Fonctions de pénalité en théorie du risque

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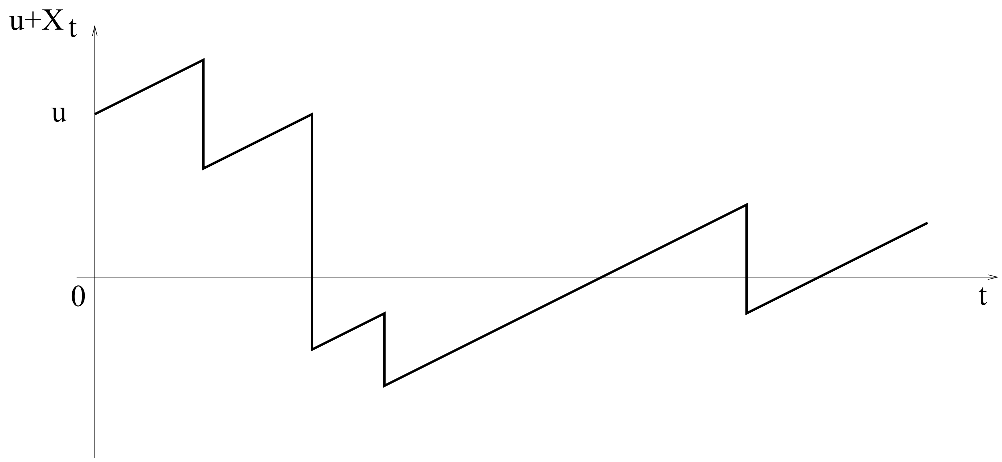
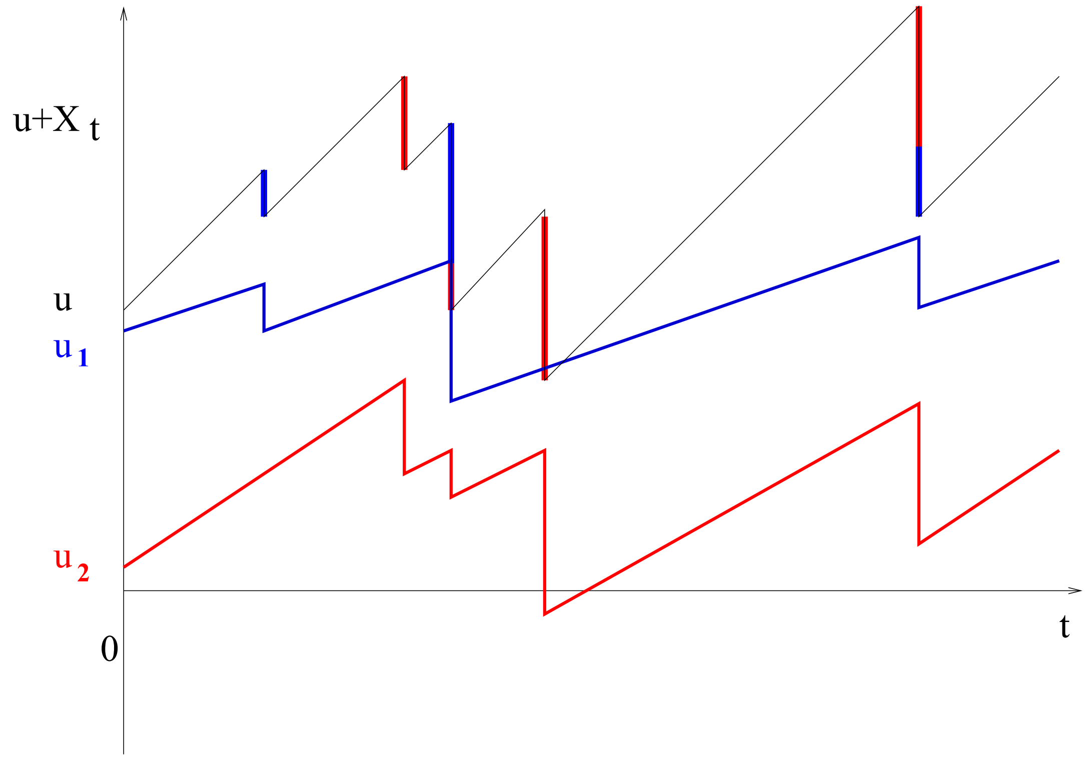


Historical model: for unidimensional risk processes $R_t = u + X_t$,

- with initial reserve u
- and with $X_t = ct - S_t$, where
  - $c > 0$ is the premium income rate,
  - $S_t = \sum_{i=1}^{N(t)} W_i$,
  - the W_i are i.i.d. nonnegative random variables, independent from $(N(t))_{t\geq 0}$,
  - with the convention that the sum is zero if $N(t) = 0$.

Probability of ruin: $\psi(u) = \mathbb{P}\left(\exists t \geq 0, \quad R_t < 0\right)$.



Two lines of business: classical, 1-dimensional surplus process (black), 2-dimensional process (1 for each line of business, blue and red).

$$u = u_1 + u_2$$

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- A general optimal reserve allocation strategy
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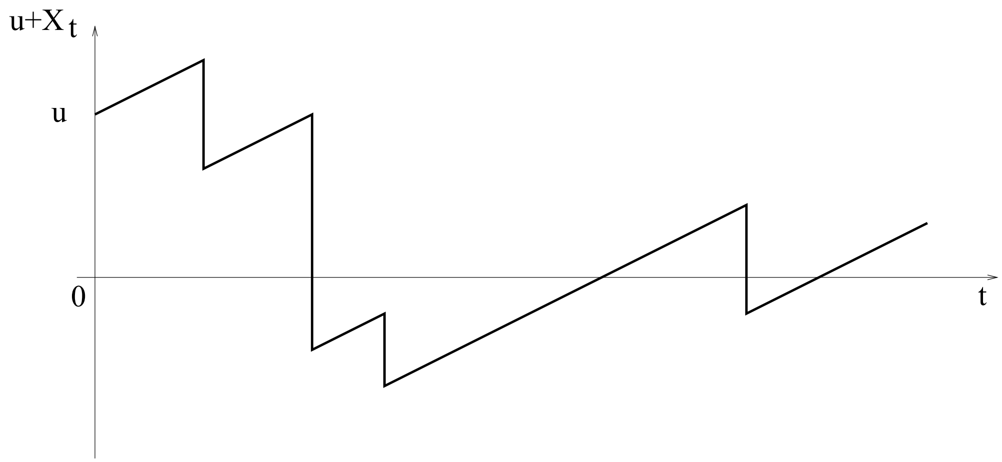

