu, \sigma^2)$. □

**Exercice 4.** Let $\xi$, $\xi_1$, $\xi_2$, $\cdots$ be random variables. Assume that for any $n$, $\xi_n$ is Gaussian $\mathscr{N}(\mu_n, \sigma_n^2)$, where $\mu_n \in \mathbb{R}$ and $\sigma_n \geq 0$, and that $\xi_n \to \xi$ in probability. Prove that $\xi_n$ converges in $L^p$, for all $p \in [1, \infty)$.

**Solution.** We use what we have proved in the previous exercise. For $a \in \mathbb{R}$, we have

$$\mathbb{E}(\mathrm{e}^{a\xi_n}) = \exp\Big(a\mu_n + \frac{a^2\sigma_n^2}{2}\Big).$$

Since $\mathrm{e}^{|x|} \leq \mathrm{e}^x + \mathrm{e}^{-x}$, we have, for all $a \geq 0$, $\sup_n \mathbb{E}(\mathrm{e}^{a|\xi_n|}) < \infty$. A fortiori, $\sup_n \mathbb{E}(|\xi_n|^{p+1}) < \infty$; hence $\sup_n \mathbb{E}(\,|\xi_n - \xi|^{p+1}) < \infty$. This implies that $(|\xi_n - \xi|^p)$ is uniformly integrable. Since $|\xi_n - \xi|^p \to 0$ in probability, the convergence takes place also in $L^1$.

**Exercice 5.** Let $(\xi, \eta, \theta)$ be an $\mathbb{R}^3$-valued Gaussian random vector. Assume $\mathbb{E}(\xi) = \mathbb{E}(\eta) = \mathbb{E}(\xi\eta) = 0$, $\sigma_\xi^2 := \mathbb{E}(\xi^2) > 0$ and $\sigma_\eta^2 := \mathbb{E}(\eta^2) > 0$.

**(i)** Prove that $\mathbb{E}(\theta \,|\, \xi, \eta) = \mathbb{E}(\theta \,|\, \xi) + \mathbb{E}(\theta \,|\, \eta) - \mathbb{E}(\theta)$.

**(ii)** Prove that $\mathbb{E}(\xi \,|\, \xi\eta) = 0$.

**(iii)** Prove that $\mathbb{E}(\theta \,|\, \xi\eta) = \mathbb{E}(\theta)$.

**Solution.** **(i)** Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$. It is clear that $(\xi, \eta, \theta - a\xi - b\eta)$, being a linear transform of the Gaussian random variable $(\xi, \eta, \theta)$, is also a Gaussian random variable. So $\theta - a\xi - b\eta$ and $(\xi, \eta)$ are independent if and only if $\mathrm{Cov}(\theta - a\xi - b\eta, \xi) = \mathrm{Cov}(\theta - a\xi - b\eta, \eta) = 0$.

We have $\mathrm{Cov}(\theta - a\xi - b\eta,\, \xi) = \mathrm{Cov}(\xi,\, \theta) - a\sigma_\xi^2$, and $\mathrm{Cov}(\theta - a\xi - b\eta,\, \eta) = \mathrm{Cov}(\eta,\, \theta) - b\sigma_\eta^2$. Choosing from now on $a := \mathrm{Cov}(\xi,\, \theta)/\sigma_\xi^2$ and $b := \mathrm{Cov}(\eta,\, \theta)/\sigma_\eta^2$, it is seen that $\theta - a\xi - b\eta$ is independent of $(\xi,\, \eta)$. Accordingly,

$$\begin{aligned} \mathbb{E}(\theta \,|\, \xi,\, \eta) &= \mathbb{E}(\theta - a\xi - b\eta \,|\, \xi,\, \eta) + a\xi + b\eta \\ &= \mathbb{E}(\theta - a\xi - b\eta) + a\xi + b\eta = \mathbb{E}(\theta) + a\xi + b\eta. \end{aligned}$$

On the other hand, $\theta - a\xi$ is independent of $\xi$: indeed, $(\xi,\, \theta - a\xi)$ is a Gaussian random vector, with $\mathrm{Cov}(\xi,\, \theta - a\xi) = 0$; hence $\mathbb{E}(\theta \,|\, \xi) = \mathbb{E}(\theta - a\xi \,|\, \xi) + a\xi = \mathbb{E}(\theta - a\xi) + a\xi = \mathbb{E}(\theta) + a\xi$. Similarly, $\mathbb{E}(\theta \,|\, \eta) = \mathbb{E}(\theta) + b\eta$. As a consequence,

$$\mathbb{E}(\theta \,|\, \xi,\, \eta) = \mathbb{E}(\theta) + a\xi + b\eta = \mathbb{E}(\theta \,|\, \xi) + \mathbb{E}(\theta \,|\, \eta) - \mathbb{E}(\theta).$$

**(ii)** Let $A \in \sigma(\xi\eta)$. By definition, there exists a Borel set $B \subset \mathbb{R}$ such that $A = \{\omega : \xi(\omega)\eta(\omega) \in B\}$. So $\mathbf{1}_A = \mathbf{1}_B(\xi\eta)$.

Since $(\xi,\, \eta)$ is a *centered* Gaussian random vector, it is distributed as $(-\xi,\, -\eta)$. Thus $\mathbb{E}[\xi\, \mathbf{1}_B(\xi\eta)] = \mathbb{E}[(-\xi)\mathbf{1}_B((-\xi)(-\eta))] = -\mathbb{E}[\xi\, \mathbf{1}_B(\xi\eta)]$, i.e., $\mathbb{E}[\xi\, \mathbf{1}_B(\xi\eta)] = 0$. In other words, $\mathbb{E}(\xi\, \mathbf{1}_A) = 0$, $\forall A \in \sigma(\xi\eta)$, which means that $\mathbb{E}(\xi \,|\, \xi\eta) = 0$.

**(iii)** We have $\mathbb{E}(\theta \,|\, \xi\eta) = \mathbb{E}(\theta - a\xi - b\eta \,|\, \xi\eta) + a\, \mathbb{E}(\xi \,|\, \xi\eta) + b\, \mathbb{E}(\eta \,|\, \xi\eta)$. By (ii), $\mathbb{E}(\xi \,|\, \xi\eta) = 0$; similarly, $\mathbb{E}(\eta \,|\, \xi\eta) = 0$. It follows that $\mathbb{E}(\theta \,|\, \xi\eta) = \mathbb{E}(\theta - a\xi - b\eta \,|\, \xi\eta)$. We have seen that $\theta - a\xi - b\eta$ is independent of $(\xi,\, \eta)$; so $\mathbb{E}(\theta - a\xi - b\eta \,|\, \xi\eta) = \mathbb{E}(\theta - a\xi - b\eta) = \mathbb{E}(\theta)$, which yields the desired identity. $\square$

**Exercice 6.** Let $(\xi_{k,\,n},\, k \geq 0,\, n \geq 0)$ be a collection of i.i.d. Gaussian $\mathscr{N}(0,\, 1)$ random variables. For all $n \geq 0$, we define the process $(X_n(t),\, t \in [0,\, 1])$ with $t \mapsto X_n(t)$ being affine on each of the intervals $[\frac{i}{2^n},\, \frac{i+1}{2^n}]$, $0 \leq i \leq 2^n - 1$, in the following way $X_0(0) := 0$, $X_0(1) := \xi_{0,\,0}$, and by induction, for $n \geq 1$,

$$\begin{aligned} X_n(\frac{2i}{2^n}) &:= X_{n-1}(\frac{2i}{2^n}), \qquad 0 \leq i \leq 2^{n-1}, \\ X_n(\frac{2j+1}{2^n}) &:= X_{n-1}(\frac{2j+1}{2^n}) + \frac{\xi_{2j+1,n}}{2^{(n+1)/2}}, \qquad 0 \leq j \leq 2^{n-1} - 1. \end{aligned}$$

Prove that for all $n \geq 0$, $(X_n(\frac{k}{2^n}),\, 0 \leq k \leq 2^n)$ is a centered Gaussian vector such that $\mathbb{E}[X_n(\frac{k}{2^n})X_n(\frac{\ell}{2^n})] = \frac{k}{2^n} \wedge \frac{\ell}{2^n}$, for $0 \leq k,\, \ell \leq 2^n$.

**Solution.** We prove by induction in $n$. The case $n = 0$ is trivial. Assume that the desired conclusion holds for $n - 1$. It is clear that $(X_n(\frac{k}{2^n}),\, 0 \leq k \leq 2^n)$ is a Gaussian random vector (which is obviously centered), being a linear function of independent Gaussian vectors $(X_{n-1}(\frac{k}{2^{n-1}}),\, 0 \leq k \leq 2^{n-1})$ and $(\xi_{k,n},\, 0 \leq k \leq 2^n)$. It remains to check the covariance. We distinguish two possible situations.

First situation: there is at least an even number among $k$ and $\ell$, say $k = 2k_1$. In this case, $X_n(\frac{k}{2^n}) = X_{n-1}(\frac{k_1}{2^{n-1}})$, and the desired identity $\mathrm{Cov}(X_{n-1}(\frac{k}{2^n}), X_{n-1}(\frac{\ell}{2^n})) = \frac{k}{2^n} \wedge \frac{\ell}{2^n}$ is trivial by the induction hypothesis if $\ell$ is even; if, however, $\ell$ is odd, say $\ell = 2\ell_1 + 1$, then $X_n(\frac{\ell}{2^n}) = \frac{1}{2}X_{n-1}(\frac{\ell_1}{2^{n-1}}) + \frac{1}{2}X_{n-1}(\frac{\ell_1+1}{2^{n-1}}) + \frac{\xi_{\ell,n}}{2^{(n+1)/2}}$; since $\xi_{\ell,n}$ is independent of $X_{n-1}(\frac{k_1}{2^{n-1}})$, we obtain:

$$\begin{aligned} &\mathrm{Cov}\Big(X_n(\frac{k}{2^n}), X_n(\frac{\ell}{2^n})\Big) \\ = \ &\frac{1}{2}\mathrm{Cov}\Big(X_{n-1}(\frac{k_1}{2^{n-1}}), X_{n-1}(\frac{\ell_1}{2^{n-1}})\Big) + \frac{1}{2}\mathrm{Cov}\Big(X_{n-1}(\frac{k_1}{2^{n-1}}), X_{n-1}(\frac{\ell_1+1}{2^{n-1}})\Big), \end{aligned}$$

which, by the induction hypothesis, is $\frac{1}{2}(\frac{k_1}{2^{n-1}} \wedge \frac{\ell_1}{2^{n-1}}) + \frac{1}{2}(\frac{k_1}{2^{n-1}} \wedge \frac{\ell_1+1}{2^{n-1}}) = \frac{k}{2^n} \wedge \frac{\ell}{2^n}$ as desired.

Second (and last) situation: both $k$ and $\ell$ odd numbers, say $k = 2k_1+1$ and $\ell = 2\ell_1+1$. In this case, we have $X_n(\frac{k}{2^n}) = \frac{1}{2}X_{n-1}(\frac{k_1}{2^{n-1}}) + \frac{1}{2}X_{n-1}(\frac{k_1+1}{2^{n-1}}) + \frac{\xi_{k,n}}{2^{(n+1)/2}}$ and $X_n(\frac{\ell}{2^n}) = \frac{1}{2}X_{n-1}(\frac{\ell_1}{2^{n-1}}) + \frac{1}{2}X_{n-1}(\frac{\ell_1+1}{2^{n-1}}) + \frac{\xi_{\ell,n}}{2^{(n+1)/2}}$. Since $\xi_{k,n}$ and $\xi_{\ell,n}$ are independent of $(X_{n-1}(t), t \in [0, 1])$, we have, by the induction hypothesis,

$$\begin{aligned} \mathrm{Cov}\Big(X_n(\frac{k}{2^n}), X_n(\frac{\ell}{2^n})\Big) \ = \ &\frac{1}{4}(\frac{k_1}{2^{n-1}} \wedge \frac{\ell_1}{2^{n-1}}) + \frac{1}{4}(\frac{k_1}{2^{n-1}} \wedge \frac{\ell_1+1}{2^{n-1}}) + \\ &+\frac{1}{4}(\frac{k_1+1}{2^{n-1}} \wedge \frac{\ell_1}{2^{n-1}}) + \frac{1}{4}(\frac{k_1+1}{2^{n-1}} \wedge \frac{\ell_1+1}{2^{n-1}}) + \frac{1}{2^{n+1}}\mathrm{Cov}(\xi_{k,n}, \xi_{\ell,n}). \end{aligned}$$

It is then easily checked that the sum of the five terms on the right-hand side is indeed $\frac{k}{2^n} \wedge \frac{\ell}{2^n}$.

By induction, we conclude that $\mathrm{Cov}[X_n(\frac{k}{2^n})X_n(\frac{\ell}{2^n})] = \frac{k}{2^n} \wedge \frac{\ell}{2^n}$. □

**Exercice 7.** Let $(B^m_t, t \in [0, 1])$, for $m \geq 0$, be a sequence of independent Brownian motions defined on $[0, 1]$. Let

$$B_t := B^{\lfloor t \rfloor}_{t-\lfloor t \rfloor} + \sum_{0 \leq m < \lfloor t \rfloor} B^m_1, \qquad t \geq 0.$$

Prove that $(B_t, t \geq 0)$ is Brownian motion.

**Solution.** Clearly, the trajectories of $B$ are a.s. continuous. It is easily checked that $B$ is a centered Gaussian process with covariance $\mathrm{Cov}(B_t, B_s) = t \wedge s$ for all $s \geq 0$ and $t \geq 0$. □

**Exercice 8.** Prove that $\mathscr{C}(\mathbb{R}_+, \mathbb{R})$, the Borel $\sigma$-field of $C(\mathbb{R}_+, \mathbb{R})$, coincides with $\sigma(X_t, t \geq 0)$, the $\sigma$-field generated by the process of projections $(X_t, t \geq 0)$.

**Solution.** For all $t \geq 0$, $X_t$ is continuous, thus measurable with respect to $\mathscr{C}(\mathbb{R}_+, \mathbb{R})$. Consequently, $\sigma(X_t, t \geq 0) \subset \mathscr{C}(\mathbb{R}_+, \mathbb{R})$.

Conversely, for all $\mathrm{w}_0 \in C(\mathbb{R}_+, \mathbb{R})$, $\delta_n(\mathrm{w}, \mathrm{w}_0) = \sup_{t \in [0,n] \cap \mathbb{Q}} |\mathrm{w}(t) - \mathrm{w}_0(t)|$ is $\sigma(X_t, t \geq 0)$-measurable, and so is $d(\mathrm{w}, \mathrm{w}_0)$. Le $F$ be a closed subset of $C(\mathbb{R}_+, \mathbb{R})$, and let $(\mathrm{w}_n)$ be a sequence that is dense in $F$ (because the space is separable), then

$$F = \{\mathrm{w} \in C(\mathbb{R}_+, \mathbb{R}) : d(\mathrm{w}, F) = 0\} = \{\mathrm{w} \in C(\mathbb{R}_+, \mathbb{R}) : \inf_n d(\mathrm{w}, \mathrm{w}_n) = 0\},$$

which is an element of $\sigma(X_t,\, t \ge 0)$. Hence, $\mathscr{C}(\mathbb{R}_+, \mathbb{R}) \subset \sigma(X_t,\, t \ge 0)$.

It is also possible to directly prove that all the open sets are $\sigma(X_t,\, t \ge 0)$-measurable, by means of the following property[2]: if a metric space is separable, then all opens sets are countable unions of open balls. $\square$

**Exercice 9.** Let $T := \inf\{t \ge 0 :\ B_t = 1\}$ (with $\inf \varnothing := \infty$). Prove that[3] $\mathbb{P}(T < \infty) \ge \frac{1}{2}$.

**Solution.** Let $t > 0$. We have $\mathbb{P}(T < \infty) \ge \mathbb{P}(T \le t) \ge \mathbb{P}(B_t \ge 1)$. Since $\mathbb{P}(B_t \ge 1) \to \frac{1}{2}$ when $t \to \infty$, we obtain: $\mathbb{P}(T < \infty) \ge \frac{1}{2}$. $\square$

**Exercice 10.** **(i)** Prove that $(-B_t,\, t \ge 0)$ is Brownian motion.

**(ii)** **(Scaling)** Prove that for any $a > 0$, $(\frac{1}{a^{1/2}}\, B_{at},\, t \ge 0)$ is Brownian motion.

**Solution.** Both are centered Gaussian processes with covariance $s \wedge t$ and with a.s. continuous trajectories. $\square$

**Exercice 11.** **(i)** Let $\xi := \int_0^1 B_t \,\mathrm{d}t$. Determine the law of $\xi$.

**(ii)** Let $\eta := \int_0^2 B_t \,\mathrm{d}t$. Determine $\mathbb{E}(B_1 \,|\, \eta)$.

**(iii)** Prove that $B_7 - B_2$ is independent of $\sigma(B_s,\, s \in [0,\, 1])$.

**(iv)** Let $\mathscr{F}_1 := \sigma(B_s,\, s \in [0,\, 1])$. Determine $\mathbb{E}(B_5 \,|\, \mathscr{F}_1)$ and $\mathbb{E}(B_5^2 \,|\, \mathscr{F}_1)$.

