Chapter 1

Convex sets and convex functions

In the sequel E denotes a finite dimensional vector space and $|\cdot|$ its euclidean norm.

1.1 Convex sets

Definition 1.1.1 *A subset C of E is called a convex set if*

$$\lambda x + (1-\lambda)y \quad \in \quad C$$

for any $x, y \in C$ and any $\lambda \in [0,1]$.

Definition 1.1.2 (Convex combination) *For $x_1, \ldots, x_n \in E$ and $\lambda_1, \cdots, \lambda_n \in [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$, the point $\sum_{i=1}^n \lambda_i x_i$ is called convex combination of the points $x_1, \ldots, x_n$.*

Proposition 1.1.1 *A subset C of E is convex if and only if the convex combination*

$$\sum_{i=1}^n \lambda_i x_i \quad \in \quad C$$

for any $x_1, \ldots, x_n \in C$ and $\lambda_1, \ldots, \lambda_n \in [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$.

Proof. The proof follows from an induction argument. □

Proposition 1.1.2 *Let $(C_i)_{i\in I}$ be a une family of convex subsets of E. Then the intersection $\bigcap_{i\in I} C_i$ is a convex subset of E.*

Proof. The proof follows from the definition of the convexity. □

Definition 1.1.3 (Convex hull) *Let A be a subset of E. The convex hull of A, denoted by $conv(A)$, is defined as the intersection of all convex subsets C of E such that $A \subset E$. By the previous Proposition $conv(A)$ is a convex subset of E.*

Proposition 1.1.3 *$conv(A)$ is the set of all convex combinations of points of A*

$$conv(A) \quad = \quad \Big\{ \sum_{i=1}^{n} \lambda_i x_i \ : \ n \geq 1 \ , \ x_i \in A \ , \ \lambda_i \geq 0 \ , \ \sum_{i=1}^{n} \lambda_i = 1 \Big\} \ .$$

Proof. Define the set

$$\tilde{A} \quad = \quad \Big\{ \sum_{i=1}^{n} \lambda_i x_i \ : \ n \geq 1 \ , \ x_i \in A \ , \ \lambda_i \geq 0 \ , \ \sum_{i=1}^{n} \lambda_i = 1 \Big\} \ .$$

From Proposition 1.1.1, we have $\tilde{A} \subset C$ for any convex set C such that $A \subset C$. Therefore $\tilde{A} \subset conv(A)$. We next notice that $\tilde{A}$ is a convex set. Therefore $conv(A) \subset \tilde{A}$. □

Definition 1.1.4 (Extremal point) *Let C be a convex subset of E. A point $x \in C$ is called an extremal point of C if*

$$x \quad \neq \quad \lambda y + (1-\lambda) z$$

for any $y, z \in C$ such that $y \neq z$ and any $\lambda \in (0,1)$. The set of extremal points of C will be denoted by $ext(C)$.

Proposition 1.1.4 *The following assertions are equivalent.*

(i) $x \in ext(C)$.

(ii) $x \neq \frac{y+z}{2}$ *for all* $y, z \in C$ *such that* $y \neq z$

(iii) $x = \frac{y+z}{2}$ *with* $y, z \in C \Rightarrow x = y = z$.

Proof. We obviously have $(i) \Rightarrow (ii) \Rightarrow (iii)$. Consider $x \in C$ satisfying (iii), and let $y, z \in C$ and $\lambda \in (0,1)$ such that $x = \lambda y + (1-\lambda)z$. If $\lambda = 1/2$ then from (iii) we have $x = y = z$. If $\lambda \in (0, 1/2)$, then $z' := 2x - z \in C$ as C is convex and $z' = 2\lambda y + (1-2\lambda)z$. Since $x = (z+z')/2$ we get from (iii) that $x = z$ and therefore $x = y$. If $\lambda \in (0, 1/2)$, we apply the same argument with y in place of z. □

Proposition 1.1.5 *(i) $x \in ext(C)$ if and only if $C \setminus \{x\}$ is convex. (ii) We have $ext(conv(A)) \subset A$ for $A \subset E$.*

Proof. (i) Let $x \in ext(C)$, $y, z \in C \setminus \{x\}$ and $\lambda \in [0,1]$. We then have $\lambda y + (1-\lambda)z \in C$. If $\lambda y + (1-\lambda)z = x$ then $y = z = x$ which is not possible. Therefore $\lambda y + (1-\lambda)z \in C \setminus \{x\}$. Conversely, suppose that $C \setminus \{x\}$ is convex. Let $y, z \in C$ with $y \neq z$ and $\lambda \in (0,1)$ such that $x = \lambda y + (1-\lambda)z$. If $y = x$ or $z = x$ we have $x = y = z$ from $x = \lambda y + (1-\lambda)z$. Therefore, $y, z \in C \setminus \{x\}$. From the convexity of $C \setminus \{x\}$ we get $x \in C \setminus \{x\}$ which is impossible. Hence $x \neq \lambda y + (1-\lambda)z$.
(ii) Let $x \in conv(A) \setminus A$. Then, there exists $n \geq 1$, $x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i x_i = x$. Without loss of generality, we suppose that $\lambda_1 \geq \cdots \geq \lambda_n > 0$ and $x_n \notin conv(x_1, \ldots, x_{n-1})$. We then notice that $n \geq 2$. Therefore, $\lambda := \sum_{i=1}^{n-1} \lambda_i \in (0,1)$ and

$$x = \lambda y + (1-\lambda)z$$

with $y = \sum_{i=1}^{n-1} \frac{\lambda_i}{\lambda} x_i \in conv(A)$ and $z = x_n \neq y$. Therefore, $x \notin ext(conv(A))$. □

Theorem 1.1.1 (Minkowski (or Krein-Milman in infinite dimension)) *Let A be a convex compact subset of E. Then $conv(ext(A)) = A$.*

1.2 Convex functions

Definition 1.2.5 (Convex function) *A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if*

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in E$ *with* $f(x) < +\infty$, $f(y) < +\infty$ *and all* $\lambda \in [0, 1]$.

We notice that for f convex function, the sets $A_f(a) = \{x \in E \ : \ f(x) \leq a\}$ and $B_f(a) = \{x \in E \ : \ f(x) < a\}$ are convex sets. Unfortunately, the reverse implication is not true. Indeed, for $f : \mathbb{R} \to \mathbb{R}$ monotone function, the sets $A_f(a)$ and $B_f(a)$ are convex sets as they are intervals of $\mathbb{R}$.

Definition 1.2.6 (Domain of a function) *The domain of a function* $f : E \to \mathbb{R} \cup \{+\infty\}$ *is the set*

$$dom(f) \quad = \quad \{x \in E \ : \ f(x) < +\infty\} \ .$$

Remark 1.2.1 *The function* $f : E \to \mathbb{R} \cup \{+\infty\}$ *is convex if and only if 1) its domain* $dom(f)$ *is convex and 2) its restriction* $f|_{dom(f)}$ *to its domain is convex.*

Definition 1.2.7 (Strictly convex function) *A function* $f : E \to \mathbb{R} \cup \{+\infty\}$ *is said to be strictly convex if it is convex and*

$$f\big(\lambda x + (1-\lambda)y\big) \quad < \quad \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in dom(f)$ *with* $x \neq y$ *and all* $\lambda \in (0, 1)$.

Proposition 1.2.6 *let* C *be a convex subset of* E *and* $f : E \to \mathbb{R} \cup \{+\infty\}$ *be a strictly convex function such that* $C \subset dom(f)$. *Let* $x_0 \in C$ *such that*

$$f(x_0) \quad = \quad \max_{x \in C} f(x) \ .$$

Then $x_0 \in ext(C)$.

Proof. Let $M = \sup_{x \in C} f(x)$. If $x_0 \in C$ such that $f(x_0) = M$ and $x \notin ext(C)$, we can find $y, z \in C$ with $x \neq z$ such that $x_0 = (y+z)/2$. We then have $M = f(x_0) < (f(y) + f(z))/2 \leq \max(f(y), f(z))$. Thus f takes a value strictly greater than M at y or z, which contradicts the definition of M. $\square$

Proposition 1.2.7 (Convex extension) *If* $f : C \to \mathbb{R}$ *is a convex function defined on the convex set* C, *it can be extended to convex function* $\tilde{f} : E \to \mathbb{R} \cup \{+\infty\}$ *by setting*

$$\tilde{f}(x) \quad = \quad \begin{cases} f(x) & \text{if } x \in C, \\ +\infty & \text{else.} \end{cases}$$

This result allows to manipulate simultaneously convex functions defined on different convex sets. In particular, if $A \subset E$ we define its characteristic function χ_A by

$$\chi_A(x) \quad = \quad \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{else.} \end{cases}$$

Then, A is convex if and only if χ_A is convex.

Definition 1.2.8 (Epigraph of a function) *Let $f : E \to \mathbb{R} \cup \{+\infty\}$. Its epigraph is the subset of $E \times \mathbb{R}$ defined by*

$$epi(f) \quad = \quad \left\{(x, t) \in E \times \mathbb{R} \; : \; f(x) \leq t\right\} .$$

The strict epigraph of f is defined by

$$stepi(f) \quad = \quad \left\{(x, t) \in E \times \mathbb{R} \; : \; f(x) < t\right\} .$$

Proposition 1.2.8 *i) The function $f : E \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if $epi(f)$ is convex.*

ii) If the function $f : E \to \mathbb{R} \cup \{+\infty\}$ is strictly convex then if its $stepi(f)$ is convex.

Proof. The proof follows from the definitions of convexity for functions and sets. □

Proposition 1.2.9 *(i) Let $(f_i)_{i \in I}$ be a family of convex functions. Its upper envelope f defined by $f = \sup_{i \in I} f_i$ is a convex function.*

(ii) If g is the pointwise limit of a sequence $(g_n)_{n \geq 0}$ of convex functions, then g is convex.

Proof. Fix x, y and $\lambda \in [0, 1]$.

(i) Since each f_i is convex we have

$$f_i(\lambda x + (1 - \lambda)y) \quad \leq \quad \lambda f_i(x) + (1 - \lambda) f_i(y)$$

for any $i \in I$. Taking the supremum we get

$$\begin{aligned}\sup_{i \in I} f_i(\lambda x + (1-\lambda)y) &\leq \sup_{i \in I} \big\{\lambda f_i(x) + (1-\lambda) f_i(y)\big\} \\ &\leq \lambda \sup_{i \in I} f_i(x) + (1-\lambda) \sup_{i \in I} f_i(y)\end{aligned}$$

(ii) From the convexity of g_n we have

$$g_n(\lambda x + (1-\lambda)y) \leq \lambda g_n(x) + (1-\lambda) g_n(y) \ .$$

Taking the limit as n goes to ∞, we get

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda) g(y) \ .$$

$\square$

We may also need to consider convex functions valued in $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. We then take the convexity property of the epigraph as definition.

Definition 1.2.9 (Convex function valued in $\bar{\mathbb{R}}$) . *The function $f : E \to \bar{\mathbb{R}}$ is said to be convex if its epigraph*

$$epi(f) = \big\{(x,t) \in E \times \mathbb{R} \ : \ f(x) \leq t\big\} \ .$$

is convex.

In the case where f is valued in $\mathbb{R} \cup \{+\infty\}$, the definition of convexity coincides with the initial one. For the case where f is valued in $\bar{\mathbb{R}}$, we have the following result.

Proposition 1.2.10 *A function $f \ : E \to \bar{\mathbb{R}}$ is convex if and only if*

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in E$ with $f(x) < +\infty$, $f(y) < +\infty$ and all $\lambda \in [0,1]$, with the convention $-\infty + t = -\infty$, for $t < +\infty$, $a \cdot (-\infty) = -\infty$, for $a > 0$, and $0 \cdot (-\infty) = 0$.

Proof. The proof follows from the definition of convexity and the conventions mentioned above. □

Remark 1.2.2 *We notice that Proposition 1.2.9 holds functions valued in* $\bar{\mathbb{R}}$*.*

We end this chapter by the following definition.

Definition 1.2.10 (Proper convex function) *A convex function is said to be proper if it does not take the value* $-\infty$*, and is not identically equal to* $+\infty$*. If not, the function is said to be unproper.*

Chapter 2

Structure of convex sets

2.1 Topological properties

Proposition 2.1.11 *Let C be a convex set. The adherence $adh(C)$ of C is convex.*

Proof. It follows from the definition of the adherence. □

Remark 2.1.2 *For F closed subset of E, $conv(F)$ is not always closed. Indeed, consider $E = \mathbb{R}^2$, $F_1 = \{(x, y) \in \mathbb{R}^2 \ : \ x > 0, \ xy = 1\}$ and F_2 the symmetric of F_1 w.r.t. the first axe. Then F_1 and F_2 are closed, so $F = F_1 \cup F_2$ is also closed. But $conv(F) = \{(x, y) \ : \ x > 0, y \in \mathbb{R}\}$ which is not closed.*

Definition 2.1.11 (Closed convex hull) *The closed convex hull of a subset A of E is the adherence of its convex hull $adh(conv(A))$.*

Proposition 2.1.12 *(i) $adh(conv(A))$ is the smallest closed convex set containing A.*

(ii)If $A_1, \ldots, A_p$ are convex compact subsets of E, then $conv(A_1, \ldots, A_p)$ is compact.

(iii) If A is a compact subset of E, $conv(A)$ is compact.

Proof. (i) It follows from the definition of the closed convex hull.
(ii) We first notice that

$$conv(A_1,\ldots,A_p) \quad = \quad \Big\{\sum_{i=1}^{p}\lambda_i x_i \ : \ x_i\in A_i,\ \lambda_i\geq 0 \text{ and } \sum_{i=1}^{p}\lambda_i=1\Big\}.$$

Therefore, $conv(A_1,\ldots,A_p)$ is compact as the image by a continuous application of the compact set $A_1\times\cdots\times A_p\times\{(\lambda_i)_{i=1}^p\in(\mathbb{R}_+)^p \ : \ \sum_{i=1}^p\lambda_i=1\}$.
(iii) It is a particular case of (ii) □

We now study the interior $int(C)$ of a convex set C. We first notice that

$$ext(C)\cap int(C) \quad = \quad \emptyset\ .$$

We start with the following lemma.

Lemma 2.1.1 *Let C be a convex subset of E and $x_0\in int(C)$. For any $x\in adh(C)$, we have $[x_0,x)\subset int(C)$.*

Proof. Let $y\in[x_0,x)$, $r>0$ and x_0 such that $B(x_0,r)\subset C$.

i) Suppose first that $x\in C$. Let f be the homothety centered at x with ratio λ such that $f(x_0)=y$. We then have

$$f(z) \quad = \quad x+\lambda(z-x) \quad = \quad (1-\lambda)x+\lambda z$$

for all $z\in E$, where λ is such that $y=f(x_0)=(1-\lambda)x+\lambda x_0$, so $0<\lambda\leq 1$. We have $f(C)\subset C$ by convexity of C, and $f(B(x_0,r))=B(y,\lambda r)$. Thus $B(y,\lambda r)\subset C$ et $y\in int(C)$.

ii) Suppose that $x\in adh(C)$ and $y\in(x_0,x)$. Let g be the homothety centered at y with ratio λ such that $g(x_0)=x$. We have $g(t)=y+\lambda(t-y)$; the condition $g(x_0)=x$ writes $x-y=\lambda(x_0-y)$, hence $\lambda<0$. Then $g(B(x_0,r))=B(x,|\lambda|r)$. Since $x\in adh(C)$, there exists $z\in C\cap B(x,|\lambda|r)$. let $u=g^{-1}(z)$. Then $u\in B(x_0,r)\subset int(C)$, and $z-y=\lambda(u-y)$, $\lambda<0$ shows that $y\in(u,z)$. From Step i) above, we get $y\in int(C)$. □

Proposition 2.1.13 *Let C be a convex subset of E such that $int(C)\neq\emptyset$. Then $int(C)$ is convex. Moreover, we have $adh(int(C))=adh(C)$ and $int(adh(C))=int(C)$.*

Proof. i) If $x, y \in int(C)$, we have $[x, y] \subset int(C)$ from the previous Lemma. Hence $int(C)$ is convex.
ii) The inclusion $adh(int(A)) \subset adh(A)$ is true for any $A \subset E$. Conversely, if $x \in adh(C)$, then x is a boundary of an open line included in C. By the previous lemma we have $x \in adh(int(C))$.

