Stochastic modelling and derivatives
– Exercises –
M2 Probability and finance

Tutorials 1 and 2

1 Notations

We define:

- The call price in Black-Scholes model:
$$\mathrm{Call}^{\mathrm{BS}}(t, T, S, K, \sigma, r, q) = Se^{-q(T-t)}N(d_+) - Ke^{-r(T-t)}N(d_-),$$
where
$$d_\pm = \frac{1}{\sigma\sqrt{T-t}} \ln\left(\frac{Se^{(r-q)(T-t)}}{K}\right) \pm \frac{1}{2}\sigma\sqrt{T-t}$$
and
$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-u^2}{2}\right) du.$$
- $B(t, T)$ is the price in $t \leq T$ of a zero-coupon bond, paying 1€ in T.
- $F_t(\phi_T, T)$ is the forward price at time t (and paid in T) for the cashflow ϕ_T delivered at T.

2 Course of 24th September 2024: Discrete-time model

Exercise 1. [Convexity] Prove that, under the no-arbitrage assumption, the European call price is a convex function of the strike.

Exercise 2. [Arbitrage] The aim of this exercise is to detect arbitrage opportunities. A trainee presents his new program to compute European call and put prices. We assume that the considered asset does not provide dividends. Assume that the price of the asset is $S_0 = 100$€ today with $r = 1\%$, $T = 1$ (1 year). The program gives the following outputs for call and put prices with different strikes:

If these prices were market prices, provide

- arbitrage opportunities between call options,

Option	Call $K = 95€$	Call $K = 100€$	Call $K = 105€$	Put $K = 95€$	Put $K = 100€$	Put $K = 105€$
Price	11€	9.5€	7.27€	6.95€	10.51€	9.83€

- ▷ arbitrage opportunities between put options,
- ▷ arbitrage opportunities between call and put options.

Exercise 3. [Payoff and strategies] For each strategy below, write and draw (including the premium) the payoff at maturity. Note that all the options used in these strategies have the same maturity. What is the financial purpose of each of these strategies?

1. **Straddle**: Long 1 Call and long 1 Put, with same strike.
2. **Strip**: Long 1 Call and long 2 Puts, with same strike.
3. **Strap**: Long 2 Calls and long 1 Put, with same strike.
4. **Butterfly**: Long 1 Call of strike $K - \delta K$, long 1 Call of strike $K + \delta K$, short 2 Calls of strike K.
5. **Strangle**: Long 1 Call of strike K_C and long 1 Put of strike K_P, with, usually (but not necessarily), $K_P < K_C$.
6. **Condor**: Long 1 Call of strike K_1, short 1 Call of strike $K_2 = K_1 + \delta K > K_1$, short 1 Call of strike $K_3 > K_2$, and long 1 Call of strike $K_4 = K_3 + \delta K$
7. **Bull call spread**: Long 1 Call of strike K_1 and short 1 Call of strike $K_2 > K_1$.
8. **Bull put spread**: Long 1 Put of strike K_1 and short 1 Put of strike $K_2 > K_1$.

Exercise 4. [Discrete time market] We consider a market with two periods (3 times $t_0 = 0 < t_1 < t_2 = T$) with:

- ▷ a risky asset denoted by S with price $S \times u$ or $S \times d$ after one period (with probability p and $1 - p$ respectively), by assuming that $0 < d < 1 + r < u$ where r is the interest rate on one period,
- ▷ a call option with strike K and maturity $T = t_2$.

Questions:

1. Assume that $S_0 = 4€$ at time $t_0 = 0$, $u = 2$, $d = 1/2$, $r = 0.25$, $K = 5€$, compute
   - ▷ the price V_{t_1} and the hedging strategy δ_{t_1}, in the two possible states at time t_1 (that is if $S_{t_1} = S_0 u$ and if $S_{t_1} = S_0 d$),
   - ▷ the price V_0 and the hedging δ_0.
2. Same questions with a put option.
3. Check the call/put parity at time $t = t_0$ and $t = t_1$.