**Solution.** (i) By definition, $\xi$ is the a.s. limit of $\xi_n := 2^{-n} \sum_{i=1}^{2^n} B_{i/2^n}$, and a fortiori, the weak limit. For each $n$, $\xi_n$ is Gaussian (because Brownian motion is a Gaussian process). By Exercice 4, $\xi$ is Gaussian, with $\mathbb{E}(\xi) = \lim_{n\to\infty} \mathbb{E}(\xi_n)$ and $\mathrm{Var}(\xi) = \lim_{n\to\infty} \mathrm{Var}(\xi_n)$.

Since $\mathbb{E}(\xi_n) = 0$, $\forall n$, we have $\mathbb{E}(\xi) = 0$.

Since $\mathrm{Var}(\xi_n) = 2^{-2n} \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} (\frac{i}{2^n} \wedge \frac{j}{2^n}) \to \int_0^1 \int_0^1 (s \wedge t) \,\mathrm{d}s \,\mathrm{d}t = \frac{1}{3}$, we have $\mathrm{Var}(\xi) = \frac{1}{3}$.

Conclusion : $\xi$ is Gaussian $\mathscr{N}(0,\, \frac{1}{3})$.

(ii) Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Exactly as in (i), we see that $aB_1 + b\eta$ is Gaussian, and centered; in other words, $(B_1,\, \eta)$ is a centered Gaussian random vector. Moreover, $\mathbb{E}(B_1) = 0 = \mathbb{E}(\eta)$, $\mathbb{E}(B_1^2) = 1$, $\mathbb{E}(\eta^2) = \frac{8}{3}$, and $\mathbb{E}(B_1\eta)$ is, by Fubini's theorem (why?), $= \int_0^2 \mathbb{E}(B_1 B_t) \,\mathrm{d}t = \int_0^2 (1 \wedge t) \,\mathrm{d}t = \frac{3}{2}$. Hence $(B_1,\, \eta)$ has the Gaussian law $\mathscr{N}(\begin{pmatrix} 0 \\ 0 \end{pmatrix},\, \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & \frac{8}{3} \end{pmatrix})$.

In particular, $\mathbb{E}(B_1 \,|\, \eta) = \frac{\mathbb{E}(B_1\eta)}{\mathbb{E}(\eta^2)}\, \eta = \frac{9}{16}\, \eta$.

(iii) Let $n \ge 1$, and let $(s_1,\, \cdots,\, s_n) \in [0,\, 1]^n$. Then $(B_7 - B_2,\, B_{s_1},\, \cdots,\, B_{s_n})$ is a centered Gaussian random vector. Since $\mathrm{Cov}(B_7 - B_2,\, B_{s_i}) = \mathrm{Cov}(B_7,\, B_{s_i}) - \mathrm{Cov}(B_2,\, B_{s_i}) = s_i - s_i = 0$ for all $i \le n$, an important property (which one?) of Gaussian random vectors tells us that

[2] Let $G$ be an open set, and let $D$ be a countable set that is dense, then for all $x \in G$, there exist $x_D \in D$ and $n_x \ge 1$ sufficiently large such that $x \in B(x_D,\, \frac{1}{n_x}) \subset G$. Thus $G = \cup_{x\in G} B(x_D,\, \frac{1}{n_x})$. The family $\{B(x_D,\, \frac{1}{n_x}),\, x \in G\}$ is countable, being a subset of $\{B(x,\, \frac{1}{n}),\, x \in D,\, n \ge 1\}$.

[3] Later on, we will see that $T < \infty$ a.s.

$B_7 - B_2$ is independent of $(B_{s_1}, \cdots, B_{s_n})$. This implies that $B_7 - B_2$ is independent of $\sigma(B_s,\, s \in [0,\,1])$.

(iv) Exactly as in the previous question, we see that $B_5 - B_1$ is indepenfent of $\mathscr{F}_1$. In particular, $\mathbb{E}(B_5 \,|\, \mathscr{F}_1) = \mathbb{E}(B_5 - B_1 \,|\, \mathscr{F}_1) + \mathbb{E}(B_1 \,|\, \mathscr{F}_1) = \mathbb{E}(B_5 - B_1) + B_1 = B_1$, et $\mathbb{E}(B_5^2 \,|\, \mathscr{F}_1) = \mathbb{E}((B_5 - B_1)^2 \,|\, \mathscr{F}_1) + 2B_1 \mathbb{E}(B_5 \,|\, \mathscr{F}_1) - B_1^2 = \mathbb{E}((B_5 - B_1)^2) + 2B_1^2 - B_1^2 = 4 + B_1^2$. □

**Exercice 12.** **(i)** Prove or disprove: for all $t > 0$, $\int_0^t B_s^2 \,\mathrm{d}s$ has the same distribution as $t^2 \int_0^1 B_s^2 \,\mathrm{d}s$.

**(ii)** Prove or disprove: the processes $(\int_0^t B_s^2 \,\mathrm{d}s,\, t \geq 0)$ and $(t^2 \int_0^1 B_s^2 \,\mathrm{d}s,\, t \geq 0)$ have the same distribution.

**Solution.** (i) The answer is yes, by the scaling property.

(ii) The answer is no: the trajectories of the second process are a.s. $C^\infty$, whereas those of the first are a.s. not $C^2$. □

**Exercice 13.** Let $T$ be a random variable having the exponential law of parameter 1, independent of $B$. Determine the law of $B_T$.

**Solution.** The measurability of $B_T$ is clear if we work in the canonical space of Brownian motion. Let us compute its characteristic function.

Let $x \in \mathbb{R}$. We have $\mathbb{E}[\mathrm{e}^{ixB_T} \,|\, T] = \mathrm{e}^{-x^2 T/2}$, so $\mathbb{E}[\mathrm{e}^{ixB_T}] = \mathbb{E}[\mathrm{e}^{-x^2 T/2}] = \frac{2}{2+x^2}$. In other words, $B_T$ has density $(1/\sqrt{2}\,)\mathrm{e}^{-\sqrt{2}\,|x|}$ ("two-sided exponential law" of parameter $\sqrt{2}\,$). □

**Exercice 14.** **(i)** Prove that $\int_0^1 \frac{B_s}{s} \,\mathrm{d}s$ is a.s. well defined.

**(ii)** Let $\beta_t := B_t - \int_0^t \frac{B_s}{s} \,\mathrm{d}s$. Prove that $(\beta_t,\, t \geq 0)$ is Brownian motion.

**Solution.** (i) By Fubini–Tonelli, $\mathbb{E}(\int_0^1 |\frac{B_s}{s}| \,\mathrm{d}s) = \int_0^1 \mathbb{E}(\,|\frac{B_s}{s}|\,) \,\mathrm{d}s = c \int_0^1 s^{-1/2} \,\mathrm{d}s < \infty$, where $c := \mathbb{E}(|B_1|) < \infty$. A fortiori, $\int_0^1 |\frac{B_s}{s}| \,\mathrm{d}s < \infty$ a.s. Consequently, $\int_0^1 \frac{B_s}{s} \,\mathrm{d}s$ is a.s. well defined.

[One can also directly prove that $\int_0^1 \frac{B_s}{s} \,\mathrm{d}s$ is a.s. well defined by means of the Hölder continuity of $B$.]

(ii) Exactly as in (i), we see that for all $t > 0$, $X_t := \int_0^t \frac{B_s}{s} \,\mathrm{d}s$ is well defined a.s. So a.s., the process $(X_t,\, t \geq 0)$ is well defined (why?), with continuous trajectories, and so is $(\beta_t := B_t - X_t,\, t \geq 0)$.

As in a previous exercise, we see that for all $n$ and all real numbers $a_1, \cdots, a_n$, $\sum_{i=1}^n a_i \beta_{t_i}$ is centered Gaussian. As a consequence, $\beta$ is a centered Gaussian process.

It remains to check the covariance. Let $t \geq s > 0$. We have $\mathbb{E}(X_t B_s) = s + s\log(\frac{t}{s})$ (why?), $\mathbb{E}(X_s B_t) = s$ and $\mathbb{E}(X_s X_t) = 2s + s\log(\frac{t}{s})$. Hence $\mathbb{E}(\beta_t \beta_s) = \mathbb{E}(B_t B_s) - \mathbb{E}(X_t B_s) - \mathbb{E}(X_s B_t) + \mathbb{E}(X_t X_s) = s$ as desired. Consequently, $\beta$ is Brownian motion. □

**Exercice 15.** Prove that $\int_0^\infty |B_s| \,\mathrm{d}s = \infty$ a.s.

**Solution.** Let $X_t := \int_0^t |B_s|\, \mathrm{d}s$, $t \ge 0$. By scaling, for all $t > 0$, $X_t$ is distributed as $t^{3/2} X_1$. For all $x > 0$, we have $\mathbb{P}\{X_\infty \ge x\} \ge \mathbb{P}\{X_t \ge x\} = \mathbb{P}\{X_1 \ge \frac{x}{t^{3/2}}\}$ which converges to $\mathbb{P}\{X_1 > 0\} = 1$ when $t \to \infty$. Since this holds for all $x > 0$, we get $X_\infty = \infty$ a.s. $\square$

**Exercice 16.** Let $T := \inf\{t \ge 0 : |B_t| = 1\}$ (with $\inf \varnothing := \infty$).

**(i)** Prove that $T < \infty$ a.s.

**(ii)** Prove that $T$ and $\mathbf{1}_{\{B_T = 1\}}$ are independent.

**Solution.** **(i)** For all $t > 0$, we have $\mathbb{P}(T < \infty) \ge \mathbb{P}(T \le t) \ge \mathbb{P}(\{B_t \ge 1\} \cup \{B_t \le -1\}) = \mathbb{P}(B_t \ge 1) + \mathbb{P}(B_t \le -1) = 2\,\mathbb{P}(B_t \ge 1)$. Since $\mathbb{P}(B_t \ge 1) \to \frac{1}{2}$ when $t \to \infty$, we get $\mathbb{P}(T < \infty) \ge 1$. In other words, $T < \infty$ a.s.

**(ii)** For bounded Borel function $f : \mathbb{R}_+ \to \mathbb{R}$ and by symmetry of Brownian motion (replacing $B$ by $-B$), we have $\mathbb{E}[f(T)\, \mathbf{1}_{\{B_T=1\}}] = \mathbb{E}[f(T)\, \mathbf{1}_{\{B_T=-1\}}]$; hence

$$\mathbb{E}[f(T)\, \mathbf{1}_{\{B_T=1\}}] = \frac{1}{2}\mathbb{E}[f(T)] = \mathbb{P}(B_T = 1)\mathbb{E}[f(T)],$$

the last identity following from the fact that $\mathbb{P}(B_T = 1) = \frac{1}{2}$ (taking $f \equiv 1$ in the previous identity). Similarly, $\mathbb{E}[f(T)\, \mathbf{1}_{\{B_T=-1\}}] = \mathbb{P}(T = -1)\mathbb{E}[f(B_T)]$. This yields the desired independence. $\square$

**Exercice 17.** Let $B := (B_t,\, t \in [0,\, 1])$ be Brownian motion defined on $[0,\, 1]$. For all $t \in [0,\, 1]$, let

$$\begin{aligned} \mathscr{F}_t &:= \sigma(B_s,\ s \in [0,\, t]), \\ \mathscr{G}_t &:= \mathscr{F}_t \vee \sigma(B_1) = \sigma(\{C;\ C \in \mathscr{F}_t \text{ or } C \in \sigma(B_1)\}). \end{aligned}$$

**(i)** Let $0 \le s < t \le 1$. Prove that

$$\mathbb{E}[(B_t - B_s) \,|\, \mathscr{G}_s] = \frac{t-s}{1-s}\,(B_1 - B_s).$$

**(ii)** Consider the process $\beta := (\beta_t,\, t \in [0,\, 1])$ defined by

$$\beta_t := B_t - \int_0^t \frac{B_1 - B_s}{1-s}\,\mathrm{d}s, \qquad t \in [0,\, 1].$$

Prove that for $0 \le s < t \le 1$, $\mathbb{E}(\beta_t \,|\, \mathscr{G}_s) = \beta_s$ a.s.

**Solution.** **(i)** Write

$$B_t - B_s = \frac{t-s}{1-s}\,(B_1 - B_s) + \frac{1-t}{1-s}\,(B_t - B_s) - \frac{t-s}{1-s}\,(B_1 - B_t).$$

Clearly, $\frac{t-s}{1-s}\,(B_1 - B_s)$ is $\mathscr{G}_s$-measurable. We now prove that $X := \frac{1-t}{1-s}\,(B_t - B_s) - \frac{t-s}{1-s}\,(B_1 - B_t)$ is independent of $\mathscr{G}_s$. It suffices to prove that for all $n$ and all $0 \le s_1 < \cdots < s_n \le s$, $X$ is independent of $(B_{s_1}, \cdots, B_{s_n}, B_1)$.

Since $(X, B_{s_1}, \cdots, B_{s_n}, B_1)$ is a Gaussian vector, it suffices to check that $\mathrm{Cov}(X, B_{s_i}) = \mathrm{Cov}(X, B_1) = 0$, $\forall i$. We have $\mathrm{Cov}(X, B_{s_i}) = \frac{1-t}{1-s}\,(s_i - s_i) - \frac{t-s}{1-s}\,(s_i - s_i) = 0$ and $\mathrm{Cov}(X, B_1) = \frac{1-t}{1-s}\,(t-s) - \frac{t-s}{1-s}\,(1-t) = 0$, as desired.

So $X$ is independent of $\mathscr{G}_s$ : we have $\mathbb{E}[X \,|\, \mathscr{G}_s] = \mathbb{E}[X] = 0$. As a consequence, $\mathbb{E}[(B_t - B_s) \,|\, \mathscr{G}_s] = \frac{t-s}{1-s}\,(B_1 - B_s)$.

**(ii)** [The integral $\int_0^1 \frac{B_1 - B_s}{1-s}\,\mathrm{d}s$ is a.s. well defined by the local Hölder continuity of Brownian sample paths.]

Let $1 \ge t > s \ge 0$. By (i), $\mathbb{E}[B_t \,|\, \mathscr{G}_s] = B_s + \frac{t-s}{1-s}\,(B_1 - B_s)$,and $\mathbb{E}[(B_1 - B_u) \,|\, \mathscr{G}_s] = B_1 - B_s - \frac{u-s}{1-s}\,(B_1 - B_s) = \frac{1-u}{1-s}\,(B_1 - B_s)$ for $u \ge s$. By Fubini's theorem (of which the application is easily justified),

$$\begin{aligned} \mathbb{E}[\beta_t \,|\, \mathscr{G}_s] &= \mathbb{E}[B_t \,|\, \mathscr{G}_s] - \int_s^t \frac{\mathbb{E}[(B_1 - B_u) \,|\, \mathscr{G}_s]}{1-u}\,\mathrm{d}u - \int_0^s \frac{B_1 - B_u}{1-u}\,\mathrm{d}u \\ &= B_s + \frac{t-s}{1-s}\,(B_1 - B_s) - \int_s^t \frac{1}{1-u}\,\frac{1-u}{1-s}\,(B_1 - B_s)\,\mathrm{d}u - \int_0^s \frac{B_1 - B_u}{1-u}\,\mathrm{d}u, \end{aligned}$$

which is nothing else but $\beta_s$. $\square$

**Exercice 18.** Let $\mathscr{F}_1 := \sigma(B_s,\, s \in [0,\,1])$, and let $a \in \mathbb{R}$. Let $\mathbb{Q}$ be the probability measure on $\mathscr{F}_1$ defined by $\mathbb{Q}(A) := \mathbb{E}(\mathrm{e}^{aB_1 - \frac{a^2}{2}}\,\mathbf{1}_A)$, $A \in \mathscr{F}_1$. Define $\gamma_t := B_t - at$, $t \in [0,\,1]$. Prove that $(\gamma_t,\, t \in [0,\,1])$ is Brownian motion under $\mathbb{Q}$.

