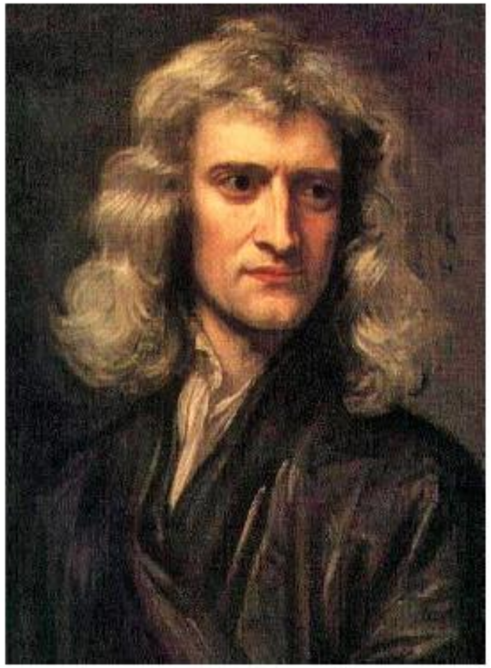
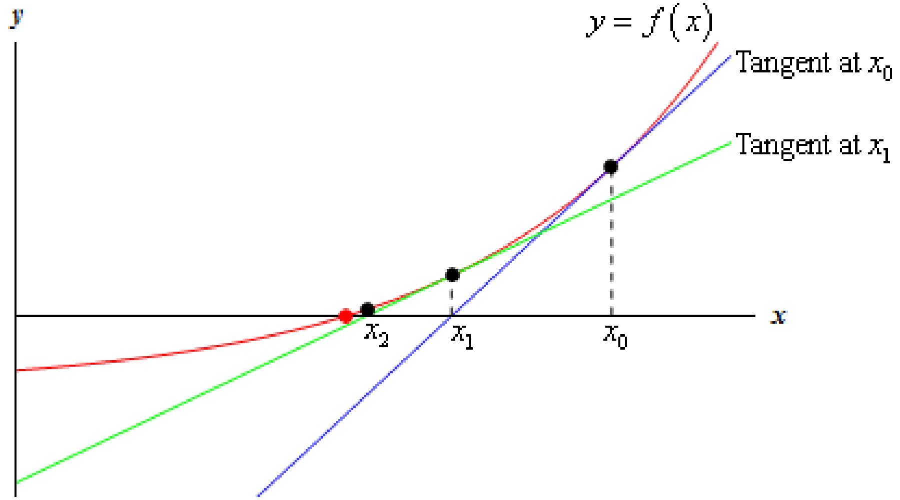
Introduction to the Newton Method



Yasemin BEKIROGLU



Outline

- When to use 
- How it works 
- Potential issues 

Newton Method

- Solving $f(x) = 0,\ f: R \longrightarrow R$

Newton Method

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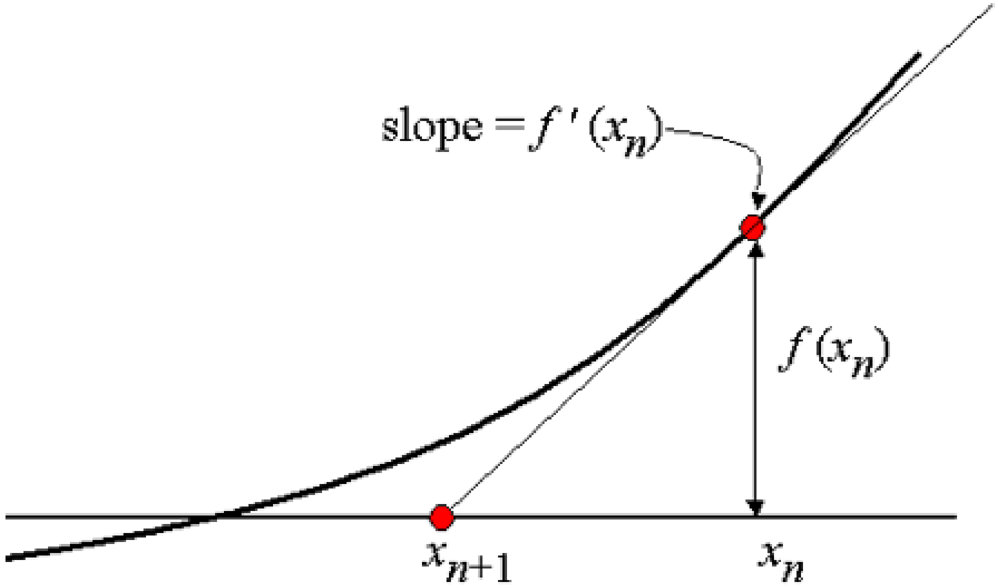
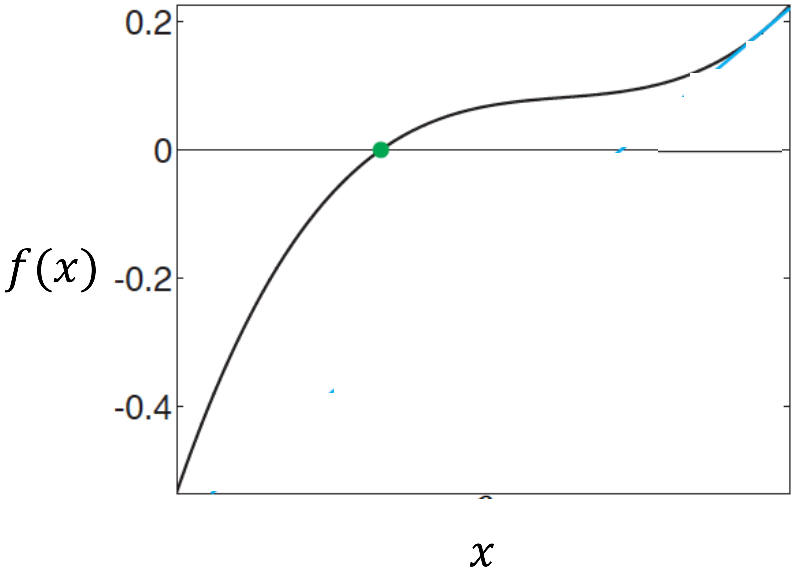
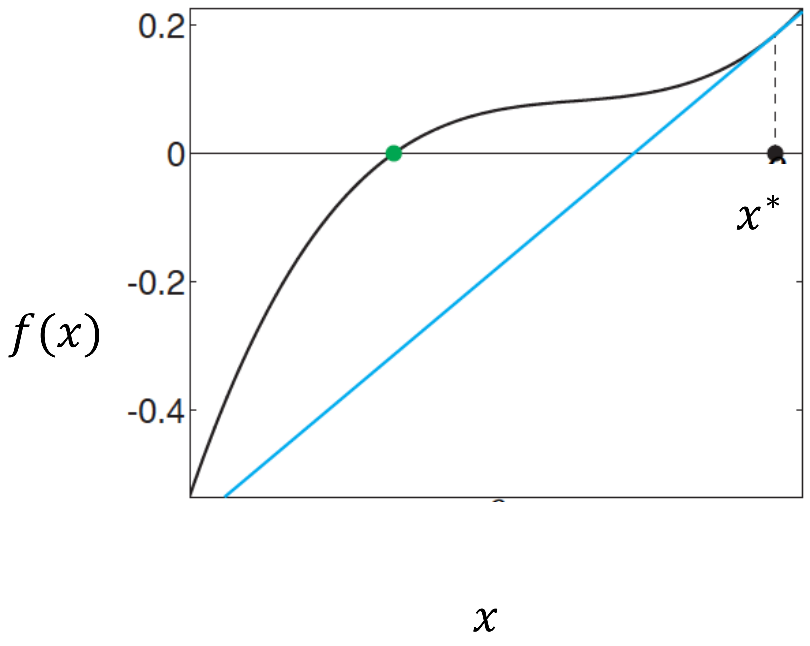
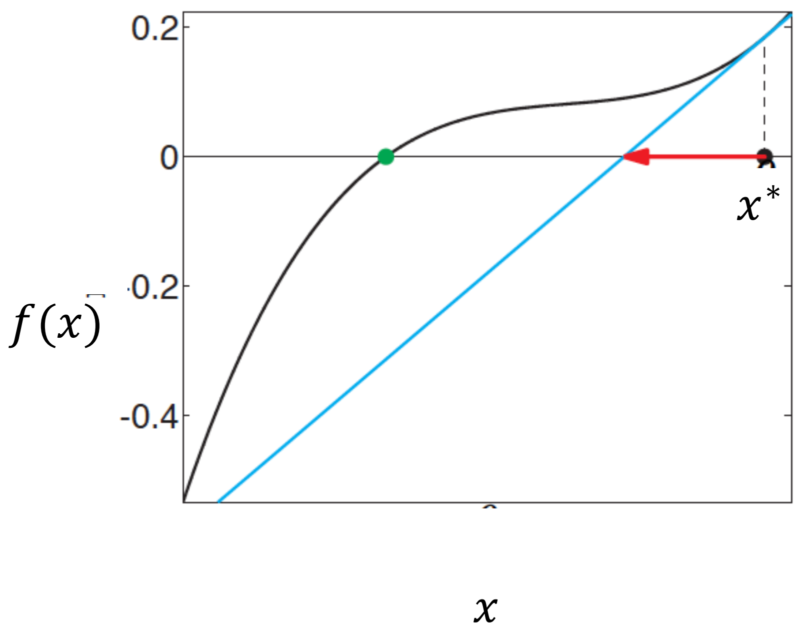


Newton Method – how it works




0.2
0
-0.2
-0.4
$f(x)$
x^*
x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$f(x)$
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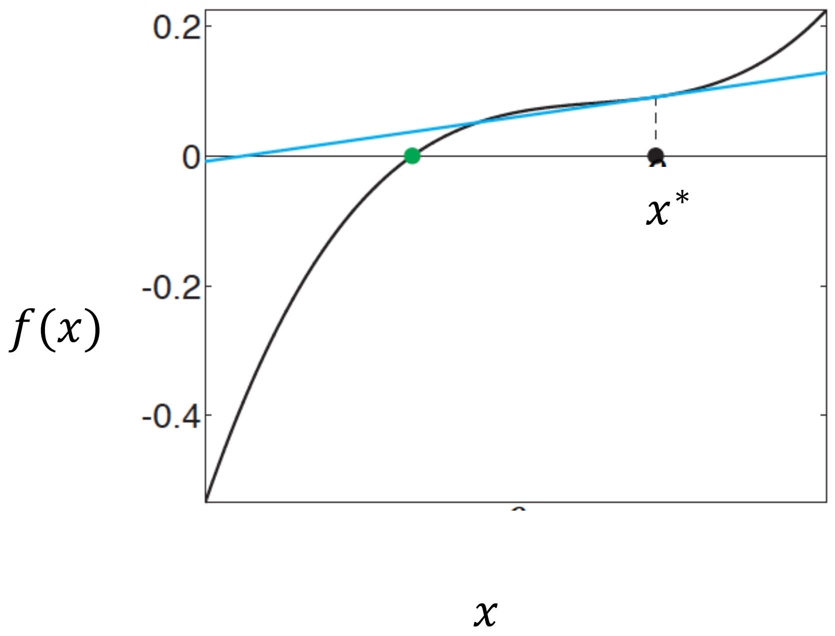
x


0.2
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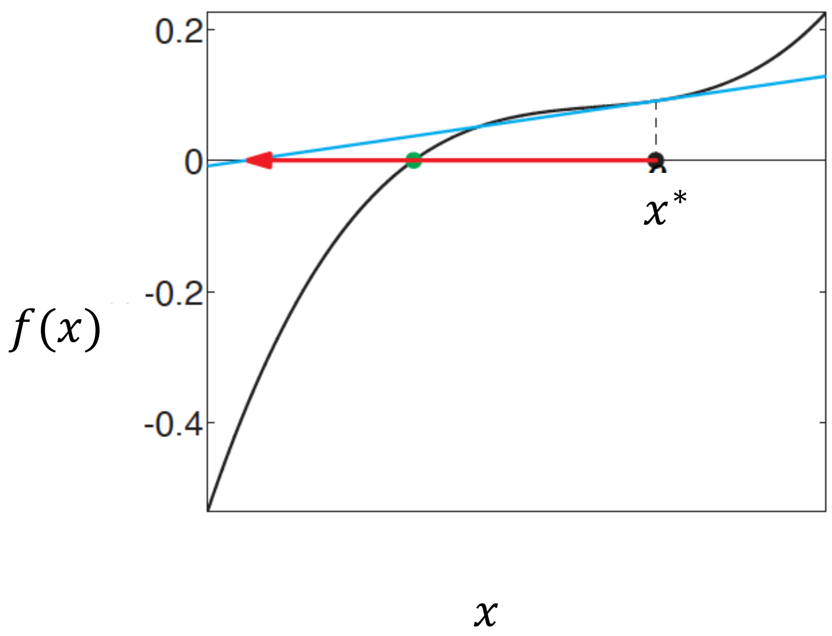
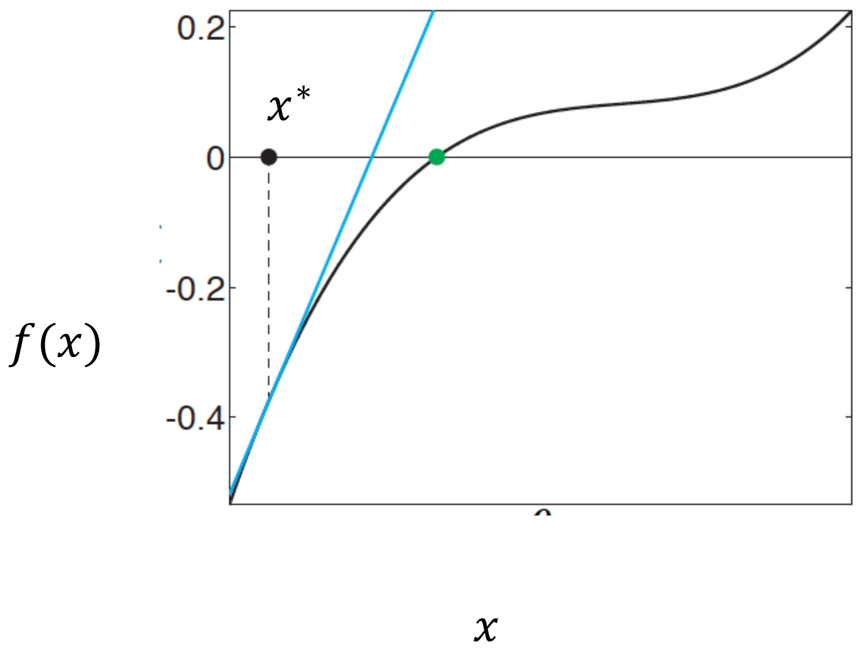
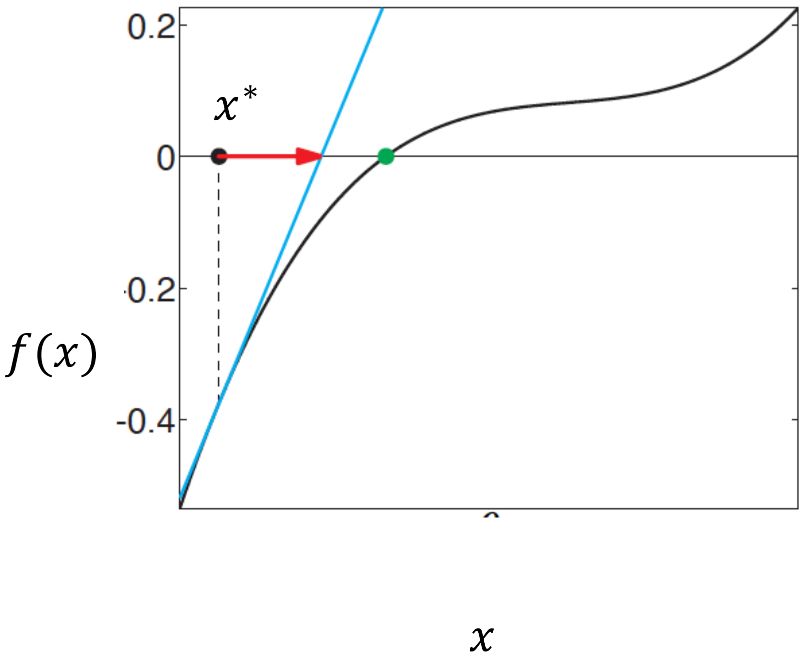
$f(x)$
x^*
x

