PSS2 - Physical Modelling

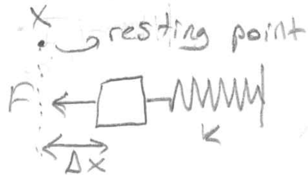
Exercises: 2.1, 2.3, 2.6, 2.9

Lectures Recap

(1) Analyze the system:
- subsystems
- inputs/outputs

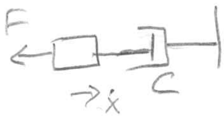
(2) Write equations → come from physics.

- conservation laws
- mechanics:
  - $\vec{F} = m\vec{a} = m\ddot{x}$, a and F are in same direction
  - $F_{spring} = -k\Delta x$

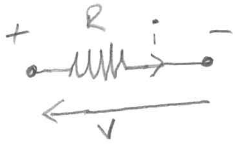


  - $F_{damping} = -bv = -b\dot{x}$

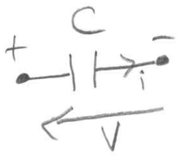
F is opposite direction to Δx and $\dot{x}$.



- electronics:
  - $V = Ri$



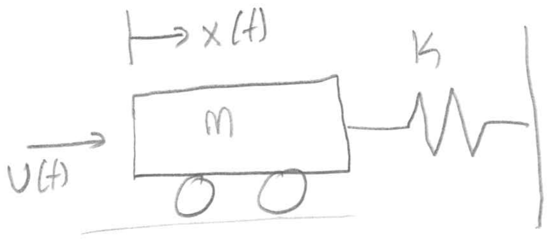
  - $i = C\dot{V}$



  - $V = L\dot{i}$

(3) Derive Models
- Choosing state variables
- State-space models

2.1



- cart at rest initially
- $x(0) = 0$
  $\dot{x}(0) = 0$
  $u(t) = \delta(t)$, starting from $t=0$

a) Newton's law: (state space form)

$$m\ddot{x} = U - kx$$

2^{nd} order system $\Rightarrow$ 2 states

$$\dot{x}_1 = x_2$$

$$x = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \dot{x}_2 = -\frac{k}{m}x_1 + \frac{U}{m}$$

$$y = x_1$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} U \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

b) Transfer Function from force U to position x?

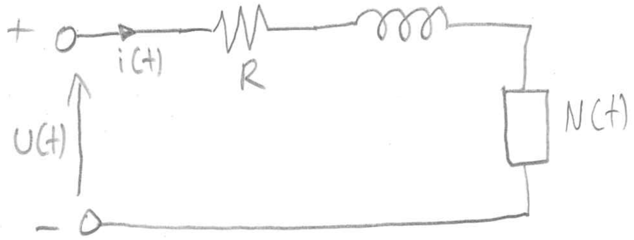
* s-transform

$$m s^2 X(s) = U(s) - k X(s)$$

$$(ms^2 + k) X(s) = U(s)$$

$$G(s) = \frac{X(s)}{U(s)} = \frac{1}{ms^2 + k}$$

2.3



$N(t) = k i^2, \ k > 0$

(voltage drop over the component)

a) Differential Equation?

Kirchoff's law: $\quad U = Ri + L\dot{i} + k i^2$

• state? $\quad x = i \Rightarrow \dot{x} = \dfrac{U - Rx - kx^2}{L}$

$$\dot{x} = \frac{-R}{L}x - \frac{k}{L}x^2 + \frac{1}{L}U$$

b) Linearize and find stationary point assuming $U(t) = U_0$

$ax^2 + bx + c = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

– stationary point: $\dot{x} = 0$

$$\dot{x} = 0 = -\frac{k}{L}x^2 - \frac{R}{L}x + \frac{1}{L}U_0$$

$$\tilde{x} = \frac{-\frac{R}{L} - \sqrt{\left(\frac{R}{L}\right)^2 + \frac{4kU_0}{L^2}}}{-\frac{2k}{L}} = -\frac{R}{2k} + \sqrt{\left(\frac{R}{2k}\right)^2 + \frac{U_0}{k}}$$