Exercise 5. [Convergence of the binomial model towards Black-Scholes model] We are given a financial market comprising a risk-free asset of price R, equal to 1 in $t = 0$, and a risky asset of price S.
We discretize the time interval $[0, T]$ in n smaller intervals $[t_i^n, t_{i+1}^n]$, with $t_i^n = iT/n$, in order to build a n-period binomial tree. We note r_n the interest rate of the risk-free asset, the value of this asset at the time t_i^n being

$$R^n_{t_i^n} = (1 + r_n)^i.$$

We note X_i^n the quantity equal to 1 plus the price return of the risky asset between times t_{i-1}^n and t_i^n. We then have, under the historical probability,

$$\mathbb{P}^n(X_i^n = u_n) = p_n = 1 - \mathbb{P}^n(X_i^n = d_n).$$

The random variables $X_1^n, ..., X_n^n$ are independent to each other. We base the quantities r_n, d_n, and u_n on the parameters r and σ:

$$r_n = \frac{rT}{n}, \qquad d_n = \left(1 + \frac{rT}{n}\right) e^{-\sigma\sqrt{\frac{T}{n}}}, \qquad \text{and } u_n = \left(1 + \frac{rT}{n}\right) e^{\sigma\sqrt{\frac{T}{n}}}.$$

1. Draw the tree representing the evolution of the risky asset.
2. What is the limit of R_T^n when n tends to infinity?
3. Is the market consistent with the no-arbitrage assumption?
4. Give an expression for $S^n_{t_i^n}$ using S_0 and $(X_1^n, ..., X_i^n)$.
5. Give the dynamic of the process X^n under the risk-neutral probability $\mathbb{Q}^n$. We then note $q_n = \mathbb{Q}^n(X_i^n = u_n)$.
6. Show that
$$q_n \overset{n\to\infty}{\longrightarrow} \frac{1}{2}, \qquad n\mathbb{E}_{\mathbb{Q}^n}[\ln(X_1^n)] \overset{n\to\infty}{\longrightarrow} \left(r - \frac{\sigma^2}{2}\right)T, \qquad \text{and } n\mathbb{V}\mathrm{ar}_{\mathbb{Q}^n}[\ln(X_1^n)] \overset{n\to\infty}{\longrightarrow} \sigma^2 T.$$
7. Using characteristic functions, prove the following convergence in distribution:
$$\sum_{i=1}^{n} \ln(X_i^n) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left[\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right].$$
8. Deduce that
$$S_T^n \underset{n\to\infty}{\overset{d}{\longrightarrow}} S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T},$$
with $W_T \sim \mathcal{N}(0, T)$.
9. Write the price of a put of strike K and maturity T in the n-period binomial model, as the expected value of some variable.
10. Deduce that the price of the put converges toward
$$P_0 = Ke^{-rT}N(-d_-) - S_0 N(-d_+),$$
when n tend to infinity.

11. Conclude that the call price tends toward

$$C_0 = S_0 N(d_+) - Ke^{-rT} N(d_-).$$

Exercise 6. [Lookback option with a binomial tree] We consider a market with two periods (3 times $t_0 = 0 < t_1 < t_2 = T$) with:

- ▷ a risky asset denoted by S with price $S \times u$ or $S \times d$ after one period, with $u = 1.1$, $d = 0.95$, $S_{t_0} = 100$, and $r = 0.05$ is the interest rate on one period,
- ▷ a European call option with strike $K_E = 105$ and maturity T.
- ▷ a lookback option of strike $K_L = 100$, maturity T, and whose payoff is $(\sup_{t \leq T} S_t - K_L)_+$.

Questions:

1. Draw the tree representing the evolution of the risky asset.
2. Describe Ω.
3. What is the risk-neutral probability in this binomial tree?
4. What is the price of the European call in this model?
5. What is the price of the lookback option in this model?

