TD Risque de crédit

04/04/202

1 . Démonstration de la formule

(a) En s'intéréssant à l'intervalle $[0, dt]$ puis en faisant tendre dt vers 0, mq

$$\varphi'(u) = \frac{\lambda}{c}\varphi(u) - \frac{\lambda}{c}\int_0^u \varphi(u-x)\, dF_X(x)$$

$$\varphi(u) = \varphi(u + c\,dt)\underbrace{(1-\lambda dt)}_{\mathbb{P}(N_{dt}=0)} + \lambda dt \int_0^{u+cdt} \varphi(u + cdt - x)\, dF_X(x) + O(dt$$

$$\frac{\varphi(u) - \varphi(u+cdt)}{cdt} = -\frac{\lambda}{c}\varphi(u+dt) + \frac{\lambda}{c}\int_0^{u+cdt}\varphi(u+cdt-x)\, dF_X(x) + \frac{O(d}{cdt}$$

$$\Rightarrow -\varphi'(u) = \lim_{dt\to 0} -\frac{\lambda}{c}\varphi(u+dt) + \frac{\lambda}{c}\int_0^{u+cdt}\varphi(u+cdt-x)\, dF_X(x) + \frac{O(dt)}{cdt}$$

$$\Rightarrow \varphi'(u) = \frac{\lambda}{c}\varphi(u) - \frac{\lambda}{c}\int_0^u \varphi(u-x)\, dF_X(x)$$

(b) En intégrant l'équation précédente, montrer que

$$\varphi(u) = \varphi(0) + \frac{\lambda}{c}\int_0^u \varphi(u-x)(1-F_X(x))\, dx$$

$$\underbrace{\int_0^u \varphi'(y)\, dy}_{=[\varphi(y)]_0^u = \varphi(u)-\varphi(0)} = \frac{\lambda}{c}\int_0^u \varphi(y)\, dy - \frac{\lambda}{c}\underbrace{\int_{y=0}^u \left[\int_{x=0}^y \varphi(y-x)\, dF_X(x)\right] dy}_{A}$$

$$A = \int_{x=0}^u \left[\int_{y=x}^u \varphi(y-x)\, dy\right] dF_X(x)$$

$$A = \int_{x=0}^u \left[\int_{T=0}^{u-x}\varphi(T)\, dT\right] dF_X(x) = \int_{T=0}^u \left[\int_{x=0}^{u-T}\varphi(T)\, dF_X(x)\right] dT$$

$$= \int_{T=0}^u \varphi(T)\left[\int_{x=0}^{u-T} dF_X(x)\right] dT = \int_T^u \varphi(T)\left[F_X(x)\right]_0^{u-T} dT$$

$$= \int_{T=0}^u \varphi(T) F_X(u-T)\, dT$$

$$\varphi(u) \simeq \varphi(0) = \frac{\lambda}{c}\left[\int_0^u \varphi(y)\,dy + \int_0^u \varphi(y)\,F_X(u-y)\,dy\right]$$

$$\Leftrightarrow \varphi(u) = \varphi(0) + \frac{\lambda}{c}\int_0^u \varphi(y)\left[1 - F_X(u-y)\right]dy$$

$$= \varphi(0) + \frac{\lambda}{c}\int_0^u \varphi(u-x)\left[1-F_X(x)\right]dx$$

(c) En déduire une expression simple pour $\varphi(0)$.

$$\varphi(0) = 1 - \frac{\lambda\mu}{c}$$

(d) On note φ_{LS} la tranformée de Laplace Stieltjes de φ $\left(\varphi_{LS}(s) = \int_{-\infty}^{\infty} e^{-su}\, d\varphi(u)\right)$ et φ_L transformée de Laplace de φ $\left(\varphi_L(s) = \int_{-\infty}^{\infty} e^{-su}\varphi(u)\,du\right)$. Montrer que

$$\varphi_{LS}(s) = s\,\varphi_L(s)$$

(c) $$\lim_{u\to+\infty}\varphi(u) = 1 = \varphi(0) + \frac{\lambda}{c}\lim_{u\to+\infty}\int_0^u \varphi(u-x)\left[1-F_X(x)\right]dx$$

$$1 = \varphi(0) + \frac{\lambda}{c}\lim_{u\to+\infty}\int_0^{+\infty}\varphi(u-x)\left[1-F_X(x)\right]dx$$

