Master 2 – Probabilités et Finance
Sorbonne Université et Ecole Polytechnique

Convexity, Optimization and Stochastic Control

Session II

Exercise I. We consider a process X defined by the stochastic differential equation : $X_0 \in \mathbf{L}^2(\mathbb{R})$,

$$dX_t = (a + bX_t)dt + \sqrt{\sigma + \theta X_t} dW_t, \ t \leq T$$

with W a scalar Brownian motion, and given scalar parameters a, b, σ, θ. The existence and uniqueness of a square integrable solution $X \in \mathbb{H}^2$ of this equation is admitted.

1. For each $t \in [0, T]$, justify that X_t is integrable and compute $\mathbb{E}[X_t]$.
2. Provide the stochastic differential equation satisfied by the process $(X_t^n, t \geq 0)$ for any positive integer n.
3. Compute $\mathbb{E}[X_t^2]$ (assuming that all local martingales appearing in the calculation are martingales).
4. We assume that $\mathbb{E}\left[e^{-\frac{\lambda^2}{2}X_T}\right] < \infty$. Justify that, for $t < T$ and $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left[e^{-\frac{\lambda^2}{2}X_T} \mid \mathcal{F}_t\right] = \psi(t, X_t)$$

   for some function ψ. Assuming that ψ is $C^{1,2}$, show that the calculation of $\mathbb{E}\left[e^{-\frac{\lambda^2}{2}X_T}\right]$ can be reduced to a partial differential equation.
5. Now, we assume that $\mathbb{E}\left[e^{-\frac{\lambda^2}{2}X_T - \mu \int_0^T X_s ds}\right] < \infty$. Show that the calculation of

$$\mathbb{E}\left[e^{-\frac{\lambda^2}{2}X_T - \mu \int_0^T X_s ds}\right]$$

   can be reduced to a partial differential equation.

Exercise II. Let S be the price process of a risky security with dynamics

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dB_t,$$

where r is a constant interest rate, δ is a constant dividend rate paid by the security, and B is a Brownian motion (under the risk-neutral measure). Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be bounded measurable map. The price of a European option with payoff $g(S_T)$ is given by

$$V(t, S_t) := \mathbb{E}_t\left[e^{-r(T-t)} g(S_T)\right],$$

with $\mathbb{E}_t$ denoting the conditional expectation on the information available at time t.

1. Justify that V is a smooth $C^{1,2}$ function.
2. Provide the partial differential equation satisfied by V.

Exercise III (Bonus). Let S be the price process of a risky security with dynamics

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t,$$

where r is a constant interest rate, and B is a Brownian motion (under the risk-neutral measure). An Asian option is defined by the payoff $(Y_T - K)^+$ with $Y_T := \int_0^T S_u du$. Given the spot price S_t and the integral

$$Y_t := \int_0^t S_u du,$$

the price at time t of the Asian option is given by

$$V(t, S_t, Y_t) := \mathbb{E}_t\left[e^{-r(T-t)}(Y_T - K)^+\right],$$

with $\mathbb{E}_t$ denoting the conditional expectation on the information available at time t.

1. Derive the dynamics of the process Y, and justify that the pair process (S, Y) is Markovian.
2. Assuming V is smooth, provide the partial differential equation satisfied by V.
3. We introduce the guess that $V(t, s, y) = s\phi(t, z(s, y))$ with $z(s, y) = \frac{y-k}{s}$. By plugging this form into the PDE satisfied by V, show that ϕ satisfies a PDE in the variables (t, z).
4. Provide a stochastic representation for the function ϕ.

