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Convexity, Optimization and Stochastic Control
Master 2 Probabilités et Finance
Session V

We denote throughout ∂_t the partial gradient with respect to t. Subscripts indicate partial gradients with respect to space variables, e.g. $f_x := \frac{\partial f}{\partial x}, f_{xx} := \frac{\partial^2 f}{\partial x^2}$, $f_{xy} := \frac{\partial^2 f}{\partial x \partial y}$, etc...

EXERCISE

Let W be a scalar Brownian motion, and $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the corresponding completed filtration. Answer the following questions with a formal justification.

1. Provide the dynamic programming equation of the stochastic control problem

$$V_0 := \inf_{\nu \in \mathcal{U}} \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\nu_t^2 + f(X_t^\nu)\right) \mathrm{d}t\right],$$

where $\mathcal{U}$ is the collection of all $\mathbb{F}-$progressively measurable processes with values in $\mathbb{R}$, and the controlled process is defined by :

EDP:

$$dX_t^\nu = Y_t^\nu \mathrm{d}W_t, \qquad dY_t^\nu = (X_t^\nu + \nu_t)\mathrm{d}t + \nu_t \mathrm{d}W_t \quad t \in [0, T]$$

$$\begin{cases} \inf_{u \in \mathcal{U}} \{\partial_t v + \mathcal{L}v + \frac{1}{2}u^2 + f(x)\} = 0 \\ u|_{t=T} = 0 \end{cases}$$

$$\mathcal{L}f(x,y) = \partial_y f \cdot (x+u) + \frac{1}{2}\partial_{xx} f \cdot y^2 + \frac{1}{2}\partial_{yy} f \cdot u^2 + \partial_{xy} f \cdot uy$$

2. What is the dynamic programming equation corresponding to the control problem

$$V_0 := \inf_{\nu \in \mathcal{U}} \mathbb{E}\Big[e^{-\int_0^T \nu_t \mathrm{d}t}\Big],$$

where $\beta \in \mathbb{R}$, $\mathcal{U}$ is the collection of all $\mathbb{F}-$progressively measurable processes with values in $\mathbb{R}$, and the controlled process is defined by :

$$\begin{cases} \inf\{\partial_t v - u v + x \partial_x v + \frac{1}{2}u^2 v_{xx}\} = 0 \\ v(T,x) = 1 \end{cases}$$

$$dX_t^\nu = X_t^\nu \mathrm{d}t + \nu_t \mathrm{d}W_t, \quad t \geq 0.$$

3. Which stochastic control problem induces the following nonlinear PDE as the corresponding dynamic programming equation : $v_{xx} = v^+_{xx} - v^-_{xx}$; $V_0 = \sup_\nu \mathbb{E}\left[\int_0^T e^{-3s} X_s^2 ds + e^{-3T} g(X_T)\right]$

$$+\partial_t v + \underbrace{8v^+_{xx} - 2v^-_{xx}}_{2v_{xx} + \sup_{u \in [0,1]}\{6u v_{xx}\}} - 3v + x^2 = 0, \quad \text{on } [0,T) \times \mathbb{R}, \quad v\big|_{t=T} = g,$$

$$dX_t = \sqrt{(2 + 6\nu_t)2}\, dW_t \qquad \nu_t \in [0,1] \ \forall t$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ is bounded continuous, and $y^+ := \max\{y, 0\}$, $y^- := (-y)^+$, $y \in \mathbb{R}$.

4. Which stochastic control problem induces the following nonlinear PDE as the corresponding dynamic programming equation :

$$V_0 = \sup_\nu \mathbb{E}\left[\int_0^\infty e^{-\beta s} f(X_s^\nu) ds\right]$$

$$-\beta v + \underbrace{5v^+_x - 2v^-_x}_{2v_x + \sup_{u \in [0,1]}\{3u v_x\}} + 2v_{xx} + f = 0, \quad \text{on } \mathbb{R}. \qquad dX_t^\nu = (2 + 3\nu_t)dt + 2dW_t$$

where $\beta > 0$ is a given scalar, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a bounded continuous function, and $y^+ := \max\{y, 0\}$, $y^- := (-y)^+$, $y \in \mathbb{R}$.

$\nu_s \in [0,1] \ \forall s$

5. Which problem induces the following nonlinear PDE as the corresponding dynamic programming equation :

$\max\{\partial_t v + 8v_{xx} + f, -v\} = 0$ $\quad V_0 = \max_{\tau} \mathbb{E}\left[\int_0^{\tau} f(X_s)ds\right]$

$$\min\left\{-\partial_t v - 8v_{xx} - f, v\right\} = 0, \quad \text{on } [0,T) \times \mathbb{R}, \quad v\big|_{t=T} = 0,$$ $dX_t = 2dW_t$

$g \equiv 0$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a given bounded continuous function.

