



# Introduction aux copules et à la dépendance stochastique dans une perspective ERM

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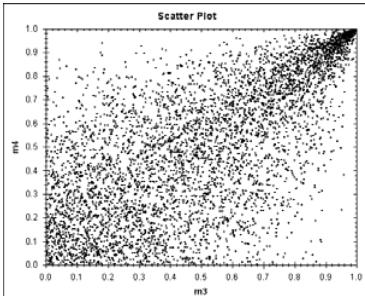
# A conservative way of modeling dependencies

The functional form of the dependency has a significant influence on the diversification benefit.

## Example:

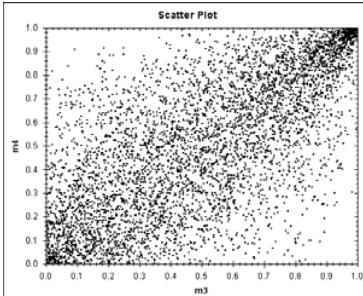
- 4 lognormal loss distributions with  $m = 10$  and  $s = 1$
- Hierarchical Clayton and Gumbel copulas, as well as Student's T and Rank Correlation are fitted to the same synthetic dataset
- The rank scatter of two sample marginals is shown, as well as the resulting diversification benefits

**Clayton  $\Theta = 2$**



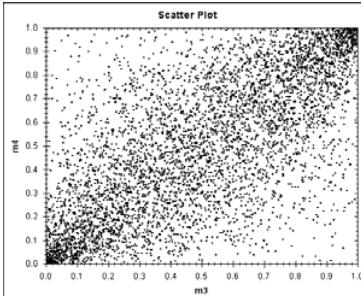
Div. Benefit: 8.2%

**Gumbel  $\Theta = 2.07$**

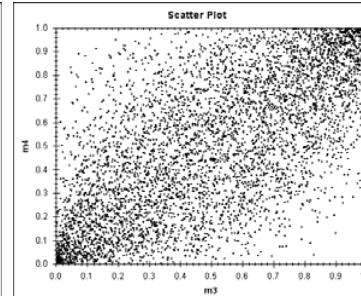


Div. Benefit: 11.7%

**Student's T  $r = 0.71$ ;  $n = 6$**  **Rank Correlation  $r = 0.7$**



Div. Benefit: 25.4%



Div. Benefit: 30.7%

The Clayton copula (used in our internal model) is asymmetric and estimates the diversification benefit conservatively. The Rank correlation is symmetric and estimates a too high diversification benefit, which is not present in the data

$$\text{SCR}_1 \approx \text{VaR}_{99.5\%}(X_1) - \mathbb{E}[X_1]$$

↳ charge sinistre annuelle branche 1.

$$\text{SCR}_{\text{avant div.}} = \sum_{i=1}^4 (\text{VaR}_{99.5\%}(X_i) - \mathbb{E}[X_i]) = [\sum_{i=1}^4 \text{VaR}_{99.5\%}(X_i)] - [\mathbb{E}[X_1 + \dots + X_4]]$$

$$\text{SCR}_{\text{diversifié}} = \text{VaR}_{99.5\%}(X_1 + \dots + X_4) - \mathbb{E}[X_1 + \dots + X_4]$$

$$\text{SCR}_{\text{diversifié}} < \text{SCR}_{\text{avant div.}}$$

Souvent mais pas toujours  
Value at Risk n'est PAS sous-additive et donc NON "cohérente" au sens d'ARTZNER et al (1999)

Envisageons un stop-loss aggregate toutes causes pour protéger le bilan de la compagnie i.e le réassureur paie :  $(X_1 + X_2 + X_3 + X_4 - b)_+$

↳ 58 m€

Prime pure de réassurance :  $\mathbb{E}[(X_1 + \dots + X_4 - b)_+]$

Loi de  $X_1$  "comme" /  
 $X_2$  — } étudiées séparément  
 $X_3$  — }  
 $X_4$  — }

J'ai besoin de  $P(X_1 \leq x_1, \dots, X_4 \leq x_4)$  pour  $(x_1, \dots, x_4) \in \mathbb{R}^4$   
 $= C(P(X_1 \leq x_1), \dots, P(X_4 \leq x_4))$

8° Cas où  $C: [0,1]^4 \rightarrow [0,1]$

en respectant  $P(X_1 \leq x_1), \dots, P(X_4 \leq x_4)$  étudiées auparavant

En général, dimension = 7, 20, 32, 108...  
 ↳ classes du risque      ↳ facteurs du risque

Sans hypothèse forte il est impossible d'estimer quelle cas où choisir en dim 10 etc à partir des données.

⚠ En assurance, il est quasiment impossible de déterminer la structure de corrélation en dim  $\geq 10$   
 Sans hypothèse extrêmement forte.

# Outline

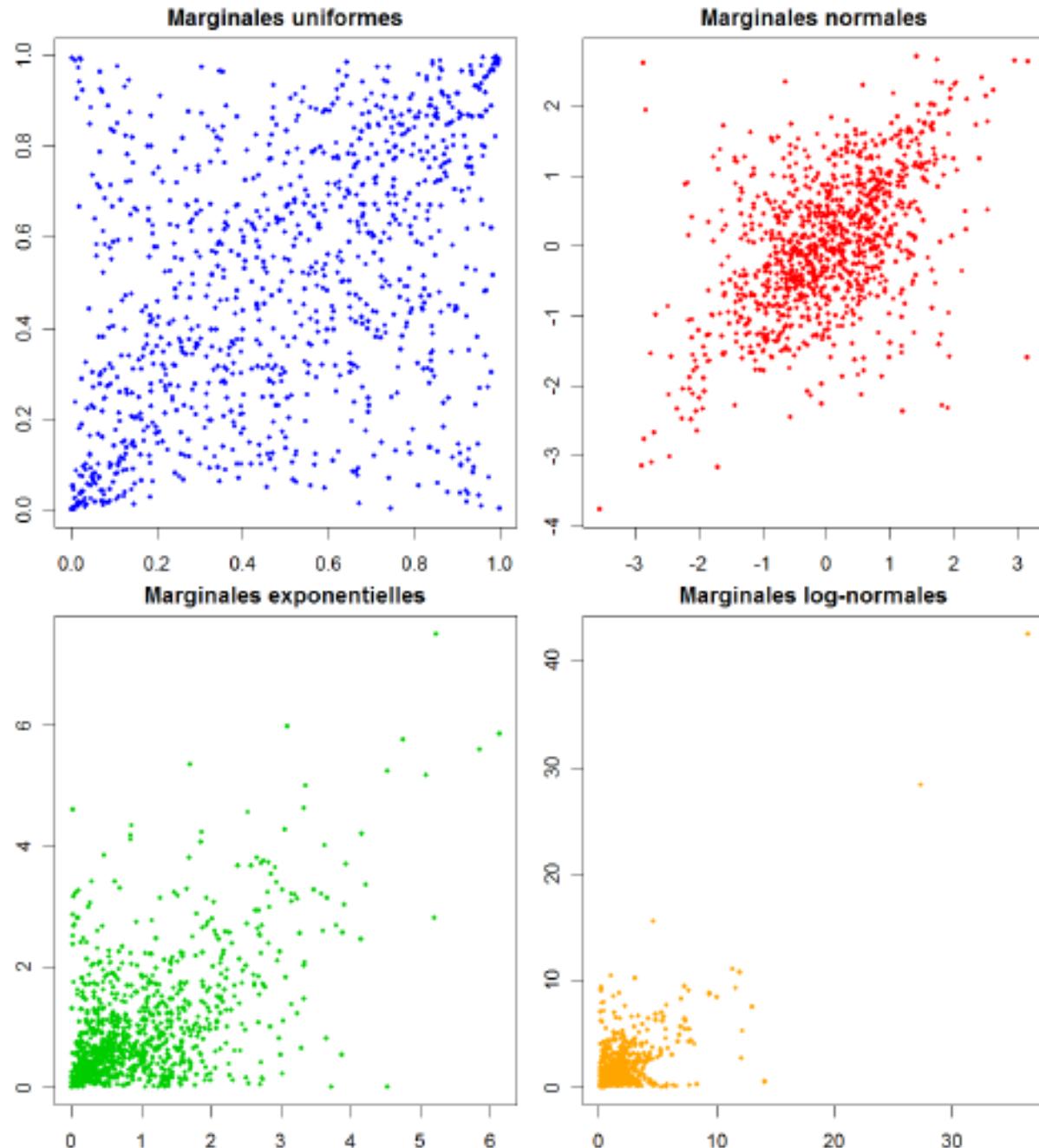
- Rappels: Définition, intérêts (cf Chap.1, Post-reading 1).
- Pourquoi se ramener à  $[0,1] \times [0,1]$ ? Lisibilité: Cf Slide 4.
- Copules usuelles
  - Elliptiques: Gaussienne, Student, ... (cf Chap. 5 du Post-reading 1)
  - Archimédiennes (cf Chap. 6 du Post-reading 1)
  - Autres copules usuelles
- Mesurer la dépendance stochastique (cf Chap. 3 du Post-reading 1)
- Concepts de dépendance (cf Chap. 3 du Post-reading 1)
- Dépendance des extrêmes (cf Post-reading 5)
- Trois problèmes concrets:
  - Corrélations DAX/CAC 40/Dow Jones (cf post-reading 5)
  - Corrélations en assurance non vie (cf post-reading 8)
  - Corrélations Hommes/Femmes en assurance-vie (cf post-reading futur)
- Quelles sont les choses qu'un CRO doit savoir sur les copules pour faire la part des choses lorsqu'on lui soumet un modèle interne?

# Quelques questions

- Faible corrélation: peut-on conclure? Slide 5 + post-reading 1)
- Dépendance des extrêmes (cf post-reading 5 + slides «Une conclusion un peu trop hâtive»)
- VaR et Risk Management (La VaR est sous-additive pour les lois elliptiques (cf exposé du vendredi): attention seulement valable si la loi jointe est elliptique: par exemple copule gaussienne et marginales gaussiennes)
- Comment modéliser les dépendances entre 23 risques? Copule Gaussienne, Student (cf post-readings 1 & 4), ou Archimédienne hiérarchique (Post-reading 8).

# Pourquoi se ramener à des marginales uniformes sur [0,1]

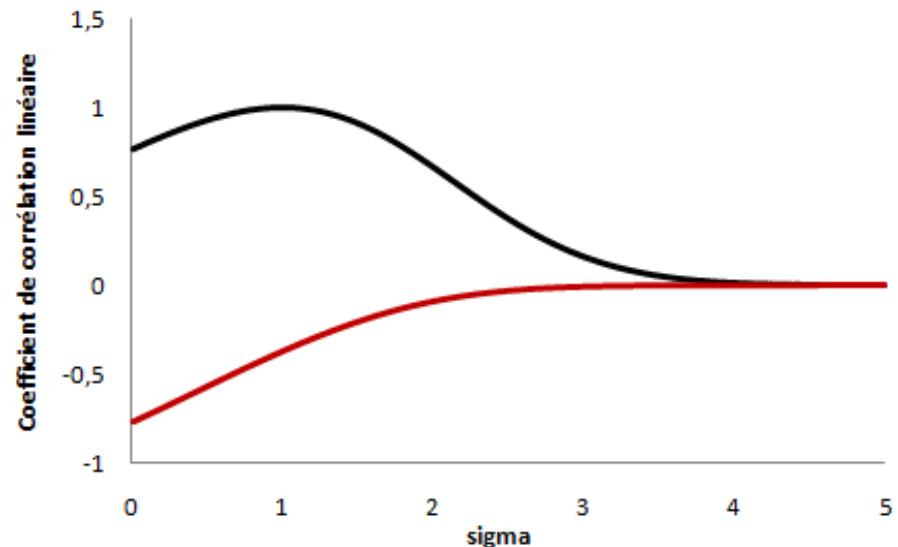
Voici le résultat de la simulation de la copule de Student avec  $\rho_{12} = 0.5$  et  $\nu = 2$  :



# Faible corrélation

Valeurs de  $r_{\min}$  et  $r_{\max}$  en fonction de  $\sigma$

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- Le coefficient de corrélation linéaire n'est pas adapté pour mesurer la dépendance entre des variables aléatoires de type log-normales ou Pareto.
- Coeff de corrélation linéaire proche de 0: ne permet pas de conclure qu'on est proche de l'indépendance.
- Utiliser des mesures de dépendance comme le tau de Kendall ou le Rho de Spearman (cf post-reading 1 Chap. 3 et Post-reading 7) , plus des mesures de dépendance des extrêmes (cf post-reading 5).

Valeurs min et max

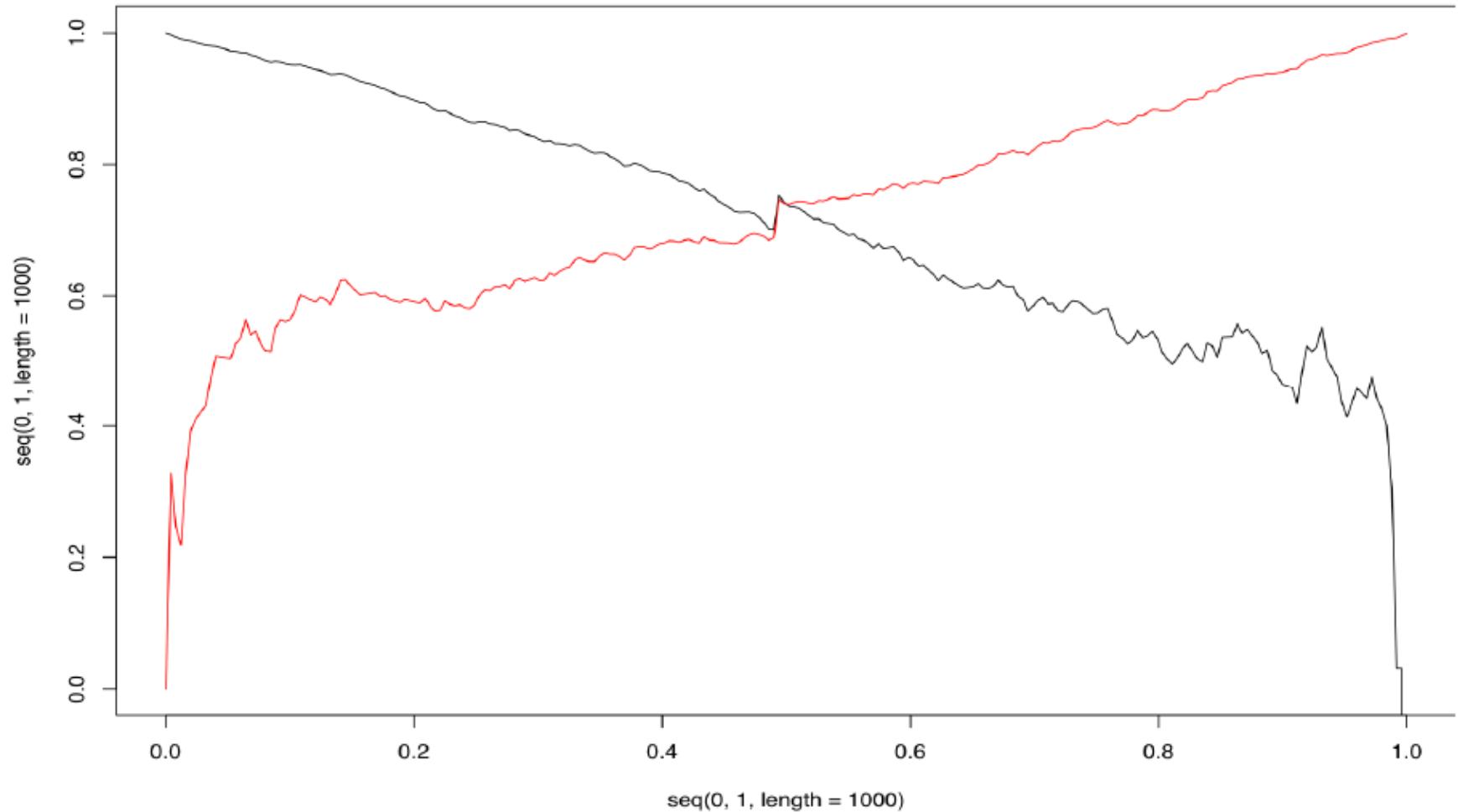
en ordonnée sur le graphique  
de la page précédente

de  $\text{corr}(X, Y)$

en fonction de  $G$

avec  $\begin{cases} X \sim LN(0, 1) \\ Y \sim LN(0, 6^2) \end{cases}$

# Une conclusion un peu trop hâtive



Si vu de ce graphe, nous pouvons caractériser la copule qui permet de modéliser la dépendance des titres AXA et VIVENDI, en considérant la copule de Student.

# Une conclusion un peu trop hâtive

- Pour la copule Student, les fonctions L et R (cf post-readings 2 & 5) ont des limites  $> 0$ , donc présentent de la dépendance forte des extrêmes.
- Pour la copule gaussienne, ces fonctions ont des limites nulles, donc pas de dépendance forte des extrêmes.
- Toutefois, avec peu de données, il peut être très difficile de différencier les deux statistiquement.

# Outline for today

- Correlation modeling
  - Copulas
  - Endogenous risk and correlation crises
- Risk aggregation
  - in Solvency II standard formula: some pitfalls
  - in internal models: example with Excel spreadsheet

## Copulas, an introduction (in dimension 2)

**Definition 1.** A copula  $C$  is a joint distribution function on  $[0, 1]^2$ , with uniform margins on  $[0, 1]$ .

Set  $C(u, v) = \mathbb{P}(U \leq u, V \leq v)$ , where  $(U, V)$  is a random pair with uniform margins.

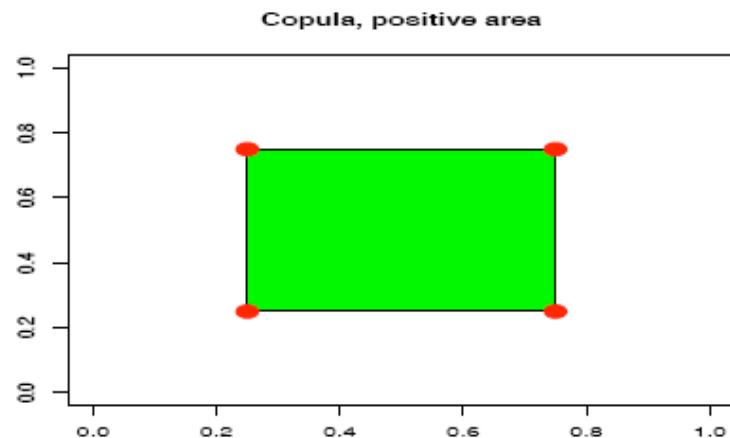
$C$  is a distribution function on  $[0, 1]^2$ , and thus  $C(0, v) = C(u, 0) = 0$ ,  $C(1, 1) = 1$ .

