TD 4 : Copula

Copule : permet de décrire la structure de dépendance entre 2 variables. Possibilité de copules multivariées

checker estimation de copule.

$\Rightarrow$ Def : fonction reliant les CDF marginales à la CDF jointe

Exercice 1

$$H_\theta(x,y) = \frac{1}{1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}}$$

a) $H_\theta(x) = \lim_{y\to\infty} H_\theta(x,y) = \frac{1}{1+e^{-x}}$

$$H_\theta(y) = \frac{1}{1+e^{-y}}$$

- $f(u) = \frac{1}{1+e^{-u}} \Rightarrow f^{-1}(u) = -\ln\left(\frac{1}{u}-1\right)$

$$C(u,v) = \frac{1}{1+\left(\frac{1}{u}-1\right)+\left(\frac{1}{v}-1\right)+(1-\theta)\left(\frac{1}{u}-1\right)\left(\frac{1}{v}-1\right)}$$

$$= \frac{uv}{v+u-uv+(1-\theta)(1-u-v+uv)}$$

$$= \frac{uv}{1-\theta(1-u-v+uv)} \quad (= uv \text{ si } \theta = 0)$$

b) $X \perp\!\!\!\perp Y \iff H_\theta(x) H_\theta(y) = \dfrac{1}{1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}}$

$\iff \theta = 0$

Exercice 2

(1) $\mathbb{P}(g(X_1) \leqslant x_1, \ldots, g(X_n) \leqslant x_n)$

$= \mathbb{P}(X_1 \leqslant g^{-1}(x_1), \ldots, X_n \leqslant g^{-1}(x_n))$

$= C(F_{X_1}(g_1^{-1}(x_1)), \ldots, F_{X_n}(g_n^{-1}(x_n)))$

$= C(\mathbb{P}(X_1 \leqslant g_1^{-1}(x_1)), \ldots, \mathbb{P}(X_n \leqslant g_n^{-1}(x_n)))$

$= C(\mathbb{P}(g(X_1) \leqslant x_1), \ldots, \mathbb{P}(g(X_n) \leqslant x_n))$

$= C(F_{g(X_1)}(x_1), \ldots, F_{g_n(X_n)}(x_n))$

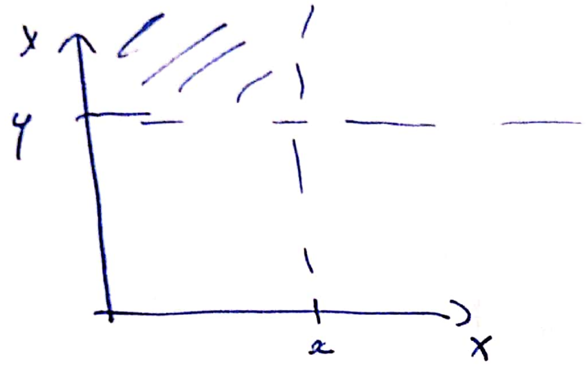
SKLAR
unicité

$\boxed{C_{g(X)} = C_X = C}$

(b) $F_{g_1,\dots g_n}(x_1,\dots,x_n) = \mathbb{P}(g_1(X_1) \leq x_1, \dots g_n(X_n) \leq x_n)$

$$(*) \quad = \mathbb{P}(X_1 \leq g_1^{-1}(x_1), \dots, X_{n-1} \leq g_{n-1}^{-1}(x_{n-1}), X_n > g_n^{-1}(x_n))$$

Or



$$\mathbb{P}(X < x, Y > y)$$
$$= \mathbb{P}(X \leq x) - \mathbb{P}(X_1 < x, Y \leq y)$$

$$(*) = \mathbb{P}(X_1 \leq g_1^{-1}(x_1), \dots, X_{n-1} \leq g_{n-1}^{-1}(x_{n-1}), X_n \leq \infty) - \mathbb{P}(X_1 \leq g_1^{-1}(x_1), \dots, X_n \leq g_n^{-1}(x_n))$$

$$(*) = C[F_{X_1}(g_1^{-1}(x_1)), \dots, 1] - C[F_{X_1}(g_1^{-1}(x_1)), \dots \; F_{X_n}(g_n^{-1}(x_n))]$$

au moment de réinverser.

$$= C[\mathbb{P}(g(X_1) \leq x_1), \dots 1] - C[\mathbb{P}(g(X_1) \leq x_1, \dots \; 1 - \mathbb{P}(g(X_n) \leq x_n))$$

$$= C[F_{g_1(x_1)}(x_1), \dots, 1] - C[F_{g_1(x_1)}(x_1), \dots, 1 - F(\text{—})]$$

Exercice 3

On note que :

$$\mathbb{P}(\overline{A \cap B}) = 1 - \mathbb{P}(A \cap B)$$

$$\mathbb{P}(\overline{A \cap B}) = \mathbb{P}(\bar{A} \cup \bar{B}) = \mathbb{P}(\bar{A}) + \mathbb{P}(\bar{B}) - \mathbb{P}(\bar{A} \cap \bar{B})$$