PROBLEM : Stochastic control under expectation constraint

Throughout this problem, W is a $d-$dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let X^ν be a process valued in $\mathbb{R}^d$ defined as the solution of the controlled stochastic differential equation

$$dX_t^\nu = b(X_t^\nu, \nu_t)\mathrm{d}t + \sigma(X_t^\nu, \nu_t)\mathrm{d}W_t, \ t \geq 0,$$

where the control ν is to be chosen in the set $\mathcal{U}$ of all $\mathbb{H}^2$ processes ν with values in some subset U of $\mathbb{R}^k$, for some given k. The coefficients $(b, \sigma) : \mathbb{R}^d \times U \longrightarrow \mathbb{R}^d \times \mathcal{M}_{\mathbb{R}}(d,d)$ satisfy all appropriate conditions so that, for all control process $\nu \in \mathcal{U}$, the controlled SDE has a unique solution X^ν satisfying $\mathbb{E}\big[\sup_{t\leq T} |X_t^\nu|^2\big] < \infty$.

Our objective is to solve the stochastic control problem

$$V_0 \ := \ \sup_{\nu\in\mathcal{A}} \mathbb{E}\left[\int_0^T f(X_t^\nu, \nu_t)\mathrm{d}t\right], \text{ with } \mathcal{A} := \left\{\nu \in \mathcal{U} : \ \mathbb{E}\big[g(X_T^\nu)\big] \leq 0\right\},$$

for some continuous functions f and g with quadratic growth $\sup_{(x,u)} \frac{|f(x,u)|+|g(x)|}{1+|x|^2+|u|^2} < \infty$.

1. Show that

$$V_0 \ \leq \ \inf_{\lambda\geq 0} V_0^\lambda, \quad \text{where} \quad V_0^\lambda := \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[\int_0^T f(X_t^\nu, \nu_t)\mathrm{d}t - \lambda g(X_T^\nu)\right].$$

2. Denote $a(x,u) := \frac{1}{2}\sigma(x,u)\sigma(x,u)^\intercal$, and

$$H(x,z,\gamma) := \sup_{u\in U} h(x,z,\gamma,u), \text{ with } h(x,z,\gamma,u) := b(x,u)\cdot z + \mathrm{Tr}[a(x,u)\gamma] + f(x,u).$$

Suppose that the partial differential equation

$$-\partial_t v(t,x) - H\big(x, Dv(t,x), D^2v(t,x)\big) = 0, \ t \in [0,T), \ x \in \mathbb{R}^d,$$

has a solution $v^\lambda \in C^{1,2}([0,T), \mathbb{R}^d) \cap C^0([0,T] \times \mathbb{R}^d)$ satisfying $v^\lambda(T,.) = -\lambda g$, together with the quadratic growth condition $\sup_{[0,T]\times\mathbb{R}^d} \frac{|v^\lambda(t,x)|}{1+|x|^2} < \infty$.

(a) Show that $v^\lambda(0, X_0) \geq V_0^\lambda$.

(b) Deduce that $\inf_{\lambda\geq 0} v^\lambda(0, X_0) \geq V_0$.

3. Assume in addition that the functions v^λ of Question 2 satisfy in addition :

(i) *there is a unique function* $\hat{u}^\lambda : [0,T] \times \mathbb{R}^d \longrightarrow U$ *such that*

$$H\big(x, Dv^\lambda(t,x), D^2 v^\lambda(t,x)\big) = h\big(x, Dv^\lambda(t,x), D^2 v^\lambda(t,x), u^\lambda(t,x)\big);$$

(ii) *the SDE* $d\hat{X}_t = b\big(\hat{X}_t, \hat{u}^\lambda(t,\hat{X}_t)\big)dt + \sigma\big(\hat{X}_t, \hat{u}^\lambda(t,\hat{X}_t)\big)dW_t$ *has a unique square integrable solution and the process* $\{\hat{\nu}_t^\lambda := \hat{u}^\lambda(t,\hat{X}_t), t \in [0,T]\}$ *is in* $\mathbb{H}^2$.

(a) Show that $v^\lambda(0, X_0) = V_0^\lambda$ with optimal control $\hat{\nu}^\lambda$.

(b) Assume further that there exists some $\lambda^* \geq 0$ such that, denoting $\nu^* := \hat{\nu}^{\lambda^*}$:

(iii) *either* $\lambda^* = 0$ *and* $\mathbb{E}\big[g\big(X_T^{\nu^*}\big)\big] \leq 0$, *or* $\mathbb{E}\big[g\big(X_T^{\nu^*}\big)\big] = 0$.

Show that $V^{\lambda^*} = \inf_{\lambda \geq 0} V_0^\lambda = V_0$, and that ν^* is an optimal control for the problems $V_0^{\lambda^*}$ and V_0.

