# Convexity, Optimization and Stochastic Control Exam

3h, documents are not allowed

January 11, 2024

## Course question

Let $E$ be a finite dimensional vector space, $f: E \to \mathbb{R} \cup \{+\infty\}$ a strictly convex function and $K \subset Int(dom(f))$ a nonempty convex and compact set. Show that $f$ admits a unique maximizer $x^*$ on $K$ and that $x^*$ is an extremal point of $K$.

## Problem

Given a finite time horizon $T > 0$, we shall consider in the sequel a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with an $\mathbb{R}^d$-valued standard Brownian motion $B = (B_t^1, \ldots, B_t^d)^\top_{0 \leq t \leq T}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ the ($\mathbb{P}$-augmentation of the) filtration generated by $B$. We also denote by $\ell$ the Lebesgue measure on $[0, T]$. In the sequel, the elements of $\mathbb{R}^d$ are considered as column vectors and $\top$ denotes the transposition of either vectors or matrices. For a vector $x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$ we denote by $\mathrm{Diag}(x)$ the diagonal matrix with $i$-th diagonal entry equal to $x^i$ for $i = 1, \ldots, d$.

**Financial market** We consider a financial market consisting of a non-risky asset $S^0$ normalized to unity, i.e. $S^0 \equiv 1$, and $d$ risky assets with price process $S = (S^1, \ldots, S^d)$ whose dynamics is defined by a stochastic differential equation. More specifically, we assume that the vector price process $S = (S^1, \ldots, S^d)^\top$, is defined by the stochastic differential equation:

$$dS_u = \mathrm{Diag}(S_u)\sigma(S_u)dB_u\,, \tag{1}$$

with $\sigma: \mathbb{R}^d \to \mathbb{R}^{d\times d}$ a continuous bounded function such that there exists a constant $L > 0$ satisfying

$$|\mathrm{Diag}(x)\sigma(x) - \mathrm{Diag}(y)\sigma(y)| \leq |x - y|\,, \quad x, y \in \mathbb{R}^d\,. \tag{2}$$

For an initial time and initial values $s = (s^1, \ldots, s^d)$, we shall denote by $S^{t,s} = (S_u^{t,s,1}, \ldots, S_u^{t,s,d})^\top_{u \in [t,T]}$ the process solution to (1) such that $S_t^{t,s} = s$. Under condition (2), the process $S^{t,s}$ is well defined for any initial condition $(t, s) \in [0, T] \times (\mathbb{R}_+^*)^d$. We shall assume that the function $\sigma$ takes invertible values and

$$\sup_{x \in \mathbb{R}^d} \left|\sigma^\top(x)\sigma(x)\right|^{-1} \quad < \quad +\infty\,. \tag{3}$$

**Portfolio and wealth process** Let $W_t$ denote the wealth at time $t$ of some investor on the financial market. We assume that the investor allocates continuously the wealth between the non-risky asset and the risky assets in a self-financing way. We shall denote by $\pi_t^i$ the proportion of wealth invested in the $i$-th risky asset. The remaining proportion of wealth $1 - \sum_{i=1}^d \pi_t^i$ is invested in the non-risky asset. Under those conditions, the process $(W_t)_{0 \le t \le T}$ satisfies

$$dW_u \quad = \quad W_u \sum_{i=1}^d \pi_u^i \frac{dS_t^i}{S_u^i} \quad = \quad W_u \pi_u^\top \sigma(S_u) dB_u \tag{4}$$

with $\pi_t = (\pi_t^1, \ldots, \pi_t^d)$ for $t \in [0, T]$. We next define the set $\mathcal{A}$ of admissible investment strategies as the set of $\mathbb{F}$-progressive, $\mathbb{R}^d$-valued processes $\pi$ such that

$$\int_0^T |\sigma^\top(S_t)\pi_t|^2 dt \quad < \quad \mathbb{P} - a.s.$$

For a given initial time $t \in [0, T]$, initial wealth $w \in \mathbb{R}$, initial stock values $s = (s^1, \ldots, s^d) \in (\mathbb{R}_+^*)^d$ and strategy $\pi \in \mathcal{A}$, we denote by $W^{t,w,s,\pi} = (W_u^{t,w,s,\pi})_{u \in [t,T]}$ the solution to (4) on $[t, T]$ with $S = S^{t,s}$ and $W_t^{t,w,s,\pi} = w$.

**Portfolio constraints** Let $K$ be a closed convex subset of $\mathbb{R}^d$ containing the origin 0, and define the set of constrained strategies $\mathcal{A}_K$ by

$$\mathcal{A}_K \quad := \quad \left\{\pi \in \mathcal{A} \,:\, \pi \in K \;\; \mathbb{P} \otimes \ell - a.e.\right\}.$$

We shall assume in the sequel that the investor is allowed to chose only investment strategy in $\mathcal{A}_K$.

