Stochastic modelling and derivatives
– Exercises –
M2 Probability and finance

24th September 2024

Exercise 1. [Convexity] Prove that, under the no-arbitrage assumption, the European call price is a convex function of the strike.

Exercise 2. [Arbitrage] The aim of this exercise is to detect arbitrage opportunities. A trainee presents his new program to compute European call and put prices. We assume that the considered asset does not provide dividends. Assume that the price of the asset is $S_0 = 100€$ today with $r = 1\%$, $T = 1$ (1 year). The program gives the following outputs for call and put prices with different strikes:

Option	Call $K = 95€$	Call $K = 100€$	Call $K = 105€$	Put $K = 95€$	Put $K = 100€$	Put $K = 105€$
Price	11€	9.5€	7.27€	6.95€	10.51€	9.83€

If these prices were market prices, provide

- ▷ arbitrage opportunities between call options,
- ▷ arbitrage opportunities between put options,
- ▷ arbitrage opportunities between call and put options.

Exercise 3. [Payoff and strategies] For each strategy below, write and draw (including the premium) the payoff at maturity. Note that all the options used in these strategies have the same maturity. What is the financial purpose of each of these strategies?

1. **Straddle**: Long 1 Call and long 1 Put, with same strike.
2. **Strip**: Long 1 Call and long 2 Puts, with same strike.
3. **Strap**: Long 2 Calls and long 1 Put, with same strike.
4. **Butterfly**: Long 1 Call of strike $K - \delta K$, long 1 Call of strike $K + \delta K$, short 2 Calls of strike K.
5. **Strangle**: Long 1 Call of strike K_C and long 1 Put of strike K_P, with, usually (but not necessarily), $K_P < K_C$.

6. **Condor**: Long 1 Call of strike K_1, short 1 Call of strike $K_2 = K_1 + \delta K > K_1$, short 1 Call of strike $K_3 > K_2$, and long 1 Call of strike $K_4 = K_3 + \delta K$
7. **Bull call spread**: Long 1 Call of strike K_1 and short 1 Call of strike $K_2 > K_1$.
8. **Bull put spread**: Long 1 Put of strike K_1 and short 1 Put of strike $K_2 > K_1$.

Exercise 4. [Discrete time market] We consider a market with two periods (3 times $t_0 = 0 < t_1 < t_2 = T$) with:

- ▷ a risky asset denoted by S with price $S \times u$ or $S \times d$ after one period (with probability p and $1 - p$ respectively), by assuming that $0 < d < 1 + r < u$ where r is the interest rate on one period,
- ▷ a call option with strike K and maturity $T = t_2$.

Questions:

1. Assume that $S_0 = 4$€ at time $t_0 = 0$, $u = 2$, $d = 1/2$, $r = 0.25$, $K = 5$€ , compute
   - ▷ the price V_{t_1} and the hedging strategy δ_{t_1}, in the two possible states at time t_1 (that is if $S_{t_1} = S_0 u$ and if $S_{t_1} = S_0 d$),
   - ▷ the price V_0 and the hedging δ_0.
2. Same questions with a put option.
3. Check the call/put parity at time $t = t_0$ and $t = t_1$.

Exercise 5. [Convergence of the binomial model towards Black-Scholes model] We are given a financial market comprising a risk-free asset of price R, equal to 1 in $t = 0$, and a risky asset of price S.
We discretize the time interval $[0, T]$ in n smaller intervals $[t_i^n, t_{i+1}^n]$, with $t_i^n = iT/n$, in order to build a n-period binomial tree. We note r_n the interest rate of the risk-free asset, the value of this asset at the time t_i^n being

$$R^n_{t_i^n} = (1 + r_n)^i.$$

We note X_i^n the quantity equal to 1 plus the price return of the risky asset between times t_{i-1}^n and t_i^n. We then have, under the historical probability,

$$\mathbb{P}^n(X_i^n = u_n) = p_n = 1 - \mathbb{P}^n(X_i^n = d_n).$$

The random variables $X_1^n, ..., X_n^n$ are independent to each other. We base the quantities r_n, d_n, and u_n on the parameters r and σ:

$$r_n = \frac{rT}{n}, \qquad d_n = \left(1 + \frac{rT}{n}\right) e^{-\sigma\sqrt{\frac{T}{n}}}, \qquad \text{and } u_n = \left(1 + \frac{rT}{n}\right) e^{\sigma\sqrt{\frac{T}{n}}}.$$

