Stochastic modelling of derivatives – Exam
Master M2 PF, SU-X 2024-2025

Duration: 3h.
No document and no electronic devices allowed.
Answers can be written either in French or in English.

Problem: Asian options

We consider a financial market Black-Scholes model with one riskless asset $S_t^0 = e^{rt}$ with constant interest rate $r \geq 0$, and a risky asset S driven under the risk-neutral measure $\mathbb{P}$ by

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

with $\sigma > 0$, and W a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_t, \mathbb{P})$. We consider an Asian option with strike $K > 0$, and maturity T, i.e., of payoff

$$H = \Big(\frac{1}{T}\int_0^T S_t dt - K\Big)_+.$$

The following three parts can be handled independently.

Part I: comparison of Asian vs European option

1) Recall why the price of the Asian option is given at time $t \in [0, T]$ by

$$V_t = \mathbb{E}\Big[e^{-r(T-t)}\Big(\frac{1}{T}\int_0^T S_t dt - K\Big)_+ \big| \mathcal{F}_t\Big].$$

2) Prove that on the event $\{\frac{1}{T}\int_0^t S_u du \geq K\}$, we have

$$V_t = \frac{e^{-r(T-t)}}{T}\int_0^t S_u du + \frac{1 - e^{-r(T-t)}}{rT} S_t - Ke^{-r(T-t)}.$$

3) We set $\tilde{S}_t = e^{-rt}S_t$ the discounted risky asset price at time t.

a) Show the inequality

$$\mathbb{E}\big[(\tilde{S}_t - Ke^{-rT})_+\big] \leq \mathbb{E}\big[e^{-rT}(S_T - K)_+\big].$$

b) Show the inequality

$$e^{-rT}\Big(\frac{1}{T}\int_0^T S_t dt - K\Big)_+ \leq \frac{1}{T}\int_0^T (\tilde{S}_t - Ke^{-rT})_+ dt,$$

and deduce that

$$V_0 \leq \mathbb{E}\big[e^{-rT}(S_T - K)_+\big],$$

i.e., the price of the Asian option is smaller than the price of the European option.

Part II: replicating strategy for Asian option

4) We denote by $(X_t)_t$ the process defined by

$$X_t = \frac{1}{S_t}\Big(\frac{1}{T}\int_0^t S_u du - K\Big).$$

Show that $(X_t)_t$ is solution to the stochastic differential equation

$$dX_t = \Big(\frac{1}{T} + (\sigma^2 - r)X_t\Big)dt - \sigma X_t dW_t.$$

5) Prove that $V_t = e^{-r(T-t)} S_t F(t, X_t)$, where F is the function defined on $[0,T] \times (0,\infty)$ by

$$F(t,x) = \mathbb{E}\Big[\Big(x + \frac{1}{T}\int_t^T S_u^t du\Big)_+\Big],$$

with $S_u^t = \exp\big((r - \frac{\sigma^2}{2})(u-t) + \sigma(W_u - W_t)\big)$.

6) Assume that F is $C^{1,2}$. By applying Itô formula, and using the $\mathbb{P}$-martingale property of $e^{-rt}V_t$, derive the PDE satisfied by F, and find a perfect replicating strategy for the Asian option, that is an adapted process θ (expressed in terms of F) representing the number of shares invested in S, s.t.

$$dV_t = rV_t dt + \theta_t(dS_t - rS_t dt).$$

Part III: Approximation of Asian option

7) We consider an approximation of the arithmetic mean by geometric mean, and define

$$\hat{V}_0 = e^{-rT}\mathbb{E}\Big[\Big(e^{\frac{1}{T}\int_0^T \log S_t dt} - K\Big)_+\Big].$$

Show that $V_0 \geq \hat{V}_0$.

8) a) Show that $\int_0^T W_t dt$ is a centered Gaussian random variable with variance $\frac{T^3}{3}$.

b) Deduce that

$$\hat{V}_0 = e^{-rT}\mathbb{E}\Big[\Big(S_0 e^{(r-\frac{\sigma^2}{2})\frac{T}{2} + \sigma\sqrt{\frac{T}{3}}Z} - K\Big)_+\Big],$$

where Z is a reduced centered Gaussian random variable.

c) Give an explicit form of $\hat{V}_0$ in terms of the cumulative distribution function N of Z.

