Cours 5 "Processus stochastiques et produits dérivés"

emmanuel.gobet@polytechnique.edu

Plan du cours

IV.1 Modelling the volatility

- ✓ Marché zone €
- ✓ Marché d'Itô dans le monde historique $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t}, \mathbb{P})$
- ✓ k mvts browniens $\widehat{W}^j \quad (1 \leq j \leq k)$
- ✓ 1 actif sans risque, rendement stochastique r_t: $\mathrm{d}S^0_t = S^0_t r_t \mathrm{d}t$
- ✓ d actifs risqués: $\mathrm{d}S^i_t = S^i_t(b^i_t \mathrm{d}t + \sum_{j=1}^k \sigma^i_j(t)\mathrm{d}\widehat{W}^j_t)$
- ✓ Vecteur (en ligne) de volatilité pour S^i: $\sigma^i_t := (\sigma^i_1(t), \cdots, \sigma^i_k(t))$
- ✓ Matrice de volatilité $\sigma_t := \begin{pmatrix} \sigma^1_t \\ \vdots \\ \sigma^d_t \end{pmatrix}$ de dimension $d \times k$

Formulation équivalente avec un MB par sous-jacent

Proposition. Il existe des MB $(B^1, \dots, B^d)$ tq

$$\mathrm{d}S_t^i = S_t^i(b_t^i \mathrm{d}t + |\sigma_t^i| \mathrm{d}B_t^i)$$

avec des MB corrélés instantanément via

$$\mathrm{d}\langle B^i, B^j \rangle_t = \frac{\sigma_t^i \cdot \sigma_t^j}{|\sigma_t^i||\sigma_t^j|} \mathrm{d}t.$$

La volatilité instantanée (vol spot) de S^i est $|\sigma_t^i|$.

Ne pas mélanger *vecteur de vol* et *vol spot*

IV.2 Self-financing portfolio and no arbitrage

Definition (Stratégie de portefeuille). On suppose les actifs négociables au comptant.

- ✓ Composition en nombre d'actifs: $\delta_t^0, \dots, \delta_t^d$
- ✓ Valeur liquidative: $V_t := \sum_{i=0}^d \delta_t^i S_t^i$
- ✓ Autofinancement (sans dividende/repo):

$$\mathrm{d}V_t := \sum_{i=0}^d \delta_t^i \mathrm{d}S_t^i$$

- ✓ Autofinancement (avec dividende/repo):

$$\mathrm{d}V_t := \sum_{i=0}^d \delta_t^i (\mathrm{d}S_t^i + q_t^i S_t^i \mathrm{d}t).$$

Il en découle

Proposition. En définissant le vecteur δ_t de nombre d'actifs risqués $\delta_t := (\delta_t^1, \dots, \delta_t^d)$, on a

$$
\begin{aligned}
\mathrm{d}V_t &= r_t V_t \mathrm{d}t + \delta_t \cdot (\mathrm{d}S_t - r_t S_t \mathrm{d}t) && \text{(sans div)} \\
&= r_t V_t \mathrm{d}t + (\delta S)_t \cdot (b_t - r_t \mathbf{1})\mathrm{d}t + (\delta S)_t \cdot \sigma_t \mathrm{d}\widehat{W}_t,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}V_t &= r_t V_t \mathrm{d}t + \delta_t \cdot (\mathrm{d}S_t - (r_t - q_t) S_t \mathrm{d}t) && \text{(avec div)} \\
&= r_t V_t \mathrm{d}t + (\delta S)_t \cdot (b_t + q_t - r_t \mathbf{1})\mathrm{d}t + (\delta S)_t \cdot \sigma_t \mathrm{d}\widehat{W}_t.
\end{aligned}
$$

Contraintes d'admissibilité sur les stratégies?

Absence d'Opportunité d'Arbitrage

Hyp. Il y a AOA entre stratégies de portefeuilles admissibles.

Theorem.

i) Un portefeuille admissible localement sans risque a le même rendement r_t que l'actif sans risque S^0:

$$\mathrm{d}V_t = V_t \mu_t \mathrm{d}t \qquad \Rightarrow \qquad \mu_t = r_t$$

ii) $\exists$ un vecteur aléatoire $\lambda_t \in \mathbb{R}^k$, appelé prime de risque tq

$$b_t = r_t \mathbf{1} + \sigma_t \lambda_t \qquad \text{(sans div)}$$

(et $b_t + q_t = r_t \mathbf{1} + \sigma_t \lambda_t$ (avec div)).