1. Draw the tree representing the evolution of the risky asset.
2. What is the limit of R_T^n when n tends to infinity?
3. Is the market consistent with the no-arbitrage assumption?

4. Give an expression for $S^n_{t^n_i}$ using S_0 and $(X^n_1, ..., X^n_i)$.

5. Give the dynamic of the process X^n under the risk-neutral probability $\mathbb{Q}^n$. We then note $q_n = \mathbb{Q}^n(X^n_i = u_n)$.

6. Show that

$$q_n \overset{n\to\infty}{\longrightarrow} \frac{1}{2}, \qquad n\mathbb{E}_{\mathbb{Q}^n}[\ln(X^n_1)] \overset{n\to\infty}{\longrightarrow} \left(r - \frac{\sigma^2}{2}\right)T, \qquad \text{and } n\mathbb{V}\mathrm{ar}_{\mathbb{Q}^n}[\ln(X^n_1)] \overset{n\to\infty}{\longrightarrow} \sigma^2 T.$$

7. Using characteristic functions, prove the following convergence in distribution:

$$\sum_{i=1}^{n} \ln(X^n_i) \xrightarrow[n\to\infty]{d} \mathcal{N}\left[\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right].$$

8. Deduce that

$$S^n_T \xrightarrow[n\to\infty]{d} S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T},$$

with $W_T \sim \mathcal{N}(0, T)$.

9. Write the price of a put of strike K and maturity T in the n-period binomial model, as the expected value of some variable.

10. Deduce that the price of the put converges toward

$$P_0 = Ke^{-rT}N(-d_-) - S_0 N(-d_+),$$

when n tend to infinity.

11. Conclude that the call price tends toward

$$C_0 = S_0 N(d_+) - Ke^{-rT}N(d_-).$$

Exercise 6. [Lookback option with a binomial tree] We consider a market with two periods (3 times $t_0 = 0 < t_1 < t_2 = T$) with:

- ▷ a risky asset denoted by S with price $S \times u$ or $S \times d$ after one period, with $u = 1.1$, $d = 0.95$, $S_{t_0} = 100$, and $r = 0.05$ is the interest rate on one period,
- ▷ a European call option with strike $K_E = 105$ and maturity T.
- ▷ a lookback option of strike $K_L = 100$, maturity T, and whose payoff is $(\sup_{t\le T} S_t - K_L)_+$.

Questions:

1. Draw the tree representing the evolution of the risky asset.
2. Describe Ω.
3. What is the risk-neutral probability in this binomial tree?
4. What is the price of the European call in this model?

5. What is the price of the lookback option in this model?

Exercise 7. [Carr formula] We assume we have access to call and put options of maturity T and strike K, whatever $K \geq 0$. We want to use these options to replicate a derivative of payoff $\psi(S_T)$, where ψ is any regular function.

1. Prove the Carr formula, that is that the cash price (which is the price paid today, *"prix au comptant"*, as opposed to the forward price) at time t of the payoff $\psi(S_T)$, which we note $C_t(\psi(S_T), T)$, follows, under the no-arbitrage assumption:

$$C_t(\psi(S_T), T) = \psi(F_t(S_T, T))B(t, T) + \int_{F_t(S_T,T)}^{+\infty} \psi''(K)\mathrm{Call}_t(T, K)dK + \int_0^{F_t(S_T,T)} \psi''(K)\mathrm{Put}_t(T, K)dK$$

2. Give a static hedging strategy for the following payoff, by using infinitely many calls and puts:

   ▷ $\psi_T = (S_T)^p$ (power underlying) for some $p > 0$,

   ▷ $\psi_T = ((S_T)^p - K)_+$ (call power) for some $p > 0$.

