Chapter 5

Duality of convex functions

5.1 Upper envelopes of affine functions

Definition 5.1.16 *An affine function $h : E \to \mathbb{R}$ is a function such that*

$$h(\lambda x + (1-\lambda)y) \quad = \quad \lambda h(x) + (1-\lambda)h(y)$$

for all $\lambda \in \mathbb{R}$ and $x, y \in E$. Equivalently h is affine if and only if there exists $g : E \to \mathbb{R}$ linear form such that $h = g + h(0)$.

If g is a linear form on E, then obviously $h = g + h(0)$ is affine. If we suppose that h is affine, g defined by $g = h - h(0)$ is linear. Indeed, we first have

$$\begin{aligned} g(\lambda x) &= h(\lambda x + (1-\lambda)0) - h(0) \\ &= \lambda h(x) - \lambda h(0) \\ &= \lambda g(x) \end{aligned}$$

for all $\lambda \in \mathbb{R}$ and all $x \in E$. We then have

$$\begin{aligned} g(x+y) &= g\big(2\frac{x+y}{2}\big) \\ &= 2g\big(\frac{x+y}{2}\big) \\ &= 2h\big(\frac{x+y}{2}\big) - 2h(0) \\ &= 2\frac{h(x)+h(y)}{2} - 2h(0) \\ &= g(x) + g(y)\ . \end{aligned}$$

For a family of function $(f_i)_{i\in I}$ from E to $\mathbb{R}$, the upper envelope is the function $\sup_{i\in I} f_i$.

Theorem 5.1.13 *A function $f: E \to \mathbb{R} \cup \{+\infty\}$ is convex and lsc if and only if it is the upper envelope of affine functions.*

Proof. The upper envelope of a family of affines functions is convex lsc and valued in $\mathbb{R} \cup \{+\infty\}$.

Conversely, let $f : E \to \mathbb{R} \cup \{+\infty\}$ convex and lsc. If f is identically equal to $+\infty$, it is the upper enveloppe of the affine functions $f_n = n$ for $n \geq 1$.

If not, $dom(f) \neq \emptyset$. Let $x_0 \in adh(dom(f))$. For $t \in \mathbb{R}$ such that $t < f(x_0)$, the set $V_f(t) = \{x \in E \ : \ f(x) > t\}$ is open and contains x_0, hence it contains the open ball $B(x_0, r)$ for some $r > 0$. Define $g = t - \chi_{B(x_0,r)}$ where we recall that $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = \infty$ if $x \notin A$ for $A \subset E$. Then g is concave, *i.e.* $-g$ is convex, finite and continuous on $B(x_0, r)$ and upper bounded by f. Since $x_0 \in adh(dom(f))$ we have $dom(f) \cap B(x_0, r) \neq \emptyset$. From Corollary 2.2.4 there exists an affine form h such that $g \leq h \leq f$. Hence h is an affine function dominated by f such that

$$t \ \leq \ h(x_0) \ < f(x_0).$$

Let $x_0 \in E \setminus adh(dom(f))$. From Theorem 2.2.5 (separation of a point from a closed convex set), there exists $u^* \in E$ such that

$$\alpha \ := \ \sup_{x \in dom(f)} \langle u^*, x\rangle \ < \ \langle u^*, x_0\rangle \ . \tag{5.1.1}$$

Let $h_{u^*,\alpha}$ be the affine function defined by

$$h_{u^*,\alpha}(x) \ = \ \langle u^*, x\rangle - \alpha$$

for $x \in E$. Then we have

$$h_{u^*,\alpha}(x) \ \leq \ 0$$

for all $x \in dom(f)$ and $h_{u^*,\alpha}(x_0) > 0$. Let h_0 be an affine form such that $h_0 \leq f$ (such h_0 is given by the case $x_0 \in adh(dom(f))$). The sequence of affine functions $(h_n)_{n\geq 1}$ defined by

$$h_n(x) \quad = \quad n h_{u^*,\alpha}(x) + h_0(x)$$

for $x \in E$ and $n \geq 1$. Since $h_0 \leq f$ we get from (5.1.1), that $h_n \leq f$ on E and

$$\lim_{n\to+\infty} h_n(x_0) \quad = \quad +\infty \quad = \quad f(x_0)\ .$$

$\square$

5.2 Fenchel-Moreau conjugate

Definition 5.2.17 (Conjuguate function) *Let $f:\ E \to \mathbb{R}$. Its conjugate function (in the sense of Fenchel-Moreau) is the function $f^*:\ E \to \mathbb{R}$ defined by*

$$f^*(x^*) \quad = \quad \sup_{x\in E}\left\{\langle x^*, x\rangle - f(x)\right\}$$

for all $x^* \in E$.

