30.11.2023

Exercice 1

$\delta = 6$ mois   Euribor 1 an.

À la date $T+\delta$: $\varphi_{T+\delta} = \left(L(T,\delta) - kL(T,2\delta) - m\right)^+$

(a) $\hat{L}(T,\delta) = 1 + \delta L(T,\delta) = \dfrac{1}{B(T,T+\delta)}$

$$\varphi_{T+\delta} = \frac{1}{\delta}\Big(\underbrace{(1+\delta L(T,\delta))}_{\hat{L}(T,\delta)} - \frac{k}{2}\underbrace{(1+2\delta L(T,2\delta))}_{\hat{L}(T,2\delta)} - \delta m - 1 + \frac{k}{2}\Big)^+ =$$

$$= \frac{1}{\delta}\Big(\hat{L}(T,\delta) - \frac{k}{2}\hat{L}(T,2\delta) - \big(\underbrace{1+\delta m - \tfrac{k}{2}}_{M}\big)\Big)^+$$

Pour quelle valeur de m_0 c'est une option d'échange?

$$1 + \delta m_0 - \frac{k}{2} = 0 \Rightarrow m_0 = \frac{1}{\delta}\left(\frac{k}{2} - 1\right)$$

(b) La dynamique des $\hat{L}_t(T,\delta)$ et $\hat{L}_t(T,2\delta)$ sous $\mathbb{Q}^{T+\delta}$?

autofinançant

$$\hat{L}_t(T,\delta) = \frac{B(t,T)}{B(t,T+\delta)} \rightarrow \text{une martingale sous } \mathbb{Q}^{T+\delta}.$$

numéraire

$\Gamma(t,T)$ - vol de $B(t,T)$

$\mathbb{Q}^{T+\delta}$ - MB

$$d\hat{L}_t(T,\delta) = \hat{L}_t(T,\delta)\left(\Gamma(t,T) - \Gamma(t,T+\delta)\right)dW_t^{T+\delta}$$

$$\overset{B_t(T,2\delta)^{-1}}{\overset{\|}{\hat{L}_t(T,2\delta)}} = \frac{B(t,T)}{B(t,T+2\delta)} = \underbrace{\frac{B(t,T)}{B(t,T+\delta)}}_{X_t} \cdot \Bigg(\underbrace{\frac{B(t,T+2\delta)}{B(t,T+\delta)}}_{Y_t}\Bigg)^{-1} = \frac{X_t}{Y_t}$$

martingales sous $\mathbb{Q}^{T+\delta}$

Vol de $X_t = \Gamma(t,T) - \Gamma(t,T+\delta)$ $\dfrac{dX_t}{X_t} = \left(\Gamma(t,T) - \Gamma(t,T+\delta)\right)dW_t^{T+\delta}$

Vol de $Y_t = \Gamma(t,T+2\delta) - \Gamma(t,T+\delta)$ $\qquad \frac{dY_t}{Y_t} = (\Gamma(t,T+2\delta)-\Gamma(t,T+\delta))dW_t^{T+\delta}$

$\langle X,Y\rangle_t =$ "$dX_t dY_t$"

$$d\left(\frac{X_t}{Y_t}\right) = \frac{X_t}{Y_t}\left(\frac{dX_t}{X_t}\right) - \frac{X_t}{Y_t}\left(\frac{dY_t}{Y_t}\right) + \frac{X_t}{Y_t}\left(\frac{d\langle Y\rangle_t}{Y_t^2}\right) + \underbrace{dX_t\, d\left(\frac{1}{Y_t}\right)}_{-\frac{X_t}{Y_t}\left(\frac{d\langle X,Y\rangle_t}{X_t Y_t}\right)}$$

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \underbrace{\frac{dX_t}{X_t} - \frac{dY_t}{Y_t}}_{(\Gamma(t,T)-\Gamma(t,T+2\delta))dW_t} + \underbrace{\frac{d\langle Y\rangle_t}{Y_t^2} - \frac{d\langle X,Y\rangle_t}{X_t Y_t}}_{(\Gamma(t,T+2\delta)-\Gamma(t,T+\delta))\left(\Gamma(t,T+2\delta)-\Gamma(t,T+\delta)-\Gamma(t,T)+\Gamma(t,T+\delta)\right)dt} \qquad \langle X,Y\rangle = \int_0^t X_s Y_s(\dots)(\dots)ds$$

alors
$$\frac{d\hat{L}_t(T,2\delta)}{\hat{L}_t(T,2\delta)} = (\Gamma(t,T+2\delta)-\Gamma(t,T+\delta))(\Gamma(t,T+2\delta)-\Gamma(t,T))dt + (\Gamma(t,T)-\Gamma(t,T+2\delta))dW_t$$

(c) Prix d'option lorsque $m = m_0$

$$\Phi_{T+\delta} = \frac{1}{\delta}\left(\hat{L}_T(T,\delta) - \frac{k}{2}\hat{L}_T(T,2\delta)\right)^+$$

