ESS101
Modelling and Simulation, 2025

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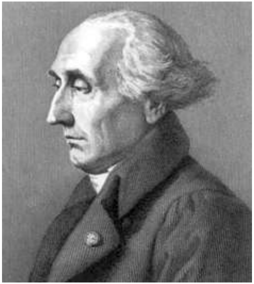
Lecture 4 - Lagrange modelling

- Generalized coordinates
- Kinetic and potential energy
- Lagrange function
- Euler-Lagrange's equation
- External Forces
- Constrained Lagrange mechanics
- Consistency condition



Euler-Lagrange's equation – summary

Kinetic, potential energies and the Lagrangian, expressed in generalized coordinates $\mathbf{q}$:

$$T = T(\mathbf{q}, \dot{\mathbf{q}}), \quad V = V(\mathbf{q}), \quad \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})$$

The Euler-Lagrange equation:

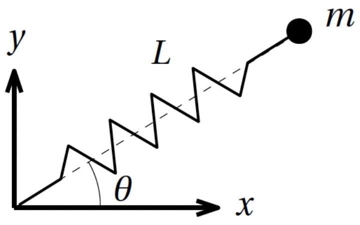
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \qquad \text{or} \qquad \frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\mathcal{L} - \nabla_{\mathbf{q}}\mathcal{L} = 0,$$

$$\nabla_{\mathbf{q}}\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}}\right)^T, \quad \nabla_{\dot{\mathbf{q}}}\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right)^T, \quad T = \frac{1}{2}\dot{\mathbf{q}}^T W(\mathbf{q})\dot{\mathbf{q}},$$

the Euler-Lagrange equation reads

$$W(\mathbf{q})\ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}\left(W(\mathbf{q})\dot{\mathbf{q}}\right)\dot{\mathbf{q}} - \nabla_{\mathbf{q}} T + \nabla_{\mathbf{q}} V = 0$$

Example:



- Resulting E-L equations using the $[q = \theta, L]$:

$$0 = \begin{bmatrix} mL^2 & 0 \\ 0 & m \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} 2mL\dot{L}\dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} mgLcos\theta \\ mgsin\theta + K(L - L_0) - mL\dot{\theta}^2 \end{bmatrix}$$

- E-L using $[q = x, y]$:

$$0 = m\ddot{\mathbf{q}} + mg \begin{bmatrix} 0 \\ 1 \end{bmatrix} + K\left(1 - \frac{L_0}{\|q\|}\right) q$$

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- Even for a simple system, modelling can be complicated, you can use symbolic computations in Matlab (e.g. differentiation)

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Some important points:

- Even for a simple system, modelling can be complicated, you can use symbolic computations in Matlab (e.g. differentiation)
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Some important points:

- Even for a simple system, modelling can be complicated, you can use symbolic computations in Matlab (e.g. differentiation)
- Modelling, setting up the basic expressions, kinetic, potential energy, E-L equation, is very straightforward.
- Complexity of the equations changes based on how you choose the generalized coordinates.
- We can include other forces (not just gravity) that externally affect the system.
- Once the generalized forces $\mathbf{Q}$ are known, they can be readily included in the Lagrange formalism using:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^T \qquad \text{or} \qquad \frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\mathcal{L} - \nabla_{\mathbf{q}}\mathcal{L} = \mathbf{Q},$$

Generalized Forces

- A force given by $\mathbf{F} \in \mathbb{R}^n$ in a given fixed reference frame $\mathcal{R}$ is applied at a specific point of the system, having a position $\mathbf{p} \in \mathbb{R}^n$ in the same reference frame $\mathcal{R}$.
- The generalized force corresponding to $\mathbf{F}$ is given by:

$$\mathbf{Q} = \frac{\partial \mathbf{p}}{\partial \mathbf{q}}^{\top} \mathbf{F} = \nabla_{\mathbf{q}} \mathbf{p} \mathbf{F} \tag{1}$$

- Extending to several forces: When a set of forces $\mathbf{F}_{1,\ldots,m}$ is applied to a set of points $\mathbf{p}_{1,\ldots,m}$, then the generalized force is given by:

$$\mathbf{Q} = \sum_{i=1}^{m} \frac{\partial \mathbf{p}_i}{\partial \mathbf{q}}^{\top} \mathbf{F}_i \equiv \sum_{i=1}^{m} \nabla_{\mathbf{q}} \mathbf{p}_i \mathbf{F}_i \tag{2}$$

