v) On suppose maintenant que $Y \sim BN(\mu; k)$ de densité :

$$f(y) = \frac{\Gamma(y+\frac{1}{k})}{y!\Gamma(\frac{1}{k})} \left(\frac{1}{1+k\mu}\right)^{1/k} \left(\frac{k\mu}{1+k\mu}\right)^{y} \qquad y = 0, 1, 2, \ldots$$

a) Montrer que la loi binomiale negative appartient à la famille exponentielle tout en spécifiant le paramètre de la moyenne θ, le paramètre de dispersion, les fonctions b et c.

$$f(y) = \frac{\Gamma(y+\frac{1}{k})}{y!\,\Gamma(\frac{1}{k})} \left(\frac{1}{1+k\mu}\right)^{1/k} \left(\frac{k\mu}{1+k\mu}\right)^{y}$$

$f(y;$

$$\ln(f(y)) = \ln(\Gamma(y+\tfrac{1}{k})) - \ln(y!\,\Gamma(\tfrac{1}{k})) - \frac{1}{k}\ln(1+k\mu) + y\ln\left(\frac{k\mu}{1+k\mu}\right)$$

$$f(y) = \exp\left(y\ln\left(\frac{k\mu}{1+k\mu}\right) - \frac{1}{k}\ln(1+k\mu) + \ln(\Gamma(y+\tfrac{1}{k})) - \ln(y!\,\Gamma(\tfrac{1}{k}))\right)$$

$$\Longrightarrow \theta = \ln\left(\frac{k\mu}{1+k\mu}\right) \Longrightarrow e^{\theta} = \frac{k\mu}{1+k\mu} \Longrightarrow \frac{e^{\theta}}{1-e^{\theta}} = k\mu$$

$$b(\theta) = \frac{1}{k}\ln(1+k\mu)$$

$$= \frac{1}{k}\ln\left(1+\frac{e^{\theta}}{1-e^{\theta}}\right)$$

$$= -\frac{1}{k}\ln(1-e^{\theta})$$

Soit $X \sim \mathcal{B}(n,p) \Rightarrow f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$\Rightarrow \ln(f_X(x)) = \ln\left(\binom{n}{x}\right) + x\ln(p) + (n-x)\ln(1-p)$

$\Rightarrow f_X(x) = e^{x\ln(p) + (n-x)\ln(1-p) + \ln\left(\binom{n}{x}\right)}$

$= \exp\left(x\left(\ln(p) - \ln(1-p)\right) + n\ln(1-p) + \ln\left(\binom{n}{x}\right)\right)$

$= \exp\left(x\ln\left(\frac{p}{1-p}\right) + n\ln(1-p) + \ln\left(\binom{n}{x}\right)\right)$

$= \exp\left(\frac{\theta y - b(\theta)}{a(\phi)} + c(y,\phi)\right)$

$\Rightarrow \theta = \ln\left(\frac{p}{1-p}\right) \Rightarrow e^\theta = \frac{p}{1-p} \Rightarrow p = \frac{e^\theta}{1+e^\theta}$ donc $1-p = \frac{1}{1+e^\theta}$

donc $n\ln(1-p) = -n\ln(1+e^\theta)$

$\Rightarrow f_X(x) = \exp\left(\theta x - n\ln(1+e^\theta) + \ln\left(\binom{n}{x}\right)\right)$

$\Rightarrow b(\theta) = n\ln(1+e^\theta)$

$a(\phi) = 1 \rightarrow \phi = 1$

$c(x,\phi) = \ln\left(\binom{n}{x}\right)$

$$f(y) = \frac{1}{\sqrt{2\pi y^3}\,\sigma} \exp\left(-\frac{1}{2y}\left(\frac{y-\mu}{\mu\sigma}\right)^2\right)$$

$$\ln(f(y)) = -\frac{1}{2y}\left(\frac{y-\mu}{\mu\sigma}\right)^2 + \ln\left(\frac{1}{\sqrt{2\pi y^3}\,\sigma}\right)$$

$$= -\frac{1}{2y}\,\frac{1}{\mu^2\sigma^2}\left(y^2 - 2\mu y + \mu^2\right) + \ln\left(\frac{1}{\sqrt{2\pi y^3}\,\sigma}\right)$$

$$\Rightarrow f(y) = \exp\left(-\frac{y}{2\mu^2\sigma^2} + \frac{1}{\mu\sigma^2} - \frac{1}{2y\sigma^2} + \ln\left(\frac{1}{\sqrt{2\pi y^3}\,\sigma}\right)\right)$$

$$\Rightarrow \theta = -\frac{1}{2\mu^2} \quad \text{et} \quad a(\phi) = \sigma^2 \Rightarrow \phi = \sigma^2$$

$$b(\theta) = -\frac{1}{\mu} = -\sqrt{-2\theta} \Rightarrow b'(\theta) = -\left(-2 \times \frac{1}{2\sqrt{-2\theta}}\right) = \frac{1}{\sqrt{-2\theta}} = \mu$$

$$b''(\theta) = \frac{1}{(-2\theta)^{3/2}} = \frac{1}{(\sqrt{-2\theta})^3} = \mu^3$$

$$c(y, \phi) = -\ln\left(\sqrt{2\pi y \phi}\right)$$

Loi Y	θ	$b(\theta)$	ϕ	$E[Y]$	$V(\mu) = \frac{Var(Y)}{\phi}$
$\mathcal{B}(n,p)$	$ln\left(\frac{p}{1-p}\right)$	$n\ln\left(1+e^{\theta}\right)$	1	np	$np(1-p)$
$\mathcal{P}(\lambda)$	$ln(\lambda)$	e^{θ}	1	λ	λ
$\mathcal{N}\left(m,\sigma^2\right)$	m	$\frac{\theta^2}{2}$	σ^2	m	1
$\Gamma(m,\nu)$	$-\frac{1}{m}$	$-ln(-\theta)$	ϕ	m	m^2
$IG(m,\sigma^2)$	$-\frac{1}{2m^2}$	$-\sqrt{-2\theta}$	$\frac{\sigma^2}{m^2}\omega$	m	m^3
$BN(\mu,k)$	$ln\left(\frac{k\mu}{1+k\mu}\right)$	$-\frac{1}{k}ln\left(1-ke^{\theta}\right)$	1	μ	$\mu(1+k\mu)$