Sous la mesure $\mathbb{Q}^{T+\delta}$

$$V_t = B(t,T+\delta)\frac{1}{\delta}\mathbb{E}_t^{\mathbb{Q}^{T+\delta}}\Big(\underbrace{\hat{L}_T(T,\delta)}_{\text{une mart sous } \mathbb{Q}^{T+\delta}} - \frac{k}{2}\hat{L}_T(T,2\delta)\Big)^+ = \frac{1}{\delta}B(t,T+\delta)\left[E_1 - \frac{k}{2}E_2\right]$$

$$\overbrace{\mathbb{E}_t^{\mathbb{Q}^{T+\delta}}\left[\hat{L}_T(T,\delta)\mathbb{1}_{\left\{\frac{\hat{L}_T(T,\delta)}{\hat{L}_T(T,2\delta)} \geq \frac{k}{2}\right\}}\right]}^{E_1} = ? \qquad \overbrace{\mathbb{E}_t^{\mathbb{Q}^{T+\delta}}\left[\hat{L}(T,2\delta)\mathbb{1}_{\left\{\frac{\hat{L}(T,\delta)}{\hat{L}(T,2\delta)} \geq \frac{k}{2}\right\}}\right]}^{E_2} = ?$$

$$d\,\frac{B(t,T+2\delta)}{B(t,T+\delta)} = \frac{B(t,T+2\delta)}{B(t,T+\delta)}(\Gamma(t,T+2\delta)-\Gamma(t,T+\delta))dW_t^{T+\delta}$$

$$\frac{B(T,T+2\delta)}{B(T,T+\delta)} = \frac{B(t,T+2\delta)}{B(t,T+\delta)}\exp\left\{-\int_t^T(\Gamma(s,T+2\delta)-\Gamma(s,T+\delta))^2ds + \int_t^T\underbrace{(\Gamma(t,T+2\delta)-\Gamma(t,T+\delta))}_{\sigma_2(s)}dW_s^{T+\delta}\right\}$$

$$\hat{L}_T(T,\delta) = \hat{L}_t(T,\delta)\exp\left\{-\frac{1}{2}\int_t^T(\Gamma(s,T)-\Gamma(s,T+\delta))^2ds + \int_t^T\overbrace{(\Gamma(s,T)-\Gamma(s,T+\delta))}^{\sigma_1(s)}dW_s^{T+\delta}\right\}$$

$\frac{B(t,T)}{B(t,T+\delta)}$

$$E_1 = \underbrace{\frac{B(t,T)}{B(t,T+\delta)} \exp\left\{-\frac{1}{2}\int_t^T \sigma_1(s)^2 ds\right\}}_{\tilde{c}_1} \mathbb{E}_t^{\mathbb{Q}^{T+\delta}}\left[e^{\int_t^T \sigma_1(s) dW_s^{T+\delta}} \mathbb{1}\left\{\int_t^T \sigma_2(s) dW_s^{T+\delta} \geq \underbrace{\ln\left(\frac{k}{2}\frac{B(t,T+\delta)}{B(t,T+2\delta)}\right) + \frac{1}{2}\int_t^T \sigma_2(s)^2 ds}_{\tilde{K}_1}\right\}\right.$$

$$= \tilde{c}_1 \mathbb{E}_t^{\mathbb{Q}^{T+\delta}}\left[e^{\int_t^T \sigma_1(s) dW_s^{T+\delta}} \mathbb{1}\left\{\int_t^T \sigma_2(s) dW_s^{T+\delta} \geq \tilde{K}_1\right\}\right] =$$

$$= \left\{ \frac{d\tilde{\mathbb{Q}}_1}{d\mathbb{Q}^{T+\delta}} = e^{\int_0^T \sigma_1(s) dW_s^{T+\delta} - \frac{1}{2}\int_0^T \sigma_1^2(s) ds} \right\} = \tilde{c}_1 \mathbb{E}_t^{\tilde{\mathbb{Q}}}\left[e^{\frac{1}{2}\int_t^T \sigma_1(s)^2 ds} \mathbb{1}\left\{\int_t^T \sigma_2(s) dW_s^{T+\delta} \geq \tilde{K}_1\right\}\right] =$$

$$\tilde{W}_t = W_t^{T+\delta} - \int_0^t \sigma_1(s) ds$$

$$= \frac{B(t,T)}{B(t,T+\delta)} \tilde{\mathbb{Q}}\left(\int_t^T \sigma_2(s) dW_s^{T+\delta} \geq \tilde{K}_1\right) = \frac{B(t,T)}{B(t,T+\delta)} \tilde{\mathbb{Q}}\left(\int_t^T \sigma_2(s) d\tilde{W}_s \geq \tilde{K}_1 - \int_0^t \sigma_1(s)\sigma_2(s) ds\right) =$$

gaussien, $\Sigma_2 := \mathrm{Var}\left[\int_t^T \sigma_2(s) d\tilde{W}_s\right] = \int_t^T \sigma_2(s)^2 ds$

$$= \frac{B(t,T)}{B(t,T+\delta)} \mathcal{N}\left(\frac{1}{\sqrt{\Sigma_2}}\left(\int_0^t \sigma_1(s)\sigma_2(s) ds - \tilde{K}_1\right)\right)$$