- the time to ruin $T_u = \inf\{t > 0, u + X_t < 0\}$,
- the severity of ruin $|u + X_{T_u}|$, or the couple $(T_u, |u + X_{T_u}|)$,
- the time in the red (below 0) from the first ruin to the first time of recovery $T'_u - T_u$, where

$$T'_u = \inf\{t > T_u, u + X_t = 0\},$$

- the maximal ruin severity $(\inf_{t>0} u + X_t)$,
- the aggregate severity of ruin until recovery $J(u) = \int_{T_u}^{T'_u} |u + X_t| dt$,...
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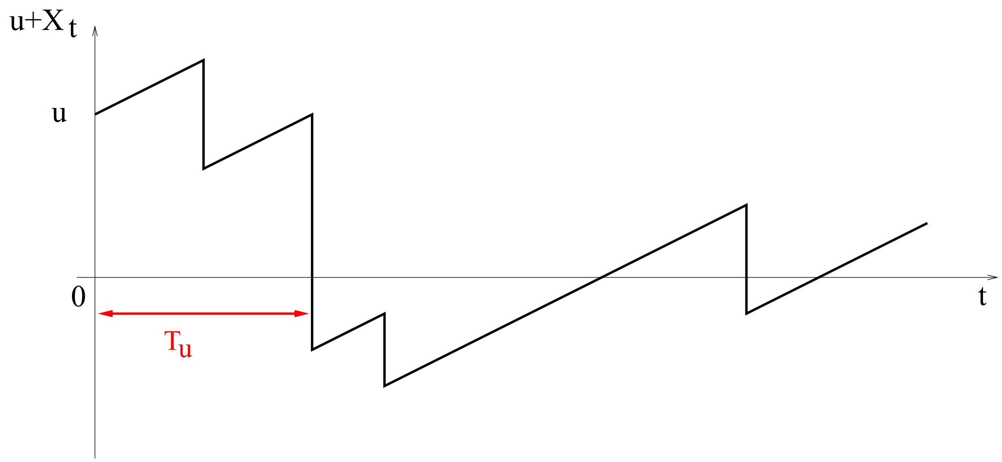

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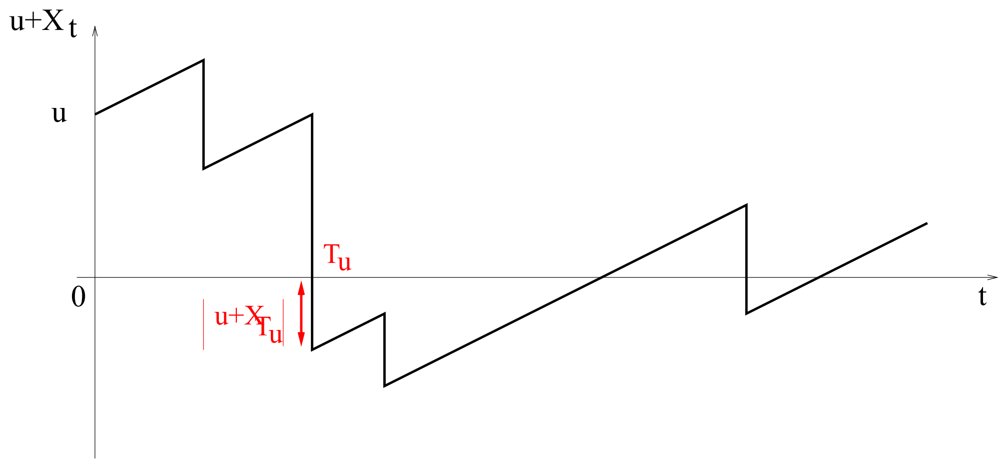

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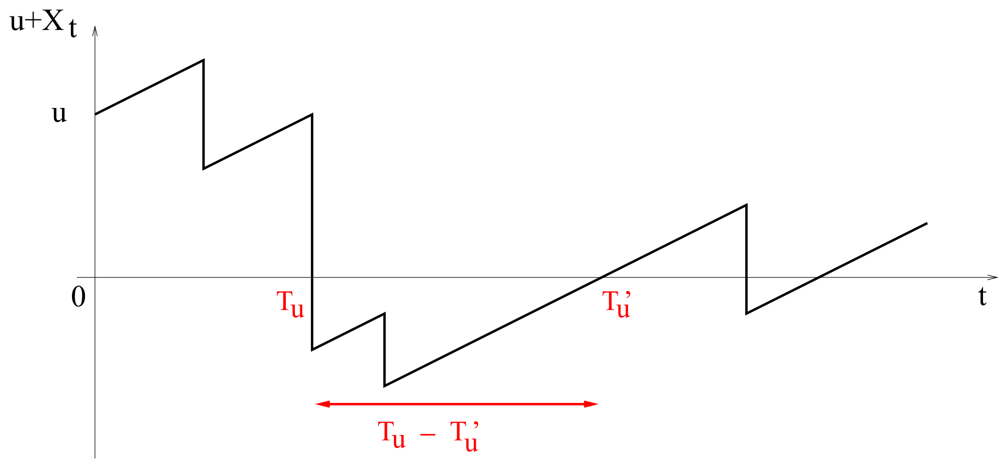