Let us notice that the adherence of the nonempty line (x_0, x) is $[x_0, x]$.
iii) The inclusion $int(A) \subset int(adh(A))$ is true for any $A \subset E$. Conversely, let $x \in int(adh(C))$ and $r > 0$ such that $B(x, r) \subset adh(C)$. Since $adh(C) = adh(int(C))$ by ii), it exists $y \in C \cap B(x, r)$. Let $z \in E$ such that $x = (z + y)/2$. We still have $z \in B(x, r)$, so $z \in adh(C)$. From the previous lemma$(z, y] \subset int(C)$. in particular $x \in int(C)$. □

2.2 Separation of convex sets

Theorem 2.2.2 (Projection on convex sets) *Let C be a closed convex subset of E and $x_0 \notin C$. there exists a unique $y_0 \in C$, called projection of x_0 on C such that*

$$|x_0 - y_0| \quad = \quad \inf_{y \in C} |x_0 - y| \ .$$

The projection of x_0 on C is characterized by the following inequality

$$\langle x_0 - y_0, y - y_0 \rangle \quad \leq \quad 0 \tag{2.2.1}$$

for all $y \in C$

Proof. Let $r = d(x_0, C) > 0$ and $B = C \cap \bar{B}(x_0, 2r)$. We then have

$$d(x_0, C) \quad = \quad \inf_{y \in C} |y - x_0| \quad = \quad \inf_{y \in B} |y - x_0|.$$

The second equality comes from $d(x_0, C) = \min\{d(x_0, B), d(x_0, C \setminus B)\}$ combined with $d(x_0, C \setminus B) \geq 2r$.

Since B is a closed and bounded subset of the finite dimension space E, B is compact. The continuous function $y \mapsto |y - x_0|$ admits a minimum over B. Let $y_0 \in B$ a minimum point, that is $|y - x_0| = d(x_0, C)$. We then have

$$|x_0 - y|^2 \quad \geq \quad |x_0 - y_0|^2,$$

for all $y \in C$. Hence, by writing $x_0 - y := (x_0 - y_0) + (y_0 - y)$ we get

$$\langle x_0 - y_0, y - y_0\rangle \leq \frac{1}{2}|y - y_0|^2.$$

Let $\theta \in (0,1)$. We apply the previous inequality to $y^\theta = \theta y + (1-\theta)y_0 \in C$ and we get

$$\theta\langle x_0 - y_0, y - y_0\rangle \leq \frac{1}{2}\theta^2|y - y_0|^2$$

Dividing by θ and sending θ to $0+$, we get (2.2.1). Conversely, for $y_0 \in C$ satisfying (2.2.1), we have

$$\begin{aligned} |x_0 - y|^2 &= |x_0 - y_0|^2 + 2\langle x_0 - y_0, y_0 - y\rangle + |y_0 - y|^2 \\ &\geq |x_0 - y_0|^2 + |y_0 - y|^2 \\ &\geq |x_0 - y_0|^2 \end{aligned}$$

and y_0 is a minimum point for the function $|x_0 - \cdot|$ over C. Moreover, the last inequality is strict whenever $y \neq y_0$, which gives the uniqueness of y_0.

□

The two following separation theorems are presented and proved in the finite dimensional framework. However, they also hold true in infinite dimension. We first recall the definition of an affine hyperplan.

Definition 2.2.12 *An affine hyperplane H is a subset of E such that the set*

$$H - x = \{y - x \ : \ y \in H\}$$

is a vector hyperplane of E for some $x \in E$.

From Riez representation Theorem, we deduce that a subset H of E is an affine hyperplane if and only if there exist a linear form f and a constant c such that

$$H = \{x \in E \ : \ f(x) = c\}.$$

Theorem 2.2.3 (Separation of a point and a closed convex set) *Let C be a closed convex subset of E and $x_0 \in E$ such that $x_0 \notin C$. There exists an affine hyperplan H strictly separating x_0 from C. That is, there exist a linear form f and a constant c such that*

$$f(x_0) \; > \; c \quad \textit{and} \quad f(y) \; < \; c \;\; \forall y \in C \, .$$

Proof. Let x_0 and C be as above. The distance $d(x_0, C) = \inf\{|y - x_0| : y \in C\}$ is strictly positive since $x_0 \notin C$ and C is closed. From Theorem 2.2.2 there exists a unique $y_0 \in C$ such that $|y_0 - x_0| = d(x_0, C)$. This y_0 being characterized by

$$\langle y - y_0, x_0 - y_0 \rangle \;\; \leq \;\; 0 \tag{2.2.2}$$

for all $y \in C$. Set $f(y) = \langle y, x_0 - y_0 \rangle$ for $y \in E$. The function f is a non zero linear form on E. The condition (2.2.2) can be stated as follows

$$f(y) \;\; \leq \;\; f(y_0) \, ,$$

for all $y \in C$. Moreover, we have

$$f(x_0) \;\; = \;\; f(y_0) + f(x_0 - y_0) \;\; = \;\; f(y_0) + |x_0 - y_0|^2 \;\; > \;\; f(y_0).$$

Let H be the hyperplane defined by

$$y \in H \;\; \Leftrightarrow \;\; f(y) = c,$$

where $c = 1/2(f(x_0) + f(y_0)) = f((x_0 + y_0)/2)$. We then have

$$f(x_0) \; > \; c \quad \text{and} \quad f(y) \; < \; c \;\; \forall y \in C \, .$$

That is, x_0 and C are included in each of the open half spaces defined by H.
□

Theorem 2.2.4 (Separation of a point from an open convex) *Let C be an open convex subset of E and $x_0 \notin C$. There exists an affine hyperplane H such that $x_0 \in H$ and $H \cap C = \emptyset$.*

Proof. We can assume w.l.o.g. that $x_0 = 0$. Let $\Gamma = \cup_{\lambda>0}\lambda C$. Γ is convex and open as the union of open convex sets λC for $\lambda > 0$. Moreover, $0 \notin \Gamma$ since $0 \notin C$. In particular, $adh(\Gamma) \neq E$, otherwise we have from Proposition 2.1.13 $E = int(E) = int(adh(\Gamma)) = int(\Gamma) = \Gamma$.

Let $y_0 \in E \setminus \Gamma$. Theorem 2.2.4 give the existence of a linear form such that

$$f(y_0) \quad < \quad f(z), \;\; z \in adh(\Gamma).$$

If $z \in \Gamma$, $tz \in \Gamma$ for all $t > 0$, so $f(y_0) < f(tz) = tf(z)$ pour all $t > 0$, and $z \in \Gamma$. Dividing the previous inequality by t and taking $t \to +\infty$, we get $0 \leq f(z)$ for all $z \in \Gamma$. Hence $\Gamma \subset \{x \; : \; f(x) \geq 0\}$. Since Γ is open we have $\Gamma \subset \{x \; : \; f(x) > 0\}$ and $C \subset \{x \; : \; f(x) > 0\}$. $\square$

Corollary 2.2.1 (Hahn-Banach Theorem) *Let C_1 and C_2 be two nonempty convex subset of E such that $C_1 \cap C_2 = \emptyset$. Suppose that C_1 is open. There exists an affine hyperplane H stricly separating C_1 and C_2. More precisely, there exists a linear form f on E and a constant c such that*

$$f(x_1) \quad < \quad c \;\; \leq \;\; f(x_2)$$

for all $x_1 \in C_1$ and $x_2 \in C_2$.

Proof. We apply Theorem 2.2.4 to the point 0 and the nonempty convex $C_1 - C_2 = \{y - z \; : \; y \in C_1, z \in C_2\}$. Indeed, $C_1 - C_2 = \cup_{z \in C_2}(C_1 - z)$ is open as the union of the open sets $C_1 - z$. Therefore, it exits a linear form f such that

$$f(y - z) \quad < \quad f(0) \quad = \quad 0$$

for all $y \in C_1$ and $z \in C_2$. Define $c := \inf_{z \in C_2} f(z)$. We then have

$$f(x_1) \leq c \leq f(x_2)$$

for all $x_1 \in C_1$ and $x_2 \in C_2$. Since C_1 is open $C_1 \subset \{f < c\}$. $\square$

Corollary 2.2.2 *Let C_1 and C_2 be two nonempty convex subset of E such that $C_1 \cap C_2 = \emptyset$. There exists an affine hyperplane H separating C_1 and C_2. More precisely, there exists a nonzero linear form f on E and a constant c such that*

$$f(x_1) \quad \leq \quad c \quad \leq \quad f(x_2)$$

for all $x_1 \in C_1$ and $x_2 \in C_2$.

Proof. The proof follows from the application of Corollary 2.2.1 to $int(C_1)$ and C_2. □

Corollary 2.2.3 *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ a convex function and $g : E \to \mathbb{R} \cup \{-\infty\}$ a concave function, i.e. $-g$ convex, such that $f \geq g$ on E. Suppose that there exists $x_0 \in E$ such that $f(x_0)$ and $g(x_0)$ are finite and g is continuous at x_0. Then, there exists an affine form h such that $f \geq h \geq g$ on E.*

Proof. Denote by $hypo(g)$ (resp. $sthypo(g)$) the hypograph (resp. stric hypograph) of g defined by

$$\begin{aligned} hypo(g) &= \{(x,t) \in E \times \mathbb{R} \ : \ t \leq g(x)\} \\ \text{(resp. } sthypo(g) &= \{(x,t) \in E \times \mathbb{R} \ : \ t < g(x)\}) \ . \end{aligned}$$

Since g is continuous at x_0, $int(hypo(g)) \neq \emptyset$. Moreover we have $int(hypo(g)) \subset sthypo(g)$. Since $f \leq g$, we get $epi(f) \cap sthypo(g) = \emptyset$. Since these two sets are convex, we get from Corollary 2.2.2 the existence of a $(u^*, a) \in E \times \mathbb{R} \setminus \{(0,0)\}$ and a constant c such that

$$\inf_{(t,x) \in epi(f)} at + \langle u^*, x \rangle \quad \geq \quad c \quad \geq \quad \sup_{(t,x) \in hypo(g)} at + \langle u^*, x \rangle \ .$$

Let $t \to +\infty$ gives $a \geq 0$.

If $a = 0$ then $\langle u^*, x - x_0 \rangle \leq 0$ for all $x \in dom(-g)$. Since $x_0 \in int(dom(-g))$, then $u^* = 0$ which is not possible. Therefore $a > 0$. We therefore get

$$af(x) + \langle u^*, x \rangle \quad \geq \quad c \quad \geq \quad ag(x) + \langle u^*, x \rangle \ ,$$

and

$$f(x) \;\; \geq \;\; \frac{1}{a}\big(c - \langle u^*, x\rangle\big) \;\; \geq \;\; g(x)$$

for all $x \in E$. Then $h = \frac{1}{a}\big(c - \langle u^*, \cdot\rangle\big)$ satisfies the required property. $\square$

Chapter 3

Regularity of convex functions

3.1 Convexity and continuity

Proposition 3.1.15 (Continuity of convex functions) *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex function and $x_0 \in int(dom(f))$. If f is upper bounded in the neighborhood of x_0, it is continuous, and also Lipschitz, in the neighborhood of x_0.*

Proof. i) We first prove the continuity of f at x_0. Let $r_0 > 0$ and $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in \bar{B}(x_0, r_0)$.

For $x \in \bar{B}(x_0, r_0)$ define y such that $y - x_0 = r_0 \frac{x-x_0}{|x-x_0|}$ and set $\lambda = \frac{|x-x_0|}{r_0}$, then $y \in \bar{B}(x_0, r_0)$ and $\lambda \in [0, 1]$. By convexity we have

$$\begin{aligned} f(x) - f(x_0) &= f(\lambda y + (1-\lambda)x_0) - f(x_0) \\ &\leq \lambda f(y) + (1-\lambda) f(x_0) - f(x_0) \\ &\leq \lambda (f(y) - f(x_0)) \\ &\leq \lambda \big(M - f(x_0)\big) = \frac{|x - x_0|}{r_0}(M - f(x_0)) \, . \end{aligned} \quad (3.1.1)$$

Define $z = x_0 - (x - x_0)$. Then, $z \in B(x_0, r_0)$ and $x_0 = (x + z)/2$. We then have

$$f(x_0) \quad = \quad f\big(\frac{x+z}{2}\big) \quad \leq \quad \frac{f(x) + f(z)}{2}$$

and

$$f(z) - f(x_0) \;\; \geq \;\; -(f(x) - f(x_0)) \; .$$

Applying (3.1.1) to z we get

$$\begin{aligned} \frac{|x - x_0|}{r_0}(M - f(x_0)) &= \frac{|z - x_0|}{r_0}(M - f(x_0)) \\ &\geq f(z) - f(x_0) \;\; \geq \;\; -(f(x) - f(x_0)) \end{aligned}$$

and

$$|f(x) - f(x_0)| \;\; \leq \;\; \frac{|x - x_0|}{r_0}(M - f(x_0))$$

for all $x \in \bar{B}(x_0, r_0)$, which gives the continuity of f at x_0.

ii) We now prove that f is Lipschitz continuous on $B(x_0, r)$, for $r < r_0$. For $x \in B(x_0, r)$, we have $\bar{B}(x, r_0 - r) \subset \bar{B}(x_0, r_0)$. Therefore f is upper bounded by M on $\bar{B}(x, r_0 - r)$. We then get

$$|f(y) - f(x)| \;\; \leq \;\; \frac{|y - x|}{r_0 - r}(M - f(x))$$

for all $y \in B(x, r_0 - r)$. Since $x \in B(x_0, r_0)$, we have $f(x) \geq f(x_0) - (M - f(x_0)) = 2f(x_0) - M$. Therefore

$$|f(y) - f(x)| \;\; \leq \;\; 2\frac{|y - x|}{r_0 - r}(M - f(x_0))$$

for all $y \in \bar{B}(x, r_0 - r)$. For $x, y \in B(x_0, r)$, we divide $[x, y]$ into consecutive sets $[x_i, x_{i+1}]$, $i = 0, \ldots, N$ with length strictly smaller than $r_0 - r$. Since $[x, y] \subset B(x_0, r)$ we get

$$\begin{aligned} |f(y) - f(x)| &\leq \sum_{i=1}^{N-1} |f(x_{i+1}) - f(x_i)| \\ &\leq \sum_{i=1}^{N-1} L|x_{i+1} - x_i| \;\; = \;\; L|x - y| \end{aligned}$$

with $L = 2\frac{M - f(x_0)}{r_0 - r}$. □

Definition 3.1.13 (Locally Lipschitz functions) *A function f is said to be locally Lipschitz on $int(dom(f))$ if it is Lipschitz in the neighborhood of all points of $int(dom(f))$.*

Corollary 3.1.5 *Let $f: E \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The following assertions are equivalent.*

i) there exists $x_0 \in dom(f)$ such that f is upper bounded at the neighborhood of x_0.

ii) There exists $x_0 \in dom(f)$ such that f is continuous at x_0.

iii) $int(dom(f)) \neq \emptyset$ and f is continuous on $dom(f)$.

iv) $int(dom(f)) \neq \emptyset$ and f is locally Lipschitz on $dom(f)$.

Proof. We obviously have $iv) \Rightarrow iii) \Rightarrow ii) \Rightarrow i)$.