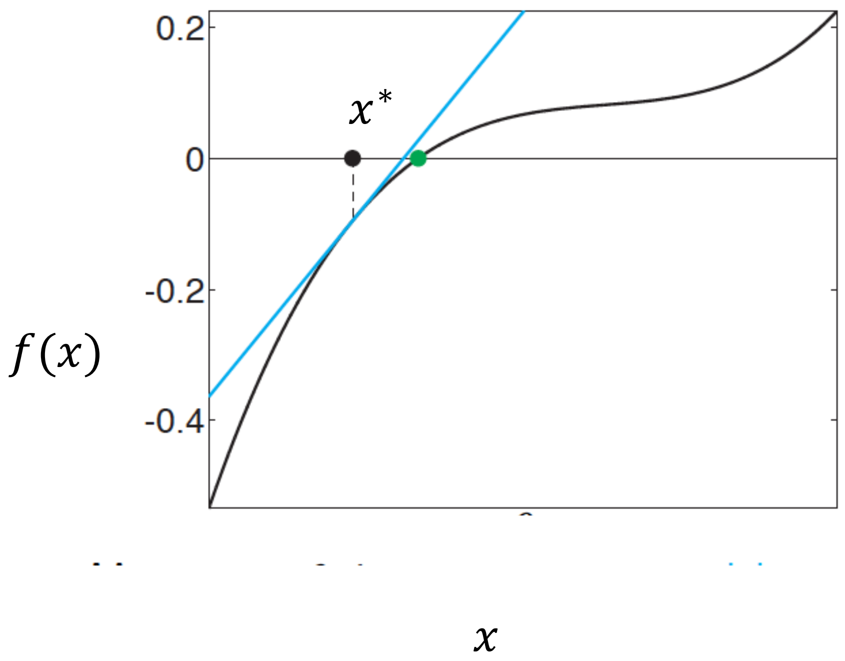


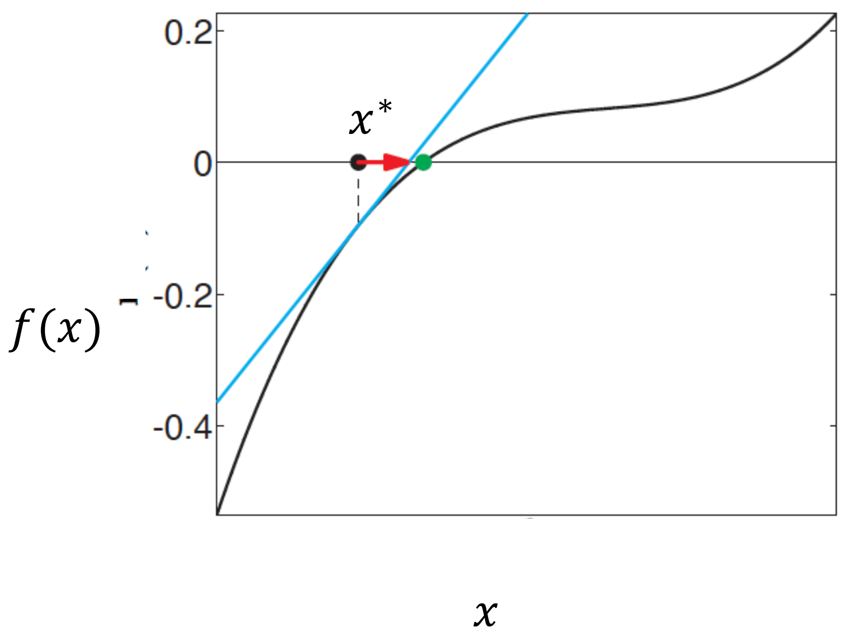
0.2
0
-0.2
-0.4
$f(x)$
x^*
x



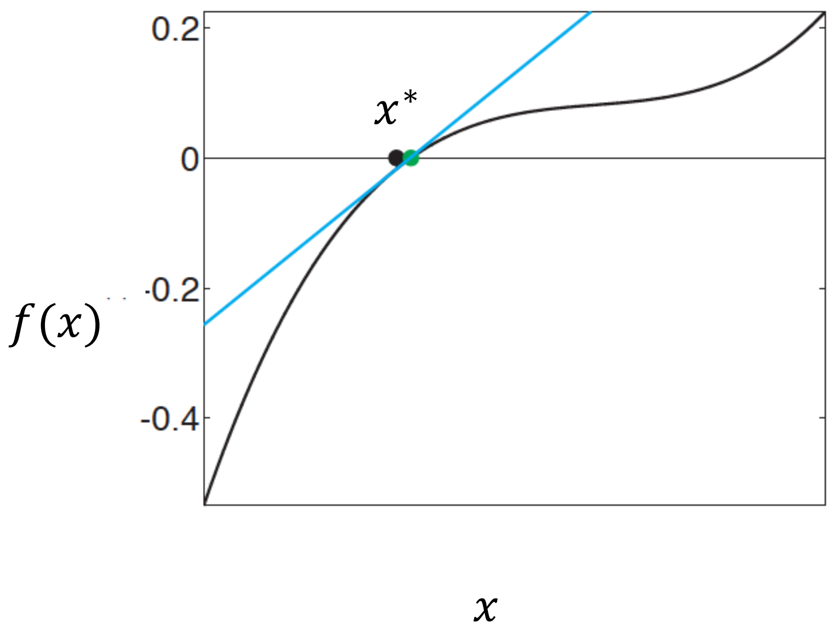
0.2
x^*
0
-0.2
$f(x)$
-0.4
x

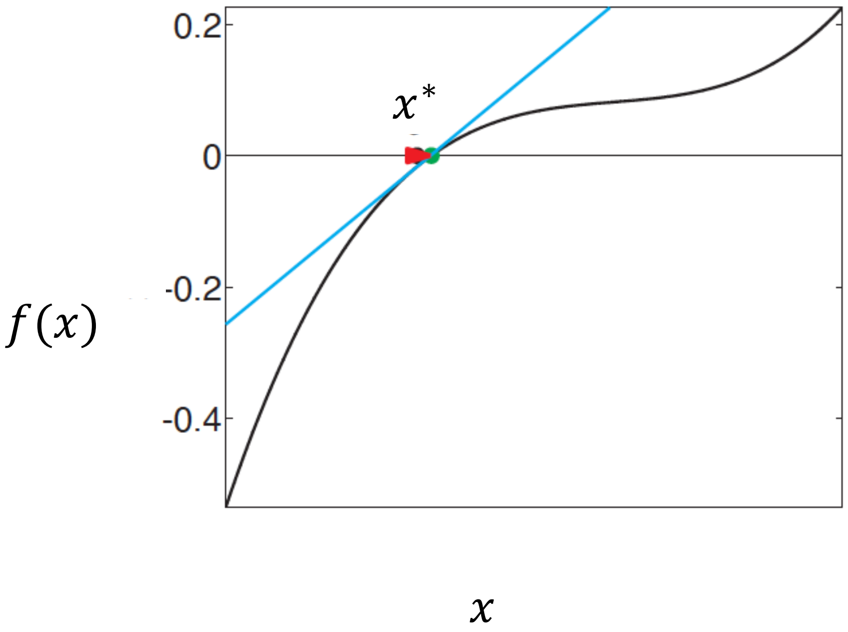

0.2
0
-0.2
-0.4
x^*
$f(x)$
x


0.2
0
-0.2
-0.4