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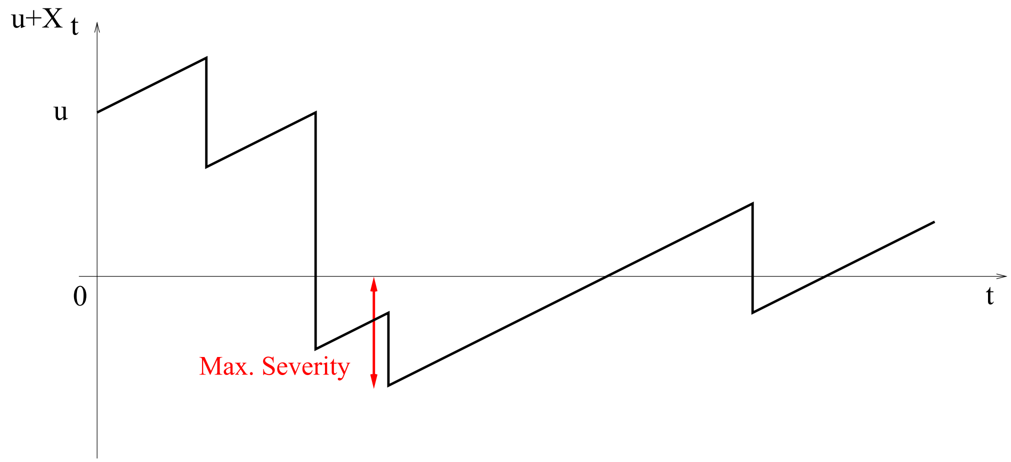

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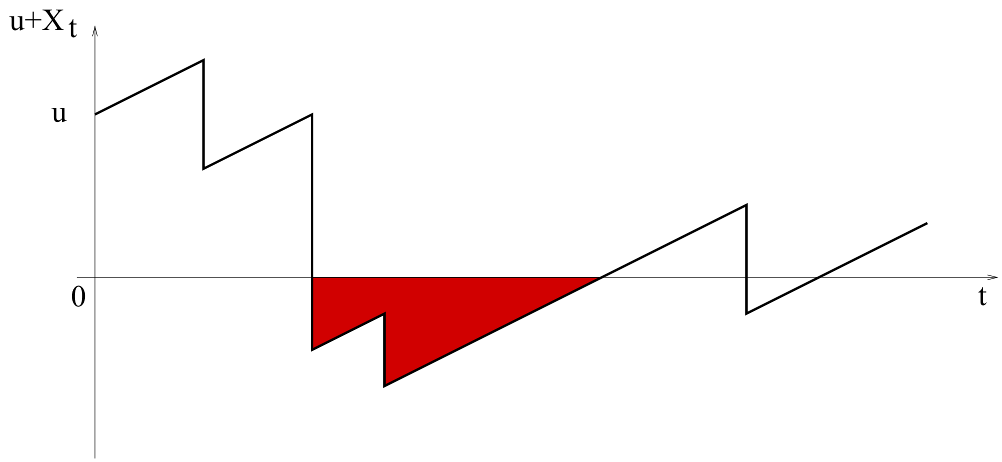

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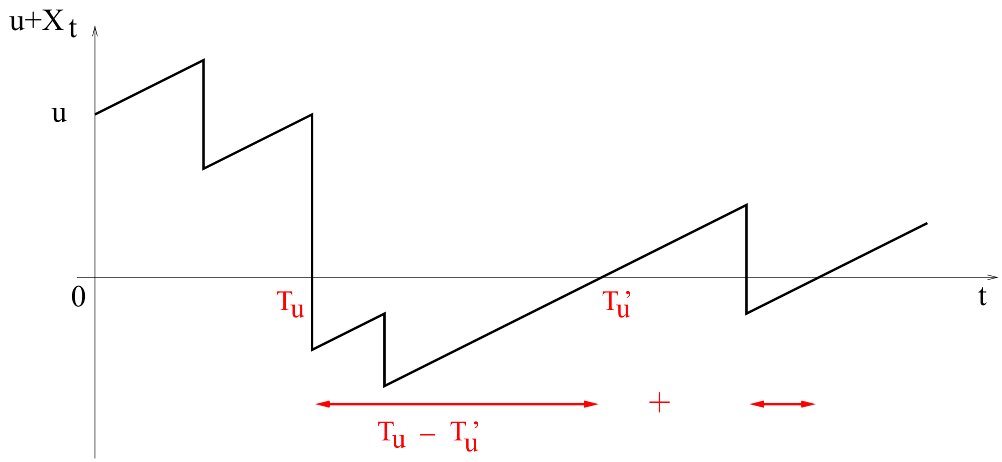