✓ 4. In this question, we assume the two following properties :

- $\mathbb{E}\big[g(X_T^{\nu^0})\big] < 0$ *for some* $\nu^0 \in \mathcal{U}$,
- $\lambda \longmapsto \mathbb{E}\big[g(X_T^{\hat{\nu}^\lambda})\big]$ *is continuous at the point* λ^* *(right-differentiable if* $\lambda^* = 0$*).*

Our goal is to justify that Condition (iii) of the previous question is a consequence of (i) and (ii).

(a) Show that $V_0^\lambda \longrightarrow +\infty$ as $\lambda \searrow +\infty$.

(b) Verify that $\lambda \longmapsto V_0^\lambda$ is convex, and deduce that V_0^λ attains a minimum at some point $\lambda^* \geq 0$. Specify the first order optimality condition.

(c) Verify that for all $\lambda \geq 0$:

$$-(\lambda - \lambda^*)\, \mathbb{E}\big[g(X_T^{\nu^*})\big] \leq V_0^\lambda - V_0^{\lambda^*} \leq -(\lambda - \lambda^*)\, \mathbb{E}\big[g(X_T^{\hat{\nu}^\lambda})\big],$$

and deduce that Condition (iii) of Question 3b holds.

✓ 5. We now consider the one dimensional ($d = 1$) example

$$V_0 \;:=\; \sup_{\nu \in \mathcal{A}} \mathbb{E}\left[-\frac{1}{2}\int_0^T \Big(a(X_t^\nu)^2 + \nu_t^2\Big)\mathrm{d}t\right], \text{ for some given } a \geq 0,$$

— with controlled state dynamics $dX_t^\nu = \nu_t dt + dW_t$, $t \geq 0$, $X_0 = 0$

— and scalar control process ν restricted to $\mathcal{A} := \big\{\nu \in \mathbb{H}^2 : \; \mathbb{E}\big[X_T^\nu\big] \leq 0\big\}$.

(a) Verify that we may find a control process ν^0 so that $\mathbb{E}\big[X_T^{\nu^0}\big] < 0$.

(b) Verify that the dynamic programming equation for the penalized problem V_λ^0, for all $\lambda \geq 0$, is solved by the following function

$$v^\lambda(t,x) := -\frac{\sqrt{a}}{2}\tanh\big(\sqrt{a}(T-t)\big)\, x^2 + B(t)\, x + C(t), \;\; t \in [0,T], x \in \mathbb{R}^d,$$

for some function $t \longmapsto (B, C)(t)$ to be determined.

(c) Deduce the expression of the value function V_0^λ, provide an optimal control $\hat{\nu}^\lambda$, together with the corresponding SDE for the state $\hat{X}^\lambda := X^{\hat{\nu}^\lambda}$, and compute $\mathbb{E}\big[\hat{X}_T^\lambda\big]$.

(d) Find the minimizer $\hat{\lambda}$ of the function $\lambda \longmapsto V_0^\lambda$, and check whether Condition (iii) of Question 3b is satisfied.

Problème

$$dX_t^\nu = b(X_t^\nu, \nu_t)\,dt + \sigma(X_t^\nu, \nu_t)\,dW_t \qquad \nu \in \mathbb{H}^2$$

On a $\mathbb{E}\left[\sup_{[0,T]} |X_t^\nu|^2\right] < \infty$

- $|b(x,u) - b(y,u)| + |\sigma(x,u) - \sigma(y,u)| \leq C|x-y| \quad \forall x, y, u$
- $|b(x,u)| + |\sigma(x,u)| \leq C(1+|x|+|u|)$

$$V_0 = \sup_{\nu \in \mathcal{A}} \mathbb{E}\left[\int_0^T f(X_s^\nu, s)\,ds\right] \quad \text{où} \quad \mathcal{A} = \{\nu \in \mathcal{U} : \mathbb{E}[g(X_T^\nu)] \leq 0\}$$

où $|f(x,u)| + |g(x)| \leq C(1+|x|^2+|u|^2)$

(1) M.q. $V_0 \leq \inf_{\lambda \geq 0} V_0^\lambda$ où $V_0^\lambda := \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[\int_0^T f(X_t^\nu, \nu_t)\,dt - \lambda g(X_T^\nu)\right]$

$$\forall \nu \in \mathcal{A} \quad \mathbb{E}\left[\int_0^T f(X_t^\nu, \nu_t)\,dt\right] - \lambda \underbrace{\mathbb{E}[g(X_T^\nu)]}_{\leq 0} \leq V_0^\lambda$$

$$\forall \nu \in \mathcal{A} \ \forall \lambda \geq 0 \qquad \mathbb{E}\left[\int_0^T f(X_t^\nu, \nu_t)\,dt\right] \leq V_0^\lambda$$

$$\sup_{\nu \in \mathcal{A}} \mathbb{E}\left[\int_0^T f(X_t^\nu, \nu_t)\,dt\right] = V_0 \leq V_0^\lambda \quad \forall \lambda \geq 0$$

Donc $V_0 \leq \inf_{\lambda \geq 0} V_0^\lambda$.