We now prove $i) \Rightarrow iv)$. Suppose f upper bounded on $\bar{B}(x_0, r_0)$. Then, $x_0 \in int(dom(f)) \neq \emptyset$. Let $y_0 \in int(dom(f))$. There exists $z_0 \in int(dom(f))$ such that $y_0 \in [x_0, z_0)$. Let h be the homothety centered at z_0 with coefficient λ such that $h(x_0) = y_0$. We then have $h(x) = z_0 + \lambda(x - z_0) = \lambda x + (1-\lambda)z_0$, with $0 < \lambda \leq 1$ and $h(\bar{B}(x_0, r_0)) = \bar{B}(y_0, \lambda r_0)$. By convexity of f we have

$$\begin{aligned} f(h(x)) &= f(\lambda x + (1-\lambda)z_0) \leq \lambda f(x) + (1-\lambda) f(z_0) \\ &\leq \max(M, f(z_0)) \end{aligned}$$

for all $x \in \bar{B}(x_0, r_0)$. This implies that f is upper bounded on $\bar{B}(y_0, \lambda r_0)$, and therefore Lipschitz continuous on $\bar{B}(y_0, \lambda r)$, for all $r < r_0$, by the previous proposition. $\square$

Corollary 3.1.6 *Let $f: E \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Then, f is continuous and even locally Lipschitz on $int(dom(f))$.*

Proof. Suppose that $int(dom(f)) \neq \emptyset$ and w.l.o.g. $0 \in int(dom(f))$. Let $r_0 > 0$ such that $\bar{B}(x_0, r_0) \subset dom(f)$. We can then find $x_1, \ldots, x_n \in \partial B(0, r_0)$ such that $(x_1, \ldots, x_n)$ is a basis of E (recall that E is finite dimensional). Then f is bounded on $conv(x_1, -x_1, \ldots, x_n, -x_n)$. Since $(x_1, \ldots, x_n)$ is a basis of E we have $int(conv(x_1, -x_1, \ldots, x_n, -x_n)) \neq \emptyset$. From the previous proposition f is Lipschitz continuous on $int(conv(x_1, -x_1, \ldots, x_n, -x_n))$ which contains 0. $\square$

Corollary 3.1.7 *Let $f: E \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Then, f is continuous at a point of its domain if and only if $int(epi(f)) \neq \emptyset$.*

Proof. The continuity assumption implies that $int(epi(f)) \neq \emptyset$. Indeed, if $f \leq M$ on $B(y_0, r)$ then $B(y_0, r) \times (M, +\infty) \subset epi(f)$. The reverse implication is a consequence of Corollary 3.1.5. $\square$

3.2 Convexity and differentiability

Definition 3.2.14 (Directional derivative) *Let $f: E \to \mathbb{R} \cup \{+\infty\}$, $x_0 \in dom(f)$ and $h \in E \setminus \{0\}$. The function f is said to be right-differentiable at x_0 in the direction h if*

$$\lim_{t \to 0+} \frac{f(x_0 + th) - f(x_0)}{t}$$

exists in $\mathbb{R} \cup \{+\infty\}$. When it exists, this limit is denoted by $f'_d(x_0, h)$ and is called directional derivative of f in the direction h at x_0.

Proposition 3.2.16 *A convex function $f: E \to \mathbb{R} \cup \{+\infty\}$ is right-differentiable at any $x_0 \in dom(f)$ in any direction h. Moreover, we have the inequality*

$$f(x) - f(x_0) \quad \geq \quad f'_d(x_0, x - x_0) \tag{3.2.2}$$

for all $x \in E$.

This result is a direct consequence of the following lemma.

Lemma 3.2.2 *Let $f: E \to \mathbb{R} \cup \{+\infty\}$ be a convex function, $x_0 \in dom(f)$ and $h \in E \setminus \{0\}$. The function $\Delta_{x_0,h}: (0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$\Delta_{x_0,h}(t) \quad = \quad \frac{f(x_0 + th) - f(x_0)}{t}, \quad t \in (0, +\infty)$$

is nondecreasing.

Proof. Define $\phi(t) = f(x_0 + th) - f(x_0)$. ϕ is a convex function from $[0, +\infty)$ to $\mathbb{R} \cup \{+\infty\}$, with $\phi(0) = 0$. Therefore

$$\phi(\alpha t) \quad = \quad \phi(\alpha t + (1-\alpha).0) \quad \leq \quad \alpha\phi(t) + (1-\alpha)\phi(0) \quad = \quad \alpha\phi(t)$$

for all $\alpha, t \in [0, +\infty)$. Setting $s = \alpha t$, we get $\phi(s) \leq (s/t)\phi(t)$ and $\phi(s)/s \leq \phi(t)/t$ for $s > 0$.

This means $\Delta_{x_0,h}(s) \leq \Delta_{x_0,h}(t)$. Therefore $\Delta_{x_0,h}$ is a nondecreasing function from $(0, +\infty)$ to $\mathbb{R} \cup \{+\infty\}$. □

Corollary 3.2.8 (Minimum of a convex function) *A convex proper function f reaches its minimum at x_0 if and only if all its directional derivatives at x_0 are nonnegatives.*

Proof. Let x_0 be a minimum point of f. Then we have

$$f(x_0 + th) - f(x_0) \quad \geq \quad 0$$

for any $t > 0$ and any $h \in E$. Sending t to $0+$ we get $f'_d(x_0, h) \geq 0$ for any $h \in E$.

Conversely, let x_0 with nonnegative directional derivatives. Using (3.2.2), we get that x_0 is a minimum point of f. □

We turn to differentiable convex functions.

Proposition 3.2.17 *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a function differentiable on $dom(f)$ that is suppose to be convex. Then f is convex if and only if*

$$f(y) - f(x) \quad \geq \quad \langle \nabla f(x), y - x \rangle$$

for all $x \in dom(f)$ and $y \in E$.

Proof. If f is convex, the inequality follows from Proposition 3.2.16.

Conversely, suppose the inequality holds. Let $x, y \in dom(f)$, $\lambda \in [0, 1]$ and $z = \lambda x + (1-\lambda)y$. We then have

$$\begin{aligned} f(x) - f(z) \quad &\geq \quad \langle \nabla f(z), x - z \rangle \ , \\ f(y) - f(z) \quad &\geq \quad \langle \nabla f(z), y - z \rangle \ . \end{aligned}$$

Hence

$$\lambda f(x) + (1-\lambda)f(y) - f(z) \geq \langle \nabla f(z), \lambda(x-z) + (1-\lambda)(y-z)\rangle = 0$$

and f is convex. □

We provide a second convexity criterium based on the gradient.

Proposition 3.2.18 *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a function differentiable on $dom(f)$ that is suppose to be convex. Then f is convex if and only if*

$$\langle \nabla f(y) - \nabla f(x), y - x\rangle \quad \geq \quad 0$$

for all $x, y \in dom(f)$.

Proof. i) Suppose that f is convex. let $x, y \in dom(f)$. From Proposition 3.2.16 we have

$$\begin{aligned} f(y) - f(x) \quad &\geq \quad \langle \nabla f(x), y - x\rangle \, , \\ f(x) - f(y) \quad &\geq \quad \langle \nabla f(y), x - y\rangle \, . \end{aligned}$$

Summing these two inequalities gives the result.

ii) Conversely, supposons that ∇f satisfies the inequality. Let $x, y \in dom(f)$ and consider the map $\phi : [0,1] \to \mathbb{R}$, $\lambda \mapsto \phi(\lambda) = f(\lambda x + (1-\lambda)y) = f(y + \lambda(x-y))$. ϕ is differentiable with derivative given by

$$\phi'(\lambda) \quad = \quad \langle \nabla f(y + \lambda(x-y)), x - y\rangle$$

For $\lambda_1 < \lambda_2$ we have

$$\phi'(\lambda_1) - \phi'(\lambda_2) \quad = \quad \langle \nabla f(y + \lambda_1(x-y)) - \nabla f(y + \lambda_2(x-y)), x - y\rangle \quad \leq \quad 0$$

since the difference between the arguments of ∇f is $(\lambda_1 - \lambda_2)(x-y)$. We then deduce that ϕ has a nondecreasing derivative and is therefore convex. In particular

$$\phi(\lambda) \quad \leq \quad \lambda\phi(1) + (1-\lambda)\phi(0) \quad = \quad \lambda f(x) + (1-\lambda)f(y)$$

which is the convexity property for f. □

We provide a more precise result on the lower bound in the case of a Lipschitz continuous gradient.

Proposition 3.2.19 (Coercivity of the gradient) *Let f be convex and differentiable function from E to $\mathbb{R}$. We suppose that ∇f L-Lipschitz continuous on E. Then*

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \;\; \geq \;\; \frac{1}{L}|\nabla f(x) - \nabla f(y)|^2$$

for $x, y \in E$.

Proof. Fix $y, z \in E$ and g a function from E to $\mathbb{R}$ satisfying the same assumptions as f. Applying the first order Taylor formula to the function $t \in [0,1] \mapsto f(ty + (1-t)z)$ and using L-Lipschitz continuity of ∇f we have

$$g(z) \;\; \leq \;\; g(y) + \langle \nabla g(y), z - y\rangle + \frac{L}{2}|z - y|^2 \; .$$

Taking $y = x$ and $z = x - \frac{1}{L}\nabla g(x)$ we get

$$\frac{1}{2L}|\nabla g(x)|^2 \;\; \leq \;\; g(x) - g(x - \frac{1}{L}\nabla f(x)) \; .$$

Suppose now that g is lower bounded. We then have

$$\frac{1}{2L}|\nabla g(x)|^2 \;\; \leq \;\; g(x) - M \tag{3.2.3}$$

for all $x \in E$, where M is a lower bound of g. We next define the functions h_1 and h_2 by

$$\begin{aligned} h_1(u) &= f(u) - \langle \nabla f(x), u\rangle \; , \\ h_2(u) &= f(u) - \langle \nabla f(y), u\rangle \; , \end{aligned}$$

for $u \in E$. These two fonctions are convex and lower bounded respectively by $h_1(x)$ and $h_2(y)$ from Proposition 3.2.17. Applying (3.2.3) to h_1 and h_2 we get

$$h_1(x) \;\leq\; h_1(y) - \frac{1}{2L}|\nabla h_1(x)|^2 \quad \text{and} \quad h_2(y) \;\leq\; h_2(x) - \frac{1}{2L}|\nabla h_2(y)|^2 \; .$$

Summing these two inequalities gives the result. □

We end this section with a second order property for convex functions.

Proposition 3.2.20 *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a function differentiable on dom(f). Suppose that the function $t \in \mathbb{R}_+ \mapsto \nabla f(x+th)$ is differentiable and*

$$\left\langle \frac{\nabla f(x+th) - \nabla f(x)}{t}, h \right\rangle \xrightarrow[t \to 0+]{} Q(x,h)$$

with $Q(x,h) \geq 0$ for all $h \in E$ and all $x \in$ dom(f). Then f is convex.

Proof. From the mean value Theorem, there exists $\theta \in [0,1]$ such that

$$\begin{aligned} \langle \nabla f(y) - \nabla f(x), y - x \rangle &= \frac{d}{dt} \langle \nabla f(y + t(x-y)), x - y \rangle|_{t=\theta} \\ &= Q(\theta x + (1-\theta)y, x - y) \quad \geq \quad 0 \end{aligned}$$

and f is convex from the previous proposition. $\square$

We deduce from this last result that for a function $f : U \to \mathbb{R}$ that is C^2 on the open convex subset U of E, such that

$$\nabla^2 f(x) \quad \geq \quad 0$$

for all $x \in U$, then f is convex on U.

Chapter 4

Optimization of differentiable functions

4.1 Optimality conditions

In this chapter we are interested in solving the problem

$$\min_{x\in K} f(x) \tag{4.1.1}$$

where $f:\ E \to \mathbb{R}$ is a differentiable convex function such that

$$\lim_{|x|\to+\infty} f(x) \quad = \quad +\infty \tag{4.1.2}$$

and K is a closed convex subset of E.

Theorem 4.1.5 *Under the previous assumptions, the optimisation problem (4.1.1) has at least one solution. If moreover f is strictly convex then (4.1.1) has a unique solution.*

Proof. Since f is differentiable and convex, it is upper-semicontinuous by Proposition 3.2.16. Since K is closed we get from (4.1.2) that (4.1.1) has at least one solution.

Suppose in addition that f is strictly convex. Then if x_1^* and x_2^* are two different solutions, we have

$$f(\lambda x_1^* + (1-\lambda)x_2^*) \quad < \quad \lambda f(x_1^*) + (1-\lambda)f(x_2^*) \quad \leq \quad \min_K f$$

for $\lambda \in (0,1)$, which contradicts the convexity of K. $\square$

We next have the following the following result.

Theorem 4.1.6 *Suppose that f is convex and differentiable and K is closed and convex. Then, x^* is solution of (4.1.1) if and only if*

$$\langle \nabla f(x^*), v - x^* \rangle \quad \geq \quad 0$$

for all $v \in K$.

Proof. This is a consequence of Proposition 3.2.16. $\square$

Lagrange multipliers We consider the case where the set K is defined by $m \geq 1$ functions $F_1, \dots, F_m$ from E to $\mathbb{R}$ as follows

$$K \quad = \quad \{x \in E \ : \ F_i(x) = 0 \text{ for } i = 1, \dots, m\} \ .$$

Theorem 4.1.7 *Suppose that f is differentiable and that the functions $F_1, \dots, F_m$ are C^1. Let $x^* \in E$ be solution to (4.1.1) such that $(\nabla F_i(x^*))_{i=1}^m$ is a family of linearly independent vectors of E. Then there exists $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that*

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla F_i(x^*) \quad = \quad 0 \ .$$

The coefficients $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ are called Lagrange multipliers.

We next turn to the case where the set K is defined by inequalities instead of equalities. More precisely, we suppose that K is given by

$$K \quad = \quad \{x \in E \ : \ F_i(x) \leq 0 \text{ for } i = 1, \dots, m\} \ . \tag{4.1.3}$$

before stating the main result, we need to introduce the following definition.

Definition 4.1.15 (Qualified constraints) *The constraints defining K in (4.1.3) are said to be qualified at $x^* \in K$ if there exists $w \in E$ such that*

$$\langle \nabla F_i(x^*), w \rangle \quad < \quad 0,$$

or

$$\langle \nabla Fi(x^*), w \rangle \quad = \quad 0 \text{ and } F_i \text{ is affine}$$

for any $i = 1, \dots, m$ such that $F_i(x^*) = 0$.

Theorem 4.1.8 *Suppose that K is given by (4.1.3). Let $x^* \in K$ such that the functions f and $(F_i)_{i=1}^m$ are differentiable at x^* and the constraints are qualified at x^*. Then, if x^* is a local minimizer of f over K, there exists $\lambda_1, \dots, \lambda_M \in \mathbb{R}_+$ called Lagrange multipliers, such that*

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla F_i(x^*) \quad = \quad 0\,,$$

and $\lambda_i = 0$ if $F_i(u) < 0$ for $i = 1, \dots, m$.

The main difference with the previous results lies in the constraints defining K which are inequalities instead of equalities. There exists another version of this theorem allowing to consider equality and inequality constraints. For the sake of clarity, this version is not presented.

Karush, Kuhn and Tucker conditions We go back to the convex framework and we suppose that the function f and the contraints are convex. We notice that if the functions $(F_i)_{i=1}^m$ are convex and differentiable, the set K defined by the (4.1.3) is closed and convex.

We also notice that an equality constraint given by an affine function F_i can be expressed as a double convex inequality constraint as follows:

$$F_i(x) \;=\; 0 \quad \Leftrightarrow \quad F_i(x) \;\leq\; 0 \text{ and } -F_i(x) \;\leq\; 0\,.$$

In this convex framework, we can state a necessary and sufficient condition of optimality as follows.

Theorem 4.1.9 *Suppose that the functions f and $(F_i)_{i=1}^m$ are convex, continuous on E and differentiable on K defined by(4.1.3). Let $x^* \in K$ be such that the constraints are qualified at x^*. Then, x^* is a global minimum of f over K if and only if there exists $\lambda_1, \ldots, \lambda_M \in \mathbb{R}_+$ called Lagrange multipliers, such that*

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla F_i(x^*) \quad = \quad 0 \, ,$$

and $\lambda_i = 0$ if $F_i(x^*) < 0$ for $i = 1, \ldots, m$.