Corollary (sur la dynamique des portefeuilles autofinançants).

$$\mathrm{d}V_t = r_t V_t \mathrm{d}t + (\delta S)_t \cdot \sigma_t (\mathrm{d}\widehat{W}_t + \lambda_t \mathrm{d}t)$$

Probabilités risques-neutres: définie par

$$\mathbb{Q}\big|\mathcal{F}_T := e^{-\int_0^T \lambda_s\cdot \mathrm{d}\widetilde{W}_s - \frac{1}{2}\int_0^T |\lambda_s|^2 \mathrm{d}s}\ \mathbb{P}\big|_{\mathcal{F}_T} =: L_T\ \mathbb{P}\big|_{\mathcal{F}_T}.$$

Proposition.

i) $W_t = \widehat{W}_t + \int_0^t \lambda_s \mathrm{d}s$ est un $\mathbb{Q}$-MB

ii) S^i actualisé (au taux instantané $r_t - q_t^i$) est une $\mathbb{Q}$-martingale locale (perception neutre au risque sous $\mathbb{Q}$)

iii) V_t actualisé (au taux instantané r_t) est une $\mathbb{Q}$-martingale locale

Corollary. Si le flux Ψ_T à la date T est réplicable par une stratégie admissible de valeur V avec $(e^{-\int_0^t r_s \mathrm{d}s} V_t)_t$ $\mathbb{Q}$-martingale, alors nécessairement

$$e^{-\int_0^t r_s \mathrm{d}s} V_t = \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_0^T r_s \mathrm{d}s}\Psi_T \mid \mathcal{F}_t\right)$$

Et donc $\mathbf{C_t(\Psi_T, T)} = \mathbb{E}_{\mathbb{Q}}\left(\mathbf{e}^{-\int_t^T \mathbf{r_s}\mathrm{d}s}\mathbf{\Psi_T} \mid \mathcal{F}_\mathbf{t}\right)$

Si plusieurs λ, plusieurs prix?

IV.3 Marché complet

Theorem (de valorisation). Supposons

i) $k = d$ et σ_t inversible

ii) la prime de risque λ_t est bornée

iii) $\mathbb{E}\left(e^{-c\int_0^T r_s \mathrm{d}s}\right) < +\infty$ pour tout $c > 0$

Alors le marché est complet: tout flux financier Ψ_T (à la date T) de carré intégrale sous $\mathbb{P}$ est réplicable par une stratégie admissible, avec $(e^{-\int_0^t r_s \mathrm{d}s} V_t)_t$ $\mathbb{Q}$-martingale.

Proof. Plus tard à l'aide des BSDE. □

IV.4 Markets with friction and non-linear valuation

Previous linear valuation/hedging problem under the form of a **backward problem**

$$\begin{cases} \mathrm{d}V_t = r_t V_t \mathrm{d}t + \sum_{i=1}^{d} \pi_t^i \sigma_t^i (\mathrm{d}\widehat{W}_t + \lambda_t \mathrm{d}t), \\ V_T = \Psi_T, \end{cases} \tag{1}$$

where $\pi_t^i = \delta^i(t) S_t^i$ is the amount in the asset i.

Included into setting of Backward Stochastic Differential Equation.

Definition (BSDE). A BSDE with terminal condition ξ and driver (or generator) f is an equation with unknown processes (Y, Z) of the form

$$-\mathrm{d}\mathbf{Y_t} = \mathbf{f}(\mathbf{t}, \mathbf{Y_t}, \mathbf{Z_t})\mathrm{d}\mathbf{t} - \mathbf{Z_t}\mathrm{d}\widehat{\mathbf{W}}_\mathbf{t}, \qquad \mathbf{Y_T} = \xi. \tag{2}$$

Here $Y_t \in \mathbb{R}, Z \in \mathbb{R}^k$ (as a row). Both ξ and f can be stochastic.

Example. For (1), take $\mathbf{f}(\mathbf{t}, \mathbf{y}, \mathbf{z}) := -\mathbf{r_t}\mathbf{y} - \mathbf{z}\lambda_\mathbf{t}$ and set $Z_t := \sum_{i=1}^d \pi_t^i \sigma_t^i = (\pi_t^1 \cdots \pi_t^d)\sigma_t = \pi_t^\top \sigma_t$ and $Y_t := V_t$.