(2) $a(x,u) = \frac{1}{2}\sigma(x,u)\sigma(x,u)^T$

$$H(x,y,z) = \sup_{u \in U} h(x,z,\gamma,u) \quad \text{où} \quad h(x,z,\gamma,u) = b(x,u)\cdot z + \mathrm{Tr}(a(x,u)\cdot\gamma) + f(x,u)$$

On suppose qu'il existe la solution v^λ régulière de

$$\begin{cases} \partial_t v(t,x) + H(x, Dv(t,x), D^2 v(t,x)) = 0 \\ v^\lambda|_{t=T} = -\lambda g \end{cases}$$

t.q. $|v^\lambda(t,x)| \leq C(1+|x|^2)$

(a) M.q. $v^\lambda(0, X_0) \geq V_0^\lambda$

Technique (à la vérification)

On considère $\vartheta \in \mathcal{U}$ et on montre que

$$v^\lambda(0, X_0) \geq \mathbb{E}\left[\int_0^T f(X_t^\vartheta, \vartheta)\,dt - \lambda g(X_T^\vartheta)\right]$$

l'idée est d'appliquer Itô sur $(v^\lambda(t, X_t^\lambda))_{t\in T}$ et utilise l'EDP:

$$\underbrace{v^\lambda(T, X_T^\lambda)}_{-\lambda g(X_T^\lambda)} = v^\lambda(0, X_0^\lambda) + \int_0^T \underbrace{(\partial_t v^\lambda + \mathcal{L}^\vartheta v^\lambda)}_{\underset{EDP}{\leq} -f(X_t^\vartheta, \vartheta_t)}(s, X_s, \vartheta_s)\,ds + \int_0^T \nabla_x v^\lambda \cdot \sigma\, dW_t$$

$$v^\lambda(0, X_0^\lambda) \geq \mathbb{E}\left[\int_0^T f(X_t^\vartheta, \vartheta_t)\,dt - \lambda g(X_T^\vartheta)\right] - \mathbb{E}\Big[\overbrace{\int_0^T \nabla_x v^\lambda \cdot \sigma\, dW_t}^{M_T}\Big]$$

(on veut $\to 0$)

Si $(M_t)_{t\leq T}$ n'est pas une martingale, il faut localiser

$$\tau^n := \inf\{t : |X_t^\vartheta| \geq n\} \wedge T \qquad M_t^{\tau^n} \text{ une mart. U.I.}$$

$\nabla_x v$ continue $\Rightarrow$ bornée sur $\{|X_t^\vartheta| \leq n\}$

$$|\sigma(X_t^\vartheta, \vartheta_t)| \leq C(1 + \underset{\hat{n}}{|X_t^\vartheta|} + \underset{\hat{\mathbb{H}}}{|\vartheta_t|})$$

→ Comme $\vartheta \in \mathbb{H}^2$, on peut vérifier $\mathbb{E}\left[\int^{T\wedge\tau_n} |v_x(s, X_s^\vartheta)\sigma(X_s, \vartheta_s)|^2 ds\right] < \infty$

(en vrai, on aurait pu juste localiser M_t par $\sigma_n \rightsquigarrow \tau_n \wedge \sigma_n$)

On obtient que $\mathbb{E}\left[v(\tau^n, X_{\tau^n}^\vartheta) + \int_0^{\tau_n} f(X_s^\vartheta, \vartheta_s) ds\right] \leq v(0, X_0)$

$$\downarrow$$

$$\mathbb{E}\left[v(T, X_T^\vartheta) + \int_0^T f(X_s^\vartheta, \vartheta_s) ds\right]$$

On sait que $v(\tau^n, X_{\tau^n}^\vartheta) \xrightarrow{p.s.} v(T, X_T^\vartheta) = -\lambda g(X_T)$

et $\int_0^{\tau_n} f(X_s^\vartheta, \vartheta_s) ds \to \int_0^T f(X_s^\vartheta, \vartheta_s) ds$

De plus, $|v(\tau^n, X_{\tau_n}^\vartheta)| \leq C(1 + |X_{\tau_n}^\vartheta|) \leq C(1 + \sup_{t \leq T} |X_t^\vartheta|^2)$

$\int_0^{\tau^n} |f(X_s^\vartheta, \vartheta_s)| ds \leq \int_0^{\tau_n} (1 + |X_s^\vartheta| + |\vartheta_s|) ds$

$\vartheta \in \mathbb{H}^2$, $|X_s^\vartheta| \leq \sup_{[0,T]} |X_s^\vartheta|$ et $\mathbb{E}[\sup_{t \leq T} |X_t^\vartheta|^2] < \infty$