Exercise IV. Let $T > 0$ and W be a $\mathbb{F}$–Brownian motion. We denote by $\mathcal{A}$ the set of $\mathbb{F}$–predictable processes $(\alpha_t)_{t\in[0,T]}$ satisfying $\|\alpha\|_\infty \leq 1$. For any $\alpha \in \mathcal{A}$, we introduce (Y^α, Z^α) two processes such that

$$Y_t^\alpha = Y_0 + \int_0^t \alpha_s \mathrm{d}W_s \text{ and } Z_t^\alpha = Z_0 + \int_0^t Y_s^\alpha \mathrm{d}W_s, \text{ with } t \in [0, T].$$

Let λ be a positive parameter verifying $2\lambda T < 1$. We define the function $[0, T] \times \mathbb{R} \times \mathbb{R} \ni (t, y, z) \mapsto v(t, y, z) \in \mathbb{R}$ by

$$v(t, Y_t^1, Z_t^1) := \mathbb{E}\big[e^{2\lambda Z_T^1}\big|\mathcal{F}_t\big].$$

1. Provide the partial differential equation satisfied by v.
2. Compute explicitly $v(t, y, z)$ for any (t, y, z).
3. Show that v is strictly convex in y and that $y\, v_{yz}(t, y, z) \geq 0$.
4. Show that

$$v(0, Y_0, Z_0) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}\big[e^{2\lambda Z_T^\alpha}\big].$$

Problem. We consider a continuous time financial market, containing a risk-free asset S^0, with a price normalized to 1, and a risky asset S with a price process defined by the SDE :

$$\frac{\mathrm{d}S_t}{S_t} = \sigma(Y_t)\left(\lambda(Y_t)\mathrm{d}t + \mathrm{d}W_t^1\right), \tag{1}$$

where Y is a state variable whose dynamics is governed by :

$$dY_t = \eta(Y_t)\, dt + \gamma(Y_t)\left[\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right]. \tag{2}$$

Here, $W = (W^1, W^2)$ denotes a $\mathbb{R}^2$–valued Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, et ρ is a constant in $]-1, 1[$. The coefficients $\eta(y)$, $\gamma(y)$, $s\lambda(y)$, $s\sigma(y)$ satisfy the usual conditions for the existence and uniqueness of strong solution of two dimensional SDE (1)-(2). In addition, we assume that the maps $\lambda(.)$, $\sigma(.)$ are bounded, and that $\inf_{y\in\mathbb{R}} \sigma(y)^2 + \gamma^2(y) > 0$. We will denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the canonical filtration of W.

A strategy θ is a $\mathbb{F}$ predictable process. For any initial capital x and any strategy θ, one defines the wealth process by :

$$X_t^{x,\theta} = x + \int_0^t \theta_r \frac{\mathrm{d}S_r}{S_r}, \qquad t \geq 0.$$

We say that θ is an admissible strategy if $X^{0,\theta}$ is well–defined and is uniformly bounded by below by a constant. We use the notation $\mathcal{A}$ for the set of admissible strategies.

Finally, consider a European option on the non-exchangeable variable Y :

$$G = g(Y_T) \quad \text{with} \quad g : \mathbb{R} \longrightarrow \mathbb{R} \text{ bounded continuous}$$

and we define the optimal investment problem by :

$$V^g(0,x,Y_0) := \sup_{\theta \in \mathcal{A}} \mathbb{E}\left[U\left(X_T^{x,\theta} - g(Y_T)\right)\right] \quad \text{where} \quad U(x) := -e^{-ax},\ a > 0,\ x \in \mathbb{R}.$$

Our goal is to determine an explicit expression for :

$$p(0,x,y) \quad := \quad \inf\left\{\pi \ : \ V^g(0,x+\pi,y) \ \geq \ V^0(0,x,y)\right\} .$$

In the following, we extend as usual the preceding quantities in the case where the initial time is t by noting $V^g(t,x,y)$ and $p(t,x,y)$. Besides, we suppose that the map V^g is $C^{1,2}\left([0,T),\mathbb{R}\times\mathbb{R}\right) \cap C\left([0,T]\times\mathbb{R}\times\mathbb{R}\right)$, for any function g satisfying the previous conditions