Furthermore  $C$  is *increasing*: since  $\mathbb{P}$  is a positive measure, for all  $u_1 \leq u_2$  and  $v_1 \leq v_2$ ,

$$\mathbb{P}(u_1 < U \leq u_2, v_1 < V \leq v_2) \geq 0,$$

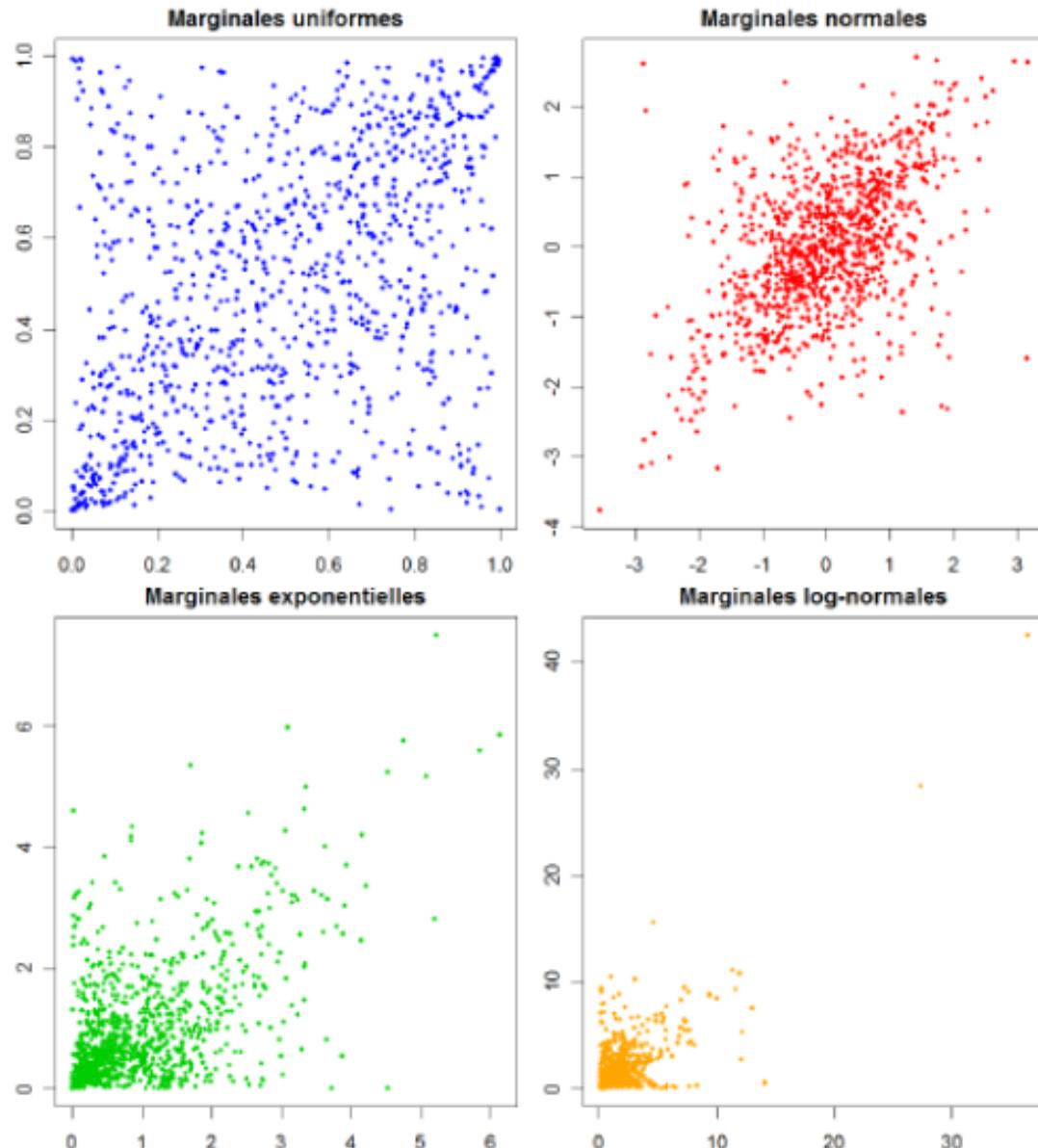
thus

$$\begin{aligned} C(u_2, v_2) - C(u_1, v_2) \\ - C(u_2, v_1) + C(u_1, v_1) \geq 0. \end{aligned}$$



# Why Uniform([0,1]) marginals?

Voici le résultat de la simulation de la copule de Student avec  $\rho_{12} = 0.5$  et  $\nu = 2$  :



$$X: \begin{cases} 0 & \text{proba } 1/2 \\ 1 & \text{proba } 1/2 \end{cases} \quad C: [0,1]^2 \longrightarrow [0,1]$$

$$F_X(x) = \begin{cases} F_X(0) & \text{prob } \frac{1}{2} \\ 1 & \text{prob } \frac{1}{2} \end{cases}$$

$$(u,v) \longrightarrow C(u,v) = P(U \leq u, V \leq v)$$

ou  $\begin{cases} U \sim \text{Unif}([0,1]) \\ V \sim \text{Unif}([0,1]) \end{cases}$

Th SKLAR: (1959)

deux v.a.  $X$  et  $Y$

il existe (au moins) une copule  $C$  t.q.

$$F_{x,y}(x,y) = C\left(\overbrace{F_x(x)}^u, \overbrace{F_y(y)}^v\right)$$

Si de plus  $F_x$  et  $F_y$  sont continues, alors  $C$  est unique et appelée copule de  $(X,Y)$

$$P(X \leq x, Y \leq y)$$

$$P\left(\underbrace{F_x(x)}_{U \sim \text{Unif}([0,1])} \leq \overbrace{F_x(x)}^u, \underbrace{F_y(y)}_{V \sim \text{Unif}([0,1])} \leq \overbrace{F_y(y)}^v\right)$$

1 dans le  
cas bijectif  
uniquement

$C$  has uniform margins, and thus

$$C(u, 1) = \mathbb{P}(U \leq u, V \leq 1) = \mathbb{P}(U \leq u) = u \text{ on } [0, 1].$$

**Proposition 2.**  *$C$  is a copula if and only if  $C(0, v) = C(u, 0) = 0$ ,  $C(u, 1) = u$  and  $C(1, v) = v$  for all  $u, v$ , with the following 2-increasingness property*

$$C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0,$$

*for any  $u_1 \leq u_2$  and  $v_1 \leq v_2$ .*

If  $C$  is twice differentiable, one can define its density as

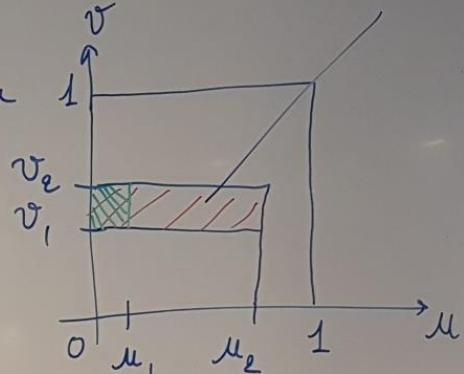
$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}.$$

$$C(\mu, \sigma) = P\left(\overset{\text{Unif}([0,1])}{U} \leq \mu, \overset{\text{Unif}([0,1])}{V} \leq \sigma\right) \leq P(V \leq \sigma) = \sigma$$

$$P(U \leq \mu_2, V \leq v_2)$$

$$C(\mu, 1) = P(U \leq \mu, V \leq 1) = P(U \leq \mu) = \mu$$

p.s. vrai



supermodulaire (ou  $\mathcal{L}$ -INCREASING)

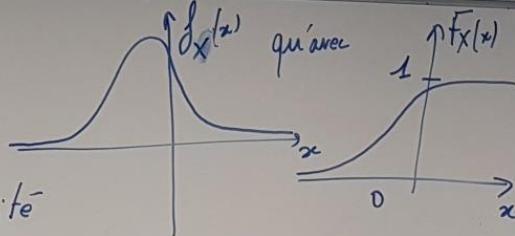
$$\forall 0 \leq \mu_1 \leq \mu_2 \leq 1, \quad \forall 0 \leq v_1 \leq v_2 \leq 1,$$

$$\frac{C(\mu_2, v_2) - C(\mu_2, v_1)}{P((U, V) \in \boxed{\text{red}})} \geq \frac{P(U \leq \mu_2, v_1 < V \leq v_2)}{P((U, V) \in \boxed{\text{green}})}$$

$$-(C(\mu_1, v_2) - C(\mu_1, v_1)) \geq 0$$

En dom 1, on préfère travailler avec

$$\text{densité } f_X(x) = F_X'(x)$$



En dom 2, on préfère travailler avec la densité

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}$$

(quand les densités existent)

$$C^\perp(u, v) = uv \quad \text{car} \quad P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y) \quad \text{qd } X \perp Y$$

$$c^\perp(u, v) = 1 \quad \forall (u, v) \in [0, 1]^2$$

$\boxed{\begin{array}{l} \text{Copule } C \\ \forall (u, v) \in [0, 1]^2, C_{\min}(u, v) \leq C(u, v) \leq C_{\max}(u, v) \end{array}}$

$U = V$  ("corrélation 100%") COMONOTONES  $\rightarrow$  copule: borne supérieure de FRÉCHET

$$P(U \leq u, V \leq v) = P(V \leq \min(u, v)) \quad C_{\max}(u, v) = \min(u, v)$$

$U = 1 - V$  ("corrélation -100%") ANTI-MONOTONE  $\rightarrow$  copule: borne inf de FRÉCHET

$$C_{\min}(u, v) = \max(0, u+v-1)$$

$$\mathbb{E}[g(X, Y)] = \iint_{\mathbb{R}^2} g(x, y) f_{X,Y}(x, y) dx dy$$

$$= \iint_{\mathbb{R}^2} g(x, y) \underbrace{C(F_x(x), F_y(y))}_{\Rightarrow \text{quand } X \perp\!\!\!\perp Y} f_x(x) f_y(y) dx dy$$

Sklar: Il existe toujours une fonction copule tq  $F_{X,Y}(x, y) = C(F_x(x), F_y(y)) \quad \forall (x, y) \in \mathbb{R}^2$   
 De plus, si  $f_x$  et  $f_y$  sont continues alors elle est unique.

Exercice: Soit  $(x, y) \in \mathbb{R}^2$ ,  $F_{X,Y}(x, y) = \frac{1}{1+e^{-x-y}}$

- 1) Déterminer  $F_x$  et  $F_y$
- 2) Déterminer  $C_{X,Y}$

$$1) F_X(x) = P(X \leq x) = \lim_{y \rightarrow \infty} F_{X,Y}(x, y) = \lim_{y \rightarrow \infty} \frac{1}{1+e^{-x-y}} = \frac{1}{1+e^{-x}}$$

$$F_Y(y) = P(Y \leq y) = \lim_{x \rightarrow \infty} F_{X,Y}(x, y) = \lim_{x \rightarrow \infty} \frac{1}{1+e^{-x-y}} = \frac{1}{1+e^{-y}}$$

$$2) F_{X,Y}(x, y) = C(F_x(x), F_y(y))$$

$$\Rightarrow \frac{1}{1+e^{-x-y}} = C\left(\frac{1}{1+e^{-x}}, \frac{1}{1+e^{-y}}\right)$$

$$\text{Posons } u = \frac{1}{1+e^{-x}} = F_x(x) \text{ et } v = \frac{1}{1+e^{-y}} = F_y(y)$$

$$\Rightarrow C(u, v) = \frac{1}{u+v-1}$$

Comment mesurer la corrélation entre  $X$  et  $Y$ ?

\* Coefficient de corrélation linéaire  $\rho_{lin} = \frac{\text{Cov}(X, Y)}{\sqrt{X} \sqrt{Y}}$

\*  $\tau$  de Kendall / autres variantes

\* Coeff de dépendance de queue (tail dependence index)

(Zoom sur corrélat. des extrêmes) à droite / à gauche

\* paramètre de corrélation Solvabilité II.

Procès en condamnation de  $\rho_{lin}$

-  $X$  et  $Y$  doivent être dans  $L^2(\Omega)$

-  $X \perp Y \Rightarrow \rho_{lin}(X, Y) = 0$  mais la réciproque est fausse!

-  $\rho_{lin}(e^X, e^Y) \neq \rho_{lin}(X, Y)$  en général.

$\rho_{lin}$  n'est pas une vraie mesure de corrélation car c'est aussi influencé par les marginales

-  $\rho_{lin}$  est bien compris entre -100% et 100% mais parfois pour certaines marginales, ses valeurs possibles sont très proche de 0 seulement.

Dans certains cas,  $r_{\text{Em}}$  peut conduire à soutenir l'hypothèse d'indépendance alors que les risques sont très fortement dépendants

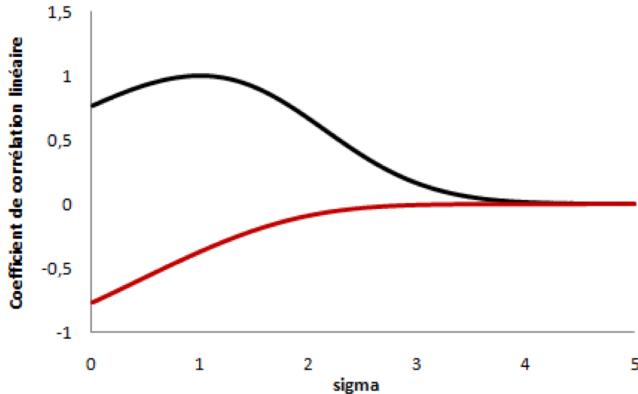
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- $r_{\text{Em}}$  est très peu robuste, notamment très sensible à une ou deux paires de valeurs extrêmes ou aberrantes.

Théorème:  $g, h \in C^1$ , copule de  $(g(X), h(Y))$  est la même que celle de  $(X, Y)$

### Valeurs de $r_{\min}$ et $r_{\max}$ en fonction de $\sigma$



1. Valeurs min et max prises par  $r_{\text{Em}}(X, Y)$

$$\text{où } \begin{cases} X \sim LN(0, 1) \\ Y \sim LN(0, \sigma^2) \end{cases}$$



Le  $\tau$  de Kendall.

(le bon élève)

- \* pas besoin de condition  $X^2$
- \* ne dépend que de la copule
- \* c'est une vraie mesure de corrélation

$$\tau = \frac{1}{4} \iint_{[0,1]^2} C(u, v) \underbrace{dC(u, v)}_{C(u, v) du dv \text{ quand c existe.}} - 1$$

Dans le cas Commomotone,  $\tau = 100\%$   
antimomotone,  $\tau = -100\%$   $\geq$  Il prend les valeurs qu'on attend

- \* Le  $\tau$  de Kendall est très robuste et est peu voire très peu sensible à une valeur extrême / aberrante.

$$\tau_{xy} = P((X-\bar{X})(Y-\bar{Y}) > 0) - P((X-\bar{X})(Y-\bar{Y}) < 0)$$

CONCORDANCE DISCORDANCE

où  $(\bar{X}, \bar{Y})$  copie  $\perp\!\!\!\perp$  de  $(X, Y)$   
Signifie que  $(X, Y)$  et  $(\bar{X}, \bar{Y})$  ont la bivarétiété et que  $(X, Y) \perp\!\!\!\perp (\bar{X}, \bar{Y})$

Exemple Coté Est des USA on est un assureur multi risque habilité (Floride et Louisiane)

	Fl	Lo
2024	1200 m€	900 m€
2025	210 m€	190 m€
2026	195 m€	$\frac{920 \text{ m€}}{200 \text{ m€}}$

$$\hat{\tau}_{\text{empirique}} = \frac{(\text{nb de paires concordantes}) - (\text{nb de paires discordantes})}{\text{nb total de paires}}$$

taux de kendall donnerait ça car on garde les rapports de Floride / Louisiane (Proba de CONCORDANCE dans ce sens)  
ordres

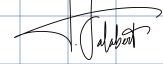
$$(X - \bar{X})(Y - \bar{Y}) > 0$$

$$(X - \bar{X})(Y - \bar{Y}) < 0$$

$$\rightarrow \hat{\zeta}_{\text{empirique}} = \frac{2-1}{3} = +1/3$$

	Concordant	Discordant
2024/2025	X	
2025/2026		X
2024/2026	X	

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## Procès en réhabilitation de Rein:

\* Le  $\zeta$  de Kendall est trop centré sur la corrélation "moyenne" et pas assez sur les extrêmes.

Pour projet (Titrisation multi-risques et risques spéciaux) : il s'agit pas de faire un exposé maths mais faire un exposé qui décrit les mécanismes de titrisation, dans le cours qui suit on aura des bases générales sur la titrisation en assurance et il ne faudra pas répéter le cours et aller plus loin sur notre sujet spécifique.

Taille marché, activité, part de marché, zoomer sur deux ou trois produits à décrire en détail, partie pricing des ILS c'est une partie où on va pouvoir trouver des documents mais très peu concret donc y passer peu de temps)

## Borders of the copula function

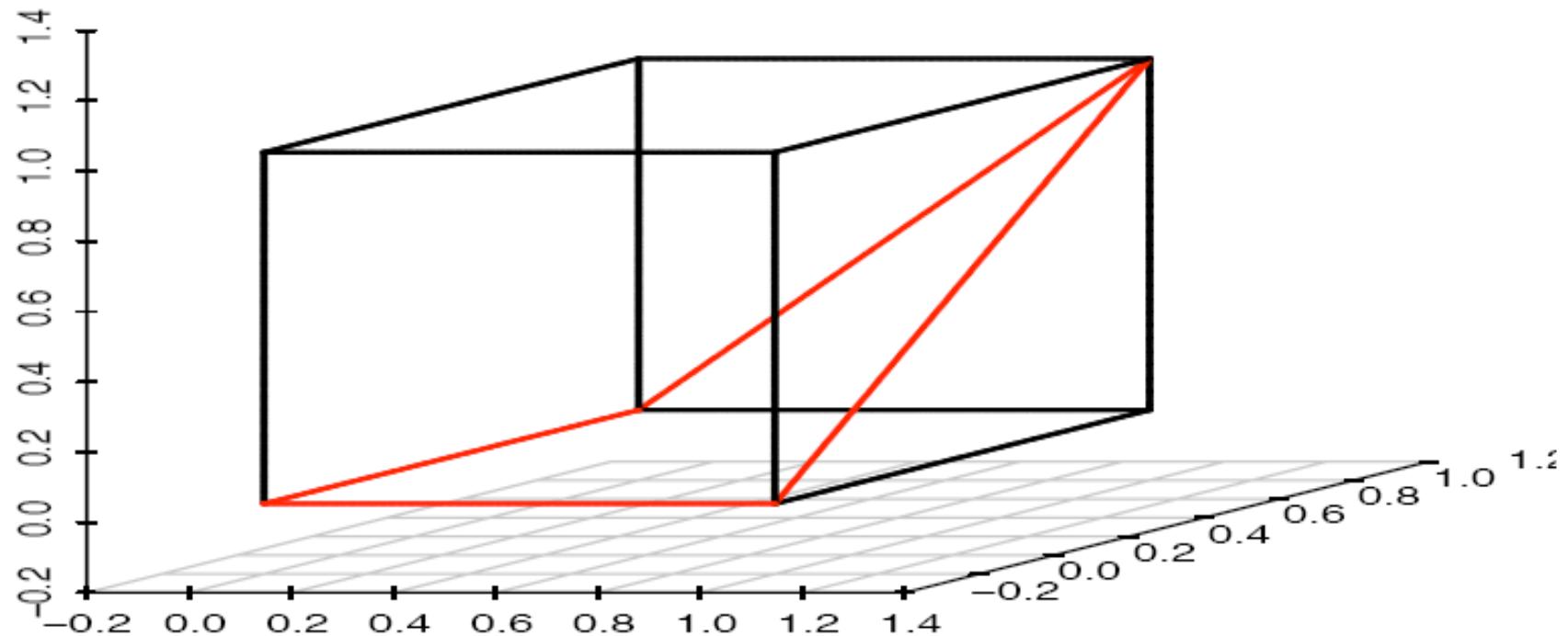
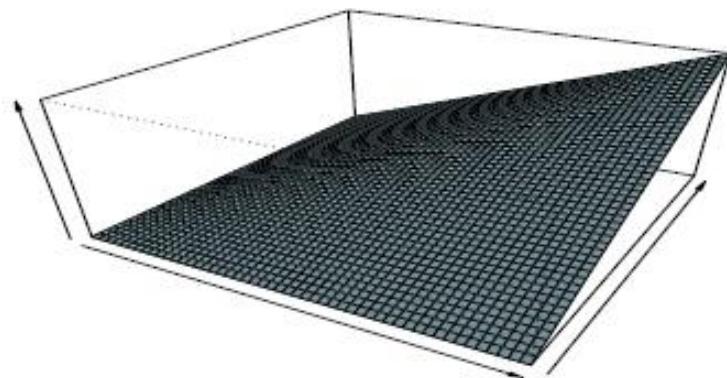
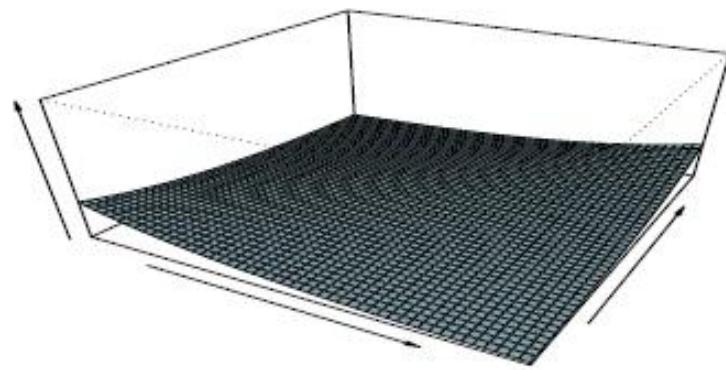


Figure 1: Value of the copula on the border of the unit square.