$$1 = \varphi(0) + \frac{\lambda}{c}\int_0^{+\infty}\left[1-F_X(x)\right]dx$$

$$1 = \varphi(0) + \frac{\lambda}{c}\mu$$

(d) $\varphi_L(s) = \int_{-\infty}^{+\infty} e^{-su}\varphi(u)\,du$

$$\varphi_{LS}(s) = \int_{-\infty}^{+\infty} e^{-su}\,d\varphi(u)$$

$$= \left[e^{-su}\right]_{-\infty}^{+\infty} + s\int_{-\infty}^{+\infty} e^{-su}\varphi(u)\,du$$

$$= s\int_{-\infty}^{+\infty} e^{-su}\varphi(u)\,du$$

$$= s\,\varphi_L(s)$$

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$$= s\int_{-\infty}^{+\infty} e^{-su}\varphi(u)\,du$$

$$= s\,\varphi_L(s)$$

(e) On note $F_e(u) = \frac{1}{\mu}\int_0^u (1 - F_X(t))\,dt$. Montrer que

$$\psi_L(s) = \frac{1 - \frac{\lambda u}{c}}{s} + \frac{\lambda u}{c} F_{e,LS}(s)\,\psi_L(s)$$

$$F_e(u) = \frac{1}{\mu}\int_0^u (1 - F_X(t))\,dt$$

$$dF_e(u) = \frac{1}{\mu}(1 - F_X(u))$$

$$\psi(u) = \psi(0) + \frac{\lambda}{c}\mu \int_0^u \psi(u-x)\,dF_e(x)$$

$$\psi_L(s) = \int_0^{+\infty} e^{-sx}\psi(0)\,dx + \frac{\lambda}{c}\mu\int_{y=0}^{u} e^{-sy}\int_{x=0}^{y}\psi(y-x)\,dF_e(x)\,dy$$

$$= +\frac{\psi(0)}{s} + \underbrace{\frac{\lambda\mu}{c}\int_{y=0}^{+\infty} e^{-sy}\int_{x=0}^{y}\psi(y-x)\,dF_e(x)\,dy}_{B}$$

$$B = \int_{y=0}^{+\infty} e^{-sy}\int_{x=0}^{y}\psi(y-x)\,dF_e(x)\,dy \times \frac{\lambda\mu}{c}$$

$$= \frac{\lambda\mu}{c}\int_{x=0}^{+\infty}\int_{y=x}^{+\infty} e^{-sy}\psi(y-x)\,dy\,dF_e(x)$$

$$= \frac{\lambda\mu}{c}\int_{x=0}^{+\infty}\int_{y=x}^{+\infty} e^{-s(y-x)}e^{-sx}\psi(y-x)\,dy\,dF_e(x)$$

$$= \frac{\lambda\mu}{c}\int_{x=0}^{+\infty} e^{-sx}\int_{T=0}^{+\infty} e^{-sT}\psi(T)\,dT\,dF_e(x)$$

$$= \frac{\lambda\mu}{c}\left[\int_{x=0}^{+\infty} e^{-sx}\,dF_e(x)\right] \times \left[\int_{T=0}^{+\infty} e^{-sT}\psi(T)\,dT\right]$$

$$= \frac{\lambda\mu}{c} F_{e,LS}(s)\,\psi_L(s)$$

D'où, $\psi_L(s) = +\frac{\psi(0)}{s} + \frac{\lambda\mu}{c} F_{e,LS}(s)\,\psi_L(s)$

(f) Montrer que

$$\varphi(u) = \varphi(0) \sum_{n=0}^{+\infty} (1-\varphi(0))^n \, F_e^{*n}(u)$$

$\varphi(0) = 1 - \dfrac{\lambda\mu}{c}$

$$\varphi_L(s) = \frac{\varphi(0)}{s} + \frac{\lambda\mu}{c} F_{e,LS}(s)\, \varphi_L(s)$$

$$\varphi_L(s)\left[1 - \frac{\lambda\mu}{c\, F_{e,LS}(s)}\right] = \frac{\varphi(0)}{s}$$

$$\varphi_{LS}(s)\left[1 - \frac{\lambda\mu}{c} F_{e,LS}(s)\right] = \varphi(0) \quad \text{(question d))}$$

$$\Rightarrow \varphi_{LS}(s) = \frac{1}{1 - \frac{\lambda\mu}{c} F_{e,LS}(s)}\, \varphi(0)$$

$\to$ série géométrique

$$= \varphi(0) \sum_{n=0}^{+\infty} \left(\frac{\lambda\mu}{c}\right)^n F_{e,LS}^{*n}$$

$$= \varphi(0) \sum_{n=0}^{+\infty} \underbrace{\left(\frac{\lambda\mu}{c}\right)}_{1-\varphi(0)}^n F_e^{*n}(u)$$