1. Give a financial interpretation of the map p. Let $\hat{\pi} \in \mathbb{R}$ and $\hat{\theta} \in \mathcal{A}$ such that $X_T^{\hat{\pi},\hat{\theta}} \geq G$ a.s. and $X_T^{\hat{\pi},\hat{\theta}}$ is bounded. Show that $p(0,x,y) \leq \hat{\pi}$, and interpret this result.
2. Show that there exists a non–negative function $F^g(t,y)$ satisfying :

   $$V^g(t,x,y) = -e^{-ax}F^g(t,y) \quad \text{for all} \quad t \leq T,\ (x,y) \in \mathbb{R}^2 ,$$

   and deduce that

   $$p(t,x,y) = p(t,y) = \frac{1}{a}\ln\left(\frac{F^g(t,y)}{F^0(t,y)}\right), \qquad (t,x,y) \in [0,T]\times\mathbb{R}^2.$$

3. We assume that the map $V^g(t,x,y)$ is a classic solution of the equation

   $$\inf_{\theta\in\mathbb{R}} -\mathcal{L}^\theta V^g(t,x,y) \ = \ 0\,, \qquad V^g(T,x,y) \ = \ U(x-g(y))$$

   where

   $$\mathcal{L}^\theta V^g \quad := \quad V_t^g + \eta V_y^g + \frac{1}{2}\gamma^2 V_{yy}^g + \lambda\sigma\theta V_x^g + \frac{1}{2}\sigma^2\theta^2 V_{xx}^g + \theta\rho\sigma\gamma V_{xy}^g$$

   Show that $F^g(t,y)$ verifies :

   $$F_t^g + \eta F_y^g + \frac{1}{2}\gamma^2 F_{yy}^g - \frac{\left(\lambda F^g + \rho\gamma F_y^g\right)^2}{2F^g} \ = \ 0, \qquad F^g(T,y) \ = \ e^{ag(y)} .$$

4. Find a parameter δ such that the function $f^g(t,y) := \left[F^g(t,y)\right]^{1/\delta}$ is the solution of a **linear** partial differential equation.
5. Deduce that

   $$f^g(t,y) \quad = \quad \mathbb{E}\left[e^{(1-\rho^2)\left(ag(\widehat{Y}_T) - \frac{1}{2}\int_t^T \lambda^2(\widehat{Y}_u)\mathrm{d}u\right)}\right]$$

   where $\widehat{Y}$ is the solution of an SDE to be specified.
6. Give a probability measure $\widehat{\mathbb{Q}}$ equivalent to $\mathbb{P}$ so that p is written as

   $$p(t,y) \quad = \quad \frac{1}{a(1-\rho^2)}\ \ln\left(\mathbb{E}^{\widehat{\mathbb{Q}}}\left[e^{a(1-\rho^2)g(\widehat{Y}_T)}\right]\right).$$

7. Find a random variable $\mathcal{F}_T$–measurable H s.t. for any function with payoff g :

   $$\frac{\partial p}{\partial y}(t,y) \quad = \quad \frac{1}{a(1-\rho^2)}\ \frac{\mathbb{E}^{\widehat{\mathbb{Q}}}\left[He^{a(1-\rho^2)g(\widehat{Y}_T)}\right]}{\mathbb{E}^{\widehat{\mathbb{Q}}}\left[e^{a(1-\rho^2)g(\widehat{Y}_T)}\right]} .$$

Exercice 1.

$X_0 \in \mathcal{L}^2(\mathbb{R})$

$$dX_t = (a+bX_t)dt + \sqrt{\sigma+\theta X_t}\,dW_t \quad t\le T$$

(a,b,σ,θ) sont pris de telle sorte que X est bien défini et $X\in\mathbb{H}^2$ i.e $\mathbb{E}[\int_0^T X_t^2\,dt]<\infty$

1) $\forall t\le T$, $X_t\in\mathcal{L}^1(\mathbb{R})$ et $\mathbb{E}[X_t]=?$

On sait que $X_t = \underbrace{X_0}_{ok} + \underbrace{\int_0^t (a+bX_s)ds}_{1} + \underbrace{\int_0^t \sqrt{\sigma+\theta X_s}\,dW_s}_{2} \quad \forall t\le T$