Donc on peut appliquer le thm. d'arrêt

(b) $\quad v^\lambda(0, X_0) \geq V_0^\lambda \geq V_0 \quad \forall \lambda \geq 0 \Rightarrow \inf_{\lambda \geq 0} v^\lambda(0, X_0) \geq V_0$

(3) En plus, on suppose que

(i) $\exists!\ \hat{u}^\lambda$: $H(x, Du^\lambda(t,x), D^2u^\lambda(t,x)) = h(x, Du^\lambda(t,x), D^2u^\lambda(t,x), u^\lambda(t,x))$

(ii) l'EDS $d\hat{X}_t = b(\hat{X}_t, \hat{u}^\lambda) dt + \sigma(\hat{X}_t, \hat{u}^\lambda) dW_t$ admet unique solution

de carré intégrable et $(\hat{\vartheta}^\lambda = \hat{u}^\lambda(t, \hat{X}_t),\ t \in [0,T])$ est dans $\mathbb{H}^2$

(a) On peut appliquer maintenant le théorème de vérification

car on peut faire le m̂ raisonnement avec $=$ au lieu de $\leq$ car il existe $\hat{u}^\lambda$ et $(X_t^{\hat{u}^\lambda})$

qui donne
- $v^\lambda(0, X_0) = V^\lambda(0, X_0)$
- $\hat{\vartheta}^\lambda$ contrôle optimale

(b) On suppose que $\exists \lambda^* \geq 0$ t.q. $(\vartheta^* = \hat{\vartheta}^{\lambda^*})$

- Soit $\lambda^* = 0$ et $\mathbb{E}[g(X_T^{\vartheta^*})] \leq 0$ soit $\mathbb{E}[g(X_T^{\vartheta^*})] = 0 \quad (*)$

M.q. $V^{\lambda^*} = \inf\limits_{\lambda \geq 0} V_0^\lambda = V_0$ et ϑ^* contrôle optimale pour $\underset{\text{(par (a))}}{V_0^{\lambda^*}}$ et V_0

On a $V_0 \leq \inf\limits_{\lambda \geq 0} V_0^\lambda \leq V^{\lambda^*}$ donc il suffit de m.q. $V^{\lambda^*} \leq V_0$

$$\sup_{\vartheta \in \mathcal{U}} \left\{ \mathbb{E}\left[\int_0^T f(X_s^\vartheta, \vartheta_s)ds - \lambda^* g(X_T^\vartheta)\right] \right\}$$

$$\|$$

$$\mathbb{E}\left[\int_0^T f(X_s^{\vartheta^*}, \vartheta_s^*)ds - \lambda^* g(X_T^{\vartheta})\right] \overset{(*)}{=} \mathbb{E}\left[\int_0^T f(X_s^{\vartheta^*}, \vartheta_s^*)ds\right]$$

Il faut m.q. $\forall \vartheta \in \mathcal{A} = \{\vartheta : \mathbb{E}[g(X_T^\vartheta)] \leq 0\}$

$$\mathbb{E}\left[\int_0^T f(X_s^{\vartheta^*}, \vartheta_s^*)ds\right] \geq \mathbb{E}\left[\int_0^T f(X_s^\vartheta, \vartheta_s)\, ds\right]$$

On sait que ϑ^* est sol. de V^{λ^*} → $\forall \vartheta \in \mathcal{U}$

$$\mathbb{E}\left[\int_0^T f(X_s^{\vartheta^*}, \vartheta_s^*)ds\right] \geq \mathbb{E}\left[\int_0^T f(X_s^\vartheta, \vartheta_s)ds - g(X_T^\vartheta)\right] \overset{\text{si } \vartheta \in \mathcal{A}}{\geq} \mathbb{E}\left[\int_0^T f(X_s^\vartheta, \vartheta_s)ds\right]$$

→ ϑ^* est un contrôle optimale pour V_0.

(4) On suppose que

- $\mathbb{E}[g(X_T^{\vartheta^0})] < 0$ $\vartheta^0 \in \mathcal{U}$
- $\lambda \mapsto \mathbb{E}[g(X_T^{\hat{\vartheta}^\lambda})]$ est continue en λ^* (dérivable à droite si $\lambda^* = 0$)

On veut montrer que (i) et (ii) ⇒ (iii)

(a) M.q. $V_0^\lambda \xrightarrow[\lambda \uparrow +\infty]{} +\infty$

On prend $\vartheta = \vartheta^0$ ⇒ $\overbrace{\mathbb{E}\left[\int_0^T f(X_s^{\vartheta^0}, \vartheta_s^0)ds\right]}^{\text{ne dépend pas de } \lambda,\ < \infty} - \lambda \overbrace{\mathbb{E}[g(X_T^{\vartheta^0})]}^{<0} \xrightarrow[\lambda \to +\infty]{} +\infty$