Exercise 7. [Carr formula] We assume we have access to call and put options of maturity T and strike K, whatever $K \geq 0$. We want to use these options to replicate a derivative of payoff $\psi(S_T)$, where ψ is any regular function.

1. Prove the Carr formula, that is that the cash price (which is the price paid today, *"prix au comptant"*, as opposed to the forward price) at time t of the payoff $\psi(S_T)$, which we note $C_t(\psi(S_T), T)$, follows, under the no-arbitrage assumption:

$$C_t(\psi(S_T), T) = \psi(F_t(S_T, T))B(t, T) + \int_{F_t(S_T,T)}^{+\infty} \psi''(K)\mathrm{Call}_t(T, K)dK + \int_0^{F_t(S_T,T)} \psi''(K)\mathrm{Put}_t(T, K)dK$$

2. Give a static hedging strategy for the following payoff, by using infinitely many calls and puts:
   - ▷ $\psi_T = (S_T)^p$ (power underlying) for some $p > 0$,
   - ▷ $\psi_T = ((S_T)^p - K)_+$ (call power) for some $p > 0$.

Exercise 8. [Trinomial model] We consider a discrete-time market, with a bond whose price is multiplied by $(1 + r)$ at each step, as well as a risky security of price S_n. This price is led by the stochastic process h_n: $h_i, ..., h_N$ are N i.i.d. random variables such that

$$h_n = \begin{cases} 1 & \text{with probability } p_1 \\ 2 & \text{with probability } p_2 \\ 3 & \text{with probability } p_3 = 1 - p_1 - p_2, \end{cases}$$

with probabilities different from zero. The price S_n is defined by $S_n = S_{n-1}(1+\mu(h_n))$, with

$$1+\mu(h_n) = \begin{cases} u & \text{if } h=1 \\ m & \text{if } h=2 \\ d & \text{if } h=3, \end{cases}$$

and $0 < d < m < u$.

1. Draw the trinomial tree.
2. Studying the martingale condition, show that this market is incomplete.
3. Alternatively, show there is no replicating strategy for a given derivative defined by its random payoff X. This will show again that the market is incomplete.
4. Show, following two distinct methods, that adding a second asset S^2 independent from S and defined in a similar way ($S^2_n = S^2_{n-1}(1+\mu^2(h_n))$) is enough to make the market complete.

Exercise 9. [Asian options] We consider three kinds of derivatives:

- ▷ European calls C_t and puts P_t of strike K,
- ▷ Asian calls $\bar{C}_t$ and puts $\bar{P}_t$ of strike K, whose payoff is $(\bar{S}-K)_+$ or $(K-\bar{S})_+$, with

  $$\bar{S} = \frac{1}{n}\sum_{i=1}^{n} S_{t_i},$$

  for $0 < t_1 < ... < t_n < T$,
- ▷ exotic European calls $\hat{C}_t$ and puts $\hat{P}_t$ with "average strike" $\bar{S}$, that is of payoff $(S_T - \bar{S})_+$ or $(\bar{S} - S_T)_+$.

All these options have the same underlying and same maturity. Find a relation between the six option prices at time 0.

3 Course of 1st October 2024: Black-Scholes model

3.1 Course-related questions

Exercise 10. In Black-Scholes model, what boundary conditions are satisfied when $S \to 0$, $S \to \infty$, $\sigma \to 0$, and $\sigma \to \infty$?

Exercise 11. Suppose two assets in a Black-Scholes world have the same volatility but different drifts. How will the price of the call options on them compare?
Then suppose one of the assets undergoes downward jumps at random times. How will this affect option prices?

Exercise 12. Starting from the expression of the call price in the Black-Scholes model, $\text{Call}^{\text{BS}}(0,T,S_0,K,\sigma,r) = S_0 N(d_+) - Ke^{-rT}N(d_-)$, prove:

1. the key relation $x \exp\left(\frac{-d_+^2(x,y)}{2}\right) = y \exp\left(\frac{-d_-^2(x,y)}{2}\right)$, for $x, y > 0$ and $d_\pm(x,y) = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x}{y}\right) \pm \frac{1}{2}\sigma\sqrt{T}$;
2. that the delta of the call is $N(d_+)$;
3. that the delta of the put is $N(d_+) - 1$.

