Chapter 8

Verification

8.1 The verification result

In this section, we give a criterion to check whether a function is the value function related to a given optimal control problem.

Theorem 8.1.6 *Suppose that there exists* $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap C^0([0,T] \times \mathbb{R}^n)$ *solution to* (7.3.6)-(7.3.7) *with polynomial growth such that the following statements hold.*

(i) There exists a measurable map $\hat{\alpha} : [0,T) \times \mathbb{R}^n \to A$ *such that*

$$\mathcal{H}\varphi(t,x) \quad = \quad (\partial_t + \mathcal{L}^{\hat{\alpha}(t,x)})\varphi(t,x) + f(x, \hat{\alpha}(t,x)) \ , \quad (t,x) \in [0,T) \times \mathbb{R}^n \ .$$

Then, $\varphi = v$ *and* $\hat{\alpha}(., X_{.}^{t,x,\hat{\alpha}})$ *is an optimal control for the initial condition* (t,x)*.*

Proof. From Itô's formula, (i) and (ii) we have

$$\varphi(t,x) \quad = \quad \mathbb{E}\Big[\varphi\big(\theta_n, X_{\theta_n}^{t,x,\hat{\alpha}}\big) + \int_t^{\theta_n} f\big(X_s^{t,x,\hat{\alpha}}, \hat{\alpha}(s, X_s^{t,x,\hat{\alpha}})\big)ds\Big] \ .$$

where

$$\theta_n \quad = \quad \inf\{s \geq 0 \ : \ |X_s^{t,x,\hat{\alpha}}| \geq n\} \wedge T$$

for $n \geq 1$. Sending n to ∞, we get

$$\varphi(t,x) \;=\; \mathbb{E}\Big[g(X_T^{t,x,\hat{\alpha}}) + \int_t^T f\big(X_s^{t,x,\hat{\alpha}}, \hat{\alpha}(s, X_s^{t,x,\hat{\alpha}})\big)ds\Big] \,.$$

This gives $\varphi(t,x) \leq v(t,x)$. We now prove the reverse inequality. Let $\alpha \in \mathcal{A}_t$, τ_n the first time s such that $|X_s^{t,x,\alpha}| \geq n$ and $\theta_n = T \wedge \tau_n$ for $n \geq 1$. We have from Itô's formula

$$\begin{aligned}
&\mathbb{E}\Big[\varphi\big(\theta_n, X_{\theta_n}^{t,x,\alpha}\big) + \int_t^{\theta_n} f\big(X_s^{t,x,\alpha}, \alpha_s\big)ds\Big] \\
=\;& \varphi(t,x) + \mathbb{E}\Big[\int_t^{\theta_n} \big((\partial_t + \mathcal{L}^\alpha)\varphi(s, X_s^{t,x,\alpha}) + f(X_s^{t,x,\alpha}, \alpha_s)\big)ds\Big] \,.
\end{aligned}$$

From (i) we have

$$(\partial_t + \mathcal{L}^\alpha)\varphi(s, X_s^{t,x,\alpha}) + f(X_s^{t,x,\alpha}, \alpha_s) \;\leq\; 0\,, \quad s \in [t,T] \,.$$

Therefore, we get from the polynomial growth of φ and Fatou's lemma

$$\begin{aligned}
\varphi(t,x) \;&\geq\; \liminf_{n\to+\infty} \mathbb{E}\Big[\varphi\big(\theta_n, X_{\theta_n}^{t,x,\alpha}\big) + \int_t^{\theta_n} f\big(X_s^{t,x,\alpha}, \alpha_s\big)ds\Big] \\
&\geq\; \mathbb{E}\Big[g\big(X_T^{t,x,\alpha}\big) + \int_t^T f\big(X_s^{t,x,\alpha}, \alpha_s\big)ds\Big] \,.
\end{aligned}$$

Finally, $\hat{\alpha}$ is optimal as it realizes $\varphi(t,x)$. □

8.2 Application: portfolio allocation problem in finite horizon

We consider an optimal investment in the framework of the Black-Scholes-Merton model over a finite horizon T. We suppose that the market is composed by two assets: a nonrisky asset S^0 and a risky one S. The nonrisky asset follows a deterministic interest rate $r > 0$ and has an initial value $S_0^0 > 0$. It therefore satisfies

$$S_t^0 \;=\; S_0^0 + \int_0^t rS_u du\,, \quad t \in [0,T] \,.$$

The risky asset is defined by its deterministic initial condition $S_0 > 0$ and the dynamics

$$dS_t = S_t\mu dt + S_t\sigma dW_t$$

where μ and σ are constants with $\sigma > 0$ and W is a one dimensional Brownian motion.