1 $\in\mathcal{L}^1$ car $\mathbb{E}[|\int_0^t (a+bX_s)ds|] \le \mathbb{E}[\int_0^t |a|+|b||X_s|ds]$

$$\underset{CS}{\le} c\Big(|a|t + |b|\underbrace{\mathbb{E}[\int_0^t |X_s|^2 ds]^{1/2}}_{\substack{<\infty \\ \text{car } X\in\mathbb{H}^2}}\Big) \quad c>0$$

2 $\in\mathcal{L}^1$ car $\mathbb{E}[|\int_0^t \sqrt{\sigma+\theta X_s}\,dW_s|] \underset{CS}{\le} \mathbb{E}[|\int_0^t \underbrace{\sqrt{\sigma+\theta X_s}}_{:=A_s}\,dW_s|^2]^{1/2} = \mathbb{E}[\int_0^t \sigma+\theta X_s\,ds]^{1/2}$

Isométrie d'Itô

car $(A_s)_{s\le T}\in\mathbb{H}^2$

car $X\in\mathbb{H}^2$

$$\le \mathbb{E}[\int_0^t |\sigma|+|\theta||X_s|\,ds]^{1/2}$$

$$\le c\big(\sqrt{\sigma T} + |\theta|\,\mathbb{E}[\int_0^T |X_s|^2 ds]^{1/2}\big) < \infty$$

$\Rightarrow X_t \in\mathcal{L}^1 \quad \forall t$

$\Rightarrow \underbrace{\mathbb{E}[X_t]}_{m_t} = \mathbb{E}[X_0 + \int_0^t a+bX_s\,ds + \int_0^t \overbrace{\sqrt{\sigma+\theta X_s}\,dW_s}^{mart}]$

$= X_0 + at + b\int_0^t \underbrace{\mathbb{E}[X_s]}_{m_s}ds \quad$ Fubini car $\mathcal{L}^1$

$$\begin{cases} m_0 = X_0 \\ \dot m_t = a + bm_t \end{cases}$$

$m_t = c(t)e^{bt} \quad c(0)=X_0$

$\dot c(t)e^{bt} = a \Rightarrow \dot c(t) = ae^{-bt}$

$\Rightarrow c(t) = X_0 + \int_0^t ae^{-bs}ds = X_0 + \frac{a}{b}(1-e^{-bt})$

$\Rightarrow m_t = X_0e^{bt} + \frac{a}{b}(e^{bt}-1) = (X_0+\frac{a}{b})e^{bt} - \frac{a}{b}$

Donc $\mathbb{E}[X_t] = (X_0+\frac{a}{b})e^{bt} - \frac{a}{b}$

2) Donner l'EDS qui satisfait le processus $(\underbrace{X_t^m}_{Y_t}, t\ge 0)$ $\forall m>0$

On applique Itô car $x\mapsto x^m \in\mathcal{C}^2$

$\Rightarrow dY_t = mX_t^{m-1}dX_t + \frac{1}{2}m(m-1)X_t^{m-2}d\langle X\rangle_t$

$= mX_t^{m-1}\big((a+bX_t)dt + \sqrt{\sigma+\theta X_t}\,dW_t\big) + \frac{1}{2}m(m-1)X_t^{m-2}(\sigma+\theta X_t)dt$

$= \big(amX_t^{m-1} + bmX_t^m + \frac{\sigma}{2}m(m-1)X_t^{m-2} + \frac{\theta}{2}m(m-1)X_t^{m-1}\big)dt + mX_t^{m-1}\sqrt{\sigma+\theta X_t}\,dW_t$

3) Calculer $\mathbb{E}[X_t^2]$ (en supposant que toutes les mart locales du calcul sont des mart).