**Solution.** The trajectories of $\gamma$ are $\mathbb{P}$-continuous and thus also $\mathbb{Q}$-continuous (the two probabilities being equivalent on $\mathscr{F}_1$). It remains to check that for $0 := t_0 < t_1 < \cdots < t_n \le 1$, $B_{t_n} - B_{t_{n-1}}, \cdots, B_{t_2} - B_{t_1}, B_{t_1}$ are independent Gaussian random variables under $\mathbb{Q}$. We consider the characteristic function. Let $(x_1, \cdots, x_n) \in \mathbb{R}^n$. Then

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{i\sum_{k=1}^n x_k(\gamma_{t_k} - \gamma_{t_{k-1}})}] &= \mathbb{E}[\mathrm{e}^{aB_1 - \frac{a^2}{2} + i\sum_{k=1}^n x_k(B_{t_k} - B_{t_{k-1}})}] \\ &= \mathrm{e}^{-\frac{a^2}{2} - ia\sum_{k=1}^n x_k(t_k - t_{k-1})}\,\mathbb{E}[\mathrm{e}^{a(B_1 - B_{t_n}) + \sum_{k=1}^n (ix_k + a)(B_{t_k} - B_{t_{k-1}})}]\,, \end{aligned}$$

which is

$$= \mathrm{e}^{-\frac{a^2}{2} - ia\sum_{k=1}^n x_k(t_k - t_{k-1})}\,\mathrm{e}^{\frac{a^2}{2}(1-t_n) + \sum_{k=1}^n \frac{(ix_k+a)^2}{2}(t_k - t_{k-1})} = \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^n x_k^2(t_k - t_{k-1})}\,.$$

This implies (i) the desired independence under $\mathbb{Q}$, and (ii) that the law of $\gamma_{t_k} - \gamma_{t_{k-1}}$ under $\mathbb{Q}$ is Gaussian $\mathscr{N}(0,\, t_k - t_{k-1})$. $\square$

**Tsinghua University** *2017–2018*

**"Advanced Probability"** (Part III: Brownian motion)

*Exercise sheet #III.2:*

*Brownian motion and the Markov property*

**Exercice 1.** Let $\mathscr{A}_1 \subset \mathscr{F}, \cdots, \mathscr{A}_n \subset \mathscr{F}$ be $\pi$-systems, satisfying $\Omega \in \mathscr{A}_i$, $\forall i$. Assume

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdots \mathbb{P}(A_n), \qquad \forall A_i \in \mathscr{A}_i.$$

Then $\sigma(\mathscr{A}_1), \cdots, \sigma(\mathscr{A}_n)$ are independent.

**Solution.** Fix $A_2 \in \mathscr{A}_2, \cdots, A_n \in \mathscr{A}_n$. Consider

$$\mathscr{M}_1 := \{C_1 \in \sigma(\mathscr{A}_1) : \mathbb{P}(C_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(C_1)\, \mathbb{P}(A_2) \cdots \mathbb{P}(A_n)\}.$$

It is easily checked by definition that $\mathscr{M}_1$ is a $\lambda$-system[4], whereas by assumption, $\mathscr{A}_1 \subset \mathscr{M}_1$, et $\mathscr{A}_1$ is a $\pi$-system. So by the $\pi$-$\lambda$ theorem, $\mathscr{M}_1 = \sigma(\mathscr{A}_1)$; in other words,

$$\mathbb{P}(C_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(C_1)\, \mathbb{P}(A_2) \cdots \mathbb{P}(A_n), \qquad \forall C_1 \in \sigma(\mathscr{A}_1),\ \forall A_2 \in \mathscr{A}_2, \cdots, \forall A_n \in \mathscr{A}_n.$$

To continue, let us fix $C_1 \in \sigma(\mathscr{A}_1)$, $A_3 \in \mathscr{A}_3, \cdots, A_n \in \mathscr{A}_n$, and consider

$$\mathscr{M}_2 := \{C_2 \in \sigma(\mathscr{A}_2) : \mathbb{P}(C_1 \cap C_2 \cap A_3 \cap \cdots \cap A_n) = \mathbb{P}(C_1)\, \mathbb{P}(C_2)\, \mathbb{P}(A_3) \cdots \mathbb{P}(A_n)\}.$$

Again, $\mathscr{M}_2$ is a $\lambda$-system, and we have proved in the previous step that it contains the $\pi$-system $\mathscr{A}_2$. Hence $\mathscr{M}_2 = \sigma(\mathscr{A}_2)$. Iterating the procedure, we arrive at:

$$\mathbb{P}(C_1 \cap \cdots \cap C_n) = \mathbb{P}(C_1) \cdots \mathbb{P}(C_n), \qquad \forall C_1 \in \sigma(\mathscr{A}_1), \cdots, \forall C_n \in \sigma(\mathscr{A}_n),$$

which means that $\sigma(\mathscr{A}_1), \cdots, \sigma(\mathscr{A}_n)$ are independent. $\square$

**Exercice 2.** **(i) (Time reversal)** Fix $a > 0$. Prove that $(B_a - B_{a-t},\, t \in [0, a])$ is Brownian motion on $[0, a]$.

**(ii) (Time inversion)** Prove that $X := (X_t,\, t \geq 0)$ defined by $X_t := t\, B_{\frac{1}{t}}$ (for $t > 0$) and $X_0 := 0$ is Brownian motion.

**Solution.** In both situations, it is easily checked that the process is centered Gaussian with covariance $s \wedge t$. For time reversal, the continuity of trajectories is obvious. For time inversion,

[4] The assumption $\Omega \in \mathscr{A}_1$ is used here to guarantee $\Omega \in \mathscr{M}_1$.

one may feel that there could be a continuity problem at 0: this however, does not cause any trouble because $X$ is, according to Kolmogorov's criterion, undistinguishable to Brownian motion. $\square$

**Exercice 3.** Prove that there exists a constant $a > 0$ (that does not depend on $\omega$) such that $\inf_{t\in[0,\,2]} B_t$ has the same distribution as $a \inf_{t\in[0,\,1]} B_t$.

**Solution.** By scaling, $\inf_{t\in[0,\,2]} B_t$ has the same distribution as $2^{1/2} \inf_{t\in[0,\,1]} B_t$. $\square$

**Exercice 4.** **(Brownian bridge)** Let $b_t = B_t - tB_1$, $t \in [0,\,1]$. It is a centered Gaussian process with a.s. continuous trajectories and with covariance $(s\wedge t) - st$. We call $b$ a Brownian bridge.

**(i)** The process $(b_t,\, t \in [0,\,1])$ is independent of the random variable $B_1$.

**(ii)** If $b$ is a Brownian bridge, so is $(b_{1-t},\, t \in [0,\,1])$.

**(iii)** If $b$ is a Brownian bridge, then $B_t = (1+t)b_{t/(1+t)}$, $t \geq 0$, is Brownian motion. Note that $b_t = (1-t)B_{t/(1-t)}$.

**Solution.** **(i)** Let $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$. Then $(b_{t_1},\, \cdots,\, b_{t_n},\, B_1)$ is a Gaussian random vector, with $\mathrm{Cov}(b_{t_i},\, B_1) = \mathrm{Cov}(B_{t_i},\, B_1) - \mathrm{Cov}(t_i\, B_1,\, B_1) = t_i - t_i = 0$, $\forall i$. So a property of Gaussian vectors tells us that $(b_{t_1},\, \cdots,\, b_{t_n})$ is independent of $B_1$.

**(ii)–(iii)** By checking covariance. $\square$

**Exercice 5.** Prove that

$$\lim_{t\to\infty} \frac{B_t}{t} = 0, \qquad \text{a.s.}$$

Hint: Use time inversion.

**Solution.** By continuity, $\lim_{t\to 0+} B_t = 0$, a.s., which yields the desired conclusion by time inversion. $\square$

**Exercice 6.** Let $(t_n)_{n\geq 1}$ be a sequence of positive real numbers decreasing towards 0. Prove that a.s., $B_{t_n} > 0$ for infinitely many $n$, and $B_{t_n} < 0$ for infinitely many $n$.

**Solution.** Let $A_n := \{B_{t_n} > 0\}$. We have $\mathbb{P}(A_n) = \frac{1}{2}$, $\forall n$, so $\mathbb{P}(\limsup_{n\to\infty} A_n) = \lim_{n\to\infty} \downarrow \mathbb{P}(\cup_{k\geq n} A_k) \geq \limsup_{n\to\infty} \mathbb{P}(A_n) = \frac{1}{2}$. On the other hand, by Blumenthal's 0–1 law, we know that $\mathbb{P}(\limsup_{n\to\infty} A_n)$ is either 0 or 1; so $\mathbb{P}(\limsup_{n\to\infty} A_n) = 1$. In other words, a.s., $B_{t_n} > 0$ for infinitely many $n$.

By considering $-B$ which is also Brownian motion, we see that a.s., $B_{t_n} < 0$ for infinitely many $n$. $\square$

**Exercice 7.** Prove that when $t \to \infty$, $(\int_0^t \mathrm{e}^{B_s}\, \mathrm{d}s)^{1/t^{1/2}} \to \mathrm{e}^{|N|}$ in law, where $N$ is a Gaussian $\mathscr{N}(0,\, 1)$ random variable.

**Solution.** By scaling, for any fixed $t > 0$, $(\int_0^t \mathrm{e}^{B_s}\, \mathrm{d}s)^{1/t^{1/2}}$ is distributed as

$$\Big( t \int_0^1 \mathrm{e}^{t^{1/2} B_u}\, \mathrm{d}u \Big)^{1/t^{1/2}} = \exp\Big( \frac{\log t}{t^{1/2}} + \frac{1}{t^{1/2}} \log \int_0^1 \mathrm{e}^{t^{1/2} B_u}\, \mathrm{d}u \Big).$$

The continuity of trajectories of $B$ implies that $\frac{1}{t^{1/2}} \log \int_0^1 \mathrm{e}^{t^{1/2} B_u}\, \mathrm{d}u \to \sup_{u\in[0,1]} B_u$ a.s., so $\exp(\frac{\log t}{t^{1/2}} + \frac{1}{t^{1/2}} \log \int_0^1 \mathrm{e}^{t^{1/2} B_u}\, \mathrm{d}u) \to \exp(\sup_{u\in[0,1]} B_u)$ a.s.

As a consequence, $(\int_0^t \mathrm{e}^{B_s}\, \mathrm{d}s)^{1/t^{1/2}} \to \exp(\sup_{u\in[0,1]} B_u)$ in law; the limit is distributed as $\mathrm{e}^{|N|}$ (by the reflection principle). $\square$

**Exercice 8.** **(i)** Prove that $0 < \sup_{t\ge 0}(|B_t| - t) < \infty$ a.s. and that $0 < \sup_{t\ge 0} \frac{|B_t|}{1+t} < \infty$ a.s.

**(ii)** Prove that $\sup_{t\ge 0}(|B_t| - t)$ and $(\sup_{t\ge 0} \frac{|B_t|}{1+t})^2$ have the same distribution.

Hint: Use the scaling property.

**(iii)** Prove that for any $p > 0$, $\mathbb{E}\{[\sup_{t\ge 0}(|B_t| - t)]^p\} < \infty$.

**(iv)** Prove that there exists a constant $C < \infty$ such that for any non-negative random variable $T$ (not necessarily a stopping time!), $\mathbb{E}(|B_T|) \le C\, [\mathbb{E}(T)]^{1/2}$.

Hint: Write, for any $a > 0$, $|B_T| = (|B_T| - aT) + aT$, and prove that $\mathbb{E}(|B_T| - aT) \le \frac{1}{a}\, \mathbb{E}[\sup_{t\ge 0}(|B_t| - t)]$.

**Solution.** **(i)** It suffices to recall that $\frac{B_t}{t} \to 0$ a.s. for $t \to \infty$ and that $\limsup_{t\to 0} \frac{B_t}{t^{1/2}} = \infty$ a.s..

**(ii)** Let $x > 0$. We have $\mathbb{P}\{\sup_{t\ge 0}(B_t - t) < x\} = \mathbb{P}\{B_t - t < x, \forall t \ge 0\}$. By scaling, the probability is

$$\begin{aligned} &= \mathbb{P}\{x^{1/2} B_{t/x} - t < x, \forall t \ge 0\} \\ &= \mathbb{P}\{x^{1/2} B_s - sx < x, \forall s \ge 0\} \\ &= \mathbb{P}\{\frac{B_s}{1+s} < x^{1/2}, \forall s \ge 0\}\,, \end{aligned}$$

from which the desired identity in law follows.

**(iii)** By (ii), it suffices to check $\mathbb{E}\{[\sup_{t\ge 0} \frac{|B_t|}{1+t}]^{2p}\} < \infty$.

By the reflection principle, $\mathbb{E}\{[\sup_{t\in[0,1]} B_t]^{2p}\} < \infty$. By symmetry, $\mathbb{E}\{[\sup_{t\in[0,1]}(-B_t)]^{2p}\} < \infty$. So $\mathbb{E}\{[\sup_{t\in[0,1]} |B_t|]^{2p}\} < \infty$. A fortiori, $\mathbb{E}\{[\sup_{t\in[0,1]} \frac{|B_t|}{1+t}]^{2p}\} < \infty$.

It remains to check $\mathbb{E}\{[\sup_{t\ge 1} \frac{|B_t|}{1+t}]^{2p}\} < \infty$. We have seen that $\mathbb{E}\{[\sup_{t\in[0,1]} |B_t|]^{2p}\} < \infty$. By inversion of time, this yields $\mathbb{E}\{[\sup_{t\ge 1} \frac{|B_t|}{t}]^{2p}\} < \infty$. A fortiori, $\mathbb{E}\{[\sup_{t\ge 1} \frac{|B_t|}{1+t}]^{2p}\} < \infty$.

**(iv)** We assume $0 < \mathbb{E}(T) < \infty$ (because otherwise, there is nothing to prove).

By scaling, $\mathbb{E}(|B_T| - aT) = \mathbb{E}(\frac{1}{a}|B_{a^2 T}| - aT) = \frac{1}{a}\, \mathbb{E}(|B_{a^2 T}| - a^2 T)$, which is obviously bounded by $\frac{1}{a}\, \mathbb{E}[\sup_{t\ge 0}(|B_t| - t)]$.

So $\mathbb{E}(|B_T|) \le \frac{K}{a} + a\, \mathbb{E}(T)$, with $K := \mathbb{E}[\sup_{t\ge 0}(|B_t| - t)] \in (0, \infty)$. Since this holds for all $a > 0$, we take $a := [\frac{K}{\mathbb{E}(T)}]^{1/2}$ to see that $\mathbb{E}(|B_T|) \le 2\, [K\, \mathbb{E}(T)]^{1/2}$. $\square$

**Exercice 9.** Let $S_t := \sup_{s\in[0,t]} B_s$, $t \geq 0$. Prove that $S_2 - S_1$ is distributed as $\max\{|N| - |\widetilde{N}|, 0\}$, where $N$ and $\widetilde{N}$ are independant Gaussian $\mathscr{N}(0, 1)$ random variables.

**Solution.** Put $\beta_s := B_{s+1} - B_1$, $s \geq 0$. By the Markov property, $\beta$ is Brownian motion, independent of $\mathscr{F}_1$, a fortiori of $(S_1, B_1)$.

Write $\widetilde{S}_t := \sup_{s\in[0,t]} \beta_s$. Then $\sup_{s\in[1,2]} B_s = \widetilde{S}_1 + B_1$; hence $S_2 = \max\{S_1, \widetilde{S}_1 + B_1\}$. In other words, $S_2 - S_1 = \max\{0, \widetilde{S}_1 - (S_1 - B_1)\}$. Since $\widetilde{S}_1$ and $S_1 - B_1$ are independent (see the previous paragraph), both having the law of $|B_1|$ (by the reflection principle, the desired identity in law follows. □

**Exercice 10.** Let $d_1 := \inf\{t \geq 1 : B_t = 0\}$ and $g_1 := \sup\{t \leq 1 : B_t = 0\}$.

**(i)** Is $d_1$ a stopping time?

**(ii)** Determine the law of $d_1$, and the law of $g_1$.

**Solution. (i)** Fix $t \geq 0$. Let us check $\{d_1 \leq t\} \in \mathscr{F}_t$.

If $t < 1$, then $\{d_1 \leq t\} = \varnothing \in \mathscr{F}_t$. If $t \geq 1$, we have

$$\{d_1 \leq t\} = \Big\{ \inf_{s\in[1,t]\cap\mathbb{Q}} |B_s| = 0 \Big\} \in \mathscr{F}_t.$$

Conclusion: $d_1$ is a stopping time.