Non-linear cases

Assume σ invertible, so that λ is uniquely given by $\lambda_t := \sigma_t^{-1}(b_t - r_t \mathbf{1}_d)$.

Example (Pricing/hedging with two interest rates [Bergman 95]). Assuming that borrowing is made at rate R_t and lending at rate $r_t \le R_t$: it is associated to the BSDE

$$\mathrm{d}\mathbf{Y_t} = \mathbf{r_t}\mathbf{Y_t}\mathrm{d}t - (\mathbf{Y_t} - \mathbf{Z_t}\sigma_{\mathbf{t}}^{-1}\mathbf{1_d})_-(\mathbf{R_t} - \mathbf{r_t})\mathrm{d}t + \mathbf{Z_t}\lambda_{\mathbf{t}}\mathrm{d}t + \mathbf{Z_t}\mathrm{d}\widehat{\mathbf{W}}_{\mathbf{t}}. \tag{3}$$

Similar to linear case, with an additional cost of borrowing cash.

Example (Pricing/hedging with short-selling constraints [Jouini etal 95]). If $\pi_t^i < 0$ (shorting the asset i) gives rise to an additional repo rate q_t^i ➠

$$\mathrm{d}\mathbf{V_t} = (\mathbf{V_t} - \sum_{\mathbf{i=1}}^{\mathbf{d}} \pi_{\mathbf{t}}^{\mathbf{i}})\mathbf{r_t}\mathrm{d}t + \sum_{\mathbf{i=1}}^{\mathbf{d}} (\pi_{\mathbf{t}}^{\mathbf{i}})_- \mathbf{q_t^i}\mathrm{d}\mathbf{t} + \sum_{\mathbf{i=1}}^{\mathbf{d}} \pi_{\mathbf{t}}^{\mathbf{i}}(\mathbf{r_t} + \sigma_{\mathbf{t}}^{\mathbf{i}}\lambda_{\mathbf{t}}\mathrm{d}t + \sigma_{\mathbf{t}}^{\mathbf{i}}\mathrm{d}\widehat{\mathbf{W}}_{\mathbf{t}} \tag{4}$$

➠ non linear BSDE.

A PRIORI ESTIMATES

Definition. In all the sequel, ξ and f satisfy standard assumptions (standard BSDE) if

✓ $\mathbb{E}\left[\xi^2\right] < +\infty$,

✓ $f : (t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^k \mapsto f(t, \omega, y, z) =: f(t, y, z)$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^k)$-measurable map,

✓ $f(., 0, 0)$ is such that $\mathbb{E}\left[\int_0^T f^2(t, 0, 0)\mathrm{d}t\right] < +\infty$,

✓ f globally Lipschitz:

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C_f(|y_1 - y_2| + |z_1 - z_2|), \qquad \forall y_1, z_1, y_2, z_2.$$

$\mathbb{H}_2(\mathbb{R}^q)$: space of predicable processes taking values in $\mathbb{R}^q$.

Norm: $\|\mathbf{U}\|^2_{2,\beta} := \mathbb{E}\left[\int_0^T e^{\beta t}|\mathbf{U}_t|^2 \mathrm{d}t\right]$ for $\beta \in \mathbb{R}$.

Theorem (a priori estimates). Let (Y^1, Z^1) and (Y^2, Z^2) two processes in $\mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^k)$ solutions to *standard* BSDEs associated to two drivers f_1 and f_2, and two terminal conditions ξ_1, ξ_2. Then, denote C the Lipschitz constant for f_1 and set

$$\delta_2 f_t = f_1(t, Y_t^2, Z_t^2) - f_2(t, Y_t^2, Z_t^2);$$

for any $\mu > 0$ and setting $\beta := C^2 + 3C + \mu^2$, we have

$$\|Y^1 - Y^2\|_{2,\beta}^2 \leq T \left[e^{\beta T} \mathbb{E}\left[|\xi_1 - \xi_2|^2\right] + \frac{1}{\mu^2} \|\delta_2 f_.\|_{2,\beta}^2 \right], \tag{5}$$

$$\|Z^1 - Z^2\|_{2,\beta}^2 \leq (C+1) \left[e^{\beta T} \mathbb{E}\left[|\xi_1 - \xi_2|^2\right] + \frac{1}{\mu^2} \|\delta_2 f_.\|_{2,\beta}^2 \right]. \tag{6}$$

Corollary (existence and uniqueness). Assume standard conditions on ξ and f, then there is a unique solution $(Y, Z) \in \mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^k)$ to (2). In addition the solution Y is continuous.

