Chapter 6

Diffusion processes

6.1 SDEs with random coefficients

We recall in this section the basic results for stochastic differential equations

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t, \quad t \in [0,T]. \tag{6.1.1}$$

Here, b and σ are $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}^n)$-progressively measurable functions from $[0,T]\times\Omega\times\mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$ respectively. In particular, for every fixed $x\in\mathbb{R}^n$, the processes $\{b_t(x),\sigma_t(x), t\in[0,T]\}$ are $\mathbb{F}$-progressively measurable.

Definition 6.1.1 *A strong solution of (6.1.1) is an $\mathbb{F}$-progressively measurable process X such that $\int_0^T(|b_t(X_t)|+|\sigma_t(X_t)|^2)dt<\infty$ a.s. and*

$$X_t = X_0 + \int_0^t b_s(X_s)ds + \int_0^t \sigma_s(X_s)dW_s, \quad t\in[0,T].$$

The main existence and uniqueness result is the following.

Theorem 6.1.1 *Let X_0 be a square integrable r.v. independent of W. Assume that the processes $b.(0)$ and $\sigma.(0)$ are in $\mathbb{H}^2$, and there exists a constant $L>0$ such that*

$$|b_t(x)-b_t(y)| + |\sigma_t(x)-\sigma_t(y)| \;\leq\; L|x-y| \tag{6.1.2}$$

for all $t\in[0,T]$ and $x,y\in\mathbb{R}^n$.

Then there exists a unique strong solution of (6.1.1) in $\mathbb{H}^2$. Moreover, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t|^2\Big] \;\leq\; C(1+\mathbb{E}[|X_0|^2]) \qquad (6.1.3)$$

for some constant $C = C(T,L)$ depending only on T and L.

Proof. We proceed in two steps. We first prove the existence and uniqueness of a solution to the SDE and then establish (6.1.3).

Step 1. Let c be a positive constant to be fixed later. We define the norm $\|.\|_{\mathbb{H}^2_c}$ by

$$\|\phi\|_{\mathbb{H}^2_c} \;=\; \mathbb{E}\Big[\int_0^T e^{-ct}|\phi_s|^2 ds\Big]^{1/2}$$

for $\phi \in \mathbb{H}^2$. We notice that the norms $\|.\|_{\mathbb{H}^2_c}$ and $\|.\|_{\mathbb{H}^2}$ are equivalent on $\mathbb{H}^2$. Consider now the map $U:\ \mathbb{H}^2 \to \mathbb{H}^2$ defined by

$$U(Y)_t \;=\; X_0 + \int_0^t b_s(Y_s)ds + \int_0^t \sigma_s(Y_s)dW_s$$

From the Lipschitz property of b and σ and since $b.(0), \sigma.(0) \in \mathbb{H}^2$, the map U is well defined on $\mathbb{H}^2$. We next notice that X is solution to (6.1.1) if and only if X is fixed point of U. Therefore, it is sufficient to prove that U is a contraction for some convenient c to get the result. For $Y, Z \in \mathbb{H}^2$, we have

$$\mathbb{E}\Big[|U(Y)_t - U(Z)_t|^2\Big] \;=\; \mathbb{E}\Bigg[\bigg|\int_0^t \big(b_s(Y_s)-b_s(Z_s)\big)ds + \int_0^t \big(\sigma_s(Y_s)-\sigma_s(Z_s)\big)dW_s\bigg|^2\Bigg].$$

From Young's inequality we get

$$\begin{aligned}\mathbb{E}\Big[|U(Y)_t - U(Z)_t|^2\Big] \;\leq\;& 2\mathbb{E}\Bigg[\bigg|\int_0^t \big(b_s(Y_s)-b_s(Z_s)\big)ds\bigg|^2\Bigg]\\ &+2\mathbb{E}\Bigg[\bigg|\int_0^t \big(\sigma_s(Y_s)-\sigma_s(Z_s)\big)dW_s\bigg|^2\Bigg].\end{aligned}$$

Using the Lipschitz properties of b and σ, Jensen's inequality and the Ito Isometry, we get

$$\mathbb{E}\Big[|U(Y)_t - U(Z)_t|^2\Big] \leq 2(T+1)L\mathbb{E}\bigg[\int_0^t \big|Y_s - Z_s\big|^2 ds\bigg].$$

Therefore we get

$$\|U(Y) - U(Z)\|_{\mathbb{H}_c^2} \leq \frac{2(T+1)L}{c}\|Y - Z\|_{\mathbb{H}_c^2}$$

and U is a $\|.\|_{\mathbb{H}_c^2}$-contraction for c large enough.