**(ii)** Let $t \geq 1$. Applying the Markov property at time 1, we get

$$\mathbb{P}\{d_1 \leq t\} = \int_{-\infty}^{\infty} \mathbb{P}\{B_1 \in \, \mathrm{d}x\} \, \mathbb{P}\{T_{-x} \leq t - 1\}.$$

Let $N$ and $\widetilde{N}$ be independent Gaussian $\mathscr{N}(0, 1)$ random variables. We know that $T_{-x}$ is distributed as $\frac{x^2}{N^2}$. Hence

$$\mathbb{P}\{d_1 \leq t\} = \mathbb{P}\Big(\frac{\widetilde{N}^2}{N^2} \leq t - 1\Big).$$

As consequence, $(d_1 - 1)^{1/2}$ has the standard Cauchy distribution. In other words,

$$\mathbb{P}(d_1 \in \, \mathrm{d}t) = \frac{1}{\pi} \frac{\mathbf{1}_{\{t>1\}}}{t(t-1)^{1/2}} \, \mathrm{d}t.$$

Let us now study the law of $g_1$. For all $t \in [0, 1)$,

$$\begin{aligned}
\mathbb{P}(g_1 \leq t) &= \int_{-\infty}^{\infty} \mathbb{P}\{B_t \in \, \mathrm{d}x\} \, \mathbb{P}\{T_{-x} > 1 - t\} \\
&= \mathbb{P}\Big(\frac{t\widetilde{N}^2}{N^2} > 1 - t\Big) \\
&= \mathbb{P}\Big(\frac{1}{1 + (\widetilde{N}/N)^2} < t\Big).
\end{aligned}$$

Thus $g_1$ is distributed as $\frac{1}{1+C^2}$, where $C$ is a standard Cauchy random variable. We have

$$\mathbb{P}(g_1 \in \,\mathrm{d}t) = \frac{1}{\pi} \frac{\mathbf{1}_{\{0<t<1\}}}{t(1-t)^{1/2}} \,\mathrm{d}t.$$

We say that $g_1$ has the **Arcsine law**, because $\mathbb{P}(g_1 \le t) = \frac{2}{\pi} \arcsin(t^{1/2})$.

Observe that we could have determined the law of $g_1$ from the law of $d_1$ by means of the scaling property: $\{g_1 < t\} = \{d_t > 1\}$, where $d_t := \inf\{s \ge t : \, B_s = 0\}$ has the same law as $td_1$. $\square$

**Exercice 11.** Define $T_1 := \inf\{t > 0 : \, B_t = 1\}$ and $\tau := \inf\{t \ge T_1 : \, B_t = 0\}$.

**(i)** Is $\tau$ a stopping time?

**(ii)** Determine the law of $\tau$.

**Solution. (i)** Let us first prove that for any finite stopping time $T \ge 0$, $\tau = \inf\{t \ge T : \, B_t = 0\}$ is a stopping time. This was proved in the previous exercise when $T$ is a constant. If $T$ takes countably many values, say $(t_n)$, then

$$\{\tau \le t\} = \bigcup_{n:\, t_n \le t} \{T = t_n\} \cap \Big\{ \inf_{s \in [t_n, t] \cap \mathbb{Q}} |B_s| = 0 \Big\} \in \mathscr{F}_t \,,$$

which means $\tau$ is a stopping time.

In the general case, for all $n$, let

$$T_n := \sum_{k=0}^{\infty} \frac{k+1}{2^n} \mathbf{1}_{\{\frac{k}{2^n} < T \le \frac{k+1}{2^n}\}},$$

which is a non-increasing stopping times tending to $T$. By what we have just proved, $\tau_n := \inf\{t \ge T_n : \, B_t = 0\}$ is a stopping time; hence

$$\{\tau \le t\} = \Big( \{T \le t\} \cap \{B_T = 0\} \Big) \cup \Big( \{T \le t\} \cap \{B_T \neq 0\} \cap \bigcup_{n=1}^{\infty} \{\tau_n \le t\} \Big),$$

which is an element of $\mathscr{F}_t$. As a conclusion, $\tau$ is a stopping time.

**(ii)** By the strong Markov property, $\tau$ is distributed as $T_1 + \widetilde{T}_{-1}$, where $\widetilde{T}_{-1}$ is an independent copy of $T_1$. So $\tau$ is distributed as $T_2$, thus also as $4T_1$. The density of $\tau$ is

$$\mathbb{P}(\tau \in \,\mathrm{d}t) = (\frac{2}{\pi t^3})^{1/2} \exp\Big(-\frac{2}{t}\Big) \,\mathrm{d}t,$$

for $t > 0$. $\square$

**Exercice 12. (i)** Study convergence in probability of $\frac{\log(1+B_t^2)}{\log t}$ (quand $t \to \infty$).

**(ii)** Study a.s. convergence of $\frac{\log(1+B_t^2)}{\log t}$.

**Solution.** **(i)** By scaling, for all fixed $t \geq 0$, $\log(1+B_t^2)$ has the same distribution as $\log(1+tB_1^2)$. Since $B_1 \neq 0$ a.s., we have $\frac{\log(1+tB_1^2)}{\log t} \to 1$ a.s. So $\frac{\log(1+B_t^2)}{\log t} \to 1$ in law. The limit being a constant, the convergence holds also in probability. Conclusion: $\frac{\log(1+B_t^2)}{\log t} \to 1$ in probability.

**(ii)** If $\frac{\log(1+B_t^2)}{\log t}$ converged a.s., it would converge a.s. to 1. But $\{t : B_t = 0\}$ is a.s. unbounded, which makes it impossible to converge a.s. to 1. Conclusion: $\frac{\log(1+B_t^2)}{\log t}$ does not converge a.s. □

**Exercice 13.** Prove, *without using inversion of time* (but using instead the law of large numbers and the reflection principle), that $\frac{B_t}{t} \to 0$ a.s. when $t \to \infty$.

**Solution.** By the strong law of large numbers, $\frac{B_n}{n} \to 0$ a.s. for $n \to \infty$. It remains to check $\frac{1}{n}\sup_{t\in[n,\,n+1]}|B_t - B_n| \to 0$ a.s.

Let $\varepsilon > 0$. Let $A_n := \{\sup_{t\in[n,\,n+1]}|B_t - B_n| > n^\varepsilon\}$. We have $\mathbb{P}(A_n) = \mathbb{P}(\sup_{s\in[0,\,1]}|B_s| > n^\varepsilon) \leq 2\,\mathbb{P}(\sup_{s\in[0,\,1]} B_s > n^\varepsilon)$. By the reflection principle, $\sup_{s\in[0,\,1]} B_s$ is distributed as $|B_1|$. So $\mathbb{P}(A_n) \leq 2\,\mathbb{P}(|B_1| > n^\varepsilon) = 4\,\mathbb{P}(B_1 > n^\varepsilon) \leq 2\exp(-\frac{n^{2\varepsilon}}{2})$, which yields $\sum_n \mathbb{P}(A_n) < \infty$. By the Borel–Cantelli lemma, $\limsup_{n\to\infty} n^{-\varepsilon}\sup_{t\in[n,\,n+1]}|B_t - B_n| \leq 1$ a.s. A fortiori, $\frac{1}{n}\sup_{t\in[n,\,n+1]}|B_t - B_n| \to 0$ a.s. □

**Exercice 14.** The aim of this exercise is to prove $T < \infty$ a.s., where $T := \inf\{t \geq 0 : B_t = (1+t)^{1/2}\}$ ($\inf\varnothing := \infty$).

Ken says : Since $T$ is $\mathscr{F}_{0+}$-measurable, we know from the Blumenthal 0–1 law that $\mathbb{P}\{T < \infty\}$ is either 0 or 1. But $\mathbb{P}\{T < \infty\} \geq \mathbb{P}\{B_1 \geq 2^{1/2}\} > 0$, so $T < \infty$ a.s.

What do you think of Ken's argument?

**Solution.** Ken's argument is wrong, because $T$ is not $\mathscr{F}_{0+}$-measurable. As a matter of fact, whenever $t > 0$, $T$ is not $\mathscr{F}_t$-measurable.

To prove $T < \infty$ a.s., it suffices to recall that $\limsup_{t\to\infty}\frac{B_t}{t^{1/2}} = \infty$ a.s. □

**Exercice 15.** **(i)** Prove that $\int_0^\infty \sin^2(B_t)\,\mathrm{d}t = \infty$ a.s.

**(ii)** More generally, prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous which is not identically 0, then $\int_0^\infty f^2(B_t)\,\mathrm{d}t = \infty$ a.s.

**Solution.** (i) We define inductively two sequences of stopping times $(\tau_i)_{i\geq 1}$ and $(T_i)_{i\geq 1}$ as follows: $\tau_1 := 0$, $T_i := \inf\{t > \tau_i : |B_t| = 1\}$ and $\tau_{i+1} := \inf\{t > T_i : B_t = 0\}$ for $i \geq 1$. The strong Markov property tells us that $\int_{\tau_i}^{T_i}\sin^2(B_t)\,\mathrm{d}t$, $i \geq 1$, are i.i.d. In particular, $\sum_{i\geq 1}\int_{\tau_i}^{T_i}\sin^2(B_t)\,\mathrm{d}t = \infty$ a.s. A fortiori, $\int_0^\infty B_t^2\,\mathrm{d}t \geq \sum_{i\geq 1}\int_{\tau_i}^{T_i}\sin^2(B_t)\,\mathrm{d}t = \infty$ a.s.

(ii) Same argument as in (i), replacing $\inf\{t > \tau_i : |B_t| = 1\}$ by $\inf\{t > \tau_i : |B_t| = a\}$, where $a > 0$ is such that $f^2(x) \in (0,\,a)$. □

**Exercice 16.** *(This exercise is not part of the examination program.)* Let $\mathscr{Z} := \{t \geq 0 : B_t = 0\}$. Prove that a.s., $\mathscr{Z}$ is closed, unbounded, with no isolated point.

**Solution.** That $\mathscr{Z}$ is a closed set comes from the continuity of $t \mapsto B_t$. We have also seen in the class that $\mathscr{Z}$ is a.s. unbounded. It remains to show that $\mathscr{Z}$ has a.s. no isolated point.

For $t \geq 0$, let $\tau_t := \inf\{s \geq t : B_s = 0\}$ which is a stopping time. Clearly, $\tau_t < \infty$ a.s., and $B_{\tau_t} = 0$. The strong Markov property telles us that $\tau_t$ is not an isolated zero point of $B$. So a.s. for all $r \in \mathbb{Q}_+$, $\tau_r$ is not an isolated zero point.

Let $t \in \mathscr{Z} \backslash \{\tau_r,\, r \in \mathbb{Q}_+\}$. It suffices to show that $t$ is not an isolated zero point. Consider a rational sequence $(r_n) \uparrow\uparrow t$. Clearly, $r_n \leq \tau_{r_n} < t$. So $\tau_{r_n} \to t$; thus $t$ is not an isolated zero point.[5] □

**Exercice 17.** **(i)** Let $[a, b]$ and $[c, d]$ be disjoint intervals of $\mathbb{R}_+$. Prove that $\sup_{t\in[a,b]} B_s \neq \sup_{t\in[c,d]} B_s$ a.s.

**(ii)** Prove that a.s., each local maximum of $B$ is a strict local maximum.

**(iii)** Prove that a.s., the set of times at which $B$ realises local maxima is countable and dense in $\mathbb{R}_+$.

**Solution.** **(i)** Let $b < c$. By the Markov property, $\sup_{t\in[c,d]} B_s - B_c$ is independent of $(B_c, \sup_{t\in[a,b]} B_s)$, and is distributed as $(d-c)^{1/2}\,|N|$, with $N$ denoting a standard Gaussian $\mathscr{N}(0,\,1)$ random variable. Since $\mathbb{P}(N = x) = 0$ for all $x \in \mathbb{R}$, we obtain the desired result.

**(ii)** By (i), a.s. for all non-negative rationals $a < b < c < d$, $\sup_{t\in[a,b]} B_s \neq \sup_{t\in[c,d]} B_s$. If $B$ had a non strict local maximum, there would be two disjoint closed intervals with rational extremity points, on which $B$ would have the same maximal value, which is impossible.

**(iii)** Let $M$ denote the set of times at which $B$ realises the local minima. Consider the mapping:

$$[a, b] \mapsto \inf\Big\{t \geq a : B_t = \sup_{s\in[a,b]} B_s\Big\},$$

for all rationals $0 \leq a < b$. According to (i), the image of this mapping contains $M$ a.s., so $M$ is a.s. countable.

Since a.s. there exists no interval on which $B$ is monotone (because $B$ is nowhere differentiable), $B$ admits a local maximum on each interval with rational extremity points: $M$ is a.s. dense. □

**Exercice 18.** **(i)** Let $a > 0$ and let $T_a := \inf\{t \geq 0 : B_t = a\}$. Recall that $\mathbb{E}[\mathrm{e}^{-\lambda T_a}] = \mathrm{e}^{-a(2\lambda)^{1/2}}$, $\forall \lambda \geq 0$. Prove that $\mathbb{P}(T_a \leq t) \leq \exp(-\frac{a^2}{2t})$, for all $t > 0$.

**(ii)** Prove that if $\xi$ is a Gaussian $\mathscr{N}(0,\,1)$ random variable, then $\mathbb{P}(\xi \geq x) \leq \frac{1}{2}\mathrm{e}^{-x^2/2}$, $\forall x > 0$.

**Solution.** **(i)** Let $\lambda > 0$. We have $\mathbb{P}(T_a \leq t) = \mathbb{P}(\mathrm{e}^{-\lambda T_a} \geq \mathrm{e}^{-\lambda t}) \leq \mathrm{e}^{\lambda t}\,\mathbb{E}(\mathrm{e}^{-\lambda T_a}) = \mathrm{e}^{\lambda t - a(2\lambda)^{1/2}}$.

[5] It is known in analysis (see page 72 of the book by Hewitt, E. and Stromberg, K.: *Real and Abstract Analysis*. Springer, New York, 1969) that a closed set with no isolated point is uncountable. So $\mathscr{Z}$ is a.s. uncountable.

Choosing $\lambda := \frac{a^2}{2t^2}$ yields the desired inequality.

**(ii)** Let $S_1 := \sup_{s\in[0,\,1]} B_s$. By (i), we have, for all $a > 0$, $\mathbb{P}(S_1 \geq a) = \mathbb{P}(T_a \leq 1) \leq \mathrm{e}^{-a^2/2}$. According to the reflection principle, $S_1$ has the law of the modulus of a standard Gaussian random variable: the desired conclusion follows immediately. □

**Exercice 19.** **(i)** Prove that for all $t > 0$ and all $\varepsilon > 0$, $\mathbb{P}\{\sup_{s\in[0,\,t]} |B_s| \leq \varepsilon\} > 0$.

**(ii)** Prove that there exists $c \in (0,\,\infty)$ such that $\mathbb{P}\{\sup_{s\in[0,\,1]} |B_s| \leq \varepsilon\} \geq \mathrm{e}^{-c/\varepsilon^2}$, $\forall \varepsilon \in (0,\,1]$.

**(iii)** Prove that for all $t > 0$ and all $x > 0$, $\mathbb{P}\{\sup_{s\in[0,\,t]} |B_s| \geq x\} > 0$.

**Solution.** (i) By scaling, $\mathbb{P}\{\sup_{s\in[0,\,t]} |B_s| \leq \varepsilon\} = \mathbb{P}\{\sup_{s\in[0,\,\frac{4t}{\varepsilon^2}]} |B_s| \leq 2\}$. So it suffices to check that for all $a > 0$, $\mathbb{P}\{\sup_{s\in[0,\,a]} |B_s| \leq 2\} > 0$.

Let $T^* := \inf\{t \geq 0 : \, |B_t| = 1\}$. Let $\delta > 0$ be such that $p := \mathbb{P}\{T^* > \delta\} > 0$. By symmetry, $\mathbb{P}\{T^* > \delta,\; B_{T^*} = 1\} = \mathbb{P}\{T^* > \delta,\; B_{T^*} = -1\} = \frac{p}{2} > 0$. It follows from the strong Markov property that $\mathbb{P}\{\sup_{s\in[0,\,a]} |B_s| \leq 2\} \geq (\frac{p}{2})^N > 0$, where $N := \lceil \frac{a}{\delta} \rceil$.

(ii) Already proved in (i).

(iii) We have $\mathbb{P}\{\sup_{s\in[0,\,t]} |B_s| \geq x\} \geq \mathbb{P}\{B_t \geq x\} = \mathbb{P}\{B_1 \geq \frac{x}{t^{1/2}}\} > 0$, as $B_1$ is a standard Gaussian random variable. □

**Exercice 20.** **(Law of the iterated logarithm)** *(This exercise is not part of the examination program.)* Let $S_t := \sup_{s\in[0,t]} B_s$, and let $h(t) := (2t \log\log t)^{1/2}$.