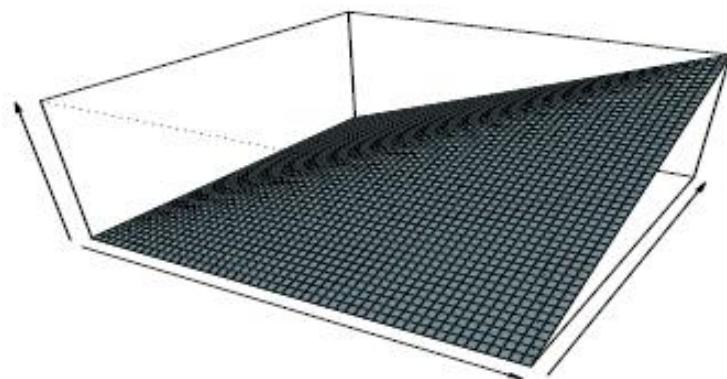
Fonction de répartition à marges uniformes



Densité d'une loi à marges uniformes



Fonction de répartition à marges uniformes



Densité d'une loi à marges uniformes

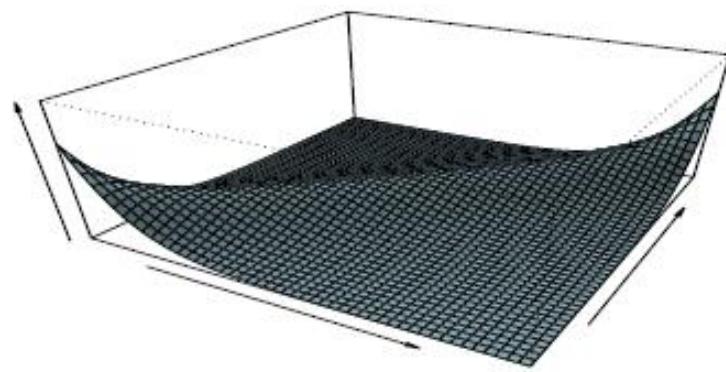
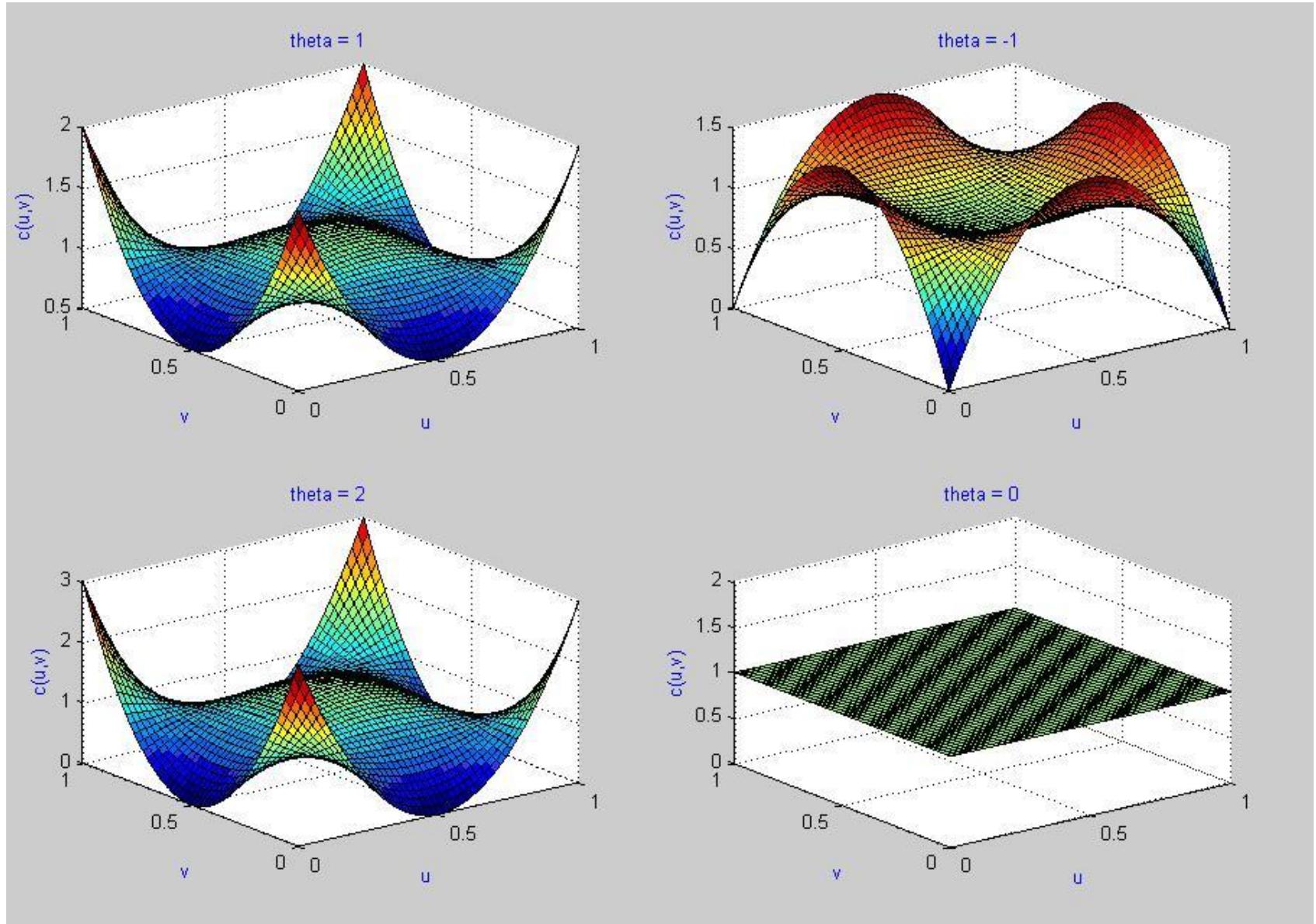
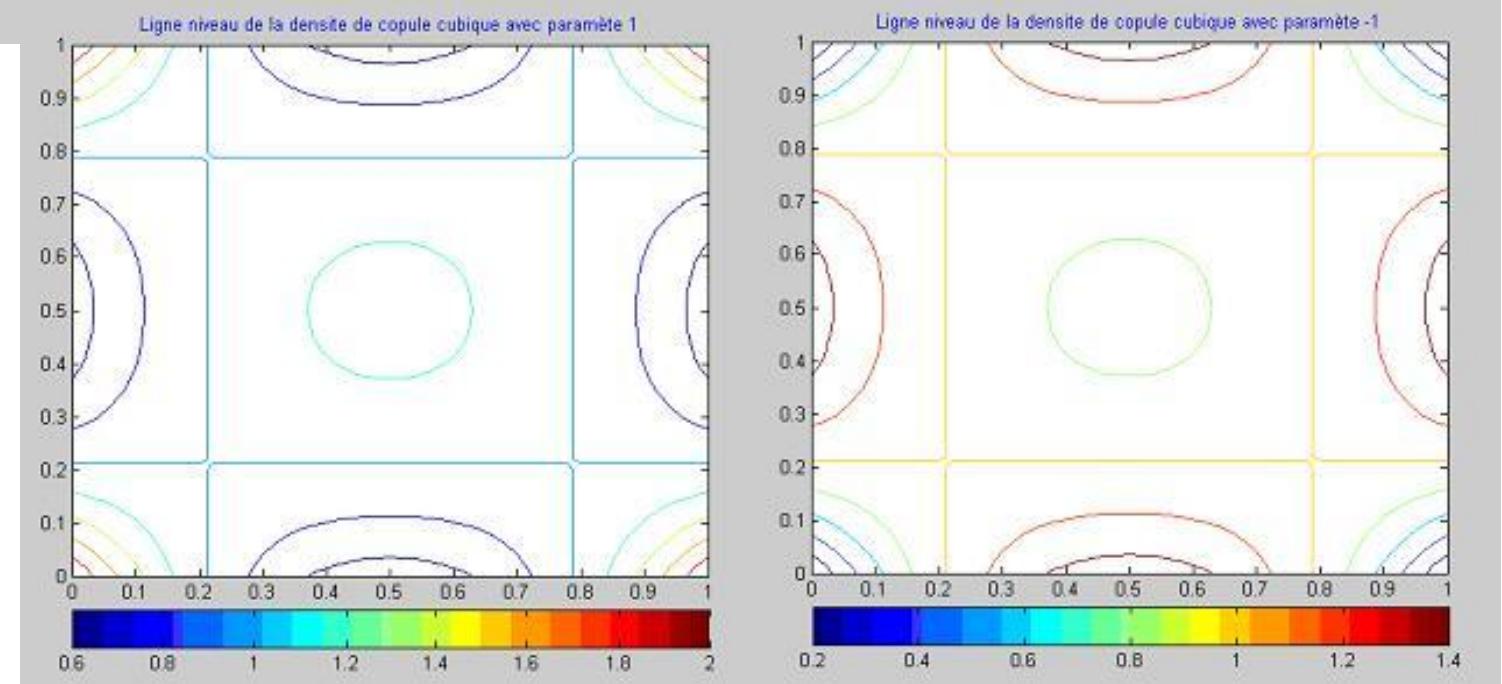
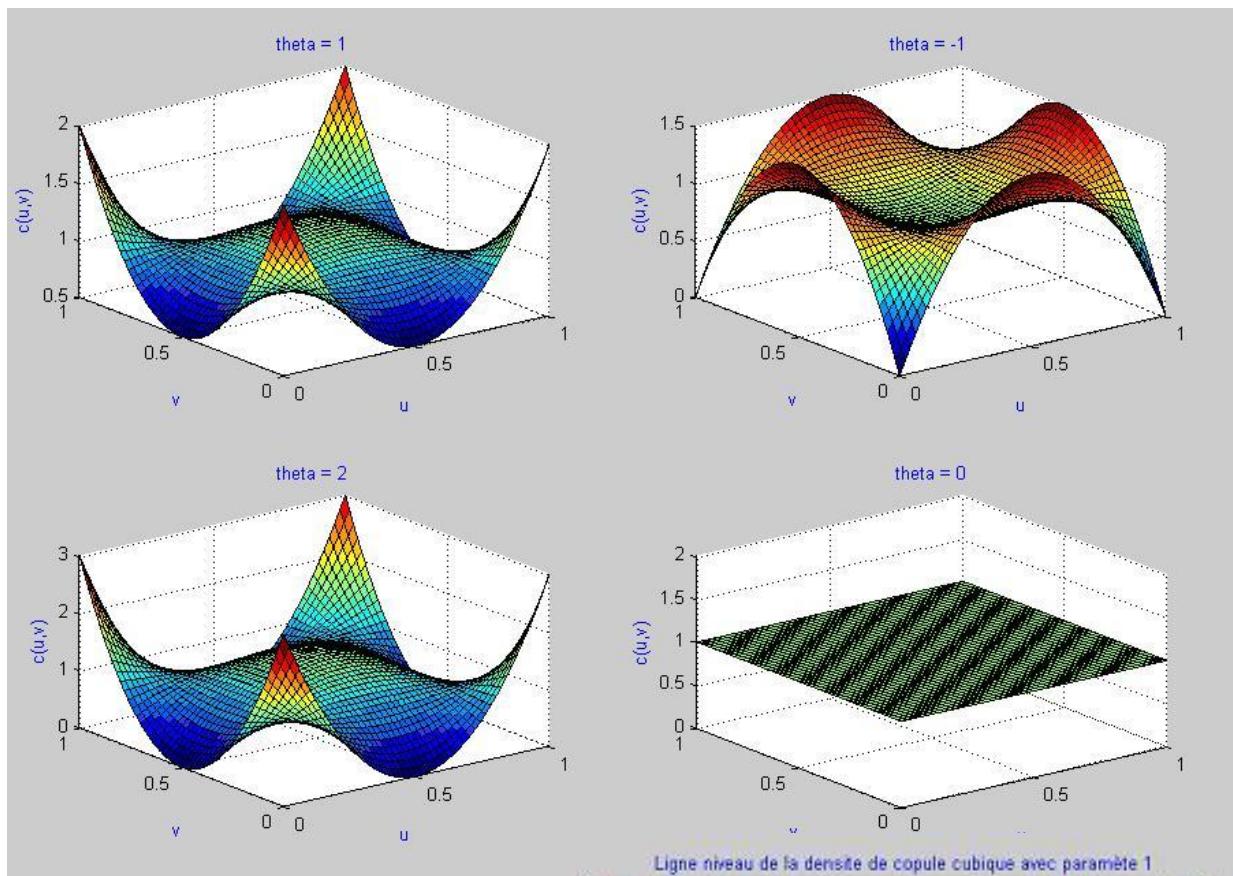


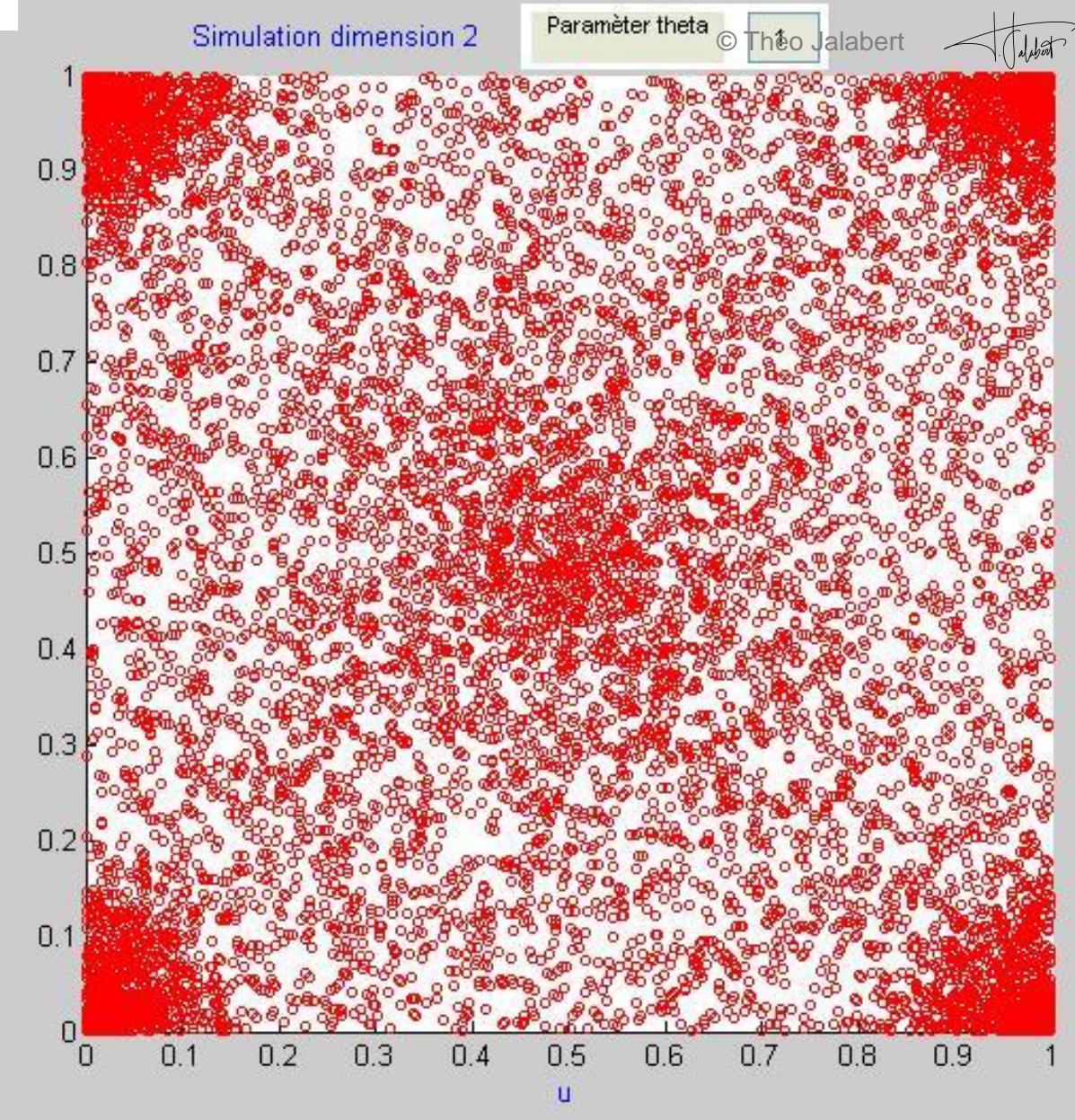
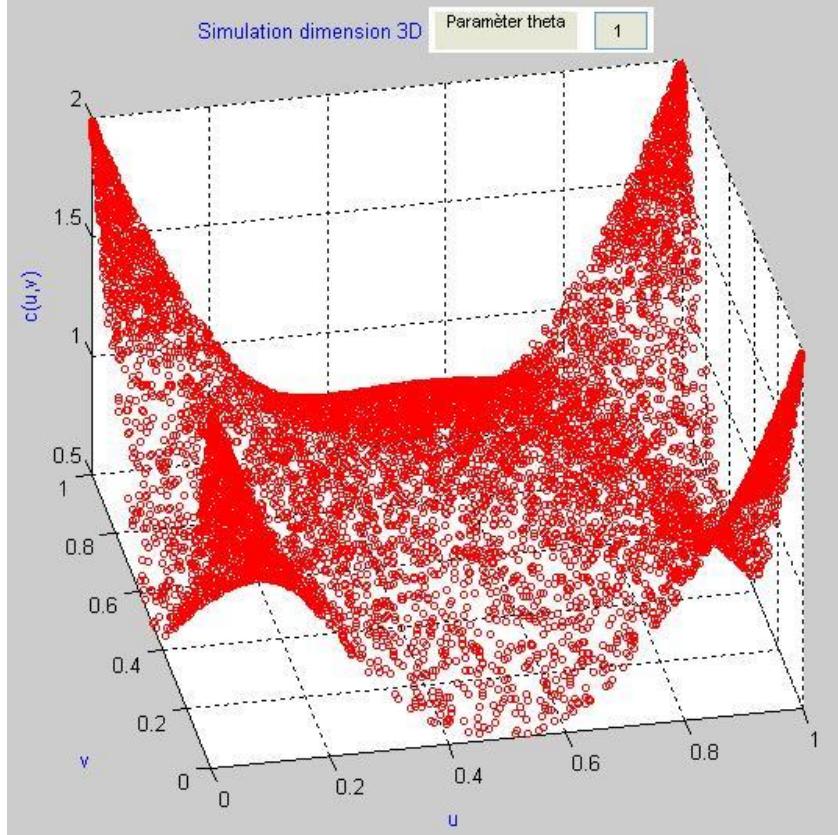
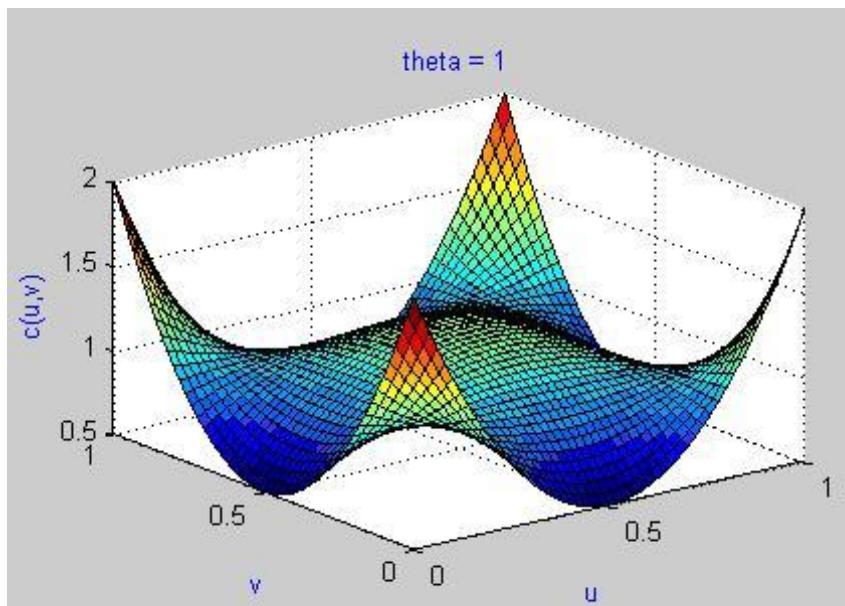
Figure 5: Distribution functions and densities.

