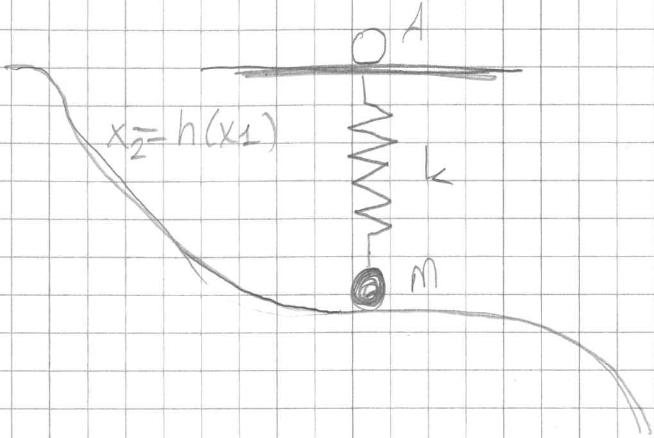
EXAM Q1



Spring potential: $\frac{1}{2} k p^2$

spring displacement

(Neutral position $x_2 = 0$)

a) Determine Lagrange Function of the system

b) Derive dynamical Model of the system.

a)

i) $q = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ii) $p = q$, $c(q) = x_2 - h(x_1) = 0$

iii) Energies: $T = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2)$ $V = mgx_2 + \frac{1}{2} k x_2^2$

iV) $\mathcal{L} = T - V - z c = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - mgx_2 - \frac{1}{2} k x_2^2 - z (x_2 - h(x_1))$

b)

V) E.L $\Rightarrow 0 = \frac{d}{dt} \nabla_{\dot{q}} \mathcal{L} - \nabla_q \mathcal{L}$

$\nabla_{\dot{q}} \mathcal{L} = \begin{bmatrix} m\dot{x}_1 \\ m\dot{x}_2 \end{bmatrix}$ $\frac{d \nabla_{\dot{q}} \mathcal{L}}{dt} = m \ddot{q}$

$\nabla_q \mathcal{L} = \begin{bmatrix} z h'(x_1) \\ -mg - kx_2 - z \end{bmatrix}$

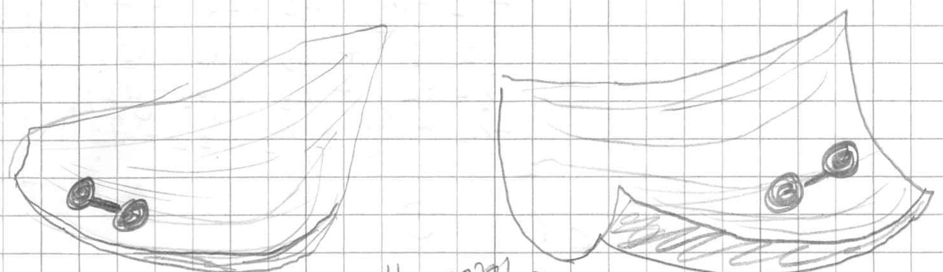
EL: $m\ddot{q} + \begin{bmatrix} -z h'(x_1) \\ mg + kx_2 + z \end{bmatrix} = 0$

$x_2 - h(x_1) = 0$

Extra: $\dot{c}(q) = \dot{x}_2 - h'(x_1)\dot{x}_1$

$\ddot{c}(q) = \ddot{x}_2 - h''(x_1)\dot{x}_1^2 - h'(x_1)\ddot{x}_1$

EXAM Q2



roller coaster

- Two balls with mass m, linked with massless rigid rod of Length L.
- Masses are gliding with surface eq:

$$z = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

a) Write lagrange function.

b) Derive eq. of motion

a) i) $q = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in \mathbb{R}^6$, where $p_1 = [x_1, y_1, z_1]^T$, $p_2 = [x_2, y_2, z_2]^T$.

ii) $[p_1, p_2] = q$, $[\dot p_1, \dot p_2] = \dot q$

$$c(q) = \begin{bmatrix} \frac{1}{2}x_1^2 + \frac{1}{2}y_1^2 - z_1 \\ \frac{1}{2}x_2^2 + \frac{1}{2}y_2^2 - z_2 \\ \frac{1}{2}(\|p_1 - p_2\|^2 - L^2) \end{bmatrix}$$

iii) $T = \frac{1}{2}m\dot p_1^T \dot p_1 + \frac{1}{2}m\dot p_2^T \dot p_2 = \frac{1}{2}m\dot q^T q$

$V = mg(z_1 + z_2)$

(to prevent confusion, λ instead of z)

iv) $\mathcal{L} = \frac{1}{2}m\dot q^T q - mg(z_1 + z_2) + \lambda^T c(q)$

b) v) $EL: 0 = \frac{d}{dt}\nabla_{\dot q}\mathcal{L} - \nabla_q \mathcal{L} = 0$

$$\frac{d}{dt}\nabla_{\dot q}\mathcal{L} = m\ddot q, \quad \nabla_q \mathcal{L} = \begin{bmatrix} \lambda_1 x_1 \\ \lambda_1 y_1 \\ -\lambda_1 - mg \\ \lambda_2 x_2 \\ \lambda_2 y_2 \\ -\lambda_2 - mg \end{bmatrix} + \lambda_3 \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \\ x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$$

EL:

$$0 = M\ddot{q} - \begin{bmatrix} \lambda_1 x_1 + \lambda_3 (x_1 - x_2) \\ \lambda_1 y_1 + \lambda_3 (y_1 - y_2) \\ -\lambda_1 - Mg + \lambda_3 (z_1 - z_2) \\ \lambda_2 x_2 + \lambda_3 (x_2 - x_1) \\ \lambda_2 y_2 + \lambda_3 (y_2 - y_1) \\ -\lambda_2 - Mg + \lambda_3 (z_2 - z_1) \end{bmatrix}$$

$$c(q) = 0$$

c)

$$C(q) \begin{bmatrix} \frac{1}{2} x_1^2 + \frac{1}{2} y_1^2 - z_1 \\ \frac{1}{2} x_2^2 + \frac{1}{2} y_2^2 - z_2 \\ \frac{1}{2} (\| p_1 - p_2 \|^2 - L^2) \end{bmatrix} = 0$$

* consistency condition: initial condition of the system must satisfy $C(q) = 0$, $\dot{C}(q) = 0$

$$\dot{C}(q) = \begin{bmatrix} x_1 \dot{x}_1 + y_1 \dot{y}_1 - \dot{z}_1 \\ x_2 \dot{x}_2 + y_2 \dot{y}_2 - \dot{z}_2 \\ (p_1 - p_2)^T (\dot{p}_1 - \dot{p}_2) \end{bmatrix} = 0$$

if initial conditions for $C(q)$ and $\dot{C}(q)$ are 0. As long as we satisfy EL eq which contains $\ddot{C}(q) = 0$, $C(q)$ and $\dot{C}(q)$ will stay 0.