**(i)** Let $\varepsilon > 0$. Prove that $\sum_n \mathbb{P}\{S_{t_{n+1}} \geq (1+\varepsilon)h(t_n)\} < \infty$, where $t_n = (1+\varepsilon)^n$. Prove that $\limsup_{t\to\infty} \frac{S_t}{h(t)} \leq 1$, a.s.

**(ii)** Prove that

$$\limsup_{t\to\infty} \frac{\sup_{s\in[0,t]} |B_s|}{h(t)} \leq 1, \qquad \text{a.s.}$$

**(iii)** Let $\theta > 1$, and let $s_n = \theta^n$. Prove that for all $\alpha \in (0, (1-\frac{1}{\theta})^{1/2})$, we have $\sum_n \mathbb{P}\{B_{s_n} - B_{s_{n-1}} > \alpha\, h(s_n)\} = \infty$. Prove that $\limsup_{t\to\infty} \frac{B_t}{h(t)} \geq \alpha - \frac{2}{\theta^{1/2}}$, a.s.

**(iv)** Prove that

$$\limsup_{t\to\infty} \frac{B_t}{h(t)} = 1, \qquad \text{a.s.}$$

**(v)** Let $X_1(t) := |B_t|$, $X_2(t) := S_t$, and $X_3(t) := \sup_{s\in[0,t]} |B_s|$. What can you say about $\limsup_{t\to\infty} \frac{X_i(t)}{h(t)}$ for $i = 1,\, 2$, ou 3 ?

**(vi)** What can you say about $\liminf_{t\to\infty} \frac{B_t}{h(t)}$ ? And about $\limsup_{t\to 0} \frac{B_t}{[2t \log\log(1/t)]^{1/2}}$ ?

**Solution.** **(i)** Let $A_n := \{S_{t_{n+1}} \geq (1+\varepsilon)h(t_n)\}$. We have

$$\mathbb{P}(A_n) = \mathbb{P}\Big(|B_1| \geq [2(1+\varepsilon)\log\log t_n]^{1/2}\Big) \leq 2\exp\Big(-(1+\varepsilon)\log\log t_n\Big),$$

as $\mathbb{P}(N \geq x) \leq \mathrm{e}^{-x^2/2}$ for all $x \geq 0$. Hence $\sum \mathbb{P}(A_n) < \infty$. By the Borel–Cantelli lemma, there exists $A \in \mathscr{F}$ with $\mathbb{P}(A) = 1$ such that for all $\omega \in A$, $\exists\, n_0 = n_0(\omega) < \infty$,

$$n \geq n_0 \implies S_{t_{n+1}} < (1+\varepsilon)(2t_n \log\log t_n)^{1/2}.$$

Therefore, for $t \in [t_n, t_{n+1}]$,

$$S_t \leq S_{t_{n+1}} < (1+\varepsilon)(2t_n \log\log t_n)^{1/2} \leq (1+\varepsilon)(2t \log\log t)^{1/2},$$

which implies $\limsup_{t\to\infty} \frac{S_t}{h(t)} \leq 1+\varepsilon$, a.s. It suffices now to let $\varepsilon \to 0$ along a sequence of rational numbers to reach the desired conclusion.

**(ii)** Since $-B$ is also Brownian motion, it follows from (i) that $\limsup_{t\to\infty} \frac{\sup_{s\in[0,t]}(-B_s)}{h(t)} \leq 1$, a.s. The desired result follows.

**(iii)** Let $E_n := \{B_{s_n} - B_{s_{n-1}} > \alpha h(s_n)\}$. The events $(E_n)$ are independent. Furthermore,

$$\begin{aligned} \mathbb{P}(E_n) &= \mathbb{P}\Big(B_1 > \alpha(\frac{2\log\log s_n}{1-\theta^{-1}})^{1/2}\Big) \\ &\sim \frac{1}{(2\pi)^{1/2}} \frac{1}{\alpha[2(\log\log s_n)/(1-\theta^{-1})]^{1/2}} \exp\Big(-\frac{\alpha^2 \log\log s_n}{1-\theta^{-1}}\Big), \end{aligned}$$

which yields $\sum_n \mathbb{P}(E_n) = \infty$ (because $\alpha < (1-\theta^{-1})^{1/2}$). By the Borel–Cantelli lemma, there exists $E \in \mathscr{F}$ with $\mathbb{P}(E) = 1$ such that for all $\omega \in E$,

$$B_{s_n} - B_{s_{n-1}} > \alpha(2s_n \log\log s_n)^{1/2}, \qquad \text{for infinitely many } n.$$

On the other hand, by (ii), a.s. for all sufficiently large $n$,

$$|B_{s_{n-1}}| \leq 2(2s_{n-1}\log\log s_{n-1})^{1/2} \leq \frac{2}{\theta^{1/2}}(2s_n \log\log s_n)^{1/2}\,.$$

The desired inequality follows.

**(iv)** By (iii), $\limsup_{t\to\infty} \frac{B_t}{h(t)} \geq 1$ a.s., which, together with (i), implies the desired result.

**(v)** The "limsup" expression is 1 a.s. (for all $i$).

**(vi)** By symmetry, $\liminf_{t\to\infty} \frac{B_t}{h(t)} = -1$ a.s.

By inversion of time, $\limsup_{t\to 0} \frac{B_t}{(2t\log\log(1/t))^{1/2}} = 1$ a.s. □

**Exercice 21.** Let $(P_t)_{t\geq 0}$ denote the semi-group of Brownian motion. Prove that if $f \in C_0$ (continuous function satisfying $\lim_{|x|\to\infty} f(x) = 0$), then $P_t f \in C_0$, $\forall t \geq 0$, and $\lim_{t\downarrow 0} P_t f = f$ uniformly on $\mathbb{R}$.

**Solution.** Let $t > 0$. We have

$$(P_t f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} f(x + t^{1/2} z) \mathrm{e}^{-z^2/2}\, \mathrm{d}z.$$

By the dominated convergence theorem (because $f$ is bounded and continuous), we have $P_t f \in C_0$.

Let us prove that $\lim_{t\downarrow 0} P_t f = f$ uniformly on $\mathbb{R}$. Write

$$(P_t f)(x) - f(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \mathrm{e}^{-z^2/2}[f(x+t^{1/2}z) - f(x)]\, \mathrm{d}z.$$

(The dominated convergence theorem allows us immediately to see that $P_t f \to f$ pointwise.) Let $\varepsilon > 0$. Since $f$ is bounded, there exists $M > 0$ such that $\int_{|z|>M} \mathrm{e}^{-z^2/2} \|f\|_\infty \, \mathrm{d}z < \varepsilon$. For $|z| \le M$, as $f$ is uniformly continuous on $\mathbb{R}$, there exists $\delta > 0$ such that for $t \le \delta$, we have $\sup_{|z|\le M} |f(x + t^{1/2}z) - f(x)| \le \varepsilon$, $\forall x \in \mathbb{R}$. Consequently, for all $t \le \delta$, $|P_t f(x) - f(x)| \le \frac{2\varepsilon}{(2\pi)^{1/2}} + \varepsilon \le 2\varepsilon$, $\forall x \in \mathbb{R}$. $\square$

**Exercice 22.** Prove that if $f \in C_c^2$ ($C^2$ function with compact support), then

$$\lim_{t \downarrow 0} \frac{(P_t f)(x) - f(x)}{t} = \frac{1}{2} f''(x), \qquad x \in \mathbb{R}.$$

**Solution.** Write

$$\frac{(P_t f)(x) - f(x)}{t} = \frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{f(x + t^{1/2}z) + f(x - t^{1/2}z) - 2f(x)}{t} \, \mathrm{e}^{-z^2/2} \, \mathrm{d}z.$$

We let $t \to 0$. Since $f \in C^2$, we have $\frac{f(x+t^{1/2}z)+f(x-t^{1/2}z)-2f(x)}{t} \to z^2 f''(x)$, and there exists a constant $K < \infty$ such that for all $t \le 1$, $\frac{f(x+t^{1/2}z)+f(x-t^{1/2}z)-2f(x)}{t} \le K z^2$ (we use, moreover, the assumption that $f$ is of compact support). Since $z^2 \mathrm{e}^{-z^2/2}$ is integrable, it follows from the dominated convergence theorem that $\frac{(P_t f)(x) - f(x)}{t} \to \frac{1}{(2\pi)^{1/2}} \int_0^\infty z^2 f''(x) \, \mathrm{e}^{-z^2/2} \, \mathrm{d}z = \frac{1}{2} f''(x)$. $\square$

**Exercice 23.** Let $f$ be a bounded Borel function on $\mathbb{R}$, and let $u(t,x) := (P_t f)(x)$ (for $t \ge 0$ and $x \in \mathbb{R}$). Prove that

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \ x \in \mathbb{R}.$$

**Solution.** Fix $t > 0$ and $x \in \mathbb{R}$. We have

$$u(t,\, x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} f(r) \, \frac{1}{t^{1/2}} \, \exp\Big( - \frac{(r-x)^2}{2t} \Big) \, \mathrm{d}r.$$

Since $f$ is bounded, we can use the dominated convergence theorem to take the partial derivative (with respect to $t$) under the integral sign:

$$\frac{\partial u(t,\, x)}{\partial t} = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} f(r) \Big( - \frac{1}{2t^{3/2}} + \frac{(r-x)^2}{2t^{5/2}} \Big) \exp\Big( - \frac{(r-x)^2}{2t} \Big) \, \mathrm{d}r.$$

Similarly, thanks again to the boundedness of $f$ and to the dominated convergence theorem, we can take the second partial derivative (with respect to $x$) under the integral sign, to see that

$$\frac{\partial^2 u(t,\, x)}{\partial x^2} = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} f(r) \, \frac{1}{t^{1/2}} \Big( - \frac{1}{t} + \frac{(r-x)^2}{t^2} \Big) \exp\Big( - \frac{(r-x)^2}{2t} \Big) \, \mathrm{d}r.$$

It is readily observed that $\frac{\partial u(t,x)}{\partial t} = \frac{1}{2} \frac{\partial^2 u(t,x)}{\partial x^2}$. $\square$

**Tsinghua University** *2017–2018*

**"Advanced Probability"** (Part III: Brownian motion)

*Exercise sheet #III.3:*

*Brownian motion and martingales*

**Exercice 1.** Let $a > 0$, and let $T_a^* := \inf\{t \geq 0 : |B_t| = a\}$. Prove that $T_a^*$ has the same distribution as $\frac{a^2}{\sup_{s\in[0,1]} B_s^2}$.

**Solution.** Let $t > 0$. Then $\mathbb{P}(T_a \leq t) = \mathbb{P}(\sup_{s\in[0,t]} |B_s| \geq a)$, which, by scaling, equals to $\mathbb{P}(t^{1/2} \sup_{u\in[0,1]} |B_u| \geq a)$. As such, $T_a$ and $\frac{a^2}{\sup_{s\in[0,1]} B_s^2}$ have the same distribution function: they have the same law. $\square$

**Exercice 2.** Let $\xi$ and $\eta$ be integrable random variables. Let $\mathscr{G} \subset \mathscr{F}$ be a sigma-algebra.

**(i)** Prove that $\mathbb{E}(\xi \,|\, \mathscr{G}) \leq \mathbb{E}(\eta \,|\, \mathscr{G})$, a.s., if and only if $\mathbb{E}(\xi \,\mathbf{1}_A) \leq \mathbb{E}(\eta \,\mathbf{1}_A)$ for all $A \in \mathscr{G}$.

**(ii)** Prove that $\mathbb{E}(\xi \,|\, \mathscr{G}) = \mathbb{E}(\eta \,|\, \mathscr{G})$, a.s., if and only if $\mathbb{E}(\xi \,\mathbf{1}_A) = \mathbb{E}(\eta \,\mathbf{1}_A)$ for all $A \in \mathscr{G}$.

**Solution.** **(i)** Without loss of generality, we may assume $\xi = 0$ (otherwise, we replace $\eta$ by $\eta + \xi$). We need to prove that $\mathbb{E}(\eta \,|\, \mathscr{G}) \geq 0$ a.s. $\Leftrightarrow$ $\mathbb{E}(\eta \,\mathbf{1}_A) \geq 0$, $\forall A \in \mathscr{G}$.

"$\Rightarrow$" Assume $\mathbb{E}(\eta \,|\, \mathscr{G}) \geq 0$ a.s. Then for all $A \in \mathscr{G}$, we have, by the definition of conditional expectation, $\mathbb{E}(\eta \,\mathbf{1}_A) = \mathbb{E}[\,\mathbf{1}_A \,\mathbb{E}(\eta \,|\, \mathscr{G})]$, which is non-negative because by assumption, $\mathbb{E}(\eta \,|\, \mathscr{G}) \geq 0$ a.s.

"$\Leftarrow$" Assume $\mathbb{E}(\eta \,\mathbf{1}_A) \geq 0$, $\forall A \in \mathscr{G}$.

Write $\theta := \mathbb{E}(\eta \,|\, \mathscr{G})$ which is $\mathscr{G}$-mesurable. Let $B := \{\omega : \theta(\omega) < 0\} \in \mathscr{G}$. By assumption, $\mathbb{E}(\eta \,\mathbf{1}_B) \geq 0$. We observe that $\mathbb{E}(\eta \,\mathbf{1}_B) = \mathbb{E}[\,\mathbb{E}(\eta \,\mathbf{1}_B \,|\, \mathscr{G})] = \mathbb{E}[\,\mathbf{1}_B \,\mathbb{E}(\eta \,|\, \mathscr{G})] = \mathbb{E}[\,\mathbf{1}_B \,\theta]$ ; as such, saying that $\mathbb{E}(\eta \,\mathbf{1}_B) \geq 0$ is equivalent to saying that $\mathbb{E}[\,\mathbf{1}_B \,\theta] \geq 0$. Since $\mathbf{1}_B \,\theta \leq 0$, this is possible only if $\mathbf{1}_B \,\theta = 0$ a.s., i.e., $\theta \geq 0$ a.s.

**(ii)** It is a consequence of (i), by considering the pair $(-\xi, -\eta)$ in place of $(-\xi, -\eta)$. $\square$

**Exercice 3.** Let $(X_n, n \geq 0)$ be a sequence of real-valued random variables and let $X_\infty$ be a real-valued random variable. Prove that $X_n \to X_\infty$ in $L^1$ (when $n \to \infty$) if and only if $X_n \to X_\infty$ in probability and $(X_n, n \geq 0)$ is uniformly integrable.

**Solution.** "$\Leftarrow$" Without loss of generality, we may assume $X_\infty = 0$ (otherwise, we consider $X_n - X_\infty$ in place of $X_t$, by observing that $(X_n - X_\infty, t \geq 0)$ is also uniformly integrable).

Let $\varepsilon > 0$. We fix $a > 0$ sufficiently large such that $\mathbb{E}(|X_n| \,\mathbf{1}_{\{|X_n|>a\}}) < \varepsilon$, $\forall n \geq 0$. Then $\mathbb{E}(|X_n|) = \mathbb{E}(|X_n| \,\mathbf{1}_{\{\varepsilon\leq|X_n|\leq a\}}) + \mathbb{E}(|X_n| \,\mathbf{1}_{\{|X_n|>a\}}) + \mathbb{E}(|X_n| \,\mathbf{1}_{\{|X_n|<\varepsilon\}}) \leq a\mathbb{P}(|X_n| \geq \varepsilon) + \varepsilon + \varepsilon.$

Letting $n \to \infty$, and since $X_n \to 0$ in probability, we get $\limsup_{n\to\infty} \mathbb{E}(|X_n|) \le 2\varepsilon$, which yields $X_t \to 0$ in $L^1$ because $\varepsilon > 0$ can be as small as possible.

"$\Rightarrow$" Assume that $X_n \to X_\infty$ in $L^1$.

Convergence in probability follows immediately from convergence in $L^1$. To prove that $(X_n,\, n \ge 0)$ is uniformly integrable, it suffices to check $(a)$ $\sup_{n\ge 1} \mathbb{E}(|X_n|) < \infty$ ; $(b)$ for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\forall B \in \mathscr{F}$, $\mathbb{P}(B) < \delta \Rightarrow \sup_{n\ge 1} \mathbb{E}(|X_n|\, \mathbf{1}_B) < \varepsilon$.