# Densité: Copule cubique



# Densité: Copule cubique

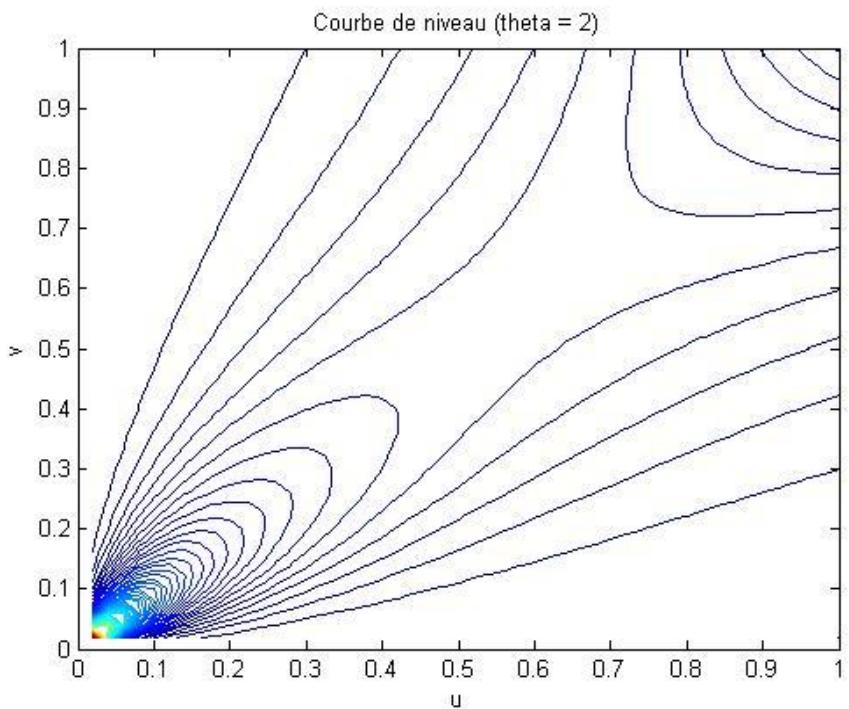
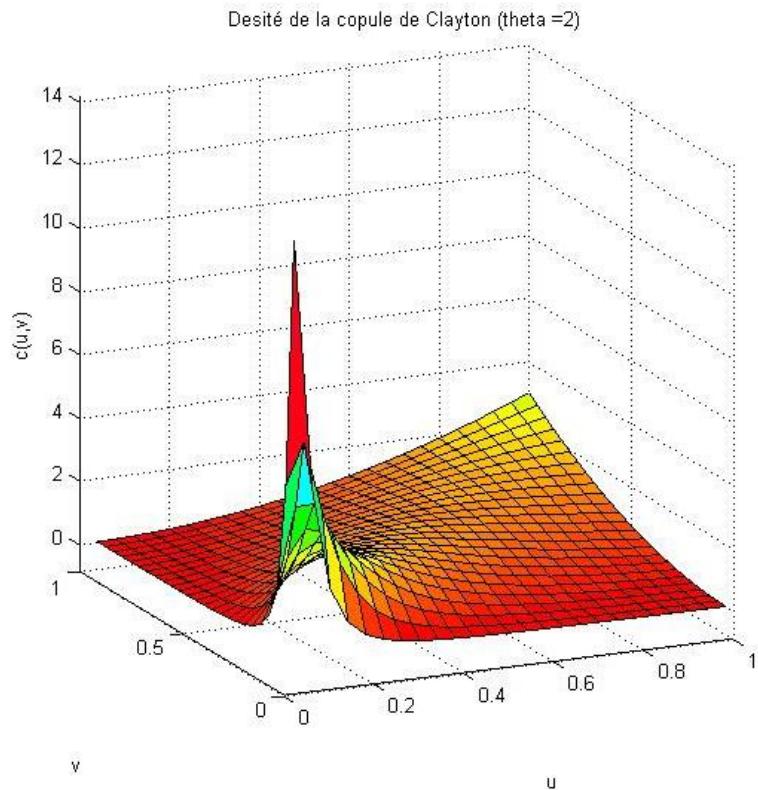




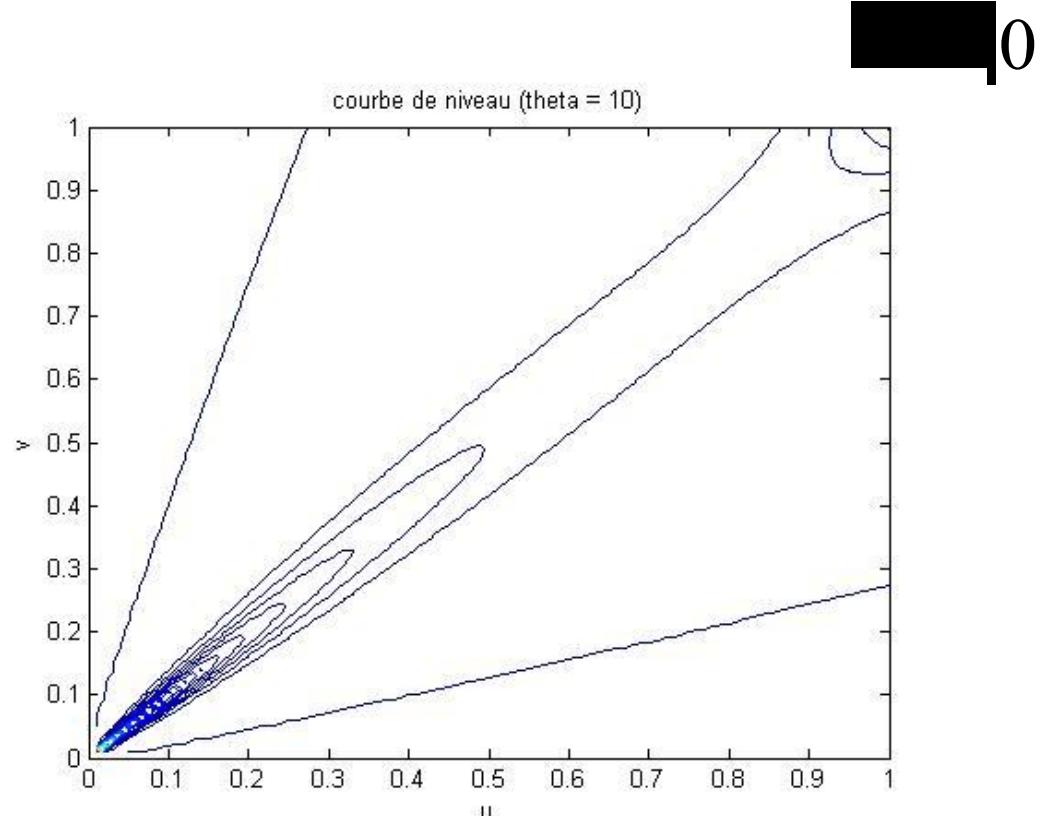
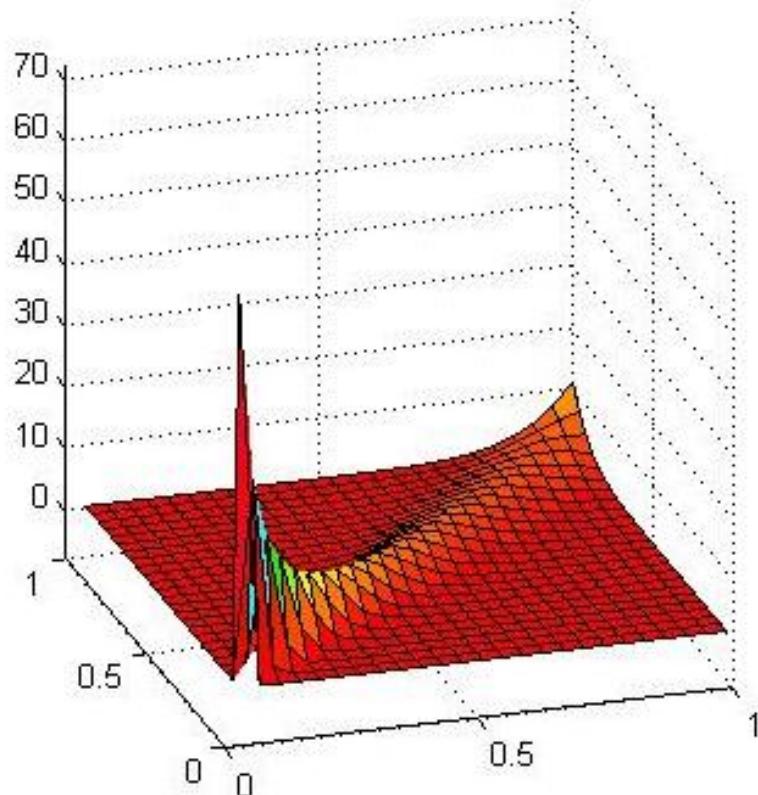
Lien entre densité et nuage de points, copule cubique

# Copule de Clayton: densité et lignes de niveau associées

2



# Copule de Clayton: densité et lignes de niveau associées



## Sklar's theorem

**Theorem 3.** (*Sklar*) Let  $C$  be a copula, and  $F_X$  and  $F_Y$  two marginal distributions, then  $F(x, y) = C(F_X(x), F_Y(y))$  is a bivariate distribution function, with  $F \in \mathcal{F}(F_X, F_Y)$ .

Conversely, if  $F \in \mathcal{F}(F_X, F_Y)$ , there exists  $C$  such that

$F(x, y) = C(F_X(x), F_Y(y))$ . Further, if  $F_X$  and  $F_Y$  are continuous, then  $C$  is unique, and given by

$$C(u, v) = F(F_X^{-1}(u), F_Y^{-1}(v)) \text{ for all } (u, v) \in [0, 1] \times [0, 1]$$

We will then define the copula of  $F$ , or the copula of  $(X, Y)$ .

In that case, the copula of  $(X, Y)$  is the distribution function of  $(F_X(X), F_Y(Y))$ .

**Proposition 4.** If  $(X, Y)$  has copula  $C$ , the copula of  $(g(X), h(Y))$  is also  $C$  for any increasing functions  $g$  and  $h$ .

$$F_{x,y}(x, y) = \frac{1}{1 + e^{-x} + e^{-y}}$$

$\downarrow$

$$\frac{1}{F_x(x)}$$

$$\text{SKLAR: } F_{X,Y}(x,y) = C\left(\underbrace{F_X(x)}_u, \underbrace{F_Y(y)}_v\right) \rightarrow C(u,v) = F_{X,Y}\left(F_X^{-1}(u), F_Y^{-1}(v)\right)$$

Si je connais  $F_X$ ,  $F_Y$  et  $C$ , alors je connais  $F_{X,Y}$

$$u = F_X(x) \Rightarrow x = F_X^{-1}(u)$$

et réciproquement:

Exemple:  $\forall (x,y) \in \mathbb{R}^2$ ,  $F_{X,Y}(x,y) = \frac{1}{1 + e^{-x} + e^{-y}} = P(X \leq x, Y \leq y)$

$$P(X \leq x) = F_X(x) = \lim_{y \rightarrow +\infty} \frac{1}{1 + e^{-x} + e^{-y}} = \frac{1}{1 + e^{-x}} \stackrel{1/F_X(x)}{\Rightarrow} F_X^{-1}(u) = \dots$$

$$F_Y(y) = \frac{1}{1 + e^{-y}} \Rightarrow F_Y^{-1}(v) = \dots$$

$$C(u,v) = \leftarrow$$

Méthode + rapide  
(- universelle)

$$F_{X,Y}(x,y) = \frac{1}{\frac{1}{F_X(x)} + \frac{1}{F_Y(y)} - 1} \Rightarrow C(u,v) = \frac{1}{\frac{1}{u} + \frac{1}{v} - 1}$$

⚠ Le modèle conjoint doit avoir une certaine cohérence

On doit beaucoup réfléchir avant de décider de choisir la copule indépendamment des marginales

# Copulas, an introduction (in dimension $d \geq 2$ )

**Definition 5.** A copula  $C$  is a joint distribution function on  $[0, 1]^d$ , with uniform margins on  $[0, 1]$ .

**Theorem 6. (Sklar)** Let  $C$  be a copula, and  $F_1, \dots, F_d$  be  $d$  marginal distributions, then  $F(x) = C(F_1(x_1), \dots, F_d(x_d))$  is a distribution function, with  $F \in \mathcal{F}(F_1, \dots, F_d)$ .

Conversely, if  $F \in \mathcal{F}(F_1, \dots, F_d)$ , there exists  $C$  such that  $F(x) = C(F_1(x_1), \dots, F_d(x_d))$ . Further, if the  $F_i$ 's are continuous, then  $C$  is unique, and given by

$$C(\mathbf{u}) = F(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)) \text{ for all } (u_i) \in [0, 1]^d$$

We will then define the copula of  $F$ , or the copula of  $\mathbf{X}$ .

In that case, the copula of  $(\mathbf{X} = (X_1, \dots, X_d))$  is the distribution function of  $\mathbf{U} = (F_1(X_1), \dots, F_d(X_d))$ .

Again, if  $C$  is differentiable, one can define its density,

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d}.$$

## Copulas in high dimension, a difficult problem

It is usually difficult to represent dependence in dimension  $d > 2$ , and it is usually studied by pairs.

In dimension  $d = 2$ , one can define the following Fréchet class  $\mathcal{F}(F_X, F_Y, F_Z)$  defined by its marginal distributions. But it can also be interested to study  $\mathcal{F}(F_{XY}, F_{XZ}, F_{YZ})$  defined by its paired distributions.

One of the problems that arises is the **compatibility of marginals**: one has to verify that

$$C_{XY}(x, y) = \int C_{X|Z}(x|z)C_{Y|Z}(y|z)dz,$$

for instance.

# Simulated plots

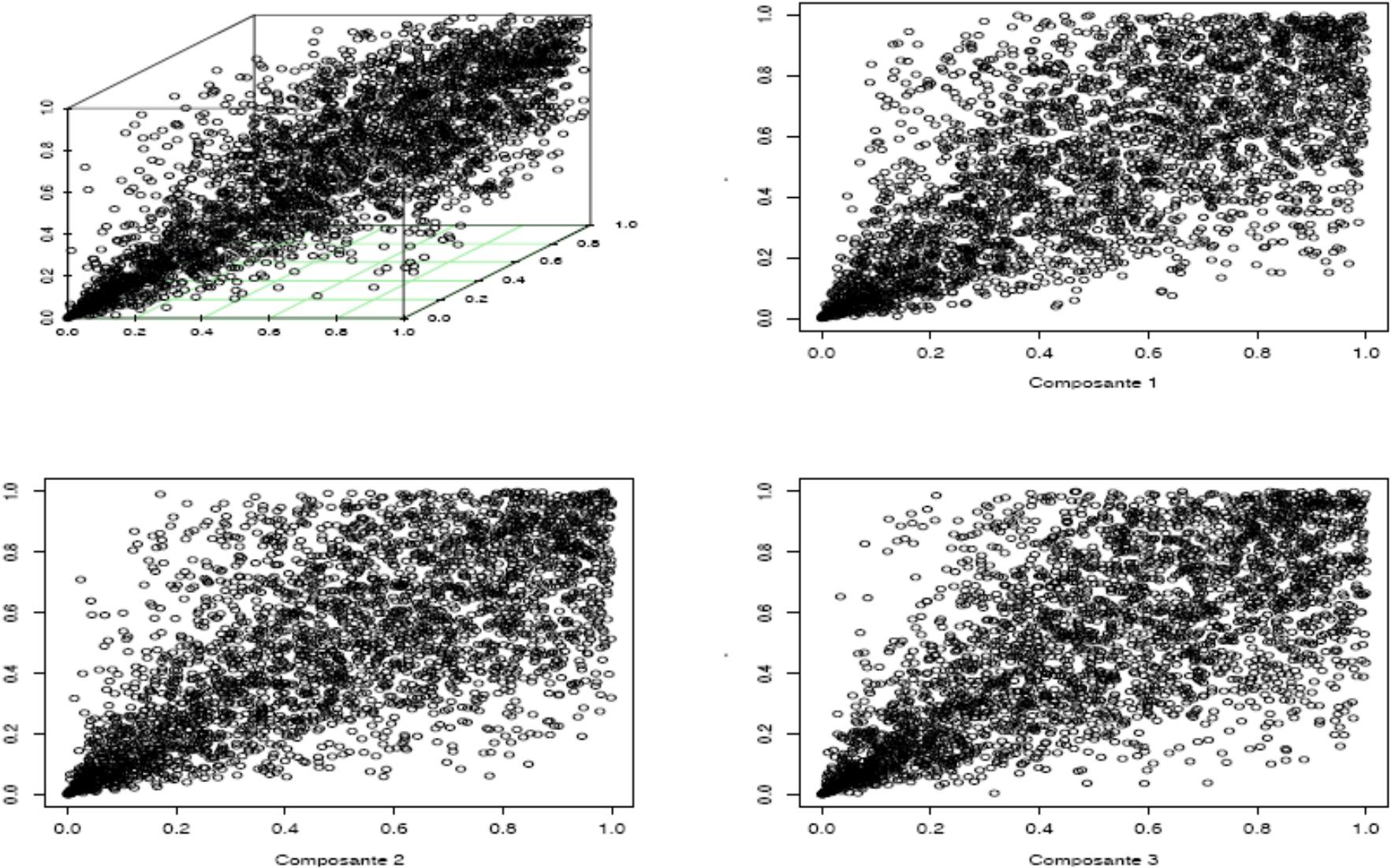


Figure 7: Scatterplot in dimension 3 including projections.

Penser à parler des mesures de corrélation générales:

- coeff. de corrélation linéaire
- tau de Kendall et rho de Spearman
- concepts de dépendance positive (cf autres slides + Post-reading 7)

## Tail concentration functions

VENTER (2002) suggest to use several Tail Concentration Functions

**Definition 24.** *For lower tails, define*

$$L(z) = \mathbb{P}(U < z, V < z)/z = C(z, z)/z = \Pr(U < z | V < z) = \Pr(V < z | U < z),$$

and for upper tails,

$$R(z) = \mathbb{P}(U > z, V > z)/(1 - z) = \Pr(U > z | V > z).$$

Joe (1990) uses the term upper tail dependence parameter for  $R = R(1) = \lim_{z \rightarrow 1} R(z)$ , and lower tail dependence parameter for  $L = L(0) = \lim_{z \rightarrow 0} L(z)$ .

$\tau$  de Kendall de  $X, Y$  — proba de concordance

$$P((\underset{\text{V}}{X} - \tilde{X})(\underset{\text{V}}{Y} - \tilde{Y}) > 0) - P((\underset{\text{V}}{X} - \tilde{X})(\underset{\text{V}}{Y} - \tilde{Y}) < 0)$$

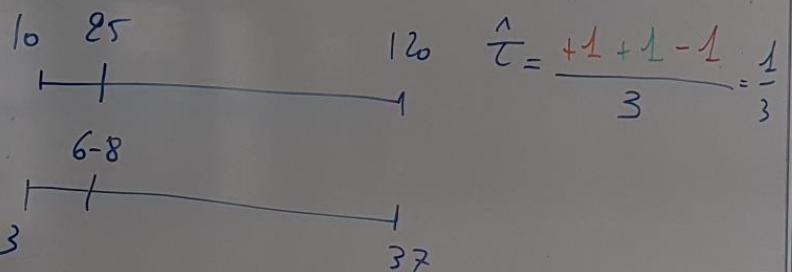
proba de discordance

où  $(\tilde{X}, \tilde{Y})$  copie  $L$  de  $(X, Y)$

$$\hat{\tau} = \frac{\text{nb de paires concordantes} - \text{nb de paires discordantes}}{\text{nb de paires}}$$

FL	LO
2020	$\frac{1}{10} \text{ m\$}$ $X$
2021	$\frac{1}{120} \text{ m\$}$ $X$
2022	$\frac{25}{60} \text{ m\$}$ $2,8 \text{ m\$}$ $24 \text{ m\$}$ $60 \text{ m\$}$

Y  
3 m \\$  
Y  
~37 m \\$  
V  
V  
A  
B  
C  
6  
3  
0

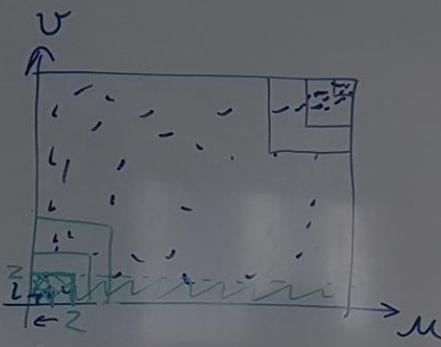


$$\tau = 4 \iint_{[0,1]^2} C(u,v) dC(u,v) - 1$$

$c(u,v) du dv$

si  $X$  et  $Y$  représentent des charges, on redoute " $X$  et  $Y$  très grands".

