Duration models / Exam of June 16, 2016

2 hours – no documents allowed

Correction

Some properties of the Makeham model

The quality of drafting, justifications as well as presentation will be taken into account in the notation.

One is interested, within the framework of mortality modelling, in the following model:

$$h(x) = \alpha + \beta \exp(\gamma x)$$

with h the hazard function of the model (used by Thatcher).

Question n°1 (2 points): Point out the definition and the principal properties of the hazard function. What is the link with the survival function?

> cf. lecture notes on page 5

Question n°2 (2 points): Compute the survival function of the Makeham model.

> Because $S(x) = \exp\left(-\int_0^x h(u)\,du\right)$, a direct calculation leads to:
>
> $$S(x) = \exp\left(-\alpha x - \frac{\beta}{\gamma}\left(e^{\gamma x} - 1\right)\right)$$
>
> See also lecture notes on page 15.

Question n°3 (3 points): Show that with this model, if q_x is small, there exist a and b such that $\ln(q_{x+1} - q_x) \approx a + b \times x$.

> By definition, $q_x = 1 - \frac{S(x+1)}{S(x)} = 1 - \exp\left(-\alpha - \frac{\beta}{\gamma}\left(e^{\gamma} - 1\right)e^{\gamma x}\right)$ and thus
>
> $$\ln(1 - q_x) = -\alpha - \frac{\beta}{\gamma}\left(e^{\gamma} - 1\right)e^{\gamma x}.$$
>
> If q_x is small, this leads to $q_x \approx \alpha + \frac{\beta}{\gamma}\left(e^{\gamma} - 1\right)e^{\gamma x}$; one can deduce form this equality that $q_{x+1} - q_x \approx \frac{\beta}{\gamma}\left(e^{\gamma} - 1\right)\left(e^{\gamma(x+1)} - e^{\gamma x}\right) = \frac{\beta}{\gamma}\left(e^{\gamma} - 1\right)^2 e^{\gamma x}$ and
>
> $$\ln(q_{x+1} - q_x) \approx \ln\left(\frac{\beta}{\gamma}\left(e^{\gamma} - 1\right)^2\right) + \gamma x$$

Which is the relation that was asked, with a= $\ln\left(\frac{\beta}{\gamma}\left(e^{\gamma}-1\right)^2\right)$ and $b=\gamma$.

Question n°4 (2 points): Use the previous expression to build an estimator of β and γ, using a non-parametric estimator of the q_x's.

If $\hat{q}_x$ is a non-parametric estimator of q_x, one can make the ordinary least square regression of $\ln\left|\hat{q}_{x+1}-\hat{q}_x\right|$ against x and use question 3 to estimate β and γ.

Question n°5 (2 points): Compute an approximation of the expectancy of a Makeham random variable.

cf. lecture notes on page 5.

Question n°6 (4 points): Recall, within a general framework, the expression of log-likelihood according to S and h. Without giving demonstration of this expression, you will provide an intuitive justification of the terms associated with each observation. How does this expression have to be modified in the presence of left truncation? You will point out what is left truncation.

See the lecture notes. One has:

$$\ln L(\theta)=\sum_{i=1}^{n}\left(d_i \ln h(t_i)+\ln S(t_i)\right).$$

In this expression the term with the survival function indicates that individual i was observed alive right before t_i and that with the hazard function, present only for the individuals whose non-censored exit was observed at this moment, is the contribution of the complete observations. In the presence of left truncation, the distribution is replaced by the conditional one (with obvious notations) $X\left|X>E\right.$ and thus the hazard function is unchanged and the survival function is replaced by the conditional survival function:

$$\ln L(\theta)=\sum_{i=1}^{n}\left(d_i \ln h(t_i)+\ln S(t_i)-\ln S(e_i)\right).$$

Question n°7 (2.5 points): Could you describe another way of estimating the parameters using a discrete model?

See the lecture notes on page 33:

$$\varphi(\theta)=\sum_{x}\frac{E_x}{\hat{q}_x(1-\hat{q}_x)}\left(q_x(\theta)-\hat{q}_x\right)^2 \text{ with } q_x=1-\exp\left(-\alpha-\frac{\beta}{\gamma}\left(e^{\gamma}-1\right)e^{\gamma x}\right)$$

Question n°8 (2.5 points): Describe the model of Brass and explain why it is more flexible than the Makeham one.

The Brass model is defined by $\mathbf{ln}\left(\frac{q_x}{1-q_x}\right) = a + b \times \mathbf{ln}\left(\frac{q_x^{ref}}{1-q_x^{ref}}\right)$ with $\left(q_x^{ref}\right)$ a given reference mortality table. It is a semi-parametric model, less constraint than the full parametric Makeham model.