PSS 3 - Lagrange Modelling

Exercises: 3.1, 3.2, 3.3

Summary

Procedure for lagrange Modelling

(i) Identify generalized coordinates q (usually positions and angles)

(ii) Describe position $p(q)$ and velocity $\dot{p}(q)$ of the mass (usually in a cartesian reference frame (x,y,z)).

$$\dot{p}(q) = \frac{\partial p}{\partial q}\dot{q} \text{ (chain rule)} \quad \text{- if } p = q \text{ then } \frac{\partial p}{\partial q} = 1 \Rightarrow \dot{p} = \dot{q}$$

⚠ q and $\dot{q}$ are independent. $\Rightarrow \frac{\partial \dot{q}}{\partial q} = 0$

(iii) Potential $(U(q))$ and kinetic $(T(q,\dot{q}))$ energy.

(iv) Form Lagrange function:

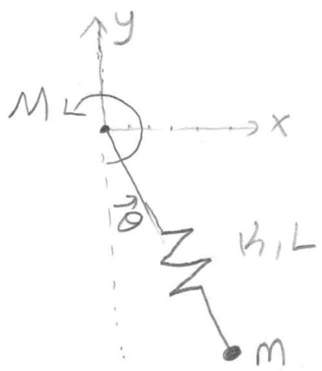
$$L(q,\dot{q}) = T(q,\dot{q}) - U(q)$$

(v) Write Euler-Lagrange equation (Dynamics)

$$\frac{d}{dt}\nabla_{\dot{q}}L - \nabla_q L = Q$$

↳ Generalized forces (external forces applied along generalized coordinates.)

3.1 2



Derive equations of motion using

a) Coordinate θ and L

b) Coordinate x and y

(external force is torque M applied at joint)

a)

(i) $q = \begin{bmatrix} \theta \\ L \end{bmatrix} \Rightarrow p_m = \begin{bmatrix} L\sin\theta \\ -L\cos\theta \end{bmatrix}$

(ii) $\dot{p} = \begin{bmatrix} L\cos\theta & \sin\theta \\ L\sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{L} \end{bmatrix} = \begin{bmatrix} L\dot{\theta}\cos\theta + \dot{L}\sin\theta \\ L\dot{\theta}\sin\theta - \dot{L}\cos\theta \end{bmatrix}$

(iii) Potential Energy:

assuming spring equilibrium is 0

$$V(q) = E_{gravity}(q) + E_{spring}(q) = -mgL\cos\theta + \frac{1}{2}kL^2$$

$$T(q,\dot{q}) = \frac{1}{2}m\dot{p}_m^T\dot{p}_m = \frac{1}{2}m\left[(L\dot{\theta}\cos\theta + \dot{L}\sin\theta)^2 + (L\dot{\theta}\sin\theta - \dot{L}\cos\theta)^2\right]$$

$$= \frac{1}{2}m(\dot{L}^2 + L^2\dot{\theta}^2)$$

(iv) $\mathcal{L} = T - U = \frac{1}{2}m(\dot{L}^2 + L^2\dot{\theta}^2) - \left(\frac{1}{2}kL^2 - mgL\cos\theta\right)$

(v) Euler-Lagrange:

$$Q = \begin{bmatrix} M \\ 0 \end{bmatrix}$$

$$\frac{d}{dt}\nabla_{\dot{q}}\mathcal{L} - \nabla_q\mathcal{L} = Q$$

$$\frac{d}{dt}\nabla_{\dot{q}}\mathcal{L} = \frac{d}{dt}\begin{bmatrix} mL^2\dot{\theta} \\ m\dot{L} \end{bmatrix} = \begin{bmatrix} 2mL\dot{L}\dot{\theta} + mL^2\ddot{\theta} \\ m\ddot{L} \end{bmatrix}$$

$$\nabla_q\mathcal{L} = \begin{bmatrix} -mgL\sin\theta \\ mL\dot{\theta}^2 - kL + mg\cos\theta \end{bmatrix}$$

The equations of motion are:

$$\begin{cases} mL^2\ddot{\theta} + 2mL\dot{L}\dot{\theta} + mgL\sin\theta = M \\ m\ddot{L} + kL - mL\dot{\theta}^2 - mg\cos\theta = 0 \end{cases}$$

3.1 b

b)

(i) generalized coordinates $q = \begin{bmatrix} x \\ y \end{bmatrix}$

(ii) mass position $p_m = q$, velocity $\dot{p} = \dot{q}$

(iii) Energies $V = mgy + \frac{1}{2}kL^2 = \frac{1}{2}k(x^2+y^2)$ (L is not part of generalized coordinates)

$$T = \frac{1}{2} m \dot{p}_m^T \dot{p}_m = \frac{1}{2} m(\dot{x}^2+\dot{y}^2)$$

(iv) Lagrange: $\mathcal{L} = T - V = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy - \frac{1}{2}k(x^2+y^2)$

(V) E.L equations:

$$\frac{d}{dt}\nabla_{\dot{q}}\mathcal{L} - \nabla_q \mathcal{L} = Q$$

$$\nabla_{\dot{q}}\mathcal{L} = \begin{bmatrix} m\dot{x} \\ m\dot{y} \end{bmatrix} \qquad \frac{d}{dt}\nabla_{\dot{q}}\mathcal{L} = \begin{bmatrix} m\ddot{x} \\ m\ddot{y} \end{bmatrix}$$

$$\nabla_q L = \begin{bmatrix} -kx \\ -ky - mg \end{bmatrix} \Rightarrow \begin{matrix} m\ddot{x} + kx = Q_x \\ m\ddot{y} + ky + mg = Q_y \end{matrix}$$

→ what is Q?