**Super-replication** We are given a European contingent claim with payoff given by $G(S_T)$ with $G : \mathbb{R}^d \to \mathbb{R}_+^*$ a continuous function satisfying the following linear growth condition:

$$\sup_{s \in \mathbb{R}^d} \frac{|G(s)|}{1 + |s|} \quad < \quad +\infty\,.$$

The aim of this problem is to study the value function $v$ defined by

$$v(t, s) \quad = \quad \inf\left\{w \in \mathbb{R}\,,\; \exists \pi \in \mathcal{A}_K \;:\; W_T^{t,s,w,\pi} \ge G(S_T^{t,s}) \;\mathbb{P} - a.s.\right\}$$

for $(t, s) \in [0, T] \times (\mathbb{R}_+^*)^d$. This value function corresponds to the super-replication price of the contingent claim $G$ under portfolio constraints $K$.

We define the support function $\delta_K$ of the convex $K$ by

$$\delta_K(x) \quad = \quad \sup_{y \in K} x^\top y \,, \qquad x \in \mathbb{R}^d$$

and the set $K^*$ as the domain of the function $\delta_K$:

$$K^* \quad = \quad \left\{ x \in \mathbb{R}^d \;:\; \delta_K(x) < +\infty \right\}.$$

1. What can we say about $\delta_K(x) - x^\top y$ for $x \in K^*$ and $y \in K$.

We shall admit the reverse implication of the previous result:

$$\delta_K(x) - x^\top y \geq 0 \;\; \forall \;\; x \in K^* \;\; \Rightarrow \;\; y \in K\,.$$

We denote by $\mathcal{A}^*$ the set of progressive processes $\nu$ with values in $K^*$ for which there exists a constant $C > 0$ (which may depend on $\nu$), such that

$$\sup_{s \in [0,T]} |\nu_s| \;\; \leq \;\; C \qquad \mathbb{P} - a.s.$$

Given $\nu \in \mathcal{A}^*$ and $(t, s, y) \in [0, T) \times (\mathbb{R}_+^*)^d \times \mathbb{R}_+^*$, we define $Y^{t,s,y,\nu}$ by

$$Y_r^{t,s,y,\nu} \quad = \quad y \exp\Big( - \int_t^r \big(\delta_K(\nu_u) + \frac{1}{2} |\sigma^{-1}(S_u^{t,s})\nu_u|^2\big) du + \int_t^r (\sigma^{-1}(S_u^{t,s})\nu_u)^\top dB_u \Big)$$

for $r \in [t, T]$.

? 2. Show that, for all $(t, s, y) \in [0, T) \times (\mathbb{R}_+^*)^d \times \mathbb{R}_+ \times \mathbb{R}_+^*$ and $\nu \in \mathcal{A}^*$,

$$\mathbb{E}\Big[ \sup_{s \in [t,T]} |Y_s^{t,s,y,\nu}|^2 \Big] \quad < \quad +\infty\,.$$

3. Show that, for all $(t, s, w, y) \in [0, T) \times (\mathbb{R}_+^*)^d \times \mathbb{R}_+ \times \mathbb{R}_+^*$ and $(\nu, \pi) \in \mathcal{A}^* \times \mathcal{A}$, the process $W^{t,w,s,\pi} Y^{t,s,y,\nu}$ is a non-negative local super-martingale.

4. Deduce that, for all $(t, s, w, y) \in [0, T) \times (\mathbb{R}_+^*)^d \times \mathbb{R}_+ \times \mathbb{R}_+^*$ and $(\nu, \pi) \in \mathcal{A}^* \times \mathcal{A}$, the process $W^{t,w,s,\pi} Y^{t,s,y,\nu}$ is a super-martingale.

We now define the value function $p :\; [0, T] \times (\mathbb{R}_+^*)^d \times \mathbb{R}_+ \to \mathbb{R}$ by

$$p(t, s, y) \quad = \quad \sup_{\nu \in \mathcal{A}^*} \mathbb{E}\Big[ Y_T^{t,s,y,\nu} G(S_T^{t,s}) \Big] \,, \qquad (t, s, y) \in [0, T] \times (\mathbb{R}_+^*)^d \times \mathbb{R}_+.$$

5. Show that for $\pi \in \mathcal{A}$ such that $W_T^{t,w,s,\pi} \geq G(S_T^{t,s})$ we have $w \geq p(t, s, 1)$.

6. Deduce that $v(t, s) \geq \bar{p}(t, s) := p(t, s, 1)$ for all $(t, s) \in [0, T] \times (\mathbb{R}_+^*)^d$.

7. Give without proof the dynamic programming principle that the function $p$ should satisfy.

We now introduce the second order operator $\mathcal{L}$ defined by

$$\mathcal{L}\varphi(t,s) = \partial_t\varphi(t,s) + \frac{1}{2}\mathrm{Tr}\big(\mathrm{Diag}(s)\sigma(s)\sigma^\top(s)\mathrm{Diag}(s)\frac{\partial^2\varphi}{\partial s^2}(t,s)\big)$$

for $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ and $(t,s)\in[0,T]\times\mathbb{R}^d$. We consider the PDE

$$\mathcal{L}\varphi(t,s,y) + \sup_{a\in K^*}\Big\{-y\delta(a)\partial_y\varphi(t,s,y) + ya^\top\mathrm{Diag}(s)\frac{\partial^2\varphi}{\partial y\partial s}(t,s,y) + \frac{1}{2}|y\sigma^{-1}(s)a|^2\frac{\partial^2\varphi}{\partial y^2}\Big\} = 0 \tag{5}$$