x^*
$f(x)$
x

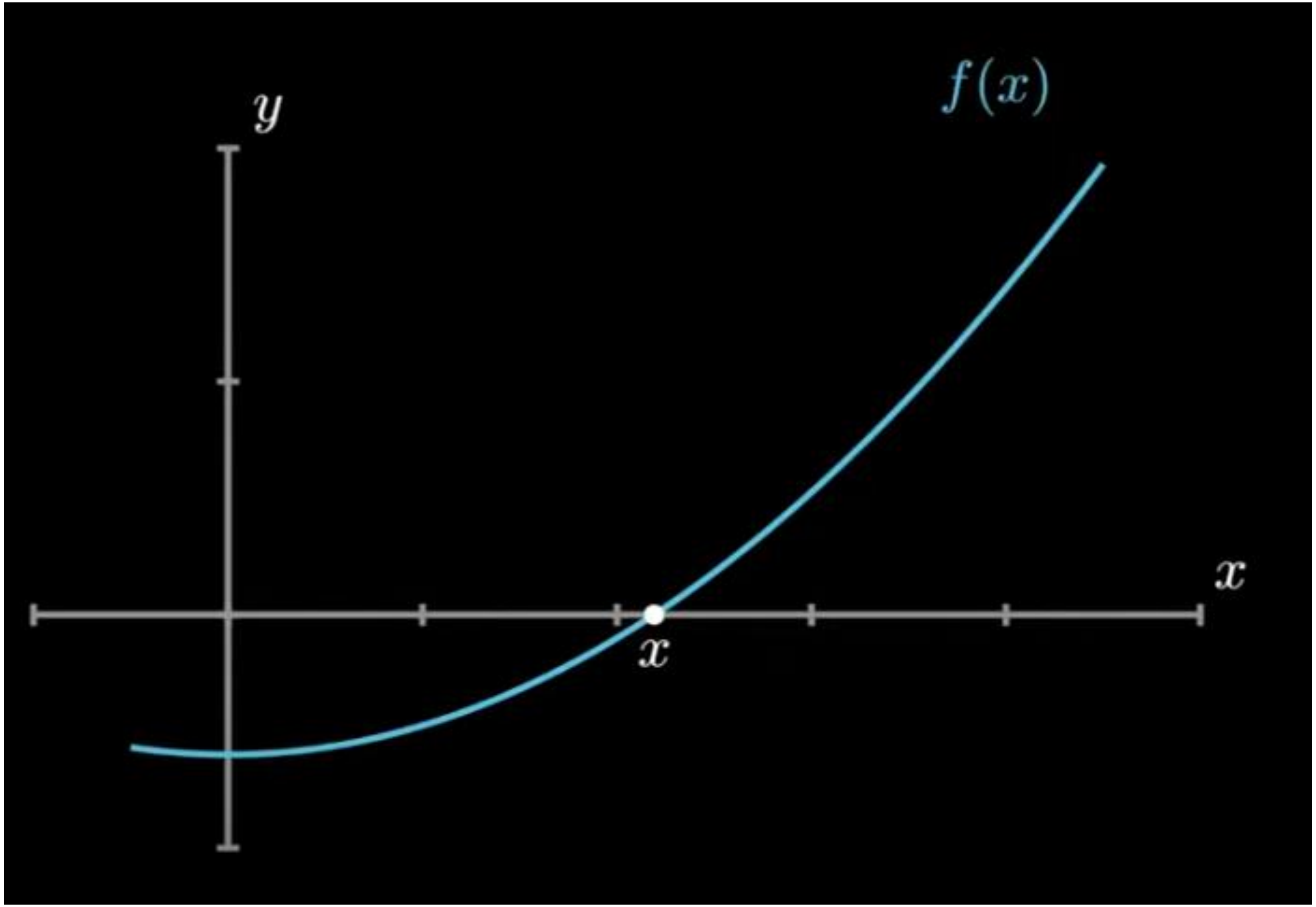


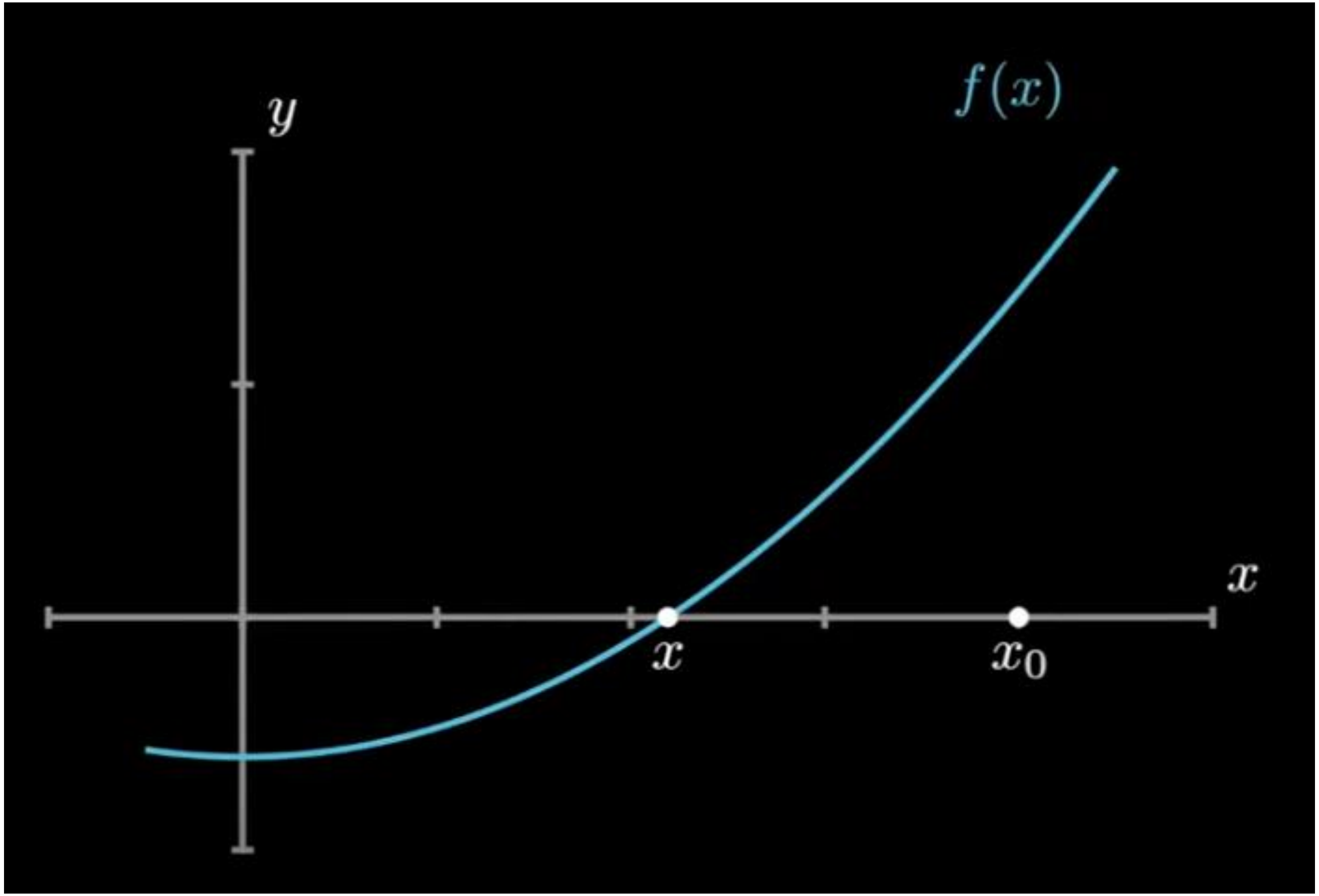
0.2
0
-0.2
-0.4
x^*
$f(x)$
x


0.2
0
-0.2
-0.4
x^*
$f(x)$
x

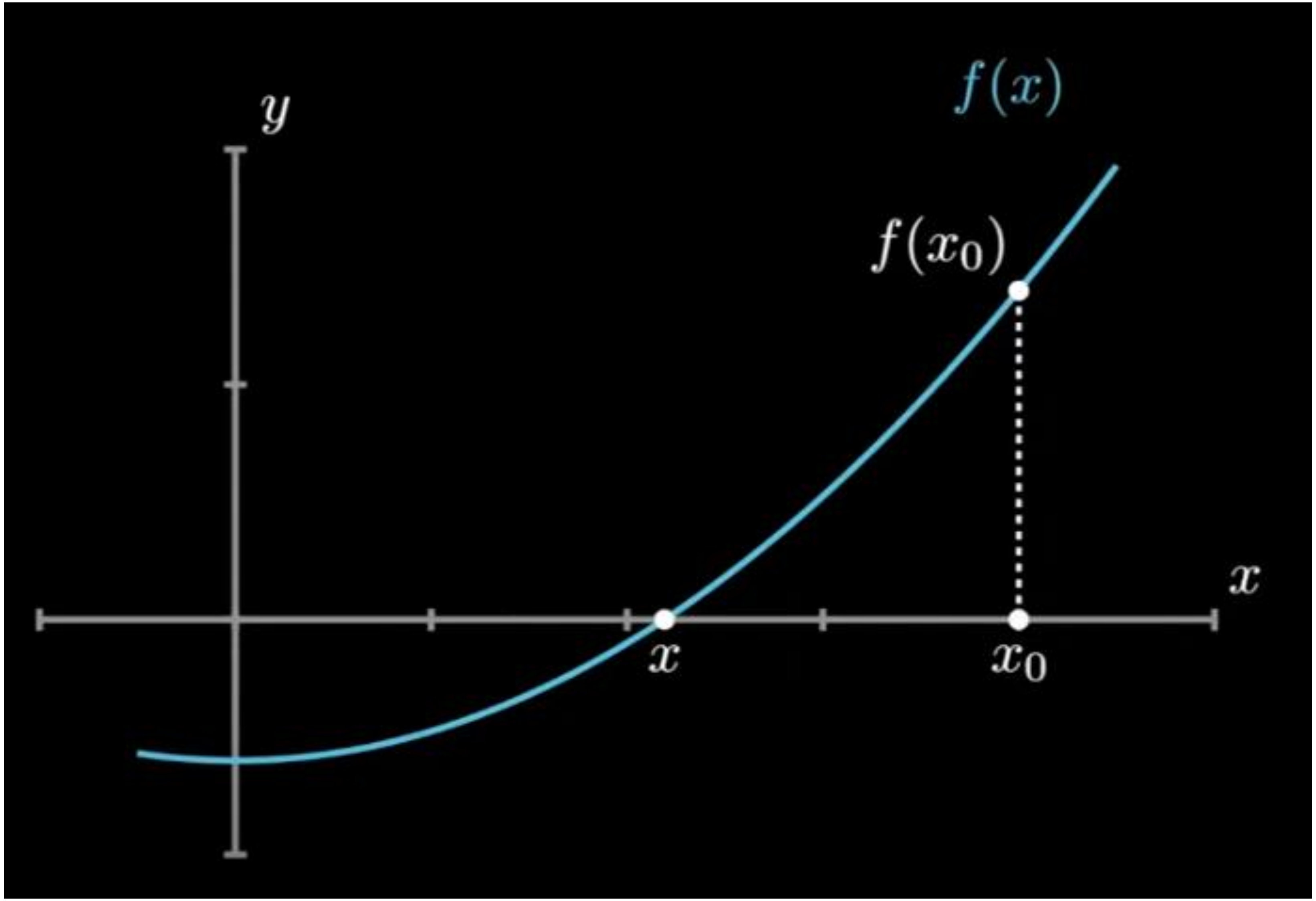


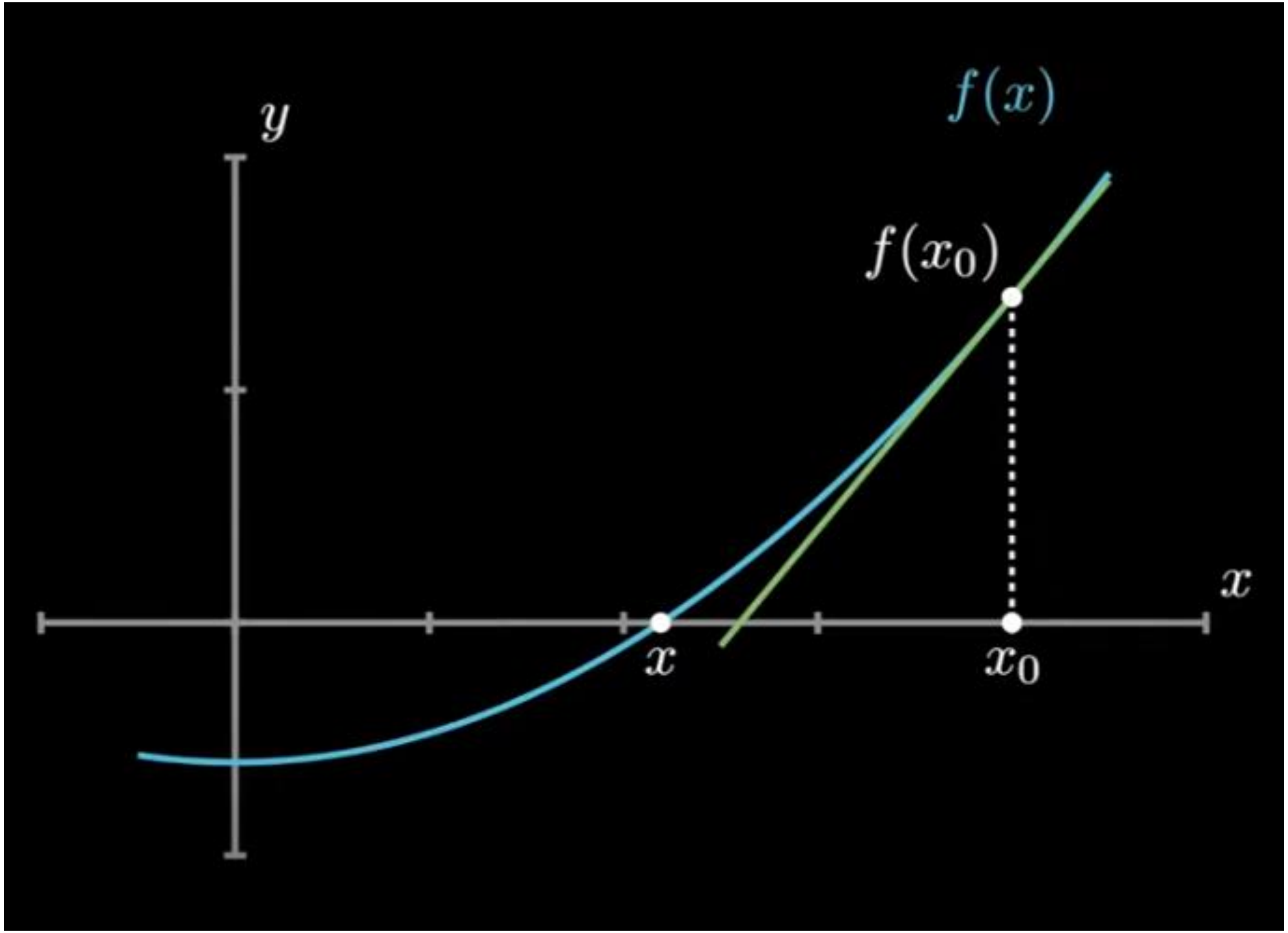
0.2
x^*
0
-0.2
$f(x)$
-0.4
x


$f(x)$