Step 2. We now prove (6.1.3). We have from Young and Jensen inequalities

$$\begin{aligned}\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s|^2\Big] &\leq \mathbb{E}\Big[\sup_{s\in[0,t]}|X_0 + \int_0^s b_u(X_u)du + \int_0^s \sigma_u(X_u)dW_u|^2\Big] \\ &\leq 3\bigg(\mathbb{E}\big[|X_0|^2\big] + t\int_0^t \mathbb{E}\big[|b_u(X_u)|^2\big]du + \mathbb{E}\Big[\sup_{s\in[0,t]}\Big|\int_0^s \sigma_u(X_u)dW_u\Big|^2\Big]\bigg).\end{aligned}$$

Using Doob's maximal inequality, we get

$$\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s|^2\Big] \leq 3\bigg(\mathbb{E}\big[|X_0|^2\big] + t\int_0^t \mathbb{E}\big[|b_u(X_u)|^2\big]du + 4\mathbb{E}\Big[\int_0^t |\sigma_u(X_u)|^2 du\Big]\bigg).$$

From Lipschitz properties of b and σ, we get a constant $C(T,L)$ such that

$$\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s|^2\Big] \leq C(T,K)\bigg(1 + \mathbb{E}\big[|X_0|^2\big] + \int_0^t \mathbb{E}\Big[\sup_{s\in[0,u]}|X_s|^2\Big]du\bigg),$$

and we get the result from Gronwall's lemma. □

The previous result can be easily extended to any initial time $t \in [0,T]$ instead of 0. In the sequel, we shall denote by $X^{t,x} = \{X_s^{t,x},\ t \leq s \leq T\}$ the process solution to

$$X_s = x + \int_t^s b_u(X_u)du + \int_t^s \sigma_u(X_u)dW_u, \quad u \in [t,T],$$

for $t \in [0,T]$ and $x \in \mathbb{R}^n$.

In addition to the estimate (6.1.3) of Theorem 6.1.1, we have the following flow continuity results of the solution of the SDE.

Theorem 6.1.2 *Suppose that assumptions of Theorem 6.1.1 hold.*
(i) There exists a constant $C > 0$ such that

$$\mathbb{E}\Big[\sup_{t\leq s\leq t'} |X_s^{t,x} - X_s^{t,x'}|^2\Big] \quad \leq \quad Ce^{Ct'}|x-x'|^2$$

for $t, t' \in [0,T]$ such that $t \leq t'$ and $x, x' \in \mathbb{R}^n$.
(ii) Moreover, if we have

$$B \quad := \quad \sup_{0\leq s<s'\leq T} (s'-s)^{-1}\mathbb{E}\int_s^{s'} \big(|b_r(0)|^2 + |\sigma_r(0)|^2\big)dr \quad < \quad +\infty$$

then, there exists a constant $C > 0$ such that

$$\mathbb{E}\Big[\sup_{t'\leq s\leq T} |X_s^{t,x} - X_s^{t',x}|^2\Big] \quad \leq \quad Ce^{CT}(B + |x|^2)(t'-t)$$

for $t, t' \in [0,T]$ such that $t \leq t'$ and $x \in \mathbb{R}^n$.

Proof. (i) To simplify notation we set $\delta x = x - x'$, $\delta X = X^{t,x} - X^{t,x'}$, $\delta b = b(X^{t,x}) - b(X^{t,x'})$ and $\delta\sigma = \sigma(X^{t,x}) - \sigma(X^{t,x'})$. We next have from Young and Jensen inequalities

$$|\delta X_s|^2 \quad \leq \quad 3\bigg(|\delta x|^2 + (s-t)\int_t^s |\delta b_s|^2 ds + \Big|\int_t^s \delta\sigma_s dW_s\Big|^2\bigg)$$

for $s \in [t,T]$. Using Doob's maximal inequality and the Lipschitz properties of b and σ we get

$$\mathbb{E}\Big[\sup_{t\leq s\leq t'} |X_s^{t,x} - X_s^{t,x'}|^2\Big] \quad \leq$$
$$3\bigg(|\delta x|^2 + L^2(t'+4)\int_t^{t'} \mathbb{E}\Big[\sup_{t\leq s\leq u} |X_s^{t,x} - X_s^{t,x'}|^2\Big]du\bigg)$$

for $t' \in [t,T]$. The result follows from Gronwall's Lemma.