Condition $(a)$ is a straightforward consequence of convergence in $L^1$. Let us check condition $(b)$. Let $B \in \mathscr{F}$. We have $\mathbb{E}(|X_n|\, \mathbf{1}_B) \le \mathbb{E}(|X_\infty|\, \mathbf{1}_B) + \mathbb{E}(|X_n - X_\infty|)$. Let $\varepsilon > 0$. There exists $n_0 < \infty$ such that $\mathbb{E}(|X_n - X_\infty|) < \frac{\varepsilon}{2}$, $\forall n \ge n_0$. On the other hand, there exists $\delta > 0$ sufficiently small such that if $\mathbb{P}(B) < \delta$, then $\mathbb{E}(|X_\infty|\, \mathbf{1}_B) < \frac{\varepsilon}{2}$, and $\max_{0\le n\le n_0} \mathbb{E}(|X_n|\, \mathbf{1}_B) \le \varepsilon$. Hence $\sup_{n\ge 0} \mathbb{E}(|X_n|\, \mathbf{1}_B) \le \varepsilon$ for all $B$ with $\mathbb{P}(B) < \delta$: condition $(b)$ is satisfied. □

**Exercice 4.** Let $(X_t,\, t \ge 0)$ be a family of of real-valued random variables and let $X_\infty$ be a real-valued random variable. Prove that if $X_t \to X_\infty$ in probability (when $t \to \infty$) and if $(X_t,\, t \ge 0)$ is uniformly integrable, then $X_t \to X_\infty$ in $L^1$.

Prove that the converse is, in general, not true.

**Solution.** The first part is proved using exactly the same argument as in the previous, replacing everywhere $n$ by $t$.

To see the converse is not true in general, it suffices to consider an example of $(X_t,\, t \in [0,\, 1])$ that is not uniformly integrable, and let $X_t := 0$ for $t > 1$. Then $X_t \to 0$ in $L^1$ but $(X_t,\, t \ge 0)$ is not uniformly integrable. □

**Exercice 5.** Let $S$ and $T$ be stopping times.

**(i)** Prove that $\mathscr{F}_S \subset \mathscr{F}_T$.

**(ii)** Prove that both $S \wedge T$ and $S \vee T$ are stopping times, and $\mathscr{F}_{S\wedge T} = \mathscr{F}_S \cap \mathscr{F}_T$. Moreover, $\{S \le T\} \in \mathscr{F}_{S\wedge T}$, $\{S = T\} \in \mathscr{F}_{S\wedge T}$, $\{S < T\} \in \mathscr{F}_{S\wedge T}$.

**(iii)** Prove that $S + T$ is a stopping time. [Hint: both $S$ and $T$ are $\mathscr{F}_{S\vee T}$-measurable.]

**Solution. (i)** Let $A \in \mathscr{F}_S$. Then $A \cap \{T \le t\} = (A \cap \{S \le t\}) \cap \{T \le t\} \in \mathscr{F}_t$.

**(ii)** We have $\{S \wedge T \le t\} = \{S \le t\} \cup \{T \le t\} \in \mathscr{F}_t$ and $\{S \vee T \le t\} = \{S \le t\} \cap \{T \le t\} \in \mathscr{F}_t$.

By (i), $\mathscr{F}_{S\wedge T} \subset \mathscr{F}_S \cap \mathscr{F}_T$. Conversely, if $A \in \mathscr{F}_S \cap \mathscr{F}_T$, then

$$A \cap \{S \wedge T \le t\} = (A \cap \{S \le t\}) \cup (A \cap \{T \le t\}) \in \mathscr{F}_t\,;$$

thus $A \in \mathscr{F}_{S\wedge T}$. Consequently, $\mathscr{F}_{S\wedge T} = \mathscr{F}_S \cap \mathscr{F}_T$.

Finally, $\{S \le T\} \cap \{T \le t\} = \{S \le t\} \cap \{T \le t\} \cap \{S \wedge t \le T \wedge t\} \in \mathscr{F}_t$, because $S \wedge t$ and $T \wedge t$ being $\mathscr{F}_{S\wedge t}$-measurable and $\mathscr{F}_{T\wedge t}$-measurable respectively, are $\mathscr{F}_t$-measurable. Hence

$\{S \le T\}$ is $\mathscr{F}_T$-measurable. Similarly, $\{S \le T\} \cap \{S \le t\} = \{S \le t\} \cap \{S \wedge t \le T \wedge t\} \in \mathscr{F}_t$, which yields $\{S \le T\} \in \mathscr{F}_S$. Therefore, $\{S \le T\} \in \mathscr{F}_S \cap \mathscr{F}_T = \mathscr{F}_{S\wedge T}$.

By exchanging $S$ and $T$, we have, $\{T \le S\} \in \mathscr{F}_{S\wedge T}$. Hence $\{S = T\} = \{S \le T\} \cap \{T \le S\} \in \mathscr{F}_{S\wedge T}$, and $\{S < T\} = \{S \le T\} \backslash \{S = T\} \in \mathscr{F}_{S\wedge T}$.

**(iii)** Since $S$ and $T$ are $\mathscr{F}_{S\vee T}$-measurable, so is $S + T$. We have $\{S + T \le t\} = \{S + T \le t\} \cap \{S \vee T \le t\} \in \mathscr{F}_t$, because $\{S + T \le t\} \in \mathscr{F}_{S\vee T}$. $\square$

**Exercice 6.** Let $T$ be a stopping time. Then

$$T_n := \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbf{1}_{\{\frac{k-1}{2^n} < T \le \frac{k}{2^n}\}} + (+\infty)\, \mathbf{1}_{\{T=\infty\}}$$

is a non-increasing sequence of stopping times such that $T_n(\omega) \downarrow T(\omega)$ for all $\omega \in \Omega$.

**Solution.** Clearly, $(T_n)$ decreases pointwise to $T$. It suffices to check that each $T_n$ is a stopping time. Since $T_n$ is $\mathscr{F}_T$-measurable, and since $T_n \ge T$, we have $\{T_n \le t\} = \{T_n \le t\} \cap \{T \le t\} \in \mathscr{F}_t$, because $\{T_n \le t\} \in \mathscr{F}_T$. $\square$

**Exercice 7.** Let $T$ Be a stopping time. Let $(X_t,\, t \ge 0)$ is an $\mathbb{R}^d$-valued adapted right-continuous (or left-continuous) process.

**(i)** Let $Y : \Omega \to \mathbb{R}^d$. Prove that $Y\, \mathbf{1}_{\{T<\infty\}}$ is $\mathscr{F}_T$-measurable if and only if $\forall\, t$, $Y\, \mathbf{1}_{\{T\le t\}}$ is $\mathscr{F}_t$-measurable.

**(ii)** Prove that for any $t$, the mapping $[0,\, t] \times \Omega \to \mathbb{R}^d$ defined by $(s,\, \omega) \mapsto X_s(\omega)$ is $\mathscr{B}([0,\, t]) \otimes \mathscr{F}_t$-measurable, where $\mathscr{B}([0,\, t])$ denotes the Borel $\sigma$-field of $[0,\, t]$.

**(iii)** Prove that $X_T\, \mathbf{1}_{\{T<\infty\}}$ is $\mathscr{F}_T$-measurable.

**Solution.** **(i)** It suffices to observe that for all $A \in \mathscr{B}(\mathbb{R}^d)$ with $0 \notin A$, $\{Y\, \mathbf{1}_{\{T\le t\}} \in A\} = \{Y \in A\} \cap \{T \le t\}$.

**(ii)** We first assume that $(X_s,\, s \ge 0)$ is right-continuous. For any $n \ge 1$, let

$$X_s^{(n)} := X_{t \wedge \frac{(\lfloor ns/t \rfloor + 1)t}{n}}, \qquad s \in [0,\, t].$$

Then $X_s^{(n)}(\omega) \to X_s(\omega)$ by the right-continuity of the trajectories. For any $A \in \mathscr{B}(\mathbb{R}^d)$,

$$\begin{aligned} &\big\{(s,\omega) :\ s \in [0,\, t],\, X_s^{(n)}(\omega) \in A\big\} \\ &\quad = \bigcup_{k=1}^{n} \Big( [\frac{(k-1)t}{n}, \frac{kt}{n}) \times \{X_{\frac{kt}{n}} \in A\} \Big) \cup \Big( \{t\} \times \{X_t \in A\} \Big) \\ &\quad \in \mathscr{B}([0,\, t]) \otimes \mathscr{F}_t\, . \end{aligned}$$

Hence $(s,\omega) \mapsto X_s(\omega)$ on $[0,\, t] \times \Omega$ is $\mathscr{B}([0,\, t]) \otimes \mathscr{F}_t$-measurable.

The proof is similar if $(X_s,\, s \ge 0)$ is left-continuous; it suffices to consider instead $X_s^{(n)} := X_{\frac{\lfloor ns/t \rfloor\, t}{n}}$.

**(iii)** We apply (i) to $Y = X_T \mathbf{1}_{\{T<\infty\}}$; so it suffices to check that for all $t$, $Y \mathbf{1}_{\{T\le t\}} = X_{T\wedge t} \mathbf{1}_{\{T\le t\}}$ is $\mathscr{F}_t$-measurable.

Note that $X_{T\wedge t}$ is the composition of the following two mappings:

$$\begin{aligned} (\Omega,\ \mathscr{F}_t) &\longrightarrow ([0,\, t]\times\Omega,\ \mathscr{B}([0,\, t])\otimes\mathscr{F}_t) \\ \omega &\longmapsto (T(\omega)\wedge t,\ \omega) \end{aligned}$$

and

$$\begin{aligned} ([0,t]\times\Omega,\ \mathscr{B}([0,\, t])\otimes\mathscr{F}_t) &\longrightarrow (\mathbb{R}^d,\ \mathscr{B}(\mathbb{R}^d)) \\ (s,\omega) &\longmapsto X_s(\omega) \end{aligned}$$

both of which are measurable. So $X_{T\wedge t}$, as well as $X_{T\wedge t} \mathbf{1}_{\{T\le t\}}$, are $\mathscr{F}_t$-measurable. □

**Exercice 8.** Let $(X_t,\ t\ge 0)$ be a submartingale. Prove that for all $t\ge 0$, we have $\sup_{s\in[0,t]} \mathbb{E}(|X_s|) < \infty$.

**Solution.** Since $(X_t^+,\, t\ge 0)$ is a submartingale, we have $\mathbb{E}(X_s^+) \le \mathbb{E}(X_t^+)$ for $s\le t$. On the other hand, $\mathbb{E}(X_s)\ge\mathbb{E}(X_0)$, which implies $\sup_{s\in[0,t]}\mathbb{E}(|X_s|) \le 2\,\mathbb{E}(X_t^+) - \mathbb{E}(X_0) < \infty$. □

**Exercice 9.** Let $(B_t, t\ge 0)$ be Brownian motion, and let $(\mathscr{F}_t)$ be its canonical filtration. Then the following processes are martingales:

**(i)** $(B_t, t\ge 0)$.

**(ii)** $(B_t^2 - t,\, t\ge 0)$.

**(iii)** For any $\theta\in\mathbb{R}$, $(\mathrm{e}^{\theta B_t - \frac{\theta^2}{2}t},\, t\ge 0)$.

**Solution. (i)** For any $t$, $\mathbb{E}(|B_t|)<\infty$ and $B_t$ is $\mathscr{F}_t$-measurable. Let $t > s\ge 0$. Since $B_t - B_s$ is independent of $\mathscr{F}_s$, we have $\mathbb{E}(B_t - B_s \,|\, \mathscr{F}_s) = \mathbb{E}(B_t - B_s)$, which vanishes because $B_t - B_s$ has the Gaussian $\mathscr{N}(0,\, t-s)$ law. So $\mathbb{E}(B_t\,|\,\mathscr{F}_s) = B_s$ a.s.

**(ii)** For any $t$, $\mathbb{E}(B_t^2)<\infty$ and $B_t^2$ is $\mathscr{F}_t$-measurable. Let $t>s$, $\mathbb{E}(B_t^2 - t\,|\,\mathscr{F}_s) = \mathbb{E}[(B_t - B_s + B_s)^2\,|\,\mathscr{F}_s)] - t$, and for all $x\in\mathbb{R}$, $\mathbb{E}[(B_t - B_s + x)^2] = \mathrm{Var}(B_t - B_s) + x^2 = t - s + x^2$, so we get $\mathbb{E}(B_t^2 - t\,|\,\mathscr{F}_s) = t - s + B_s^2 - t = B_s^2 - s$ a.s.

**(iii)** For any $t$, $\mathbb{E}(\mathrm{e}^{\theta B_t - \frac{\theta^2}{2}t}) < \infty$ and $\mathrm{e}^{\theta B_t - \frac{\theta^2}{2}t}$ is $\mathscr{F}_t$-measurable. Let $t>s$. We have $\mathbb{E}[\mathrm{e}^{\theta B_t - \frac{\theta^2}{2}t}\,|\,\mathscr{F}_s] = \mathrm{e}^{\frac{\theta^2}{2}2(t-s)}\mathrm{e}^{\theta B_s - \frac{\theta^2}{2}t} = \mathrm{e}^{\theta B_s - \frac{\theta^2}{2}s}$. □

**Exercice 10.** Let $(X_t,\, t\ge 0)$ be a process with independent increments, and let $(\mathscr{F}_t)$ be its canonical filtration.

**(i)** If for all $t$, $\mathbb{E}(|X_t|)<\infty$, then $\widetilde{X}_t := X_t - \mathbb{E}(X_t)$ is a martingale.

**(ii)** If for all $t$, $\mathbb{E}(X_t^2)<\infty$, then $Y_t := \widetilde{X}_t^2 - \mathbb{E}(\widetilde{X}_t^2)$ is a martingale.

**(iii)** Let $\theta\in\mathbb{R}$. If $\mathbb{E}(\mathrm{e}^{\theta X_t})<\infty$ for all $t\ge 0$, then $(Z_t := \frac{\mathrm{e}^{\theta X_t}}{\mathbb{E}[\mathrm{e}^{\theta X_t}]},\, t\ge 0)$ is a martingale.

**Solution.** Similar to the solution to the previous exercise. □

**Exercice 11.** Let $X := (X_t,\ t \ge 0)$ be a martingale such that $\sup_{t\ge 0} \mathbb{E}(|X_t|) < \infty$.

**(i)** Prove that for all $t \ge 0$, $\mathbb{E}(X_n^+ \,|\, \mathscr{F}_t)$ converges (when $n \to \infty$) a.s. to a real-valued random variable, denoted by $\alpha_t$ .

**(ii)** Prove that $(\alpha_t,\ t \ge 0)$ is a martingale.

**(iii)** Prove that $X$ is the difference of two non-negative martingales.

**Solution.** **(i)** Fix $t \ge 0$. Let $\xi_n := \mathbb{E}(X_n^+ \,|\, \mathscr{F}_t)$.

For $m > n \ge t$, $\xi_n = \mathbb{E}\{\, [\mathbb{E}(X_m \,|\, \mathscr{F}_n)]^+ \,|\, \mathscr{F}_t\} \le \mathbb{E}\{\, \mathbb{E}(X_m^+ \,|\, \mathscr{F}_n) \,|\, \mathscr{F}_t\} = \mathbb{E}\{X_m^+ \,|\, \mathscr{F}_t\} = \xi_m$. So the sequence $(\xi_n)_{n\ge t}$ is a.s. non-decreasing. In particular, it converges a.s., whose limit is denoted by $\alpha_t$.

By the monotone convergence theorem, $\mathbb{E}(\alpha_t) = \lim_{n\to\infty} \uparrow \mathbb{E}(\xi_n)$. We observe that $\mathbb{E}(\xi_n) = \mathbb{E}(X_n^+) \le \sup_{t\ge 0} \mathbb{E}(|X_t|)$, which implies $\mathbb{E}(\alpha_t) \le \sup_{t\ge 0} \mathbb{E}(|X_t|) < \infty$. In particular, $\alpha_t < \infty$ a.s.

**(ii)** We have seen that for any $t$, $\alpha_t$ is integrable, and is clearly $\mathscr{F}_t$-measurable (being the pointwise limit of $\mathscr{F}_t$-measurable random variables). Let us check the characteristic identity.

Let $s < t$, and let $A \in \mathscr{F}_s$. Since $\alpha_t$ is the limit of the non-decreasing sequence $(\xi_n)$, it follows from the monotone convergence theorem that $\mathbb{E}(\alpha_t \mathbf{1}_A) = \lim_{n\to\infty} \uparrow \mathbb{E}(\xi_n \mathbf{1}_A)$. For $n \ge t$, we have $\mathbb{E}(\xi_n \mathbf{1}_A) = \mathbb{E}(X_n^+ \mathbf{1}_A)$, thus $\mathbb{E}(\alpha_t \mathbf{1}_A) = \lim_{n\to\infty} \uparrow \mathbb{E}(X_n^+ \mathbf{1}_A)$. Similarly, $\mathbb{E}(\alpha_s \mathbf{1}_A) = \lim_{n\to\infty} \uparrow \mathbb{E}(X_n^+ \mathbf{1}_A)$. It follows that $\mathbb{E}(\alpha_t \mathbf{1}_A) = \mathbb{E}(\alpha_s \mathbf{1}_A)$. Since $A \in \mathscr{F}_s$ is arbitrary, we deduce that $\mathbb{E}(\alpha_t \,|\, \mathscr{F}_s) = \alpha_s$ a.s.