y
x
x

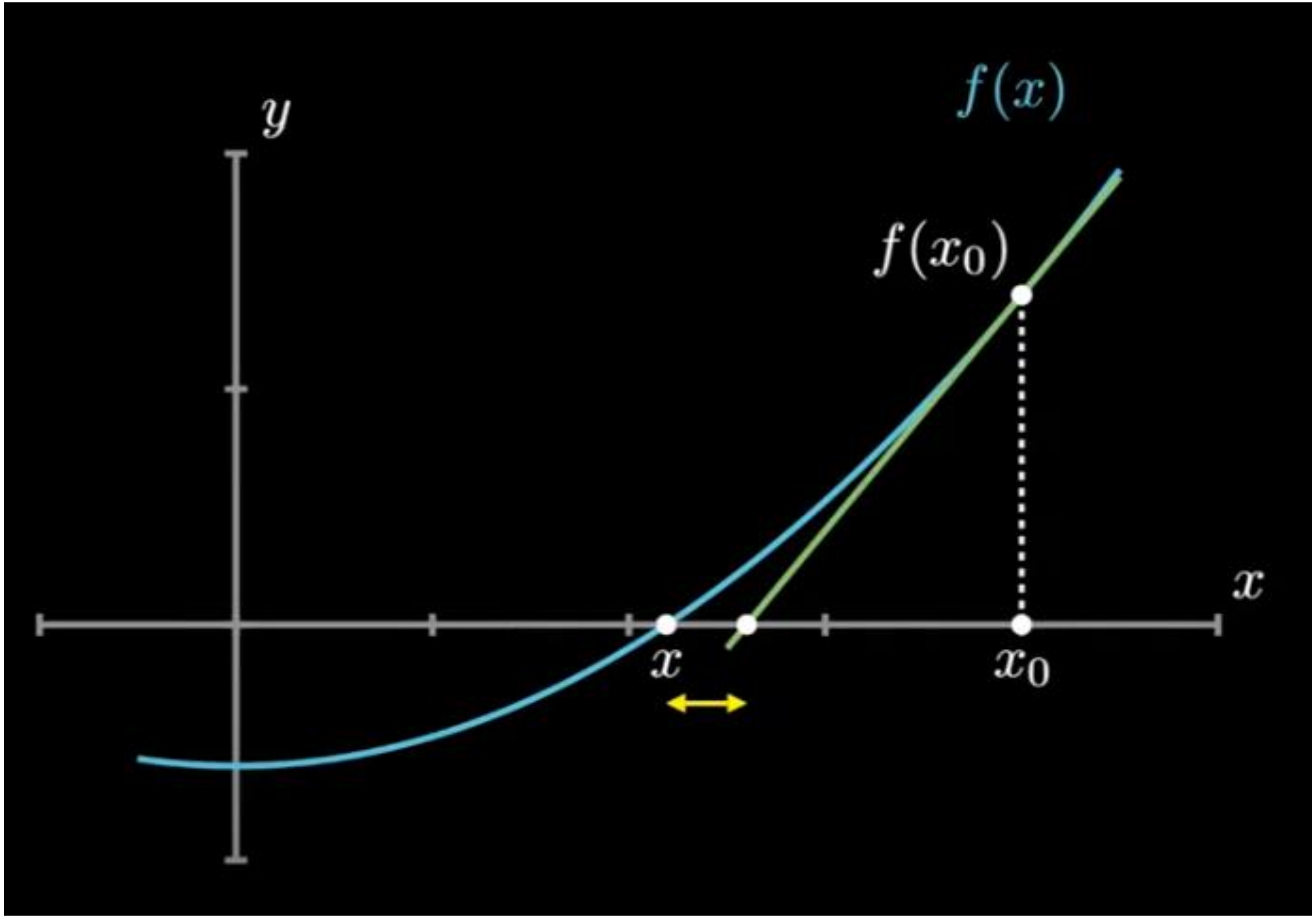


$f(x)$
y
x
x
x_0

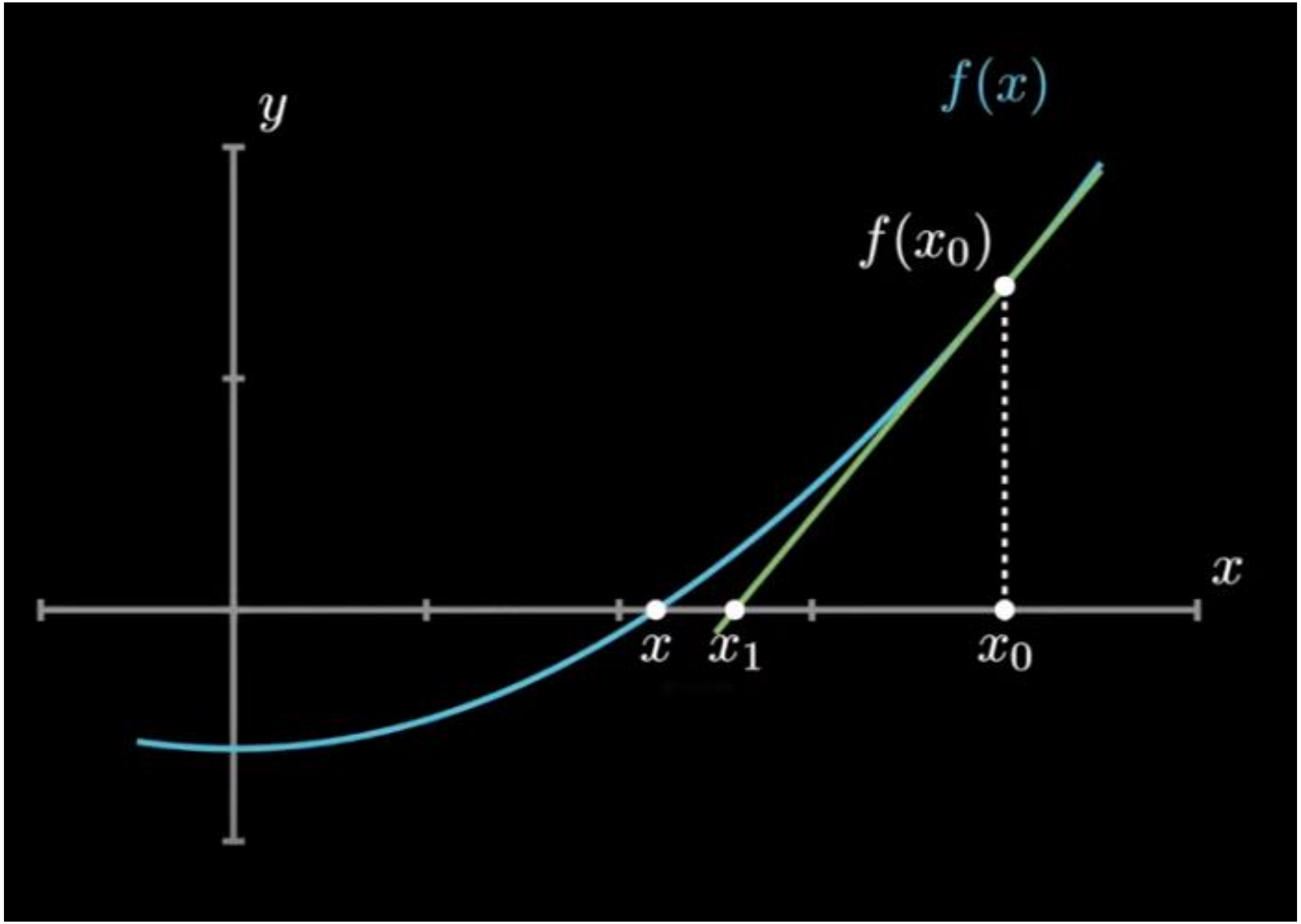

$f(x)$
y
$f(x_0)$
x
x
x_0

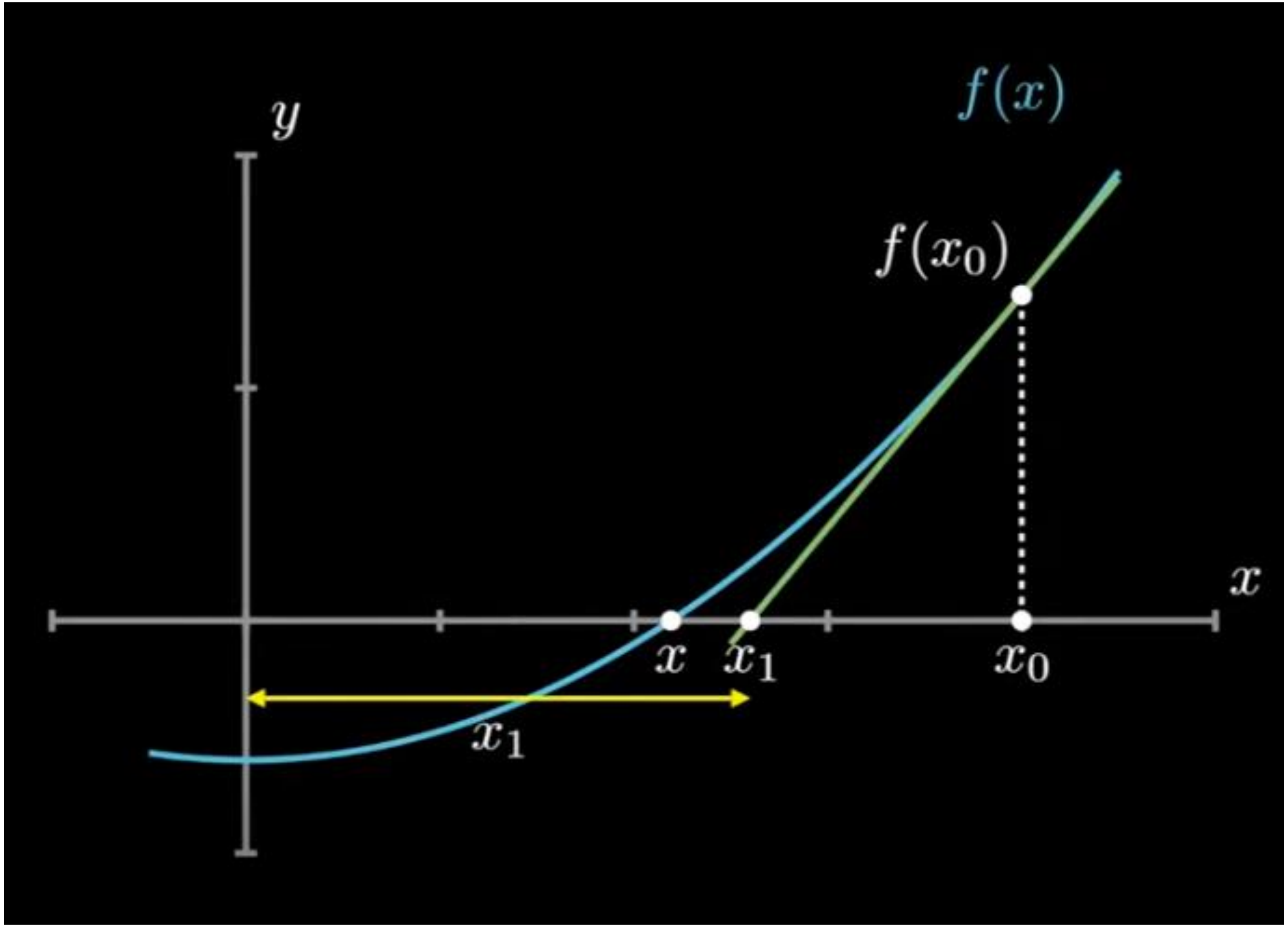


$f(x)$
y
$f(x_0)$
x
x
x_0

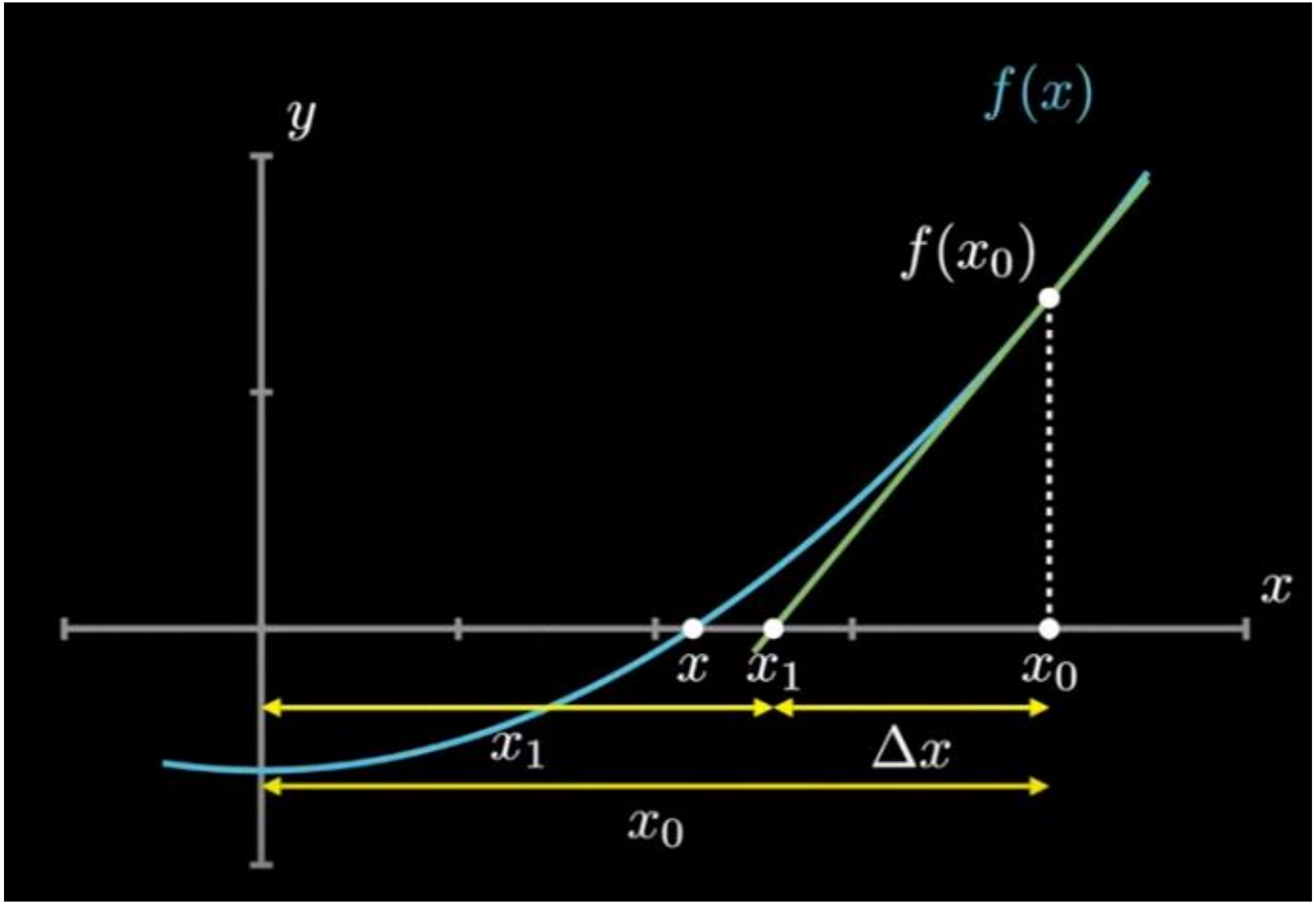


$f(x)$
y
$f(x_0)$
x
x
x_0