$X_t^2 = X_0^2 + \int_0^t 2X_s\{(a+bX_s)ds + \sqrt{\sigma+\theta X_s}\,dW_s\} + \int_0^t(\sigma+\theta X_s)ds$

$\Rightarrow \mathbb{E}[X_t^2] = \mathbb{E}[X_0^2 + \int_0^t \sigma + (2a+\theta)X_s + 2bX_s^2\,ds + \underbrace{\int_0^t 2X_s\sqrt{\sigma+\theta X_s}\,dW_s}_{\text{On admet vraie mart} \Rightarrow \mathbb{E}[\cdot]=0}]$

$\Rightarrow$ Par Fubini, $\underbrace{\mathbb{E}[X_t^2]}_{f(t)} = \mathbb{E}[X_0^2] + \int_0^t (2a+\theta)\underbrace{\mathbb{E}[X_s]}_{m_s} + 2b\underbrace{\mathbb{E}[X_s^2]}_{f(s)}ds$

$$\Rightarrow \begin{cases} f'(t) = m_t(2a+\theta) + 2bf(s) \quad t\le T \\ f(0) = \mathbb{E}[X_0^2] \end{cases}$$

$f(t) = c(t)e^{2bt} \quad \begin{cases} \dot c e^{2bt} = \sigma + (2a+\theta)m(t) \\ c(0) = X_0^2 \end{cases} \quad \Rightarrow c(t) = X_0^2 + \int_0^t e^{-2bs}(\sigma + (2a+\theta)m(s))ds$

$\Rightarrow f(t) = X_0^2e^{2bt} + \int_0^t e^{2b(t-s)}(\sigma + (2m+\theta)m(s))ds$

4) $\mathbb{E}[e^{-\frac{\lambda^2}{2}X_T} | \mathcal{F}_t] = \psi(t, X_t)$? pour un certain ψ ?

On a que $\mathbb{E}[e^{-\frac{\lambda^2}{2}X_T} | \mathcal{F}_t] = \mathbb{E}[e^{-\frac{\lambda^2}{2}(X_T - X_t) - \frac{\lambda^2}{2}X_t} | \mathcal{F}_t]$

$$= e^{-\frac{\lambda^2}{2}X_t}\,\mathbb{E}[e^{-\frac{\lambda^2}{2}(X_T - X_t)} | \mathcal{F}_t]$$

On observe que $X_t = X_t^{0,X_0} = X_t^{s,X_s}$ $\forall s \leq t$ (Propriété de Markov) ← prop 6.2.1 du Cours

où $X_t^{r,x} = x + \int_r^t (a + bX_s^{r,x})ds + \int_r^t \sqrt{\sigma + \delta X_s^{r,x}}\,dW_s$

et $X_t^{r,x} = F(r, x, (W_s - W_r)_{s\leq t})$

$$\Rightarrow \mathbb{E}[e^{-\frac{\lambda^2}{2}X_T} | \mathcal{F}_t] = e^{-\frac{\lambda^2}{2}X_t}\,\mathbb{E}[e^{-\frac{\lambda^2}{2}(X_T^{t,X_t} - X_t)} | \mathcal{F}_t]$$

$$= \mathbb{E}[e^{-\frac{\lambda^2}{2}(F(t, X_t, \underbrace{(W_s - W_t)_{s\leq T}}_{\perp\!\!\!\perp \mathcal{F}_t})} | \mathcal{F}_t]$$

Connu sous $\mathcal{F}_t$ car X_t est $\mathcal{F}_t$ mes

$$\Rightarrow \psi(t, x) = \mathbb{E}[e^{-\frac{\lambda^2}{2}F(t, x, (W_s - W_t)_{s\leq T})}]$$

$$= \mathbb{E}[e^{-\frac{\lambda^2}{2}X_T^{t,x}}]$$

Rappel : si Z est $\mathcal{G}$ mes et $Y \perp\!\!\!\perp \mathcal{G}$ alors $\mathbb{E}[H(Z,Y)|\mathcal{G}] = h(Z)$

où $h(z) = \mathbb{E}[H(z, Y)]$

En supposant que ψ est $C^{1,2}$ mq le calcul $\mathbb{E}[e^{-\frac{\lambda^2}{2}X_T}]$ peut être réduit à une EDP

L'idée est de trouver une méthode par EDP pour calculer $\mathbb{E}[e^{-\frac{\lambda^2}{2}X_T}]$