Exercise 8. [Trinomial model] We consider a discrete-time market, with a bond whose price is multiplied by $(1 + r)$ at each step, as well as a risky security of price S_n. This price is led by the stochastic process h_n: $h_i, ..., h_N$ are N i.i.d. random variables such that

$$h_n = \begin{cases} 1 & \text{with probability } p_1 \\ 2 & \text{with probability } p_2 \\ 3 & \text{with probability } p_3 = 1 - p_1 - p_2, \end{cases}$$

with probabilities different from zero. The price S_n is defined by $S_n = S_{n-1}(1 + \mu(h_n))$, with

$$1 + \mu(h_n) = \begin{cases} u & \text{if } h = 1 \\ m & \text{if } h = 2 \\ d & \text{if } h = 3, \end{cases}$$

and $0 < d < m < u$.

1. Draw the trinomial tree.
2. Studying the martingale condition, show that this market is incomplete.
3. Alternatively, show there is no replicating strategy for a given derivative defined by its random payoff X. This will show again that the market is incomplete.
4. Show, following two distinct methods, that adding a second asset S^2 independent from S and defined in a similar way ($S^2_n = S^2_{n-1}(1 + \mu^2(h_n))$) is enough to make the market complete.

Exercise 9. [Asian options] We consider three kinds of derivatives:

▷ European calls C_t and puts P_t of strike K,

- Asian calls $\bar{C}_t$ and puts $\bar{P}_t$ of strike K, whose payoff is $(\bar{S} - K)_+$ or $(K - \bar{S})_+$, with

$$\bar{S} = \frac{1}{n} \sum_{i=1}^{n} S_{t_i},$$

  for $0 < t_1 < ... < t_n < T$,

- exotic European calls $\hat{C}_t$ and puts $\hat{P}_t$ with "average strike" $\bar{S}$, that is of payoff $(S_T - \bar{S})_+$ or $(\bar{S} - S_T)_+$.

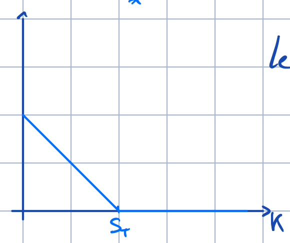
All these options have the same underlying and same maturity. Find a relation between the six option prices at time 0.

Exercice 1:

Exercise 1. [Convexity] Prove that, under the no-arbitrage assumption, the European call price is a convex function of the strike.

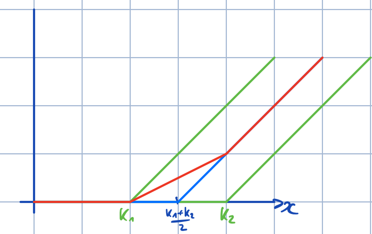
$\forall \lambda \in [0,1], \forall K_1 < K_2$, on veut montrer que $CALL_T(T, \lambda K_1 + (1-\lambda)K_2) \leq \lambda CALL_T(T,K_1) + (1-\lambda) CALL_T(T,K_2)$

$(S_T \underbrace{-\lambda K_1 - (1-\lambda)K_2}_{*})_+ \leq \underbrace{\lambda (S_T - K_1)_+ + (1-\lambda)(S_T - K_2)_+}_{*}$



Le payoff est une fonct° cvx du strike

Si $\lambda = 1/2$



Sous AOA, $\forall t \leq T$, $CALL_t(T, \lambda K_1 + (1-\lambda)K_2) \leq \lambda CALL_t(T,K_1) + (1-\lambda) CALL_t(T,K_2)$

Exercice 2:

Exercise 2. [Arbitrage] The aim of this exercise is to detect arbitrage opportunities. A trainee presents his new program to compute European call and put prices. We assume that the considered asset does not provide dividends. Assume that the price of the asset is $S_0 = 100€$ today with $r = 1\%$, $T = 1$ (1 year). The program gives the following outputs for call and put prices with different strikes:

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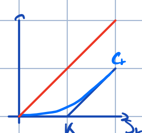
If these prices were market prices, provide

- ▷ arbitrage opportunities between call options,
- ▷ arbitrage opportunities between put options,
- ▷ arbitrage opportunities between call and put options.