4.2 Gradient descent

For a convex and differentiable function f, there exist several gradient descent methods allowing to construct minimizing sequences. In this section, we shall present a classical method using the gradient ∇f of f to approximate solutions of the minimization problem

$$\min_{x \in E} f(x). \tag{4.2.4}$$

We consider $\gamma \in (0, +\infty)$ called step size, $x_0 \in E$ and we define the sequence $(x_n)_{n \geq 0}$ by

$$x_{n+1} \quad = \quad x_n - \gamma \nabla f(x_n) \, , \quad n \geq 0 \, . \tag{4.2.5}$$

Denote by T the map from E to E defined by

$$T(x) \quad = \quad x - \gamma \nabla f(x) \quad = \quad (I - \gamma \nabla f)(x)$$

for $x \in E$, the sequence $(x_n)_{n \in \mathbb{N}}$ is given by

$$x_{n+1} = T(x_n) \, , \quad n \geq 0.$$

Without any assumption on ∇f, this algorithm may diverge, especially if γ is taken too large. However, if we suppose that ∇f is Lipschitz continuous, we are able to make this algorithm converge to a minimizer by choosing correctly γ as the following result shows.

Theorem 4.2.10 *Suppose that f is convex differentiable with grandient L-Lipschitz. Supoose also that f has a global minimum at $x^* \in E$. Then for any $x_0 \in E$, t and any $\gamma \in (0, \frac{2}{L})$ the sequence $(x_n)_{n\geq 0}$ defined by (4.2.5) converges to x^*.*

To prove this result, we first need the following lemma. In the sequel, for a given map T from E to E, a point $x \in E$ such that $T(x) = x$ is called a fixed point for the map T

Lemma 4.2.3 *Let T be a map from E to E, 1-Lipschitz continuous and admitting at least one fixed point. Let $x_0 \in E$ and $(x_n)_{n\geq 0}$ the sequence defined by (4.2.5). If $\lim_{n\to+\infty} |x_{n+1} - x_n| = 0$, $(x_n)_{n\geq 0}$ converges to a fixed point of T.*

Proof. Let y be a fixed point for T. Since T is 1-Lipschitz continuous, the sequence $(|x_n - y|)_{n\in\mathbb{N}}$ is nonincreasing and hence bounded. Therefore $(x_n)_{n\in\mathbb{N}}$ is bounded. Since E is finite a dimension space, the sequence $(x_n)_{n\in\mathbb{N}}$ admits a subsequence converging to $z \in E$. Since $\lim_{n\infty} |x_{n+1} - x_n| = 0$, we have $Tz = z$ and z is a fixed point for T. The sequence $|z - x_n|$ is then nonincreasing and admits a subsequence converging to 0, therefore, it converges to 0. □

To prove Theorem 4.2.10, we show that the operator T satisfies the assumptions of Lemma 4.2.3.

Lemma 4.2.4 *Let f be a convex differentiable function whose gradient ∇f is L-Lipschitz continuous. For $\gamma \in (0, \frac{2}{L}]$, the map $T = Id - \gamma\nabla f$ is 1-Lipschitz continuous.*

Proof. Define the operator $S = Id - T = \gamma\nabla f$. We then have

$$|x - y|^2 - |Tx - Ty|^2 \quad = \quad 2\langle Sx - Sy, x - y\rangle - |Sx - Sy|^2.$$

From Proposition 3.2.18 we have

$$\langle Sx - Sy, x - y\rangle \quad \geq \quad \frac{1}{\gamma L}|Sx - Sy|^2.$$

Therefore we get for $\gamma \leq \frac{2}{L}$

$$|x-y|^2 - |Tx - Ty|^2 \ \geq \ 0$$

and T is 1-Lipschitz continuous. $\square$

Proof of Theorem 4.2.10 We first notice that being a minimizer of f is equivalent to being a fixed point for T, since $T(x) = x$ is equivalent to $\nabla f(x) = 0$ and f is convex.

From Lemmata 4.2.3 and 4.2.4, it suffices to prove that $\lim_{n\infty} |x_{n+1} - x_n| = 0$.

For that we have by an application of first order Taylor formula and Lipschitz property of ∇f

$$f(z) \ \leq \ f(y) + \langle \nabla f(y), z - y\rangle + \frac{L}{2}|z-y|^2$$

for $y, z \in E$. We then apply this inequality with $y = x_n$ and $z = x_{n+1}$. Since $x_{n+1} - x_n = -\gamma \nabla f(x_n)$ we have

$$f(x_{n+1}) \ \leq \ f(x_n) - \frac{1}{\gamma}|x_{n+1} - x_n|^2 + \frac{L}{2}|x_{n+1} - x_n|^2 \ .$$

Since $\gamma < \frac{2}{L}$ we get

$$f(x_{n+1}) + \Big(\frac{1}{\gamma} - \frac{L}{2}\Big)|x_{n+1} - x_n|^2 \ \leq \ f(x_n) \ .$$

Hence, the sequence $(f(x_n))_{n\in\mathbb{N}}$ is nondecreasing. As it is lower bounded, it converges and hence

$$\lim_{n\to+\infty} |x_{n+1} - x_n| \ = \ 0 \ .$$

Chapter 5

Duality of convex functions

5.1 Upper envelopes of affine functions

Definition 5.1.16 *An affine function $h : E \to \mathbb{R}$ is a function such that*

$$h(\lambda x + (1-\lambda)y) \;=\; \lambda h(x) + (1-\lambda)h(y)$$

for all $\lambda \in \mathbb{R}$ and $x, y \in E$. Equivalently h is affine if and only if there exists $g : E \to \mathbb{R}$ linear form such that $h = g + h(0)$.

If g is a linear form on E, then obviously $h = g + h(0)$ is affine. If we suppose that h is affine, g defined by $g = h - h(0)$ is linear. Indeed, we first have

$$\begin{aligned} g(\lambda x) &= h(\lambda x + (1-\lambda)0) - h(0) \\ &= \lambda h(x) - \lambda h(0) \\ &= \lambda g(x) \end{aligned}$$

for all $\lambda \in \mathbb{R}$ and all $x \in E$. We then have

$$\begin{aligned} g(x+y) &= g\big(2\frac{x+y}{2}\big) \\ &= 2g\big(\frac{x+y}{2}\big) \\ &= 2h\big(\frac{x+y}{2}\big) - 2h(0) \\ &= 2\frac{h(x)+h(y)}{2} - 2h(0) \\ &= g(x) + g(y)\ . \end{aligned}$$

For a family of function $(f_i)_{i\in I}$ from E to $\mathbb{R}$, the upper envelope is the function $\sup_{i\in I} f_i$.

Theorem 5.1.13 *A function $f: E \to \mathbb{R}\cup\{+\infty\}$ is convex and lsc if and only if it is the upper envelope of affine functions.*

Proof. The upper envelope of a family of affines functions is convex lsc and valued in $\mathbb{R}\cup\{+\infty\}$.

Conversely, let $f : E \to \mathbb{R}\cup\{+\infty\}$ convex and lsc. If f is identically equal to $+\infty$, it is the upper enveloppe of the affine functions $f_n = n$ for $n \geq 1$.

If not, $dom(f) \neq \emptyset$. Let $x_0 \in adh(dom(f))$. For $t \in \mathbb{R}$ such that $t < f(x_0)$, the set $V_f(t) = \{x \in E \ : \ f(x) > t\}$ is open and contains x_0, hence it contains the open ball $B(x_0, r)$ for some $r > 0$. Define $g = t - \chi_{B(x_0,r)}$ where we recall that $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = \infty$ if $x \notin A$ for $A \subset E$. Then g is concave, *i.e.* $-g$ is convex, finite and continuous on $B(x_0, r)$ and upper bounded by f. Since $x_0 \in adh(dom(f))$ we have $dom(f) \cap B(x_0, r) \neq \emptyset$. From Corollary 2.2.4 there exists an affine form h such that $g \leq h \leq f$. Hence h is an affine function dominated by f such that

$$t \ \leq \ h(x_0) \ \ < f(x_0).$$

Let $x_0 \in E \setminus adh(dom(f))$. From Theorem 2.2.5 (separation of a point from a closed convex set), there exists $u^* \in E$ such that

$$\alpha \ := \ \sup_{x\in dom(f)} \langle u^*, x\rangle \ < \ \langle u^*, x_0\rangle \ . \tag{5.1.1}$$

Let $h_{u^*,\alpha}$ be the affine function defined by

$$h_{u^*,\alpha}(x) \ = \ \langle u^*, x\rangle - \alpha$$

for $x \in E$. Then we have

$$h_{u^*,\alpha}(x) \ \leq \ 0$$

for all $x \in dom(f)$ and $h_{u^*,\alpha}(x_0) > 0$. Let h_0 be an affine form such that $h_0 \leq f$ (such h_0 is given by the case $x_0 \in adh(dom(f))$). The sequence of affine functions $(h_n)_{n\geq 1}$ defined by

$$h_n(x) \quad = \quad nh_{u^*,\alpha}(x) + h_0(x)$$

for $x \in E$ and $n \geq 1$. Since $h_0 \leq f$ we get from (5.1.1), that $h_n \leq f$ on E and

$$\lim_{n\to+\infty} h_n(x_0) \quad = \quad +\infty \quad = \quad f(x_0)\ .$$

□

5.2 Fenchel-Moreau conjugate

Definition 5.2.17 (Conjuguate function) *Let $f:\ E \to \mathbb{R}$. Its conjugate function (in the sense of Fenchel-Moreau) is the function $f^*:\ E \to \mathbb{R}$ defined by*

$$f^*(x^*) \quad = \quad \sup_{x\in E}\big\{\langle x^*, x\rangle - f(x)\big\}$$

for all $x^* \in E$.

The supremum of lsc convex functions dominated by f is also lsc and convex. This motivates the following definition.

Definition 5.2.18 (Convex lsc regularization) *The convex lsc regularization of a function $f:\ E \to \mathbb{R}$ is the greatest lsc convex function $\hat{f}\ : E \to \mathbb{R}$ dominated by f.*

Remark 5.2.4 *From the previous definition the functions f and $\hat{f}$ have the same affine lower-bound functions. From Theorem 5.1.13, $\hat{f}$ is the supremum of affine functions dominated by f.*

Proposition 5.2.21 *The conjuguate function f^* of a function f is either a convex lsc function from E to $\mathbb{R} \cup \{+\infty\}$, or identically equal to $-\infty$.*

Proof. If f is identically equal to $+\infty$, its conjuguate function is identically equal to $-\infty$. If not, we can restrict the sup in the definition of f^* to $x \in dom(f)$. Hence, f^* is the upper envelope of continuous affine functions. Hence it is convex lsc and do not take the value $-\infty$. □

Remark 5.2.5 *The value $f^*(x^*)$ is the smallest constant b such that the affine function $h_{x^*,b} = \langle x^*, . \rangle - b$ is dominated by f (with the convention $\inf \emptyset = +\infty$). In particular f^* is the constant $+\infty$ if and only if f does not dominate any affine function. Indeed, $\langle x^*, . \rangle - b \leq f$ if and only if $\langle x^*, x \rangle - f(x) \leq b$ for all $x \in E$, which is equivalent to $f^*(x^*) \leq b$.*

Proposition 5.2.22 *i) $f^*(0) = -\inf_{x \in E} f(x)$.*

ii) $f \leq g \Rightarrow f^* \geq g^*$.

iii) For $\lambda > 0$, we have $(\lambda f)^*(x^*) = \lambda f^*(\frac{x^*}{\lambda})$.

iv) For $\lambda \neq 0$, we have $(D_\lambda f)^* = D_{1/\lambda} f^*$,(where $D_\lambda f$is the dilatation of f defined by $D_\lambda f(x) = f(\lambda x)$).

v) For $\alpha \in \mathbb{R}$, we have $(f + \alpha)^* = f^* - \alpha$.

vi) For $a \in E$, we have $(\tau_a f)^* = f^* + \langle ., a \rangle$ (where $\tau_a f$ is the translation of f defined by $\tau_a f(x) = f(x - a)$).

vii) $(\inf_{i \in I} f_i)^* = \sup_{i \in I} f_i^*$.

viii) $(\sup_{i \in I} f_i)^* \leq \inf_{i \in I} f_i^*$.

Proof. The proof is let as an exercice. □

5.3 Bi-conjugate

Definition 5.3.19 *The biconjugate function of $f : E \to \mathbb{R}$ is defined as the conjugate of the conjugate f^* of f and is denoted by $(f^*)^*$.*

Theorem 5.3.14 (Bi-conjugate function theorem) *The bi-conjugate $(f^*)^*$ is equal to the convexe lsc regularization of f if f is lower bounded by an affine function. If not, $(f^*)^*$ is identically equal to $-\infty$.*

Proof. From Remark 5.2.5, the affine function $h_{x^*,b} = \langle x^*, .\rangle - b$ is dominated by f if and only if $(x^*, b) \in epi(f^*)$. If such a lower bound afine function exists we have from Remark 5.2.4

$$\hat{f}(x) = \sup_{(x^*,b)\in epi(f^*)} h_{x^*,b}(x) = \sup_{x^*\in dom(f^*)} \langle x, x^*\rangle - f^*(x^*) = (f^*)^*(x)$$

for all $x \in E$. □

Corollary 5.3.9 *Let $f : E \to \mathbb{R}$ a convex proper lsc function. We then have $(f^*)^* = f$. The equality also holds if f is identically equal to $-\infty$.*

Proof. If $f : E \to \mathbb{R}$ is convex proper lsc, it dominates an affine function and we have $\hat{f} = f$ and the result follows from the previous theorem. □

Chapter 6

Diffusion processes

6.1 SDEs with random coefficients

We recall in this section the basic results for stochastic differential equations

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t, \quad t \in [0,T]. \tag{6.1.1}$$

Here, b and σ are $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}^n)$-progressively measurable functions from $[0,T] \times \Omega \times \mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$ respectively. In particular, for every fixed $x \in \mathbb{R}^n$, the processes $\{b_t(x), \sigma_t(x), t \in [0,T]\}$ are $\mathbb{F}$-progressively measurable.

Definition 6.1.1 *A strong solution of (6.1.1) is an $\mathbb{F}$-progressively measurable process X such that $\int_0^T(|b_t(X_t)| + |\sigma_t(X_t)|^2)dt < \infty$ a.s. and*

$$X_t = X_0 + \int_0^t b_s(X_s)ds + \int_0^t \sigma_s(X_s)dW_s, \quad t \in [0,T].$$

The main existence and uniqueness result is the following.

Theorem 6.1.1 *Let X_0 be a square integrable r.v. independent of W. Assume that the processes $b.(0)$ and $\sigma.(0)$ are in $\mathbb{H}^2$, and there exists a constant $L > 0$ such that*

$$|b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| \quad \leq \quad L|x-y| \tag{6.1.2}$$

for all $t \in [0,T]$ and $x, y \in \mathbb{R}^n$.

Then there exists a unique strong solution of (6.1.1) *in* $\mathbb{H}^2$. *Moreover, we have*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t|^2\Big] \quad\leq\quad C(1+\mathbb{E}[|X_0|^2]) \tag{6.1.3}$$

for some constant $C = C(T,L)$ *depending only on* T *and* L.

Proof. We proceed in two steps. We first prove the existence and uniqueness of a solution to the SDE and then establish (6.1.3).