Dans le cours de Proba Num, on voit qu'on peut le faire par Monte Carlo

On sait que $\psi : [0,T] \times \mathbb{R} \to \mathbb{R}$ de classe $C^{1,2}([0,T[\times\mathbb{R}) \cap C([0,T]\times\mathbb{R})$ est solut° de :

FEYNMANN-KAC (*) $\begin{cases} \partial_t v(t,x) + (a + bx)\partial_x v(t,x) + \frac{1}{2}(\sigma + \delta x)\partial^2_{xx} v(t,x) = 0 & t < T,\ x \in \mathbb{R} \\ v(T, x) = e^{-\frac{\lambda^2}{2}x} \end{cases}$

À absolument connaître par cœur.

Si l'EDP (*) admet une solution $W : [0,T] \times \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}[e^{-\frac{\lambda^2}{2}X_T}] = \mathbb{E}[W(0, X_0)]$$

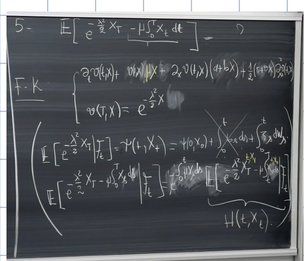
$\mathcal{L}(X_0)$ connue

On peut approximer une solution de (*) par différence finie

5) On suppose $\mathbb{E}[e^{-\frac{\lambda^2}{2}X_T - \mu\int_0^T X_s ds}] < \infty$

Mq le calcul de $\mathbb{E}[e^{-\frac{\lambda^2}{2}X_T - \mu\int_0^T X_s ds}]$ peut être réduit à une EDP

F.K (**) $\begin{cases} \partial_t v(t,x) + v(t,x)\mu x + \partial_x v(t,x)(a + bx) + \frac{1}{2}(\sigma + \delta x)\partial^2_{xx} v(t,x) = 0 \\ v(T, x) = e^{-\frac{\lambda^2}{2}x} \end{cases}$



La vers° dynamique serait

$$V(t, X_t) = \mathbb{E}[e^{-\frac{\lambda^2}{2}X_T}\,\underbrace{e^{-\int_t^T \mu X_s ds}}_{e^{-\int_t^T \beta(s, X_s) ds}} | \mathcal{F}_t]$$

$\overset{F.K}{\Rightarrow}$ $\begin{cases} \partial_t v - kv + b\partial_x v + \frac{1}{2}\sigma^2\partial_x^2 v = 0 \\ v(T, x) = e^{-\frac{\lambda^2}{2}x_T} \end{cases}$

Donc si W solut° de (**), alors on a que $\mathbb{E}[e^{-\frac{\lambda^2}{2}X_T - \mu\int_0^T X_t dt}] = \mathbb{E}[W(0, X_0)]$

Exercice 2 :

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma dB_t$$

$V(t,S_t) := \mathbb{E}_t[e^{-r(T-t)}g(S_T)] = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}g(S_T)|\mathcal{F}_t]$ $|S_t$ d'où le fait que c'est une fct° de S_t — $g : \mathbb{R} \to \mathbb{R}$ mesurable bornée.

1) Mq V est une fct° de $C^{1,2}([0,T[\times\mathbb{R})$

On sait que $V(t,x) = \mathbb{E}[e^{-r(T-t)}g(S_T^{t,x})]$

Si g est régulière, on dérive sous l'espérance pour vérifier la régularité de V.

$$S_T^{t,x} = x e^{(r-\delta)(T-t) - \frac{\sigma^2}{2}(T-t) + \sigma(W_T - W_t)}$$

$$\Rightarrow V(t,x) = \mathbb{E}[e^{-r(T-t)}g(x e^{((r-\delta)-\frac{\sigma^2}{2})(T-t)+\sigma\overbrace{(W_T-W_t)}^{\sim\mathcal{N}(0,T-t)}})]$$ par chgt de variable

$$= \int_{\mathbb{R}} e^{-r(T-t)} g(e^{\ln(x)+((r-\delta)-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,y}) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

Par chgt de variable et en dérivant sous l'intégrale, on vérifie que $V \in C^{1,2}([0,T[\times\mathbb{R})$