(ii) We set $X = X^{t,x}$, $X' = X^{t',x}$, $\delta t = t'-t$, $\delta X = X'-X$, $\delta b = b(X') - b(X)$ and $\delta\sigma = \sigma(X') - \sigma(X)$. Following the same arguments as in (i) we get

$$\mathbb{E}\Big[\sup_{t'\leq s\leq u} |\delta X_s|^2\Big] \quad \leq \quad 3\bigg(\mathbb{E}|\delta X_{t'}|^2 + L^2(T+4)\int_{t'}^u \mathbb{E}\Big[\sup_{t\leq s\leq r} |\delta X_s|^2\Big]dr\bigg) \tag{6.1.4}$$

for $u \in [t', T]$. We now concentrate on the first term of the left hand side. We have by the same arguments

$$\begin{aligned}
\mathbb{E}|\delta X_{t'}|^2 &= \mathbb{E}|X_{t'}^{t,x} - x|^2 \\
&\leq 2\Big(T\int_t^{t'} \mathbb{E}|b(X_u^{t,x})|^2 du + \int_t^{t'} \mathbb{E}|\sigma(X_u^{t,x})|^2 du\Big) \\
&\leq 6(T+1)\Big(\int_t^{t'} \big(L^2\mathbb{E}|X_u^{t,x} - x|^2 + L^2|x|^2 + \mathbb{E}[|b_u(0)|^2 + |\sigma_u(0)|^2]\big)du \\
&\leq 6(T+1)\bigg((t'-t)(L^2|x|^2 + B) + L^2\int_t^{t'} \mathbb{E}|X_u^{t,x} - x|^2 du\bigg) .
\end{aligned}$$

By Gronwall's lemma we get

$$\mathbb{E}|\delta X_{t'}|^2 \quad \leq \quad C(t'-t)(|x|^2 + B)e^{C(t'-t)} .$$

Pugging this last inequality in (6.1.4), we get the result using Gronwall's lemma another time. □

6.2 Markov SDEs

In this section, we restrict the coefficients b and σ to be deterministic functions of (t, x). In this context, we suppose that b and σ are continuous functions Lipschitz in x uniformly in t, that is (6.1.2) holds true. We recall that $X^{t,x}$ denotes the solution of the stochastic differential equation

$$X_s = x + \int_t^s b_u(X_u)du + \int_t^s \sigma_u(X_u)dW_u, \quad u \in [t, T],$$

for $t \in [0, T]$ and $x \in \mathbb{R}^n$. We then have the two following properties.

- From the pathwise uniqueness, we have the flow property:

  $$X_s^{t,x} \quad = \quad X_s^{u, X_u^{t,x}}$$

  for $t \leq u \leq s$. We notice that this flow property also holds for u being a stopping time valued in $[t, s]$.

- Since $X^{t,x}$ is adapted to the filtration generated by the process $\{W_s - W_t,\ s \geq t\}$ we get from the previous flow continuity and Doob's representation Theorem the existence of a measurable function F such that

$$X_s^{t,x} \quad = \quad F(t, x, s, W_u - W_t,\ u \in [t, s])$$

for $s \leq T$.

Proposition 6.2.1 (Markov property) *We have*

$$\mathbb{E}\big[\Phi(X_r^{t,x}, u \leq r \leq s)|\mathcal{F}_u\big] \quad = \quad \mathbb{E}\big[\Phi(X_r^{t,x}, u \leq r \leq s)|X_u^{t,x}\big]$$

for any $u \in [t, s]$ *and any measurable bounded (or nonnegative) function* $\Phi : C([u, s]) \to \mathbb{R}$.