$S_0 = 100€$, $r = 1\%$, $T = 1$

* Parité CALL-PUT: $C_t(T,K) - P_t(T,K) = S_t - K \overbrace{B(t,T)}^{e^{-r(T-t)}}$

* Bornes d'arbitrage:
  * $(S_t - KB(t,T))_+ \leq C_t(T,K) \leq S_t$
  * $(KB(t,T) - S_t)_+ \leq P_t(T,K) \leq KB(t,T)$



OA possibles: * BULL SPREAD
  * $K \mapsto CALL(T,K)$ ↘
  * $K \mapsto PUT(T,K)$ ↗

* BUTTERFLY SPREAD
  * $\left. \begin{array}{l} K \mapsto CALL_t(T,K) \\ K \mapsto PUT_t(T,K) \end{array} \right\}$ Convexe

* CALENDAR SPREAD
  * $\left. \begin{array}{l} T \mapsto CALL_t(T,K) \\ T \mapsto PUT_t(T,K) \end{array} \right\}$ croissantes si $B(t,T) \leq 1$

Résultats à calculer...

Arbitrage 1:

On va acheter le moins cher i.e $P_0(K=105)$

— vendre le plus cher i.e $P_0(K=100)$

Autofinancement:

$\rightarrow$ On achète 10.51

$\rightarrow$ On vend 9.83

À maturité on reçoit $10.51 \times (105-S_T)_+ - 9.83(100-S_T)_+$

Arbitrage 2:

On aurait dû avoir $P_0(K=100) \leq \frac{1}{2}(P_0(95)+P_0(105))$

Stratégie autofinancée:

Long 10.51 PUT(95) + PUT(105)

i.e calculer $\rightarrow$

valeur $\rightarrow$ Short $\frac{33.56}{2}$ PUT(100)

À la maturité, on reçoit $10.51[(95-S_T)_+ + (105-S_T)_+] - \frac{33.56}{2}(100-S_T)_+$

Arbitrage CALL-PUT:

On a $C_t(T,K) + KB(t,T) < S_t + P_t(T,K)$

Stratégie autofinancée:

Long 106.95 [CALL(95) + 95 ZC]

Short 165.05 [action + PUT(95)]

À maturité, on reçoit:

$106.95((S_T-95)_+ + 95) - 165.05(S_T + (95-S_T)_+)$

Film Rockfeller $\rightarrow$ Short Straddle

Exercice 3:

Exercise 3. [Payoff and strategies] For each strategy below, write and draw (including the premium) the payoff at maturity. Note that all the options used in these strategies have the same maturity. What is the financial purpose of each of these strategies?

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8. **Bull put spread**: Long 1 Put of strike K_1 and short 1 Put of strike $K_2 > K_1$.

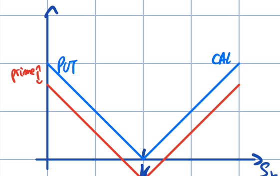
— payoff sans prime

— payoff avec prime

① STRADDLE

$\rightarrow$ Long CALL(K,T)

Long PUT(K,T)

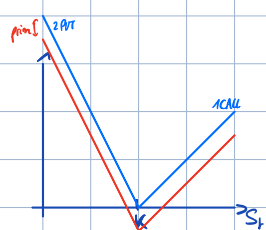


Payoff $= (S_T-K)_+ + (K-S_T)_+$

② STRIP

$\rightarrow$ Long CALL(K,T)

Long 2 PUT(K,T)