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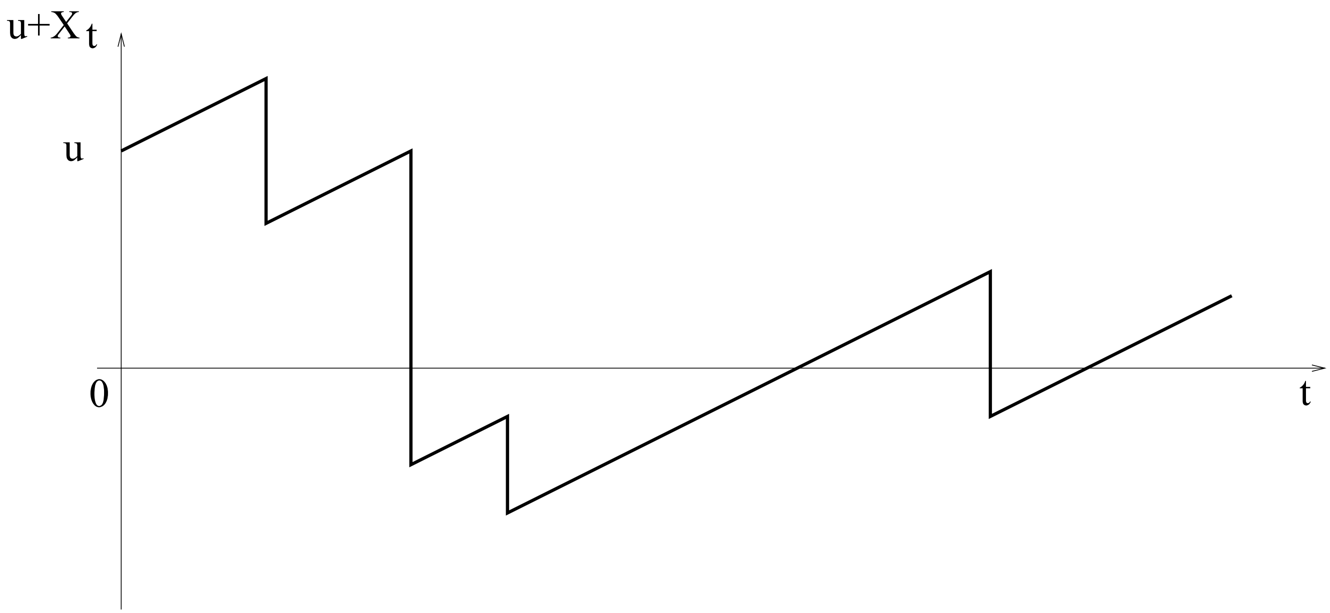
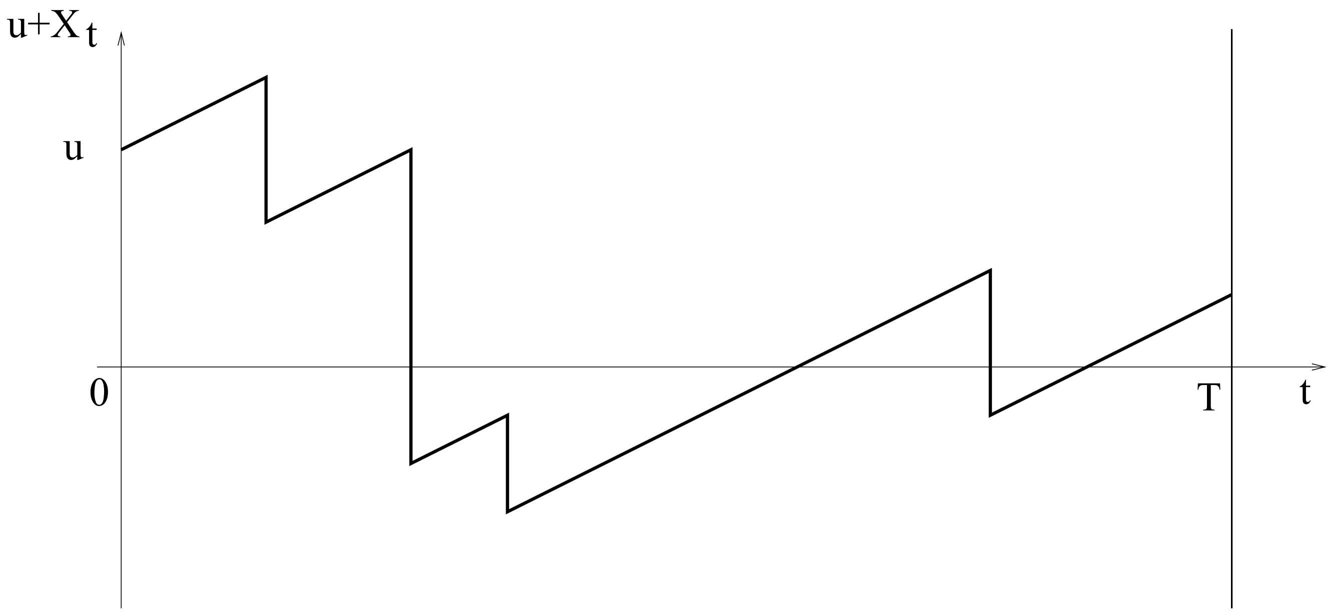


- Consider risk measures based on some fixed time interval $[0, T]$ (T may be infinite).
- Simple penalty function (expected penalty to pay due to insolvency until time horizon T) :

$$\mathbb{E}\left(I_T(u)\right) = \mathbb{E}\left(\int_0^T 1_{\{u+X_t<0\}}|u+X_t|dt\right).$$

- Note that the probability $\mathbb{P}\left(I_T(u)=0\right)$ is the probability of non ruin within finite time T.



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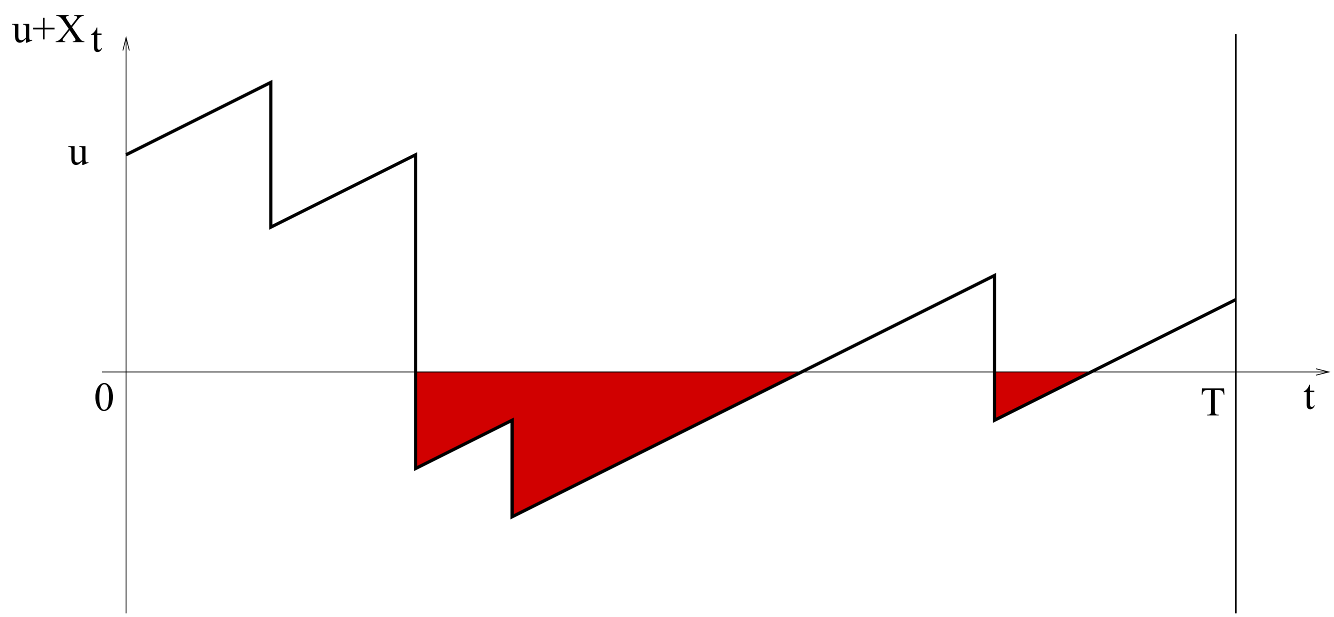