Generalized Forces - Example: Double pendulum

$$\mathbf{p}_1 = \begin{bmatrix} x + Lsin\theta_1 \\ -Lcos\theta_1 \end{bmatrix} \mathbf{p}_2 = \begin{bmatrix} x + Lsin\theta_1 + Lsin\theta_2 \\ -Lcos\theta_1 - Lcos\theta_2 \end{bmatrix} \mathbf{q} = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\frac{\partial \mathbf{p}_1}{\partial \mathbf{q}}^\top = \begin{bmatrix} 1 & 0 \\ L\cos\theta_1 & L\sin\theta_1 \\ 0 & 0 \end{bmatrix}, \qquad \frac{\partial \mathbf{p}_2}{\partial \mathbf{q}}^\top = \begin{bmatrix} 1 & 0 \\ L\cos\theta_1 & L\sin\theta_1 \\ L\cos\theta_2 & L\sin\theta_2 \end{bmatrix} \tag{4}$$

If forces $\mathbf{F}_{1,2} \in \mathbb{R}^2$ are applied to masses m_1, m_2 respectively, then the generalized force then reads as:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ L\cos\theta_1 & L\sin\theta_1 \\ 0 & 0 \end{bmatrix} \mathbf{F}_1 + \begin{bmatrix} 1 & 0 \\ L\cos\theta_1 & L\sin\theta_1 \\ L\cos\theta_2 & L\sin\theta_2 \end{bmatrix} \mathbf{F}_2 \tag{5}$$

Constrained Lagrange Mechanics

- So far we allowed the generalized coordinates q to move independently from each other.
  - i.e. any element of the set $\mathbb{R}^{n_\mathbf{q}}$ is admissible for the vector of generalized coordinates $\mathbf{q}$.
- In real systems, we usually have constraints, e.g. a limit on length or positions.

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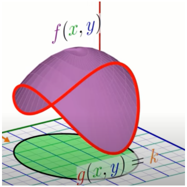
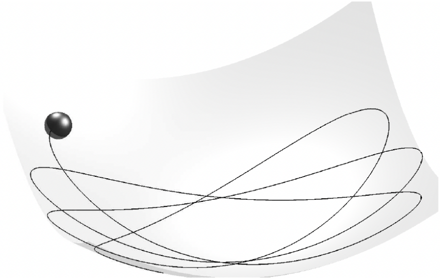


Figur: Example: A function f's boundaries given another constraining function g.

Constrained Lagrange Mechanics

- An example where the generalized coordinates are not free to move independently: Consider a bowl in 3D, described by the scalar equation $\mathbf{c}(\mathbf{p}) = 0$, where $\mathbf{p} \in \mathbb{R}^3$ are cartesian coordinates and

$$\mathbf{c}(\mathbf{p}) = \mathbf{p}_3 - \frac{1}{4}\left(\mathbf{p}_2^2 + \mathbf{p}_1^2\right) \in \mathbb{R} \tag{6}$$



- A mass m moving on the surface of the bowl will be forced to slide on the surface - its position is not free to move anywhere.
- constrained to move in the space described by $c(p)$, only positions that satisfy $c(p) = 0$ are allowed.

Constrained Lagrange Mechanics

- For the constraint function $\mathbf{c}(\mathbf{q}) = 0$,

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}) - \mathbf{z}^\top \mathbf{c}(\mathbf{q}) \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^\top \tag{8}$$

$$\mathbf{c}(\mathbf{q}) = 0 \tag{9}$$

- The term $\nabla_{\mathbf{q}} \mathbf{c} \cdot \mathbf{z}$ is similar to a force keeping the system on $\mathbf{c}$.

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$$\frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\mathcal{L} - \nabla_{\mathbf{q}}\mathcal{L} = \mathbf{Q}$$
$$\mathbf{c}(\mathbf{q}) = 0$$

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Constrained Lagrange Mechanics

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$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^\top \tag{12}$$

$$\mathbf{c}\left(\mathbf{q}\right) = 0 \tag{13}$$

- The term $\nabla_{\mathbf{q}} \mathbf{c} \cdot \mathbf{z}$ is similar to a force keeping the system on $\mathbf{c}$ - i.e. $\nabla_{\mathbf{q}} \mathbf{c} \cdot \mathbf{z}$ enforces the constraints c.