← other root will be negative
→ $i > 0$ due to voltage direction

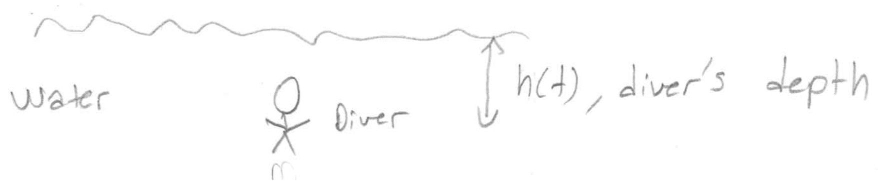
– $\Delta\dot{x} = A\Delta x + B\Delta U$

$A = \left.\dfrac{\partial f}{\partial x}\right|_{x_0, U_0}$ $\quad B = \left.\dfrac{\partial f}{\partial U}\right|_{x_0, U_0}$

– $A = -\dfrac{R}{L} - \dfrac{2k}{L}x \qquad B = \dfrac{1}{L}$

$$\Delta\dot{x} = \left(-\frac{R}{L} - \frac{2k}{L}x_0\right)\Delta x + \frac{1}{L}\Delta U$$

2.6



- F_{lift} = outer force that lifts diver (u)
- m : diver's mass, V: diver's volume, ρ: density of water.
- lifting force from the water : $g(\rho V - m)$
- Friction force proportional to velocity : $F = -b\dot{h}$ introduced
- $p(t)$, pressure relative to atmospheric pressure
- $\dot{p}(t) = k(\rho g h(t) - p(t))$
- $q(t) = p(t) - \rho g h(t)$

a) Flift as input, q(t) as output, determine a model.

$$m\ddot{h} = -b\dot{h} + g(\rho V - m) + u, \quad \dot{p} = k\rho g h - kp$$
$$q = p - \rho g h$$

$x_1 = h$
$x_2 = \dot{h}$
$x_3 = p$
$y = q$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{b}{m}x_2 + \frac{g(\rho V - m)}{m} + \frac{1}{m}u$$
$$\dot{x}_3 = k\rho g x_1 - k x_3$$
$$y = x_3 - \rho g x_1$$

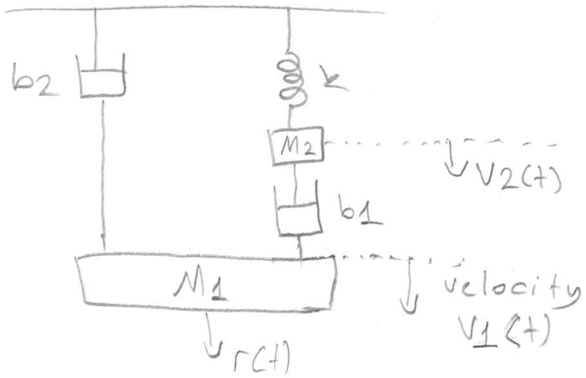
b) Stationary point?

$\dot{x}_1 = 0 \Rightarrow x_2 = 0$, $x_1 = h_0$ st depth

$\dot{x}_3 = 0 \Rightarrow x_3 = \rho g h_0 = p_0$ st. pressure

$\dot{x}_2 = 0 \Rightarrow u = mg - \rho g V$

2.9



$r(t)$ = input

– Model it in state space form

– $d(t)$ distance from spring equilibrium $\left(\int_0^t V_2\,dt\right)$

* Newton's law

- $M_1 \dot{V}_1 = r(t) - b_2 V_1 - b_1 (V_1 - V_2)$
- $M_2 \dot{V}_2 = b_1 (V_1 - V_2) - k\, d(t)$

– assumption: no gravity (or objects on horizontal plane)

$$X_1 = V_1$$
$$X_2 = V_2$$
$$X_3 = d(t) \Rightarrow \dot{X}_3 = X_2$$

$$U = r(t)$$

$$\dot{X} = \begin{bmatrix} \frac{-b_2 - b_1}{m_1} & b_1 & 0 \\ b_1/m_2 & -b_1/m_2 & -k/m_2 \\ 0 & 1 & 0 \end{bmatrix} X + \begin{bmatrix} 1/M_1 \\ 0 \\ 0 \end{bmatrix} U$$

$$y = V_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$