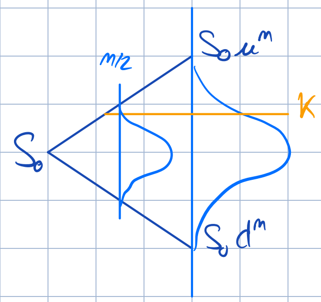
TD 4 : BIS

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t$$

$$u = \exp\left(\sigma\sqrt{\tfrac{T}{m}}\right)$$
$$d = \exp\left(-\sigma\sqrt{\tfrac{T}{m}}\right)$$



option $(S_T - K)\mathbb{1}_{\{S_T > K\}}$

Sur l'arbre il faut aller assez loin car potentiellement on tarife mal.

Soient S_0, σ, T et m ↳ nombre de pas

$$u = \exp\left(\sigma\sqrt{\tfrac{T}{m}}\right) \qquad R = e^{r\frac{T}{m}}$$
$$d = \exp\left(-\sigma\sqrt{\tfrac{T}{m}}\right) = 1/u$$

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{S_T}{e^{rT}} \middle| \mathcal{F}_t\right] = \frac{S_t}{e^{rt}}$$ martingale (proba risque neutre)

$$S_0 = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_T}{e^{rT}} \middle| \mathcal{F}_0\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_1}{R}\right] = \frac{1}{R}(p u S_0 + (1-p) S_0 d)$$

$$\Rightarrow p = \frac{R-d}{u-d}$$

$$C_0 = \mathbb{E}_{\mathbb{Q}}\left[\max(S_T - K, 0)e^{-rT}\right]$$ c'est ce que l'on veut déterminer

Ex : Chemin avec j u et $m-j$ d $\Rightarrow S_T = S_0 u^j d^{m-j}$

$$\Rightarrow C_0 = \sum_{j=0}^{m} \binom{m}{j} p^j (1-p)^{m-j} \max(S_0 u^j d^{m-j} - K, 0)\, e^{-rT}$$

On veut j tel que : $S_0 u^j d^{m-j} > K$

$$\iff u^{-j} d^{j-m} < \frac{S_0}{K}$$
$$\iff -j\ln(u) - (m-j)\ln(d) < \ln\left(\frac{S_0}{K}\right)$$
$$\iff -j\sigma\sqrt{\tfrac{T}{m}} + (m-j)\sigma\sqrt{\tfrac{T}{m}} < \ln\left(\frac{S_0}{K}\right)$$
$$\iff -2j\sigma\sqrt{\tfrac{T}{m}} + m\sigma\sqrt{\tfrac{T}{m}} < \ln\left(\frac{S_0}{K}\right)$$
$$\iff j > \frac{m}{2} - \frac{\ln\left(\frac{S_0}{K}\right)}{2\sigma\sqrt{T/m}} = \alpha$$

$$\Rightarrow C_0 = e^{-rT}\sum_{j>\alpha}\binom{m}{j} p^j (1-p)^{m-j}(S_0 u^j d^{m-j} - K)$$

$$u_1 = \sum_{j>\alpha}\binom{m}{j} p^j (1-p)^{m-j} \qquad u_2 = \sum_{j>\alpha}\binom{m}{j} p^j (1-p)^{m-j} u^j d^{m-j}$$

$$\Rightarrow C_0 = e^{-rT}(S_0 u_2 - K u_1)$$

On remarque que $u_1 = \mathbb{P}(X > \alpha)$ où $X \sim \mathcal{B}(n,p) \xrightarrow{n\to\infty} \mathcal{N}(np, np(1-p))$

$$\text{et } \mathbb{P}\left(\frac{X-np}{\sqrt{np(1-p)}} > \frac{\alpha-np}{\sqrt{np(1-p)}}\right) = \mathbb{P}\left(-\frac{X-np}{\sqrt{np(1-p)}} \leq \frac{np-\alpha}{\sqrt{np(1-p)}}\right)$$

$$= \mathbb{P}\left(Y \leq \frac{np-\alpha}{\sqrt{np(1-p)}}\right) \quad \text{où } Y \sim \mathcal{N}(0,1).$$

Donc $u_1 = \mathcal{N}\left(\frac{np-\alpha}{\sqrt{np(1-p)}}\right)$ ← f.d.r d'une $\mathcal{N}(0,1)$.

$$\frac{np-\alpha}{\sqrt{np(1-p)}} = \frac{np - \frac{n}{2} + \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}}{\sqrt{np(1-p)}} = \frac{\sqrt{n}(p-1/2)}{\sqrt{p(1-p)}} + \frac{\ln(S_0/K)}{2\sigma\sqrt{T}\sqrt{p(1-p)}}$$