Constrained Lagrange Mechanics

- For the constraint function $\mathbf{c}(\mathbf{q}) = 0$,

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}) - \mathbf{z}^\top \mathbf{c}(\mathbf{q}) \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^\top \tag{15}$$

$$\mathbf{c}(\mathbf{q}) = 0 \tag{16}$$

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- $\mathbf{z}$ ought to be chosen such that the accelerations $\ddot{\mathbf{q}}$ enforce the constraints $\mathbf{c}(\mathbf{q}) = 0$.

Constrained Lagrange Mechanics

- For the constraint function $\mathbf{c}(\mathbf{q}) = 0$,

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}) - \mathbf{z}^\top \mathbf{c}(\mathbf{q}) \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^\top \tag{18}$$

$$\mathbf{c}\left(\mathbf{q}\right) = 0 \tag{19}$$

- The term $\nabla_{\mathbf{q}}\mathbf{c} \cdot \mathbf{z}$ is similar to a force keeping the system on $\mathbf{c}$ - i.e. $\nabla_{\mathbf{q}}\mathbf{c} \cdot \mathbf{z}$ enforces the constraints c.
- $\mathbf{z}$ ought to be chosen such that the accelerations $\ddot{\mathbf{q}}$ enforce the constraints $\mathbf{c}\left(\mathbf{q}\right) = 0$.
- Running simulations requires to compute the system accelerations $\ddot{\mathbf{q}}$ as a function of the positions and velocities $\mathbf{q}$, $\dot{\mathbf{q}}$ and of the external forces $\mathbf{Q}$. In the unconstrained case, this is feasible as long as matrix $W\left(\mathbf{q}\right)$ is full rank.

Constrained Lagrange Mechanics

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$$\nabla_{\mathbf{q}}\mathcal{L} = \nabla_{\mathbf{q}} T - \nabla_{\mathbf{q}} V - \boldsymbol{\nabla}_{\mathbf{q}}\mathbf{c}\cdot\mathbf{z} \tag{20}$$

$$W(\mathbf{q})\,\ddot{\mathbf{q}} + \frac{\partial}{\partial\mathbf{q}}\left(W(\mathbf{q})\,\dot{\mathbf{q}}\right)\dot{\mathbf{q}} - \nabla_{\mathbf{q}} T + \nabla_{\mathbf{q}} V + \boldsymbol{\nabla}_{\mathbf{q}}\mathbf{c}\cdot\mathbf{z} = \mathbf{Q} \tag{21a}$$

$$\mathbf{c}(\mathbf{q}) = 0 \tag{21b}$$

- And the accelerations $\ddot{\mathbf{q}}$ can be explicitly expressed as:

$$\ddot{\mathbf{q}} = W(\mathbf{q})^{-1}\left[\mathbf{Q} + \nabla_{\mathbf{q}} T - \nabla_{\mathbf{q}} V - \boldsymbol{\nabla}_{\mathbf{q}}\mathbf{c}\cdot\mathbf{z} - \frac{\partial}{\partial\mathbf{q}}\left(W(\mathbf{q})\,\dot{\mathbf{q}}\right)\dot{\mathbf{q}}\right] \tag{22a}$$

- The Lagrange equation deliver the acceleration $\ddot{\mathbf{q}}$ as a function of $\mathbf{q}$, $\dot{\mathbf{q}}$, $\mathbf{Q}$ <u>and</u> of the Lagrange multipliers $\mathbf{z}$.
- A simulation can be produced for given initial conditions $\mathbf{q}(0)$, $\dot{\mathbf{q}}(0)$, given external forces $\mathbf{Q}(t)$ (given at all time) <u>and</u> the Lagrange multipliers $\mathbf{z}(t)$.
- To compute a simulation, calculate the Lagrange multipliers $\mathbf{z}$ at every time instant of the simulation.

Constrained Euler-Lagrange

We consider systems with constraints

$$\mathbf{c}(\mathbf{q}) = 0, \quad \mathbf{c} : \mathrm{R}^{n_q} \to \mathrm{R}^{n_c},$$

so that the motion is constrained i.e. the generalized coordinates are not free. To handle the constraints, the Lagrangian is modified:

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}) - \mathbf{z}^T \mathbf{c}(\mathbf{q})$$