6.3 Connection with PDEs

Let $X^{t,x}$ be the unique solution to the SDE

$$X_s = x + \int_t^s b_u(X_u)du + \int_t^s \sigma_u(X_u)dW_u, \quad s \in [t, T], \tag{6.3.5}$$

for $t \in [0, T]$ and $x \in \mathbb{R}^n$. We next define the operator $\mathcal{L}$ by

$$\mathcal{L}\varphi(t, x) \quad = \quad \lim_{h \to 0} \frac{\mathbb{E}[\varphi(X_{t+h}^{t,x})] - \varphi(x)}{h}$$

for any function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\mathcal{L}\varphi$ is well defined. $\mathcal{L}$ is called the generator of the diffusion. From Ito's formula, $\mathcal{L}\varphi$ is well defined for any φ bounded C^2 with bounded derivatives and we have

$$\mathcal{L}\varphi(t, x) \quad = \quad b_t(x).\nabla\varphi(x) + \frac{1}{2}\mathrm{Tr}(\sigma_t(x)\sigma_t(x)^\top \nabla^2\varphi(x))$$

for $t \in [0, T]$ and $x \in \mathbb{R}^n$. The generator provides a connection between diffusion processes and linear partial differential equations.

Proposition 6.3.2 *Suppose that the function v defined by*

$$v(t,x) \quad = \quad \mathbb{E}[g(X_T^{t,x})] \ , \quad (t,x) \in [0,T] \times \mathbb{R}^n \ ,$$

is in $C^{1,2}([0,T] \times \mathbb{R}^n)$. Then, v solves the partial differential equation:

$$\partial_t v + \mathcal{L}v \quad = \quad 0 \ \text{on} \ [0,T) \times \mathbb{R}^n$$

with terminal condition

$$v(T,.) \quad = \quad g \ \text{on} \ \mathbb{R}^n \ .$$

Proof. Fix $(t,x) \in [0,T] \times \mathbb{R}^n$ and let $\tau := T \wedge \inf\{s > t : |X_s^{t,x} - x| \geq 1\}$. By the tower property and the Markov property of $X^{t,x}$ we have

$$v(t,x) \quad = \quad \mathbb{E}\big[v(s \wedge \tau, X_{s\wedge\tau}^{t,x})\big]$$

for any $s \in [t,T]$. Since $v \in C^{1,2}([0,T] \times \mathbb{R}^n)$ we have from Ito's formula

$$\begin{aligned} 0 \quad &= \quad \mathbb{E}\Big[\int_t^{s\wedge\tau} \big(\partial_t v + \mathcal{L}v\big)(u, X_u^{t,x})du\Big] + \mathbb{E}\Big[\int_t^{s\wedge\tau} \nabla v.\sigma(u, X_u^{t,x})dW_u\Big] \\ &= \quad \mathbb{E}\Big[\int_t^{s\wedge\tau} \big(\partial_t v + \mathcal{L}v\big)(u, X_u^{t,x})du\Big] \end{aligned}$$

where the last inequality comes from the boundedness of $X^{t,x}$ on $[t,\tau]$. We now take $s = t + h$ with $h > 0$ and we get

$$0 \quad = \quad \mathbb{E}\Big[\frac{1}{h}\int_t^{(t+h)\wedge\tau} \big(\partial_t v + \mathcal{L}v\big)(u, X_u^{t,x})du\Big] \ .$$

From the mean value theorem and the dominated convergence theorem, we get the result by sending h to $0+$. □

Feynman-Kac representation of Cauchy problem We consider the following linear partial differential equation called Cauchy problem

$$\begin{cases} \partial_t v + \mathcal{L}v - kv + f = 0, & \text{on } [0,T) \times \mathbb{R}^n, \\ v(T,.) = g \ , & \text{on } \mathbb{R}^n \ . \end{cases} \qquad (6.3.6)$$

with k and f two functions from $[0,T] \times \mathbb{R}^n$ to $\mathbb{R}$. The next results provides a representation of this purely deterministic problem by means of stochastic differential equations.

