TD 3

Exercice 1 :

$$S = \sum_{i=1}^{N} X_i \qquad \text{et} \qquad N \sim \mathcal{P}(\lambda) \quad \text{et } \mathbb{P}(X_i = 1456) = \mathbb{P}(X_i = 2912) = \frac{1}{2}$$

$$\mathbb{P}(S < 1856) = \mathbb{P}(N = 0 \cup (N = 1 \cap X_1 = 1456)$$

$$= \mathbb{P}(N=0) + \mathbb{P}(N=1)\mathbb{P}(X_1 = 1456)$$

$$= \frac{0{,}53^0}{0!} e^{-0{,}53} + \frac{0{,}53}{1} e^{-0{,}53} \times \frac{1}{2} \simeq 0{,}74$$

Exercice 2

(a) $$p_0 = \mathbb{P}(X = 0) = \mathbb{P}\Big(\bigcap_{i,j} \{X_{ij} = 0\}\Big)$$

$$= \prod_{i,j} \mathbb{P}(X_{ij} = 0)$$

$$= \prod_{i,j} (1 - q_j)^{n_{ij}}$$

(b) $G_X(s) = \mathbb{E}[s^X] = \prod_{ij} \mathbb{E}[X^{ijl}]^{n_{ij}}$

$$G_{X_{ijl}}(s) = \mathbb{E}[s^{X_{ijl}}]$$

$$= \underbrace{1-\theta_j}_{\text{pas de sinistre}} \;\text{ou}\; + \theta_j \underbrace{\overbrace{\sum_{k=1}^{m_i} p_k^{(i)} s^k}^{G_i}}_{1 \text{ sinistre}}$$

Donc $G_X(s) = \prod_{ij} \left(1-\theta_j+\theta_j G_i(s)\right)^{n_{ij}}$

$$G_i^{(n)}(0) = n!\, p_n^{(i)} \quad \text{pour } n \leq m_i$$

(c) On passe par le log

$$\ln(G_X(s)) = \sum_{ij} n_{ij} \ln(1-\theta_j + \theta_j G_i(s))$$

ie $$(\ln(G_X(s)))' = \sum_{ij} n_{ij} \frac{\theta_j G_i'(s)}{1-\theta_j+\theta_j G_i(s)}$$

or $$(\ln(u))' = \frac{u'}{u} \Rightarrow u' = u(\ln(u))'$$

D'où $$G_X'(s) = \sum_{ij} n_{ij} \underbrace{\frac{\theta_j G_X(s) G_i'(s)}{1-\theta_j+\theta_j G_i(s)}}_{V_{ij}(s)}$$

(d) $$P_h = \frac{1}{h!} G_x^{(h)}(0) = \frac{1}{h!} (G')^{(h-1)}(0)$$

$$= \frac{1}{h!} \sum n_{ij} V^{(k-1)}(0)$$

$$= \frac{1}{h} \sum n_{ij} \frac{V^{(k-1)}(0)}{(h-1)!}$$

(e) On a $$V_{ij}(s) = \frac{G_x(s) G_i'(s) \theta_j}{1-\theta_j + \theta_j G_i(s)}$$

$$\Rightarrow (1-\theta_j) V_{ij}(s) + \theta_j G_i(s) V_{ij}(s) = \theta_j G_x(s) G_i'(s)$$

$$\Rightarrow V_{ij}(s) = \frac{\theta_j}{1-\theta_j} \left(-G_i(s) V_{ij}(s) + G_x(s) G_i'(s) \right)$$

<u>Rappel</u> : $$(uv)^n = \sum_{h=0}^{n} \binom{n}{h} u^h v^{n-h}$$

$$V_{ij}^{(k-1)}(s) = \frac{\theta_j}{1-\theta_j} \sum_{l=0}^{k-1} \binom{k-1}{l} G_i^{(l+1)}(s) G_x^{(k-1-l)}(s) - \binom{k-1}{l} V^{(k-1-l)} G_i^{(l)}(n)$$

$$\Rightarrow V_{ij}^{(k-1)}(0) = \frac{\theta_j}{1-\theta_j} \sum_{l=0}^{k-1} \binom{k-1}{l} p^{(i)} l! - \binom{k}{l} V_{ij}(k-l) \times (k-1-l)! \times p^{(i)} l!$$

$$= \frac{\theta_j}{1-\theta_j} \sum_{l=0}^{k-1} p_l^{(i)} \times ((k-1)!\, p_{k-l} - (k-1)!\, V_{ij}(k-l))$$

$$\Rightarrow V_{ij}(n) = \frac{V_{ij}^{(n-1)}}{(k-1)!} = \frac{\theta_j}{1-\theta_j} \sum_{l=1}^{\min(m_i,n)} p_l^{(i)} (l\, p_{n-l} - V_{ij}(n-l)) \mathbb{1}_{n \in [1,m]}$$