Chapter 2

Structure of convex sets

2.1 Topological properties

Proposition 2.1.11 *Let C be a convex set. The adherence $adh(C)$ of C is convex.*

Proof. It follows from the definition of the adherence. $\square$

Remark 2.1.2 *For F closed subset of E, $conv(F)$ is not always closed. Indeed, consider $E = \mathbb{R}^2$, $F_1 = \{(x, y) \in \mathbb{R}^2 \ : \ x > 0, \ xy = 1\}$ and F_2 the symmetric of F_1 w.r.t. the first axe. Then F_1 and F_2 are closed, so $F = F_1 \cup F_2$ is also closed. But $conv(F) = \{(x, y) \ : \ x > 0, y \in \mathbb{R}\}$ which is not closed.*

Definition 2.1.11 (Closed convex hull) *The closed convex hull of a subset A of E is the adherence of its convex hull $adh(conv(A))$.*

Proposition 2.1.12 *(i) $adh(conv(A))$ is the smallest closed convex set containing A.*

(ii)If $A_1, \ldots, A_p$ are convex compact subsets of E, then $conv(A_1, \ldots, A_p)$ is compact.

(iii) If A is a compact subset of E, $conv(A)$ is compact.

Proof. (i) It follows from the definition of the closed convex hull.
(ii) We first notice that

$$conv(A_1, \dots, A_p) \;=\; \Big\{ \sum_{i=1}^{p} \lambda_i x_i \;:\; x_i \in A_i,\ \lambda_i \geq 0 \text{ and } \sum_{i=1}^{p} \lambda_i = 1 \Big\}.$$

Therefore, $conv(A_1, \dots, A_p)$ is compact as the image by a continuous application of the compact set $A_1 \times \cdots \times A_p \times \{(\lambda_i)_{i=1}^p \in (\mathbb{R}_+)^p \;:\; \sum_{i=1}^p \lambda_i = 1\}$.
(iii) It is a particular case of (ii) $\square$

We now study the interior $int(C)$ of a convex set C. We first notice that

$$ext(C) \cap int(C) \;=\; \emptyset\,.$$

We start with the following lemma.

Lemma 2.1.1 *Let C be a convex subset of E and $x_0 \in int(C)$. For any $x \in adh(C)$, we have $[x_0, x) \subset int(C)$.*

Proof. Let $y \in [x_0, x)$, $r > 0$ and x_0 such that $B(x_0, r) \subset C$.

i) Suppose first that $x \in C$. Let f be the homothety centered at x with ratio λ such that $f(x_0) = y$. We then have

$$f(z) \;=\; x + \lambda(z - x) \;=\; (1 - \lambda)x + \lambda z$$

for all $z \in E$, where λ is such that $y = f(x_0) = (1 - \lambda)x + \lambda x_0$, so $0 < \lambda \leq 1$. We have $f(C) \subset C$ by convexity of C, and $f(B(x_0, r)) = B(y, \lambda r)$. Thus $B(y, \lambda r) \subset C$ et $y \in int(C)$.

ii) Suppose that $x \in adh(C)$ and $y \in (x_0, x)$. Let g be the homothety centered at y with ratio λ such that $g(x_0) = x$. We have $g(t) = y + \lambda(t - y)$; the condition $g(x_0) = x$ writes $x - y = \lambda(x_0 - y)$, hence $\lambda < 0$. Then $g(B(x_0, r)) = B(x, |\lambda| r)$. Since $x \in adh(C)$, there exists $z \in C \cap B(x, |\lambda| r)$. let $u = g^{-1}(z)$. Then $u \in B(x_0, r) \subset int(C)$, and $z - y = \lambda(u - y)$, $\lambda < 0$ shows that $y \in (u, z)$. From Step i) above, we get $y \in int(C)$. $\square$

Proposition 2.1.13 *Let C be a convex subset of E such that $int(C) \neq \emptyset$. Then $int(C)$ is convex. Moreover, we have $adh(int(C)) = adh(C)$ and $int(adh(C)) = int(C)$.*

Proof. i) If $x, y \in int(C)$, we have $[x, y] \subset int(C)$ from the previous Lemma. Hence $int(C)$ is convex.
ii) The inclusion $adh(int(A)) \subset adh(A)$ is true for any $A \subset E$. Conversely, if $x \in adh(C)$, then x is a boundary of an open line included in C. By the previous lemma we have $x \in adh(int(C))$.