$\iff$   $U$  et  $V$  très proches de 1



si  $X$  et  $Y$  représentent des valeurs éco, des rendements, des dates de défaut, on redoute

$X$  et  $Y$  très petits  $\iff$   $U$  et  $V$  très proches de 0

$$\lim_{z \rightarrow 0} P(U < z | V < z) = L \text{ ou } \lambda_L$$

$L(z)$   
↑  
left

$\nearrow$   
lower

coeff de dépendance  
FORTE

des extrêmes  
à gauche      à droite

$$\lim_{z \uparrow 1} P(U > z | V > z) = R \text{ ou } \lambda_R$$

$R(z)$   
↑  
right

↑  
upper

La copule Gaussienne ne présente PAS de dépendance FORTE des extrêmes (paramètre  $\neq 1$ ) mais elle présente de la dép. faible des extrêmes pour paramètre  $\neq 0$

On dit que une copule ( $\copule(X, Y)$ ) présente de la dépendance FORTE de extrêmes à gauche si  $|L| > 0$ .  
à droite       $|R| > 0$ .

copules elliptiques  
lignes de niveau  
densité jointe des vecteurs  
gaussiens / Student

- Gaussienne

log rendements Renault

$$X_1 = \rho X_0 + \sqrt{1-\rho^2} X_1^\perp$$

propre à Renault

$$X_2 = \rho X_0 + \sqrt{1-\rho^2} X_2^\perp$$

propre à Ford

Ford      secteur automobile

- Student

$$(F_1^{-1}(u), F_2^{-1}(v))$$

$$X_0, X_1^\perp, X_2^\perp \sim N(0,1)$$

et mutuellement  $\perp$

$$(U, V) = (\Phi(X_1), \Phi(X_2))$$

pour changer /  
adapter les marginales

$$\text{Copule Gausienne}$$

$$\text{Copule Gausienne}$$

copules Archimédiennes

CLAYTON

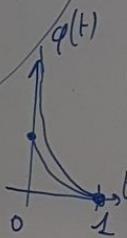
dép. forte à gauche  
mais pas à droite  
(U, V)

(SCOR)

copule de survie  
de Clayton  
(1-U, 1-V)

Gumbel

dép forte à droite  
mais pas à gauche



21 familles "classiques"

livre NELSEN

"An introduction to copulas"

copules elliptiques

lignes de niveau  
densité jointe des vecteurs  
gaussiens / Student-

- Gaussienne

Log rendements Renault

$$X_1 = \rho X_0 + \sqrt{1-\rho^2} X_1^\perp$$

propre à Renault

$$X_2 = \rho X_0 + \sqrt{1-\rho^2} X_2^\perp$$

propre à Ford

Ford      seconde automobile

- Student

$$(F_1^{-1}(u), F_2^{-1}(v))$$

$$(U, V) = (\Phi(X_1), \Phi(X_2))$$

pour changer/  
adapter les marginales

$Unif(0,1)$

$Unif(0,1)$

copule Gaussienne

© Théo Malabert  
copules Archimédiennes (HIERARCHIES)

CLAYTON

dép. forte à gauche  
mais pas à droite

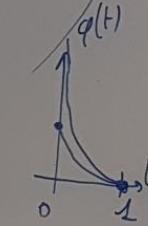
(SCOR)

copule de survie  
de Clayton

Poneto

$(U, V)$  forte ouragan

$$\begin{cases} X_1 = \frac{1}{Z} X_1^\perp & \text{propre à FC} \\ X_2 = \frac{1}{Z} X_2^\perp & \text{propre à LO} \end{cases}$$



$$X_1^\perp, X_2^\perp \sim Exp(1)$$

$X_1^\perp, X_2^\perp, Z$  mutuellement  $\perp$

$$Q = L_Z^{-1}$$

$$P(X_1 > x_1, X_2 > x_2) = \int e^{-z(x_1+x_2)} dF(z)$$

$$\frac{1}{Z} X_1^\perp, X_1^\perp > z x_1$$

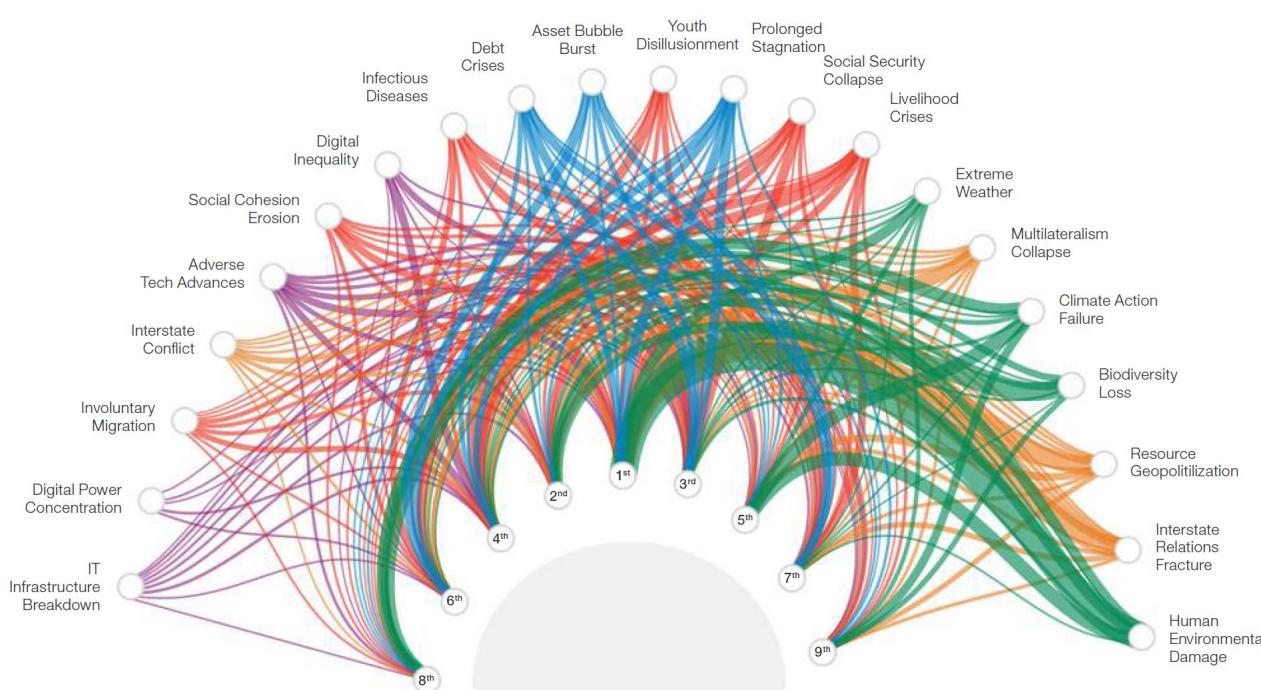
$$= L_Z^{(x_1+x_2)}$$

$$P(X_1 > x_1) = L_Z^{(x_1)}$$

$$x_1 = L_Z^{-1}(P(X_1 > x_1))$$

Les facteurs communs X0 et 1/Z du slide précédent jouent le rôle de source de corrélation comme les éléments sur le cercle extérieur ici

8 <sup>th</sup>	6 <sup>th</sup>	4 <sup>th</sup>	2 <sup>nd</sup>	1 <sup>st</sup>	3 <sup>rd</sup>	5 <sup>th</sup>	7 <sup>th</sup>	9 <sup>th</sup>
Extreme Weather	Debt Crises	Social Cohesion Erosion	Infectious Diseases	Climate Action Failure	Livelihood Crises	Biodiversity Loss	Prolonged Stagnation	Human Environmental Damage



$X_1$  rdt actions

$X_2$  coût tremblement de terre à Monaco

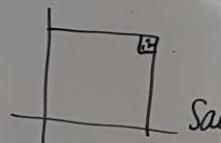
$$(X_1, -X_2) \quad \boxed{?}$$

ou  $(-X_1, X_2)$



$X_3$ : tx d'intérêt (épargne)

peut être désastreux si hausse rapide  
ou baisse prolongée



super robuste

INTERDIT ENERN  
sauf pour comparer  
des modèles

$$\bar{\tau} = 17\%$$

$$\mathcal{R}_{\min} = 99\%$$

pas robuste  
très sensible donnée absente

copule  $C_1$   $\begin{cases} 1 \\ 0 \end{cases}$   
engendant les  
5%, 1%, 0,5% pris  
jours

$$\alpha C_1 + (1-\alpha)C_2$$

copule

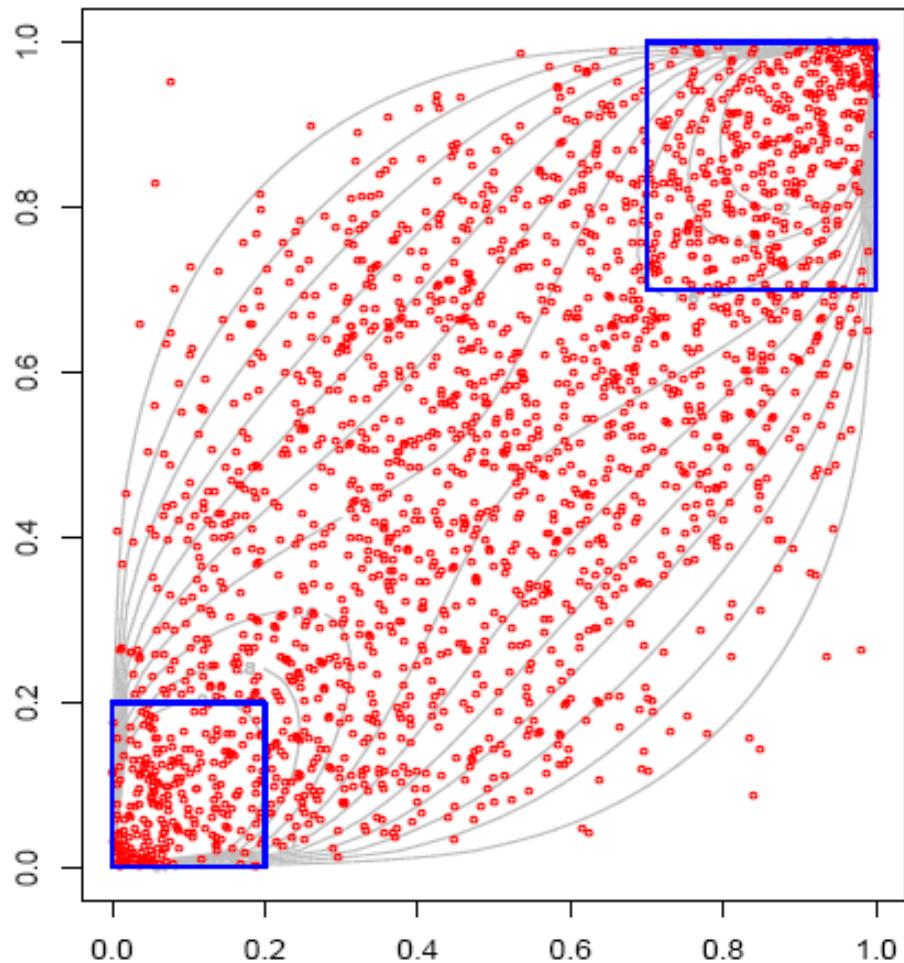
$$\alpha \in [0,1]$$

$$\approx 4500 \text{ jours}$$

jan 1999 - 2010  
ou 2011

# Gaussian copula

Gaussian copula



L and R concentration functions

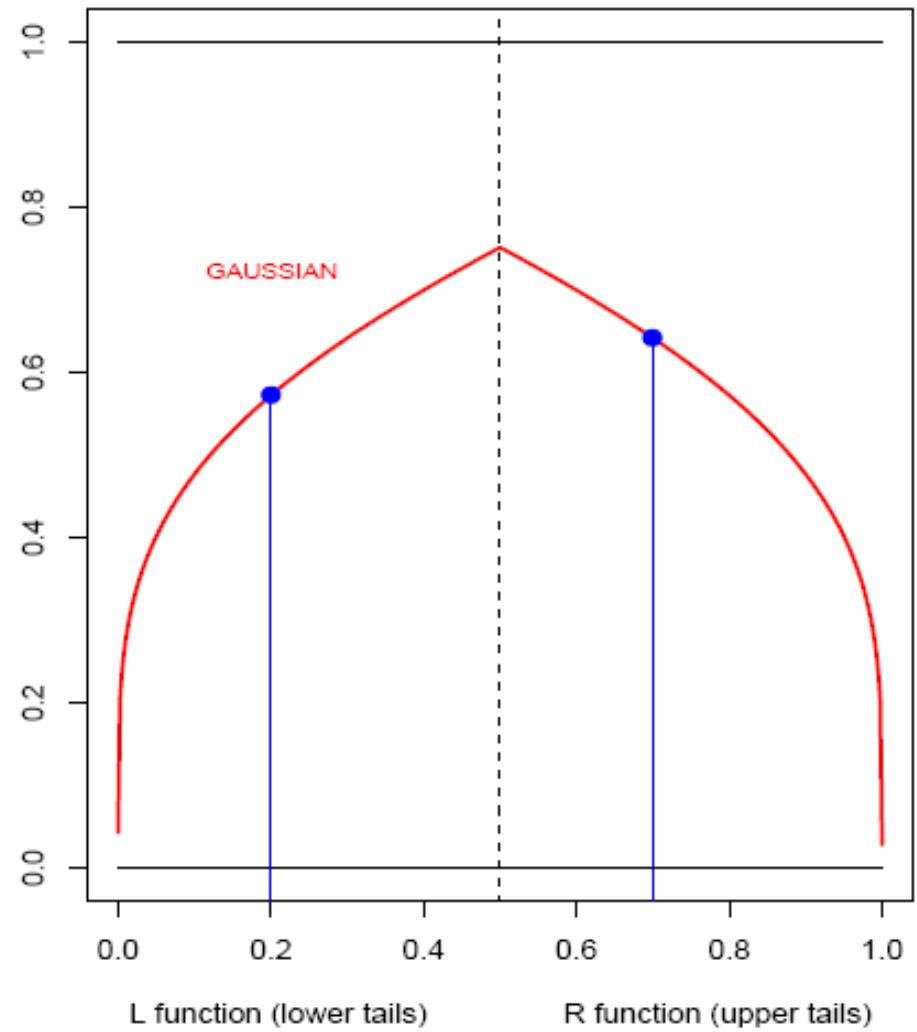


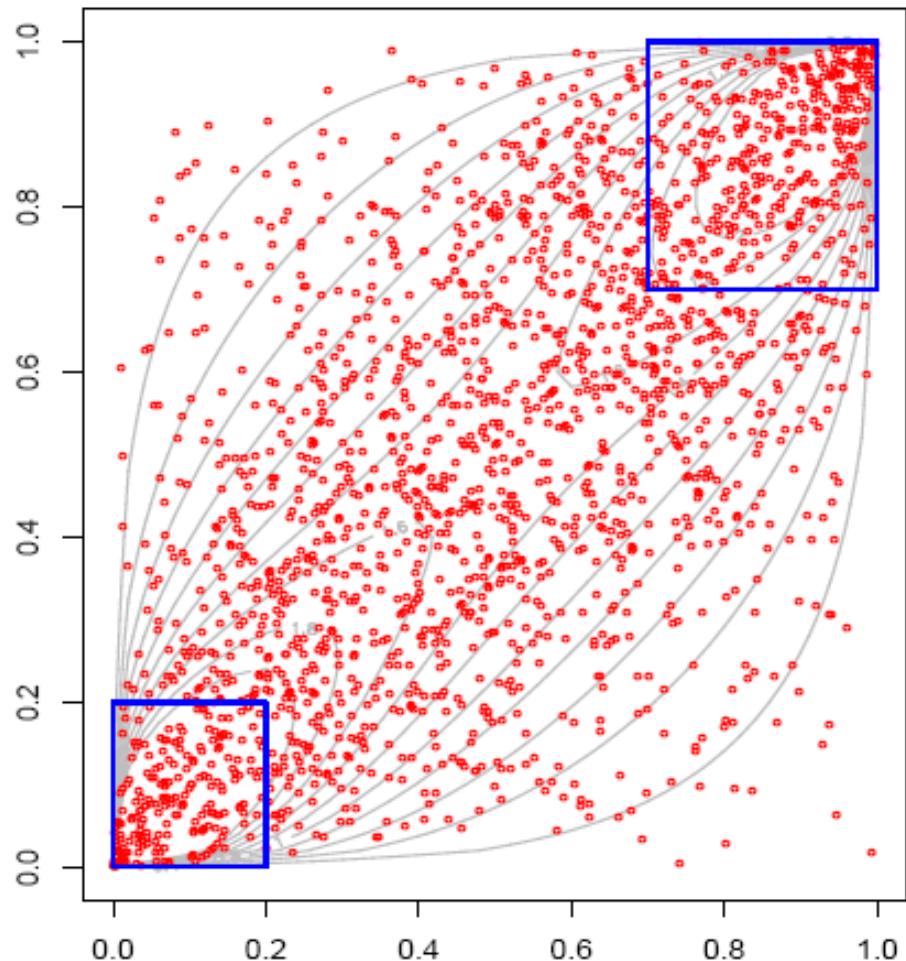
Figure 32: L and R cumulative curves.

# Student copula

© Théo Jalabert



Student t copula



L and R concentration functions

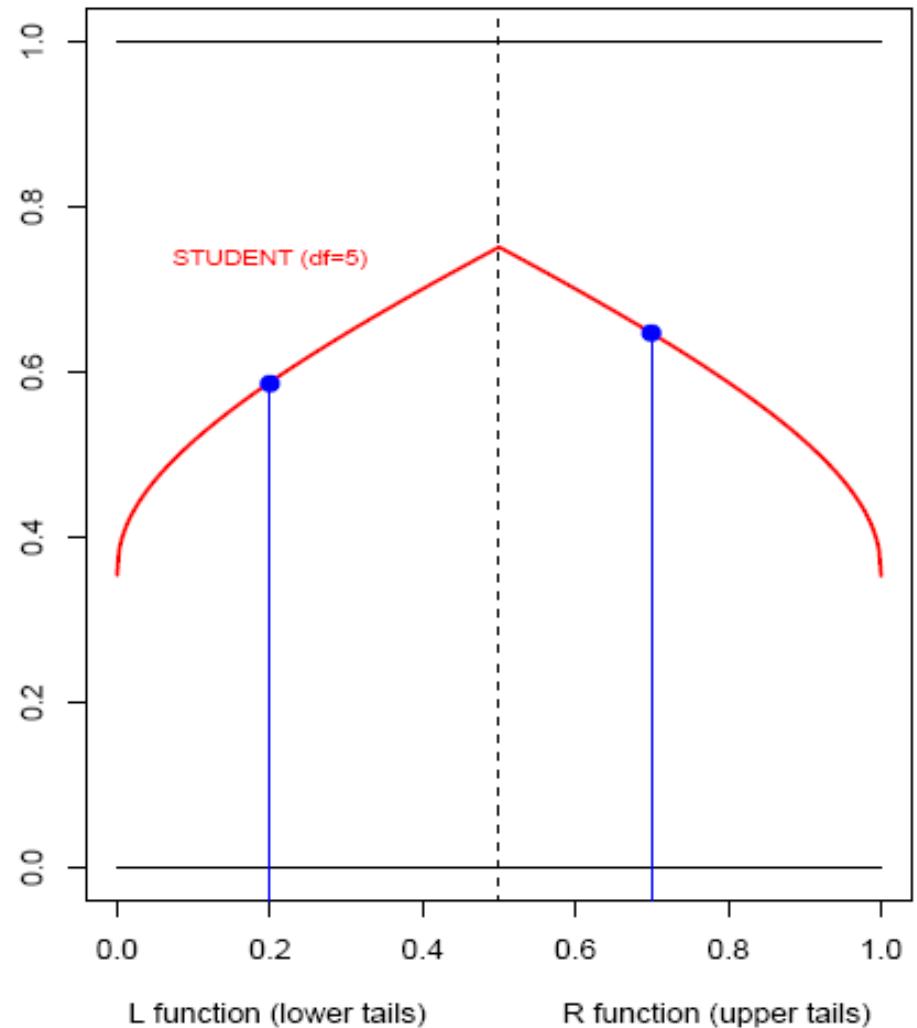
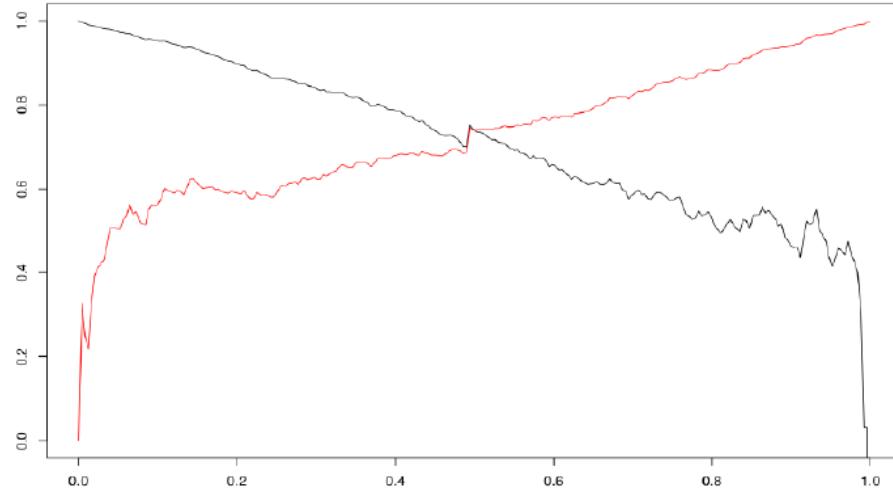
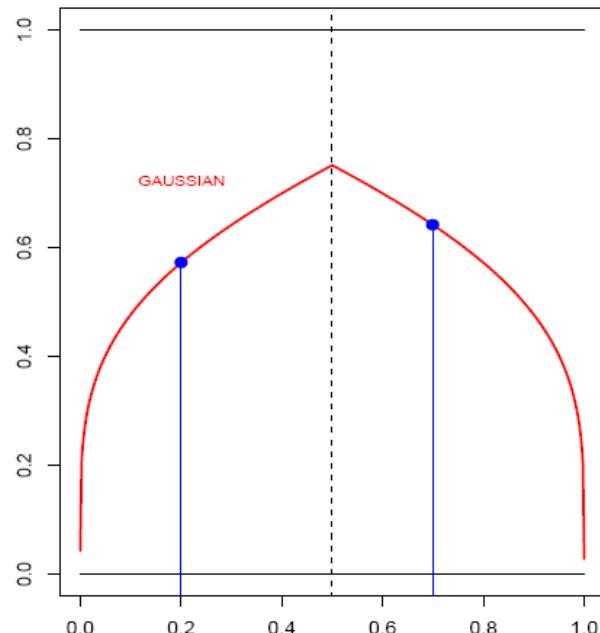


Figure 35: L and R cumulative curves.