Donc, $\frac{1}{\delta} B(t,T+\delta) E_1 = \frac{1}{\delta} B(t,T) \mathcal{N}\left(\frac{\int_0^t \sigma_1(s)\sigma_2(s) - \frac{1}{2}\sigma_2^2(s) ds + \ln\left(\frac{2B(t,T+2\delta)}{kB(t,T+\delta)}\right)}{\sqrt{\int_t^T \sigma_2(s)^2 ds}}\right)$

$$E_2 = \frac{B(t,T)}{B(t,T+2\delta)} e^{\frac{1}{2}\int_t^T \sigma_2(s)^2 - \sigma_1(s)^2 ds} \mathbb{E}_t^{\mathbb{Q}^{T+\delta}}\left[e^{\int_t^T (\sigma_1(s)-\sigma_2(s)) dW_s^{T+\delta}} \mathbb{1}\left\{\int_t^T \sigma_2(s) dW_s^{T+\delta} \geq \tilde{K}_1\right\}\right] \circleddash$$

$$\frac{d\hat{L}_t(T,2\delta)}{\hat{L}_t(T,2\delta)} = \big(\overbrace{\Gamma(t,T+2\delta) - \Gamma(t,T+\delta)}^{\sigma_2(t)}\big)\big(\overbrace{\Gamma(t,T+2\delta) - \Gamma(t,T)}^{\sigma_2(t)-\sigma_1(t)}\big) dt + \big(\underbrace{\Gamma(t,T) - \Gamma(t,T+2\delta)}_{\sigma_1(t)-\sigma_2(t)}\big) dW_t^{T+\delta}$$

$$\left(\hat{L}(T,2\delta) = \frac{B(t,T)}{B(t,T+2\delta)} \exp\left\{\int_t^T \overbrace{\left(\sigma_2(s)(\sigma_2(s)-\sigma_1(s)) - \frac{1}{2}(\sigma_2(s)-\sigma_1(s))^2\right)}^{\frac{1}{2}(\sigma_2(s)^2-\sigma_1(s)^2)} ds + \int_t^T (\sigma_1(s)-\sigma_2(s)) dW_s^{T+\delta}\right\}\right)$$

$$\circleddash \left\{ \frac{d\tilde{\tilde{\mathbb{Q}}}}{d\mathbb{Q}^{T+\delta}} = \exp\left\{\int_t (\sigma_1(s)-\sigma_2(s)) dW_s^{T+\delta} - \frac{1}{2}\int_t^T (\sigma_1(s)-\sigma_2(s))^2 ds\right\} \quad \tilde{\tilde{W}}_t = W_t^{T+\delta} - \int_0^t (\sigma_1(s)-\sigma_2(s)) ds \right\} =$$

$$= \frac{B(t,T)}{B(t,T+2\delta)} \exp\left\{\frac{1}{2}\int_t^T (\sigma_2(s)-\sigma_1(s))(\cancel{\sigma_1(s)}+\sigma_2(s)+\sigma_2(s)-\cancel{\sigma_1(s)}) ds\right\} \tilde{\tilde{\mathbb{Q}}}\left(\int_t^T \sigma_2(s) dW_s^{T+\delta} \geq \tilde{K}_1\right)$$

$$= \frac{B(t,T)}{B(t,T+2\delta)} e^{\int_t^T \sigma_2(s)(\sigma_2(s)-\sigma_1(s)) ds} \tilde{\tilde{\mathbb{Q}}}\left(\int_t^T \sigma_2(s) d\tilde{\tilde{W}}_s \geq \tilde{K}_1 + \int_t^T \sigma_2(s)(\sigma_1(s)-\sigma_2(s)) ds\right) =$$

$$= \frac{B(t,T)}{B(t,T+2\delta)} e^{\int_t^T \sigma_2(s)(\sigma_2(s)-\sigma_1(s))\,ds} \mathcal{N}\left(\frac{-\tilde{K}_1 + \int_t^T \sigma_2(s)(\sigma_1(s)-\sigma_2(s))\,ds}{\sqrt{\Sigma_2}}\right)$$

$$-\frac{k}{2}\cdot\frac{1}{\delta} B(t,T+\delta) E_2 = -\frac{k}{2\delta} \frac{B(t,T+\delta)B(t,T)}{B(t,T+2\delta)} e^{\int_t^T \sigma_2(s)(\sigma_2(s)-\sigma_1(s))\,ds} \mathcal{N}\left(\frac{\int_t^T \sigma_1(\sigma_2-\sigma_1) - \frac{\sigma_2^2}{2}\,ds + \ln\left(\frac{2B(t,T+2\delta)}{kB(t,T+\delta)}\right)}{\sqrt{\int_t^T \sigma_2(s)^2 ds}}\right)$$