Step 1. Let c be a positive constant to be fixed later. We define the norm $\|.\|_{\mathbb{H}^2_c}$ by

$$\|\phi\|_{\mathbb{H}^2_c} \quad=\quad \mathbb{E}\Big[\int_0^T e^{-ct}|\phi_s|^2ds\Big]^{1/2}$$

for $\phi\in\mathbb{H}^2$. We notice that the norms $\|.\|_{\mathbb{H}^2_c}$ and $\|.\|_{\mathbb{H}^2}$ are equivalent on $\mathbb{H}^2$. Consider now the map $U:\ \mathbb{H}^2\rightarrow\mathbb{H}^2$ defined by

$$U(Y)_t \quad=\quad X_0+\int_0^t b_s(Y_s)ds+\int_0^t\sigma_s(Y_s)dW_s$$

From the Lipschitz property of b and σ and since $b.(0),\sigma.(0)\in\mathbb{H}^2$, the map U is well defined on $\mathbb{H}^2$. We next notice that X is solution to (6.1.1) if and only if X is fixed point of U. Therefore, it is sufficient to prove that U is a contraction for some convenient c to get the result. For $Y,Z\in\mathbb{H}^2$, we have

$$\mathbb{E}\Big[|U(Y)_t-U(Z)_t|^2\Big] \quad=$$

$$\mathbb{E}\Big[\Big|\int_0^t\big(b_s(Y_s)-b_s(Z_s)\big)ds+\int_0^t\big(\sigma_s(Y_s)-\sigma_s(Z_s)\big)dW_s\Big|^2\Big].$$

From Young's inequality we get

$$\begin{aligned}\mathbb{E}\Big[|U(Y)_t-U(Z)_t|^2\Big] \quad&\leq\quad 2\mathbb{E}\Big[\Big|\int_0^t\big(b_s(Y_s)-b_s(Z_s)\big)ds\Big|^2\Big]\\ &\quad+2\mathbb{E}\Big[\Big|\int_0^t\big(\sigma_s(Y_s)-\sigma_s(Z_s)\big)dW_s\Big|^2\Big].\end{aligned}$$

Using the Lipschitz properties of b and σ, Jensen's inequality and the Ito Isometry, we get

$$\mathbb{E}\Big[|U(Y)_t - U(Z)_t|^2\Big] \quad \leq \quad 2(T+1)L\mathbb{E}\left[\int_0^t \big|Y_s - Z_s\big|^2 ds\right].$$

Therefore we get

$$\|U(Y) - U(Z)\|_{\mathbb{H}^2_c} \quad \leq \quad \frac{2(T+1)L}{c}\|Y - Z\|_{\mathbb{H}^2_c}$$

and U is a $\|.\|_{\mathbb{H}^2_c}$-contraction for c large enough.

Step 2. We now prove (6.1.3). We have from Young and Jensen inequalities

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s|^2\Big] \quad &\leq \quad \mathbb{E}\Big[\sup_{s\in[0,t]}|X_0 + \int_0^s b_u(X_u)du + \int_0^s \sigma_u(X_u)dW_u|^2\Big] \\
&\leq \quad 3\left(\mathbb{E}\big[|X_0|^2\big] + t\int_0^t \mathbb{E}\big[|b_u(X_u)|^2\big]du + \mathbb{E}\Big[\sup_{s\in[0,t]}\Big|\int_0^s \sigma_u(X_u)dW_u\Big|^2\Big]\right).
\end{aligned}$$

Using Doob's maximal inequality, we get

$$\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s|^2\Big] \quad \leq \quad 3\left(\mathbb{E}\big[|X_0|^2\big] + t\int_0^t \mathbb{E}\big[|b_u(X_u)|^2\big]du + 4\mathbb{E}\Big[\int_0^t |\sigma_u(X_u)|^2du\Big]\right).$$

From Lipschitz properties of b and σ, we get a constant $C(T, L)$ such that

$$\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s|^2\Big] \quad \leq \quad C(T,K)\left(1 + \mathbb{E}\big[|X_0|^2\big] + \int_0^t \mathbb{E}\Big[\sup_{s\in[0,u]}|X_s|^2\Big]du\right),$$

and we get the result from Gronwall's lemma. □

The previous result can be easily extended to any initial time $t \in [0, T]$ instead of 0. In the sequel, we shall denote by $X^{t,x} = \{X^{t,x}_s,\ t \leq s \leq T\}$ the process solution to

$$X_s = x + \int_t^s b_u(X_u)du + \int_t^s \sigma_u(X_u)dW_u, \quad u \in [t, T],$$

for $t \in [0, T]$ and $x \in \mathbb{R}^n$.

In addition to the estimate (6.1.3) of Theorem 6.1.1, we have the following flow continuity results of the solution of the SDE.

Theorem 6.1.2 *Suppose that assumptions of Theorem 6.1.1 hold.*
(i) There exists a constant $C > 0$ such that

$$\mathbb{E}\Big[\sup_{t\le s\le t'}|X_s^{t,x} - X_s^{t,x'}|^2\Big] \;\le\; Ce^{Ct'}|x-x'|^2$$

for $t, t' \in [0,T]$ such that $t \le t'$ and $x, x' \in \mathbb{R}^n$.
(ii) Moreover, if we have

$$B \;:=\; \sup_{0\le s<s'\le T}(s'-s)^{-1}\mathbb{E}\int_s^{s'}\big(|b_r(0)|^2 + |\sigma_r(0)|^2\big)dr \;<\; +\infty$$

then, there exists a constant $C > 0$ such that

$$\mathbb{E}\Big[\sup_{t'\le s\le T}|X_s^{t,x} - X_s^{t',x}|^2\Big] \;\le\; Ce^{CT}(B+|x|^2)(t'-t)$$

for $t, t' \in [0,T]$ such that $t \le t'$ and $x \in \mathbb{R}^n$.

Proof. (i) To simplify notation we set $\delta x = x - x'$, $\delta X = X^{t,x} - X^{t,x'}$, $\delta b = b(X^{t,x}) - b(X^{t,x'})$ and $\delta\sigma = \sigma(X^{t,x}) - \sigma(X^{t,x'})$. We next have from Young and Jensen inequalities

$$|\delta X_s|^2 \;\le\; 3\left(|\delta x|^2 + (s-t)\int_t^s|\delta b_s|^2ds + \Big|\int_t^s\delta\sigma_s dW_s\Big|^2\right)$$

for $s \in [t,T]$. Using Doob's maximal inequality and the Lipschitz properties of b and σ we get

$$\mathbb{E}\Big[\sup_{t\le s\le t'}|X_s^{t,x} - X_s^{t,x'}|^2\Big] \;\le\; 3\left(|\delta x|^2 + L^2(t'+4)\int_t^{t'}\mathbb{E}\Big[\sup_{t\le s\le u}|X_s^{t,x} - X_s^{t,x'}|^2\Big]du\right)$$

for $t' \in [t,T]$. The result follows from Gronwall's Lemma.

(ii) We set $X = X^{t,x}$, $X' = X^{t',x}$, $\delta t = t'-t$, $\delta X = X'-X$, $\delta b = b(X')-b(X)$ and $\delta\sigma = \sigma(X') - \sigma(X)$. Following the same arguments as in (i) we get

$$\mathbb{E}\Big[\sup_{t'\le s\le u}|\delta X_s|^2\Big] \;\le\; 3\left(\mathbb{E}|\delta X_{t'}|^2 + L^2(T+4)\int_{t'}^u\mathbb{E}\Big[\sup_{t\le s\le r}|\delta X_s|^2\Big]dr\right) \quad (6.1.4)$$

for $u \in [t', T]$. We now concentrate on the first term of the left hand side. We have by the same arguments

$$\begin{aligned}
\mathbb{E}|\delta X_{t'}|^2 &= \mathbb{E}|X_{t'}^{t,x} - x|^2 \\
&\leq 2\Big(T\int_t^{t'} \mathbb{E}|b(X_u^{t,x})|^2 du + \int_t^{t'} \mathbb{E}|\sigma(X_u^{t,x})|^2 du\Big) \\
&\leq 6(T+1)\Big(\int_t^{t'} \big(L^2\mathbb{E}|X_u^{t,x} - x|^2 + L^2|x|^2 + \mathbb{E}[|b_u(0)|^2 + |\sigma_u(0)|^2]\big)du \\
&\leq 6(T+1)\left((t'-t)(L^2|x|^2 + B) + L^2\int_t^{t'} \mathbb{E}|X_u^{t,x} - x|^2 du\right).
\end{aligned}$$

By Gronwall's lemma we get

$$\mathbb{E}|\delta X_{t'}|^2 \leq C(t'-t)(|x|^2 + B)e^{C(t'-t)}.$$

Pugging this last inequality in (6.1.4), we get the result using Gronwall's lemma another time. □

6.2 Markov SDEs

In this section, we restrict the coefficients b and σ to be deterministic functions of (t, x). In this context, we suppose that b and σ are continuous functions Lipschitz in x uniformly in t, that is (6.1.2) holds true. We recall that $X^{t,x}$ denotes the solution of the stochastic differential equation

$$X_s = x + \int_t^s b_u(X_u)du + \int_t^s \sigma_u(X_u)dW_u, \quad u \in [t, T],$$

for $t \in [0, T]$ and $x \in \mathbb{R}^n$. We then have the two following properties.

- From the pathwise uniqueness, we have the flow property:

  $$X_s^{t,x} = X_s^{u, X_u^{t,x}}$$

  for $t \leq u \leq s$. We notice that this flow property also holds for u being a stopping time valued in $[t, s]$.

- Since $X^{t,x}$ is adapted to the filtration generated by the process $\{W_s - W_t,\ s \geq t\}$ we get from the previous flow continuity and Doob's representation Theorem the existence of a measurable function F such that

$$X_s^{t,x} \quad = \quad F(t, x, s, W_u - W_t,\ u \in [t, s])$$

for $s \leq T$.

Proposition 6.2.1 (Markov property) *We have*

$$\mathbb{E}\big[\Phi(X_r^{t,x}, u \leq r \leq s)|\mathcal{F}_u\big] \quad = \quad \mathbb{E}\big[\Phi(X_r^{t,x}, u \leq r \leq s)|X_u^{t,x}\big]$$

for any $u \in [t, s]$ and any measurable bounded (or nonnegative) function $\Phi : C([u, s]) \to \mathbb{R}$.

6.3 Connection with PDEs

Let $X^{t,x}$ be the unique solution to the SDE

$$X_s = x + \int_t^s b_u(X_u)du + \int_t^s \sigma_u(X_u)dW_u, \quad s \in [t, T], \tag{6.3.5}$$

for $t \in [0, T]$ and $x \in \mathbb{R}^n$. We next define the operator $\mathcal{L}$ by

$$\mathcal{L}\varphi(t, x) \quad = \quad \lim_{h \to 0} \frac{\mathbb{E}[\varphi(X_{t+h}^{t,x})] - \varphi(x)}{h}$$

for any function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\mathcal{L}\varphi$ is well defined. $\mathcal{L}$ is called the generator of the diffusion. From Ito's formula, $\mathcal{L}\varphi$ is well defined for any φ bounded C^2 with bounded derivatives and we have

$$\mathcal{L}\varphi(t, x) \quad = \quad b_t(x).\nabla\varphi(x) + \frac{1}{2}\mathrm{Tr}(\sigma_t(x)\sigma_t(x)^\top \nabla^2\varphi(x))$$

for $t \in [0, T]$ and $x \in \mathbb{R}^n$. The generator provides a connection between diffusion processes and linear partial differential equations.

Proposition 6.3.2 *Suppose that the function v defined by*

$$v(t,x) \quad = \quad \mathbb{E}[g(X_T^{t,x})] \;, \quad (t,x) \in [0,T] \times \mathbb{R}^n \;,$$

is in $C^{1,2}([0,T] \times \mathbb{R}^n)$. Then, v solves the partial differential equation:

$$\partial_t v + \mathcal{L}v \quad = \quad 0 \text{ on } [0,T) \times \mathbb{R}^n$$

with terminal condition

$$v(T,.) \quad = \quad g \text{ on } \mathbb{R}^n \;.$$

Proof. Fix $(t,x) \in [0,T] \times \mathbb{R}^n$ and let $\tau := T \wedge \inf\{s > t : |X_s^{t,x} - x| \geq 1\}$. By the tower property and the Markov property of $X^{t,x}$ we have

$$v(t,x) \quad = \quad \mathbb{E}\big[v(s \wedge \tau, X_{s\wedge\tau}^{t,x})\big]$$

for any $s \in [t,T]$. Since $v \in C^{1,2}([0,T] \times \mathbb{R}^n)$ we have from Ito's formula

$$\begin{aligned} 0 \quad &= \quad \mathbb{E}\Big[\int_t^{s\wedge\tau} \big(\partial_t v + \mathcal{L}v\big)(u, X_u^{t,x})du\Big] + \mathbb{E}\Big[\int_t^{s\wedge\tau} \nabla v.\sigma(u, X_u^{t,x})dW_u\Big] \\ &= \quad \mathbb{E}\Big[\int_t^{s\wedge\tau} \big(\partial_t v + \mathcal{L}v\big)(u, X_u^{t,x})du\Big] \end{aligned}$$

where the last inequality comes from the boundedness of $X^{t,x}$ on $[t,\tau]$. We now take $s = t + h$ with $h > 0$ and we get

$$0 \quad = \quad \mathbb{E}\Big[\frac{1}{h}\int_t^{(t+h)\wedge\tau} \big(\partial_t v + \mathcal{L}v\big)(u, X_u^{t,x})du\Big] \;.$$

From the mean value theorem and the dominated convergence theorem, we get the result by sending h to $0+$. □

Feynman-Kac representation of Cauchy problem We consider the following linear partial differential equation called Cauchy problem

$$\begin{cases} \partial_t v + \mathcal{L}v - kv + f = 0, & \text{on } [0,T) \times \mathbb{R}^n, \\ v(T,.) = g \;, & \text{on } \mathbb{R}^n \;. \end{cases} \tag{6.3.6}$$

with k and f two functions from $[0,T] \times \mathbb{R}^n$ to $\mathbb{R}$. The next results provides a representation of this purely deterministic problem by means of stochastic differential equations.

Theorem 6.3.3 *Assume that the coefficients b and σ satisfy the assumptions of Theorem 6.1.1. Assume further that the function k is uniformly lower bounded, and f has quadratic growth in x uniformly in t. Let $v \in C^{1,2}([0,T)\times \mathbb{R}^n) \cap C^0([0,T] \times \mathbb{R}^n)$ be a solution of (6.3.6) with quadratic growth in x uniformly in t. Then*

$$v(t,x) \quad = \quad \mathbb{E}\Big[\int_t^T \beta_s^{t,x} f(s,X_s^{t,x})ds + \beta_T^{t,x} g(X_T^{t,x})\Big]$$

where $X^{t,x}$ is the unique solution to (6.3.5) and

$$\beta_s^{t,x} \quad = \quad e^{-\int_t^s k(u,X_u^{t,x})du}\ , \qquad s \in [t,T]\ ,$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$.