3.2 Problems

Exercise 13. [Option on a futures contract] We consider a futures contract on an underlying S and maturity T. The future price at time t is $F_t(S,T)$. In Black-Scholes framework, what is the price of a call of maturity $\tau < T$, on this futures.

Exercise 14. [Binary options] Assume that the interest rate has a constant value r. The value of a stock follows the dynamic

$$dS_t = S_t(\mu dt + \sigma dW_t),\ \ S_0 = x_0.$$

The stock does not pay dividends. A binary option pays out 1€ if the stock price is greater than or equal to the strike K at maturity T, and 0 otherwise.

1. What is the price of this binary option in Black-Scholes model?
2. What is its delta? In particular, what happens to the number of stocks in the portfolio as the time to maturity goes to zero while the option stays at the money?
3. Let $\varepsilon \in (0,K)$. For a given maturity $T > 0$, consider a static portfolio in which the holder has longed ε^{-1} calls with strike $K - \varepsilon$ and shorted ε^{-1} calls with strike K at time 0. Prove that this is a *superhedging* portfolio for the binary option, i.e. the value of the portfolio at maturity *exceeds* the payoff of the binary.
   What is the interest of such static strategy compared to the dynamic one?

Exercise 15. [Power option] We are given an option of payoff $h(S_t) = S_T^n$. Show that the Black-Scholes price is of the form $v(t,x) = \phi(t,T)x^n$, with the expression of $\phi(t,T)$ to be found, following the two following distinct methods:

1. the risk-neutral pricing rule;
2. starting from Black-Scholes PDE, find the ODE in ϕ and solve it.

Exercise 16. [Forward-start call option] We study a variant of call option in which the strike is fixed in the future. We discuss how to price and hedge the contract.
Assume that the interest rate has a constant value r. A forward start call with maturity T and parameter $\theta \in (0,T)$ is an option that pays out $(S_T - S_\theta)_+$ at time T, where S is the price of the underlying following the Black-Scholes model.

1. What is the value of the option in the time interval $[\theta, T]$?
2. What is the value of the option in the time interval $[0, \theta]$?
3. Compute the Delta and the Gamma of this option.

4. Describe the hedging strategy on $[0,T]$.

Exercise 17. [Chooser option] A chooser option is a derivative providing its holder with the right to choose at a future date τ either a call or a put of maturity $T > \tau$ and strike K. We note $C_t(K,T)$ (respectively $P_t(K,T)$) the price of a call (resp. put) at time t and of strike K and maturity T. We assume the underlying stock doesn't pay any dividend.

1. What is the payoff of a chooser option?
2. Show that the no-arbitrage price of the chooser option, at time $t = 0$ is

$$\Pi_0 = C_0(K,T) + \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(K - S_T)\mathbb{1}_{C_\tau(K,T) < P_\tau(K,T)}\right].$$

3. Prove that this price may also write $\Pi_0 = C_0(K,T) + P_0(Ke^{-r(T-\tau)},\tau) = P_0(K,T) + C_0(Ke^{-r(T-\tau)},\tau)$.
4. Using a replication point of view, interpret the value of this derivative at time τ.

Exercise 14. [Binary options] Assume that the interest rate has a constant value r. The value of a stock follows the dynamic

$$dS_t = S_t(\mu dt + \sigma dW_t), \; S_0 = x_0.$$

The stock does not pay dividends. A binary option pays out 1€ if the stock price is greater than or equal to the strike K at maturity T, and 0 otherwise.