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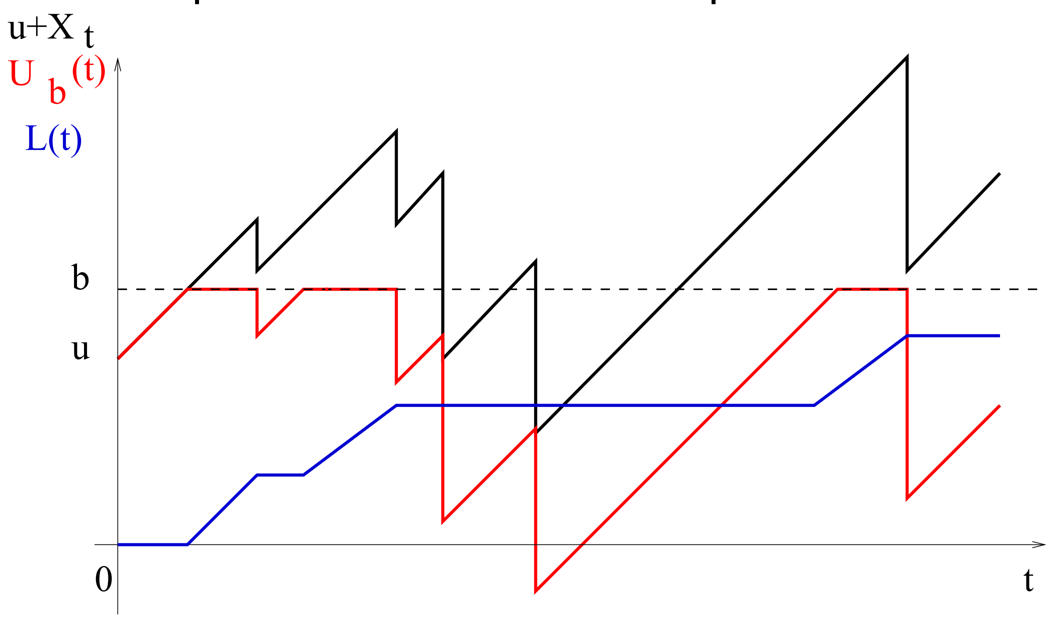
- From an economical point of view, it seems more consistent to consider

$$\mathbb{E}I_{g,h}(u) = \mathbb{E}\left(\int_0^T \left(1_{\{u+X_t\geq 0\}}g(|u+X_t|) - 1_{\{u+X_t\leq 0\}}h(|u+X_t|)\right)dt\right)$$

  - $0 \leq g \leq h$
  - g corresponds to a reward function for positive reserves,
  - and h is a penalty function in case of insolvency.
- These risk measures may be differentiated with respect to the initial reserve u.
- Fubini's theorem.

Other profit indicator: dividends paid until ruin.



Horizontal barrier strategy for dividend paiement (at level b):

modified surplus process $U_b(t)$ (red) and dividend process $L(t)$ (blue).

$\mathbb{P}(L(T_u) > 0)$ is the probability to reach $u + (b - u)$ from u before ruin

- *win first* probability (Rullière and L., 2004): $\mathtt{WF}(u, v) = \mathbb{P}(T_u > T_u^v)$,
- where $T_u = \inf\{t, R_t < 0\}$ and $T_u^v = \inf\{t, R_t \geq u + v\}$.
- Property: For $v, w \geq 0$, $\mathtt{WF}(u, v + w) = \mathtt{WF}(u, v)\mathtt{WF}(u + v, w)$.

Or: discounted expected dividends paid until ruin minus a penalty function if ruin occurs too early.

Gerber-Shiu penalty function:

$$m_\delta(u) = E\left[e^{-\delta t} w\left(R(\tau^-), \mid R(\tau) \mid\right) 1_{\{\tau < +\infty\}} \mid R(0) = u\right],$$

- where $(R(t))$ is the reserve process and starts at $u \in \mathbb{R}$,
- $\delta > 0$ is the discounting factor,
- w is a bounded continuous function from $[0, +\infty[^2$ to $\mathbb{R}$ (it is often named the penalty function),
- τ is the time to ruin (starting from u),
- $R(\tau^-)$ corresponds to the level of the surplus just before ruin,
- and $\mid R(\tau) \mid = -R(\tau)$ corresponds to the severity of ruin.

