Microstructure

Mathematical tools for metaorders

We need to answer the following questions:

- How to model market impact?
- How to test strategies?
- How to estimate interdependence between stocks?

Almgren and Chriss framework provides a strategy for the order execution in terms of mean–variance optimization:

$$\mathbb{E}[C] + \lambda \mathbb{V}\mathrm{ar}[C] \to \min$$

where λ is a risk aversion parameter to be determined. There are two components of market impact in the model — permanent (other participants revise prices seeing us selling) and temporary (fees, liquidity in the order book).

For permanent market impact (here $\tau = t_k - t_{j-1}$ and n_k is a number of stocks sold during the time imterval):

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\xi_k - \tau g(n_k/\tau), \quad \xi_k \sim \mathcal{N}(0,1).$$

For the permanent impact mean price of selling between t_{k-1} and t_k is:

$$\tilde{S}_k = S_{k-1} - h(n_k/\tau)$$

It a simple analytically solvable model g is assumed to be linear and h is assumed to be affine.

Cost of selling X stocks is determined by

$$C = XS_0 - \sum_{k=1}^{N} n_k \tilde{S}_k$$

This optimization provides a time-homogeneous optimal strategy.

In order to find an optimal strategy via Almgren and Chriss approach for two or more assets one may need an **estimate of the covariance** between them.

Here, at the micro structural level the Hayashi–Yoshida estimator can be used for asynchronous prices data. Namely, for two processes X and Y with observation times (T_i^X) and (T_i^Y) we denote by $I_i^X = (T_i^X, T_{i+1}^X]$ and $I_i^Y = (T_i^Y, T_{i+1}^Y]$. The Hayashi–Yoshida estimator is defined by

$$U_n = \sum_{i,j} \Delta X(I_i^X) \Delta Y(I_j^X) \mathbb{1}_{\{I_i^X \cap I_j^Y \neq \emptyset\}}$$

It works well (i.e. estimates the covariance $\int_0^1 \rho_t \sigma_t^X \sigma_t^Y dt$) when applied to the estimated values of efficient prices (cf. model with uncertainty zones).

However, Almgren and Chriss approach doesn't say anything about optimal execution strategies between t_{k-1} and t_k. To do it in efficient way, one should have a **good model for a limit order book (LOB)** taking into account participants' behavior and orders' dynamics. Such a model will allow to simulate market impact and costs of the strategies via Monte Carlo.

How to asses the quality of the tick value?

Signature plot:

$$RV_n(k) = \sum_{i=1}^{\lfloor n/k \rfloor - 1} (P_{\frac{k(i+1)}{n}} - P_{\frac{ki}{n}})^2$$

For $\dfrac{n}{k}$ large enough it is close to the quadratic variation. However, it practice often decreasing due to microstructural effects. For a good tick value, signature plot remains flat and the process resembles a continuous semimartingale.

Link between tick size and **market making**: market makers provide liquidity, make money getting the spread and taking the volatility risk. Market takers pay the spread but no not take the volatility risk. On can show that

- Market and limit orders will have the same average cost equal to zero
- Market maker's P&L = 0

Therefore $\mathcal{S} \sim c\sigma_1$ where $\mathcal{S}$ denotes the spread and σ_1 is a volatility per trade.

Model with uncertainty zones allows us to estimate an implicit spread. Let X_t denote unobservable efficient price, α is a tick value, (t_i) are times of price changes, P_t observable price.

Transaction times: $t_{i+1} = \inf\{t > t_i : X_t = X_{t_i}^{(\alpha)} \pm \alpha(0.5 + \eta)\}$.

One can estimate α counting the numbers of continuations N^c and alternations N^a of price. For continuation distance to the barrier is equal to α, for alternation it is $2\eta\alpha$. Applying stopping theorem, one obtains and estimator

$$\hat{\eta} = \frac{N^c}{2N^a},$$

which can be used to reconstruct the efficient price:

$$\hat{X}_{t_i} = P_{t_i} - \alpha(0.5 - \hat{\eta})\text{sign}(P_{t_i} - P_{t_{i-1}}).$$

Intuitions about η

- $2\eta\alpha$ us an implicit unobservable spread.
- If η is too small –> uncertainty zone is small –> strong mean reversion of the observed price –> decreasing signature plot. It means that <u>tick size is too big</u>.
- $\eta \sim \frac{1}{2}$ –> last traded price can be seen as a sampled Brownian motion –> no microstructure effects –> flat signature plot –> uncertainty zone ≈ 1 tick –> optimal.

The following linear model describes well the spread:

$$\eta\alpha \sim \frac{\sigma}{\sqrt{M}} + kS$$

where M is a number of trades and σ is a price process volatility.

Conclusions

- Too small α encourages free-riding (directional HFT) and market makers cannot fix their quotes.
- Too large α implies price sloppiness and favors speed –> investments in inrastructure.
- Optimal situation:
    - Spread ≈ 1 tick
    - $\eta \approx \frac{1}{2}$
    - Cost of market orders = Cost of limit orders = 0

How to asses the quality of the order book model?

Aims of the order book model:

- Reproduce the behavior of market participants at different limits (market participants' intellegence)
- Provide a market simulator enabling, for example, to compute execution costs
- Correctly model behavior at different limits
- Explain the empirical shape of the book, bid-ask spread, and the joint laws of the limits.
- Empirical data also contains the intensities of limit orders insertion / cancellation and market order consumption for each limit and an ergodic distribution of the price process.
- Provide a diffusive behavior at large time scales.