9) Justify that

$$\Big(\frac{1}{T}\int_0^T S_t dt - K\Big)_+ - \Big(e^{\frac{1}{T}\int_0^T \log S_t dt} - K\Big)_+ \leq \frac{1}{T}\int_0^T S_t dt - e^{\frac{1}{T}\int_0^T \log S_t dt}$$

and prove the inequality

$$V_0 - \hat{V}_0 \leq S_0 e^{-rT}\Big[\frac{e^{rT} - 1}{rT} - e^{\frac{rT}{2} - \frac{\sigma^2 T}{12}}\Big].$$

Exercise 1 – Implied density vs implied volatility

We are given, at time t, market prices $C(t, S_t, K, T)$ for calls on a same underlying of price S_t, where T is the maturity and K the strike. We assume a constant risk-free rate r and no dividends. We recall that in Black-Scholes framework the price writes $C^{\mathrm{BS}}(t, S_t, K, T, \sigma) = S_t N(d_+(S_t, K, \sigma)) - Ke^{-r(T-t)} N(d_-(S_t, K, \sigma))$, with N the standard Gaussian cdf and $d_\pm(x, y, \sigma) = \frac{1}{\sigma\sqrt{T-t}}\left[\ln(x/y) + (r \pm \sigma^2/2)(T-t)\right]$.

1. Prove that $\partial C^{\mathrm{BS}}/\partial K = -e^{-r(T-t)} N(d_-(S_t, K, \sigma))$ and that $\partial C^{\mathrm{BS}}/\partial \sigma = S_t\sqrt{T-t}N'(d_+(S_t, K, \sigma))$.
2. Prove Breeden-Litzenberger formula, which provides the density of S_T conditional on S_t under the risk-neutral probability:
$$q_t(S_T) = e^{r(T-t)} \left.\frac{\partial^2 C(t, S_t, K, T)}{\partial K^2}\right|_{K=S_T}.$$
3. We write $V(K)$ the implied volatility in Black-Scholes model for the call of strike K, supposed here not to depend on the maturity. We assume that V is two times differentiable. Show that
$$\frac{\partial d_\pm(S_t, K, V(K))}{\partial K} = -\frac{1}{KV(K)\sqrt{T-t}} - \frac{V'(K)}{V(K)} d_\mp(S_t, K, V(K)).$$
4. Write the Breeden-Litzenberger implied density using V and its derivatives (in other words, option prices must not appear explicitly in your final formula). In the particular case where $S_T = S_t e^{r(T-t)}$, give a more concise expression of the form $\alpha_1(T-t)^{-1/2} + \alpha_2(T-t)^{1/2} + \alpha_3(T-t)^{3/2}$, with the α_i to be determined.

Exercise 2 – Parkinson's measure of volatility

1. Considering the cumulated sum $X_n = \sum_{i=1}^n U_i$, with iid random variables U_i such that $\mathbb{P}(U_i = 1) = \mathbb{P}(U_i = -1) = 1/2$, show, using the law of total probability, that
$$\mathbb{P}(M_n \geq a) = 2\mathbb{P}(X_n > a) + \mathbb{P}(X_n = a),$$
where $a \geq 0$ is an integer and $M_n = \max_{0 \leq i \leq n} X_i$. Hint: you will have to compare first $\mathbb{P}(M_n \geq a, X_n < a)$ and $\mathbb{P}(M_n \geq a, X_n > a)$.
2. Deduce that, for any real number $\alpha \geq 0$, we have
$$\mathbb{P}\left(\max_{0 \leq i \leq n} \frac{1}{\sqrt{n}} X_i \geq \alpha\right) \overset{n\to\infty}{\longrightarrow} 2N(-\alpha),$$
with N the standard Gaussian cdf.
3. Conclude that, for a Brownian motion W_t, we have
$$\mathbb{P}\left(\sup_{t\in[0,1]} W_t \leq \alpha\right) = \sqrt{\frac{2}{\pi}} \int_0^\alpha \exp\left(-\frac{u^2}{2}\right) du.$$
4. We describe the price dynamics of a stock using the Bachelier model: $S_t = S_0 + \sigma W_t$. We want to define an estimator of σ in Parkinson's style:
$$\widehat{\sigma}_n = \frac{\gamma}{n} \sum_{i=1}^n \left(H_{(i-1)\tau, i\tau} - L_{(i-1)\tau, i\tau}\right),$$
where $H_{t-\tau,t} = \sup_{s\in[t-\tau,t]} S_s$ is the high price in a time interval of duration τ, $L_{t-\tau,t} = \inf_{s\in[t-\tau,t]} S_s$ is the corresponding low price. What must be the value of the scalar γ so that $\widehat{\sigma}_n$ is a consistent estimator of σ?