Tools to compute m_δ: integro-differential equations, renewal equations, other methods.

$$m_\delta(u) = \int_0^{+\infty} e^{-\delta t} f_T(t) \left[\int_0^{u+ct} m_\delta(u + ct - y) + \int_{u+ct}^{+\infty} w(u + ct, y - (u + ct))\right] f_W(y) dy dt.$$

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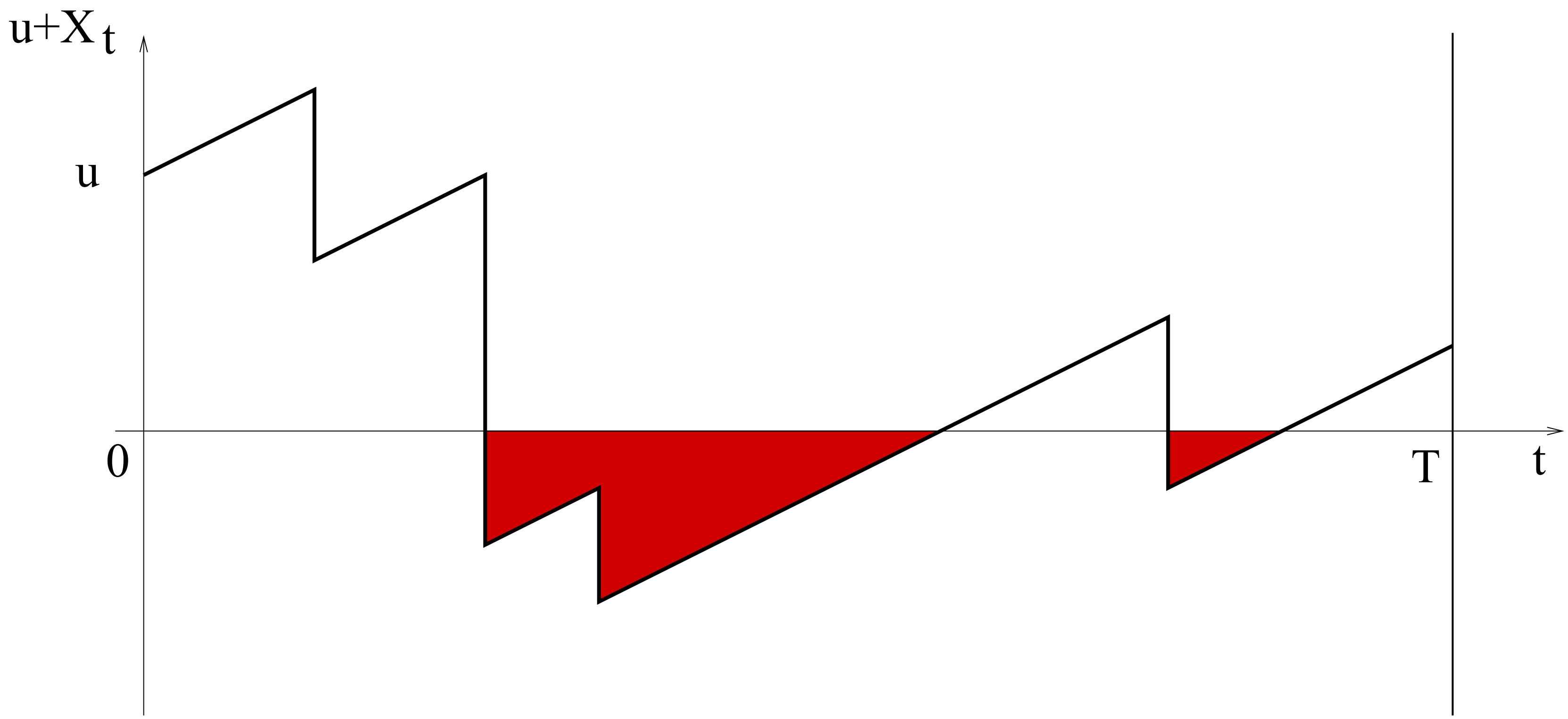
Differentiation theorems



$$I(u,T) = \int_0^T 1_{\{u+X_t<0\}} |u + X_t| dt$$

Th. (L., 2004b):

Let $(X_t)_{t\in[0,T)}$ be a compound renewal risk process starting at 0 and with almost surely time-integrable sample paths. Let T be a fixed time horizon. Let f be defined by $f(u) = \mathbb{E}(I(u,T))$ for $u \in \mathbb{R}$, where

$$I(u,T) = \left(\int_0^T 1_{\{u+X_t<0\}} |u + X_t| dt \right).$$

- For $u \in \mathbb{R}$, f is differentiable at u, and $f'(u) = -\mathbb{E}\tau(u,T)$, where

$$\tau(u,T) = \left(\int_0^T 1_{\{u+X_t<0\}} dt \right).$$

Remark: I give here a simplified version of this Theorem which may be extended to a larger class of risk processes and of reward-penalty functions.

- Th. (L., 2004b): Let $X_t = ct - S_t$, where S_t is a compound renewal process. Define h by $h(u) = \mathbb{E}(\tau(u,T))$ for $u \in \mathbb{R}$. h is differentiable on $\mathbb{R}_+^*$, and for $u > 0$,

$$h'(u) = -\frac{1}{c}\mathbb{E}N^0(u,T),$$

where $N^0(u,T) = \text{Card}\left(\{t \in [0,T], \quad u + ct - S_t = 0\}\right)$.

- This remains valid with $T = +\infty$ if X_t has a positive drift and $\tau(u)$ is integrable. In the compound Poisson case, for $u \geq 0$,

$$h'(u) = -\frac{1}{c}\frac{1}{1-\psi(0)}\psi(u)$$

.

- $\mathbb{E}I(.,T)$ is thus well strictly convex, which will be very important for minimization.

Differentiation theorems

- Theorem: In the Poisson-Exponential($1/\mu$) case, $\psi(u) = \frac{1-\mu R}{\mu R} e^{-Ru}$, with $R = \frac{1}{\mu}\left(1 - \frac{\lambda\mu}{c}\right)$. Hence, for $T = +\infty$,

$$\mathbb{E}\tau(u) = \frac{1-\mu R}{c\mu R^2} e^{-Ru}$$ Gerber (Hans, not Martin!), Dos Reis (1993)

and

$$\mathbb{E}I_\infty(u) = \frac{1-\mu R}{c\mu R^3} e^{-Ru}$$ L. (2004b)

- Proof: Integration of the well-known formula for $\psi(u)$. The considered functions tend to 0 as $u \to +\infty$.

- It is possible to derive $\mathbb{E}I_\infty(u)$ explicitly for Γ and *phase-type*-distributed claim amounts.