(3.118) $\delta W = \tau \delta \theta$ → M

(3.116) $\delta W = F^T \delta q$ → Q

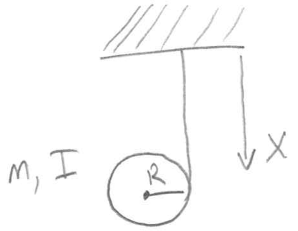
$$Q^T \delta q = M \delta\theta = M \frac{d\theta}{dq}\delta q \Rightarrow Q = M \nabla_q \theta$$

$$\theta(q) = \arctan\left(\frac{x}{-y}\right) \qquad \left(\arctan'(x) = \frac{1}{1+x^2}\right)$$

$$\nabla_q \theta = \begin{bmatrix} \left(-\frac{1}{y}\right)\frac{1}{1+\frac{x^2}{y^2}} \\ \left(\frac{x}{y^2}\right)\frac{1}{1+\frac{x^2}{y^2}} \end{bmatrix} = \begin{bmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{bmatrix}$$

$$Q_x = \frac{-yM}{x^2+y^2} \qquad Q = \frac{xM}{x^2+y^2}$$

3.2



M: mass
I: inertia

- Write model using lagrange

(i) Generalized coordinates $q: X$

(ii) $p_M = q$, $\dot{p}_M = \dot{q}$

(iii) $V = -mgX$, $T = \frac{1}{2} m \dot{p}^T \dot{p} + \frac{1}{2} I \omega^T \omega$ → angular velocity

$T = \frac{1}{2} m \dot{x}^2 + \underbrace{\frac{1}{2} I \left(\frac{\dot{x}}{R}\right)^2}_{\text{inertia of cylinder}}$ $\underbrace{(\dot{x} = R\omega)}_{\text{Reminder}}$

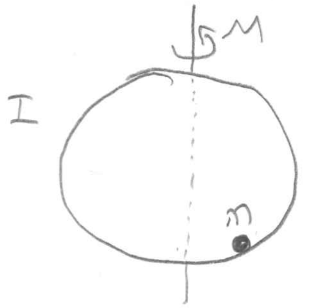
(iv) $\mathcal{L} = T - V = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I \frac{\dot{x}^2}{R^2} + mgX$

(v) E.L. $\frac{d}{dt} \nabla_{\dot{q}} \mathcal{L} - \nabla_q L = 0$

$\nabla_{\dot{q}} \mathcal{L} = m\dot{x} + \frac{I\dot{x}}{R^2}$ $\frac{d}{dt} \nabla_{\dot{q}} \mathcal{L} = m\ddot{x} + \frac{I}{R^2} \ddot{x}$, $\nabla_q \mathcal{L} = mg$

$\left(m + \frac{I}{R^2}\right)\ddot{x} - mg = 0$ } equation of motion

3.3



I: inertia
M: torque
m: mass

- Determine lagrange's equations in minimum coordinates (two angles)

R: radius

Coordinat axises

Side view

Top View

(i) $q = (\varphi, \theta)$

(ii) p_M ?

- φ:

$$z = -R\cos\varphi, \quad (xy) = \overbrace{+R\sin\varphi}^{\text{length on xy-plane}}$$

- θ: $x = R\sin\varphi\cos\theta, \quad y = R\sin\varphi\sin\theta$

$$p_M = \begin{bmatrix} R\sin\varphi\cos\theta \\ R\sin\varphi\sin\theta \\ -R\cos\varphi \end{bmatrix}, \quad \dot{p}_M = \begin{bmatrix} R\dot{\varphi}\cos\varphi\cos\theta - R\dot{\theta}\sin\varphi\sin\theta \\ R\dot{\varphi}\cos\varphi\sin\theta + R\dot{\theta}\sin\varphi\cos\theta \\ R\dot{\varphi}\sin\varphi \end{bmatrix}$$

(iii) $V = -mgR\cos\varphi$, $T = \frac{1}{2}m\dot{p}_M^T\dot{p}_M + \frac{1}{2}I\dot{\theta}^2 = \underbrace{\frac{1}{2}m(R^2\dot{\varphi}^2 + R^2\dot{\theta}^2\sin^2\varphi) + \frac{1}{2}I\dot{\theta}^2}_{\text{lots of } \sin^2+\cos^2=1 \text{ cancellation}}$

(iv) $L = \frac{1}{2}m(R^2\dot{\varphi}^2 + R^2\dot{\theta}^2\sin^2\varphi) + \frac{1}{2}I\dot{\theta}^2 + mgR\cos\varphi$

(v) $\frac{d}{dt}\nabla_{\dot{q}}L - \nabla_q L = \begin{bmatrix} 0 \\ M \end{bmatrix}$ Torque around z axis

$$\frac{d}{dt}\nabla_{\dot{q}}L = \frac{d}{dt}\begin{bmatrix} mR^2\dot{\varphi} \\ mR^2\dot{\theta}\sin^2\varphi + I\dot{\theta} \end{bmatrix} = \begin{bmatrix} mR^2\ddot{\varphi} \\ mR^2\ddot{\theta}\sin^2\varphi + 2mR^2\dot{\theta}\dot{\varphi}\sin\varphi\cos\varphi + I\ddot{\theta} \end{bmatrix}$$

$$\nabla_q L = \begin{bmatrix} mR^2\dot{\theta}^2\sin\varphi\cos\varphi - mgR\sin\varphi \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} mR^2\ddot{\varphi} - mR^2\dot{\theta}^2\sin\varphi\cos\varphi + mgR\sin\varphi &= 0 \\ mR^2\ddot{\theta}\sin^2\varphi + 2mR^2\dot{\theta}\dot{\varphi}\sin\varphi\cos\varphi + I\ddot{\theta} &= M \end{aligned}\right\}$$

Equations Motion (E-L)