$$\text{La réponse} = \frac{1}{\delta} B(t,T) \mathcal{N}\left(\frac{\int_0^t \sigma_1(s)\sigma_2(s) - \frac{1}{2}\sigma_2^2(s)\,ds + \ln\left(\frac{2B(t,T+2\delta)}{kB(t,T+\delta)}\right)}{\sqrt{\int_t^T \sigma_2(s)^2 ds}}\right)$$

$$-\frac{k}{2\delta} \frac{B(t,T+\delta)B(t,T)}{B(t,T+2\delta)} e^{\int_t^T \sigma_2(s)(\sigma_2(s)-\sigma_1(s))\,ds} \mathcal{N}\left(\frac{\int_t^T \sigma_1(\sigma_2-\sigma_1) - \frac{\sigma_2^2}{2}\,ds + \ln\left(\frac{2B(t,T+2\delta)}{kB(t,T+\delta)}\right)}{\sqrt{\int_t^T \sigma_2(s)^2 ds}}\right)$$

(4) Le cas général: $\Phi_{t+\delta} = \frac{1}{\delta}\left(\hat{L}(T,\delta) - \frac{k}{2}\hat{L}(T,2\delta) - M\right)^+$

$$V_t = \frac{1}{\delta} B(t,T+\delta)\left[\mathbb{E}\left[\hat{L}(T,\delta)\mathbb{1}_{\{\underbrace{\hat{L}(T,\delta)-\frac{k}{2}\hat{L}(T,2\delta) \geq M}_{A}\}}\right] - \frac{k}{2}\mathbb{E}\left[\hat{L}(T,2\delta)\mathbb{1}_A\right] - \mathbb{Q}^{T+\delta}(A)\cdot M\right]$$

On fait le m̂ calcul (que en (c)), mais il faut savoir calculer $\mathbb{Q}^{T+\delta}(A)$, $\tilde{\mathbb{Q}}(A)$, $\tilde{\tilde{\mathbb{Q}}}(A)$

$$\hat{L}(T,2\delta) = \frac{B(t,T)}{B(t,T+2\delta)} \exp\left\{\int_t^T \left(\overbrace{\sigma_2(s)(\sigma_2(s)-\sigma_1(s)) - \frac{1}{2}(\sigma_2(s)-\sigma_1(s))^2}^{\frac{1}{2}(\sigma_2(s)^2-\sigma_1(s)^2)}\right)ds + \int_t^T (\sigma_1(s)-\sigma_2(s))\,dW_s^{T+\delta}\right\}$$

$$\hat{L}_T(T,\delta) = \frac{B(t,T)}{B(t,T+\delta)} \exp\left\{-\frac{1}{2}\int_t^T (\Gamma(s,T)-\Gamma(s,T+\delta))^2 ds + \int_t^T (\overbrace{\Gamma(s,T)-\Gamma(s,T+\delta)}^{\sigma_1(s)})\,dW_s^{T+\delta}\right\}$$

$$\hat{L}(T,\delta) - \frac{k}{2}\hat{L}(T,2\delta) = \underbrace{\left(\frac{B(t,T)}{B(t,T+\delta)} e^{-\frac{1}{2}\int_t^T \sigma_1(s)^2 ds}\right)}_{A_1(t)} e^{Z_1} + \underbrace{\left(\frac{B(t,T)}{B(t,T+2\delta)} e^{\frac{1}{2}\int_t^T \sigma_2(s)^2 - \sigma_1(s)^2 ds}\right)}_{A_2(t)} e^{Z_2}$$

$$\mathbb{V}\text{ar}[Z_1] = \int_t^T \sigma_1(s)^2 ds \qquad \mathbb{V}\text{ar}[Z_2] = \int_t^T (\sigma_1(s)-\sigma_2(s))^2 ds$$

$$\mathbb{Q}^{T+\delta}(A) = \mathbb{Q}\left(e^{\xi_1} + e^{\xi_2} \geq M\right) \quad \text{où} \quad \xi_1 \sim \mathcal{N}(\ln A_1, \Sigma_1) \quad \xi_2 \sim \mathcal{N}(\ln A_2, \Sigma_2)$$

loi lognormale

$$\text{cov}(\xi_1,\xi_2) = \text{cov}(Z_1,Z_2) = \int_t^T \sigma_1(s)(\sigma_1(s)-\sigma_2(s))\,ds$$

Pour $\tilde{\mathbb{Q}}, \tilde{\tilde{\mathbb{Q}}}$ on change le MB pour $\tilde{W}_t$ et $\tilde{\tilde{W}}_t$ et obtient $\hat{A}_{1,2}$ et $\tilde{\tilde{A}}_{1,2}$