[We note that for question (i) and (ii), it suffices to have a submartingale $X$ satisfying $\sup_{t\ge 0} \mathbb{E}(X_t^+) < \infty$.]

**(iii)** By considering $-X$ in place of $X$, we see that $\mathbb{E}(X_n^- \,|\, \mathscr{F}_t)$ converges a.s. (when $n \to \infty$) to a limit, denoted by $\beta_t$, and that $(\beta_t,\ t \ge 0)$ is a non-negative martingale. We have $X_t = \alpha_t - \beta_t$, $\forall t \ge 0$. □

**Exercice 12.** Let $\xi$ be a real-valued random variable. Let $X_t := \mathbb{P}(\xi \le t \,|\, \mathscr{F}_t)$. Prove that $(X_t,\ t \ge 0)$ is a submartingale.

**Solution.** Let $0 \le s < t$. Let us check that $\mathbb{E}(X_t \,|\, \mathscr{F}_s) \ge X_s$ a.s.

By definition, $X_t \ge \mathbb{P}(\xi \le s \,|\, \mathscr{F}_t)$; so $\mathbb{E}[X_t \,|\, \mathscr{F}_s] \ge \mathbb{E}[\,\mathbb{P}(\xi \le s \,|\, \mathscr{F}_t) \,|\, \mathscr{F}_s] = \mathbb{P}(\xi \le s \,|\, \mathscr{F}_s) = X_s$. □

**Exercice 13.** Let $(X_t,\ t \ge 0)$ be a submartingale. Prove that $\sup_{t\ge 0} \mathbb{E}(X_t^+) < \infty$ if and only if $\sup_{t\ge 0} \mathbb{E}(|X_t|) < \infty$.

**Solution.** "$\Leftarrow$" Obvious.

"$\Rightarrow$" Suppose $\sup_{t\geq 0}\mathbb{E}(X_t^+) < \infty$. Since $|X_t| = 2X_t^+ - X_t$ and $\mathbb{E}(X_t) \geq \mathbb{E}(X_0)$, we have $\sup_{t\geq 0}\mathbb{E}(|X_t|) \leq 2\sup_{t\geq 0}\mathbb{E}(X_t^+) - \mathbb{E}(X_0) < \infty$. $\square$

**Exercice 14.** Let $(X_t,\ t \geq 0)$ be a martingale. If there exists $\xi \in L^1(\mathbb{P})$ such that for all $t \geq 0$, $\mathbb{E}(\xi \,|\, \mathscr{F}_t) = X_t$ a.s., we say that $(X_t,\ t \geq 0)$ is closed by $\xi$.

Prove that a right-continuous martingale is closed if and only if it is uniformly integrable.

**Solution.** If $X$ is closed by $\xi$, then $X_t = \mathbb{E}(\xi \,|\, \mathscr{F}_t)$ is uniformly integrable.

Conversely, we assume that $X$ is right-continuous and uniformly integrable. Then $X_t \to X_\infty$ a.s. and in $L^1$, with $X_t = \mathbb{E}(X_\infty \,|\, \mathscr{F}_t)$. By definition, this means $X$ is closed by $X_\infty$. $\square$

**Exercice 15. (Discrete backwards submartingales)** Let $(\mathscr{F}_n,\, n \leq 0)$ be a sequence of sub-$\sigma$-fields of $\mathscr{F}$ satisfying $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ for all $n \leq 0$. Let $(X_n,\, n \leq 0)$ be such that $\forall\, n$, $X_n$ is $\mathscr{F}_n$-measurable et integrable, and that $\mathbb{E}(X_{n+1} \,|\, \mathscr{F}_n) \geq X_n$ a.s. We call $(X_n,\, n \leq 0)$ a backward submartingale.

**(i)** Let $a < b$. Let $U_n(X;\, a,\, b)$ be the number of up-crossings along $[a,\, b]$ by $X_n, \cdots, X_{-1}$, $X_0$. Prove that $\mathbb{E}[U_n(X;\, a,\, b)] \leq \frac{\mathbb{E}[(X_0-a)^+]}{b-a}$.

**(ii)** Prove that $X_n \to X_{-\infty}$ a.s. when $n \to -\infty$.

**(iii)** Assume from now on that $\inf_{n\leq 0}\mathbb{E}(X_n) > -\infty$. Prove that $X_n \to X_{-\infty}$ in $L^1$.

Hint: Only uniform integrability needs proved. By considering $X_n - \mathbb{E}(X_0 \,|\, \mathscr{F}_n)$, you can argue that $X_n$ may be assumed to take values in $(-\infty,\, 0]$.

**(iv)** Prove that $X_{-\infty} \leq \mathbb{E}(X_0 \,|\, \mathscr{F}_{-\infty})$ a.s., where $\mathscr{F}_{-\infty} := \bigcap_{n\leq 0}\mathscr{F}_n$.

**(v) (P. Lévy)** Let $\xi$ be a real-valued random variable with $\mathbb{E}(|\xi|) < \infty$. Prove that $\mathbb{E}(\xi \,|\, \mathscr{F}_n) \to \mathbb{E}(\xi \,|\, \mathscr{F}_{-\infty})$ a.s. and in $L^1$, as $n \to -\infty$.

**Solution. (i)** It follows from the usual inequality for the number of up-crossings.

**(ii)** By (i) and the monotone convergence theorem, $\mathbb{E}[U_\infty(X;\, a,\, b)] \leq \frac{\mathbb{E}[(X_0-a)^+]}{b-a}$, where $U_\infty(X;\, a,\, b)$ denotes the number of up-crossings along the interval $[a,\, b]$ by $(X_n,\, n \leq 0)$. A fortiori, $U_\infty(X;\, a,\, b) < \infty$ a.s.; hence $\mathbb{P}(U_\infty(X;\, a,\, b) < \infty,\ \forall a < b \text{ rationals}) = 1$. This yields the a.s. existence of $\lim_{n\to-\infty} X_n$.

**(iii)** In view of a.s. convergence proved in (ii), it only remains to prove that $(X_n,\, n \leq 0)$ is uniformly integrable. Since $(\mathbb{E}[X_0 \,|\, \mathscr{F}_n],\ n \leq 0)$ is uniformly integrable, it suffices, for the proof of convergence in $L^1$, to verify that the submartingale $(X_n - \mathbb{E}[X_0 \,|\, \mathscr{F}_n],\ n \leq 0)$ is uniformly integrable. As such, we can assume, without loss of generality, that $X_n \leq 0$ for all $n \leq 0$.

When $n \to -\infty$, $\mathbb{E}(X_n) \to A = \inf_{n\leq 0}\mathbb{E}(X_n) \in ]-\infty, 0]$. Let $\varepsilon > 0$. There exists $N < \infty$ such that $\mathbb{E}(X_{-N}) - A \leq \varepsilon$, and a fortiori $\mathbb{E}(X_{-N}) - \mathbb{E}(X_n) \leq \varepsilon$, $\forall\, n \leq 0$. Let $a > 0$. We have, for $n \le -N$,

$$\begin{aligned}
\mathbb{E}[\,|X_n|\,\mathbf{1}_{\{|X_n|>a\}}\,] &= -\mathbb{E}[\,X_n\,\mathbf{1}_{\{X_n<-a\}}\,] \\
&= -\mathbb{E}(X_n) + \mathbb{E}[\,X_n\,\mathbf{1}_{\{X_n\ge -a\}}\,] \\
&\le -\mathbb{E}(X_n) + \mathbb{E}[\,X_{-N}\,\mathbf{1}_{\{X_n\ge -a\}}\,] \\
&= -\mathbb{E}(X_n) + \mathbb{E}(X_{-N}) - \mathbb{E}[\,X_{-N}\,\mathbf{1}_{\{X_n<-a\}}\,] \\
&\le \varepsilon + \mathbb{E}[\,|X_{-N}|\,\mathbf{1}_{\{|X_n|>a\}}\,].
\end{aligned}$$

By the Markov inequality, $\mathbb{P}(|X_n| > a) \le \frac{-\mathbb{E}(X_n)}{a} \le \frac{-A}{a} = \frac{|A|}{a}$. Hence we can choose $a$ so large that $\mathbb{E}[\,|X_{-N}|\,\mathbf{1}_{\{|X_n|>a\}}\,] \le \varepsilon$. Then

$$\sup_{n\le -N} \mathbb{E}[\,|X_n|\,\mathbf{1}_{\{|X_n|>a\}}\,] \le 2\varepsilon.$$

On the other hand, we can choose $a$ sufficiently large such that $\mathbb{E}[\,|X_n|\,\mathbf{1}_{\{|X_n|>a\}}\,] \le \varepsilon$ for $n = 0$, $-1, \cdots, -N$. Consequently, $(X_n,\, n \le 0)$ is uniformly integrable (and $\mathbb{E}(|X_{-\infty}|) < \infty$).

**(iv)** Since $X_n \le \mathbb{E}(X_0 \,|\, \mathscr{F}_n)$, we have, for all $A \in \mathscr{F}_{-\infty}$ ($A$ is, a fortiori, an element of $\mathscr{F}_n$),

$$\mathbb{E}[X_n\,\mathbf{1}_A] \le \mathbb{E}[X_0\,\mathbf{1}_A].$$

Since $X_n \to X_{-\infty}$ in $L^1$, by letting $n \to -\infty$, we get $\mathbb{E}[X_{-\infty}\,\mathbf{1}_A] \le \mathbb{E}[X_0\,\mathbf{1}_A]$. Since $X_{-\infty}$ is $\mathscr{F}_n$-measurable (for all $n \le 0$) hence ($\mathscr{F}_{-\infty}$)-measurable, this implies that $X_{-\infty} \le \mathbb{E}(X_0 \,|\, \mathscr{F}_{-\infty})$, a.s.

**(v)** Let $X_n := \mathbb{E}(\xi \,|\, \mathscr{F}_n)$, $n \le 0$, which is a backward martingale. By (ii) and (iii), $X_n \to X_{-\infty}$ a.s. and in $L^1$, where

$$X_{-\infty} = \mathbb{E}[X_0 \,|\, \mathscr{F}_{-\infty}] = \mathbb{E}[\mathbb{E}(\xi \,|\, \mathscr{F}_0) \,|\, \mathscr{F}_{-\infty}] = \mathbb{E}[\xi \,|\, \mathscr{F}_{-\infty}], \qquad \text{a.s.},$$

as desired. □

**Exercice 16.** Let $(X_t,\, t \ge 0)$ be a continuous and non-negative martingale. Let $T := \inf\{t \ge 0 :\, X_t = 0\}$ (with $\inf \varnothing := \infty$). Prove that a.s. on $\{T < \infty\}$, we have $X_t = 0$, $\forall t \ge T$.

**Solution.** Fix $n \ge 1$. We apply the optional sampling theorem to the uniformly integrable martingale $(X_{t\wedge n},\, t \ge 0)$ and to the pair of stopping times $T$ and $T + t$, to see that $\mathbb{E}(X_{(T+t)\wedge n} \,|\, \mathscr{F}_T) = X_{T\wedge n}$. Let $n \to \infty$. By the conditional Fatou's lemma, $\mathbb{E}(X_{T+t} \,|\, \mathscr{F}_T) \le X_T$, hence $\mathbb{E}(X_{T+t}\,\mathbf{1}_{\{T<\infty\}} \,|\, \mathscr{F}_T) \le X_T\,\mathbf{1}_{\{T<\infty\}} = 0$. This is possible only if $X_{T+t}\,\mathbf{1}_{\{T<\infty\}} = 0$ a.s., i.e., $X_{T+t} = 0$ a.s. on $\{T < \infty\}$.

Summarizing: a.s. on $\{T < \infty\}$, we have $X_{T+t} = 0$, $\forall t \in \mathbb{R}_+ \cap \mathbb{Q}$. The continuity of $X$ tells us that we can remove the restriction $t \in \mathbb{Q}$. □

**Exercice 17.** Let $(X_t,\ t \geq 0)$ be a right-continuous submartingale, and let $S$ and $T$ be bounded stopping times. Prove that

$$\mathbb{E}(X_T \,|\, \mathscr{F}_S) \geq X_{T\wedge S}, \qquad \text{a.s.}$$

**Solution.** We have

$$\begin{aligned}
\mathbb{E}[X_T \,|\, \mathscr{F}_S] &= \mathbb{E}[X_{T\wedge S}\,\mathbf{1}_{\{T\leq S\}} \,|\, \mathscr{F}_S] + \mathbb{E}[X_{T\vee S}\,\mathbf{1}_{\{T>S\}} \,|\, \mathscr{F}_S] \\
&= X_{T\wedge S}\,\mathbf{1}_{\{T\leq S\}} + \mathbf{1}_{\{T>S\}}\mathbb{E}[X_{T\vee S} \,|\, \mathscr{F}_S] \\
&\geq X_{T\wedge S}\,\mathbf{1}_{\{T\leq S\}} + \mathbf{1}_{\{T>S\}} X_S = X_{T\wedge S},
\end{aligned}$$

as desired. □

**Exercice 18.** Let $(X_t,\ t \geq 0)$ be a right-continuous martingale. Let $T$ be a stopping time.

**(i)** Prove that $(X_{T\wedge t},\ t \geq 0)$ is a right-continuous martingale.

**(ii)** Prove that if $(X_t,\ t \geq 0)$ is uniformly integrable, then so is $(X_{T\wedge t},\ t \geq 0)$.

**Solution.** **(i)** The right-continuity of the trajectories is obvious. Let us prove that $(X_{T\wedge t},\ t \geq 0)$ is a martingale with respect to $(\mathscr{F}_t)$.

For $t \geq 0$, it is clear that $\mathbb{E}(|X_{T\wedge t}|) < \infty$ (a consequence of the optional sampling theorem) and that $X_{T\wedge t}$ is $\mathscr{F}_t$-measurable (being $\mathscr{F}_{T\wedge t}$-measurable). Let $t > s \geq 0$. Applying the previous exercise gives $\mathbb{E}(X_{T\wedge t} \,|\, \mathscr{F}_s) = X_{(T\wedge t)\wedge s}$, which is $X_{T\wedge s}$.

**(ii)** If $(X_t,\ t \geq 0)$ is uniformly integrable, then the optional sampling theorem says that $X_{T\wedge t} = \mathbb{E}(X_\infty \,|\, \mathscr{F}_{T\wedge t})$, which yields the uniform integrability of $(X_{T\wedge t},\ t \geq 0)$ by recalling that for any integrable random variable $\xi$, $(\mathbb{E}(\xi \,|\, \mathscr{G}),\ \mathscr{G} \subset \mathscr{F}\ \sigma\text{-field})$ is uniformly integrable. □

**Exercice 19.** Let $(X_t,\, t \geq 0)$ be a non-negative and right-continuous *super*martingale. Recall that $X_t \to X_\infty$ a.s. in this case. Prove that if $\mathbb{E}(X_\infty) = \mathbb{E}(X_0)$, then $(X_t,\, t \geq 0)$ is a uniformly integrable martingale.

**Solution.** By the conditional Fatou's lemma, $\mathbb{E}(X_\infty \,|\, \mathscr{F}_t) \leq X_t$ a.s. Taking expectation on both sides gives $\mathbb{E}(X_\infty) \leq \mathbb{E}(X_t)$ which is $\leq \mathbb{E}(X_0)$ because $X$ is a *super*martingale. By assumption, $\mathbb{E}(X_\infty) = \mathbb{E}(X_0)$, which is possible only if $\mathbb{E}(X_\infty \,|\, \mathscr{F}_t) = X_t$ a.s., i.e., only if is a uniformly integrable martingale. □

**Exercice 20.** Let $X = (X_t,\, t \geq 0)$ be a non-negative continuous submartingale. We write $S_t := \sup_{s\in[0,\,t]} X_s$, $t \geq 0$.

**(i)** Prove that for all $\lambda > 0$ and all $t \geq 0$, $\lambda\,\mathbb{P}(S_t > 2\lambda) \leq \mathbb{E}[\,X_t\,\mathbf{1}_{\{X_t>\lambda\}}]$.