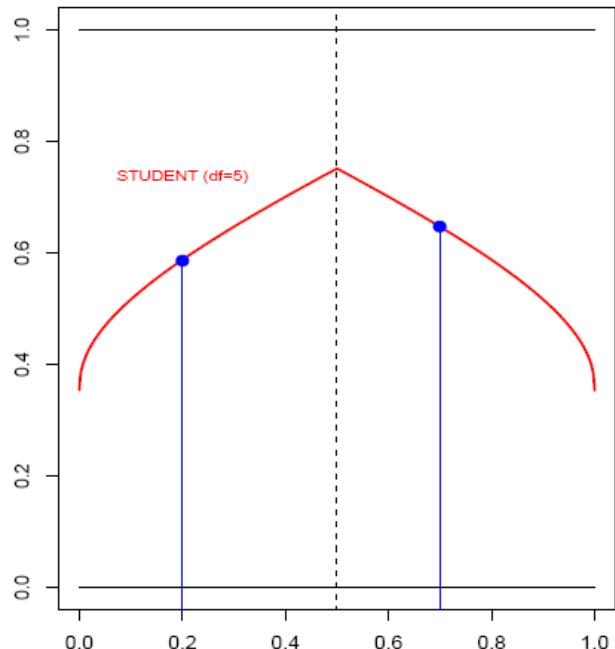
# Wired paper and choice between Gaussian and Student copulas



Empirical copula,  
L and R functions,  
AXA & Vivendi  
stock returns



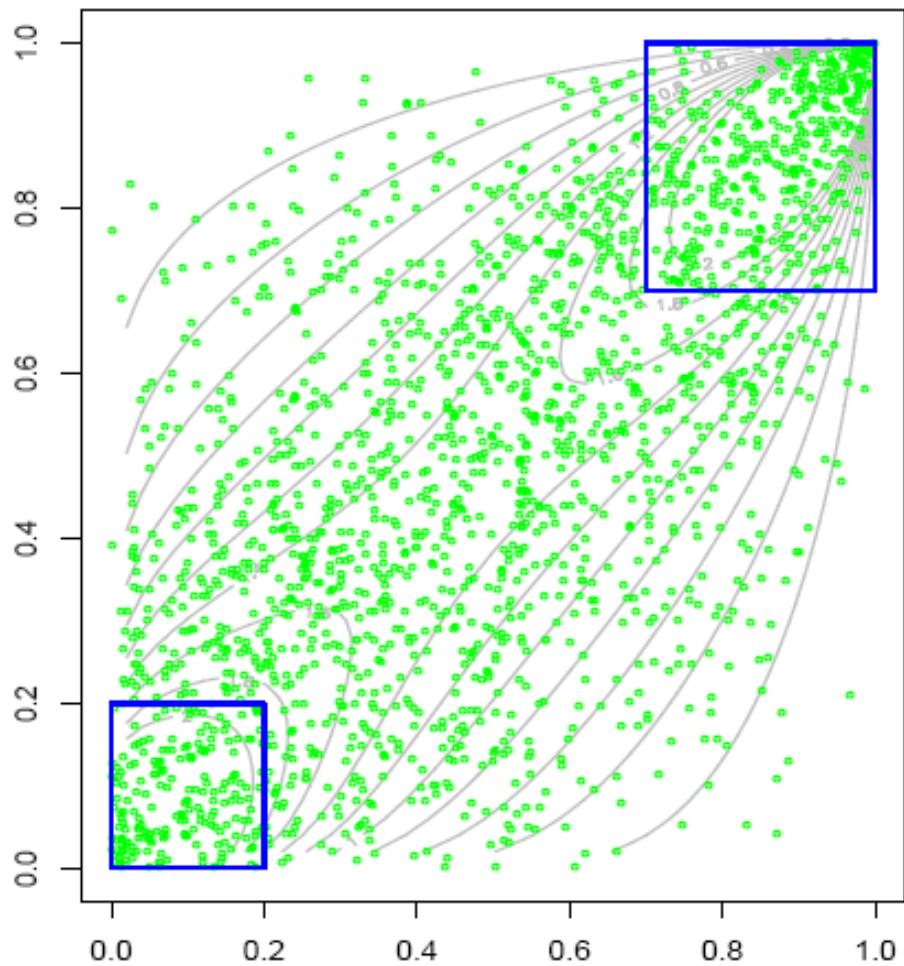
or



?

# Gumbel copula

Gumbel copula



L and R concentration functions

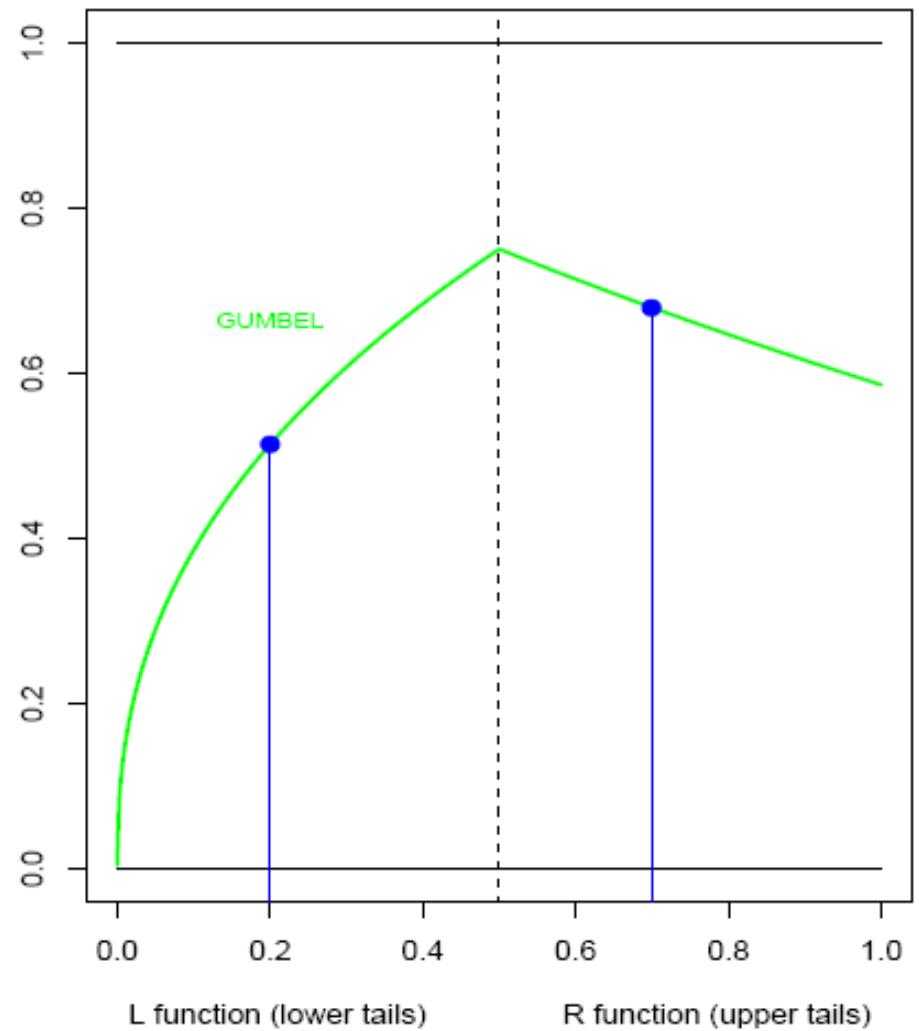
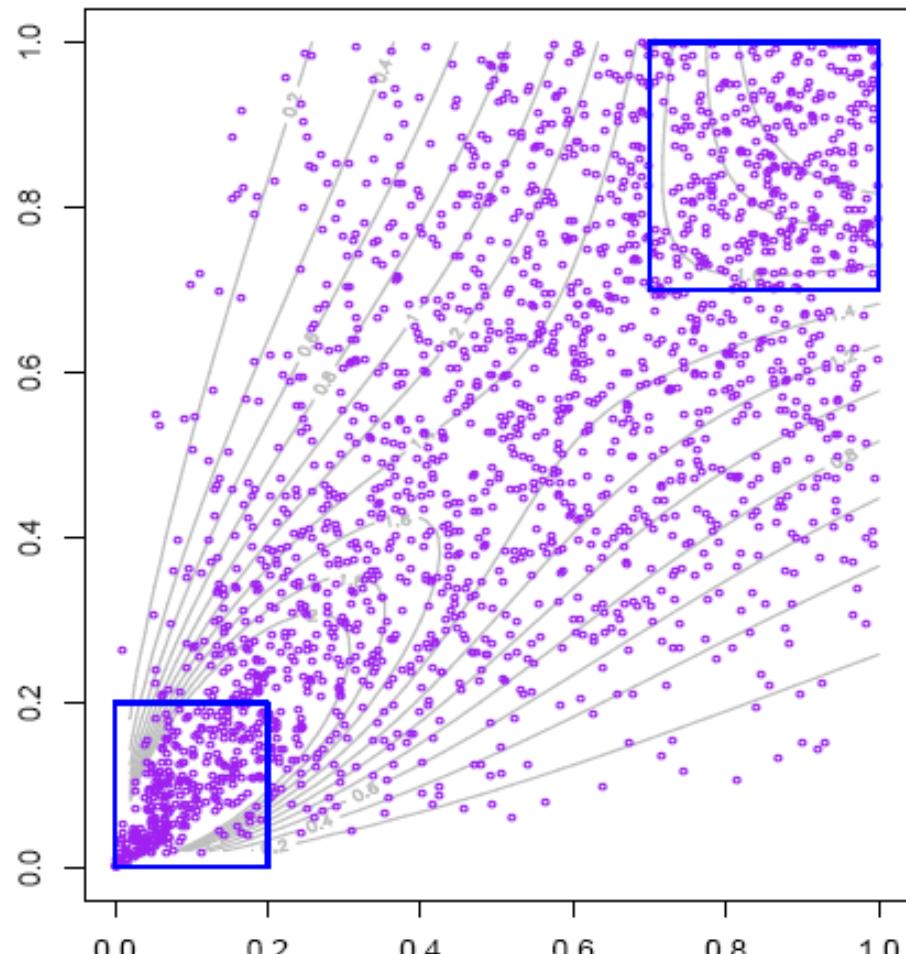


Figure 33: L and R cumulative curves.

# Clayton copula

Clayton copula



L and R concentration functions

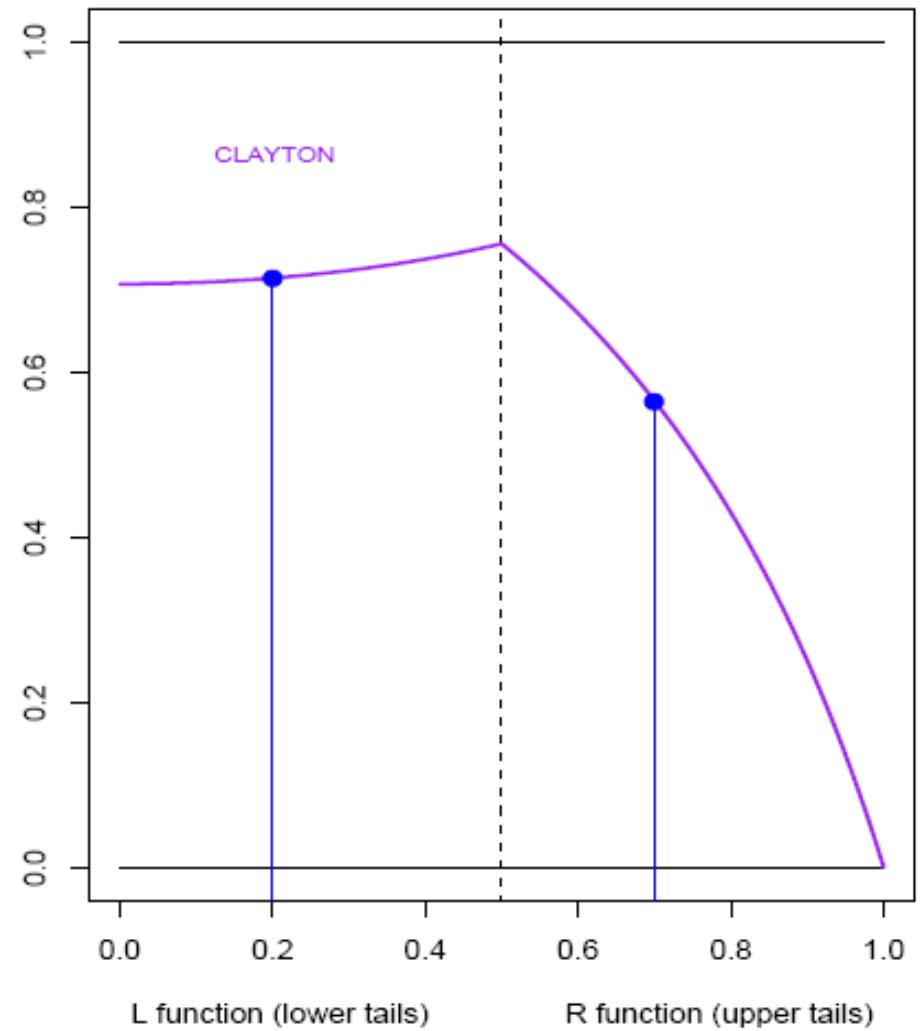
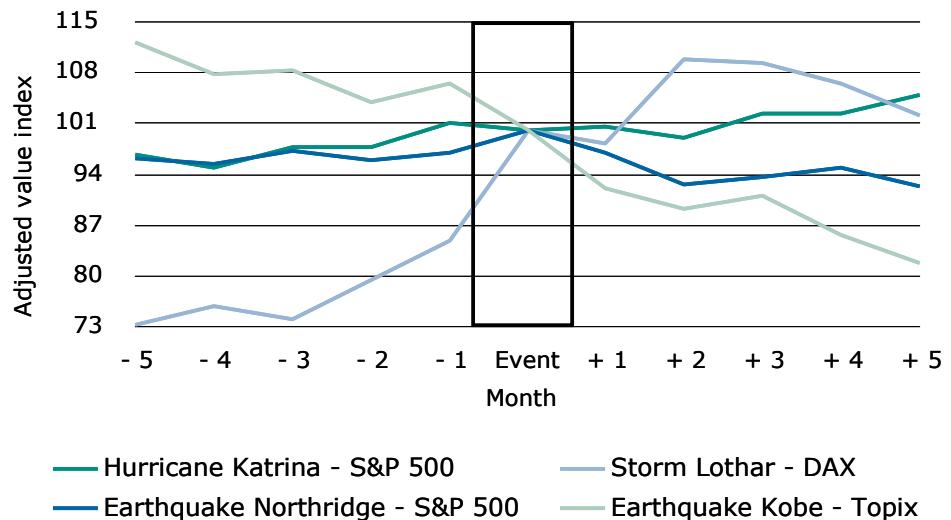


Figure 34: L and R cumulative curves.

# Réponse à question: les risques cat sont-ils relativement indépendants des risques financiers?



(Source: Horizon 21 2008)

But potentially some dependency in the extremes for some risks...

# Playing with Gaussian and Student copulas

- Use Excel Spreasheet
- Change correlation parameter and see modification of scatterplot of simulated couples.
- Compare Gaussian and Student copulas outcomes.
- (to be continued after the break: risk aggregation)

# Outline for today

- Correlation modeling
  - Copulas
  - Endogenous risk and correlation crises
- Risk aggregation
  - in Solvency II standard formula: some pitfalls
  - in internal models: example with Excel spreadsheet

# Endogenous risk and correlation crises

How can independent risks become suddenly strongly correlated?

Endogenous uncertainty:

- Uncertainty is generated/modified by response of individual entities to events
- Feedback loop: outcomes → forecasts → decisions → outcomes → revised forecasts → revised decisions → ... (Millennium Bridge)
- Statistical relationships are endogenous to the model, and may undergo structural shifts (Goodhart's Law: *Any observed statistical regularity will tend to collapse once pressure is placed upon it for control purposes*)
- Relevant when individual entities are similar in terms of forecasts and likely reactions to events
- Relevant when outcomes are sensitive to concerted actions

# Endogenous risk and correlation crises

How can independent risks become suddenly strongly correlated?

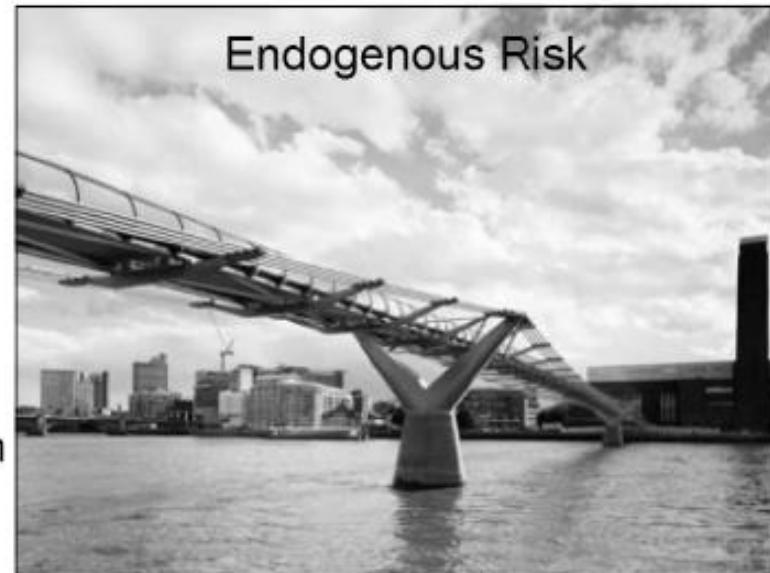
## Endogenous Risk

- Risk from shocks generated and amplified *within* the system

in contrast to...

## Exogenous Risk

- Risk from shocks from *outside* the system



## Millennium Bridge

- New design
- Tested with extensive simulations
- All angles covered
- No endogenous shocks



## Endogenous Risk

## What Endogeneity?

- Pedestrians had some problems
- Bridge closed

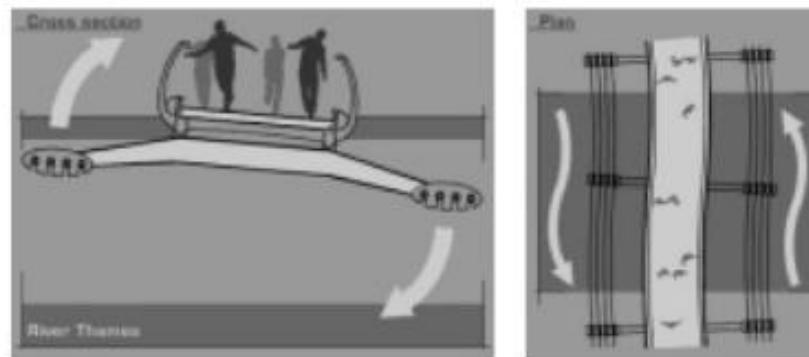


# Endogenous risk and correlation crises

How can independent risks become suddenly strongly correlated?