À la fin, on arrive à $\tilde{\tilde{\mathbb{Q}}}\left(e^{\tilde{\tilde{\xi}}^1} + e^{\tilde{\tilde{\xi}}^2} \geq \mathcal{M}\right)$ où $\tilde{\tilde{\xi}}_1 \sim \mathcal{N}(\ln \tilde{\tilde{A}}_1, \Sigma_1)$ sous $\tilde{\tilde{\mathbb{Q}}}$

$\tilde{\tilde{\xi}}_2 \sim \mathcal{N}(\ln \tilde{\tilde{A}}_2, \Sigma_2)$

Deuxième question

$\boxed{\text{BGM} = \text{LMM}}$ $\quad m = 0 \quad \Phi_{T+\delta} = \left(L(T,\delta) - k\, L(T,2\delta)\right)^+$

(a)
$$\frac{dL_t(T,\delta)}{L_t(T,\delta)} = \varkappa_t(T,\delta)\, dW_t^{T+\delta}$$

(b) $\mathbb{E}^{\mathbb{Q}^{T+2\delta}}\left[L(T,\delta)\right]$ en fct de $L(T,\delta)$ et $L_T(T+\delta,\delta)$

$\|$ $\quad$ mart sous $\mathbb{Q}^{T+\delta}$

$$\mathbb{E}^{\mathbb{Q}^{T+\delta}}\left[\frac{d\mathbb{Q}^{T+2\delta}}{d\mathbb{Q}^{T+\delta}} L_T(T,\delta)\right] = \frac{B(0,T+\delta)}{B(0,T+2\delta)}\, \mathbb{E}^{\mathbb{Q}^{T+\delta}}\left[\frac{L_T(T,\delta)}{1+\delta L_T(T+\delta,\delta)}\right]$$

$$\frac{d\mathbb{Q}^{T+2\delta}}{d\mathbb{Q}^{T+\delta}} = \frac{B(T,T+2\delta)}{B(T,T+\delta)}\, \frac{B(0,T+\delta)}{B(0,T+2\delta)} = \frac{B(0,T+\delta)}{B(0,T+2\delta)} \cdot \frac{1}{1+\delta L_T(T+\delta,\delta)}$$

$$1 + \delta L_T(T+\delta,\delta) = \frac{1}{B_T(T+\delta,\delta)} = \frac{B(T,T+\delta)}{B(T,T+2\delta)}$$

en fonct de $L(T,\delta)$ et $L_T(T+\delta,\delta)$

$$\mathbb{E}^{\mathbb{Q}^{T+2\delta}}\left[L(T,\delta)\right] = \frac{B(0,T+\delta)}{B(0,T+2\delta)}\, \mathbb{E}^{\mathbb{Q}^{T+\delta}}\left[\frac{L_T(T,\delta)}{1+\delta L_T(T+\delta,\delta)}\right]$$

(c) Proposer une approximation pour la dynamique de $L_t(T,\delta)$ sous $\mathbb{Q}^{T+2\delta}$

$$\frac{dL_t(T,2\delta)}{L_t(T,2\delta)} = \varkappa_t(T,2\delta)\, dW_t^{T+2\delta}$$

Approximation (assez stupide) : $\dfrac{dL_t(T,\delta)}{L_t(T,\delta)} = \varkappa_t(T,\delta)\, dW_t^{T+2\delta}$

Exercice 2

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ $\qquad$ (B^1_t, B^2_t) indép

$W^1_t = B^1_t$

$W^2_t = \rho B^1_t + \sqrt{1-\rho^2}\, B^2_t$

$r_t = \alpha + Z^1_t Z^2_t$ $\qquad$ $Z^1_t = z_1 + W^1_t$

$Z^2_t = z_2 + \varepsilon W^2_t$

I. On pose $\varepsilon = 1$

$$r_t = \alpha + (z_1 + W^1_t) z_2$$

(1) $r_t = \alpha + z_1 z_2 + z_2 W^1_t$

(2) Taux spot forward $f(t,T)$

$$B(t,T) = e^{-\int_t^T f(t,u)\,du}$$

$$B(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s ds}\right] = \mathbb{E}\left[\exp\left\{-(\lambda+z_1z_2)(T-t) - z_2\int_t^T W_s^1 ds\right\}\right]$$

$$\int_t^T W_s^1 ds = \int_t^T \underbrace{(W_s^1 - W_t^1)}_{\tilde{W}_{s-t} \perp\!\!\!\perp W_t^1} ds + W_t^1(T-t) \overset{\mathcal{L}}{=} \underbrace{\int_0^{T-t} \tilde{W}_u du}_{\mathcal{N}\left(0, \frac{(T-t)^3}{3}\right)} + \underbrace{W_t^1(T-t)}_{\mathcal{N}\left(0, t(T-t)^2\right)}$$