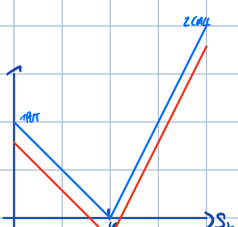


Payoff $= (S_T-K)_+ + 2(K-S_T)_+$

③ STRAP

$\rightarrow$ Long 2CALL(K,T)

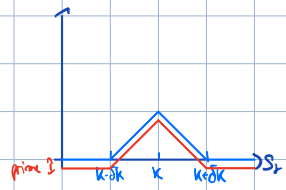
Long 1 PUT(K,T)



Payoff $= 2(S_T-K)_+ + (K-S_T)_+$

④ BUTTERFLY

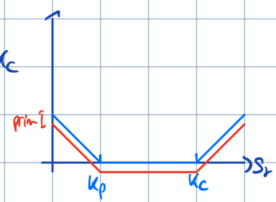
→ Long CALL($K-\delta K, T$)
Long CALL($K+\delta K, T$)
Short 2 CALL(K, T)



Payoff = $(S_T-(K-\delta K))_+ - 2(S_T-K)_+ + (S_T-(K+\delta K))_+$

⑤ STRANGLE

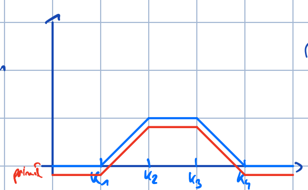
→ Long CALL(K_c, T) $K_p < K_c$
Long PUT(K_p, T)



Payoff : $(K_p-S_T)_+ + (S_T-K_c)_+$

⑥ CONDOR :

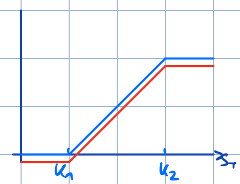
→ Long CALL(K_1, T)
Short CALL($\underbrace{K_1+\delta K}_{K_2}, T$) $K_4 > K_3 > K_2 > K_1$
Short CALL(K_3, T)
Long CALL($\underbrace{K_3+\delta K}_{K_4}, T$)



Payoff = $(S_T-K_1)_+ - (S_T-K_2)_+ - (S_T-K_3)_+ + (S_T-K_4)_+$

⑦ BULL CALL SPREAD

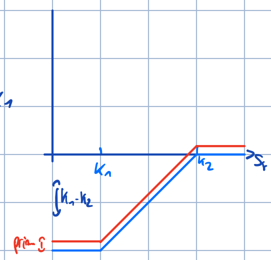
→ Long CALL(K_1, T) $K_2 > K_1$
Short CALL(K_2, T)



Payoff : $(S_T-K_1)_+ - (S_T-K_2)_+$

⑧ BULL PUT SPREAD

→ Long PUT(K_1, T) $K_2 > K_1$
Short PUT(K_2, T)



Payoff : $(K_1-S_T)_+ - (K_2-S_T)_+$

Exercice 4 :

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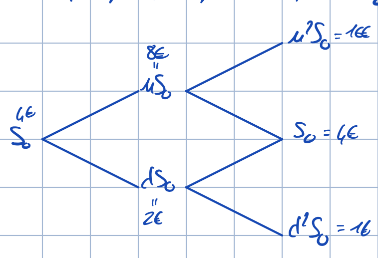
- ▷ a risky asset denoted by S with price $S \times u$ or $S \times d$ after one period (with probability p and $1-p$ respectively), by assuming that $0 < d < 1 + r < u$ where r is the interest rate on one period,
- ▷ a call option with strike K and maturity $T = t_2$.

Questions:

1. Assume that $S_0 = 4€$ at time $t_0 = 0$, $u = 2$, $d = 1/2$, $r = 0.25$, $K = 5€$, compute
   - ▷ the price V_{t_1} and the hedging strategy δ_{t_1}, in the two possible states at time t_1 (that is if $S_{t_1} = S_0u$ and if $S_{t_1} = S_0d$),
   - ▷ the price V_0 and the hedging δ_0.
2. Same questions with a put option.
3. Check the call/put parity at time $t = t_0$ and $t = t_1$.