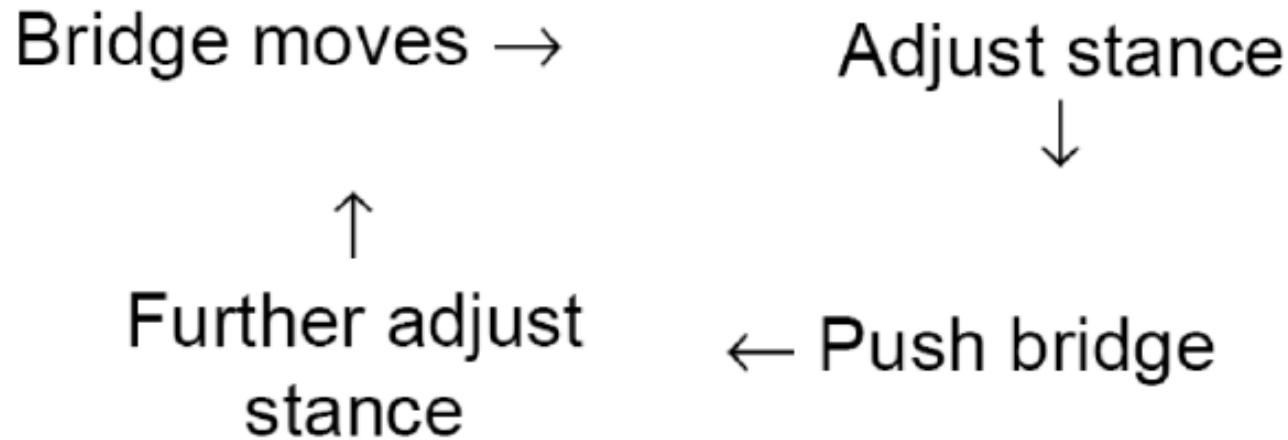
## Diagnosis

- Trouble at 1 hertz (one complete cycle per second)
- Walking pace: two steps per second (2 hertz)
- But sideways motion every two steps (1 hertz)

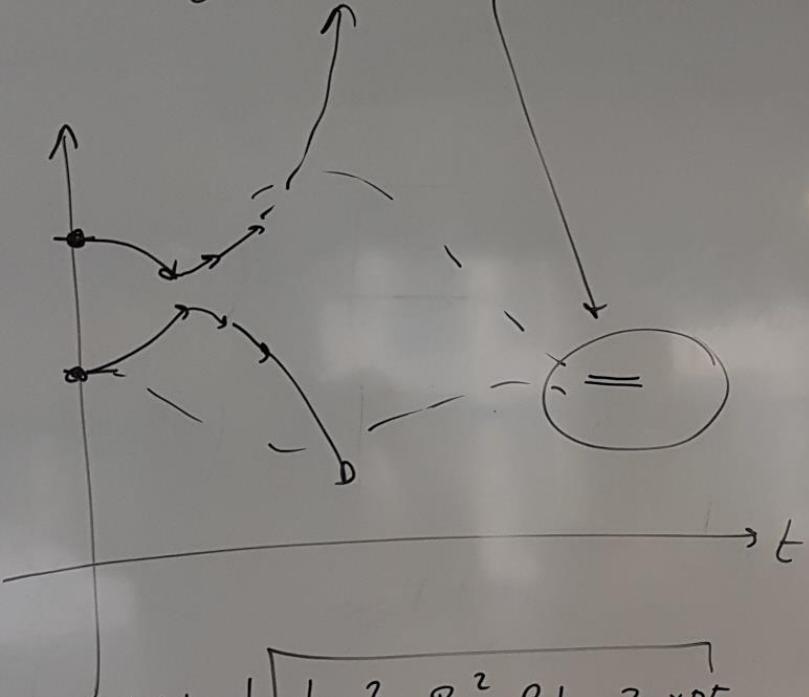


# Endogenous risk and correlation crises

How can independent risks become suddenly strongly correlated?

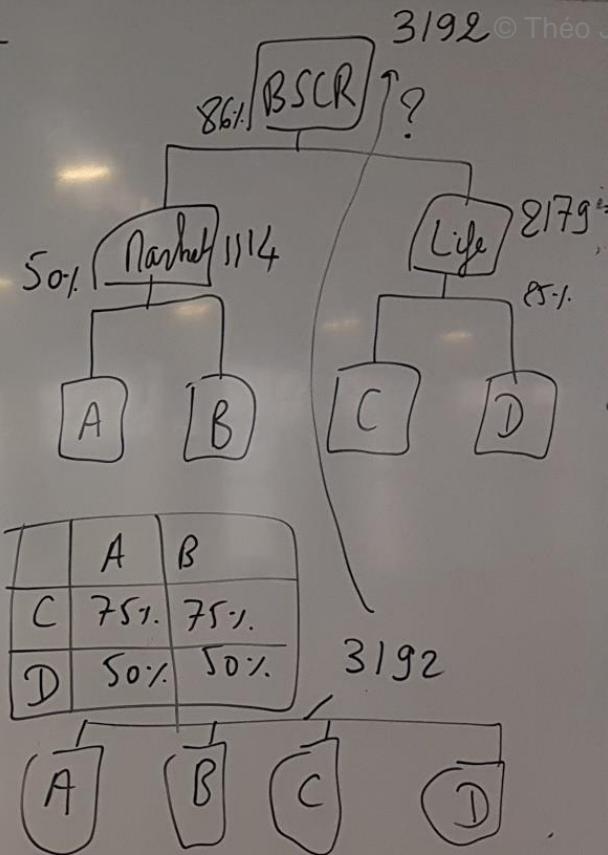


# CONVERGENCE TRADING



$$\text{III}_h = \sqrt{1000^2 + 200^2 + 2 \times 1000 \times 200 \times 0.5}$$

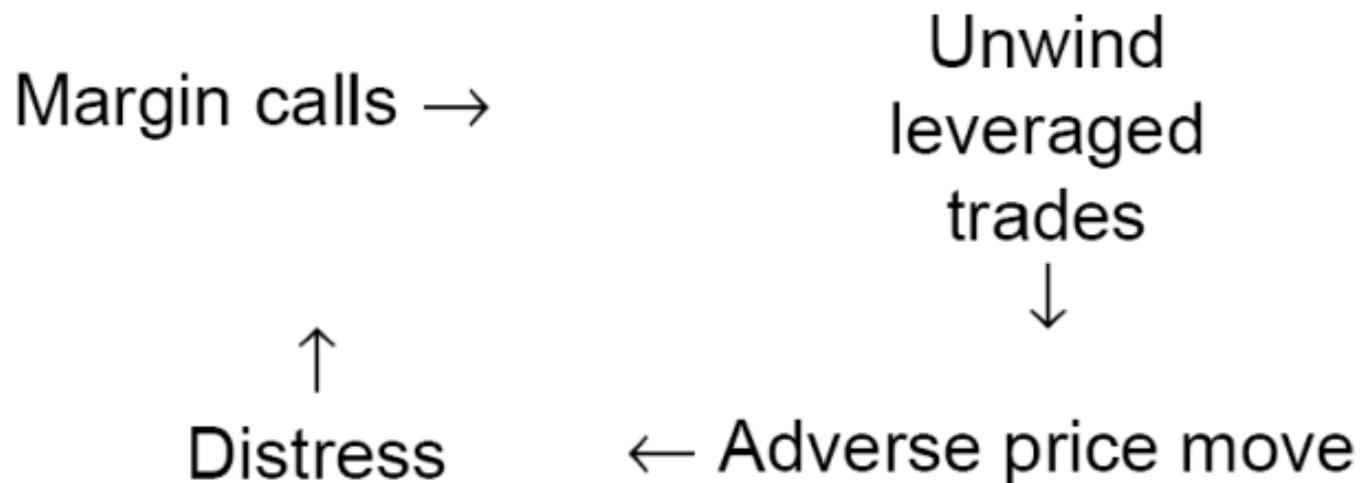
$$(3192) = \text{III}_h^2 + 2179^2 + 2 \times \text{III}_h \times 2179 \times ?$$



# Endogenous risk and correlation crises

Analogy with LTCM

## Feedback Revisited



This analogy is drawn from Danielsson and Shin (2003).

# Endogenous risk and correlation crises

## Impact of dependence between claim amounts or LoBs

- In practice, claim amounts are influenced by common factors.
- This leads to stochastic correlation models.
- This correlation may change during time, due to endogenous risk or external shocks (see [Fisher et al., 2008]), parameter uncertainty (see [Meyers, 1999]), ...
- What happens
  - ▶ if claim amounts are linked with Gaussian copulas?
  - ▶ if some claim amounts of a certain kind are comonotonic?
  - ▶ if dependence between claim amounts (or Lobs) changes over time?

[Biard, R., Lefèvre, C., and Loisel, S. \(2008\).](#)

Impact of correlation crises in risk theory: Asymptotics of finite-time ruin probabilities for heavy-tailed claim amounts when some independence and stationarity assumptions are relaxed.

*Insurance: Mathematics and Economics*, 43(3):412 – 421.

[Fisher, W., Loisel, S., and Wang, S. \(2008\).](#)

On some key research issues in Enterprise Risk Management related to economic capital and diversification effect at group level.

*Bulletin Français d'Actuariat*, 15(9):32–37.

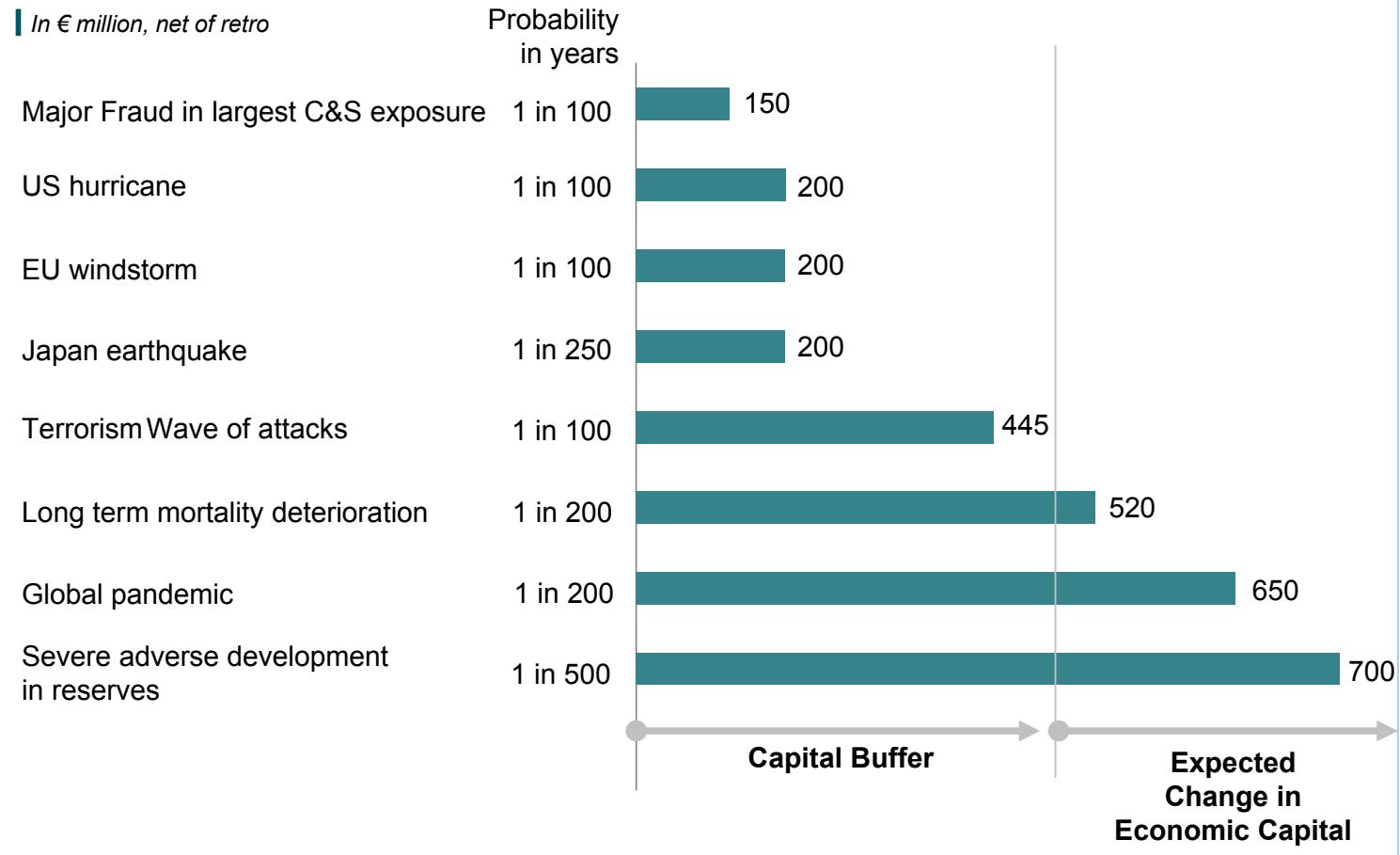
# Outline for today

- Correlation modeling
  - Copulas
  - Endogenous risk and correlation crises
- Risk aggregation
  - in Solvency II standard formula: some pitfalls
  - in internal models: example with Excel spreadsheet



# The Capital Buffer absorbs the single worst case scenarios

## Buffer capital checked against single worst-case scenarios (examples)

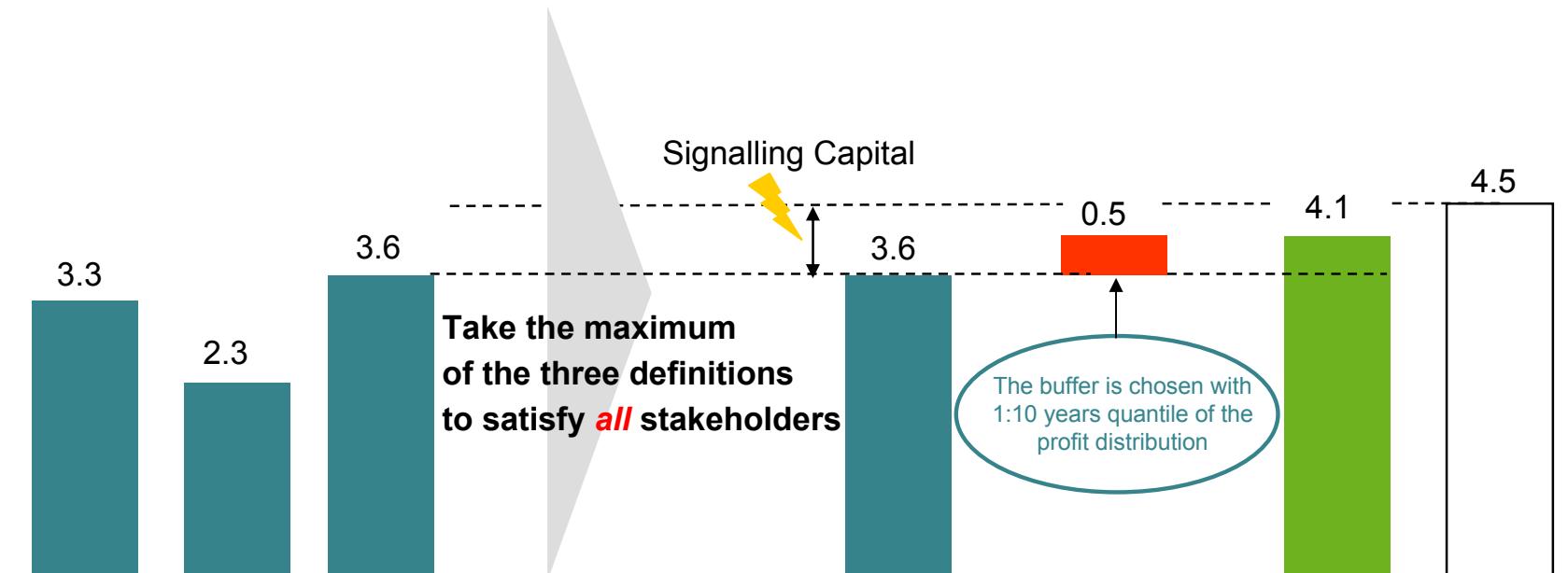




# The internal capital requirement satisfies all stakeholders

## Internal RBC, Required Capital, Buffer and Target Capital

In € billion, based on figures for 2008



Internal model RBC  
Regulatory requirements

"A" range rating capital requirement\*

\* currently undergoing rating agency reviews

**Required Capital**

Buffer Capital

Target Capital

**Available Capital\*\***

\*\* All capital is computed at  $t_i$  with data at  $t_0$

# Solvency Capital Requirement (SCR)

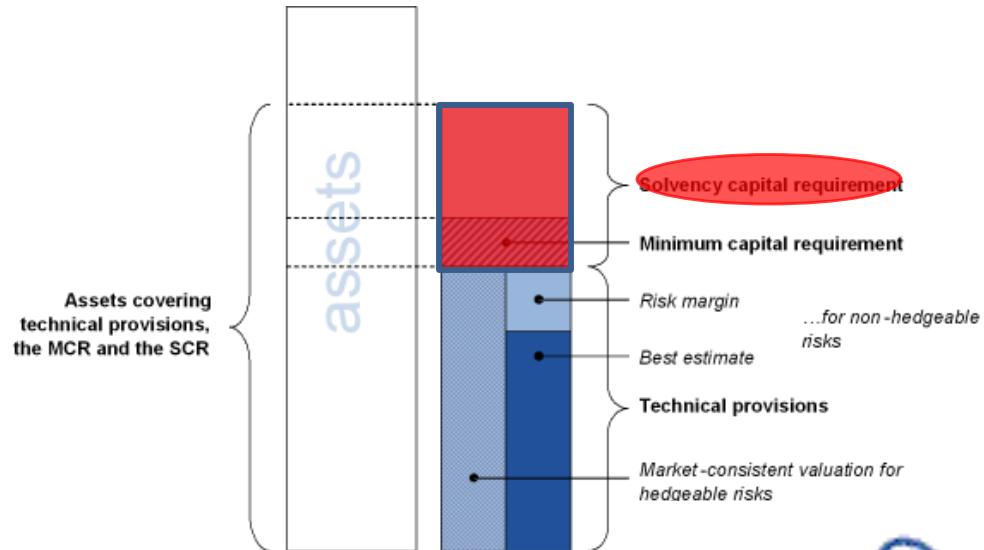
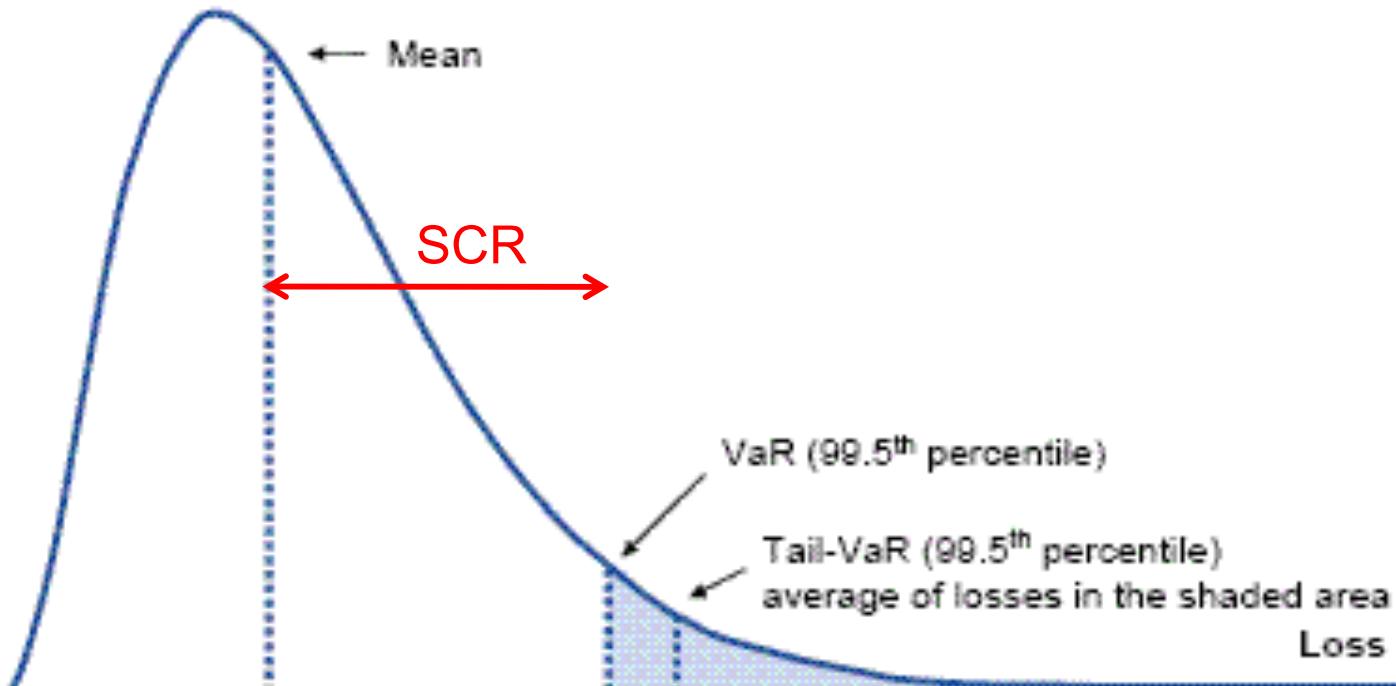