Let us notice that the adherence of the nonempty line (x_0, x) is $[x_0, x]$.
iii) The inclusion $int(A) \subset int(adh(A))$ is true for any $A \subset E$. Conversely, let $x \in int(adh(C))$ and $r > 0$ such that $B(x, r) \subset adh(C)$. Since $adh(C) = adh(int(C))$ by ii), it exists $y \in C \cap B(x, r)$. Let $z \in E$ such that $x = (z + y)/2$. We still have $z \in B(x, r)$, so $z \in adh(C)$. From the previous lemma$(z, y] \subset int(C)$. in particular $x \in int(C)$. □

2.2 Separation of convex sets

Theorem 2.2.2 (Projection on convex sets) *Let C be a closed convex subset of E and $x_0 \notin C$. there exists a unique $y_0 \in C$, called projection of x_0 on C such that*

$$|x_0 - y_0| \quad = \quad \inf_{y \in C} |x_0 - y| \ .$$

The projection of x_0 on C is characterized by the following inequality

$$\langle x_0 - y_0, y - y_0 \rangle \quad \leq \quad 0 \tag{2.2.1}$$

for all $y \in C$

Proof. Let $r = d(x_0, C) > 0$ and $B = C \cap \bar{B}(x_0, 2r)$. We then have

$$d(x_0, C) \quad = \quad \inf_{y \in C} |y - x_0| \quad = \quad \inf_{y \in B} |y - x_0|.$$

The second equality comes from $d(x_0, C) = \min\{d(x_0, B), d(x_0, C \setminus B)\}$ combined with $d(x_0, C \setminus B) \geq 2r$.

Since B is a closed and bounded subset of the finite dimension space E, B is compact. The continuous function $y \mapsto |y - x_0|$ admits a minimum over B. Let $y_0 \in B$ a minimum point, that is $|y - x_0| = d(x_0, C)$. We then have

$$|x_0 - y|^2 \quad \geq \quad |x_0 - y_0|^2,$$

for all $y \in C$. Hence, by writing $x_0 - y := (x_0 - y_0) + (y_0 - y)$ we get

$$\langle x_0 - y_0, y - y_0 \rangle \leq \frac{1}{2}|y - y_0|^2.$$

Let $\theta \in (0,1)$. We apply the previous inequality to $y^\theta = \theta y + (1-\theta)y_0 \in C$ and we get

$$\theta \langle x_0 - y_0, y - y_0 \rangle \leq \frac{1}{2}\theta^2 |y - y_0|^2$$

Dividing by θ and sending θ to $0+$, we get (2.2.1). Conversely, for $y_0 \in C$ satisfying (2.2.1), we have

$$\begin{aligned} |x_0 - y|^2 &= |x_0 - y_0|^2 + 2\langle x_0 - y_0, y_0 - y \rangle + |y_0 - y|^2 \\ &\geq |x_0 - y_0|^2 + |y_0 - y|^2 \\ &\geq |x_0 - y_0|^2 \end{aligned}$$

and y_0 is a minimum point for the function $|x_0 - \cdot|$ over C. Moreover, the last inequality is strict whenever $y \neq y_0$, which gives the uniqueness of y_0.

□

The two following separation theorems are presented and proved in the finite dimensional framework. However, they also hold true in infinite dimension. We first recall the definition of an affine hyperplan.

Definition 2.2.12 *An affine hyperplane H is a subset of E such that the set*

$$H - x = \{y - x \ : \ y \in H\}$$

is a vector hyperplane of E for some $x \in E$.

From Riez representation Theorem, we deduce that a subset H of E is an affine hyperplane if and only if there exist a linear form f and a constant c such that

$$H = \{x \in E \ : \ f(x) = c\}.$$

Theorem 2.2.3 (Separation of a point and a closed convex set) *Let C be a closed convex subset of E and $x_0 \in E$ such that $x_0 \notin C$. There exists an affine hyperplan H strictly separating x_0 from C. That is, there exist a linear form f and a constant c such that*

$$f(x_0) \;>\; c \quad \textit{and} \quad f(y) \;<\; c \;\; \forall y \in C\,.$$

Proof. Let x_0 and C be as above. The distance $d(x_0, C) = \inf\{|y - x_0| : y \in C\}$ is strictly positive since $x_0 \notin C$ and C is closed. From Theorem 2.2.2 there exists a unique $y_0 \in C$ such that $|y_0 - x_0| = d(x_0, C)$. This y_0 being characterized by

$$\langle y - y_0, x_0 - y_0\rangle \;\le\; 0 \tag{2.2.2}$$

for all $y \in C$. Set $f(y) = \langle y, x_0 - y_0\rangle$ for $y \in E$. The function f is a non zero linear form on E. The condition (2.2.2) can be stated as follows

$$f(y) \;\le\; f(y_0)\,,$$

for all $y \in C$. Moreover, we have

$$f(x_0) \;=\; f(y_0) + f(x_0 - y_0) \;=\; f(y_0) + |x_0 - y_0|^2 \;>\; f(y_0).$$

Let H be the hyperplane defined by

$$y \in H \;\Leftrightarrow\; f(y) = c,$$

where $c = 1/2(f(x_0) + f(y_0)) = f((x_0 + y_0)/2)$. We then have

$$f(x_0) \;>\; c \quad \text{and} \quad f(y) \;<\; c \;\; \forall y \in C\,.$$

That is, x_0 and C are included in each of the open half spaces defined by H.
□

Theorem 2.2.4 (Separation of a point from an open convex) *Let C be an open convex subset of E and $x_0 \notin C$. There exists an affine hyperplane H such that $x_0 \in H$ and $H \cap C = \emptyset$.*

Proof. We can assume w.l.o.g. that $x_0 = 0$. Let $\Gamma = \cup_{\lambda>0}\lambda C$. Γ is convex and open as the union of open convex sets λC for $\lambda > 0$. Moreover, $0 \notin \Gamma$ since $0 \notin C$. In particular, $adh(\Gamma) \neq E$, otherwise we have from Proposition 2.1.13 $E = int(E) = int(adh(\Gamma)) = int(\Gamma) = \Gamma$.