Proof. We introduce the sequence of stopping times $(\tau_n)_n$ defined by

$$\tau_n \quad = \quad \Big(T - \frac{1}{n}\Big) \wedge \inf\{s > t\ :\ |X_s^{t,x} - x|\ \geq\ n\}\ , \qquad n \geq 1\ .$$

We notice that that $\tau_n \uparrow T-$ $\mathbb{P}$-a.s. as $n \uparrow +\infty$. We then have from Ito's formula

$$\begin{aligned} d\big(\beta_s^{t,x} v(s,X_s^{t,x})\big) \quad &= \quad \beta_s^{t,x}(\partial_t v + \mathcal{L}v - kv)(s,X_s^{t,x})ds \\ &\quad + \beta_s^{t,x} \nabla v.\sigma(s,X_s^{t,x})dW_s\ , \end{aligned}$$

for $s \in [t,T)$. Since v is a solution of (6.3.6), we get

$$d\big(\beta_s^{t,x} v(s,X_s^{t,x})\big) \quad = \quad -\beta_s^{t,x} f(s,X_s^{t,x})ds + \beta_s^{t,x}\nabla v.\sigma(s,X_s^{t,x})dW_s\ .$$

Therefore, we get

$$\begin{aligned} v(t,x) - \mathbb{E}\big[\beta_{\tau_n}^{t,x} v(\tau_n, X_{\tau_n}^{t,x})\big] \quad &= \quad \mathbb{E}\Big[\int_t^{\tau_n} \beta_s^{t,x} f(s,X_s^{t,x})ds\Big] \\ &\quad -\mathbb{E}\Big[\int_t^{\tau_n} \beta_s^{t,x}\nabla v.\sigma(s,X_s^{t,x})dW_s\Big]\ . \end{aligned}$$

We then notice that the expectation of the stochastic integral term vanishes as the stochastic integrand is bounded since the function v is continuous, the function σ and satisfies (6.1.2) and the function k is lower bounded. Therefore, we get

$$v(t,x) \quad = \quad \mathbb{E}\left[\beta^{t,x}_{\tau_n} v(\tau_n, X^{t,x}_{\tau_n}) + \int_t^{\tau_n} \beta^{t,x}_s f(s, X^{t,x}_s) ds\right] . \qquad (6.3.7)$$

Using the continuity of v, the terminal condition of (6.3.6) and the continuity of $X^{t,x}$ we have

$$\beta^{t,x}_{\tau_n} v(\tau_n, X^{t,x}_{\tau_n}) + \int_t^{\tau_n} \beta^{t,x}_s f(s, X^{t,x}_s) ds \xrightarrow[n\to+\infty]{\mathbb{P}-a.s.} \beta^{t,x}_T g(X^{t,x}_T) + \int_t^T \beta^{t,x}_s f(s, X^{t,x}_s) ds .$$

Moreover, since k is lower bounded and the functions f and v have quadratic growth in x uniformly in t, there exists a constant C such that

$$\left|\beta^{t,x}_{\tau_n} v(\tau_n, X^{t,x}_{\tau_n}) + \int_t^{\tau_n} \beta^{t,x}_s f(s, X^{t,x}_s) ds\right| \quad \leq \quad C\Big(1 + \sup_{s\in[t,T]} |X^{t,x}_s|^2\Big) ,$$

for any $n \geq 1$. We can then apply the dominated convergence theorem and we get the result by sending n to ∞ in (6.3.7). $\square$

Chapter 7

Optimal control of diffusion processes and dynamic programming

7.1 Stochastic control problem

To define the control problem, we introduce the following elements.

Set of controls We fix a subset A of some $\mathbb{R}^p$ for $p \geq 1$. We suppose that A is bounded and we denote by $\mathcal{A}$ the set of $\mathbb{F}$-progressive processes $(\alpha_t)_{t\in[0,T}$ valued in A.

Controlled diffusion process We fix two functions $b, \sigma : \mathbb{R}^n \times A \to \mathbb{R}^n, \mathbb{R}^{d\times n}$. We suppose that b and σ are continuous and there exists a constant L such that

$$|b(x,a) - b(x',a')| + |\sigma(x,a) - \sigma(x',a')| \leq L\big(|x-x'| + |a-a'|\big) ,$$

for all $(x,a),(x',a') \in \mathbb{R}^n \times A$. From Theorem 6.1.1, we have existence and uniqueness of the controlled process $X^{t,x,\alpha}$ defined by

$$X_s^{t,x,\alpha} = x + \int_t^s b(X_u^{t,x,\alpha}, \alpha_u)du + \int_t^s \sigma(X_u^{t,x,\alpha}, \alpha_u)dW_u \qquad (7.1.1)$$

for any $(t, x) \in [0, T] \times \mathbb{R}^n$ and any control process $\alpha \in \mathcal{A}$. Following the same arguments as in the proof of Theorem 6.1.2, we get the following result.

Proposition 7.1.3 *For $p \geq 1$, there exists a constant C_p such that*

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[t,T]} |X_s^{t,x,\alpha}|^p\Big] &\leq C_p(1 + |x|^p) , \\
\mathbb{E}\Big[\sup_{s\leq t\leq T} |X_s^{t,x,\alpha} - X_s^{t,x',\alpha}|^p\Big] &\leq C_p|x - x'|^p , \\
\mathbb{E}\Big[\sup_{t\leq s\leq T} |X_s^{t,x,\alpha} - X_{s\vee(t+h)}^{t+h,x,\alpha}|^p\Big] &\leq C_p h^{\frac{p}{2}}\big(1 + |x|^p\big) , \\
\mathbb{E}\Big[\sup_{s\leq t\leq T} |X_s^{t,x,\alpha} - X_s^{t,x,\alpha'}|^p\Big] &\leq C_p\mathbb{E}\Big[\int_t^T |\alpha_s - \alpha'_s|^p ds\Big] ,
\end{aligned}$$

for all $t \in [0, T]$, $h \in [0, T - t]$, $x, x' \in \mathbb{R}^n$ and $\alpha, \alpha' \in \mathcal{A}$.

Reward functions and gain We fix two reward functions $f : \mathbb{R}^n \times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$. We assume that f and g are locally Lipschitz continuous, that is, for any $N > 0$, there exists a constant L_N such that

$$|f(x, a) - f(x', a')| + |g(x) - g(x')| \leq L_N\big(|x - x'| + |a - a'|\big)$$

for any $(x, a), (x', a') \in \mathbb{R}^n \times A$ such that $|x| \leq N$ and $|x'| \leq N$. We also assume that f and g have a polynomial growth, that is, there exist a constant C and an integer p such that

$$|f(x, a)| + |g(x)| \leq C\big(1 + |x|^p\big)$$

for all $(x, a) \in \mathbb{R}^n \times A$. We next define the gain functional $J : [0, T] \times \mathbb{R}^n \times \mathcal{A} \to \mathbb{R}$ by

$$J(t, x, \alpha) = \mathbb{E}\Big[g(X_T^{t,x,\alpha}) + \int_t^T f(X_s^{t,x,\alpha}, \alpha_s)ds\Big]$$

for $(t, x, \alpha) \in [0, T] \times \mathbb{R}^n \times \mathcal{A}$. We observe that $J(t, x, \alpha)$ is well defined under the polynomial growth assumption from Proposition 7.1.3.

We now define the value function v of the considered stochastic control problem by

$$v(t, x) = \sup_{\alpha\in\mathcal{A}_t} J(t, x, \alpha) \tag{7.1.2}$$

for $(t,x) \in [0,T] \times \mathbb{R}^n$ where

$$\mathcal{A}_t = \{\alpha \in \mathcal{A} \ : \ \alpha \text{ independent of } \mathcal{F}_t\} , \quad t \in [0,T] .$$

Our goal is to characterize the function v in terms of a partial differential equation called the Hamilton-Jacobi-Bellman (HJB) equation.

Remark 7.1.1 *It can be proved that v satisfies*

$$v(t,x) = \sup_{\alpha \in \mathcal{A}} J(t,x,\alpha)$$

for $(t,x) \in [0,T] \times \mathbb{R}^n$. Indeed, as the reward $J(t,x,.)$ involves only randomness independent of $\mathcal{F}_t$, one can show by a conditioning property that a control in $\mathcal{A}$ provides the same reward as a control in $\mathcal{A}_t$. This property is admitted.

7.2 Dynamic programming principle

To state the dynamic programming principle, we first provide a regularity result on the function v.

Proposition 7.2.4 *For a compact set $\Theta \subset [0,T] \times \mathbb{R}^n$ there exists a real map $\lambda_\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lambda_\Theta(r) \to 0$ as $r \to 0$ and*

$$|J(t,x,\alpha) - J(s,y,\alpha)| \leq \lambda_\Theta\big(|t-s| + |x-y|\big) \tag{7.2.3}$$

for all $(t,x),(s,y) \in \Theta$ and $\alpha \in \mathcal{A}$. The value function v is locally uniformly continuous and has polynomial growth.

Proof. Since f and g are locally Lipschitz continuous with polynomial growth, we get from Proposition 7.1.3 that there exist a constant C, an integer p such that for each N, there exists a constant C_N such that

$$\begin{aligned} & |J(t,x,\alpha) - J(s,y,\alpha)| \\ \leq \ & C_N|t-s|^{\frac{1}{2}}(1+|x|) + L_N\big(|x-y| + |s-t|\big) \\ & +C(1+|x|^p + |y|^p)\mathbb{P}\Big(\sup_{u\in[t,T]} \min\{|X_u^{t,x,\alpha}|, |X_{u\vee s}^{s,y,\alpha}|\} \geq N\Big) , \end{aligned}$$

for any $t, s \in [0,T]$ $t \le s$, any $x, y \in \mathbb{R}^n$ such that $(t,x),(s,y) \in \Theta$ and any $\alpha \in \mathcal{A}$. Using Markov inequality, we get

$$\begin{aligned}
& \sup_{\alpha\in\mathcal{A}} |J(t,x,\alpha) - J(s,y,\alpha)| \\
\le\ & C_N|t-s|^{\frac{1}{2}}(1+|x|) + L_N\big(|x-y| + |s-t|\big) \\
& +C(1+|x|^p+|y|^p)/N\ .
\end{aligned}$$

which gives 7.2.3 and the local uniform continuity of v from Remark 7.1.1. The polynomial growth of v follows from the polynomial growth of f and g and Proposition 7.1.3. □

We can now state the DPP.

Theorem 7.2.4 (Dynamic programming principle) *The value function v satisifies*

$$v(t,x) \ = \ \sup_{\alpha\in\mathcal{A}_t} \mathbb{E}\Big[v\big(\theta^\alpha, X^{t,x,\alpha}_{\theta^\alpha}\big) + \int_t^{\theta^\alpha} f(X_s^{t,x,\alpha},\alpha_s)ds\Big]$$

for any $(t,x) \in [0,T]\times\mathbb{R}^n$ and any family of stopping times $\{\theta^\alpha,\ \alpha\in\mathcal{A}\}$ valued in $[t,T]$.

Proof. To alleviate notations, we omit the dependence of θ in α. We proceed in two steps by proving that each term of the equality is greater than the other.

Step 1. Fix $(t,x)\in[0,T]\times\mathbb{R}^n$ and $\alpha\in\mathcal{A}_t$. By the conditioning property for controlled diffusions, we can find for $\mathbb{P}$-a.a. $\omega\in\Omega$ a control $\alpha^\omega\in\mathcal{A}_{\theta(\omega)}$ such that

$$\begin{aligned}
& \mathbb{E}\Big[g(X_T^{t,x,\alpha}) + \int_t^T f(X_s^{t,x,\alpha},\alpha_s)ds\Big|\mathcal{F}_\theta\Big](\omega) \\
=\ & J(\theta(\omega), X^{t,x,\alpha}_{\theta(\omega)}(\omega),\alpha^\omega) + \int_t^{\theta(\omega)} f(X_s^{t,x,\alpha},\alpha_s)ds\ .
\end{aligned}$$

Since $J\le v$, we get from the tower property

$$v(t,x) \ \le \ \sup_{\alpha\in\mathcal{A}_t}\mathbb{E}\Big[v(\theta,X_\theta^{t,x,\alpha}) + \int_t^\theta f(X_s^{t,x,\alpha},\alpha_s)ds\Big]\ .$$

Step 2. We now prove the reverse inequality. Fix $(t_0, x_0) \in [0,T) \times \mathbb{R}^n$, $\alpha \in \mathcal{A}_{t_0}$, a ball $B_0 = B((t_0,x_0),r)$ of radius r centered on (t_0,x_0) and fix a compact subset Θ of $[0,T]\times\mathbb{R}^n$ such that $B_0 \subset \Theta$. Let $(B_n)_{n\geq 1}$ be a partition of Θ and $(t_n,x_n)_{n\geq 1}$ be a sequence such that $(t_n,x_n) \in B_n$ for each $n \geq 1$. By definition, for each $n \geq 1$, we can find $\alpha^n \in \mathcal{A}_{t_n}$ such that

$$J(t_n, x_n, \alpha^n) \quad \geq \quad v(t_n, x_n) - \varepsilon \tag{7.2.4}$$

with $\varepsilon > 0$. Moreover, from the local uniform continuity of $J(.,\alpha)$ and v, we can chose $(t_n, x_n, B_n)_{n\geq 1}$ such that

$$B_n \quad \subset \quad [t_n - \eta, t_n] \times B(x_n, \eta)$$

for some $\eta > 0$ and

$$|v(.) - v(t_n,x_n)| + |J(.,\alpha_n) - J(t_n,x_n,\alpha_n)| \quad \leq \quad \varepsilon \text{ on } B_n\ . \tag{7.2.5}$$

Let us now define the stopping time

$$\vartheta \quad = \quad \inf\{s \in [t_0,T]\ :\ (s, X_s^{t_0,x_0,\alpha}) \notin B_0\} \wedge \theta$$

where θ is a given stopping time valued in $[t_0,T]$. We next define the control $\bar{\alpha}$ by

$$\bar{\alpha}_t \quad = \quad \alpha_t \mathbb{1}_{t<\vartheta} + \mathbb{1}_{t\geq\vartheta}\Big(\sum_{n\geq 1} \alpha_t^n \mathbb{1}_{(\vartheta, X_\vartheta^{t_0,x_0,\alpha})\in B_n}\Big)\ , \quad t \geq 0\ .$$

Since α^n is independent of $\mathcal{F}_{t_n}$ for each $n \geq 1$, we get from (7.2.4) and (7.2.5)

$$\begin{aligned}
J(t_0,x_0,\bar{\alpha}) \quad &\geq \quad \mathbb{E}\Big[J(\vartheta, X_\vartheta^{t_0,x_0,\bar{\alpha}}, \bar{\alpha}) + \int_{t_0}^{\vartheta} f(X_s^{t_0,x_0,\bar{\alpha}}, \bar{\alpha}_s)ds\Big] \\
&\geq \quad \mathbb{E}\Bigg[\sum_{n\geq 1}\Big(J(t_n,x_n,\alpha^n) - \varepsilon + \int_{t_0}^{\vartheta} f(X_s^{t_0,x_0,\alpha}, \alpha_s)ds\Big)\mathbb{1}_{(\vartheta, X_\vartheta^{t_0,x_0,\alpha})\in B_n}\Bigg] \\
&\geq \quad \mathbb{E}\Bigg[\sum_{n\geq 1}\Big(v(t_n,x_n) - 2\varepsilon + \int_{t_0}^{\vartheta} f(X_s^{t_0,x_0,\alpha}, \alpha_s)ds\Big)\mathbb{1}_{(\vartheta, X_\vartheta^{t_0,x_0,\alpha})\in B_n}\Bigg] \\
&\geq \quad \mathbb{E}\Big[v(\vartheta, X_\vartheta^{t_0,x_0,\alpha}) + \int_{t_0}^{\vartheta} f(X_s^{t_0,x_0,\alpha}, \alpha_s)ds\Big] - 3\varepsilon\ .
\end{aligned}$$

Therefore, we get

$$v(t_0,x_0) \;\; \geq \;\; \mathbb{E}\Big[v(\vartheta, X_\vartheta^{t_0,x_0,\alpha}) + \int_{t_0}^{\vartheta} f(X_s^{t_0,x_0,\alpha},\alpha_s)ds\Big] \; .$$

Letting $r \to +\infty$, we get $\vartheta \to \theta$ and by dominated convergence, we get

$$v(t_0,x_0) \;\; \geq \;\; \mathbb{E}\Big[v(\theta, X_\theta^{t_0,x_0,\alpha}) + \int_{t_0}^{\theta} f(X_s^{t_0,x_0,\alpha},\alpha_s)ds\Big] \; .$$

□

7.3 Dynamic programming equation

We prove in this section that, if v is smooth enough, it solves a PDE called the Hamilton-Jacobi-Bellman (HJB) equation. More precisely, define the second order local operator $\mathcal{L}^a$, for $a \in A$, by

$$\mathcal{L}^a\varphi(t,x) \;\; = \;\; \partial_t\varphi(t,x) + b(x,a).\nabla\varphi(t,x) + \frac{1}{2}\mathrm{Tr}\big(\sigma(x,a)\sigma^\top(x,a)\nabla^2\varphi(t,x)\big)$$

for $(t,x) \in [0,T) \times \mathbb{R}^n$ and $\varphi \in C^{1,2}([0,T)\times\mathbb{R}^n)$. Then, the HJB equation takes the following form

$$\sup_{a\in A}\big\{\mathcal{L}^a v(t,x) + f(x,a)\big\} \;\; = \;\; 0 \; , \quad (t,x) \in [0,T) \times \mathbb{R}^n \qquad (7.3.6)$$

together with the terminal condition

$$v(T,x) \;\; = \;\; g(x) \; , \quad x \in \mathbb{R}^n \; . \qquad (7.3.7)$$

To simplify notations, we denote by $\mathcal{H}$ the operator defined by

$$\mathcal{H}\varphi(t,x) \;\; = \;\; \sup_{a\in A}\big\{\mathcal{L}^a\varphi(t,x) + f(x,a)\big\}$$

for $(t,x) \in [0,T) \times \mathbb{R}^n$ and $\varphi \in C^{1,2}([0,T)\times\mathbb{R}^n)$.