Montrer que $\begin{cases} p(1-p) \xrightarrow[n\to\infty]{} 1/4 \\ \sqrt{n}(p-\frac{1}{2}) \xrightarrow[n\to\infty]{} \frac{(r-\sigma^2/2)\sqrt{T}}{2\sigma} \end{cases}$ $\qquad e^x = \sum_0^\infty \frac{x^k}{k!}$

$$p = \frac{e^{rT/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}} \Rightarrow p(1-p) = \frac{1}{(e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}})^2}\left(e^{rT/n} - e^{-\sigma\sqrt{T/n}}\right)\left(e^{\sigma\sqrt{T/n}} - e^{rT/n}\right)$$

1) $n \in \mathcal{V}(\infty)$, $p(1-p) = \frac{1}{(1+\sigma\sqrt{T/n} - 1 + \sigma\sqrt{T/n})^2}\left(1 - \left(1 - \sigma\sqrt{\frac{T}{n}}\right)\right)\left(1 + \sigma\sqrt{T/n} - 1\right)$

$$= \frac{1}{4\sigma^2 T/n}\left(\sigma\sqrt{T/n}\right)\left(\sigma\sqrt{T/n}\right) = 1/4$$

2) $\sqrt{n}\left(\frac{1 + r\frac{T}{n} - \left(1 - \sigma\sqrt{\frac{T}{n}} + \sigma^2\frac{T}{n}\right)}{1 + \sigma\sqrt{\frac{T}{n}} - 1 + \sigma\sqrt{\frac{T}{n}}} - \frac{1}{2}\right)_{+o(\dots)} = \sqrt{n}\left(\frac{r\frac{T}{n} + \sigma\sqrt{\frac{T}{n}} - \sigma^2\frac{T}{n}}{2\sigma\sqrt{T/n}} - \frac{1}{2}\right)$

n au voisinage de ∞

$$= \frac{r\sqrt{T}}{2\sigma} - \frac{\sigma^2\sqrt{T}}{2\sigma}$$

$$= \frac{(r - \sigma^2/2)\sqrt{T}}{2\sigma}$$

Donc $\frac{np-\alpha}{\sqrt{np(1-p)}} \sim \frac{\ln(S_0/K)}{\sigma\sqrt{T}} + \frac{(r - \frac{\sigma^2}{2})\sqrt{T}}{2\sigma}$

Donc $u_1 = \mathcal{N}\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S_0}{K}\right) + (r - \frac{\sigma^2}{2})T\right)\right)$

$$u_2 = \sum_{j > \alpha} \binom{m}{j} p^j (1-p)^{m-j} u^j d^{m-j}$$
$$= \sum_{j>\alpha} \binom{m}{j} (pu)^j ((1-p)d)^{m-j}$$

Posons $p^* = \frac{pu}{pu+(1-p)d}$ et $1-p^* = \frac{(1-p)d}{pu+(1-p)d}$

$$\Longrightarrow u_2 = \underbrace{(pu+(1-p)d)}_{R}{}^m \sum_{j>\alpha} \binom{m}{j} p^{*j} (1-p^*)^{m-j}$$

$S_0 = \mathbb{E}_Q[\frac{S_1}{R}] = \frac{puS_0 + (1-p)S_0 d}{R} \Rightarrow R = pu + (1-p)d$

$$p^{*j} = \frac{(pu)^j}{(pu+(1-p)d)^j} \quad \text{et} \quad (1-p^*)^{m-j} = \frac{((1-p)d)^{m-j}}{(pu+(1-p)d)^j}$$

$$\Longrightarrow u_2 = e^{rT} \mathcal{N}\left(\frac{mp^*-\alpha}{\sqrt{mp^*(1-p^*)}}\right)$$

$$\frac{mp^*-\alpha}{\sqrt{mp^*(1-p^*)}} = \frac{\ln(S_0/K)}{2\sigma\sqrt{T}\sqrt{p^*(1-p^*)}} + \frac{\sqrt{m}(p^*-1/2)}{\sqrt{p^*(1-p^*)}}$$

Montrer que $p^*(1-p^*) \xrightarrow[m\to\infty]{} 1/4$

$\sqrt{m}(p^* - 1/2) \xrightarrow[m\to\infty]{} \frac{(r + \sigma^2/2)\sqrt{T}}{2\sigma}$