- What has to be minimized is

$$A(u_1, \ldots, u_K) = \sum_{k=1}^{K} \mathbb{E} I_T^k(u_k)$$

under the constraint $u_1 + \cdots + u_K = u$, where

$$\mathbb{E} I_T^k(u_k) = \mathbb{E}\left[\int_0^T |R_t^k| 1_{\{R_t^k < 0\}} dt\right]$$

with $R_t^k = u_k + X_t^k$

- This does not depend on the dependence structure.

- From previous differentiation theorems, A is strictly convex. On the compact space

$$\mathcal{S} = \{(u_1, \ldots, u_K) \in (\mathbb{R}^+)^K, \quad u_1 + \cdots + u_K = u\},$$

A admits a unique minimum.

A general optimal reserve allocation strategy

- Lagrange multipliers $\Rightarrow$ optimal allocation:

  there is a subset $J \subset [1, K]$ such that

  - for $j \notin J$, $u_j = 0$,
  - and for $j, k \in J$, $\mathbb{E}\tau_j = \mathbb{E}\tau_k$.

- In the Poisson-Exponential($\frac{1}{\mu}$) case, recall that

$$\mathbb{E}I_u = \frac{1 - \mu R}{c\mu R^3} e^{-Ru}.$$

Consider a two-line model, with the following parameters:

$\mu_1 = \mu_2 = 1$, $c_1 = c_2 = 1$, $R_2 = 0.4$ and $u = 10$.

Three values of $R_1 \Rightarrow$ different optimal allocation strategies.

A general optimal reserve allocation strategy

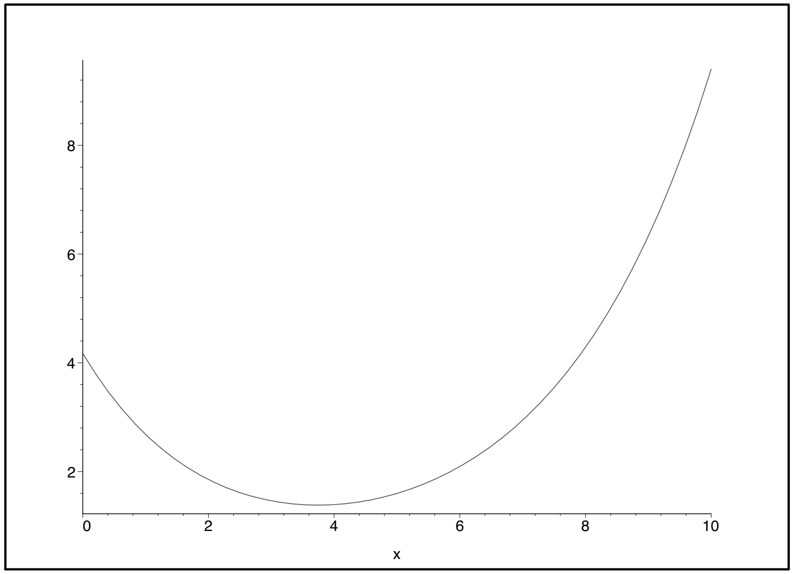


Figure 1: Graph of $A(x, 10-x)$.

When $R_1 = 0.5 > R_2$,

line of business 1 is safer than line 2.

$\Rightarrow \qquad u_1 < u_2$.

The optimal allocation is about

$(u_1 = 3.5, \quad u_2 = 6.5.)$

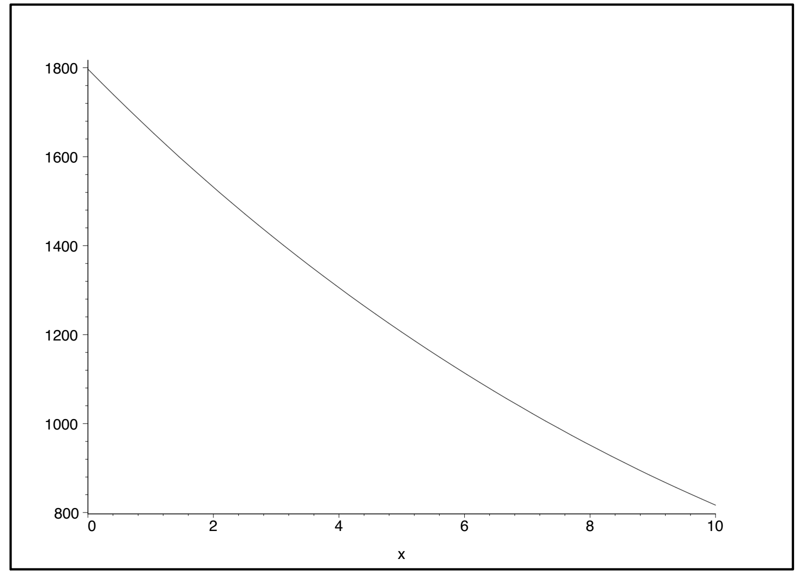


Figure 2: Graph of $A(x, 10-x)$.

When $R_1 = 0.08 < R_2$,

line of business 1 is much riskier than line 2.

$\Rightarrow \qquad u_1 = u = 10$ and $u_2 = 0$.

Transfer of the whole reserve to line 1.

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Asymptotics of the penalty function

Noël's first love: joint work with

Romain Biard (Lyon), Claudio Macci (Rome), and Noël Veraverbeke (Hasselt U.)

- Goal: derive the asymptotic behavior of the expected area in red to facilitate numerical optimization
- In the regular variation case, one may use the classical result of Embrechts and Veraverbeke (1982).
- In the super-exponential case, one may use the ones of Macci (2008).

Infinite time horizon, regular variation case

- In the sub-exponential case, Embrechts and Veraverbeke (1982) have shown that

$$\psi(u) \sim \frac{\mu}{c-\lambda\mu}\int_u^{+\infty}(1-F_W(x))dx \text{ as } u \to +\infty.$$

- In the α-regularly varying case with $\alpha > 1$ (this means that

$$1 - F_W(x) = x^{-\alpha} l(x) \text{ as } x \to +\infty,$$

where l is a slowly varying function), this corresponds to

$$\psi(u) = \frac{\mu}{c-\lambda\mu}\frac{1}{\alpha-1}u^{-\alpha+1}l(u).$$

- From the previous analysis, we get that

$$E\left[I_{+\infty}(u)\right] = \frac{1}{c}\frac{1}{1-\psi(0)}\frac{\mu}{c-\lambda\mu}\frac{1}{(\alpha-1)(\alpha-2)(\alpha-3)}u^{-\alpha+3}l(u)$$

for $\alpha > 3$.

