Corrigé Séance 3 TD Théorie des Options

Exercice 1 :
Soient K_1 et K_3, avec $(K_1 < K_3)$ et $\lambda \in]0,1[$. Montrons que

$$C_0(\lambda K_1 + (1-\lambda)K_3) \leq \lambda C_0(K_1) + (1-\lambda)C_0(K_3).$$

On pose $K_2 = \lambda K_1 + (1-\lambda)K_3$. On a alors $K_1 < K_2 < K_3$. Supposons

$$C_0(K_2) > \lambda C_0(K_1) + (1-\lambda)C_0(K_3).$$

	$t=0$	$t=T$
Vente $C_0(K_2)$	$C_0(K_2)$	$-(S_T-K_2)_+$
Achat $\lambda C_0(K_1)$	$-\lambda C_0(K_1)$	$\lambda(S_T-K_1)_+$
Achat $(1-\lambda)C_0(K_3)$	$-(1-\lambda)C_0(K_3)$	$(1-\lambda)(S_T-K_3)_+$
Placement du reste	$-(C_0(K_2)-\lambda C_0(K_1)$ $-(1-\lambda)C_0(K_3))$	$(C_0(K_2)-\lambda C_0(K_1)-(1-\lambda)C_0(K_3))e^{rT}$

En $t=0$, $X_0=0$ et en $t=T$

$$X_T = -(S_T-K_2)_+ + \lambda(S_T-K_1)_+ + (1-\lambda)(S_T-K_3)_+ + (C_0(K_2)-\lambda C_0(K_1)-(1-\lambda)C_0(K_3))e^{rT}.$$

Si $S_T < K_1$, $X_T = (C_0(K_2)-\lambda C_0(K_1)-(1-\lambda)C_0(K_3))e^{rT} \geq 0$.
Si $K_1 \leq S_T < K_2$, $X_T = \lambda(S_T-K_1) + (C_0(K_2)-\lambda C_0(K_1)-(1-\lambda)C_0(K_3))e^{rT} \geq 0$.
Si $K_2 \leq S_T < K_3$, $X_T = (1-\lambda)(K_3-S_T) + (C_0(K_2)-\lambda C_0(K_1)-(1-\lambda)C_0(K_3))e^{rT} \geq 0$.
Si $S_T > K_3$, $X_T = (C_0(K_2)-\lambda C_0(K_1)-(1-\lambda)C_0(K_3))e^{rT} \geq 0$.
On a donc construit une OA: le prix du call est une fonction convexe du strike.

Exercice 3 :
La parité call-put s'écrit:

$$C_0 - P_0 = S_0 - Ke^{-rT}.$$

- Or :

$$C_0 = \hat{C}_0 \quad et \quad \hat{P}_0 \geq P_0$$

D'où

$$\hat{C}_0 - \hat{P}_0 = C_0 - \hat{P}_0 \leq C_0 - P_0 = S_0 - Ke^{-rT}.$$

- $S_0 - K \leq \hat{C}_0 - \hat{P}_0$:
Supposons $S_0 - K - \hat{C}_0 + \hat{P}_0 > 0$

	$t=0$	$t=\tau$ date d'ex du PUT
Vente S	S_0	$-S_\tau$
Vente $\hat{P}$	$\hat{P}_0$	$-(K-S_\tau)_+$
Achat $\hat{C}$	$-\hat{C}_0$	$\hat{C}_\tau$ (pas optimal exercer CALL avant T)
Placement $S-\hat{C}_0+\hat{P}_0$	$-(S_0+\hat{P}_0-\hat{C}_0)$	$(S_0+\hat{P}_0-\hat{C}_0)e^{r\tau}$
	0	X_τ

où $X_\tau = \hat{C}_\tau - S_\tau - (K-S_\tau)_+ + (S_0+\hat{P}_0-C_0)e^{r\tau}$
avec $\hat{C}_\tau \geq (S_\tau - K)_+$
donc $X_\tau \geq (S_\tau-K)_+ - S_\tau - (K-S_\tau)_+ + (S_0+\hat{P}_0-C_0)e^{r\tau}$
Si $S_\tau > K$:

$$\begin{aligned} X_\tau &\geq S_\tau - K - S_\tau + (S_0+\hat{P}_0-C_0)e^{r\tau} \\ &\geq S_0 + \hat{P}_0 - C_0 - K > 0 \end{aligned}$$

Si $S_\tau < K$:

$$\begin{aligned} X_\tau &\geq -S_\tau - K + S_\tau + (S_0 + \hat{P}_0 - C_0)e^{r\tau} \\ &> 0 \end{aligned}$$

OA absurde,
Finalement $S_0 - K \leq \hat{C}_0 - \hat{P}_0 \leq S_0 - Ke^{-rT}$.