Theorem 6.3.3 *Assume that the coefficients b and σ satisfy the assumptions of Theorem 6.1.1. Assume further that the function k is uniformly lower bounded, and f has quadratic growth in x uniformly in t. Let $v \in C^{1,2}([0,T)\times\mathbb{R}^n) \cap C^0([0,T]\times\mathbb{R}^n)$ be a solution of (6.3.6) with quadratic growth in x uniformly in t. Then*

$$v(t,x) \;=\; \mathbb{E}\Big[\int_t^T \beta_s^{t,x} f(s,X_s^{t,x})ds + \beta_T^{t,x} g(X_T^{t,x})\Big]$$

where $X^{t,x}$ is the unique solution to (6.3.5) and

$$\beta_s^{t,x} \;=\; e^{-\int_t^s k(u,X_u^{t,x})du}\,, \quad s\in[t,T]\,,$$

for $(t,x)\in[0,T]\times\mathbb{R}^d$.

Proof. We introduce the sequence of stopping times $(\tau_n)_n$ defined by

$$\tau_n \;=\; \Big(T-\frac{1}{n}\Big)\wedge \inf\{s>t \;:\; |X_s^{t,x}-x| \;\geq\; n\}\,, \quad n\geq 1\,.$$

We notice that that $\tau_n \uparrow T-$ $\mathbb{P}$-a.s. as $n\uparrow+\infty$. We then have from Ito's formula

$$\begin{aligned} d\big(\beta_s^{t,x} v(s,X_s^{t,x})\big) \;&=\; \beta_s^{t,x}(\partial_t v + \mathcal{L}v - kv)(s,X_s^{t,x})ds \\ &\quad + \beta_s^{t,x}\nabla v.\sigma(s,X_s^{t,x})dW_s\,, \end{aligned}$$

for $s\in[t,T)$. Since v is a solution of (6.3.6), we get

$$d\big(\beta_s^{t,x} v(s,X_s^{t,x})\big) \;=\; -\beta_s^{t,x} f(s,X_s^{t,x})ds + \beta_s^{t,x}\nabla v.\sigma(s,X_s^{t,x})dW_s\,.$$

Therefore, we get

$$\begin{aligned} v(t,x) - \mathbb{E}\big[\beta_{\tau_n}^{t,x} v(\tau_n, X_{\tau_n}^{t,x})\big] \;&=\; \mathbb{E}\Big[\int_t^{\tau_n} \beta_s^{t,x} f(s,X_s^{t,x})ds\Big] \\ &\quad - \mathbb{E}\Big[\int_t^{\tau_n} \beta_s^{t,x}\nabla v.\sigma(s,X_s^{t,x})dW_s\Big]\,. \end{aligned}$$

We then notice that the expectation of the stochastic integral term vanishes as the stochastic integrand is bounded since the function v is continuous, the function σ and satisfies (6.1.2) and the function k is lower bounded. Therefore, we get

$$v(t,x) \;=\; \mathbb{E}\left[\beta^{t,x}_{\tau_n} v(\tau_n, X^{t,x}_{\tau_n}) + \int_t^{\tau_n} \beta^{t,x}_s f(s, X^{t,x}_s) ds\right] . \qquad (6.3.7)$$

Using the continuity of v, the terminal condition of (6.3.6) and the continuity of $X^{t,x}$ we have

$$\beta^{t,x}_{\tau_n} v(\tau_n, X^{t,x}_{\tau_n}) + \int_t^{\tau_n} \beta^{t,x}_s f(s, X^{t,x}_s) ds \xrightarrow[n\to+\infty]{\mathbb{P}-a.s.} \beta^{t,x}_T g(X^{t,x}_T) + \int_t^T \beta^{t,x}_s f(s, X^{t,x}_s) ds .$$

Moreover, since k is lower bounded and the functions f and v have quadratic growth in x uniformly in t, there exists a constant C such that

$$\left|\beta^{t,x}_{\tau_n} v(\tau_n, X^{t,x}_{\tau_n}) + \int_t^{\tau_n} \beta^{t,x}_s f(s, X^{t,x}_s) ds\right| \;\leq\; C\Big(1 + \sup_{s\in[t,T]} |X^{t,x}_s|^2\Big) ,$$

for any $n \geq 1$. We can then apply the dominated convergence theorem and we get the result by sending n to ∞ in (6.3.7). $\square$