

TD 2

Exercice 1

$$X = \frac{U}{a} + \frac{V}{b} \qquad Y = \frac{U}{a} - \frac{V}{b}$$

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}\left(\frac{U}{a} + \frac{V}{b}, \frac{U}{a} - \frac{V}{b}\right)$$

$$= \frac{1}{a^2} \mathrm{Var}(U) + 2\mathrm{cov}\left(\frac{V}{b}, \frac{U}{a}\right) - \frac{1}{b^2} \mathrm{Var}(V)$$

$$= \frac{2}{ab} \underbrace{\mathrm{Cov}(V,U)}_{=0}$$

0 car (U,V) soit gaussien

Donc $X \perp\!\!\!\perp Y$

D'où $F_{X,Y}(x,y) = C(F_X(x), F_Y(y)) = F_X(x) F_Y(y)$ $\quad (C(u,v) = uv)$

Exercice 2

cf. Cours

Exercice 3

(1) a. $F_{X_2}(x_2) = \mathbb{P}(X_2 \leq x_2)$

$$= \mathbb{P}(X_1 + X_3 \leq x_2)$$

$$= \int_0^{x_2} \mathbb{P}(X_1 + X_3 \leq x_2 \mid X_1 = x_1)\, f_{X_1}(x_1)\, dx_1$$

$$= \int_0^{x_2} \overbrace{\left(1 - e^{-2(x_2 - x_1)}\right)}^{= F_{X_3}(x_2 - x_1)} e^{-x_1}\, dx_1$$

$$= 1 - e^{-x_2} - e^{-2x_2} \int_0^{x_2} e^{x_1}\, dx_1$$

$$= 1 - 2e^{-x} + e^{-2x} = \left(1 - e^{-x_2}\right)^2$$

b. $F_{(X_1,X_2)}(x_1, x_2) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2)$

$$= \int_0^{x_1 \wedge x_2} \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2 \mid X_1 = a)\, f_{X_1}(a)\, da$$

$$= \int_0^{x_1 \wedge x_2} \mathbb{P}(a \leq x_1, X_3 \leq x_3 - a)\, f_{X_1}(a)\, da$$

$$= \int_0^{x_1 \wedge x_2} \mathbb{P}(X_3 \leq x_2 - a)\, f_{X_1}(a)\, da$$

$$= \int_0^{x_1 \wedge x_2} \left(1 - e^{-2x_2 - 2a}\right) e^{-a}\, da = \dots = 1 - e^{-(x_1 \wedge x_2)} - 2e^{-2x_2} - e^{(x_1 \wedge x_2) - 2a}$$

(c). • $F_{X_1}^{-1}(u) = -\ln(1-u)$

$F_{X_2}^{-1}(v) = -\ln(1-\sqrt{v})$

• $\min(F_{X_1}^{-1}(u), F_{X_2}^{-1}(v)) = -\ln(1-\min(u,\sqrt{v}))$

• $C(u,v) = (1-\sqrt{v})^2 + \min(u,\sqrt{v}) - \dfrac{(1-\sqrt{v})^2}{1-\min(u,\sqrt{v})}$

(2) 4. $\rho(X,Y) = 12 \iint_{[0,1]^2} C(u,v)\,du\,dv - 3 = 12 I - 3$

On enlève le min en scindant le domaine de définition

$$I = \int_0^1 \int_0^{u^2} (1-\sqrt{v})^2 + \sqrt{v} - \frac{(1-\sqrt{v})^2}{1-\sqrt{v}}\,dv\,du + \int_0^1 \int_{u^2}^1 (1-\sqrt{v})^2 + u - \frac{(1-\sqrt{v})^2}{1-u}\,dv\,du$$