8. Suppose that the function $p$ is smooth. Show that $p$ is a solution to (5).

9. Deduce that $\bar{p}$ satisfies

$$\sup_{a\in K^*}\{y\mathcal{L}\bar{p}(t,s) - y\delta_K(a)\bar{p}(t,s) + ya^\top\mathrm{Diag}(s)\frac{\partial\bar{p}}{\partial s}(t,s)\} = 0$$

for any $y>0$.

10. Conclude that $\bar{p}$ satisfies

$$\max\Big\{\mathcal{L}\bar{p}(t,s),\ \sup_{a\in K_1^*}\{a^\top\mathrm{Diag}(s)\frac{\partial\bar{p}}{\partial s}(t,s) - \delta_K(a)\bar{p}(t,s)\}\Big\} \le 0$$

where $K_1^* = \{a\in K^* \ : \ |a|=1\}$.

We now study the boundary condition at $t=T$.

11. For a sequence $(t_n,s_n)_n$ of $[0,T)\times(\mathbb{R}_+^*)^d$ converging to $(T,s)$ consider the sequence of controls of the form

$$\nu_s^n = \frac{1}{T-t_n}a\mathbb{1}_{[t_n,T]}(s)$$

for $a\in K^*$, and use an appropriate Girsanov transformation, show that

$$\liminf_{(t',s')\to(T-,s)}\bar{p}(t',s') =: \bar{p}(T-,s) \ \ge\ \hat{G}(s) := \sup_{a\in K^*} e^{-\delta_K(a)}G(se^a) \ \ge\ 0\,,$$

where $se^a = (s^1e^{a^1},\dots,s^de^{a^d})$ for $s=(s^1,\dots,s^d)$ and $a=(a^1,\dots,a^d)$.

Suppose that $\hat{G}$ is differentiable on $(\mathbb{R}_+^*)^d$.

12. By using the fact that $K^*$ is a convex cone, show that $\hat{G}(s)\ge e^{-\lambda\delta_K(a)}\hat{G}(se^{\lambda a})$ for all $a\in K^*$, $\lambda>0$ and $s\in(\mathbb{R}_+^*)^d$.

• 13. Deduce that $\inf_{a\in K^*}\{\delta_K(a)\hat{G}(s) - a^\top\mathrm{Diag}(s)\frac{\partial\hat{G}}{\partial s}(s)\}\ge 0$ for all $s\in(\mathbb{R}_+^*)^d$.

14. What can you say about $\hat{G}(s)^{-1}\mathrm{Diag}(s)\frac{\partial \hat{G}}{\partial s}(s)$ for $s \in (\mathbb{R}_+^*)^d$?

From now on, we assume that $\sigma$ is constant and there exists a function with linear growth $V \in C^{1,2}([0,T) \times \mathbb{R}^d) \cap C^0([0,T] \times \mathbb{R}^d)$ solution to

$$\mathcal{L}V = 0 \text{ on } [0,T) \times \mathbb{R}^d \quad \text{and} \quad V(T,.) = \hat{G} \text{ on } \mathbb{R}^d .$$

15. Under the assumption that $\bar{p} \in C^{1,2}([0,T) \times \mathbb{R}^d) \cap C^0([0,T] \times \mathbb{R}^d)$, show that $\bar{p} \geq V$ on $[0,T] \times \mathbb{R}^d$.

16. Give an expression of $V$ as an expectation of a function of the process $S^{t,x}$.

17. Suppose that $\frac{\partial \hat{G}}{\partial s}$ is bounded. Compute $\frac{\partial \hat{V}}{\partial s}(t,s)$ for $(t,s) \in [0,T) \times (\mathbb{R}_+^*)^d$.

18. Deduce that $\inf_{a \in K^*}\{\delta_K(a)V(s) - a^\top \mathrm{Diag}(s)\frac{\partial V}{\partial s}(s)\} \geq 0$ for all $s \in (\mathbb{R}_+^*)^d$.

19. Show that, if

$$\int_t^T \left| \frac{\mathrm{Diag}(S_u^{t,x})\frac{\partial V}{\partial s}(u, S_u^{t,x})}{V(u, S_u^{t,x})} \right|^2 du \quad < \quad +\infty \quad \mathbb{P} - a.s.$$

for all $(t,s) \in [0,T] \times (\mathbb{R}_+^*)^d$ then $V \geq v$.

20. Conclude that in this case $V = v = \bar{p}$.