1) $p^* = \frac{pu}{pu+(1-p)d} = \frac{pe^{\sigma\sqrt{T/m}}}{pe^{\sigma\sqrt{T/m}} + (1-p)e^{-\sigma\sqrt{T/m}}} \Rightarrow p^*(1-p^*) = \frac{1}{(pe^{\sigma\sqrt{T/m}}+(1-p)e^{-\sigma\sqrt{T/m}})^2} pe^{\sigma\sqrt{T/m}}(1-p)e^{-\sigma\sqrt{T/m}}$

Donc $p^*(1-p)^* = \frac{p(1-p)}{(pe^{\sigma\sqrt{T/m}}+(1-p)e^{-\sigma\sqrt{T/m}})^2} \underset{\text{au } \mathcal{V}(\infty)}{=} \frac{p(1-p)}{(p\times 1 + (1-p)\times 1)} = p(1-p) \xrightarrow[m\to\infty]{} 1/4.$

2) $$\sqrt{m}(p^* - 1/2) = \sqrt{m}\left[\frac{(e^{r T/m} - e^{-\sigma\sqrt{T/m}})e^{\sigma\sqrt{T/m}}}{(e^{\sigma\sqrt{T/m}} - e^{-\sigma\sqrt{T/m}})e^{rT/m}} - 1/2\right]$$
$$= \sqrt{m}\left[\frac{(e^{\sigma\sqrt{T/m}} - e^{-rT/m})}{(e^{\sigma\sqrt{T/m}} - e^{-\sigma\sqrt{T/m}})} - 1/2\right]$$
$$\underset{m \in \mathcal{V}(\infty)}{=} \sqrt{m}\left[\frac{(1+\sigma\sqrt{T/m} + \frac{\sigma^2}{2}T/m - (1 - rT/m))}{(1+\sigma\sqrt{T/m} - (1-\sigma\sqrt{T/m}))} - 1/2\right]$$
$$= \sqrt{m}\left[\frac{rT/m + \sigma\sqrt{T/m} + \frac{\sigma^2}{2}T/m}{2\sigma\sqrt{T/m}} - 1/2\right]$$
$$= \sqrt{m}\left[\frac{r\sqrt{T/m} + \sigma + \frac{\sigma^2}{2}\sqrt{T/m} - \sigma}{2\sigma}\right] = \frac{(r+\frac{\sigma^2}{2})\sqrt{T}}{2\sigma}$$

D'où $\sqrt{m}(p^* - 1/2) \xrightarrow[m\to\infty]{} \frac{(r+\frac{\sigma^2}{2})\sqrt{T}}{2\sigma}$

Donc $u_2 \xrightarrow[m\to\infty]{} e^{rT}\mathcal{N}\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T\right)\right)$

$$C_0 = e^{-rT}(S_0 u_2 - K \underset{\substack{\uparrow\\ u_1(\infty)}}{u_1}) = S_0\,\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)$$

avec $\begin{cases} d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T\right) \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases}$

$dS_t = rS_t\,dt + \sigma S_t\,dW_t$ Sous $\mathbb{Q}$

$S_t = S_0 \exp\left(\left(r-\frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\,X\right)$ $\quad X \sim \mathcal{N}(0,1)$

$$C_0 = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}(S_T-K)\mathbb{1}_{\{S_T>K\}}\right] = e^{-rT}\,\mathbb{E}_{\mathbb{Q}}\left[S_T\,\mathbb{1}_{\{S_T>K\}}\right] - Ke^{-rT}\,\mathbb{Q}(S_T>K)$$

$$= e^{-rT}\,\mathbb{E}_{\mathbb{Q}}\left[S_0\exp\left(\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T\right)\mathbb{1}_{\{S_T>K\}}\right] - e^{-rT}K\,\mathbb{Q}(S_T>K)$$

$$= S_0\,\underbrace{\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\frac{\sigma^2}{2}T+\sigma W_T\right)\mathbb{1}_{\{S_T>K\}}\right]}_{\mathcal{N}(d_1)} - e^{-rT}K\,\underbrace{\mathbb{Q}(S_T>K)}_{\mathcal{N}(d_2)}$$

$$\mathbb{Q}\left(S_0\exp\left(\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}X\right)>K\right)$$
$$= \mathbb{Q}\left(X > \frac{\ln(K/S_0) - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$
$$= \mathbb{Q}\left(X < \frac{\ln(S_0/K) + \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$\ln\left(\frac{S_t}{S_0}\right) = \left(r-\frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\,X \sim \mathcal{N}\left(\left(r-\frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$$