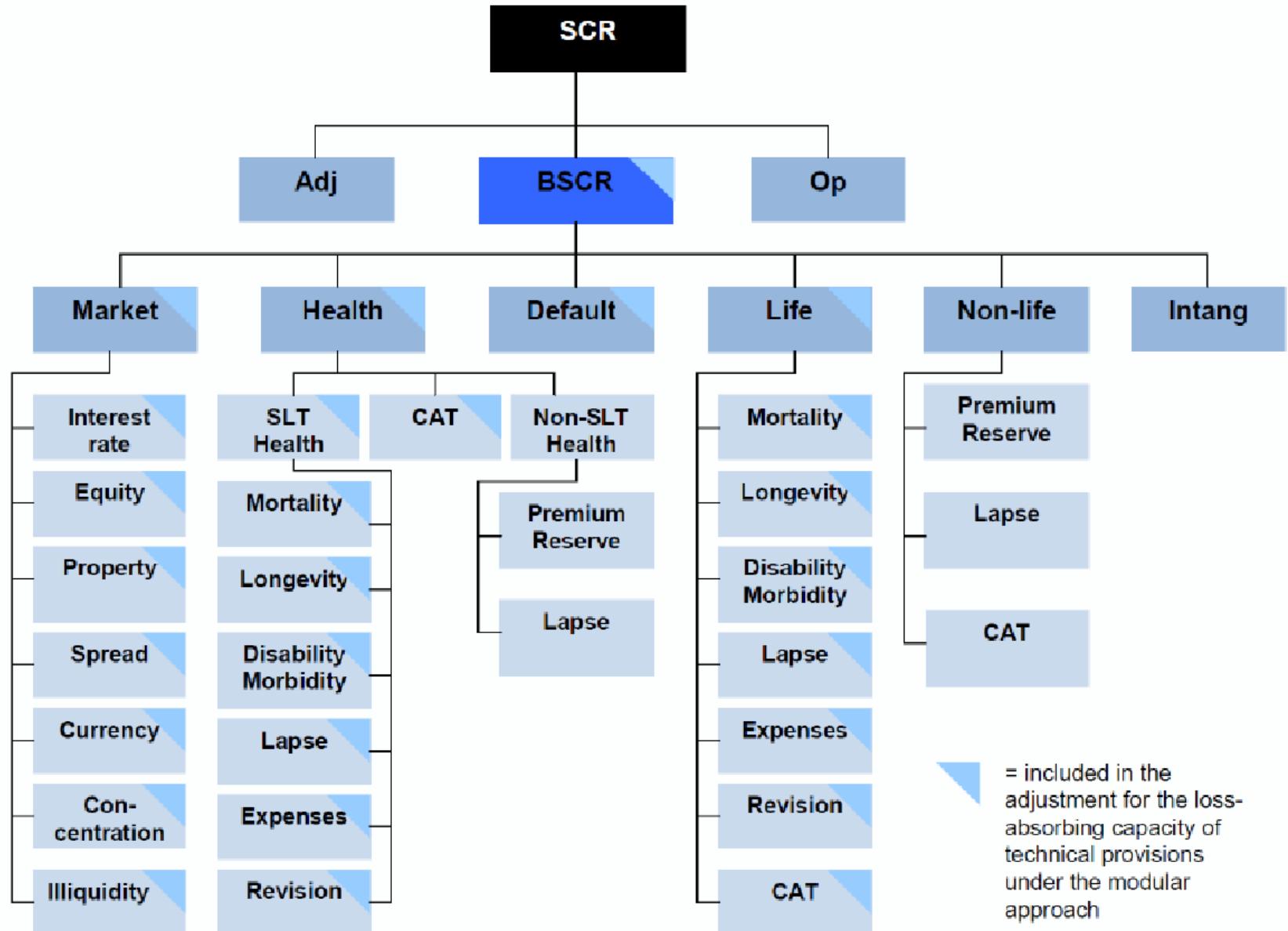
Figure (Source: CP 20, p. 9)

2.2 Pillar 1 is made up of a number of different combination, should provide a structured way whether the insurer has adequate financial carries.

Standard formula:  
 Lognormal loss distribution assumption  
 1-Year Time Horizon



# Modular Structure of SCR



▼ = included in the adjustment for the loss-absorbing capacity of technical provisions under the modular approach

# Risk aggregation

$$SCR_{glob} = \sqrt{\sum_i \sum_j \rho_{ij} SCR_i SCR_j}.$$

The  $\rho_{ij}$  are correlation parameters given by QIS5 matrices.

Advantage of the above formula:

it takes diversification effect into account: the global SCR is always smaller than the sum of the SCR per LoB

Some obvious drawbacks:

- multi-level risk aggregation problems
- Gap with internal models

Incentive to develop internal risk models.

# Multi-level risk aggregation: exercise

Suppose we have the following risks and required capital for each risk on a stand alone basis:

Table 1

Risk	Capital
A	1000
B	200
C	2000
D	500

$$SCR_{glob} = \sqrt{\sum_i \sum_j \rho_{ij} SCR_i SCR_j}.$$

And the following correlation factors between those risks:

Table 2

	A	B	C	D
A	1			
B	0.50	1		
C	0.75	0.75	1	
D	0.50	0.50	0.25	1

- 1) Compute the risk capitals for A+B, for C+D and for A+B+C+D.
- 2) Derive the implicit correlation coefficient between (A+B) and (C+D).

# Multi-level risk aggregation: solution

Suppose we have the following risks and required capital for each risk on a stand alone basis:

Table 1

Risk	Capital
A	1000
B	200
C	2000
D	500

$$SCR_{glob} = \sqrt{\sum_i \sum_j \rho_{ij} SCR_i SCR_j}$$

Risk	Capital
A+B	1114
C+D	2179
A+B+C+D	3192

And the following correlation factors between those risks:

Table 2

	A	B	C	D
A	1			
B	0.50	1		
C	0.75	0.75	1	
D	0.50	0.50	0.25	1

correlation factor between (A+B) and (C+D)

$$0.865 \quad (= \frac{3192^2 - 1114^2 - 2179^2}{2 \times 1114 \times 2179})$$

- 1) Compute the risk capitals for A+B, for C+D and for A+B+C+D.
- 2) Derive the implicit correlation coefficient between (A+B) and (C+D).

# Multi-level risk aggregation: exercise

Suppose the correlation factors stay the same but during a year the capitals change from the ones in table 1 into (this could arise from changes in business volume or of product mix):

Table 4

Risk	Capital
A	1100
B	300
C	1800
D	800

	A	B	C	D
A	1			
B	0.50	1		
C	0.75	0.75	1	
D	0.50	0.50	0.25	1

- 1) Recompute the risk capitals for A+B, for C+D and for A+B+C+D.
- 2) Derive the new implicit correlation coefficient between (A+B) and (C+D).

# Multi-level risk aggregation: solution

Suppose the correlation factors stay the same but during a year the capitals change from the ones in table 1 into (this could arise from changes in business volume or of product mix):

Table 4

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- 1) Recompute the risk capitals for A+B, for C+D and for A+B+C+D.
- 2) Derive the new implicit correlation coefficient between (A+B) and (C+D).

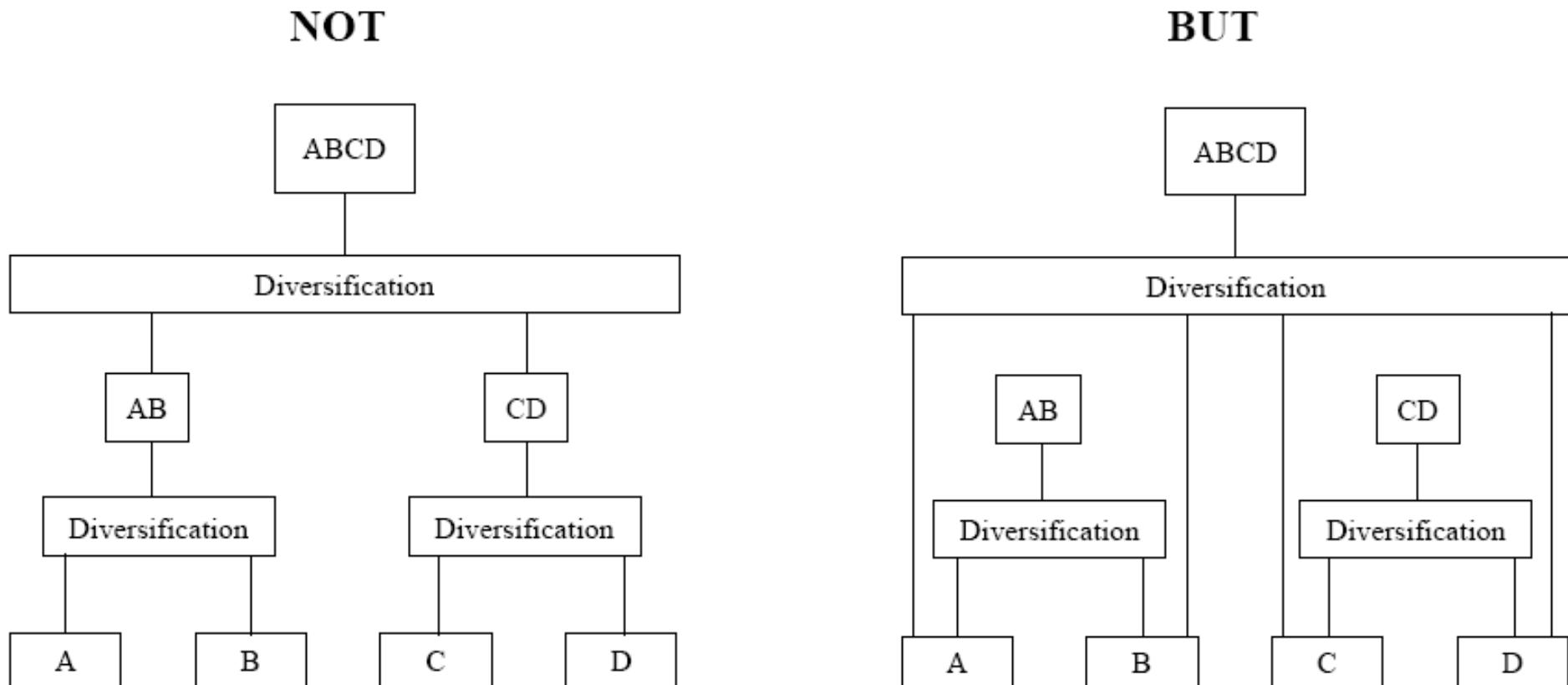
Risk	Capital
A+B	1277
C+D	2145
A+B+C+D	3336

From these capitals we can derive that while the dependencies between the risks did not changed at all the correlation factor between (A+B) and (C+D) changed from: **0.865** into **0.895** to get the total capital for A+B+C+D: 3336.

# Multi-level risk aggregation: conclusion\*

*Only correlation factors set at the lowest level are unequivocal and stable over time. Thus capitals at higher levels should always be derived starting at a lower level. Deriving a higher level aggregation from a lower level aggregation above the sub risk level risks will result in misleading outcomes at the entity or group level.*

So, the Bottom up approach is



\*Suggested post-reading: Multi-level risk aggregation by D. Filipovic, ASTIN Bulletin (2009).

# Bridging the gap between Standard formula and internal models

Risk aggregation in the QIS approach

$$SCR_{glob} = \sqrt{\sum_i \sum_j \rho_{ij} SCR_i SCR_j}.$$

This requires several assumptions (consider 3 risks here): the economic value after 1 year must be affine

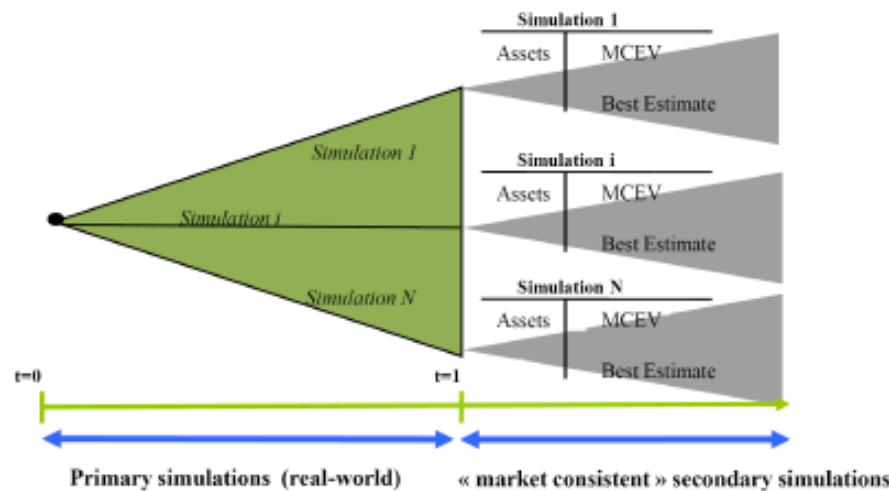
$$EcV_1 = C + aX + bY + dZ,$$

- where  $X, Y, Z$  are the risk factors,
- and  $(X, Y, Z)$  has an elliptic distribution.

In internal models, the last assumption can generally be translated into:  
(Gaussian marginal distributions and Gaussian copula), or  
(Student marginal distributions and Student copula).

# Bridging the gap between Standard formula and internal models

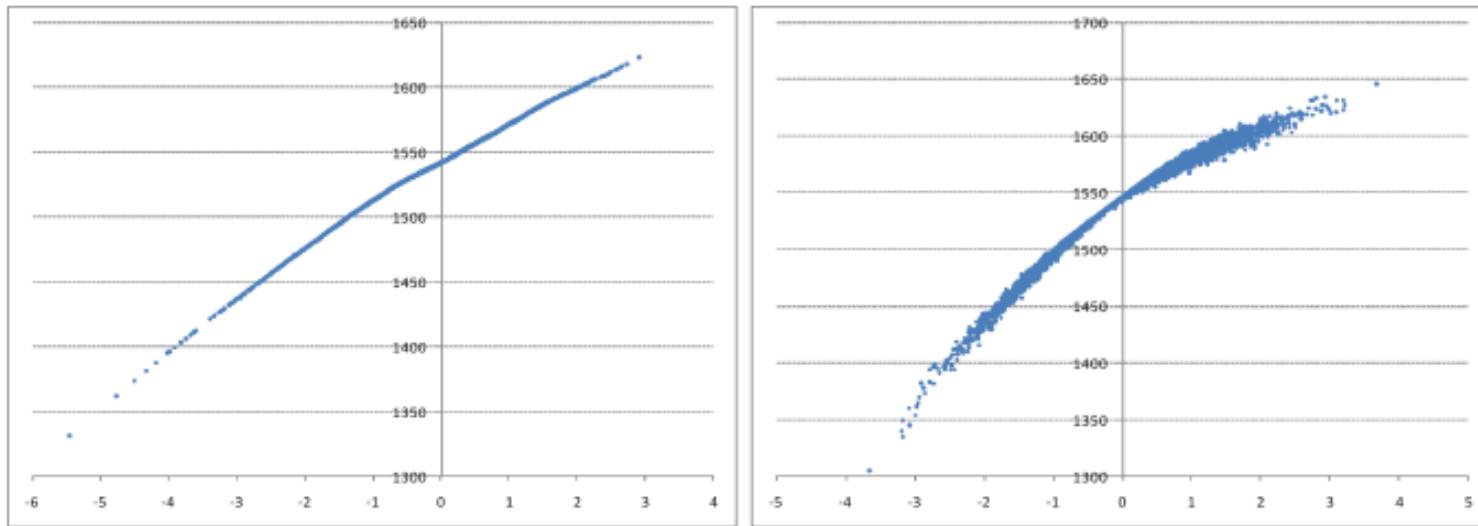
## Solvency II: some internal models



In internal risk models, the SCR is equal to  $E(EcV_1) - VaR_{0.5\%}(EcV_1)$ , where  $EcV_1$  is the economic value of the company after 1 year: in many internal models, first simulate what happens during one year (historical universe), and then determine "market-consistent" value of liabilities (and assets) given what has happened during the first year.

# Bridging the gap between Standard formula and internal models

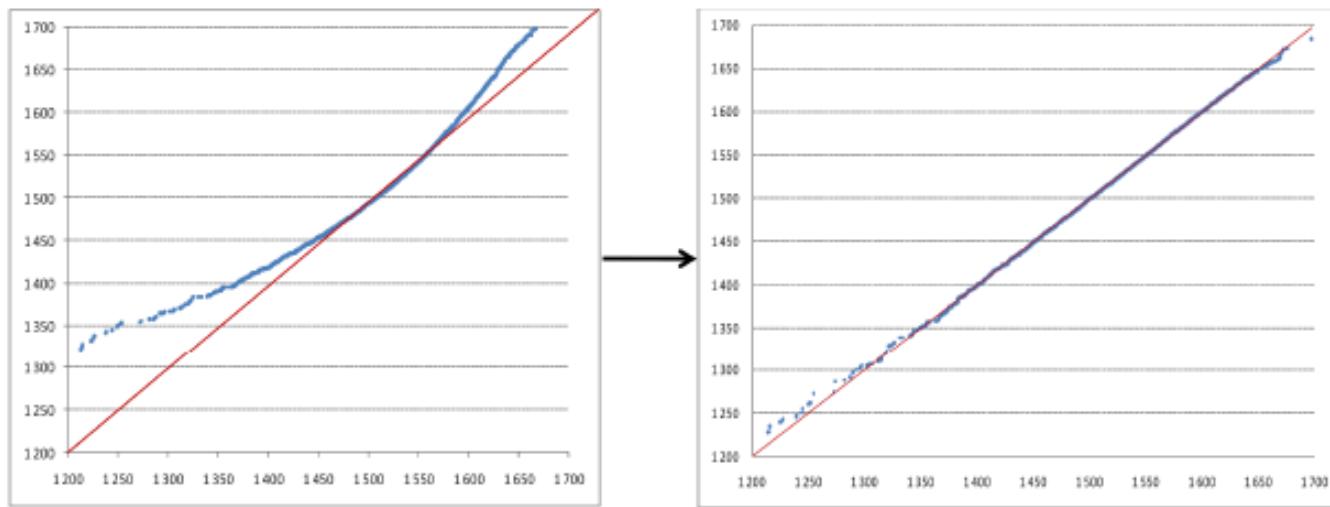
Basic example with two risk factors (savings)



Linearity in the asset factor is acceptable, but not in the interest rate factor.  
 $SCR_{IM} = 242.4$ ,  $SCR_{SF} = 205.3$ , relative error 15.3%.  
Correlation in internal model is 21%, implicit correlation to have  
 $SCR_{SF} = SCR_{IM}$  is 71%!

# Bridging the gap between Standard formula and internal models

Solvency II: Pillar I bottom-up approach



Discrepancy for  $EcV$  (left) can be fixed in some cases with higher order approximations (but not in all cases), see [Devineau and Loisel, 2009].

[Devineau, L. and Loisel, S. \(2009\).](#)

Risk aggregation in Solvency II in internal models and in the standard formula.

# Outline for today

- Correlation modeling
  - Copulas
  - Endogenous risk and correlation crises
- Risk aggregation
  - in Solvency II standard formula: some pitfalls
  - in internal models: example with Excel spreadsheet

# Risk aggregation

- Value at Risk and Tail-Value at Risk definitions
- TVaR is sub-additive
- VaR is not sub-additive in general,
- But it may be sub-additive for certain classes of random variables
- VaR and TVaR are comonotone-additive
- More general distortion risk measures may be defined, including the **Wang**-transform!

# Risk aggregation in internal models

- In-class experiments with Excel Spreadsheet.
- Gaussian copula
- Lognormal risks
- Risk measure: VaR or TVaR
- Impact of dependence between risks on diversification benefit