1. What is the price of this binary option in Black-Scholes model?
2. What is its delta? In particular, what happens to the number of stocks in the portfolio as the time to maturity goes to zero while the option stays at the money?
3. Let $\varepsilon \in (0,K)$. For a given maturity $T > 0$, consider a static portfolio in which the holder has longed ε^{-1} calls with strike $K - \varepsilon$ and shorted ε^{-1} calls with strike K at time 0. Prove that this is a *superhedging* portfolio for the binary option, i.e. the value of the portfolio at maturity *exceeds* the payoff of the binary.
   What is the interest of such static strategy compared to the dynamic one?

Exercice 14 : $dS_t = S_t(\mu dt + \sigma dW_t)$, $S_0 = x_0$

1) Sous $\mathbb{Q}$ proba risque neutre :

$$\frac{dS_t}{S_t} = r\,dt + \sigma\, d\hat{W}_t$$

Par Feynman-Kac, $P_t(x) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} \mathbb{1}_{\left\{x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma \underbrace{(\hat{W}_T - \hat{W}_t)}_{\mathcal{N}(0,T-t)}} \geq K\right\}}\right]$

prix

$$= e^{-r(T-t)}\, \mathbb{Q}\left(\frac{\hat{W}_T - \hat{W}_t}{\sqrt{T-t}} \geq -d_2\right) \quad \text{avec } d_2 = \frac{\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$= e^{-r(T-t)}\, \mathcal{N}(d_2)$$

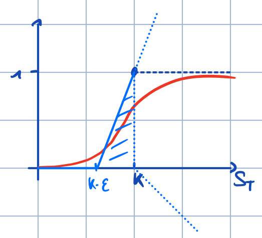
2) $\Delta(x,t) = \dfrac{\partial P_t(x)}{\partial x}$

$$= e^{-r(T-t)}\, \mathcal{N}'(d_2)\, \frac{\partial d_2}{\partial x}$$

$$= \frac{e^{-r(T-t)}\, \mathcal{N}'(d_2)}{x\sigma\sqrt{T-t}}$$

A la monnaie, $d \xrightarrow[t \to T]{} 0$, $\Delta \to \infty$

Ça devient compliqué en terme de couverture



3) cf graphique

$$V(x,t) = \frac{1}{\varepsilon}\, CALL_t(x, K-\varepsilon, T) - \frac{1}{\varepsilon}\, CALL_t(x,K,T)$$

ptf de réplic°

$$V(x,t) \xrightarrow{\varepsilon \to 0} -\frac{\partial}{\partial K} CALL_t(x,K,T) = \mathcal{N}(d_2)\, e^{-r(T-t)}$$

Exercise 17. [Chooser option] A chooser option is a derivative providing its holder with the right to choose at a future date τ either a call or a put of maturity $T > \tau$ and strike K. We note $C_t(K,T)$ (respectively $P_t(K,T)$) the price of a call (resp. put) at time t and of strike K and maturity T. We assume the underlying stock doesn't pay any dividend.

1. What is the payoff of a chooser option?
2. Show that the no-arbitrage price of the chooser option, at time $t = 0$ is
$$\Pi_0 = C_0(K,T) + \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(K - S_T)\mathbb{1}_{C_\tau(K,T) < P_\tau(K,T)}\right].$$
3. Prove that this price may also write $\Pi_0 = C_0(K,T) + P_0(Ke^{-r(T-\tau)}, \tau) = P_0(K,T) + C_0(Ke^{-r(T-\tau)}, \tau)$.
4. Using a replication point of view, interpret the value of this derivative at time τ.