Finite time horizon, regular variation case

In the Pareto case, with very large initial reserve u one would expect that one large claim would be responsible for ruin and for the main contribution to the penalty function

$$E(I_T(u)).$$

This is a well-known heuristic result for ruin probabilities, but does it remain true for the expected time-integrated negative part of the risk process? Denote by T_u the time to ruin. Using the decomposition

$$E(I_T(u)) = E\left(I_T(u) \mid T_u \leq T\right)\psi(u,T),$$

the result we expect is that one large claim is likely to cause ruin. Given that this claim occurs, the conditional distribution of this large claim instant is uniform on the interval $[0,T]$ (with average $T/2$), and the average severity at ruin is of the same order as

$$e(u) \sim \frac{1}{\alpha-1}u.$$

Finite time horizon, regular variation case

Consequently, with this approach, it is tempting to say that at the first order, given that ruin occurs before T the risk process stays below zero during an average time $T/2$ at a level equivalent to $-\frac{1}{\alpha-1}u$, which corresponds to an average surface in red

$$\frac{T}{2}\frac{1}{\alpha-1}u.$$

This would lead to the following equivalent:

$$E(I_T(u)) = E\left(I_T(u) \mid T_u \leq T\right)\psi(u,T) \sim \left[\frac{T}{2}\frac{1}{\alpha-1}u\right]\left[\lambda T u^{-\alpha}\right],$$

which may be rewritten as

$$E(I_T(u)) \sim \frac{\lambda T^2}{2(\alpha-1)}u^{-\alpha+1} \tag{1}$$

as $u \to +\infty$.

Finite time horizon, regular variation case

- It is likely that considering initial reserve u and time horizon Tu would give more interesting results than for u and T, as it is often the case for asymptotic ruin probabilities.

- After some computations, one gets the following asymptotic behavior for the average number of visits in zero:

$$EN^0(u,Tu) = \left(1 - (1+cT)^{-\alpha+1} + (T(c-\lambda\mu)(\alpha-1))\,(1+cT)^{-\alpha}\right)$$

$$\frac{\lambda}{c-\lambda\mu}\frac{1}{\alpha-1}\frac{1}{1-\psi(0)}u^{-\alpha+1}l(u).$$

- One of the first steps is to adapt a result of Biard, Lefèvre and Loisel (2008) to the case where the time horizon is Tu (instead of T).

Finite time horizon, light-tailed claim amounts

If the Cramer-Lundberg adjustment coefficient R exists, then a convexity argument enables us to show that:

$$Ce^{-Ru}\left(1 - e^{-R(c-\lambda\mu)T}\right) \sim E\left[I_{+\infty}(u)\right] - E\left[I_{+\infty}\left(E\left[U(T)\right]\right)\right]$$

$$\leq E\left[I_T(u)\right] \leq E\left[I_{+\infty}(u)\right] \sim Ce^{-Ru}.$$

Consider now the super-exponential case, i.e. assume that $E[e^{\theta W_1}] < \infty$ for all $\theta > 0$. The aim is to present a large deviation principle (LDP) based on the results in Macci (2008). Define $\Lambda : \mathbb{R} \to \mathbb{R}$ by $\Lambda(\theta) = c\theta + \lambda(E[e^{-\theta W_1}] - 1)$; let Λ^* be Fenchel-Legendre transform of Λ, i.e. the function $\Lambda^*(x) = \sup_{\theta \in \mathbb{R}}\{\theta x - \Lambda(\theta)\}$. We recall that $\Lambda'(0) = c - \lambda E[W_1]$, and the net profit condition is $\Lambda'(0) > 0$.

Proposition 1 *Assume $\Lambda'(0) \geq -1/T$. Then $\left\{\frac{1}{u^2} I(u, Tu) : u > 0\right\}$ satisfies the LDP with good rate function J defined by*

$$J(z) = \begin{cases} T\Lambda^*\left(\frac{1}{T}\left(-\frac{z}{T} - \sqrt{\left(\frac{z}{T}\right)^2 + \frac{2z}{T}} - 1\right)\right) & if\ z > 0 \\ 0 & if\ z = 0 \\ \infty & if\ z < 0. \end{cases}$$

This means that

$$-\inf_{z \in E^\circ} J(z) \leq \liminf_{u \to \infty} \frac{1}{u} \log P\left(\frac{1}{u^2} I_{Tu}(u) \in E\right) \leq \limsup_{u \to \infty} \frac{1}{u} \log P\left(\frac{1}{u^2} I_{Tu}(u) \in E\right) - \inf_{z \in \overline{E}} J(z)$$

for all measurable sets E (E° is the interior of E and $\overline{E}$ is the closure of E).

References

[1] Biard, R., Lefèvre, C. and Loisel, S. (2008) Impact of correlation crises in risk theory: asymptotics of finite-time ruin probabilities for heavy-tailed claim amounts when some independence and stationarity assumptions are relaxed, *to appear in Insurance: Math. Econom.*

[2] Biard, R., Loisel, S., Macci, C. and Veraverbeke, N. (2008) Asymptotic behavior of the finite-time expected time-integrated negative part of some risk processes. *Work still in progress*.

[3] Embrechts, P. and Veraverbeke, N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance: Math. Econom.*, 1(1):55–72.

[4] Loisel, S. (2005). Differentiation of some functionals of risk processes, and optimal reserve allocation. *J. Appl. Probab.*, 42(2):379–392.

[5] Macci, C. (2008). Large deviations for the time-integrated negative parts of some processes. *Statistics and Probability Letters*, 78(1):75–83.