Section: Decentralized Finance & Blockchain Technology
Master 2, Probabilité et Finance
January 2025

Part 1 - Blockchain

Bitcoin

1. Formula for Bitcoin Supply.

   Write the formula that determines the total supply of Bitcoin issued over time, given the following parameters:
   - The initial number of bitcoins issued per block is 50.
   - The number of blocks between halvings is 210,000.
   - The total number of halvings expected is 32.
   - The maximum Bitcoin supply that can ever be issued approaches 21 million.

   The process starts at block 0, with the number of bitcoins issued per block halving (divided by 2) at each subsequent halving.

2. Bitcoin Supply After 10 Halvings.

   - Using the formula from Question 1, calculate the total supply of Bitcoin after 10 halvings.
   - How many blocks have been mined by this point?

   Make sure to show all the steps in your calculation.

3. Market Analysis

Using the law of supply and demand, explain how Bitcoin's market value should theoretically change when a halving occurs. Additionally, discuss why the expected market behavior, as predicted by this theory, has not been observed in Bitcoin markets during the past four halvings.

Ethereum

4. Gas Fees on the Ethereum Network

   Which of the following statements correctly describes gas fees on the Ethereum network?
   - [ ] Gas fees are proportional to the amount of tokens transferred on the Ethereum blockchain.
   - [x] The gas required for a transaction depends on the complexity of the execution.
   - [x] Gas fees are determined by the gas price and the transaction's gas consumption.
   - [ ] Gas fees are determined by the gas price only.
   - [x] The gas price depends on the number of Ethereum accounts performing transactions at a given time (network congestion).
   - [ ] The gas price is independent of the number of Ethereum accounts performing transactions at a given time (network congestion).

Part 2 - Crypto Markets

Centralized Exchanges & Orderbooks

5. Orderbook Basics

Centralized exchanges often use an order book mechanism to match buy and sell orders. Consider the following order book for a hypothetical cryptocurrency pair, BTC-USD.

Order Book Example for BTC-USD on Coinbase:

Price (USD)	Quantity (BTC)	Side
30,000	1	Sell
29,900	2	Sell
29,800	0.5	Sell
29,700	1	Buy
29,600	3	Buy
29,500	0.8	Buy

Based on the above order book data, answer the following questions:

- What is the current best ask?
- What is the current best bid?
- What is the spread between the best bid and best ask?
- If a sell market order for 1 BTC is placed, at which price will it be executed?
- What are the three key liquidity measures used to assess the liquidity of a market that utilizes an order book? *Reminder: Liquidity refers to how easily an asset can be bought or sold without significantly impacting its price.*

6. Orderbook Liquidity and Execution Cost

In this question, we ignore the existence of trading fees.
Based on the order book data provided above, answer the following questions:

- If a buy market order for 2.8 BTC is placed, calculate the impact (in percentage) on the price due to slippage.
- At which price will this order be executed?

Decentralized Exchanges (Univwap V2, Automated Market Makers)

Question 7 (Liquidity Provision):

Uniswap V2 is an automated market maker that operates as a constant product market maker, where: $k = x.y$ Use this rule in this exercise.

Consider a liquidity pool containing 3500 tokens X and 1500000 tokens Y.

The current price in the liquidity pool is $p_0 = 428.57$

Assume that no swaps have taken place in the pool, so that the total amount of supplied liquidity S coincides with L, the actual liquidity available in the pool. Then, we have S = L = 75497.7

A liquidity provider (LP) wishes to deposit $\Delta x = 450$ tokens X into the pool.

- Calculate the number of tokens Y (Δy) that the LP needs to deposit at the same time.
- Determine the resulting liquidity in the pool L_0 after both quantities of tokens have been deposited.
- Calculate the additional liquidity ΔS marked to the LP and the percentage of the supplied liquidity in the pool that the LP owns.

Question 8 (Trade, Swap):

Consider a trader wishing to transfer tokens Y into the pool and receive $\Delta x = 550$ tokens X in exchange, after the LP added liquidity to the pool (question 7). Assuming the price is still $p_0 = 428.57$:

- Calculate the amount Δy of tokens Y to transfer into the pool.
- Determine the new price in the pool (p_1) after the swap operation.
- Calculate the liquidity in the pool (L_1) after the swap operation and compare it with the supplied liquidity (S).