The supremum of lsc convex functions dominated by f is also lsc and convex. This motivates the following definition.

Definition 5.2.18 (Convex lsc regularization) *The convex lsc regularization of a function $f:\ E \to \mathbb{R}$ is the greatest lsc convex function $\hat{f}\ : E \to \mathbb{R}$ dominated by f.*

Remark 5.2.4 *From the previous definition the functions f and $\hat{f}$ have the same affine lower-bound functions. From Theorem 5.1.13, $\hat{f}$ is the supremum of affine functions dominated by f.*

Proposition 5.2.21 *The conjuguate function f^* of a function f is either a convex lsc function from E to $\mathbb{R}\cup\{+\infty\}$, or identically equal to $-\infty$.*

Proof. If f is identically equal to $+\infty$, its conjuguate function is identically equal to $-\infty$. If not, we can restrict the sup in the definition of f^* to $x \in dom(f)$. Hence, f^* is the upper envelope of continuous affine functions. Hence it is convex lsc and do not take the value $-\infty$. $\square$

Remark 5.2.5 *The value $f^*(x^*)$ is the smallest constant b such that the affine function $h_{x^*,b} = \langle x^*, .\rangle - b$ is dominated by f (with the convention $\inf \emptyset = +\infty$). In particular f^* is the constant $+\infty$ if and only if f does not dominate any affine function. Indeed, $\langle x^*, .\rangle - b \leq f$ if and only if $\langle x^*, x\rangle - f(x) \leq b$ for all $x \in E$, which is equivalent to $f^*(x^*) \leq b$.*

Proposition 5.2.22 *i) $f^*(0) = -\inf_{x\in E} f(x)$.*

ii) $f \leq g \Rightarrow f^* \geq g^*$.

iii) For $\lambda > 0$, we have $(\lambda f)^*(x^*) = \lambda f^*(\frac{x^*}{\lambda})$.

iv) For $\lambda \neq 0$, we have $(D_\lambda f)^* = D_{1/\lambda} f^*$,(where $D_\lambda f$is the dilatation of f defined by $D_\lambda f(x) = f(\lambda x)$).

v) For $\alpha \in \mathbb{R}$, we have $(f + \alpha)^* = f^* - \alpha$.

vi) For $a \in E$, we have $(\tau_a f)^* = f^* + \langle ., a\rangle$ (where $\tau_a f$ is the translation of f defined by $\tau_a f(x) = f(x - a)$).

vii) $(\inf_{i\in I} f_i)^* = \sup_{i\in I} f_i^*$.

viii) $(\sup_{i\in I} f_i)^* \leq \inf_{i\in I} f_i^*$.

Proof. The proof is let as an exercice. $\square$

5.3 Bi-conjugate

Definition 5.3.19 *The biconjugate function of $f : E \to \mathbb{R}$ is defined as the conjugate of the conjugate f^* of f and is denoted by $(f^*)^*$.*

Theorem 5.3.14 (Bi-conjugate function theorem) *The bi-conjugate $(f^*)^*$ is equal to the convexe lsc regularization of f if f is lower bounded by an affine function. If not, $(f^*)^*$ is identically equal to $-\infty$.*

Proof. From Remark 5.2.5, the affine function $h_{x^*,b} = \langle x^*, .\rangle - b$ is dominated by f if and only if $(x^*, b) \in epi(f^*)$. If such a lower bound afine function exists we have from Remark 5.2.4

$$\hat{f}(x) = \sup_{(x^*,b)\in epi(f^*)} h_{x^*,b}(x) = \sup_{x^*\in dom(f^*)} \langle x, x^*\rangle - f^*(x^*) = (f^*)^*(x)$$

for all $x \in E$. $\square$

Corollary 5.3.9 *Let $f : E \to \mathbb{R}$ a convex proper lsc function. We then have $(f^*)^* = f$. The equality also holds if f is identically equal to $-\infty$.*

Proof. If $f : E \to \mathbb{R}$ is convex proper lsc, it dominates an affine function and we have $\hat{f} = f$ and the result follows from the previous theorem. $\square$