Donc $$\int_t^T W_s^1 ds \sim \mathcal{N}\left(0, \frac{(T-t)^2(T+2t)}{3}\right)$$

$$B(t,T) = e^{-(\lambda+z_1z_2)(T-t) + \frac{1}{6} z_2^2 (T-t)^2 (T+2t)}$$

$$+\int_t^T f(t,u)du = +(\lambda+z_1z_2)(T-t) - \frac{1}{6} z_2^2 (t-T)^2 (T+2t)$$

$$\Big\downarrow \partial_T$$

$$f(t,T) = (\lambda+z_1z_2) - \frac{1}{3} z_2^2 (T-t)(T+2t) - \frac{1}{6} z_2^2 (T-t)^2 =$$

$$= (\lambda+z_1z_2) - \frac{1}{6} z_2^2 (T-t)(3T+3t) = (\lambda+z_1z_2) - \frac{1}{2} z_2^2 (T-t)(T+t)$$

$$\underline{f(t,T) = (\lambda+z_1z_2) - \frac{1}{2} z_2^2 (T^2 - t^2)}$$

En déduire la courbe de taux d'aujourd'hui. $T \to \infty$?

$$B(0,T) = e^{-R(0,T)T}$$

$$-R(0,T)\cdot T = -(\lambda+z_1z_2)T + \frac{1}{6} z_2^2 T^3$$

$$R(0,T) = (\lambda+z_1z_2) - \frac{1}{6} z_2^2 T^2 \qquad R(0,T) \xrightarrow{T\to+\infty} -\infty$$

Maintenant $\varepsilon \neq 0$.

On suppose que $f(t,T) = \varphi(Z_t^1, Z_t^2, T-t)$

$$\Phi(z^1, z^2, T) = \int_0^T \varphi(z^1, z^2, s) ds \qquad r_t = \lambda + Z_t^1 Z_t^2$$

$$Z_t^1 = z_1 + W_t^1$$
$$Z_t^2 = z_2 + \varepsilon W_t^2$$

$$\partial_T \Phi(z^1, z^2, T) = \varphi(z^1, z^2, T)$$

(a) Ecrire l'EDP satisfaite par φ. $B(t,T;r_t)$

$$\underline{e^{-\int_0^t r_s ds} B(t,T)} = e^{-\int_0^t r_s ds}\, e^{-\int_t^T f(t,u)du} = \exp\left\{-\int_0^t r_s ds - \Phi(Z^1_t, Z^2_t, T-t)\right\}$$

$$\int_t^T f(t,u)du = \int_t^T \varphi(Z^1_t, Z^2_t, u-t)\,du = \{s = u-t\} = \int_0^{T-t} \varphi(Z^1_t, Z^2_t, v)\,dv = \Phi(Z^1_t, Z^2_t, T-t)$$

Sous $\mathbb{Q}$

$$d\left[e^{-\int_0^t r_s ds} B(t,T)\right] = \underbrace{[\dots]}_{=0} dt$$

$$\|$$

$$d\left[\exp\Big\{\underbrace{-\int_0^t r_s ds - \Phi(Z^1_t, Z^2_t, T-t)}_{X_t}\Big\}\right] = \exp\{\dots\}\Big[\underbrace{-r_t}_{-(\alpha+z_1 z_2)} + \underbrace{\partial_\tau \Phi}_{\varphi(z^1, z^2, T-t)} - \frac{1}{2}\partial_{z_1 z_1}\Phi - \frac{1}{2}\partial_{z_2 z_2}\Phi\,\varepsilon^2 -$$

$$- \partial_{z_1 z_2}\Phi \cdot \varepsilon + \frac{1}{2}(\partial_{z_1}\Phi)^2 + \frac{1}{2}\varepsilon^2(\partial_{z_2}\Phi)^2 + \rho\varepsilon\, \partial_{z_1}\Phi\, \partial_{z_2}\Phi\Big] dt + [\dots] dW_t$$

$(\dots)dt$ dans dX_t $= 0$

$$d[e^{X_t}] = e^{X_t} dX_t + \frac{1}{2} e^{X_t} d\langle X\rangle_t$$

$$dX_t = [\dots]dt - \partial_{z_1}\Phi \cdot dW^1_t - \partial_{z_2}\Phi \cdot \varepsilon\, dW^2_t$$

$$d\langle X\rangle_t = \left[(\partial_{z_1}\Phi)^2 + \varepsilon^2(\partial_{z_2}\Phi)^2 + 2\rho(\partial_{z_1}\Phi)(\partial_{z_2}\Phi)\cdot\varepsilon\right] dt$$

(EDP)

$$\partial_\tau \Phi - \frac{1}{2}\partial_{z_1 z_1}\Phi - \frac{1}{2}\partial_{z_2 z_2}\Phi\,\varepsilon^2 - \partial_{z_1 z_2}\Phi\cdot\varepsilon + \frac{1}{2}(\partial_{z_1}\Phi)^2 + \frac{1}{2}\varepsilon^2(\partial_{z_2}\Phi)^2 + \rho\varepsilon\,\partial_{z_1}\Phi\,\partial_{z_2}\Phi = \alpha + z_1 z_2$$