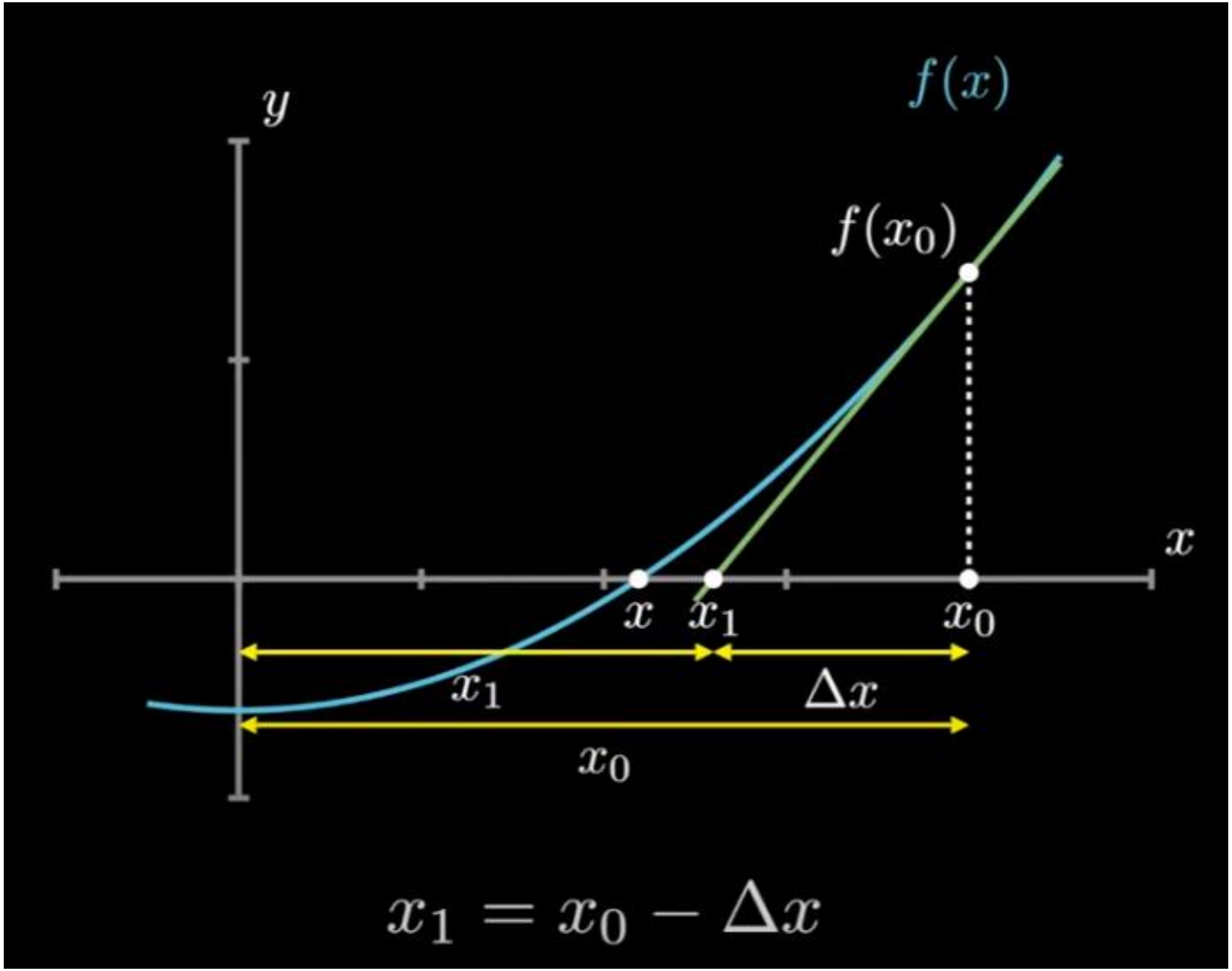

$f(x)$
y
$f(x_0)$
x
x
x_1
x_0

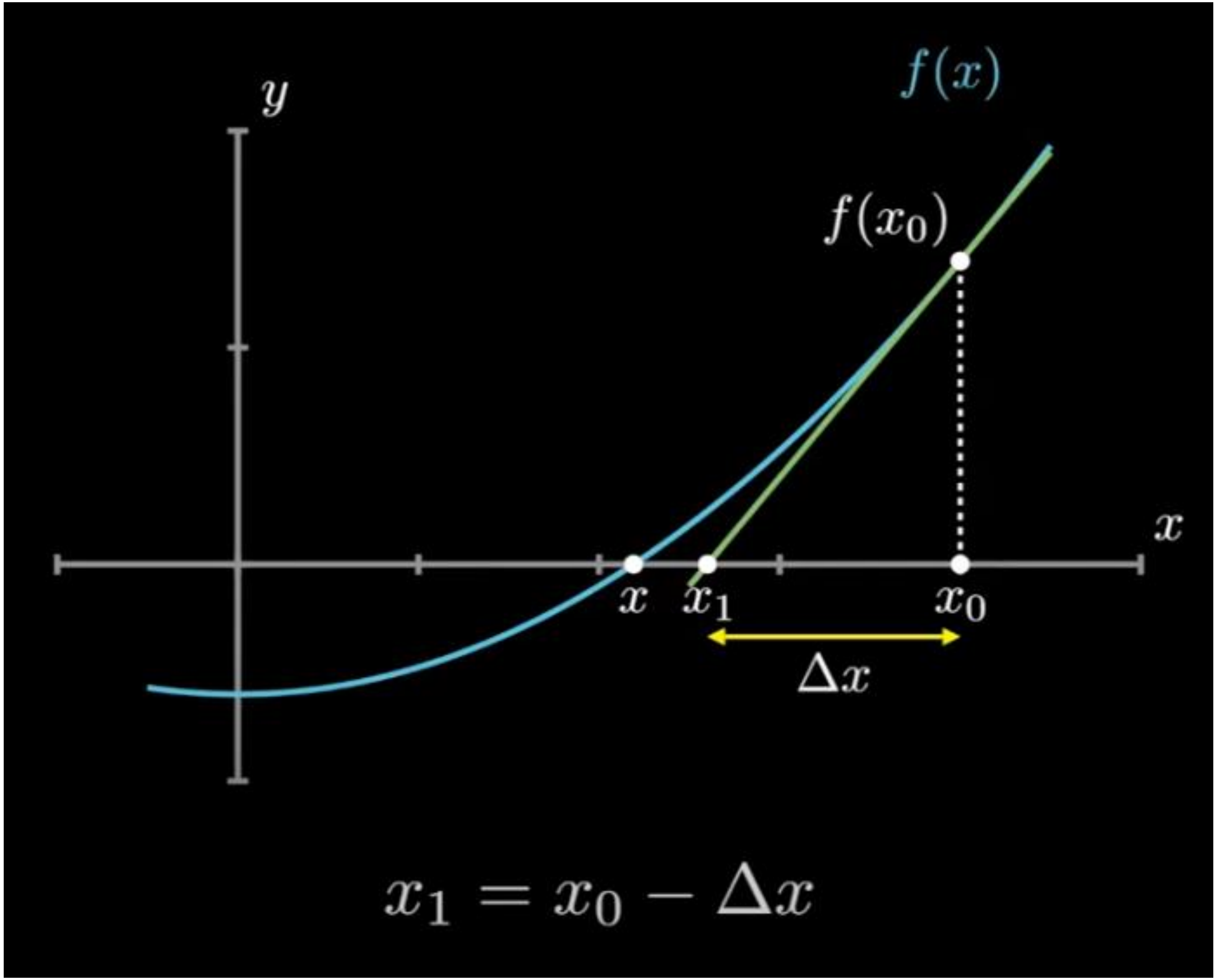


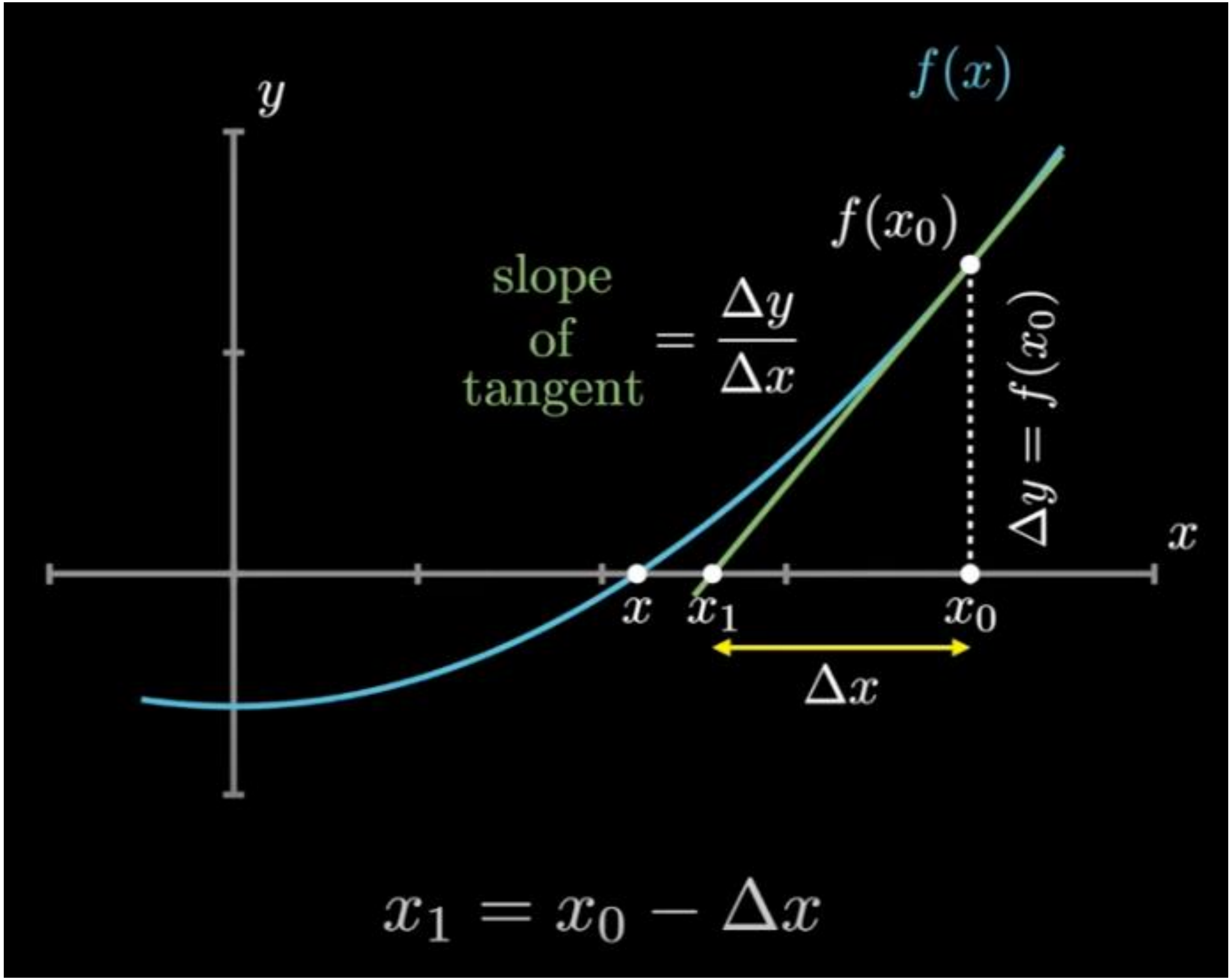
$f(x)$
y
$f(x_0)$
x
x
x_1
x_0
x_1



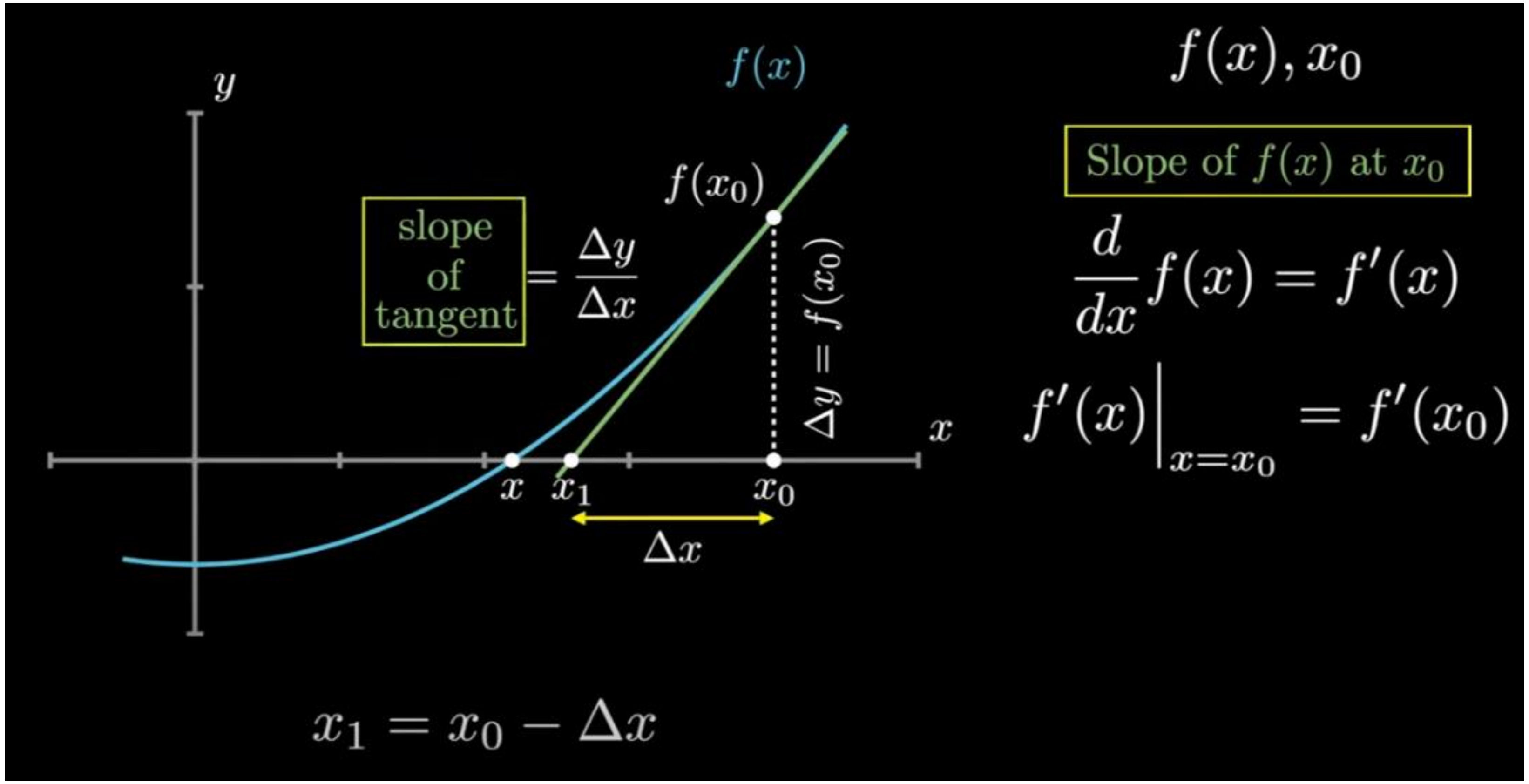
$f(x)$
y
$f(x_0)$
x
x
x_1
x_0
x_1
Δx
x_0