2) Donner l'EDP satisfaite par V

$V \in C^{1,2} \cap C$

$$\text{FK} \to \begin{cases} \partial_t v(t,x) - r v(t,x) + (r-\delta)x\partial_x v(t,x) + \sigma^2 x^2 \frac{1}{2}\partial^2_{xx} v(t,x) = 0 \\ V(T,x) = g(x) \end{cases}$$

Exercice 4 : $\|\alpha\|_\infty \leq 1$

$$Y_t^\alpha = Y_0 + \int_0^t \alpha_s dW_s \quad \text{et} \quad Z_t^\alpha = Z_0 + \int_0^t Y_s^\alpha dW_s \quad t \in [0,T]$$

$\lambda > 0$ tq $2\lambda T < 1$ et $v : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $(t,y,z) \mapsto v(t,y,z)$ tq $v(t,Y_t^1,Z_t^1) := \mathbb{E}[e^{2\lambda Z_T^1}|\mathcal{F}_t]$ $|Y_t^1, Z_t^1$

$$d\begin{pmatrix} Y_t^1 \\ Z_t^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ Y_t & 0 \end{pmatrix}\begin{pmatrix} dW_t \\ d\tilde{W}_t \end{pmatrix}$$

$P_t \in \mathbb{R}^d$ $\quad dP_t = b_t dt + \sigma_t dW_t \Longleftrightarrow$

$b_t \in \mathbb{R}^d, \sigma_t \in \mathbb{R}^{d\times q}, W \in \mathbb{R}^q$

(Y^1, Z^1) vérifie l'EDS, la déf se justifie par la prop de Markov

1- (EDP) :

$$\begin{cases} \partial_t v + 0\times v + 0\times\partial_y v + 0\times\partial_z v + \frac{1}{2}1^2\partial^2_{yy}v + \frac{1}{2}y^2\partial^2_{zz}v + \underbrace{y\times 1}_{*}\,\partial^2_{zy}v \\ v(T,y,z) = e^{2\lambda z} \end{cases}$$

Si $Y_t^\alpha = Y_0 + \int_0^t \alpha_s d\tilde{W}_s$ avec $\tilde{W} \perp W$

$\Rightarrow * = 0$

Si $Y_t^\alpha = Y_0 + \int_0^t \alpha_s d\tilde{W}_s + c\,dB_t$ avec $B \perp W$

$\Rightarrow \frac{1}{2}\times 1^2 \partial^2_{yy}v$ devient $\frac{1}{2}(1^2+c^2)\partial^2_{yy}v$

Retenir Feynmann KAC et HJB pour exam!!!

Exercise IV. Let $T > 0$ and W be a $\mathbb{F}$–Brownian motion. We denote by $\mathcal{A}$ the set of $\mathbb{F}$–predictable processes $(\alpha_t)_{t\in[0,T]}$ satisfying $\|\alpha\|_\infty \leq 1$. For any $\alpha \in \mathcal{A}$, we introduce (Y^α, Z^α) two processes such that

$$Y_t^\alpha = Y_0 + \int_0^t \alpha_s \mathrm{d}W_s \text{ and } Z_t^\alpha = Z_0 + \int_0^t Y_s^\alpha \mathrm{d}W_s, \text{ with } t \in [0,T].$$

Let λ be a positive parameter verifying $2\lambda T < 1$. We define the function $[0,T]\times\mathbb{R}\times\mathbb{R} \ni (t,y,z) \mapsto v(t,y,z) \in \mathbb{R}$ by

$$v(t, Y_t^1, Z_t^1) := \mathbb{E}[e^{2\lambda Z_T^1}|\mathcal{F}_t].$$

1. Provide the partial differential equation satisfied by v.
2. Compute explicitly $v(t,y,z)$ for any (t,y,z).
3. Show that v is strictly convex in y and that $y\, v_{yz}(t,y,z) \geq 0$.
4. Show that

$$v(0, Y_0, Z_0) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}[e^{2\lambda Z_T^\alpha}].$$