(b) Vérifier que $\lambda \mapsto V_0^\lambda$ est convexe $\to \exists$ point de min $\lambda^* \geq 0$

Condition de 1re ordre?

continue + coersive (a)

Soit $\alpha \in (0,1)$, $\lambda_1, \lambda_2 \geq 0$

$$V_0^{\alpha\lambda_1+(1-\alpha)\lambda_2} = \sup_{\vartheta\in\mathcal{U}} \mathbb{E}\left[\int_0^T f(X_s^\vartheta, \vartheta_s)ds - (\alpha\lambda_1+(1-\alpha)\lambda_2)g(X_T^\vartheta)\right] =$$

$$= \sup_{\vartheta\in\mathcal{U}} \Big\{ \alpha \underbrace{\mathbb{E}\Big[\int_0^T f(X_s^\vartheta, \vartheta_s)ds - \lambda_1 g(X_T^\vartheta)\Big]}_{\leq V_0^{\lambda_1}} + (1-\alpha)\underbrace{\mathbb{E}\Big[\int_0^T f(X_s^\vartheta, \vartheta_s)ds - \lambda_2 g(X_T^\vartheta)\Big]}_{\leq V_0^{\lambda_2}} \Big\} \leq$$

$$\leq \alpha V_0^{\lambda_1} + (1-\alpha)V_0^{\lambda_2}$$

(On peut dire aussi que V_0^λ est sup de fct cnvxes $\to$ convexe)

Condition de 1re ordre: $\frac{\partial^d}{\partial x}V_0^{\lambda^*} \geq 0$ et $\frac{\partial^g}{\partial x}V_0^{\lambda^*} \geq 0$ si $\lambda^* > 0$

Si $\lambda^* = 0$ $\quad \frac{d^+}{d\lambda}V_0^0 \geq 0$

(c) Vérifier que $\forall \lambda \geq 0$ $\quad -(\lambda-\lambda^*)\mathbb{E}[g(X_T^{\vartheta^*})] \leq V_0^\lambda - V_0^{\lambda^*} \leq -(\lambda-\lambda^*)\mathbb{E}[g(X_T^{\hat\vartheta^\lambda})]$

En déduire (iii)

$$V_0^\lambda - V_0^{\lambda^*} = \mathbb{E}\left[\int_0^T f(X_s^{\hat\vartheta^\lambda}, \hat\vartheta_s^\lambda) - f(X_s^{\vartheta^*}, \vartheta_s^*)ds - \lambda g(X_T^{\hat\vartheta^\lambda}) - \lambda^* g(X_T^{\vartheta^*})\right]$$

De plus, on a $\mathbb{E}\left[\int_0^T f(X_s^{\vartheta^*}, \vartheta_s^*)ds - \lambda g(X_T^{\vartheta^*})\right] \leq \mathbb{E}\left[\int_0^T f(X_s^{\hat\vartheta^\lambda}, \hat\vartheta_s^\lambda)ds - \lambda g(X_T^{\hat\vartheta^\lambda})\right]$

$$\mathbb{E}\left[\int_0^T f(X_s^{\hat\vartheta^\lambda}, \hat\vartheta_s^\lambda) - f(X_s^{\vartheta^*}, \vartheta_s^*)ds\right] \geq -\lambda\mathbb{E}[g(X_T^{\vartheta^*})] + \lambda\mathbb{E}[g(X_T^{\hat\vartheta^\lambda})]$$

Donc $V_0^\lambda - V_0^{\lambda^*} \geq -(\lambda-\lambda^*)\mathbb{E}[g(X_T^{\vartheta^*})]$

Pour $V_0^{\lambda^*}$ on a $\mathbb{E}\left[\int_0^T f(X_s^{\hat\vartheta^\lambda}, \hat\vartheta_s^\lambda)ds - \lambda^* g(X_T^{\hat\vartheta^\lambda})\right] \leq \mathbb{E}\left[\int_0^T f(X_s^{\vartheta^*}, \vartheta_s^*)ds - \lambda^* g(X_T^{\vartheta^*})\right]$

$$\mathbb{E}\left[\int_0^T f(X_s^{\hat\vartheta^\lambda}, \hat\vartheta_s^\lambda) - f(X_s^{\vartheta^*}, \vartheta_s^*)ds\right] \leq -\lambda^*\mathbb{E}[g(X_T^{\vartheta^*})] + \lambda^*\mathbb{E}[g(X_T^{\hat\vartheta^\lambda})]$$

Donc $V_0^\lambda - V_0^{\lambda^*} \leq -(\lambda-\lambda^*)\mathbb{E}[g(X_T^{\hat{\vartheta}^\lambda})]$

$$-(\lambda-\lambda^*)\mathbb{E}[g(X_T^{\vartheta^*})] \leq V_0^\lambda - V_0^{\lambda^*} \leq -(\lambda-\lambda^*)\mathbb{E}[g(X_T^{\hat{\vartheta}^\lambda})]$$