① $S_0 = 4€$, $t_0 = 0$, $u = 2$, $d = 1/2$, $r = 0.25$, $K = 5€$



CALL

$C_0(S_0)$

$C_1(uS_0) = \frac{1}{1+r}(q\,C_{2,u} + (1-q)C_{2,d}) = 4.4€$

$C_1(dS_0) = 0€$

$C_2(u^2S_0) = (u^2S_0 - K)_+ = 11€$

$C_2(udS_0) = (S_0 - K)_+ = 0€$

$C_2(d^2S_0) = 0€$

Ptf réplicant :

$V_0(S_0) = \delta_0 S_0 + \Pi_0$
$= C_0(S_0)$

$V_1(uS_0) = \delta_0 u S_0 + \Pi_0(1+r)$
réallocation $= \delta_{1,u} u S_0 + \Pi_{1,u}$ $= 4.4€$

$V_1(dS_0) = \delta_0 dS_0 + \Pi_0(1+r)$
$= \delta_{1,d} S_0 + \Pi_{1,d}$
$= 0€$

$$\begin{cases} \delta_0 \times 8 + \Pi_0 \times 1.25 = 4.4 \\ \delta_0 \times 2 + \Pi_0 \times 1.25 = 0 \end{cases} \Rightarrow \begin{cases} \delta_0 = 11/15 = 0.7\overline{3} \\ \Pi_0 = -\frac{88}{75} = -1.17\overline{3} \end{cases}$$

observat° de la valeur

$V_2(u^2S_0) = \delta_{1,u} u^2 S_0 + \Pi_{1,u}(1+r) = C_2(u^2S_0) = 11€$

$V_2(udS_0) = \delta_{1,u} ud S_0 + \Pi_{1,u}(1+r) = 0€$

$$\Rightarrow \begin{cases} \delta_{1,u} \times 16 + \Pi_{1,u} \times 1.25 = 11 \\ \delta_{1,u} \times 4 + \Pi_{1,u} \times 1.25 = 0 \end{cases} \Rightarrow \begin{cases} \delta_{1,u} = 11/12 \\ \Pi_{1,u} = -\frac{44}{15} = -2.93 \end{cases}$$

$V_2(duS_0) = \delta_{1,d} ud S_0 + \Pi_{1,d}(1+r) = 0€$

$V_2(d^2S_0) = \delta_{1,d} d^2 S_0 + \Pi_{1,d}(1+r) = 0€$

$$= \begin{cases} \delta_{1,d} \times 4 + \Pi_{1,d} \times 1.25 = 0 \\ \delta_{1,d} \times 1 + \Pi_{1,d} \times 1.25 = 0 \end{cases} \Rightarrow \begin{cases} \delta_{1,d} = 0 \\ \Pi_{1,d} = 0 \end{cases}$$

$\Rightarrow V_0(S_0) = 0.73 \times 4 - 1.17$
$= 1.75$

m̂ chose avec un PUT

Proba martingale $\mathbb{Q}$ $\quad q = \mathbb{Q}(u) \quad 1-q = \mathbb{Q}(d)$

$$S_0 = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_1}{1+r} \,\middle|\, S_0\right]$$
$$= \frac{uS_0 q + (1-q)dS_0}{1+r} \Rightarrow q = \frac{1+r-d}{u-d} = 1/2$$

$$C_0(S_0) = \mathbb{E}_{\mathbb{Q}}\left[\frac{C_1(S_1)}{1+r}\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{C_2(S_2)}{(1+r)^2}\right]$$