Theorem (linear BSDE). Assume that $\mathbf{f(t,y,z)} = \varphi_{\mathbf{t}} + \beta_{\mathbf{t}}\mathbf{y} + \gamma_{\mathbf{t}} \cdot \mathbf{z}$ with predictable coefficients s.t. $\varphi \in \mathbb{H}_2(\mathbb{R})$ and β, γ are bounded. Then for any square integrable ξ, the BSDE solution to the data (ξ, f) is explicitly given by

$$Y_t = \mathbb{E}\left[\xi \frac{H_T}{H_t} + \int_t^T \varphi_s \frac{H_s}{H_t} \mathrm{d}s \mid \mathcal{F}_t\right], \tag{7}$$

where $H_t := \exp\left(\int_0^t (\beta_u - \frac{1}{2}|\gamma_u|^2 \mathrm{d}u) + \int_0^t \gamma_u \mathrm{d}\widehat{W}_u\right)$.

Corollary (linear pricing rule, where $\varphi_t \equiv 0$, $\beta_t = -r_t$, $\gamma_t = -\lambda_t^\top$). Define the new probability measure $\mathbb{Q}^\lambda$ by

$$\mathbb{Q}^\lambda|_{\mathcal{F}_\mathbf{T}} = \mathbf{e}^{-\int_\mathbf{0}^\mathbf{T} \lambda_\mathbf{u}^\top \mathrm{d}\widehat{\mathbf{W}}_\mathbf{u} - \frac{\mathbf{1}}{\mathbf{2}} \int_\mathbf{0}^\mathbf{T} |\lambda_\mathbf{u}|^\mathbf{2} \mathrm{d}\mathbf{u}} \mathbb{P}|_{\mathcal{F}_\mathbf{T}}, \tag{8}$$

then

$$\mathbf{Y_t} = \mathbb{E}_{\mathbb{Q}^\lambda}\left(\mathbf{e}^{-\int_\mathbf{t}^\mathbf{T} \mathbf{r_s} \mathrm{d}\mathbf{s}} \xi \mid \mathcal{F}_\mathbf{t}\right) ! \tag{9}$$

We retrieve the risk-neutral valuation using the risk-neutral measure $\mathbb{Q}^\lambda$.
But the BSDE tools provide the price but also the hedging strategy through the Z-process.

Theorem (comparison). Consider two standard BSDEs such that

- ✓ $\xi_1 \geq \xi_2$ a.s.
- ✓ $\delta_2 f_t = f_1(t, Y_t^2, Z_t^2) - f_2(t, Y_t^2, Z_t^2) \geq 0, \mathrm{d}t \otimes \mathrm{d}\mathbb{P}$ a.e.

Then, $Y_t^1 \geq Y_t^2$ for any $t \in [0, T]$ a.s.

Application: For linear and non-linear pricing rule, two payoffs $\boldsymbol{\xi_1 \geq \xi_2}$ generate two prices $\mathbf{Y_t^1} \geq \mathbf{Y_t^2}$!

Exercise. Assume a Black-Scholes model with drift μ and volatility $\sigma \neq 0$, and two interest rates for borrowing and lending $R > r$. Prove that

- ✓ the call price is given by the usual Black-Scholes formula with the interest R.
- ✓ the put price is given by the usual Black-Scholes formula with the interest r.

Give a financial interpretation.

Résultats techniques

Proposition. Soit

$$\mathbb{Q}^L\big|\mathcal{F}_T := L_T \;\; \mathbb{Q}\big|_{\mathcal{F}_T}$$

avec $L_t > 0$, $\mathbb{Q}$-martingale, avec $L_0 = 1$.

1. Si Z_T est $\mathcal{F}_T$-mesurable, et $\mathbb{E}_{\mathbb{Q}}(|Z_T|) < +\infty$, alors

$$\mathbb{E}_{\mathbb{Q}^L}\left(\frac{|Z_T|}{L_T}\right) < +\infty.$$

2. (formule de Bayes généralisée)

$$\mathbb{E}_{\mathbb{Q}^L}\left(\frac{Z_T}{L_T} \mid \mathcal{F}_t\right) = \frac{1}{L_t}\mathbb{E}_{\mathbb{Q}}\left(Z_T \mid \mathcal{F}_t\right).$$

3. M est $\mathbb{Q}$-martingale si et seulement si $\frac{M}{L}$ est $\mathbb{Q}^L$-martingale