Si $\lambda^* = 0$ $\quad 0 \leq V_0^\lambda - V_0^{\lambda^*} \leq -\lambda \mathbb{E}[g(X_T^{\hat{\vartheta}^\lambda})] \Rightarrow \mathbb{E}[g(X_T^{\hat{\vartheta}^\lambda})] \leq 0 \quad \forall \lambda$

$\lambda > 0$ $\qquad \downarrow \lambda \downarrow 0$ (continuité en λ^*)

$$\mathbb{E}[g(X_T^{\vartheta^*})] \leq 0$$

Si $\lambda^* > 0$ $\quad -(\lambda-\lambda^*)\mathbb{E}[g(X_T^{\hat{\vartheta}^\lambda})] \leq 0 \quad \forall \lambda$

$$\left.\begin{array}{l} \lambda > \lambda^* \Rightarrow \mathbb{E}[g(X_T^{\hat{\vartheta}^\lambda})] \leq 0 \\ \lambda < \lambda^* \Rightarrow \mathbb{E}[g(X_T^{\hat{\vartheta}^\lambda})] \geq 0 \end{array}\right\} \text{cont. en } \lambda^* \Rightarrow \lambda \to \lambda^* \Rightarrow \mathbb{E}[g(X_T^{\vartheta^*})] = 0.$$

(5) $\quad V_0 := \sup_{\vartheta \in \mathcal{A}} \mathbb{E}\left[-\frac{1}{2}\int_0^T (a(X_t^\vartheta)^2 + \vartheta_t^2)dt\right] \quad a \geq 0$

$$dX_t^\vartheta = \vartheta_t dt + dW_t \quad t \geq 0$$

$$\vartheta \in \mathcal{A} = \{\vartheta \in \mathbb{H}^2 : \mathbb{E}[X_T^\vartheta] \leq 0\}$$

(a) Vérifier qu'il existe ϑ^0 : $\mathbb{E}[X_T^{\vartheta^0}] < 0$

$$\vartheta^0 \equiv -1 \Rightarrow X_T^{\vartheta^0} = -T + W_t \quad \mathbb{E}[X_T^{\vartheta^0}] = -T < 0$$

(b) Vérifier que la sol. de V_0^λ peut être écrite dans la forme

$$v^\lambda(t,x) = -\frac{\sqrt{a}}{2}\, th(\sqrt{a}(T-t))x^2 + B(t)x + C(t), \quad t \in [0,T], \ x \in \mathbb{R}^d$$

$$V_0^\lambda = \sup_{\vartheta \in \mathbb{H}^2}\left\{\mathbb{E}\left[-\frac{1}{2}\int_0^T (a\cdot(X_t^\vartheta)^2 + \vartheta_t^2)dt - \lambda X_T^\vartheta\right]\right\}$$

$$f(x,u) = -\frac{1}{2}(ax^2 + u^2) \qquad g(x) = x$$

$$\mathcal{L}v(t,x) = \partial_t v + u\,\partial_x v + \frac{1}{2}\partial_{xx} v$$

HJB $\sup_u \{ \partial_t v + u \partial_x v + \frac{1}{2}\partial_{xx} v - \frac{1}{2} a x^2 - \frac{1}{2} u^2 \} = 0$

$$u^* = \partial_x v$$

$$\begin{cases} \partial_t v^\lambda + \frac{1}{2}\partial_{xx} v^\lambda + \frac{1}{2}(\partial_x v^\lambda)^2 - \frac{1}{2} a x^2 = 0 \\ v^\lambda|_{t=T} = -\lambda x \quad \to B(T) = -\lambda, \quad C(T) = 0 \end{cases}$$

$$(th(\theta))' = \left(\frac{sh(\theta)}{ch(\theta)}\right)' = \frac{ch(\theta)^2 - sh(\theta)^2}{ch(\theta)^2} = \frac{1}{ch(\theta)^2}$$

$$v^\lambda(t,x) = -\frac{\sqrt{a}}{2} th(\sqrt{a}(T-t)) x^2 + B(t) x + C(t)$$

$$\partial_t v^\lambda = -\frac{\sqrt{a}}{2} \cdot \frac{-\sqrt{a}\, x^2}{ch(\sqrt{a}(T-t))^2} + \dot{B} x + \dot{C} = \frac{a x^2}{2} ch(\sqrt{a}(T-t))^{-2} + \dot{B} x + \dot{C}$$

$$\partial_{xx} v^\lambda = -\sqrt{a}\, th(\sqrt{a}(T-t))$$

$$(\partial_x v^\lambda)^2 = \left(B(t) - \sqrt{a}\, th(\sqrt{a}(T-t)) x\right)^2 = B(t)^2 - 2\sqrt{a} B(t) th(\sqrt{a}(T-t)) x + a\, th(\underbrace{\sqrt{a}(T-t)}_{\theta})^2 x^2 =$$

$$sh^2(\theta) = ch(\theta)^2 - 1$$

$$= \left\{ th(\theta)^2 = \frac{sh(\theta)^2}{ch(\theta)^2} = 1 - \frac{1}{ch(\theta)^2} \right\} = B(t)^2 - 2\sqrt{a} B(t)\, th(\sqrt{a}(T-t)) x + a x^2 - \frac{a x^2}{ch(\theta)^2}$$