$f(x)$
y
$f(x_0)$
x
x
x_1
x_0
x_1
Δx
x_0
$x_1 = x_0 - \Delta x$

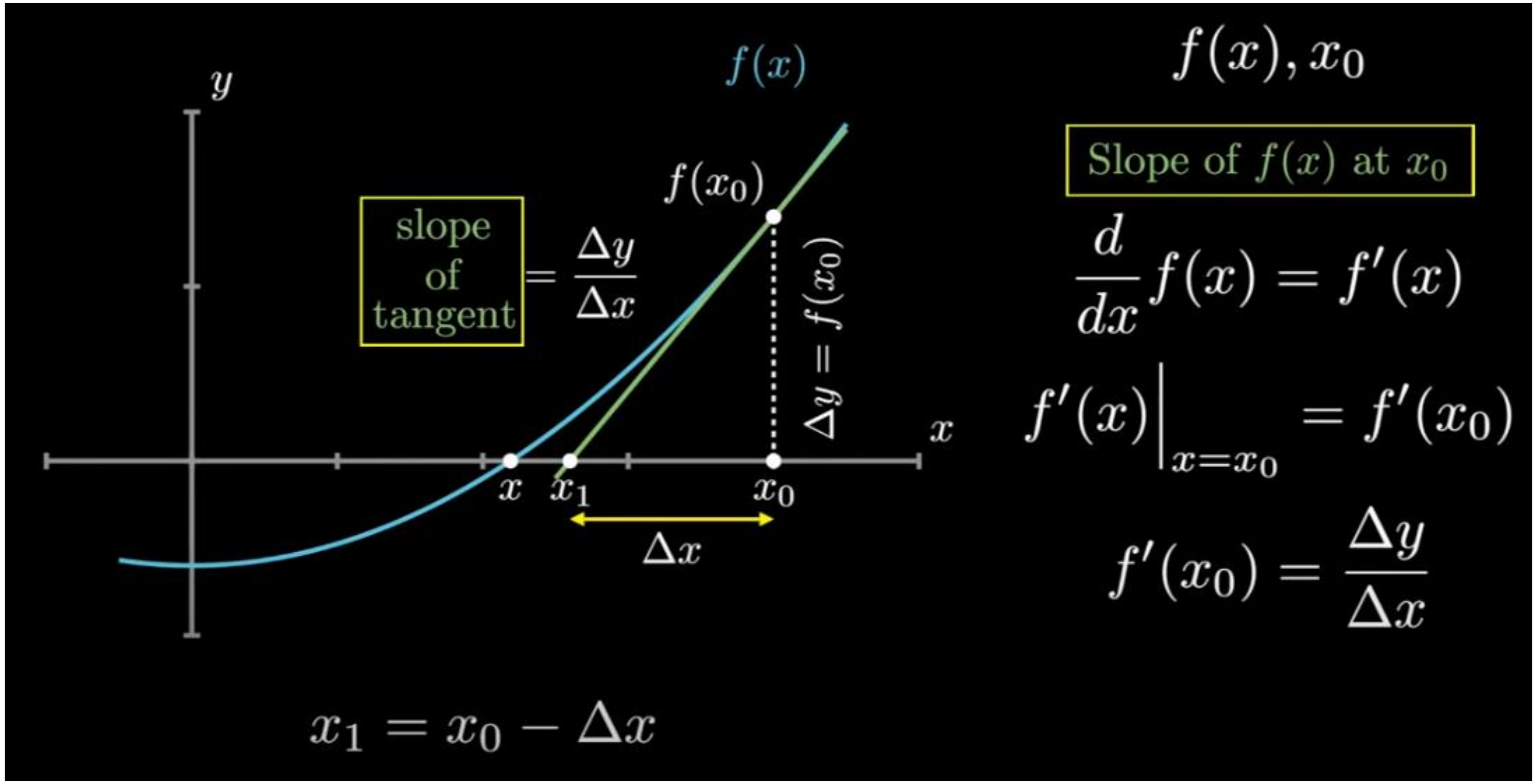

$f(x)$
y
$f(x_0)$
x
x
x_1
x_0
Δx
$x_1 = x_0 - \Delta x$



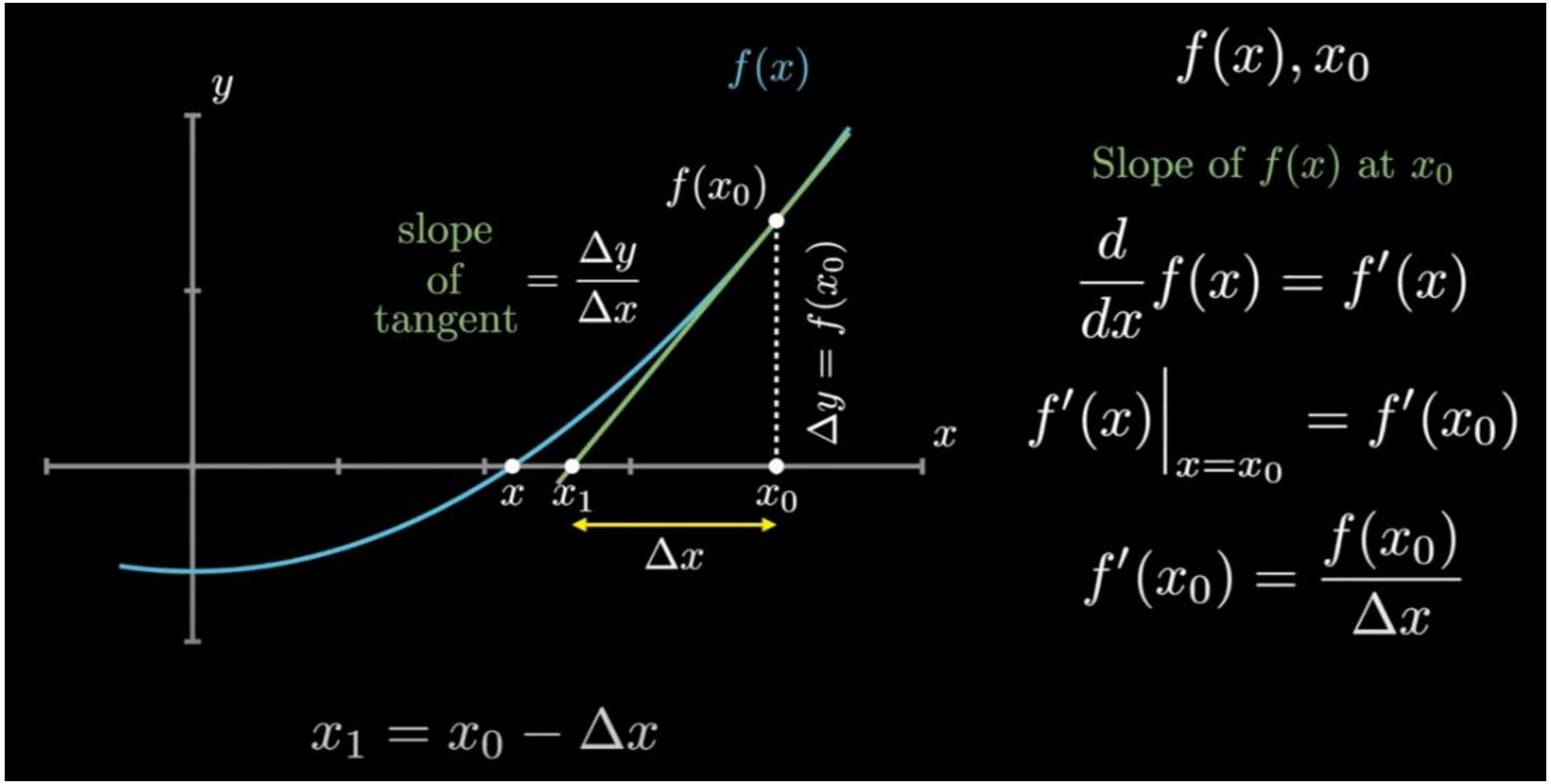
$f(x)$
y
$f(x_0)$
slope of tangent $= \frac{\Delta y}{\Delta x}$
$\Delta y = f(x_0)$
x
x
x_1
x_0
Δx
$x_1 = x_0 - \Delta x$



$f(x), x_0$
Slope of $f(x)$ at x_0
$\frac{d}{dx}f(x) = f'(x)$
$f'(x)\Big|_{x=x_0} = f'(x_0)$
y
$f(x)$
$f(x_0)$
slope of tangent $= \frac{\Delta y}{\Delta x}$
$\Delta y = f(x_0)$
x
x
x_1
x_0
Δx
$x_1 = x_0 - \Delta x$



$f(x), x_0$
Slope of $f(x)$ at x_0
$\frac{d}{dx} f(x) = f'(x)$
$f'(x)\Big|_{x=x_0} = f'(x_0)$
$f'(x_0) = \frac{\Delta y}{\Delta x}$
$f(x)$
y
x
$f(x_0)$
slope of tangent $= \frac{\Delta y}{\Delta x}$
$\Delta y = f(x_0)$
x
x_1
x_0
Δx
$x_1 = x_0 - \Delta x$


$f(x)$
y
x
$f(x_0)$
slope of tangent $= \frac{\Delta y}{\Delta x}$
$\Delta y = f(x_0)$
x
x_1
x_0
Δx
$x_1 = x_0 - \Delta x$
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Slope of $f(x)$ at x_0
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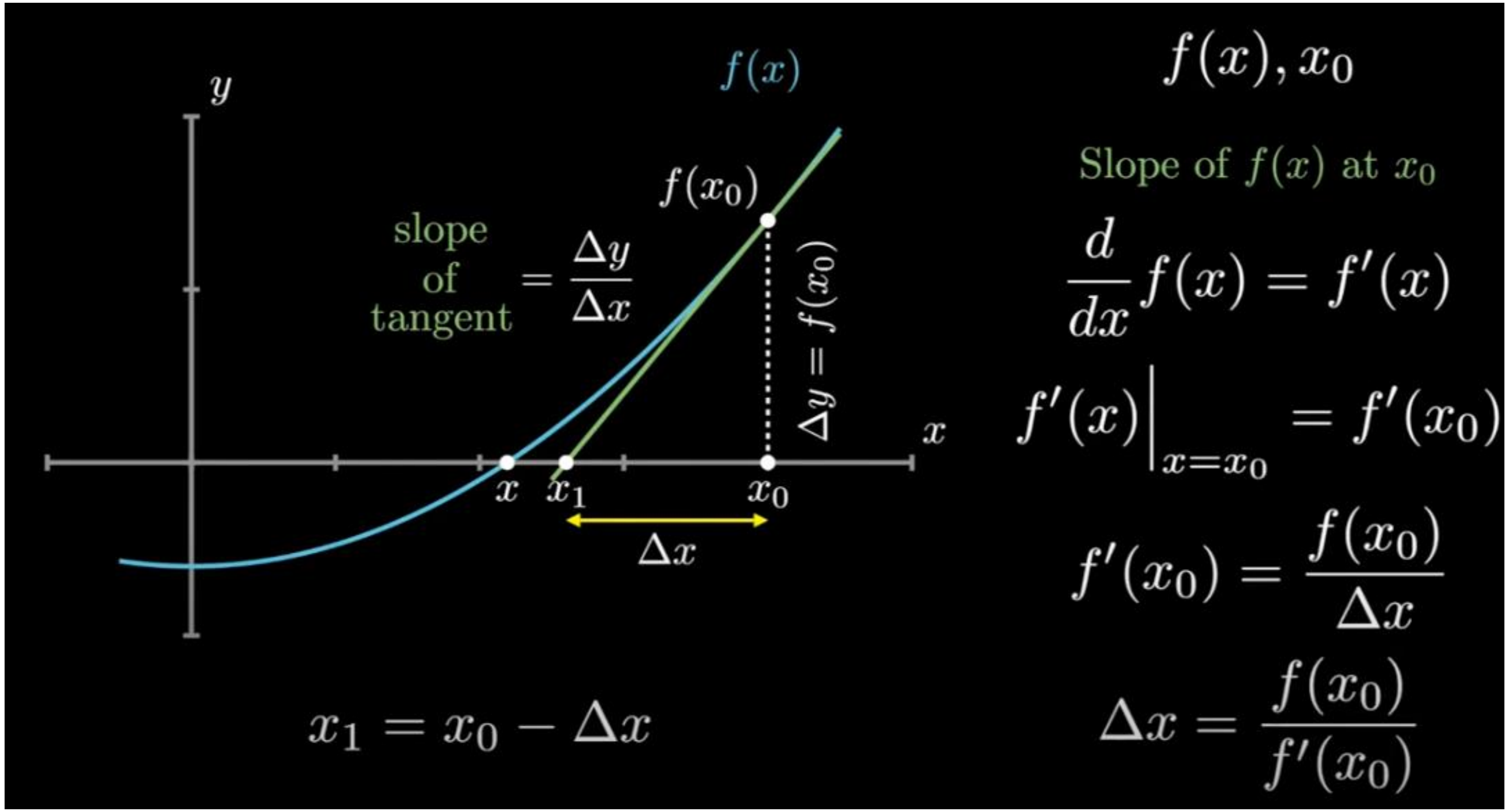
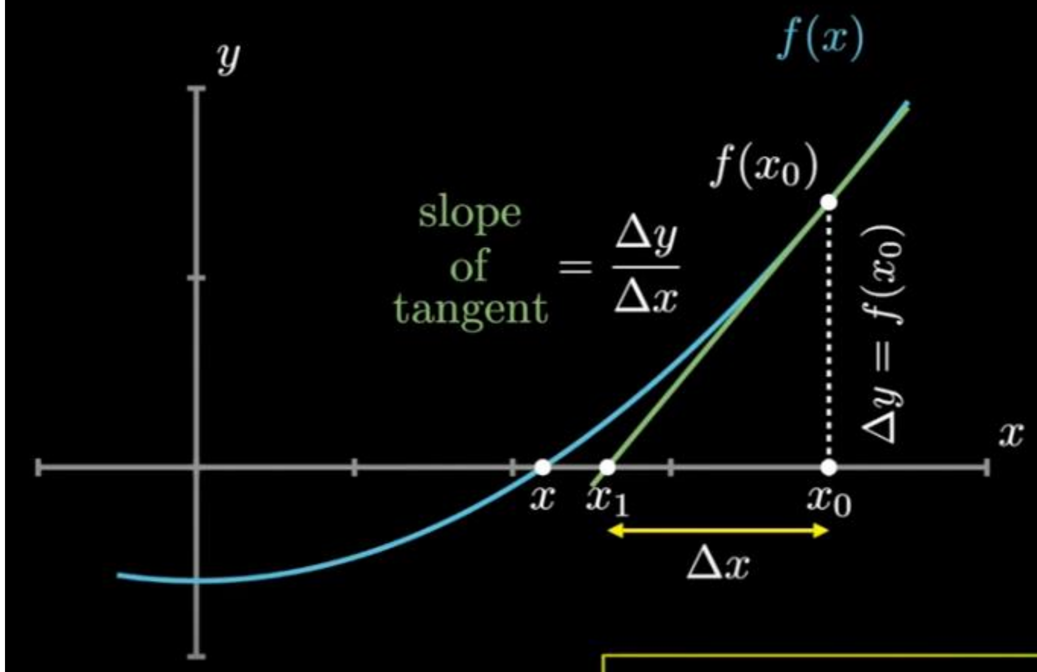
$f'(x_0) = \frac{f(x_0)}{\Delta x}$