An agent invests at any time t a proportion α_t of his wealth in the stock S and $1-\alpha_t$ in S^0. The self-financing wealth process X^α evolves according to

$$\begin{aligned} dX_t &= \frac{X_t\alpha_t}{S_t}dS_t + \frac{X_t(1-\alpha_t)}{S_t^0}dS_t^0 \\ &= X_t(r+(\mu-r)\alpha_t)dt + X_t\alpha_t\sigma dW_t\ , \quad t\in[0,T]\ . \end{aligned}$$

We denote by $\mathcal{A}$ the set of progressively measurable processes α valued in A, that is supposed to be closed and convex, and such that $\mathbb{P}$-a.s. $\int_0^T|\alpha_s|^2ds < +\infty$. This integrability condition ensures the existence and uniqueness of a stong solution to the SDE governing the wealth process controlled by $\alpha\in\mathcal{A}$. Given a portfolio strategy $\alpha\in\mathcal{A}$, we denote by $X^{t,x,\alpha}$ the corresponding wealth process starting from an initial capital $X_t^{t,x,\alpha} = x \geq 0$ at time $t\in[0,T]$. We suppose that the preferences of the agent are described by a utility function U of CRRA type given by

$$U(x) = \frac{x^p}{p}\ , \quad x\geq 0$$

with $p\in(0,1)$. The agent aims at maximizing the expected utility from terminal wealth at horizon T. The value function of the utility maximization problem is then defined by

$$v(t,x) = \sup_{\alpha\in\mathcal{A}}\mathbb{E}\big[U(X_T^{t,x,\alpha})\big],$$

for $(t,x)\in[0,T]\times\mathbb{R}_+$. The HJB equation is then given by

$$\sup_{a\in A}\big\{(\partial_t+\mathcal{L}^a)v(t,x)\big\} = 0\ , \quad (t,x)\in[0,T)\times\mathbb{R}^n \qquad (8.2.1)$$

with

$$\mathcal{L}^a v(t,x) = x(a\mu+(1-a)r)\partial_x v(t,x) + \frac{1}{2}x^2a^2\sigma^2\partial_{xx}^2 v(t,x)$$

for $(t,x) \in [0,T) \times \mathbb{R}_+$. The terminal condition is then given by

$$v(T,x) = U(x) , \quad x \in \mathbb{R}_+ \tag{8.2.2}$$

We look for an explicit smooth solution φ to (8.2.1)-(8.2.2). We propose a candidate solution in the form

$$\varphi(t,x) = \phi(t)U(x) , \quad (t,x) \in [0,T) \times \mathbb{R}^n ,$$

for some positive function ϕ. By substituting into the HJB equation, we derive that ϕ should satisfy the ordinary differential equation

$$\begin{aligned} \phi'(t) + \rho\phi(t) &= 0 , \quad t \in [0,T) \\ \phi(T) &= 1, \end{aligned}$$

where

$$\rho = p \sup_{a \in A} \{a(\mu - r) + r - \frac{1}{2}a^2(1-p)\sigma^2\} .$$

We then obtain $\phi(t) = \exp(\rho(T-t))$. Hence, the function given by

$$\varphi(t,x) = \exp(\rho(T-t))U(x) , \quad (t,x) \in [0,T] \times \mathbb{R}_+ ,$$

is a smooth solution to the HJB PDE. Furthermore, the function $a \in A \mapsto a(\mu - r) + r - \frac{1}{2}a^2(1-p)\sigma^2$ is strictly concave on the closed convex set A . Thus it reaches its maximum at some constant $\hat{a} \in A$. Moreover, the SDE associated to the constant control $\hat{a}$

$$dX_s = X_s(r + (\mu - r)\hat{a})ds + X_s\hat{a}\sigma dW_s ,$$

admits a unique solution for a given initial condition (t,x). From the verification theorem, we have $\varphi = v$. In the case where $A = \mathbb{R}$, $\hat{a}$ and ρ can be explicitly computed and we have

$$\hat{a} = \frac{\mu - r}{\sigma^2(1-p)}$$

and

$$\rho = \frac{(\mu - r)^2}{\sigma^2(1-\rho)} \frac{p}{1-p} + rp .$$

8.3 Application: investment-consumption problem

We use the framework of the previous section for the model on asset prices. A control is a pair of progressively measurable processes (α, c) valued in $A \times \mathbb{R}_+$ for some closed convex subset A of $\mathbb{R}$ such that $\mathbb{P}$-a.s. $\int_0^T |\alpha_t|^2 dt + \int_0^T c_t dt < +\infty$. We denote by $\mathcal{A} \times \mathcal{C}$ the set of control processes. The quantity α_t represents the proportion of wealth invested in stock S, and c_t is the time rate consumption per unit of wealth. Given $(\alpha, c) \in \mathcal{A} \times \mathcal{C}$, there exists a unique solution, denoted by $X^{t,x}$, to the SDE governing the wealth process