We can use the following inequality: for all $a > 0$, $a\,\mathbb{P}(S_t > a) \leq \mathbb{E}[\,X_t\,\mathbf{1}_{\{S_t>a\}}]$ (this follows from the maximal inequality for discrete-time submartingales and the continuity of the trajectories).

**(ii)** Prove that $\frac{1}{2}\,\mathbb{E}[S_t] \le 1 + \mathbb{E}[\, X_t \log_+ X_t\,]$, wher $\log_+ x := \log \max(x, 1)$.

**(iii)** Let $(Y_t,\, t \ge 0)$ be a continuous and uniformly integrable martingale. We assume that $\mathbb{E}[\,|Y_\infty|\, \log_+ |Y_\infty|\,] < \infty$. Prove that $\sup_{t\ge 0} |Y_t|$ is integrable.

**Solution.** **(i)** For all $a > 0$, $a\,\mathbb{P}(S_t \ge a) \le \mathbb{E}[\, X_t\, \mathbf{1}_{\{S_t \ge a\}}]$. So

$$\begin{aligned} 2\lambda\,\mathbb{P}(S_t \ge 2\lambda) &\le \mathbb{E}[\, X_t\, \mathbf{1}_{\{S_t\ge 2\lambda\}}] \le \mathbb{E}[\, X_t\, \mathbf{1}_{\{X_t > \lambda\}}] + \mathbb{E}[\, X_t\, \mathbf{1}_{\{X_t \le \lambda,\, S_t \ge 2\lambda\}}] \\ &\le \mathbb{E}[\, X_t\, \mathbf{1}_{\{X_t > \lambda\}}] + \lambda\,\mathbb{P}(S_t \ge 2\lambda), \end{aligned}$$

from which the desired inequality follows.

**(ii)** We have

$$\begin{aligned} \frac{1}{2}\,\mathbb{E}[S_t] &= \int_0^\infty \mathbb{P}(S_t \ge 2\lambda)\,\mathrm{d}\lambda \le 1 + \int_1^\infty \mathbb{P}(S_t \ge 2\lambda)\,\mathrm{d}\lambda \\ &\le 1 + \int_1^\infty \mathbb{E}[\,\lambda^{-1} X_t\, \mathbf{1}_{\{X_t > \lambda\}}]\,\mathrm{d}\lambda. \end{aligned}$$

By Fubini's theorem, the last integral equals $\mathbb{E}[\,\int_1^{X_t} \lambda^{-1} X_t\, \mathbf{1}_{\{X_t \ge 1\}}\,\mathrm{d}\lambda\,] = \mathbb{E}[\, X_t\, \log_+ X_t\,]$. We obtain the desired result.

**(iii)** By assumption, $Y_t = \mathbb{E}(Y_\infty \,|\, \mathscr{F}_t)$. Since $x \mapsto |x| \log_+ |x| =: \varphi(x)$ is convex, Jensen's inequality says that $\varphi(Y_t) \le \mathbb{E}[\,\varphi(Y_\infty) \,|\, \mathscr{F}_t\,]$; hence $\sup_{t\ge 0} \mathbb{E}[\,\varphi(Y_t)\,] \le \mathbb{E}[\,\varphi(Y_\infty)\,] < \infty$. By (ii) (applied to $X_t := |Y_t|$, $t \ge 0$, which is a non-negative submartingale), $\frac{1}{2}\,\mathbb{E}(\sup_{s\in[0,\,t]} |Y_s|) \le 1 + \mathbb{E}[\,\varphi(Y_t)\,] \le 1 + \mathbb{E}[\,\varphi(Y_\infty)\,]$. It follows from the monotone convergence theorem that $\mathbb{E}(\sup_{t\ge 0} |Y_t|) \le 2 + 2\,\mathbb{E}[\,\varphi(Y_\infty)\,] < \infty$. $\square$

**Exercice 21.** For any martingale $X := (X_t,\, t \ge 0)$, we say that it is square-integrable if $\mathbb{E}(X_t^2) < \infty$, $\forall t \ge 0$, and that it is bounded in $L^2$ if $\sup_{t\ge 0} \mathbb{E}(X_t^2) < \infty$.

**(i)** Prove that if $X$ is a right-continuous martingale and is bounded in $L^2$, then it is uniformly integrable, with $\mathbb{E}(\sup_{t\ge 0} X_t^2) < \infty$.

**(ii)** Let $X$ and $Y$ be right-continuous martingales that are bounded in $L^2$. Let $S$ and $T$ be stopping times. Prove that $\mathbb{E}(X_S Y_T) = \mathbb{E}(X_{S\wedge T} Y_{S\wedge T})$.

**(iii)** Let $X$ and $Y$ be right-continuous and square-integrable martingales. Let $S$ and $T$ be bounded stopping times. Prove that $\mathbb{E}(X_S Y_T) = \mathbb{E}(X_{S\wedge T} Y_{S\wedge T})$.

**Solution.** **(i)** That $\mathbb{E}(\sup_{t\ge 0} X_t^2) < \infty$ is a consequence of Doob's inequality. In particular, $\mathbb{E}(\sup_{t\ge 0} |X_t|) < \infty$ ; a fortiori, $X$ is uniformly integrable.

**(ii)** Since $|X_S| \le \sup_{t\ge 0} |X_t|$, we have $\mathbb{E}(X_S^2) < \infty$. Similarly, $\mathbb{E}(Y_T^2) < \infty$. Hence by the Cauchy–Schwarz inequality, $\mathbb{E}(|X_S Y_T|) < \infty$.

Applying the optional sampling theorem to the uniformly integral martingale $Y$ gives

$$\begin{aligned} \mathbb{E}(X_S Y_T\, \mathbf{1}_{\{S \le T\}} \,|\, \mathscr{F}_S) &= X_S\, \mathbf{1}_{\{S\le T\}}\, \mathbb{E}(Y_{T\vee S} \,|\, \mathscr{F}_S) \\ &= X_S\, \mathbf{1}_{\{S \le T\}}\, Y_S \\ &= X_{S\wedge T} Y_{S\wedge T}\, \mathbf{1}_{\{S\le T\}}, \end{aligned}$$

from which it follows that

$$\mathbb{E}(X_S Y_T\, \mathbf{1}_{\{S\le T\}}) = \mathbb{E}(X_{S\wedge T} Y_{S\wedge T}\, \mathbf{1}_{\{S\le T\}}).$$

On the other hand, $X_S Y_T\, \mathbf{1}_{\{S>T\}} = X_{S\wedge T} Y_{S\wedge T}\, \mathbf{1}_{\{S>T\}}$. Hence

$$\mathbb{E}(X_S Y_T\, \mathbf{1}_{\{S>T\}}) = \mathbb{E}(X_{S\wedge T} Y_{S\wedge T}\, \mathbf{1}_{\{S>T\}}).$$

Consequently, $\mathbb{E}(\,X_S Y_T) = \mathbb{E}(X_{S\wedge T} Y_{S\wedge T})$.

**(iii)** The same proof as in (ii), except in two places:

- to justify the integrability of $X_S Y_T$, let $a > 0$ be such that $S \le a$, then $\mathbb{E}(X_S^2) \le \mathbb{E}(\sup_{u\in[0,\,a]} X_u^2) \le 4\mathbb{E}(X_a^2) < \infty$, and similarly, $\mathbb{E}(Y_T^2) < \infty$, so $\mathbb{E}(|X_S Y_T|) < \infty$ ;
- to justify $\mathbb{E}(Y_{T\vee S}\,|\,\mathscr{F}_S) = Y_S$, we apply the optional sampling theorem to $Y$ and to the pair of *bounded* stopping times $T \vee S$ and $S$. □

**Exercice 22.** Let $S \le T$ be bounded stopping times. Prove that $\mathbb{E}[(B_T - B_S)^2] = \mathbb{E}(B_T^2) - \mathbb{E}(B_S^2) = \mathbb{E}(T - S)$.

**Solution.** Since $S$ and $T$ are bounded, Doob's inequality implies that $\mathbb{E}(B_s^2) < \infty$ and that $\mathbb{E}(B_T^2) < \infty$. We have

$$\begin{aligned}\mathbb{E}[(B_T - B_S)^2] &= \mathbb{E}(B_S^2) + \mathbb{E}(B_T^2) - 2\mathbb{E}[\,\mathbb{E}(B_S B_T\,|\,\mathscr{F}_S)\,]\\ &= \mathbb{E}(B_S^2) + \mathbb{E}(B_T^2) - 2\mathbb{E}[\,B_S\mathbb{E}(B_T\,|\,\mathscr{F}_S)\,],\end{aligned}$$

because $B_S$ is $\mathscr{F}_S$)-measurable. Applying the optional sample theorem to $B$ and to the pair of *bounded* stopping times $S$ and $T$ yields $\mathbb{E}(B_T\,|\,\mathscr{F}_S) = B_S$, which, in turn, implies that

$$\mathbb{E}[(B_T - B_S)^2] = \mathbb{E}(B_S^2) + \mathbb{E}(B_T^2) - 2\mathbb{E}[\,B_S^2] = \mathbb{E}(B_T^2) - \mathbb{E}(B_S^2).$$

We now apply the optional sample theorem to $(B_t^2 - t,\ t \ge 0)$ and to the pair of *bounded* stopping times $T$ and 0, to see that $\mathbb{E}(B_T^2 - T) = 0$; thus $\mathbb{E}(B_T^2) = \mathbb{E}(T)$. Similarly, $\mathbb{E}(B_S^2) = \mathbb{E}(S)$. Hence $\mathbb{E}(B_T^2) - \mathbb{E}(B_S^2) = \mathbb{E}(T - S)$. □

**Exercice 23.** **(i)** Let $(X_t,\, t \ge 0)$ be a non-negative and continuous martingale such that $X_t \to 0$, a.s. $(t \to \infty)$. Prove that for all $x > 0$, $\mathbb{P}(\sup_{t\ge 0} X_t \ge x\,|\,\mathscr{F}_0) = 1 \wedge \frac{X_0}{x}$, a.s.

**(ii)** Let $B$ be Brownian motion. Determine the law of $\sup_{t\ge 0}(B_t - t)$.

**Solution.** **(i)** Let $T := \inf\{t \ge 0 :\ X_t \ge x\}$ which is a stopping time. Clearly, $(X_{t\wedge T},\ t \ge 0)$ is a continuous martingale, and is uniformly integrable (because $|X_{t\wedge T}| \le x + X_0$), closed by $X_T$ (with the notation $X_\infty := 0$). By the optional sampling theorem, $\mathbb{E}(X_T\,|\,\mathscr{F}_0) = X_0$. We observe that

$$\begin{aligned}\mathbb{E}[X_T\,|\,\mathscr{F}_0] &= \mathbb{E}[X_T\,\mathbf{1}_{\{T<\infty\}}\,|\,\mathscr{F}_0] + \mathbb{E}[X_\infty\,\mathbf{1}_{\{T=\infty\}}\,|\,\mathscr{F}_0]\\ &= \mathbb{E}[(x\vee X_0)\,\mathbf{1}_{\{T<\infty\}}\,|\,\mathscr{F}_0]\\ &= (x\vee X_0)\,\mathbb{P}[T<\infty\,|\,\mathscr{F}_0],\end{aligned}$$

which yields

$$\mathbb{P}[T < \infty \mid \mathscr{F}_0] = \frac{X_0}{x \vee X_0} = 1 \wedge \frac{X_0}{x}.$$

It suffices then to remark that $\{T < \infty\} = \{\sup_{t\geq 0} X_t \geq x\}$.

**(ii)** Let $X_t := \mathrm{e}^{2(B_t - t)}$ which is a continuous martingale. Since a.s. $\frac{B_t}{t} \to 0$ $(t \to \infty)$, we have $B_t - t = (\frac{B_t}{t} - 1)t \to -\infty$, a.s., and thus $X_t \to 0$ a.s. By (i), $\mathbb{P}\{\sup_{t\geq 0} X_t \geq x\} = 1 \wedge \frac{1}{x}$, $x > 0$, which means $\mathbb{P}\{\sup_{t\geq 0}(B_t - t) \geq a\} = \mathrm{e}^{-2a}$, $a > 0$. In other words, $\sup_{t\geq 0}(B_t - t)$ has the exponential law of parameter 2 (i.e., with mean $\frac{1}{2}$). $\square$

**Exercice 24.** Let $\gamma \neq 0$, $a > 0$ and $b > 0$. Let $T_x := \inf\{t > 0 : B_t + \gamma t = x\}$, $x = -a$ or $b$. Compute $\mathbb{P}(T_{-a} > T_b)$.

Hint: You can use the martingale $(\mathrm{e}^{-2\gamma(B_t + \gamma t)}, t \geq 0)$.

**Solution.** Consider the martingale $(X_t := \mathrm{e}^{-2\gamma B_t - 2\gamma^2 t}, t \geq 0)$. Since $\mathrm{e}^{-2\gamma B_{t\wedge T_{a,b}} - 2\gamma^2(t\wedge T_{a,b})} \leq \mathrm{e}^{2|\gamma|(a+b)}$, we see that $(X_{T_{a,b}\wedge t}, t \geq 0)$ is a continuous and bounded martingale, closed by $X_{T_{a,b}}$. Applying the optional sample theorem to this uniformly integrable martingale, we obtain:

$$\begin{aligned}
1 &= \mathbb{E}[\mathrm{e}^{-2\gamma B_{T_{a,b}} - 2\gamma^2 T_{a,b}}] \\
&= \mathbb{E}[\mathrm{e}^{2\gamma a}\, \mathbf{1}_{\{T_{-a} < T_b\}}] + \mathbb{E}[\mathrm{e}^{-2\gamma b}\, \mathbf{1}_{\{T_{-a} > T_b\}}] \\
&= \mathrm{e}^{2\gamma a} - \mathrm{e}^{2\gamma a}\, \mathbb{P}(T_{-a} > T_b) + \mathrm{e}^{-2\gamma b}\, \mathbb{P}(T_{-a} > T_b),
\end{aligned}$$

which yields[6] $\mathbb{P}(T_{-a} > T_b) = \frac{\mathrm{e}^{2\gamma a} - 1}{\mathrm{e}^{2\gamma a} - \mathrm{e}^{-2\gamma b}}$. $\square$

**Exercice 25. (First Wald identity)** Let $T$ be a stopping time such that $\mathbb{E}(T) < \infty$. Prove that $B_T$ is integrable and that $\mathbb{E}(B_T) = 0$.

**Solution.** Both $(B_{t\wedge T}, t \geq 0)$ and $(B_{t\wedge T}^2 - t \wedge T, t \geq 0)$ are continuous martingales, with $\mathbb{E}(B_{t\wedge T}^2) = \mathbb{E}(t \wedge T) \leq \mathbb{E}(T)$; hence $\sup_t \mathbb{E}(B_{t\wedge T}^2) \leq \mathbb{E}(T) < \infty$. Consequently, $(B_{t\wedge T}, t \geq 0)$ is a uniformly integrable martingale, closed by $B_T$ (in particular, $B_T$ is integrable). Applying the optional sampling theorem to this uniformly integrable martingale yields $\mathbb{E}(B_T) = \mathbb{E}(B_{0\wedge T}) = 0$. $\square$

**Exercice 26. (Second Wald identity)** Let $T$ be a stopping time such that $\mathbb{E}(T) < \infty$. Prove that $B_T$ has a finite second moment and that $\mathbb{E}(B_T^2) = \mathbb{E}(T)$.

**Solution.** By Doob's inequality,

$$\mathbb{E}\left[\sup_{t\geq 0} B_{t\wedge T}^2\right] \leq 4 \sup_{t\geq 0} \mathbb{E}\left[B_{t\wedge T}^2\right] \leq 4\mathbb{E}(T) < \infty,$$

[6] Letting $a \to \infty$, we see that $\mathbb{P}(T_b < \infty)$ is 1 if $\gamma > 0$, and is $\mathrm{e}^{2\gamma b}$ if $\gamma < 0$, which is in agreement with the previous exercise, because $\mathbb{P}(T_b < \infty) = \mathbb{P}\{\sup_{t\geq 0}(B_t + \gamma t) \geq b\}$.

so $(B_{t\wedge T}^2,\, t \geq 0)$ is uniformly integrable. Since $(t \wedge T,\, t \geq 0)$ is also uniformly integrable (being bounded by $T$), $(B_{t\wedge T}^2 - t \wedge T,\, t \geq 0)$ is a continuous and uniformly integrable martingale, closed by $B_T^2 - T$ (in particular, $B_T$ has a finite second moment). Applying the optional sampling theorem to this uniformly integrable martingale yields $\mathbb{E}(B_T^2 - T) = 0$. In other words, $\mathbb{E}(B_T^2) = \mathbb{E}(T)$. $\square$