Exercice 17 :

1) $H_T = (S_T-K)^+ \mathbb{1}_{\{C_\tau \geq P_\tau\}} + (K-S_T)^+ \mathbb{1}_{\{P_\tau > C_\tau\}}$

2) $\Pi_0 = \mathbb{E}_0^{\mathbb{Q}}[e^{-rT} H_T]$ avec $H_T = (S_T-K)^+ - \underbrace{(S_T-K)^+ \mathbb{1}_{P_\tau > C_\tau} + (K-S_T)^+ \mathbb{1}_{P_\tau > C_\tau}}_{(K-S_T)\mathbb{1}_{C_\tau < P_\tau}}$

Donc $\Pi_0 = C_0 + \mathbb{E}_0^{\mathbb{Q}}[e^{-rT}(K-S_T)\mathbb{1}_{C_\tau < P_\tau}]$

3) $$\begin{aligned}\mathbb{E}_0^{\mathbb{Q}}[e^{-rT}(K-S_T)\mathbb{1}_{C_\tau<P_\tau}] &= \mathbb{E}_0^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[e^{-rT}(K-S_T)\mathbb{1}_{C_\tau<P_\tau} \mid \mathcal{F}_\tau]] \\ &= \mathbb{E}_0^{\mathbb{Q}}[\mathbb{1}_{C_\tau<P_\tau}\mathbb{E}^{\mathbb{Q}}[e^{-rT}(K-S_T) \mid \mathcal{F}_\tau]] \\ &= \mathbb{E}_0^{\mathbb{Q}}[\mathbb{1}_{C_\tau<P_\tau}(e^{-rT}K - e^{-r\tau}S_\tau)]\end{aligned}$$

Parité CALL-PUT : $P_\tau - C_\tau = -S_\tau + Ke^{-r(T-\tau)}$

$\Rightarrow \{C_\tau < P_\tau\}$ est équivalent à $\{Ke^{-rT} > S_\tau e^{-r\tau}\}$

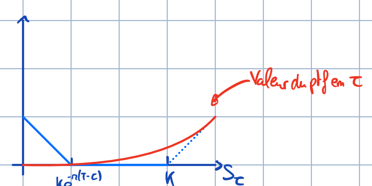
Donc $$\begin{aligned}\mathbb{E}_0^{\mathbb{Q}}[\mathbb{1}_{C_\tau<P_\tau}(e^{-rT}K - e^{-r\tau}S_\tau)] &= \mathbb{E}_0^{\mathbb{Q}}[e^{-r\tau}(e^{-r(T-\tau)}K - S_\tau)^+] \\ &= P_0(e^{-r(T-\tau)}K, \tau)\end{aligned}$$

Parité CALL-PUT : $(e^{-r(T-\tau)}K - S_\tau)_+ = (e^{-r(T-\tau)}K - S_\tau) + (S_\tau - e^{-r(T-\tau)}K)_+$

Donc $$\begin{aligned}\Pi_0 &= C_0(K,T) + \mathbb{E}_0^{\mathbb{Q}}[e^{-r\tau}(e^{-r(T-\tau)}K - S_\tau)] + \mathbb{E}_0^{\mathbb{Q}}[e^{-r\tau}(S_\tau - e^{-r(T-\tau)}K)_+] \\ &= \underbrace{C_0(K,T) + e^{-rT}K - \underbrace{\mathbb{E}_0^{\mathbb{Q}}[e^{-r\tau}S_\tau]}_{=S_0}}_{= P_0(K,T) \text{ par parité CALL-PUT}} + C_0(e^{-r(T-\tau)}K, \tau)\end{aligned}$$

4) Réplication statique entre 0 et τ :

- Long $C(K,T)$
- Long $P(e^{-r(T-\tau)}K, \tau)$



En τ :

⊛ Si $S_\tau > K$, ptf : $C(K,T)$

⊛ Si $S_\tau < Ke^{-r(T-\tau)}$, ptf : $C(K,T) + \underbrace{\text{liquidités}}_{Ke^{-r(T-\tau)} - S_\tau} = C(K,T) + Ke^{-r(T-\tau)} - S_\tau \underset{\text{parité CALL-PUT}}{=} P(K,T)$

⊛ Si $S_\tau \in [Ke^{-r(T-\tau)}, K] \longrightarrow$ ptf $=$ CALL(K,T)

Or, $C_\tau(K,T) - P_\tau(K,T) = S_\tau - Ke^{-r(T-\tau)} \geq 0$ parité CALL-PUT