Cheatsheet

Derivative of a vector w.r.t a scalar:

If $x(t) \in R^n$ is a vector function of a scalar variable t:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Its derivative is the component wise derivative:

$$\frac{d}{dt} x(t) = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

A scalar $f(x)$ differentiated w.r.t a vector $x \in R^n$:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \in R^{1 \times n} \quad \text{(row vector)}$$

Gradient $\nabla_x f \in R^{n \times 1}$ (column vector)

$$\nabla_x f = \left(\frac{\partial f}{\partial x} \right)^T$$ $\nabla_x f$: gradient of f w.r.t x

$$\nabla_x f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \in R^{n \times 1}$$

Jacobian Matrix

A vector valued function

$$f(x_1, x_2, \dots, x_n) = \begin{bmatrix} f_1(x_1, x_2, \dots x_n) \\ f_2(x_1, x_2, \dots x_n) \\ \vdots \\ f_m(x_1, x_2 \dots, x_n) \end{bmatrix}$$

$f(x) \in R^m$, $x \in R^n$

$f: R^n \to R^m$: f maps n-dimensional input to a m-dimensional output.

The Jacobian of f is the mxn matrix of first order partial derivatives:

$$J(f) = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

(Jacobian matrix)

Chain rule:

$$z(t) = f(x(t), y(t))$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Quadratic form:

$\alpha = x^T A x$, where A is a symmetrix matrix ($A = A^T$) and x is $n \times 1$, A is $n \times n$, A does not depend on x, then

$$\frac{\partial \alpha}{\partial x} = 2x^T A$$