(b) $\Phi = a_0(\tau) + a_1(\tau) z_1 + a_2(\tau) z_2 + a_{12} z_1 z_2 + a_{11} z_1^2 + a_{22} z_2^2$

$$\Phi|_{\tau=0} = 0$$

$$\left[\begin{array}{l} \dot a_0 + \dot a_1 z_1 + \dot a_2 z_2 + \dot a_{12} z_1 z_2 + \dot a_{11} z_1^2 + \dot a_{22} z_2^2 - a_{11} - \varepsilon^2 a_{22} - a_{12}\varepsilon + \\ + \frac{1}{2}(a_1 + a_{12} z_2 + 2a_{11} z_1)^2 + \frac{1}{2}\varepsilon^2(a_2 + a_{12} z_1 + 2a_{22} z_2)^2 + \qquad \forall z_1, z_2 \\ + \rho\varepsilon(a_1 + a_{12} z_2 + 2a_{11} z_1)(a_2 + a_{12} z_1 + 2a_{22} z_2) = \alpha + z_1 z_2 \end{array}\right]$$

Les EDO sur $a_i(\tau)$:

$$
\begin{cases}
\dot a_0 = a_{11} + \varepsilon^2 a_{22} + a_{12}\varepsilon - \frac{1}{2}a_1 - \frac{1}{2}\varepsilon^2 a_2^2 - \rho\varepsilon a_1 a_2 + \alpha \\
\dot a_1 + 2a_1 a_{11} + \varepsilon^2 a_2 a_{12} + \rho\varepsilon a_1 a_{12} + 2\rho\varepsilon a_2 a_{11} = 0 \\
\dot a_2 + a_1 a_{12} + 2\varepsilon^2 a_2 a_{22} + \rho\varepsilon a_2 a_{12} + 2\rho\varepsilon a_1 a_{22} = 0 \\
\dot a_{12} + 2a_{11} a_{12} + 2\varepsilon^2 a_{22} a_{12} + 4\rho\varepsilon a_{11} a_{22} + \rho\varepsilon a_{12}^2 = 1 \\
\dot a_{11} + 2a_{11}^2 + \frac{1}{2}\varepsilon^2 a_{12}^2 + 2\rho\varepsilon a_{12} a_{11} = 0 \\
\dot a_{22} + \frac{1}{2}a_{12}^2 + 2\varepsilon^2 a_{22}^2 + 2\rho\varepsilon a_{12} a_{22} = 0
\end{cases}
$$

$$a_i(0) = 0 \quad i \in \{0, 1, 2, 12, 11, 22\}$$

(c) M.q. la volatilité d'un ZC est une forme affine de Z^1, Z^2.

Dynamique de Z^1, Z^2 sous $\mathbb{Q}^T$?

$$dB(t,T) = d\left[e^{-\Phi(Z_t^1, Z_t^2, T-t)}\right] = e^{-\Phi}\left[(\dots)dt - \left(\partial_{z_1}\Phi\, dW_t^1 + \varepsilon\, \partial_{z_2}\Phi\, dW_t^2\right)\right]$$

fcts affines

$$\begin{pmatrix} \partial_{z_1}\Phi = a_1 + a_{12} z_2 + 2a_{11} z_1 \\ \partial_{z_2}\Phi = a_2 + a_{12} z_1 + 2a_{22} z_2 \end{pmatrix}$$

Sous $\mathbb{Q}^T$ $\quad \frac{B_t}{B(t,T)} = e^{\int_0^t r_s ds + \Phi}$ est une martingale

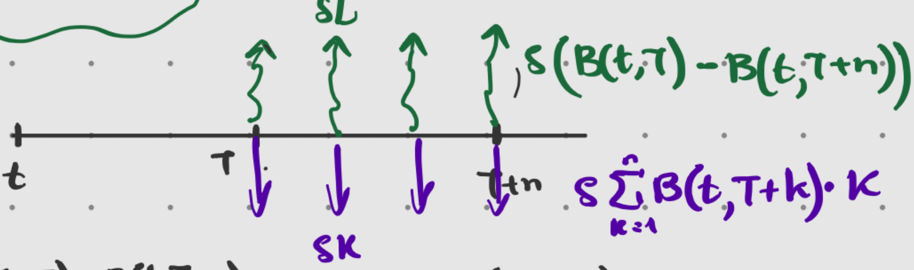
$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_t = \frac{B(t,T)}{B(0,T)} \cdot \frac{B_0}{B_t} = \mathcal{E}\left(-\int \partial_{z_1}\Phi\, dW_t^1 - \varepsilon \int \partial_{z_2}\Phi\, dW_t^2\right)$$

$$\Rightarrow W_t^{T,1} = W_t^1 + \int \partial_{z_1}\Phi\, ds$$

$$W_t^{T,2} = W_t^2 + \varepsilon \int \partial_{z_2}\Phi\, ds$$

$$\begin{cases} dZ_t^1 = dW_t^{T,1} + \partial_{z_1}\Phi\, dt \\ dZ_t^2 = \varepsilon\, dW_t^{T,2} + \varepsilon\, \partial_{z_2}\Phi\, dt \end{cases}$$