$f(x)$
y
x
$f(x_0)$
slope of tangent $= \frac{\Delta y}{\Delta x}$
$\Delta y = f(x_0)$
x
x_1
x_0
Δx
$x_1 = x_0 - \Delta x$
$f(x), x_0$
Slope of $f(x)$ at x_0
$\frac{d}{dx} f(x) = f'(x)$
$f'(x)\Big|_{x=x_0} = f'(x_0)$
$f'(x_0) = \frac{f(x_0)}{\Delta x}$
$\Delta x = \frac{f(x_0)}{f'(x_0)}$



$$f(x), x_0$$

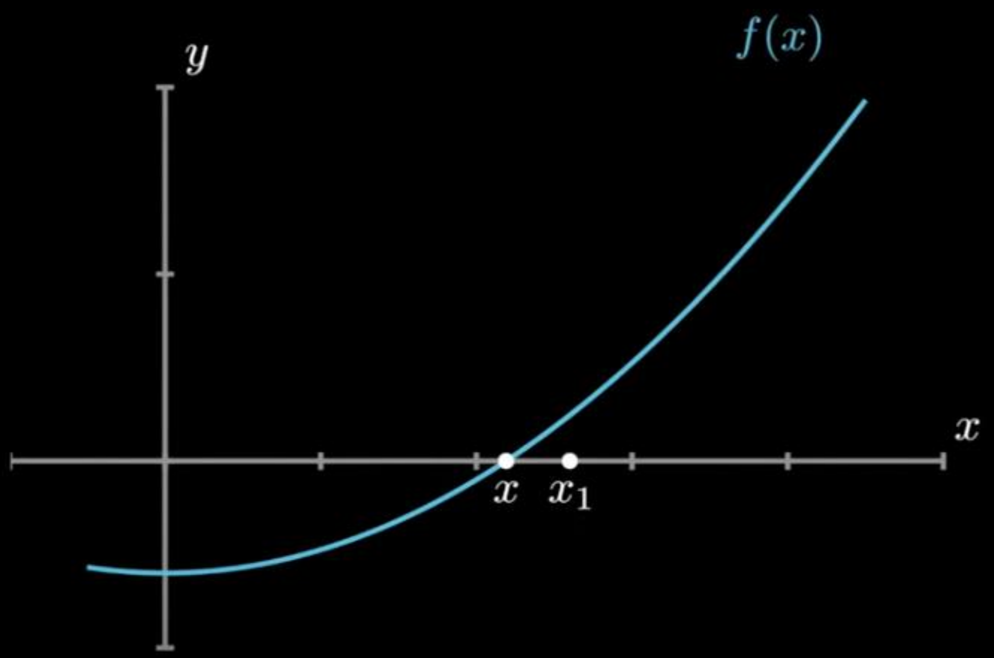
Slope of $f(x)$ at x_0

$$\frac{d}{dx} f(x) = f'(x)$$

$$f'(x)\Big|_{x=x_0} = f'(x_0)$$

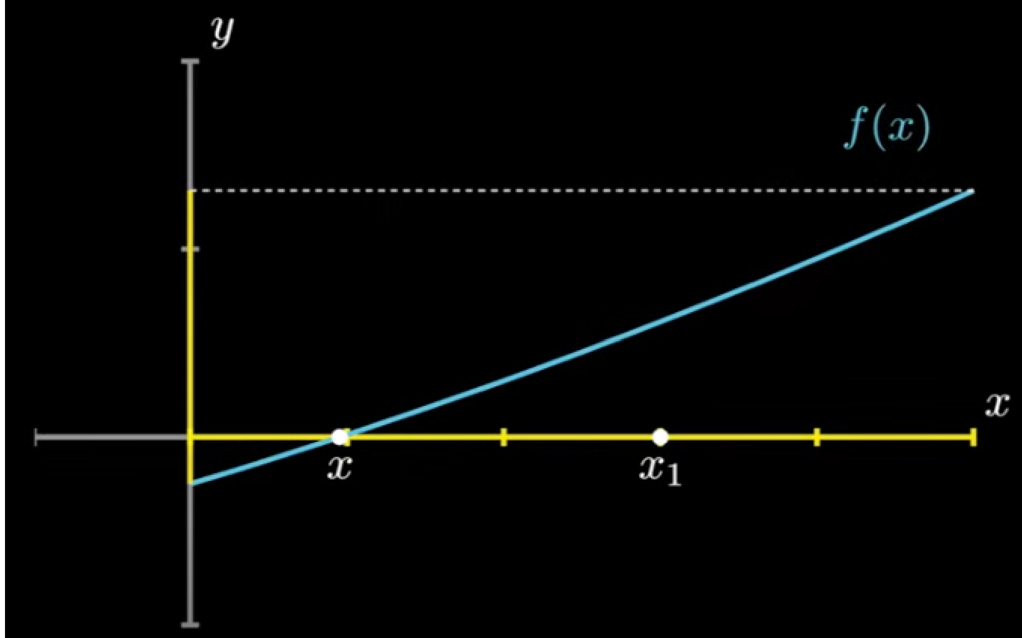
$$f'(x_0) = \frac{f(x_0)}{\Delta x}$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$



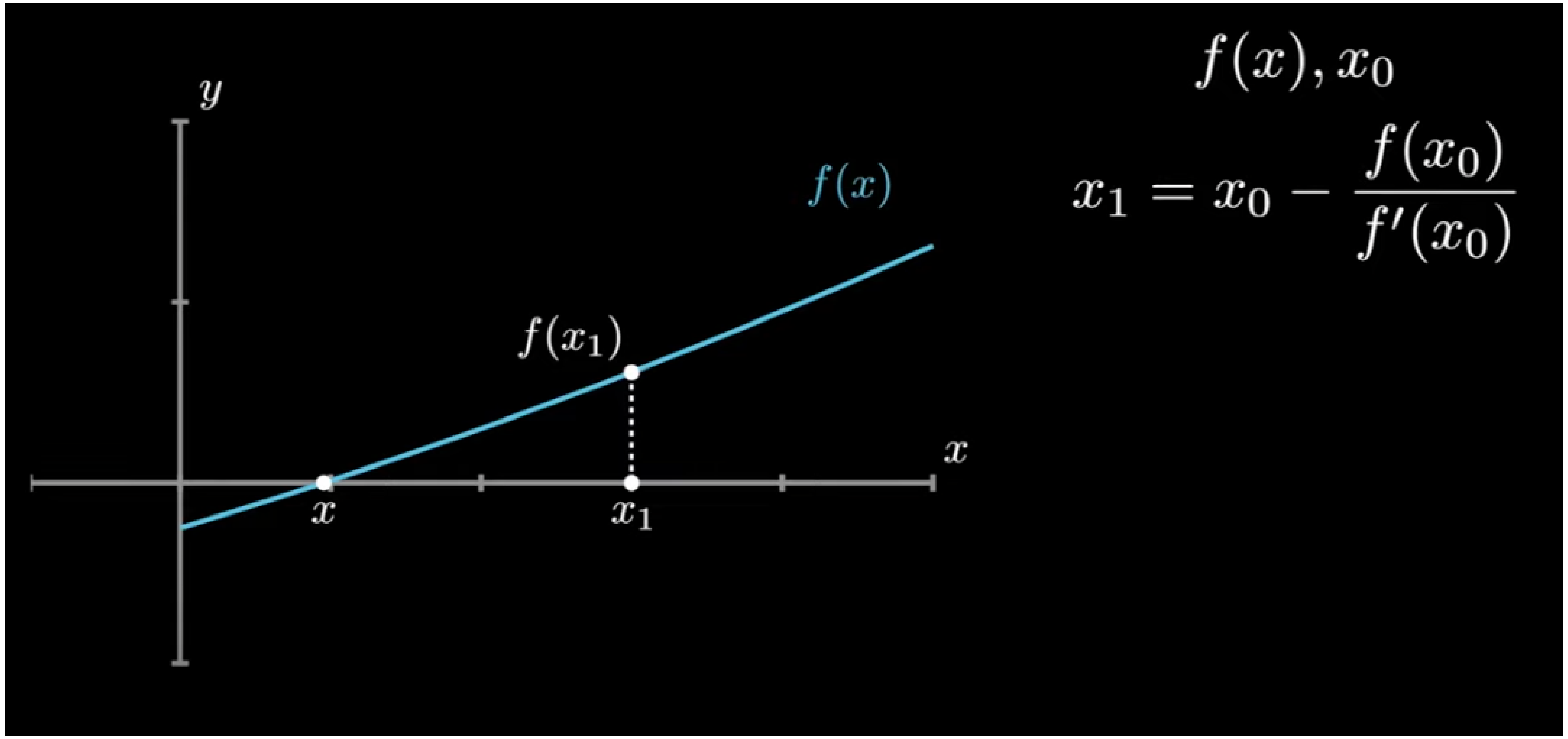
$$f(x), x_0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

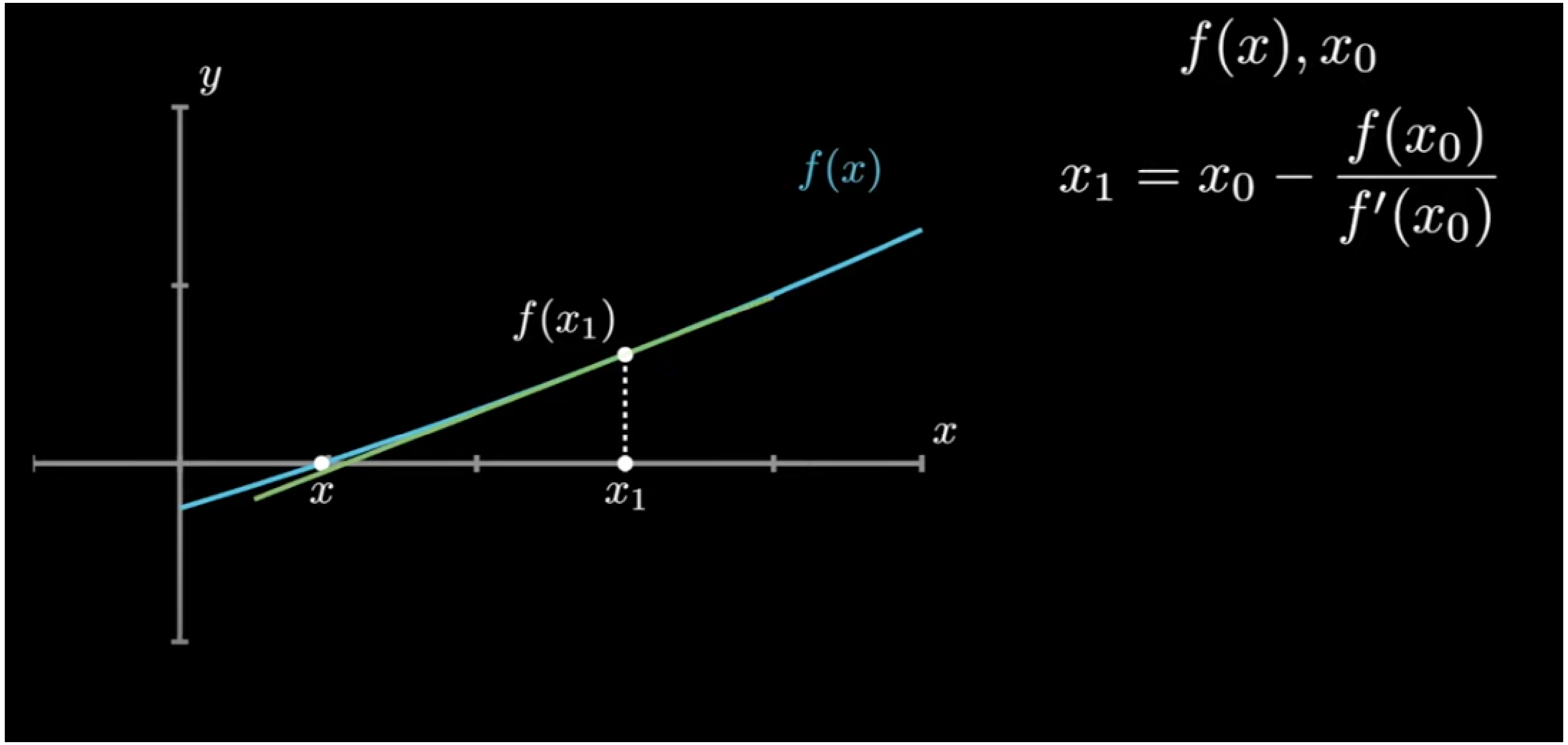


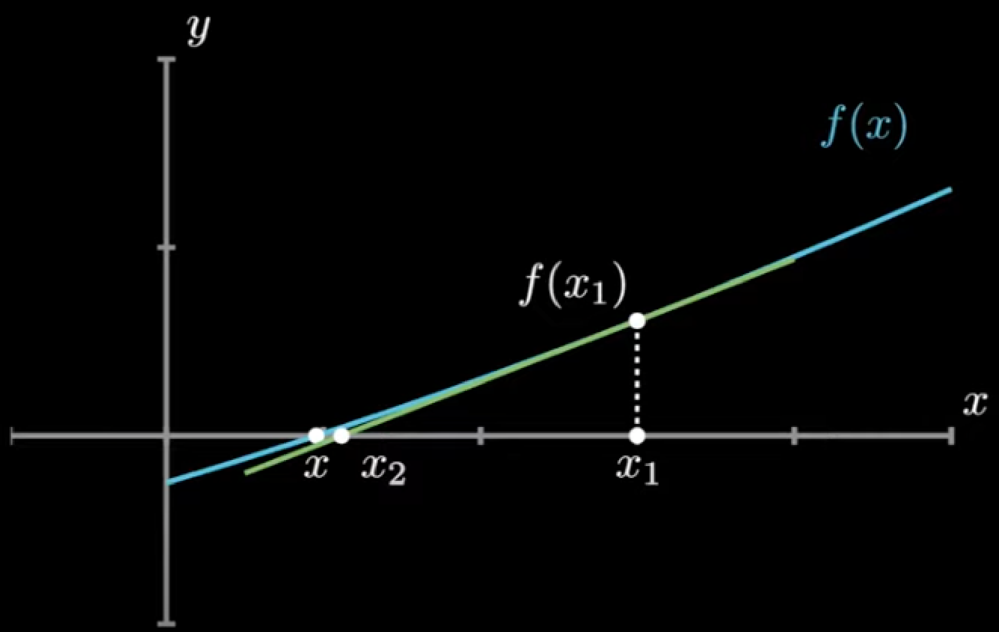
$$f(x), x_0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$



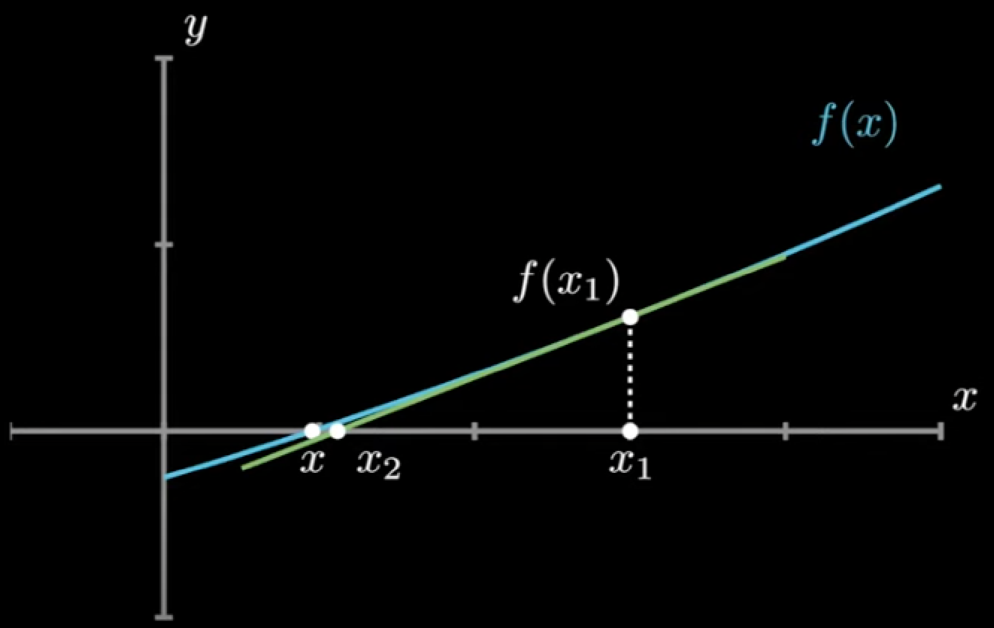
$f(x), x_0$
$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$
y
$f(x)$
$f(x_1)$
x
x
x_1