Theorem 7.3.5 *Suppose that $v \in C^0([0,T]\times\mathbb{R}^n) \cap C^{1,2}([0,T)\times\mathbb{R}^n)$. Then, v is a solution to* (7.3.6)-(7.3.7).

Proof. Fix $(t, x) \in [0, T) \times \mathbb{R}^d$ and assume that $\mathcal{H}v(t, x) \neq 0$. We then distinguish two cases and work toward a contradiction in each of them.

Case 1: $\mathcal{H}v(t, x) > 0$. Let $a \in A$ such that

$$\mathcal{L}^a\varphi(t, x) + f(x, a) \quad > \quad 0 \; .$$

By continuity of the involved functions, there exists a compact neighborhood $V \subset [0, T) \times \mathbb{R}^n$ of (t, x) and $\eta > 0$ such that

$$\mathcal{L}^a\varphi(.) + f(., a) \quad > \quad \eta \text{ on } V \; . \tag{7.3.8}$$

Let $\alpha = a$, be the constant control of $\mathcal{A}$ equal to a, and θ be the first exit time of $(s, X_s^{t,x,\alpha})$ from V. From Itô's formula we have

$$\begin{aligned} & \mathbb{E}\Big[v\big(\theta, X_\theta^{t,x,\alpha}\big) + \int_t^\theta f(X_s^{t,x,\alpha}, \alpha_s)ds\Big] \\ = \quad & v(t, x) + \mathbb{E}\Big[\int_t^\theta \big(\mathcal{L}^a v(s, X_s^{t,x,\alpha}) + f(X_s^{t,x,\alpha}, a)\big)ds\Big] \\ \geq \quad & v(t, x) + \eta\mathbb{E}[\theta] \; . \end{aligned}$$

In view of (7.3.8) and since $\theta > 0$ a.s., this contradict the DPP.

Case 2: $\mathcal{H}v(t, x) < 0$. Still using the continuity of the involved functions, this implies

$$\mathcal{H}v \quad < \quad 0$$

on $V := B((t, x), r) \subset [0, T) \times \mathbb{R}^n$. Moreover, for r small enough, we also have

$$\mathcal{H}w \quad \leq \quad 0 \text{ on } V \tag{7.3.9}$$

where the function w is defined by

$$w(s, y) \quad = \quad v(s, y) + (s - t)^2 + |y - x|^2 \; , \quad (s, y) \in [0, T) \times \mathbb{R}^n \; .$$

For $\alpha \in \mathcal{A}_t$, let θ be the first exit time of $(s, X_s^{t,x,\alpha})$ from V. Using Itô's formula and (7.3.9), we have

$$\begin{aligned}
& \mathbb{E}\Big[w\big(\theta, X_\theta^{t,x,\alpha}\big) + \int_t^\theta f(X_s^{t,x,\alpha}, \alpha_s)ds\Big] \\
= \; & w(t,x) + \mathbb{E}\Big[\int_t^\theta \big(\mathcal{L}^{\alpha_s} w(s, X_s^{t,x,\alpha}) + f(X_s^{t,x,\alpha}, \alpha_s)\big)ds\Big] \\
\leq \; & v(t,x) \; .
\end{aligned}$$

From the definition of w and θ, it follows that

$$\begin{aligned}
v(t,x) \;\; \geq \;\; & \mathbb{E}\Big[(\theta - t)^2 + |X_\theta^{t,x,\alpha} - x|^2\Big] \\
& + \mathbb{E}\Big[v\big(\theta, X_\theta^{t,x,\alpha}\big) + \int_t^\theta f(X_s^{t,x,\alpha}, \alpha_s)ds\Big] \; .
\end{aligned}$$

We then notice from the definition of θ that

$$(\theta - t)^2 + |X_\theta^{t,x,\alpha} - x|^2 \;\; \geq \;\; r^2 \;\; > \;\; 0 \; .$$

Therefore, we get

$$v(t,x) \;\; \geq \;\; r^2 + \mathbb{E}\Big[v\big(\theta, X_\theta^{t,x,\alpha}\big) + \int_t^\theta f(X_s^{t,x,\alpha}, \alpha_s)ds\Big] \; ,$$

for any control $\alpha \in \mathcal{A}_t$. This contradicts the DPP. $\square$

Chapter 8

Verification

8.1 The verification result

In this section, we give a criterion to check whether a function is the value function related to a given optimal control problem.

Theorem 8.1.6 *Suppose that there exists $\varphi \in C^{1,2}([0,T)\times\mathbb{R}^n)\cap C^0([0,T]\times\mathbb{R}^n)$ solution to* (7.3.6)-(7.3.7) *with polynomial growth such that the following statements hold.*

(i) There exists a measurable map $\hat{\alpha} : [0,T)\times\mathbb{R}^n \to A$ such that

$$\mathcal{H}\varphi(t,x) \;=\; (\partial_t + \mathcal{L}^{\hat{\alpha}(t,x)})\varphi(t,x) + f(x,\hat{\alpha}(t,x))\ , \quad (t,x)\in[0,T)\times\mathbb{R}^n\ .$$

Then, $\varphi = v$ and $\hat{\alpha}(.,X_.^{t,x,\hat{\alpha}})$ is an optimal control for the initial condition (t,x).

Proof. From Itô's formula, (i) and (ii) we have

$$\varphi(t,x) \;=\; \mathbb{E}\Big[\varphi\big(\theta_n, X_{\theta_n}^{t,x,\hat{\alpha}}\big) + \int_t^{\theta_n} f\big(X_s^{t,x,\hat{\alpha}}, \hat{\alpha}(s, X_s^{t,x,\hat{\alpha}})\big)ds\Big]\ .$$

where

$$\theta_n \;=\; \inf\{s\geq 0\ :\ |X_s^{t,x,\hat{\alpha}}|\geq n\}\wedge T$$

for $n \geq 1$. Sending n to ∞, we get

$$\varphi(t,x) \;=\; \mathbb{E}\Big[g(X_T^{t,x,\hat\alpha}) + \int_t^T f\big(X_s^{t,x,\hat\alpha}, \hat\alpha(s, X_s^{t,x,\hat\alpha})\big)ds\Big]\,.$$

This gives $\varphi(t,x) \leq v(t,x)$. We now prove the reverse inequality. Let $\alpha \in \mathcal{A}_t$, τ_n the first time s such that $|X_s^{t,x,\alpha}| \geq n$ and $\theta_n = T \wedge \tau_n$ for $n \geq 1$. We have from Itô's formula

$$\begin{aligned}
&\mathbb{E}\Big[\varphi\big(\theta_n, X_{\theta_n}^{t,x,\alpha}\big) + \int_t^{\theta_n} f\big(X_s^{t,x,\alpha}, \alpha_s\big)ds\Big]\\
&= \;\varphi(t,x) + \mathbb{E}\Big[\int_t^{\theta_n} \big((\partial_t + \mathcal{L}^\alpha)\varphi(s, X_s^{t,x,\alpha}) + f(X_s^{t,x,\alpha}, \alpha_s)\big)ds\Big]\,.
\end{aligned}$$

From (i) we have

$$(\partial_t + \mathcal{L}^\alpha)\varphi(s, X_s^{t,x,\alpha}) + f(X_s^{t,x,\alpha}, \alpha_s) \;\leq\; 0\,, \quad s \in [t,T]\,.$$

Therefore, we get from the polynomial growth of φ and Fatou's lemma

$$\begin{aligned}
\varphi(t,x) \;&\geq\; \liminf_{n\to+\infty} \mathbb{E}\Big[\varphi\big(\theta_n, X_{\theta_n}^{t,x,\alpha}\big) + \int_t^{\theta_n} f\big(X_s^{t,x,\alpha}, \alpha_s\big)ds\Big]\\
&\geq\; \mathbb{E}\Big[g\big(X_T^{t,x,\alpha}\big) + \int_t^T f\big(X_s^{t,x,\alpha}, \alpha_s\big)ds\Big]\,.
\end{aligned}$$

Finally, $\hat\alpha$ is optimal as it realizes $\varphi(t,x)$. □

8.2 Application: portfolio allocation problem in finite horizon

We consider an optimal investment in the framework of the Black-Scholes-Merton model over a finite horizon T. We suppose that the market is composed by two assets: a nonrisky asset S^0 and a risky one S. The nonrisky asset follows a deterministic interest rate $r > 0$ and has an initial value $S_0^0 > 0$. It therefore satisfies

$$S_t^0 \;=\; S_0^0 + \int_0^t rS_u du\,, \quad t \in [0,T]\,.$$

The risky asset is defined by its deterministic initial condition $S_0 > 0$ and the dynamics

$$dS_t = S_t \mu dt + S_t \sigma dW_t$$

where μ and σ are constants with $\sigma > 0$ and W is a one dimensional Brownian motion.

An agent invests at any time t a proportion α_t of his wealth in the stock S and $1-\alpha_t$ in S^0. The self-financing wealth process X^α evolves according to

$$\begin{aligned} dX_t &= \frac{X_t \alpha_t}{S_t} dS_t + \frac{X_t(1-\alpha_t)}{S_t^0} dS_t^0 \\ &= X_t(r + (\mu - r)\alpha_t)dt + X_t \alpha_t \sigma dW_t \ , \quad t \in [0,T] \ . \end{aligned}$$

We denote by $\mathcal{A}$ the set of progressively measurable processes α valued in A, that is supposed to be closed and convex, and such that $\mathbb{P}$-a.s. $\int_0^T |\alpha_s|^2 ds < +\infty$. This integrability condition ensures the existence and uniqueness of a stong solution to the SDE governing the wealth process controlled by $\alpha \in \mathcal{A}$. Given a portfolio strategy $\alpha \in \mathcal{A}$, we denote by $X^{t,x,\alpha}$ the corresponding wealth process starting from an initial capital $X_t^{t,x,\alpha} = x \geq 0$ at time $t \in [0,T]$. We suppose that the preferences of the agent are described by a utility function U of CRRA type given by

$$U(x) = \frac{x^p}{p} \ , \quad x \geq 0$$

with $p \in (0,1)$. The agent aims at maximizing the expected utility from terminal wealth at horizon T. The value function of the utility maximization problem is then defined by

$$v(t,x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\big[U(X_T^{t,x,\alpha})\big],$$

for $(t,x) \in [0,T] \times \mathbb{R}_+$. The HJB equation is then given by

$$\sup_{a \in A} \big\{(\partial_t + \mathcal{L}^a)v(t,x)\big\} = 0 \ , \quad (t,x) \in [0,T) \times \mathbb{R}^n \qquad (8.2.1)$$

with

$$\mathcal{L}^a v(t,x) = x(a\mu + (1-a)r)\partial_x v(t,x) + \frac{1}{2}x^2 a^2 \sigma^2 \partial_{xx}^2 v(t,x)$$

for $(t, x) \in [0, T) \times \mathbb{R}_+$. The terminal condition is then given by

$$v(T, x) \;=\; U(x)\;, \quad x \in \mathbb{R}_+ \tag{8.2.2}$$

We look for an explicit smooth solution φ to (8.2.1)-(8.2.2). We propose a candidate solution in the form

$$\varphi(t, x) = \phi(t)U(x)\;, \quad (t, x) \in [0, T) \times \mathbb{R}^n\;,$$

for some positive function ϕ. By substituting into the HJB equation, we derive that ϕ should satisfy the ordinary differential equation

$$\begin{aligned} \phi'(t) + \rho\phi(t) &= 0\;, \quad t \in [0, T) \\ \phi(T) &= 1, \end{aligned}$$

where

$$\rho \;=\; p \sup_{a \in A}\{a(\mu - r) + r - \frac{1}{2}a^2(1 - p)\sigma^2\}\;.$$

We then obtain $\phi(t) = \exp(\rho(T - t))$. Hence, the function given by

$$\varphi(t, x) \;=\; \exp(\rho(T - t))U(x)\;, \quad (t, x) \in [0, T] \times \mathbb{R}_+\;,$$

is a smooth solution to the HJB PDE. Furthermore, the function $a \in A \mapsto a(\mu - r) + r - \frac{1}{2}a^2(1 - p)\sigma^2$ is strictly concave on the closed convex set A . Thus it reaches its maximum at some constant $\hat{a} \in A$. Moreover, the SDE associated to the constant control $\hat{a}$

$$dX_s \;=\; X_s(r + (\mu - r)\hat{a})ds + X_s\hat{a}\sigma dW_s\;,$$

admits a unique solution for a given initial condition (t, x). From the verification theorem, we have $\varphi = v$. In the case where $A = \mathbb{R}$, $\hat{a}$ and ρ can be explicitly computed and we have

$$\hat{a} \;=\; \frac{\mu - r}{\sigma^2(1 - p)}$$

and

$$\rho \;=\; \frac{(\mu - r)^2}{\sigma^2(1 - \rho)}\frac{p}{1 - p} + rp\;.$$

8.3 Application: investment-consumption problem

We use the framework of the previous section for the model on asset prices. A control is a pair of progressively measurable processes (α, c) valued in $A \times \mathbb{R}_+$ for some closed convex subset A of $\mathbb{R}$ such that $\mathbb{P}$-a.s. $\int_0^T |\alpha_t|^2 dt + \int_0^T c_t dt < +\infty$. We denote by $\mathcal{A} \times \mathcal{C}$ the set of control processes. The quantity α_t represents the proportion of wealth invested in stock S, and c_t is the time rate consumption per unit of wealth. Given $(\alpha, c) \in \mathcal{A} \times \mathcal{C}$, there exists a unique solution, denoted by $X^{t,x}$, to the SDE governing the wealth process

$$dX_s \;=\; X_s(r + (\mu - r)\alpha_t - c_t)ds + X_s\alpha_t\sigma dW_s \;,\;\; s \in [t, T] \;,$$

given the initial condition $X_t = x \geq 0$. The agent's investment-consumption problem is to maximize over strategies (α, c) the expected utility from intertemporal consumption up to the time horizon T. Given a utility function u for consumption, we then consider the corresponding value function:

$$v(t, x) \;=\; \sup_{(\alpha,c)\in\mathcal{A}\times\mathcal{C}} \mathbb{E}\Big[u(X_T^{t,x}) + \int_t^T u(c_s X_s^{t,x})ds\Big] \;, \quad (t, x) \in [0, T] \times \mathbb{R}_+ \;.$$