Exercice 7:
1.a) Le portefeuille X_T est construit par:
- emprunt de $(K+D)e^{-rT}$
- Achat put : $-P_0$
- A chat sous-jacent: $-S_0$
- Vente call: $+C_0$

En $t = 0$, on a alors $X_0 = C_0 + (K+D)e^{-rT} - P_0 - S_0$.

1.b) En $t = T$, $X_T = (S_T - K)_+ - (K+D) + (K - S_T)_+ + (S_T + D)$.
Si $S_T < K$, $\Rightarrow -(K+D) + K - S_T + S_T + D = 0$,
Si $S_T \geq K$, $\Rightarrow -(S_T - K) - (K+D) + S_T + D = 0$.

1.c) A T solde nul, donc investissement nul.

1.d) $\Rightarrow C_0 + (K+D)e^{-rT} - P_0 - S_0 = 0$, sinon arbitrage.

2) Deux égalités (deux fois la relation de parité CALL-PUT).
$15 + (40+D)e^{-r} - 1.71 - 52 = 0, \Rightarrow (40+D)e^{-r} = 38.71$

$$8.36 + (50+D)e^{-r} - 4.5 - 52 = 0, \Rightarrow (50+D)e^{-r} = 48.14$$

$$\Rightarrow 10e^{-r} = 9.43 \Rightarrow e^{-r} = 0.943 \Rightarrow r = 5.86\% \text{ et } D = 1.05 \text{ €}.$$

Exercice 8:
Achat 200 CALL + Vente 40 actions.
a) $V = 200C - 400S_0$
$\Delta(V) = 0$, $\Gamma(V) = 3$, $\Delta(S_0) = 1$ et $\Gamma(S_0) = 0$.

$\Delta(V) = 0 = 200\Delta(C) - 40\Delta(S_0)$

$\Rightarrow 5\Delta(C) = 1 \Rightarrow \Delta(C) = 0.2$

$3 = 200\Gamma(C) \Rightarrow \Gamma(C) = 0.015$

b) $V^\star = V + \alpha P_0$
$\Gamma(V^\star) = \Gamma(V) + \alpha\Gamma(P_0) = 3 + \alpha 0.015$

$$\text{Si } \Gamma(V^\star) = 0 \Rightarrow \alpha = -\frac{3}{0.015} = -200$$

$\Rightarrow$ Vente de 200 PUT (position courte).

c) $\Delta(V^\star) = \Delta(V) - 200\Delta(P_0) = 100$

d) $V^{\star\star} = V^{\star} + \beta S_0$, $\Delta(V^{\star\star}) = \Delta(V^{\star}) + \beta \Rightarrow \beta = -100$

$\Gamma(V^{\star\star}) = \Gamma(V^{\star}) - 100 \times 0 = 0$

Conclusion :

$$V^{\star\star} = \left.\begin{matrix} achat & 200 & CALL \\ vente & 200 & PUT \\ vente & 140 & actions \end{matrix}\right\} \begin{matrix} \Delta - neutre \\ \Gamma - neutre \end{matrix}$$

Exercice 9:

(a) Relation de parité Call- Put: $P'_0 = C'_0 - S_0 + K_1 e^{-rT}$. On obtient alors:

$$P_0^1 = 8.772 - 42 + 40 \exp\left(-0.05 \times \frac{9}{12}\right) = 5.30$$

et

$$\begin{aligned} &\Delta(P^1) = \Delta(C^1) - 1 = -0.339 \\ &\Gamma(P^1) = \Gamma(C^1) = 0.02 \end{aligned}$$

On procède de la même manière pour le PUT 2:

$$\begin{aligned} &P_0^2 = 9.861 \\ &\Delta(P^2) = \Delta(C^2) - 1 = -0.502 \\ &\Gamma(P^2) = 0.026 \end{aligned}$$

(b) (i) Achat CALL 1 + 2 PUT P^1 + PUT P^2. On obtient

$$R_T = -C_0^1 - 2P_0^1 - P_0^2 + (S_T - K_1)_+ + 2(K_1 - S_T)_+ + (K_2 - S_T)_+.$$

- Si $S_T < 40$: $R_T = 98.767 - 3S_T$.
- Si $40 \leq S_T < 48$: $R_T = -21.233$.
- Si $48 \leq S_T$: $R_T = S_T - 69.233$.

(ii) Les points morts sont les valeurs de S_T telles que $R_T = 0$. On obtient alors les valeurs 32.922 et 69.233.

(iii) $-29.233 + (S_T - 40)_+ + 2(40 - S_T)_+ + (48 - S_T)_+ > -5.3 + (40 - S_T)_+ \Rightarrow S^{\star} = 32.0335.$

(iv) $\Delta = 0.661_2 \times 0.339 - 0.502 = -0519$ et $\Gamma = 0.02 + 2 \times 0.02 + 0.026 = 0.086.$