$x = \sqrt{v} \Rightarrow v = x^2$

$dv = 2x\,dx$

$$= \int_0^1 \int_0^{u^2} v\,dv\,du + \int_0^1 \int_u^1 (1-x)^2 + u - \frac{(1-x)^2}{1-u}\,2x\,dx\,du$$

$$= \frac{1}{10} + \int_0^1 \int_0^1 2x - 4x^2 + 2x^3 + u - \frac{1}{1-u}(2x - 4x + 2x^3)\,dx\,du = \frac{59}{180}$$

$$\boxed{\rho(X,Y) = \frac{14}{15}}$$

2)b. C i) On note $A = (X_1 - X_1')(X_2 - X_2')$

$$\tau = \mathbb{P}(A > 0) - \mathbb{P}(A < 0)$$

$$= \mathbb{P}(A > 0) - 1 + \mathbb{P}(A > 0$$

$$= 2\mathbb{P}(A > 0) - 1$$

- $\mathbb{P}(A > 0) = \mathbb{P}\Big[\big((X_1 - X_1' > 0) \cap (X_2 - X_2' > 0)\big) \cup \big((X_1 - X_1' < 0) \cap (X_2 - X_2' < 0)\big)\Big]$

$$= 2\mathbb{P}(\underbrace{X_1 < X_1' \ , X_2 \leq X_2'}_{B})$$

- $\mathbb{P}(B) = \mathbb{P}((X_1 \leq X'), (X_1 + X_3 \leq X_1' + X_3'))$

$$= \int_0^\infty \int_{x_1}^\infty \mathbb{P}(X_1 + X_3 \leq X_1' + X_3' \mid X_1 = x_1, X_1' = x_1') \times f_{X_1}(x) \times f_{X_1'}(x_1') dx_1 \, dx_1'$$

$$= \int_0^\infty \int_{x_1}^\infty \int_0^\infty \mathbb{P}(x_1 + X_3 \leq x_1' + X_3' \mid X_3' = x_3') f_{X_3'}(x_3') f_{X_1}(x_1) f_{X_1'}(x_1') dx_3' \, dx_1 \, dx_1'$$

$$= \int_0^\infty \int_{x_1}^\infty \int_0^\infty F_{X_3}(x_1' + x_3' - x_1) f_{X_3'}(x_3') f_{X_1}(x_1) f_{X_1'}(x_1') dx_3' \, dx_1' \, dx_1$$

$$= \int_0^\infty \int_{x_1}^\infty \int_0^\infty (1 - e^{-2(x_1' + x_3' - x_1)}) \times 2e^{-2x_3'} \times e^{-x_1} \times e^{-x_1'} dx_3 \, dx_1' \, dx_1$$

$$= \frac{5}{12}$$

Donc $\boxed{\tau = 4 \times \frac{5}{12} - 1 = \frac{2}{3}}$

2)b.(ii). $\mathbb{E}[f_X(X_1,X_2)] = \frac{5}{12}$.

Reste : à faire

2)b.(iii)

Pour $\sqrt{v} \leq u$ $\quad \frac{\partial C}{\partial u} = 0 \quad \frac{\partial^2 C(u,v)}{\partial u \partial v} = 0$

Pour $u \leq \sqrt{v}$ $\quad I = \int_0^1 \int_{u^2}^1 (1-\sqrt{v})^2 + u - \frac{(1-\sqrt{v})^2}{1-u} \, dC(u,v)$

$$\frac{\partial C(u,v)}{\partial u} = 1 - \left(\frac{1-\sqrt{v}}{1-u}\right)^2 \Rightarrow \frac{\partial^2}{\partial u \partial v} = \frac{1-\sqrt{v}}{\sqrt{v}(1-u)^2}$$

$$\Rightarrow I = \int_0^1 \int_{u^2}^1 \left[(1-\sqrt{v})^2 + u - \frac{(1-\sqrt{v})^2}{1-u}\right]\left[\frac{1-\sqrt{v}}{\sqrt{v}(1-u)^2}\right] dv\, du$$

chgt de var, puis comme le 2.a