1) Avec $A = X_1 \leq x_1$ et $B = X_2 \leq x_2$

$$1 - \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) = \mathbb{P}(X_1 \geq x_1) + \mathbb{P}(X_2 \geq x_2) - \mathbb{P}(X_1 \geq x_1, X_2 \geq x_2)$$

$$1 - F(x_1, x_2) = \bar{F}_1(x_1) + \bar{F}_2(x_2) - \bar{F}(x_1, x_2)$$

$$\bar{F}(x_1, x_2) = \bar{F}_1(x_1) + \bar{F}_2(x_2) + F(x_1, x_2) - 1$$

$$= 1 - F_1(x_1) + 1 - F_2(x_2) + F(x_1, x_2) - 1$$

$$= 1 + F(x_1, x_2) - F_1(x_1) - F_2(x_2)$$

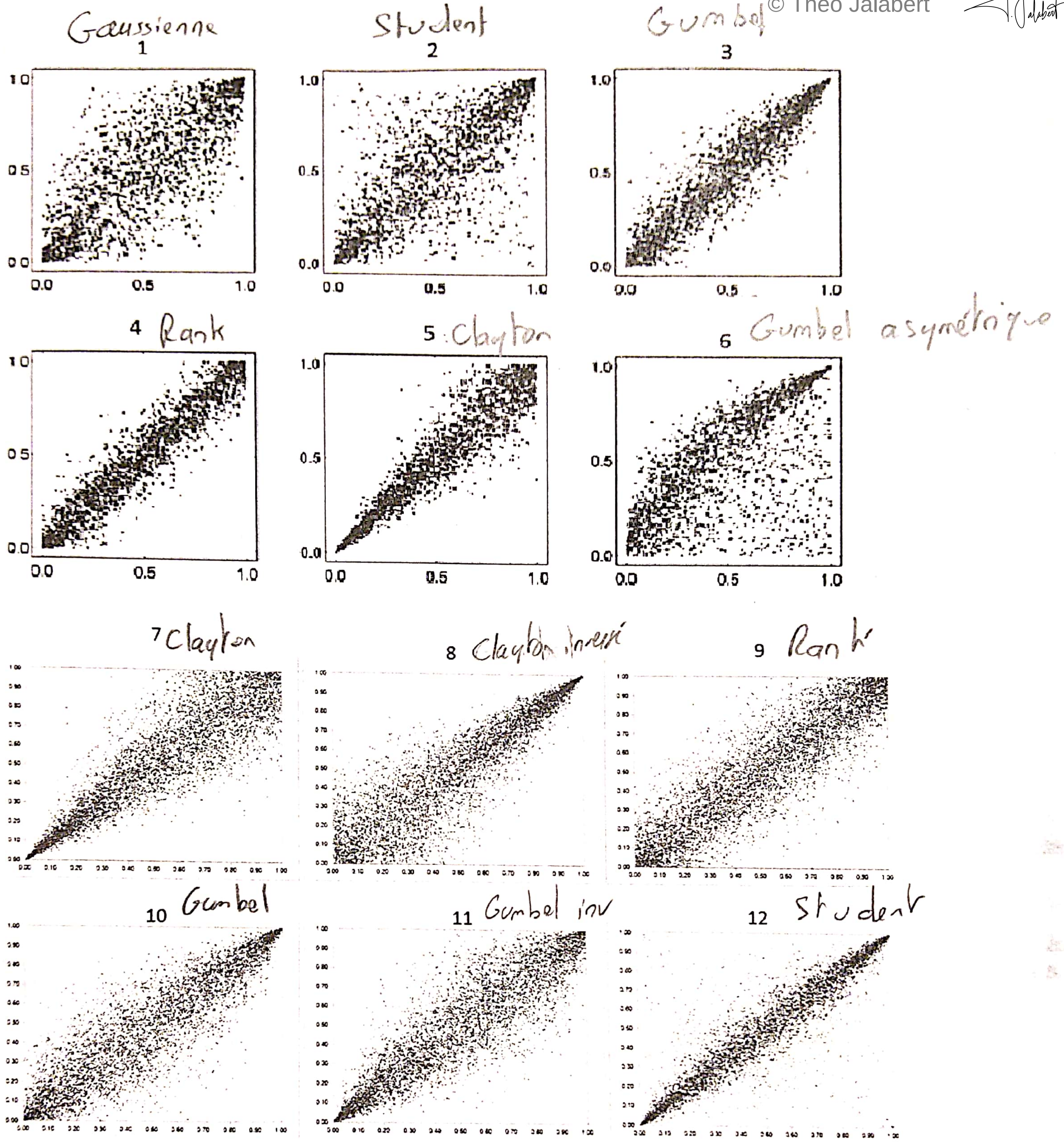
$$\bar{F}_1(x_1)\bar{F}_2(x_2) = (1 - F_1(x_1))(1 - F_2(x_2))$$

$$= 1 - F_1(x_1) - F_2(x_2) + F_1(x_1)F_2(x_2)$$

Donc $F(x_1, x_2) \geqslant F_1(x_1) F_2(x_2)$

$$C(F_1(x_1) F_2(x_2)) > C_I(F_1(x_1), F_2(x_2))$$

$$\Rightarrow C \geqslant C_I$$



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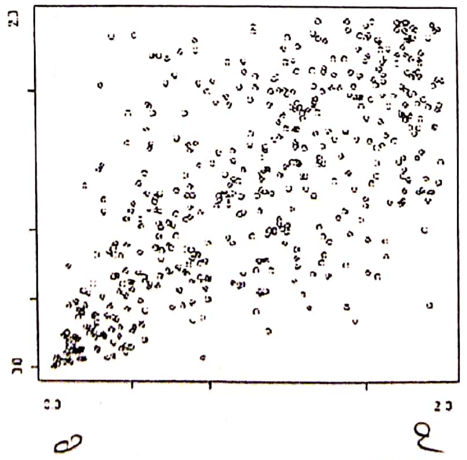

pas copule.