The HJB equation associated to this control problem is

$$\sup_{(a,c)\in A\times\mathbb{R}_+} \{(\partial_t + \mathcal{L}^{(a,c)})v + u(cx)\} \;=\; 0$$

with terminal condition $v(T, .) = u$, where

$$\begin{aligned}\mathcal{L}^{(a,c)}v(t, x) \;&=\; x(a\mu + (1 - a)r - c)\partial_x v(t, x) \\ &\quad + \frac{1}{2}x^2a^2\sigma^2\partial^2_{xx}v(t, x)\end{aligned}$$

for $(t, x) \in [0, T) \times \mathbb{R}_+$. By defining $\tilde{u}(z) = \sup_{C\geq 0}[u(C) - Cz]$, $z \geq 0$, the Legendre transform of u, this HJB equation may be written as

$$\sup_{a\in A}\{(\partial_t + \mathcal{L}^a)v\} + \tilde{u}(\partial_x v) \;=\; 0$$

with $\mathcal{L}^a$ defined as in the previous section. If we take $u(x) = x^p/p$ then $\tilde{u}(z) = z^{-q}/q$ with $q = p/(1-p)$. We next look for a candidate solution of the form

$$\varphi(t,x) \quad = \quad \phi(t)u(x)$$

for $(t,x) \in [0,T) \times \mathbb{R}_+$. By substituting into the HJB equation, we derive that ϕ should satisfy the ordinary differential equation

$$\begin{aligned} \phi'(t) + \rho\phi(t) + \frac{p}{q}\phi(t)^{-q} &= 0\ , \quad t \in [0,T) \\ \phi(T) &= 1, \end{aligned}$$

where

$$\rho \quad = \quad p \sup_{a \in A}\{a(\mu - r) + r - \frac{1}{2}a^2(1-p)\sigma^2\}\ .$$

We notice that the ODE can be rewritten

$$\begin{aligned} (\phi^{q+1}(t))' + \tilde{\rho}\phi^{q+1}(t) + \frac{p(q+1)}{q} &= 0\ , \quad t \in [0,T)\ , \\ \phi^{q+1}(T) &= 1\ , \end{aligned}$$

with $\tilde{\rho} = \rho(q+1)$. Therefore we get

$$\phi(t)^{q+1} \quad = \quad (1 + \frac{p}{q\rho})e^{\tilde{\rho}(T-t)} - \frac{p}{q\rho}\ , \quad t \in [0,T]\ .$$

We then have that the optimizer $\hat{a}$ in the variable a is the same as that of the previous section. We next compute the optimizer an the variable c and we have

$$\begin{aligned} \hat{c}^*(t,x) &= \frac{1}{x}\big(\phi(t)u'(x)\big)^{\frac{1}{p-1}} \\ &= \Big((1 + \frac{p}{q\rho})e^{\tilde{\rho}(T-t)} - \frac{p}{q\rho}\Big)^{-1} \end{aligned}$$

We observe that $c^*(t,x) = c^*(t) \in [0,1]$. Therefore the related SDE

$$dX_s \quad = \quad X_s(r + (\mu - r)\hat{a} - c^*(t))ds + X_s\hat{a}\sigma dW_s\ ,\ s \in [t,T]\ ,$$

admits a unique solution. We can then apply the verification theorem to get that

$$v(t,x) \quad = \quad \Big((1 + \frac{p}{q\rho})e^{\tilde{\rho}(T-t)} - \frac{p}{q\rho}\Big)^{\frac{1}{q+1}}\frac{x^p}{p}\ , \quad (t,x) \in [0,T] \times \mathbb{R}_+\ .$$

Chapter 9

Viscosity solutions for the dynamic programming equation

9.1 Viscosity solutions for parabolic PDEs

Let F be an operator from $[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ into $\mathbb{R}$ where $\mathbb{S}^n$ stands for the set of n-dimensional symmetric matrices. In this chapter, we will be mostly interested by the case

$$F(t,x,u,q,p,M) \quad = \quad -\sup_{a\in A}\big\{q+\mathcal{L}^a[t,x,u,p,M]+f(x,a)\big\}$$

where

$$\mathcal{L}^a[t,x,u,p,M] \quad = \quad b(x,a).p+\frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^\top(x,a)M\big)$$

for $(t,x,u,q,p,M)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ where the coefficients b,σ and f and the set A are those of the previous chapters and satisfy the related assumptions. This case is called the HJB case.

In the sequel, we assume that F is an elliptic operator, that is, F is nonincreasing in the variable M. We are interested in the PDE

$$F\big(t,x,u(t,x),\partial_t u(t,x),\nabla u(t,x),\nabla^2 u(t,x)\big) \quad = \quad 0 \tag{9.1.1}$$

for $(t,x)\in[0,T)\times\mathbb{R}^n$, together with the terminal condition

$$u(T,x) \quad = \quad g(x)\ , \quad x\in\mathbb{R}^n\ . \tag{9.1.2}$$

Our objective is to give a notion of weak solution to such a PDE.

Definition 9.1.2 *(i) A continuous function $u : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ is called a viscosity subsolution to* (9.1.1) *if for any function $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$ and $(t,x) \in [0,T)\times\mathbb{R}^n$ such that*

$$0 \quad = \quad (u-\varphi)(t,x) \quad = \quad \max_{[0,T]\times\mathbb{R}^n}(u-\varphi)$$

we have

$$F(t,x,\varphi(t,x),\partial_t\varphi(t,x),\nabla\varphi(t,x),\nabla^2\varphi(t,x)) \quad \leq \quad 0 \ .$$

(ii) A continuous function $u : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ is called a viscosity supersolution to (9.1.1) *if for any function $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$ and $(t,x) \in [0,T)\times\mathbb{R}^n$ such that*

$$0 \quad = \quad (u-\varphi)(t,x) \quad = \quad \min_{[0,T]\times\mathbb{R}^n}(u-\varphi)$$

we have

$$F(t,x,\varphi(t,x),\partial_t\varphi(t,x),\nabla\varphi(t,x),\nabla^2\varphi(t,x)) \quad \geq \quad 0 \ .$$

(iii) A function $u : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ is called a viscosity solution to (9.1.1) *if it is a viscosity subsolution to* (9.1.1) *and a viscosity supersolution to* (9.1.1)*.*

9.2 Viscosity properties of the value function

We consider the HJB case for PDE (9.1.1) and the value function v defined by (7.1.2).

Theorem 9.2.7 *The value function v is a viscosity solution to* (9.1.1) *and satisfies* (9.1.2)*.*

Proof. The fact that v satisfies (9.1.2) comes from its definition. We turn to the viscosity properties w.r.t. equation to (9.1.1).

Step 1. Subsolution property. Let $(\hat{t},\hat{x}) \in [0,T)\times\mathbb{R}^n$ and $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$ be such that

$$0 \;=\; (v-\varphi)(\hat{t},\hat{x}) \;=\; \max_{[0,T]\times\mathbb{R}^n}(v-\varphi)\;. \tag{9.2.3}$$

We proceed by contradiction and assume that

$$-\sup_{a\in A}\big\{(\partial_t+\mathcal{L}^a)\varphi(\hat{t},\hat{x})+f(\hat{x},a)\big\} \;>\; 0\;.$$

By continuity of the coefficients, we can find $r\in(T-\hat{t},T)$ such that

$$-\sup_{a\in A}\big\{(\partial_t+\mathcal{L}^a)\varphi(t,x)+f(x,a)\big\} \;>\; 0 \tag{9.2.4}$$

for $(t,x)\in B:=B(\hat{t},r)\times B(\hat{x},r)$. Without loss of generality, we can assume that the minimum in (9.2.5), is strict and that

$$\max_{\partial B}(v-\varphi) \;\leq\; -\eta\;.$$

for some $\eta>0$. Let ν be the first exit time of $(s,X_s^{t,x,\hat{\alpha}})_{s\geq t}$ from B. From the DPP, there exists a control $\hat{\alpha}\in\mathcal{A}_{\hat{t}}$ such that

$$v(\hat{t},\hat{x}) \;\leq\; \mathbb{E}\Big[v(\theta,X_\theta^{t,x,\hat{\alpha}})+\int_t^\theta f\big(X_s^{t,x,\hat{\alpha}},\hat{\alpha}_s\big)ds\Big]+\frac{\eta}{2}\;.$$

From the previous inequalities, we get

$$\varphi(\hat{t},\hat{x}) \;\leq\; \mathbb{E}\Big[\varphi(\theta,X_\theta^{t,x,\hat{\alpha}})-\eta+\int_t^\theta f\big(X_s^{t,x,\hat{\alpha}},\hat{\alpha}_s\big)ds\Big]+\frac{\eta}{2}\;.$$

Applying Itô's Lemma to φ, we get

$$0 \;\leq\; -\frac{\eta}{2}+\mathbb{E}\Big[\int_t^\theta \big((\partial_t+\mathcal{L}^{\hat{\alpha}_s})\varphi\big(s,X_s^{t,x,\hat{\alpha}}\big)+f\big(X_s^{t,x,\hat{\alpha}},\hat{\alpha}_s\big)\big)ds\Big]\;.$$

Using (9.2.4) we get

$$0 \;\leq\; -\frac{\eta}{2}\;.$$

Step 2. Supersolution property. Let $(t,x) \in [0,T)\times\mathbb{R}^n$ and $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$ be such that

$$0 \quad = \quad (v-\varphi)(t,x) \quad = \quad \min_{[0,T]\times\mathbb{R}^n}(v-\varphi)\ . \tag{9.2.5}$$

Fix $a \in A$ and denote by $\hat{\alpha}$ the control constant equal to a. From the dynamic programming principle we have

$$v(t,x) \quad \geq \quad \mathbb{E}\Big[v\big(\theta, X_\theta^{t,x,\hat{\alpha}}\big) + \int_t^{t+h} f\big(X_s^{t,x,\hat{\alpha}},a\big)ds\Big]$$

for $h \in (0,T-\hat{t})$. Using (9.2.5), we get

$$\varphi(t,x) \quad \geq \quad \mathbb{E}\Big[\varphi\big(t+h, X_{t+h}^{t,x,\hat{\alpha}}\big) + \int_t^{t+h} f\big(X_s^{t,x,\hat{\alpha}},a\big)ds\Big]$$

Applying Itô's formula and diving by $h>0$, we get

$$0 \quad \geq \quad \frac{1}{h}\mathbb{E}\Big[\int_t^{t+h}\big[(\partial_t+\mathcal{L}^a)\varphi\big(X_s^{t,x,\hat{\alpha}},a\big) + f\big(X_s^{t,x,\hat{\alpha}},a\big)\big]ds\Big]$$

Sending h to $0+$, we get by the mean value theorem

$$(\partial_t+\mathcal{L}^a)\varphi\big(t,x\big) + f\big(x,a\big) \quad \leq \quad 0\ .$$

Since this holds for all $a \in A$, we get the supersolution property. □

9.3 Comparison and uniqueness

We first have the following result which allow to compare a supersolution and a subsolution as soon as we can compare their values at the terminal time T.

Theorem 9.3.8 *Let U (resp. V) $\in C^{1,2}([0,T)\times\mathbb{R}^n)\cap C^0([0,T]\times\mathbb{R}^n)$ be a subsolution (resp. supersolution) with polynomial growth to* (9.1.1) *in the HJB case. If $U(T,.) \leq V(T,.)$ on $\mathbb{R}^n$, then $U \leq V$ on $[0,T]\times\mathbb{R}^n$.*

Proof. We present the proof in the regular case. The proof for viscosity solutions uses an additional result called Ishii's lemma which gives a similar condition to second order optimality condition. We proceed in 3 steps.
Step 1. Let $\tilde{U}(t,x) = e^{\lambda t}U(t,x)$ and $\tilde{V}(t,x) = e^{\lambda t}V(t,x)$. Then a straightforward calculation shows that $\tilde{U}$ (resp. $\tilde{V}$) is a subsolution (resp. supersolution) to

$$\tilde{F}\big(t,x,w,\partial_t w(t,x),\nabla w(t,x),\nabla^2 w(t,x)\big) \;=\; 0 \tag{9.3.6}$$

with

$$\begin{aligned}
F\big(t,x,u,q,p,M\big) &= -\sup_{a\in A}\big\{q+\tilde{\mathcal{L}}^a[t,x,u,p,M]+\tilde{f}(t,x,a)\big\}\ ,\\
\tilde{\mathcal{L}}^a[t,x,u,p,M] &= b(x,a).p+\frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^\top(x,a)M\big)-\lambda u\ ,\\
\tilde{f}(t,x,a) &= e^{\lambda t}f(x,a)
\end{aligned}$$

for $(t,x,u,q,p,M)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ and $a\in A$.

Step 2. From the polynomial growth of U and V, we may choose an integer $p\geq 1$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n}\frac{|\tilde{U}(t,x)|+|\tilde{V}(t,x)|}{1+|x|^p} \;<\; +\infty\ ,$$

and we consider the function $\phi(t,x) = e^{-\lambda t}(1+|x|^{2p}) =: e^{-\lambda t}\psi(x)$. From the linear growth condition on b and σ, a straightforward computation shows that there exists some constant $c>0$ s.t.

$$\begin{aligned}
&-\partial_t\phi(t,x)+\lambda\phi(t,x)-\sup_{a\in A}b(x,a).\nabla\phi(t,x)+\frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(x,a)\nabla^2\phi(t,x))\\
\geq\;& e^{-\lambda t}\psi(x)(\lambda-c)\ .
\end{aligned}$$

By taking $\lambda\geq c$, the function $\tilde{V}_\varepsilon := \tilde{V}+\varepsilon\phi$ is a supersolution to (9.1.1) for any $\varepsilon>0$. Moreover, from the growth conditions on U,V and ϕ, we have

$$\lim_{|x|\to+\infty}\sup_{t\in[0,T]}(\tilde{U}-\tilde{V}_\varepsilon)(t,x) \;=\; -\infty\ . \tag{9.3.7}$$

for all $\varepsilon>0$.

Step 3. We finally argue by contradiction to show that $\tilde{U} - \tilde{V}_\varepsilon \leq 0$ on $[0,T] \times \mathbb{R}^n$ for all $\varepsilon > 0$, which gives the required result by sending ε to 0. On the contrary, by continuity of $\tilde{U} - \tilde{V}_\varepsilon$, and from (9.3.7), there exists $\varepsilon > 0$, $(t,x) \in [0,T] \times \mathbb{R}^n$ such that

$$\sup_{[0,T]\times\mathbb{R}^n} (\tilde{U} - \tilde{V}_\varepsilon) \;=\; (\tilde{U} - \tilde{V}_\varepsilon)(t,x) \;>\; 0\,. \tag{9.3.8}$$

Since $(\tilde{U} - \tilde{V}_\varepsilon)(T,.) \leq (\tilde{U} - \tilde{V})(T,.) \leq 0$ on $\mathbb{R}^n$, we have $t < T$. Therefore, the first and second order optimality conditions (9.3.8) give

$$\begin{aligned}
\partial_t(\tilde{U} - \tilde{V}_\varepsilon)(t,x) &\leq 0\,,\\
\nabla(\tilde{U} - \tilde{V}_\varepsilon)(t,x) &= 0\,,\\
\nabla^2(\tilde{U} - \tilde{V}_\varepsilon)(t,x) &\leq 0\,.
\end{aligned}$$

By writing that $\tilde{U}$ (resp. $\tilde{V}_\varepsilon$) is a subsolution (resp. supersolution) to (9.3.6), and recalling that $\tilde{F}$ is nondecreasing in its last argument, we then deduce that

$$\begin{aligned}
& \lambda(\tilde{U} - \tilde{V}_\varepsilon)(t,x)\\
\leq\; & \tilde{F}(t,x,\partial_t\tilde{U}(t,x),\nabla\tilde{U}(t,x),\nabla^2\tilde{U}(t,x))\\
& -\tilde{F}(t,x,\partial_t\tilde{V}_\varepsilon(t,x),\nabla\tilde{V}_\varepsilon(t,x),\nabla^2\tilde{V}_\varepsilon(t,x))\\
\leq\; & 0
\end{aligned}$$

which contradicts (9.3.8). □

Corollary 9.3.1 *The value function v is the unique viscosity solution to* (9.1.1) *in the HJB case satisfying* (9.1.1) *and having polynomial growth.*

Proof. We simply need to prove that v has polynomial growth. It actually follows from the polynomial growth assumption on f and g and Proposition (7.1.3). □