Let $y_0 \in E \setminus \Gamma$. Theorem 2.2.4 give the existence of a linear form such that

$$f(y_0) \quad < \quad f(z), \;\; z \in adh(\Gamma).$$

If $z \in \Gamma$, $tz \in \Gamma$ for all $t > 0$, so $f(y_0) < f(tz) = tf(z)$ pour all $t > 0$, and $z \in \Gamma$. Dividing the previous inequality by t and taking $t \to +\infty$, we get $0 \leq f(z)$ for all $z \in \Gamma$. Hence $\Gamma \subset \{x \;:\; f(x) \geq 0\}$. Since Γ is open we have $\Gamma \subset \{x \;:\; f(x) > 0\}$ and $C \subset \{x \;:\; f(x) > 0\}$. □

Corollary 2.2.1 (Hahn-Banach Theorem) *Let C_1 and C_2 be two nonempty convex subset of E such that $C_1 \cap C_2 = \emptyset$. Suppose that C_1 is open. There exists an affine hyperplane H stricly separating C_1 and C_2. More precisely, there exists a linear form f on E and a constant c such that*

$$f(x_1) \quad < \quad c \;\leq\; f(x_2)$$

for all $x_1 \in C_1$ and $x_2 \in C_2$.

Proof. We apply Theorem 2.2.4 to the point 0 and the nonempty convex $C_1 - C_2 = \{y - z \;:\; y \in C_1, z \in C_2\}$. Indeed, $C_1 - C_2 = \cup_{z \in C_2}(C_1 - z)$ is open as the union of the open sets $C_1 - z$. Therefore, it exits a linear form f such that

$$f(y - z) \quad < \quad f(0) \quad = \quad 0$$

for all $y \in C_1$ and $z \in C_2$. Define $c := \inf_{z \in C_2} f(z)$. We then have

$$f(x_1) \leq c \leq f(x_2)$$

for all $x_1 \in C_1$ and $x_2 \in C_2$. Since C_1 is open $C_1 \subset \{f < c\}$. □

Corollary 2.2.2 *Let C_1 and C_2 be two nonempty convex subset of E such that $C_1 \cap C_2 = \emptyset$. There exists an affine hyperplane H separating C_1 and C_2. More precisely, there exists a nonzero linear form f on E and a constant c such that*

$$f(x_1) \quad \leq \quad c \quad \leq \quad f(x_2)$$

for all $x_1 \in C_1$ and $x_2 \in C_2$.

Proof. The proof follows from the application of Corollary 2.2.1 to $int(C_1)$ and C_2. □

Corollary 2.2.3 *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ a convex function and $g : E \to \mathbb{R} \cup \{-\infty\}$ a concave function, i.e. $-g$ convex, such that $f \geq g$ on E. Suppose that there exists $x_0 \in E$ such that $f(x_0)$ and $g(x_0)$ are finite and g is continuous at x_0. Then, there exists an affine form h such that $f \geq h \geq g$ on E.*

Proof. Denote by $hypo(g)$ (resp. $sthypo(g)$) the hypograph (resp. stric hypograph) of g defined by

$$\begin{aligned} hypo(g) &= \{(x,t) \in E \times \mathbb{R} \ : \ t \leq g(x)\} \\ \text{(resp. } sthypo(g) &= \{(x,t) \in E \times \mathbb{R} \ : \ t < g(x)\}) \ . \end{aligned}$$

Since g is continuous at x_0, $int(hypo(g)) \neq \emptyset$. Moreover we have $int(hypo(g)) \subset sthypo(g)$. Since $f \leq g$, we get $epi(f) \cap sthypo(g) = \emptyset$. Since these two sets are convex, we get from Corollary 2.2.2 the existence of a $(u^*, a) \in E \times \mathbb{R} \setminus \{(0,0)\}$ and a constant c such that

$$\inf_{(t,x)\in epi(f)} at + \langle u^*, x\rangle \quad \geq \quad c \quad \geq \quad \sup_{(t,x)\in hypo(g)} at + \langle u^*, x\rangle \ .$$

Let $t \to +\infty$ gives $a \geq 0$.

If $a = 0$ then $\langle u^*, x - x_0\rangle \leq 0$ for all $x \in dom(-g)$. Since $x_0 \in int(dom(-g))$, then $u^* = 0$ which is not possible. Therefore $a > 0$. We therefore get

$$af(x) + \langle u^*, x\rangle \quad \geq \quad c \quad \geq \quad ag(x) + \langle u^*, x\rangle \ ,$$

and

$$f(x) \;\geq\; \frac{1}{a}\big(c - \langle u^*, x\rangle\big) \;\geq\; g(x)$$

for all $x \in E$. Then $h = \frac{1}{a}\big(c - \langle u^*, \cdot\rangle\big)$ satisfies the required property. $\square$