TD2

Question 1

(a) $\phi_{N_1}(t) = \mathbb{E}\left[e^{tN_1}\right]$

$$= \sum_{k \in \mathbb{N}} e^{tk}(1-p)\,p^k$$

$$= (1-p) \sum_{k \in \mathbb{N}} (pe^t)^k$$

$$= \frac{1-p}{1-pe^t} \quad \text{pour } t < \ln\left(\frac{1}{p}\right)$$

De plus on sait que $\mathbb{E}[N_1^k] = \dfrac{\partial^{(k)} \phi_{N_1}(0)}{\partial t^k}$

- $\mathbb{E}[N_1] = \phi'_{N_1}(0) = \dfrac{p}{1-p}$
- $\mathbb{E}[N_1^2] = \dfrac{p(1+p)}{(1-p)^2}$
- $\mathrm{Var}(N_1) = \dfrac{p}{(1-p)^2}$

(b) Soit $N = \sum_{p=1}^{\alpha} X_p$

- $\mathbb{E}[N] = \alpha\,\mathbb{E}[X_1] = \dfrac{\alpha p}{1-p}$

- $\mathbb{V}[N] = \alpha\,\mathrm{Var}(X_1)$ car les X_p sont idp^t^

$= \dfrac{\alpha p}{(1-p)^2}$

- $\Phi_N(t) = \mathbb{E}[e^{tN}] = \mathbb{E}\left[e^{t\sum_{p=1}^{\alpha} X_p}\right]$

car idp^t^

$= \prod_{p=1}^{\alpha} \mathbb{E}[e^{tX_p}]$

$= \left(\dfrac{1-p}{1-pe^t}\right)^{\alpha}$

(c) $G_N(t) = \mathbb{E}[t^N] = \mathbb{E}\left[t^{\sum_{p=1}^{\alpha} X_p}\right]$

car idp^t^

$= \prod_{p=1}^{\alpha} \mathbb{E}[t^{X_p}]$

$= \prod_{p=1}^{\alpha} \sum_{k\in\mathbb{N}} t^k (1-p) p^k = \prod_{p=1}^{\alpha} (1-p)\dfrac{1}{1-pt} = \left(\dfrac{1-p}{1-pt}\right)^{\alpha}$

(d) $$\frac{p_N(k)}{p_N(k-1)} = \frac{\binom{\alpha+k-1}{k}\cancel{(1-p)^\alpha} p^k}{\binom{\alpha+k-2}{k-1}\cancel{(1-p)^\alpha} p^{k-1}} = \frac{(\alpha+k-1)!\,(k-1)!\,\cancel{(\alpha-1)!}}{(\alpha+k-2)!\,k!\,\cancel{(\alpha-1)!}}\,p$$

$$= \frac{\alpha+k-1}{k}\,p$$

D'où $p_N(k) = \left(p + p\,\frac{\alpha-1}{k}\right) p_N(k-1) = \left(a + \frac{b}{k}\right) p_N(k-1)$

Donc $\begin{cases} a = p \\ b = p(\alpha-1) \end{cases}$

Question 2

$$\mathbb{E}\left[\sum_{\ell=1}^{n} X_\ell \,\middle|\, \sum_{i=1}^{n} X_i = j\right] = j$$

Donc $n\,\mathbb{E}\left[X_1 \,\middle|\, \sum_{i=1}^{n} X_i = j\right] = j$

Donc $\mathbb{E}\left[X_1 \,\middle|\, \sum_{i=1}^{n} X_i = j\right] = \frac{j}{n}$

Question 3

$$\mathbb{P}\left(X_1 = i \,\middle|\, \sum_{k=1}^{n} X_k = j\right) = \frac{\mathbb{P}\left(X_1 = i, \sum_{\ell=1}^{n} X_\ell = j\right)}{\mathbb{P}\left(\sum X_i = j\right)}$$

$$= \frac{\mathbb{P}\left(X_1 = i \cap \sum_{k=2}^{n} X_k = j-i\right)}{\mathbb{P}\left(\sum^{n} X_i = j\right)}$$

Donc $\mathbb{P}\left(X_1 = i \mid \sum_{k=1}^{n} X_k = j\right) = \dfrac{\mathbb{P}\left(X_1 = i \cap \sum_{k=1}^{n-1} X_k = j-i\right)}{\mathbb{P}\left(\sum X_i = j\right)}$

$$= \frac{p_X(i)\, p_X^{*(n-1)}(j-i)}{p_X^{*(n)}(j)}$$

Question 4

$$\mathbb{P}(S=0) = \mathbb{P}\left(\sum_{k=1}^{N} X_k = 0\right)$$

$$= \sum_{n\in\mathbb{N}} \mathbb{P}\left(\sum_{i=1}^{N} X_i = 0 \mid N=n\right)\mathbb{P}(N=n)$$

$\downarrow$ indépendance

$$= \sum_{n\in\mathbb{N}} \mathbb{P}\left(\sum_{i=1}^{n} X_i = 0\right)\mathbb{P}(N=n)$$

$$= \sum_{n\in\mathbb{N}} \mathbb{P}(X_1 = 0)^n\, \mathbb{P}(N=n)$$

$$= G_N(p_X(0))$$

Question 5

(a) Soit $j \geq 1$

$$p_S(j) = \sum_{n=1}^{\infty} \mathbb{P}(N=n)\,\mathbb{P}(S=j \mid N=n)$$

$$= \sum_{n=1}^{\infty} \left(a + \frac{b}{n}\right) p_N(n-1)\, p_X^{*n}(j)$$

(b) $$\sum_{n=1}^{\infty} p_X^{*(n)}(j)\, p_N(n-1) = \sum_{n=1}^{\infty} \sum_{i=1}^{j} p_X(i)\, p_X^{(n-1)}(j-i)\, p_N(n-1)$$

$$= \sum_{i=1}^{j} p_X(i) \sum_{n=1}^{\infty} p_X^{(n-1)}(j-i)\, p_N(n-1)$$

$$= \sum_{i=1}^{j} p_X(i) \underbrace{\sum_{n=0}^{\infty} p_N(n)\, p_X^{(n)}(j-i)}_{p_S(j-i)}$$