$$dX_s \quad = \quad X_s(r + (\mu - r)\alpha_t - c_t)ds + X_s \alpha_t \sigma dW_s \ , \ \ s \in [t, T] \ ,$$

given the initial condition $X_t = x \geq 0$. The agent's investment-consumption problem is to maximize over strategies (α, c) the expected utility from intertemporal consumption up to the time horizon T. Given a utility function u for consumption, we then consider the corresponding value function:

$$v(t, x) \quad = \quad \sup_{(\alpha,c) \in \mathcal{A} \times \mathcal{C}} \mathbb{E}\Big[u(X_T^{t,x}) + \int_t^T u(c_s X_s^{t,x})ds\Big] \ , \quad (t, x) \in [0, T] \times \mathbb{R}_+ \ .$$

The HJB equation associated to this control problem is

$$\sup_{(a,c) \in A \times \mathbb{R}_+} \{(\partial_t + \mathcal{L}^{(a,c)})v + u(cx)\} \quad = \quad 0$$

with terminal condition $v(T, .) = u$, where

$$\begin{aligned} \mathcal{L}^{(a,c)} v(t, x) \quad &= \quad x(a\mu + (1 - a)r - c)\partial_x v(t, x) \\ & \quad\quad + \frac{1}{2} x^2 a^2 \sigma^2 \partial^2_{xx} v(t, x) \end{aligned}$$

for $(t, x) \in [0, T) \times \mathbb{R}_+$. By defining $\tilde{u}(z) = \sup_{C \geq 0}[u(C) - Cz]$, $z \geq 0$, the Legendre transform of u, this HJB equation may be written as

$$\sup_{a \in A}\{(\partial_t + \mathcal{L}^a)v\} + \tilde{u}(\partial_x v) \quad = \quad 0$$

with $\mathcal{L}^a$ defined as in the previous section. If we take $u(x) = x^p/p$ then $\tilde{u}(z) = z^{-q}/q$ with $q = p/(1-p)$. We next look for a candidate solution of the form

$$\varphi(t,x) \;=\; \phi(t)u(x)$$

for $(t,x) \in [0,T) \times \mathbb{R}_+$. By substituting into the HJB equation, we derive that ϕ should satisfy the ordinary differential equation

$$\begin{aligned}\phi'(t) + \rho\phi(t) + \frac{p}{q}\phi(t)^{-q} &= 0\,, \quad t \in [0,T)\\ \phi(T) &= 1,\end{aligned}$$

where

$$\rho \;=\; p\sup_{a\in A}\{a(\mu - r) + r - \frac{1}{2}a^2(1-p)\sigma^2\}\,.$$

We notice that the ODE can be rewritten

$$\begin{aligned}(\phi^{q+1}(t))' + \tilde{\rho}\phi^{q+1}(t) + \frac{p(q+1)}{q} &= 0\,, \quad t \in [0,T)\,,\\ \phi^{q+1}(T) &= 1\,,\end{aligned}$$

with $\tilde{\rho} = \rho(q+1)$. Therefore we get

$$\phi(t)^{q+1} \;=\; (1 + \frac{p}{q\rho})e^{\tilde{\rho}(T-t)} - \frac{p}{q\rho}\,, \quad t \in [0,T]\,.$$

We then have that the optimizer $\hat{a}$ in the variable a is the same as that of the previous section. We next compute the optimizer an the variable c and we have

$$\begin{aligned}\hat{c}^*(t,x) &= \frac{1}{x}\big(\phi(t)u'(x)\big)^{\frac{1}{p-1}}\\ &= \Big((1 + \frac{p}{q\rho})e^{\tilde{\rho}(T-t)} - \frac{p}{q\rho}\Big)^{-1}\end{aligned}$$

We observe that $c^*(t,x) = c^*(t) \in [0,1]$. Therefore the related SDE

$$dX_s \;=\; X_s(r + (\mu - r)\hat{a} - c^*(t))ds + X_s\hat{a}\sigma dW_s\,,\; s \in [t,T]\,,$$

admits a unique solution. We can then apply the verification theorem to get that

$$v(t,x) \;=\; \Big((1 + \frac{p}{q\rho})e^{\tilde{\rho}(T-t)} - \frac{p}{q\rho}\Big)^{\frac{1}{q+1}}\frac{x^p}{p}\,, \quad (t,x) \in [0,T] \times \mathbb{R}_+\,.$$