Parité CALL-PUT

CALL 1.75 ; 4.4, 0 ; 11, 0, 0

PUT 0.96 ; 0.4, 2 ; 0, 1, 4

Quelque soit le noeud de l'arbre, $C_t(S_t) - P_t(S_t) = S_t - KB(t,T)$

Exercice 5 :

$$R^m_{t_i} = (1+r_m)^i$$

$$\mathbb{P}^m(X_i^m = u_m) = p_m = 1 - \mathbb{P}(X_i^m = d_m) \qquad r_m = \frac{rT}{m} \quad d_m = \left(1 + \frac{rT}{m}\right)e^{-\sigma\sqrt{\frac{T}{m}}} \quad u_m = \left(1 + \frac{rT}{m}\right)e^{\sigma\sqrt{\frac{T}{m}}}$$

(1)

S_0 ; $u_m S_0$ - - - ; $d_m S_0$

$$u_m^m S_0 = (1+r_m)^m e^{m\sigma\sqrt{\frac{T}{m}}} S_0$$

$$u_m^{m-1} d\, S_0 = (1+r_m)^m e^{(m-2)\sigma\sqrt{\frac{T}{m}}} S_0$$

$\vdots$

$$d_m^{m-1} u_m S_0 = (1+r_m)^m e^{-(m-2)\sigma\sqrt{\frac{T}{m}}} S_0$$

$$d_m^m S_0 = (1+r_m)^m e^{-m\sigma\sqrt{\frac{T}{m}}} S_0$$

(2) $$\lim_{m\to\infty} R_T^m = \lim_{m\to\infty} (1+r_m)^m = \lim_{m\to\infty} e^{m\ln\left(1+\frac{rT}{m}\right)}$$

$$m \in \mathcal{V}(\infty),\ m\ln\left(1+\frac{rT}{m}\right) \sim rT \Rightarrow \lim_{m\to\infty} R_T^m = e^{rT}$$

③ On doit vérifier la condition de non arbitrage i.e $d_n < 1+r_n < u_n$

$r_n = \frac{rT}{n}$ $\quad d_n = \left(1+\frac{rT}{n}\right)e^{-\sigma\sqrt{\frac{T}{n}}}$ $\quad u_n = \left(1+\frac{rT}{n}\right)e^{\sigma\sqrt{\frac{T}{n}}}$

Or $\sigma, T, n > 0 \Rightarrow \sigma\sqrt{\frac{T}{n}} > 0 \Rightarrow e^{\sigma\sqrt{\frac{T}{n}}} > 1$ et $e^{-\sigma\sqrt{\frac{T}{n}}} < 1$

$\Rightarrow d_n < \underbrace{\left(1+\frac{rT}{n}\right)}_{1+r_n} < u_n$

④ $S^n_{t^n_i} = \left(\prod_{i=1}^{n} X^n_i\right) S_0$

⑤ Sous $\mathbb{Q}^n$ on a $\frac{1}{1+r_n}\mathbb{E}_{\mathbb{Q}^n}[S^n_{t^n_i} \mid \mathcal{F}_{t^n_{i-1}}] = S^n_{t^n_{i-1}}$

Or $S^n_{t^n_i} = S^n_{t^n_{i-1}} \times X^n_i \Rightarrow \frac{1}{1+r_n}\mathbb{E}_{\mathbb{Q}^n}[S^n_{t^n_{i-1}} X^n_i \mid \mathcal{F}_{t^n_{i-1}}] = S^n_{t^n_{i-1}}$

$$\Rightarrow \frac{1}{1+r_n}\mathbb{E}_{\mathbb{Q}^n}[X^n_i \mid \mathcal{F}_{t^n_{i-1}}] = 1$$

Donc la condit° de martingalité devient $\mathbb{E}_{\mathbb{Q}^n}[X^n_i \mid \mathcal{F}_{t^n_{i-1}}] = 1+r_n$

On a $\mathbb{E}_{\mathbb{Q}^n}[X^n_i \mid \mathcal{F}_{t^n_{i-1}}] = q_n u_n + (1-q_n)d_n = 1+r_n$

$\Rightarrow q_n(u_n - d_n) + d_n = 1+r_n \Rightarrow q_n = \frac{(1+r_n)-d_n}{u_n-d_n}$

⑥