Exo 3



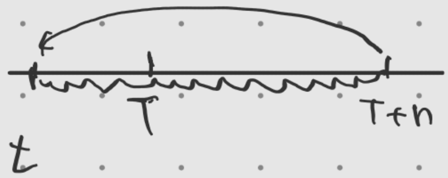
(a) $$SW_t(T,n) = \frac{B(t,T) - B(t,T+n)}{\sum_{k=1}^{n} B(t,T+k)} = \frac{1 + B_t(T,T+n)}{\sum_{k=1}^{n} B_t(T+k,T+n)}$$

ZC forward $B_t(T,T+\delta) = \dfrac{B(t,T+\delta)}{B(t,T)}$

$A_t = \sum_{k=1}^{n} B(t,T+k)$ annuité (fwd?) $\rightarrow$ SW_t est une martingale sous $\mathbb{Q}^A$

$\uparrow$ martingale sous $\mathbb{Q}^T$ $\Rightarrow$ $\mathbb{Q}^T \rightsquigarrow \mathbb{Q}^A$

(b) L'approximation $B_t(T,T+n) \sim \left(\dfrac{1}{1+SW_t}\right)^n$



$$SW_t = \underbrace{B_t(T,T+n)^{-1/n}}_{\text{lognormale si vol est déterministe}} - 1$$

Sous $\mathbb{Q}^T$ $B_t(T,T+n) = \dfrac{B(t,T+n)}{B(t,T)}$ est une mart. $\Rightarrow$

$$\Rightarrow \frac{dB_t(T,T+n)}{B_t(T,T+n)} = \left(\Gamma(t,T+n) - \Gamma(t,T)\right) dW_t^T$$

$$dSW_t = -\frac{1}{n} B_t(T,T+n)^{-1/n} \frac{dB_t(T,T+n)}{B_t(T,T+n)} + \frac{1}{2n}\left(1+\frac{1}{n}\right) B_t(\cdot\cdot)^{-1/n} \frac{d\langle B_t(\cdot\cdot)\rangle_t}{B_t(\cdot\cdot)^2}$$

$$dSW_t = \frac{n+1}{2n^2}(1+SW_t)\overbrace{\left(\Gamma(t,T+n)-\Gamma(t,T)\right)}^{\Sigma_t(T,T+n)}{}^2 dt - \frac{1}{n}(1+SW_t)\left(\Gamma(t,T+n)-\Gamma(t,T)\right) dW_t^T$$

(c) Une option sur le taux de swap: $(SW_T-K)^+$ en T

Pourquoi ce n'est pas une swaption? Pour la swaption on a le payoff $\Rightarrow A_T(SW_T-K)^+$!

Prix d'option (sous $\mathbb{Q}^T$): $B(t,T)\,\mathbb{E}^T(SW_T-K)^+ =$

$$= B(t,T)\,\mathbb{E}^T\left((1+SW_T)-(K+1)\right)^+ = B(t,T)\cdot BS\left(\begin{array}{lll} S_t = (1+SW_t)\exp\left\{\frac{n+1}{2n^2}\overbrace{\int_0^T \Sigma_s^2 ds}^{=:V^2}\right\} & & T=T \\ \sigma^2 T = \frac{1}{n^2}\int_0^T \Sigma_s^2 ds & K=K+1 & r=0 \end{array}\right) =$$

$$= B(t,T)\left((1+SW_t)e^{\frac{n+1}{2n^2}\cdot V^2}\mathcal{N}(d_+) - (1+K)\mathcal{N}(d_-)\right) \quad \text{où} \quad d_\pm = \frac{\ln\left(\frac{S_t}{K+1}\right) \pm \frac{1}{2}\cdot\frac{1}{n}V}{\frac{1}{n^2}V^2}$$

(4) Stratégie de couverture :

$$\text{Prix} = B(t,T)\left(1+SW_t\right) e^{\frac{n+1}{2n^2} v^2} \mathcal{N}(d_+) - B(t,T) \underbrace{(1+K)\mathcal{N}(d^-)}_{\text{en oblig. } B(t,T)}$$

$$1+SW_t = B_t(T,T+n)^{-1/n} = \left(\frac{B(t,T)}{B(t,T+n)}\right)^{\frac{1}{n}}$$

$$B(t,T)\left(1+SW_t\right) = \frac{B(t,T)^{1+\frac{1}{n}}}{B(t,T+n)^{1/n}}$$

- on peut répliquer ça (j'espère) en utilisant $B(t,T)$ et $B(t,T+n)$?