TD 1

Exercice 1

Mesure d'Escher : $\mathbb{E}_{S_h} = \dfrac{\mathbb{E}[Xe^{hX}]}{\mathbb{E}[e^{hX}]}$

- $\mathbb{E}_{S_{1/2}}[X] = \dfrac{6e^3}{e^3+2} \simeq 5,5$

- $\mathbb{E}_{S_{1/2}}[Y] = \dfrac{3\frac{e^{3/2}}{3} + 6\frac{e^3}{3}}{\frac{1}{3} + \frac{e^{3/2}}{3} + \frac{e^3}{3}} = \dfrac{3(e^{3/2} + 2e^3)}{1 + e^{3/2} + e^3} \simeq 5,2$

D'après l'énoncé $X \leq Y$ p.s

Donc la mesure n'est pas cohérente (pas monotone)

Exercice 2

a. $VaR_\alpha(X) = F_X^{-1}(\alpha) = 2\alpha$ $\quad \left(F_X(x) = \frac{1}{2}x\right)$

$VaR_\alpha(Y) = F_Y^{-1}(\alpha) = -\ln(1-\alpha)$ $\quad \left(F_Y(y) = 1 - e^{-y}\right)$

Pour comparer X et Y on pose

$$g : [0,1[\longrightarrow \mathbb{R}$$
$$x \longmapsto VaR_x(X) - VaR_x(Y)$$

$g(x) = 2x + \ln(1-x)$ $\qquad g'(x) = 2 - \dfrac{1}{1-x}$

x	0		$1/2$		β		1
$g'(x)$		$+$	0	$-$			
g	0	$\nearrow$		$\searrow$	0	$\searrow$	$-\infty$

$\beta \simeq 0,8$

Donc pour $\alpha \geqslant \beta$

$VaR_\alpha(X) \leqslant VaR_\alpha(Y)$

b. $\cdot$ $TVaR_\alpha(X) = \dfrac{1}{1-\alpha}\displaystyle\int_\alpha^1 VaR_z(X)\,dz = 1+\alpha$

$\cdot$ $TVaR_\alpha(Y) = \dfrac{1}{1-\alpha}\displaystyle\int_\alpha^1 VaR_z(Y)\,dz = \dfrac{1}{1-\alpha}\displaystyle\int_\alpha^1 -\ln(1-z)\,dz$

chgt de variable

$$= \frac{1}{1-\alpha}\int_{1-\alpha}^0 -\ln(u)(-du) = \frac{1}{1-\alpha}\Big[u\ln(u)-u\Big]_{1-\alpha}^0$$

$$= 1-\ln(1-\alpha)$$

On pose $g(x) = VaR_x(X) - VaR_x(Y)$

$$= 1+x+\ln(1-x)-1$$

$$g'(x) = \frac{-x}{1-x} \leqslant 0 \quad \text{et} \quad g(0)=0$$

Donc pour $\alpha \in [0,1[$, $\boxed{TVaR_\alpha(X) \leqslant TVaR_\alpha(Y)}$

Exercice 3

1) $$VaR_\alpha(X) = F_X^{-1}(\alpha)$$
$$= -\frac{1}{\alpha-1} - 1$$
$$= \frac{1}{1-\alpha} - 1 = \boxed{\frac{\alpha}{1-\alpha}}$$

$$F_X(x) = 1 - \frac{1}{1+x}$$
$$F_X^{-1}(x) = -\frac{1}{x-1} - 1$$

2) $$\mathbb{P}(X+Y \leq t) = \int_{-\infty}^{+\infty} \mathbb{P}(X+Y \leq t \mid Y=y) f_Y(y)\,dy \quad (\text{car } X \perp Y)$$
$$= \int_0^t \mathbb{P}(X+y \leq t) f_Y(y)\,dy$$
$$= \int_0^t F_X(t-y) f_Y(y)\,dy$$
$$= \int_0^t \left(1 - \frac{1}{t-v+1}\right) \frac{1}{(1+v)^2}\,dv$$

Méthode brutale.
prod. convolution.
$$f_{X+Y}(x) = \int_0^x f_X(y) f_Y(1-y)\,dy$$

Or $$\frac{t-v}{(1+t-v)(1+v)^2} = \frac{-1/(1+t)^2}{1+v} + \frac{t+1/t+2}{(1+v)^2} + \frac{-1/(1+t)^2}{1+t-v}$$

$$\mathbb{P}(X+Y \leq t) = 1 - \left[\frac{2}{t+2} + 2\frac{\ln(1+t)}{(t+2)^2}\right]$$

3) $F_{X+Y}(2\,VaR_\alpha(X)) = \alpha - \underbrace{\frac{(1-\alpha)^2}{2}\ln\left(\frac{1+\alpha}{1-\alpha}\right)}_{\geq 0}$

$\leq \alpha$

Donc $2VaR_\alpha(X) < F^{-1}_{X+Y}(\alpha)$

$$\boxed{VaR_\alpha(X) + VaR_\alpha(Y) < VaR(X+Y)}$$