Dans l'EDP

$$x^2 \frac{a}{2} \cancel{ch(\theta)^{-2}} - \frac{\sqrt{a}}{2} th(\theta) + \frac{1}{2} B(t)^2 + \sqrt{a} B(t)\, th(\theta) x + \cancel{\frac{a x^2}{2}} - \cancel{\frac{a x^2}{2 ch(\theta)^2}} - \cancel{\frac{a x^2}{2}} = 0$$
$$+ \dot{B} x + \dot{C}$$

$$\begin{cases} \dot{B}(t) + \sqrt{a} B(t)\, th(\theta) = 0 \quad B(T) = -\lambda \\ \dot{C}(t) - \frac{\sqrt{a}}{2} th(\theta) + \frac{1}{2} B(t)^2 = 0 \quad C(T) = 0 \end{cases} \rightsquigarrow \begin{matrix} B(t) = \dots \\ C(t) = \dots \end{matrix}$$

$$\frac{d}{dt} \log |B(t)| = -\sqrt{a}\, th(\sqrt{a}(T-t))$$

$$\log|B(T)| - \log|B(t)| = -\sqrt{a} \int_t^T th(\sqrt{a}(T-s))\, ds$$

$$\log|B(t)| = \log\lambda + \sqrt{a}\int^{T}_{t} th(\sqrt{a}(T-s))ds$$

$$B(t) = -\lambda e^{-\sqrt{a}\int_t^T th(\sqrt{a}(T-s))ds}$$ on peut calculer ça, mais

$$\dot{C}(t) = \frac{\sqrt{a}}{2} th(\sqrt{a}(T-t)) - \frac{\lambda^2}{2} e^{-2\sqrt{a}\overbrace{\int_t^T th(\sqrt{a}(T-s))ds}^{\Psi(t)}}$$

$$C(t) = -\frac{\sqrt{a}}{2}\Psi(t) + \frac{\lambda^2}{2}\int_t^T e^{-2\sqrt{a}\Psi(s)} ds$$

$$v^\lambda(t,x) = -\frac{\sqrt{a}}{2} th(\sqrt{a}(T-t))x^2 + B(t)x + C(t)$$

$$V_0^\lambda = v^\lambda(0,0) = C(0) = -\frac{\sqrt{a}}{2}\Psi(0) + \frac{\lambda^2}{2}\int_0^T e^{-2\sqrt{a}\Psi(s)} ds$$

$$\hat{\vartheta}^\lambda : \quad \hat{\vartheta}^\lambda_t = \partial_x v(t,x) = -\sqrt{a}\, th(\sqrt{a}(T-t))x - \lambda e^{-\sqrt{a}\Psi(t)}$$

$$d\hat{X}^\lambda_t = -\left(\sqrt{a}\, th(\sqrt{a}(T-t))\hat{X}^\lambda_t + \lambda e^{-\sqrt{a}\Psi(t)}\right)dt + dW_t$$

$$m(t) = \mathbb{E}\hat{X}^\lambda_t = -\int_0^t \left(\sqrt{a}\, th(\sqrt{a}(T-t))m(s) + \lambda e^{-\sqrt{a}\Psi(s)}\right)ds$$

$$\begin{cases} \dot{m}(t) = -\sqrt{a}\, th(\sqrt{a}(T-t))m(s) + \lambda e^{-\sqrt{a}\Psi(s)} \\ m(0) = 0 \end{cases}$$

EDO linéaire $\rightsquigarrow m(t) = \mathbb{E}[\hat{X}^\lambda_t]$

(d) Minimiser $\lambda \mapsto V_0^\lambda$ $\quad \lambda \mapsto A + B\lambda^2$, $B > 0$ $\to$ $\underline{\lambda^* = 0}$

Vérifier la condition (iii). soit $\lambda^* = 0$ et $\mathbb{E}[(X_T^{\vartheta^*})] \leq 0$ soit $\mathbb{E}[(X_T^{\vartheta^*})] = 0$

On a $\lambda^* = 0$ et $\mathbb{E}[\hat{X}^0_T] = 0$ $(m(t) \equiv 0$ est l'unique solution si $\lambda = 0)$

$\to$ la condition est vérifiée.