$f(x), x_0$
$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$
y
$f(x)$
$f(x_1)$
x
x
x_1



$$f(x), x_0$$

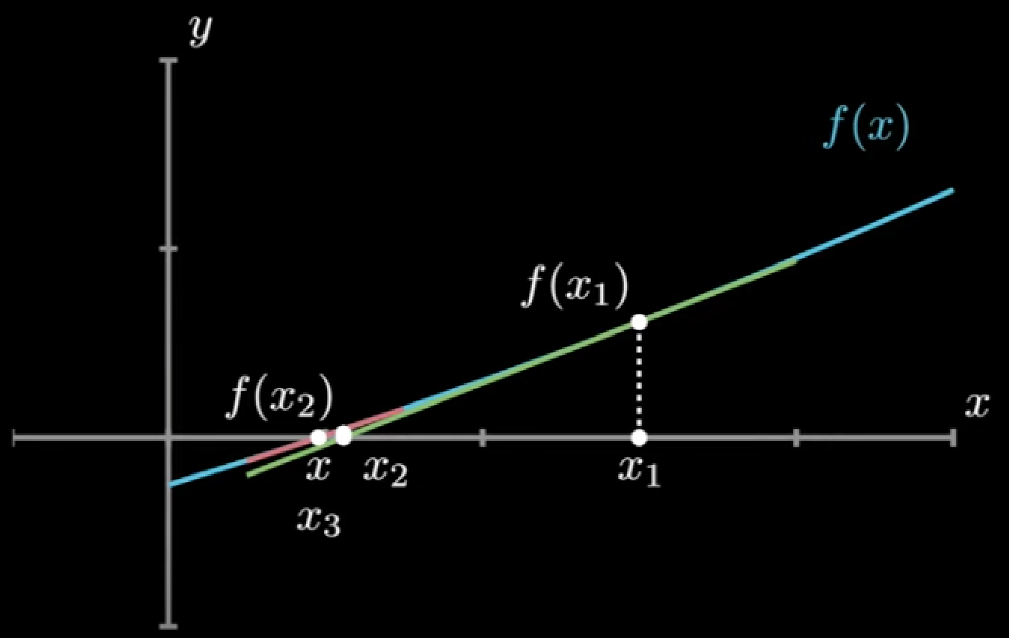
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$



$$f(x), x_0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

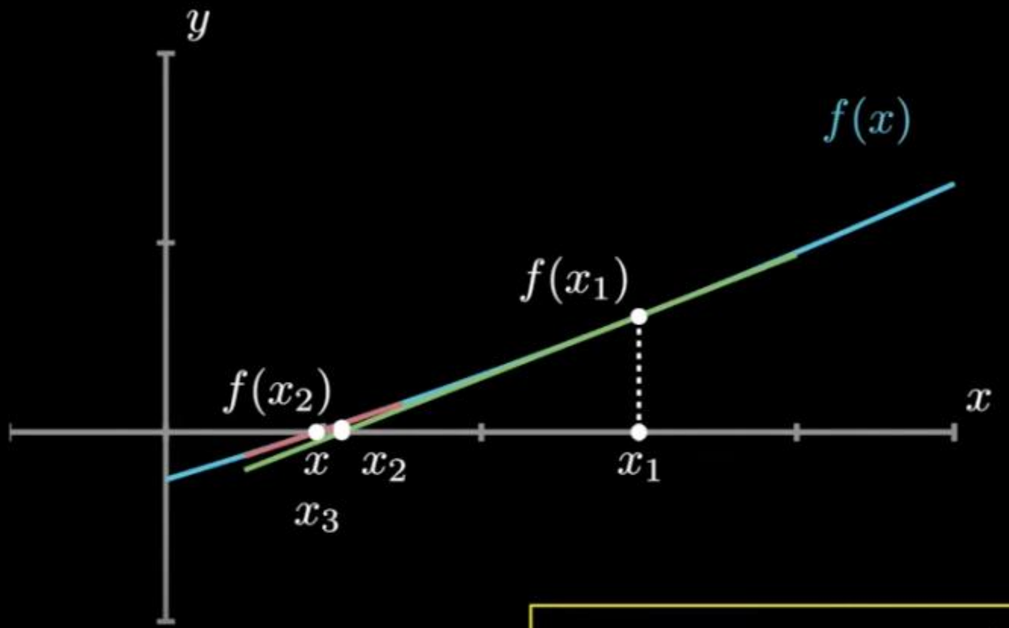


$$f(x), x_0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$



$$f(x), x_0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

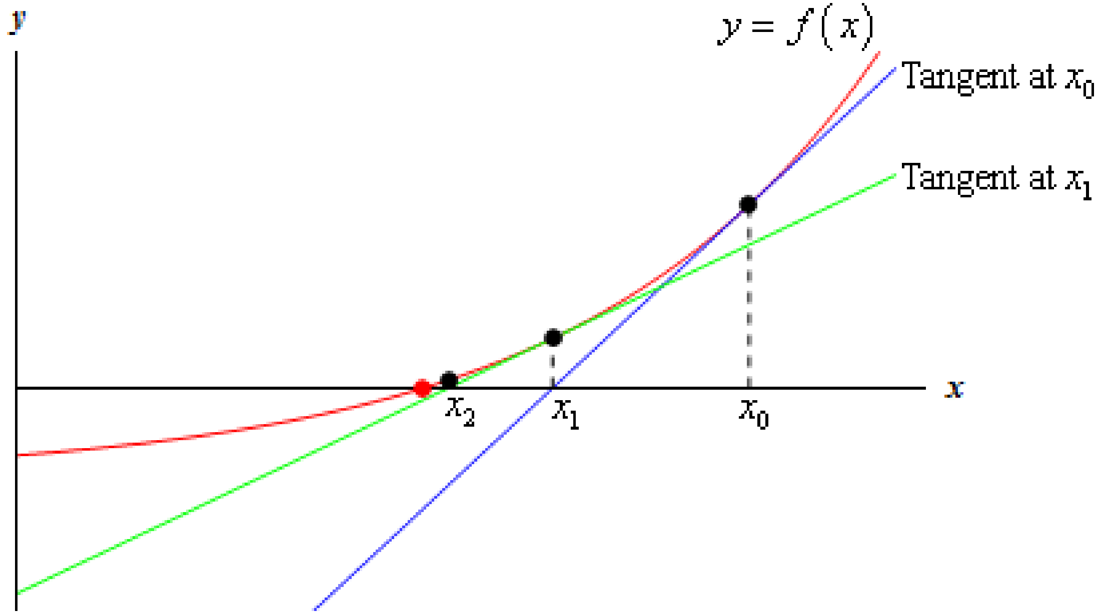
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Newton Method

- Solving $f(x) = 0, \ f: R \longrightarrow R$

- **Solve for x iteratively – approximate solution**

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$



Newton Method

We make use of linear approximation, first-order Taylor approximation, of $f(x)$ and solve for x_+, given a guess - x

$$f(x_+) \approx f(x) + f'(x)(x_+ - x)$$

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$$f(x_+) \approx f(x) + f'(x)(x_+ - x) = 0$$

Newton Method

We make use of linear approximation, first-order Taylor approximation, of $f(x)$ and solve for x_+, given a guess - x

$$f(x_+) \approx f(x) + f'(x)(x_+ - x)$$
$$f(x_+) \approx f(x) + f'(x)(x_+ - x) = 0$$

$$x_+ = x - \underbrace{\frac{f(x)}{f'(x)}}_{\Delta x}$$

Algorithm

- Given an initial solution x_0, iterate until a stopping criteria is fullfilled:

Algorithm

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$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Algorithm

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$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Algorithm

- Given an initial solution x_0, iterate until a stopping criteria is fullfilled:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\vdots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Algorithm

- Given an initial solution x_0, iterate until a stopping criteria is fullfilled:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\vdots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Stopping criteria can be:

$$|f(x_n)| < \varepsilon$$

$$|x_n - x_{n-1}| < \delta$$

Example: $x^2 = 2$

		$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$	x_{n+1}
x_n	x				–
x_1					
x_2					
x_3					
x_4					

True solution

1.4142135623731

Example: $x^2 = 2$

		$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$	x_{n+1}
x_n	x	$f(x) = x^2 - 2$			–
x_1					
x_2					
x_3					
x_4					

True solution

1.4142135623731

Example: $x^2 = 2$

		$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$	x_{n+1}
x_n	x	$f(x) = x^2 - 2$	$f'(x) = 2x$		–
x_1					
x_2					
x_3					
x_4					

True solution
1.4142135623731

Example: $x^2 = 2$

		$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$	x_{n+1}
x_n	x	$f(x) = x^2 - 2$	$f'(x) = 2x$	$x - \frac{x^2 - 2}{2x}$	–
x_1					
x_2					
x_3					
x_4					

True solution

1.4142135623731

Example: $x^2 = 2$

		$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$	x_{n+1}
x_n	x	$f(x) = x^2 - 2$	$f'(x) = 2x$	$x - \frac{x^2 - 2}{2x}$	–
x_1	1				
x_2					
x_3					
x_4					

True solution

1.4142135623731

Example: $x^2 = 2$

		$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$	x_{n+1}
x_n	x	$f(x) = x^2 - 2$	$f'(x) = 2x$	$x - \frac{f(x)}{f'(x)}$	–
x_1	1	-1	2	$1 - \frac{-1}{2} = 3/2$	$\underline{1}.5000000000000$
x_2					
x_3					
x_4					

True solution
1.4142135623731

(The computed value is correct up to the final underlined digit.)

Example: $x^2 = 2$

	$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$	x_{n+1}
x_n	x	$f(x) = x^2 - 2$	$f'(x) = 2x$	$x - \frac{f(x)}{f'(x)}$	–
x_1	1	-1	2	$1 - \frac{-1}{2} = 3/2$	$\underline{1}.5000000000000$
x_2	$\frac{3}{2}$				
x_3					
x_4					

True solution
1.4142135623731

(The computed value is correct up to the final underlined digit.)

Example: $x^2 = 2$

	$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$	x_{n+1}
x_n	x	$f(x) = x^2 - 2$	$f'(x) = 2x$	$x - \frac{f(x)}{f'(x)}$	–
x_1	1	-1	2	$1 - \frac{-1}{2} = 3/2$	$\underline{1}.5000000000000$
x_2	$\frac{3}{2}$	$\frac{1}{4}$	3	$\frac{3}{2} - \frac{1/4}{3} = \frac{17}{2}$	$\underline{1.41}66666666667$
x_3					
x_4					

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(The computed value is correct up to the final underlined digit.)

Example: $x^2 = 2$

		$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$	x_{n+1}
x_n	x	$f(x) = x^2 - 2$	$f'(x) = 2x$	$x - \frac{f(x)}{f'(x)}$	–
x_1	1	-1	2	$1 - \frac{-1}{2} = 3/2$	$\underline{1}.5000000000000$
x_2	$\frac{3}{2}$	$\frac{1}{4}$	3	$\frac{3}{2} - \frac{1/4}{3} = \frac{17}{2}$	$\underline{1.41}66666666667$
x_3	$\frac{17}{12}$				
x_4					

True solution
1.4142135623731

(The computed value is correct up to the final underlined digit.)

Example: $x^2 = 2$

		$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$	x_{n+1}
x_n	x	$f(x) = x^2 - 2$	$f'(x) = 2x$	$x - \frac{f(x)}{f'(x)}$	–
x_1	1	-1	2	$1 - \frac{-1}{2} = 3/2$	$\underline{1}.5000000000000$
x_2	$\frac{3}{2}$	$\frac{1}{4}$	3	$\frac{3}{2} - \frac{1/4}{3} = \frac{17}{2}$	$\underline{1.41}66666666667$
x_3	$\frac{17}{12}$	$\frac{1}{144}$	$\frac{17}{6}$	$\frac{17}{12} - \frac{1/144}{17/6} = \frac{577}{408}$	$\underline{1.41421}56862745$
x_4					

True solution
1.4142135623731

(The computed value is correct up to the final underlined digit.)

Example: $x^2 = 2$

x_n	x	$f(x) = x^2 - 2$	$f'(x) = 2x$	$x - \frac{f(x)}{f'(x)}$	–
x_1	1	-1	2	$1 - \frac{-1}{2} = 3/2$	$\underline{1}.5000000000000$
x_2	$\frac{3}{2}$	$\frac{1}{4}$	3	$\frac{3}{2} - \frac{1/4}{3} = \frac{17}{2}$	$\underline{1.41}66666666667$
x_3	$\frac{17}{12}$	$\frac{1}{144}$	$\frac{17}{6}$	$\frac{17}{12} - \frac{1/144}{17/6} = \frac{577}{408}$	$\underline{1.414215}6862745$
x_4	$\frac{577}{408}$	$\frac{1}{166464}$	$\frac{577}{204}$	$\frac{665857}{470832}$	$\underline{1.41421356237}47$

Column labels above the table: $f(x)$, $f'(x)$, $x - \frac{f(x)}{f'(x)}$, x_{n+1}

True solution
1.4142135623731

(The computed value is correct up to the final underlined digit.)

Example: $x^2 = 2$

		$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$	x_{n+1}
x_n	x	$f(x) = x^2 - 2$	$f'(x) = 2x$	$x - \frac{f(x)}{f'(x)}$	–
x_1	1	-1	2	$1 - \frac{-1}{2} = 3/2$	$\underline{1}.5000000000000$
x_2	$\frac{3}{2}$	$\frac{1}{4}$	3	$\frac{3}{2} - \frac{1/4}{3} = \frac{17}{2}$	$\underline{1.41}66666666667$
x_3	$\frac{17}{12}$	$\frac{1}{144}$	$\frac{17}{6}$	$\frac{17}{12} - \frac{1/144}{17/6} = \frac{577}{408}$	$\underline{1.41421}56862745$
x_4	$\frac{577}{408}$	$\frac{1}{166464}$	$\frac{577}{204}$	$\frac{665857}{470832}$	$\underline{1.41421356237}47$

True solution
1.4142135623731

roughly doubling the number of decimal points in each round

(The computed value is correct up to the final underlined digit.)