The Euler-Lagrange equation now becomes

$$\frac{d}{dt} \nabla_{\dot{\mathbf{q}}} \mathcal{L} - \nabla_{\mathbf{q}} \mathcal{L} = \mathbf{Q}$$
$$\mathbf{c}(\mathbf{q}) = 0$$

or, with the previously used expression for T,

$$W(\mathbf{q})\ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}\big(W(\mathbf{q})\dot{\mathbf{q}}\big)\dot{\mathbf{q}} - \nabla_{\mathbf{q}} T + \nabla_{\mathbf{q}} V + \nabla_{\mathbf{q}} \mathbf{c}\, \mathbf{z} = \mathbf{Q}$$
$$\mathbf{c}(\mathbf{q}) = 0$$

Constrained E-L

- The constraints specify conditions on the system position, not on its accelerations.
- To chose the $\mathbf{z}$ adequately, we need to make the impact of the accelerations on $\mathbf{c}(\mathbf{q})$ appear explicitly.
- We take two time derivatives of the constraints $\mathbf{c}(\mathbf{q})$.
- If $\mathbf{c}(\mathbf{q}) = 0$ is enforced throughout the trajectory of the system (i.e. at every time instant), then:

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{c}(\mathbf{q}) = 0 \tag{23}$$

  also hold at all time for any $k \geq 0$.
- $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{c}(\mathbf{q}) = 0$ is a condition where the accelerations appear explicitly.

Constrained Euler-Lagrange, cont'd

In order to be able to solve the Euler-Lagrange equations with constraints, we differentiate the constraint equation $\mathbf{c}(\mathbf{q}) = 0$ twice:

$$
\begin{aligned}
0 &= \dot{\mathbf{c}}(\mathbf{q}) = \frac{\partial \mathbf{c}}{\partial \mathbf{q}} \dot{\mathbf{q}} \\
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Replacing the constraint equation $\mathbf{c}(\mathbf{q}) = 0$ with $\ddot{\mathbf{c}}(\mathbf{q}) = 0$ now gives:

Constrained Euler-Lagrange, cont'd

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\end{aligned}
$$

Replacing the constraint equation $\mathbf{c}(\mathbf{q}) = 0$ with $\ddot{\mathbf{c}}(\mathbf{q}) = 0$ now gives:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^\top \tag{24a}
$$

$$
\ddot{\mathbf{c}}\left(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\right) = 0 \tag{24b}
$$

Constrained Euler-Lagrange, cont'd

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\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^\top \tag{25a}
$$

$$
\ddot{\mathbf{c}}\left(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\right) = 0 \tag{25b}
$$

In the explicit form:

Constrained Euler-Lagrange, cont'd

In order to be able to solve the Euler-Lagrange equations with constraints, we differentiate the constraint equation $\mathbf{c}(\mathbf{q}) = 0$ twice:

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Replacing the constraint equation $\mathbf{c}(\mathbf{q}) = 0$ with $\ddot{\mathbf{c}}(\mathbf{q}) = 0$ now gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^\top \tag{26a}$$
$$\ddot{\mathbf{c}}\left(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\right) = 0 \tag{26b}$$

Or in the explicit form:

$$W\left(\mathbf{q}\right)\ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}\left(W\left(\mathbf{q}\right)\dot{\mathbf{q}}\right)\dot{\mathbf{q}} - \nabla_{\mathbf{q}} T + \nabla_{\mathbf{q}} V + \nabla_{\mathbf{q}}\mathbf{c} \cdot \mathbf{z} = \mathbf{Q} \tag{27a}$$
$$\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} = 0 \tag{27b}$$

Constrained Euler-Lagrange, cont'd

Using $\ddot{\mathbf{c}}(\mathbf{q}) = 0$:

$$W(\mathbf{q})\,\ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}\left(W(\mathbf{q})\,\dot{\mathbf{q}}\right)\dot{\mathbf{q}} - \nabla_{\mathbf{q}} T + \nabla_{\mathbf{q}} V + \nabla_{\mathbf{q}}\mathbf{c} \cdot \mathbf{z} = \mathbf{Q} \tag{28a}$$

$$\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} = 0 \tag{28b}$$

Via algebraic manipulations:

$$\underbrace{\begin{bmatrix} W(\mathbf{q}) & \nabla_{\mathbf{q}}\mathbf{c} \\ \nabla_{\mathbf{q}}\mathbf{c}^\top & 0 \end{bmatrix}}_{:=M(\mathbf{q})} \begin{bmatrix} \ddot{\mathbf{q}} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} - \frac{\partial}{\partial \mathbf{q}}\left(W(\mathbf{q})\,\dot{\mathbf{q}}\right)\dot{\mathbf{q}} + \nabla_{\mathbf{q}} T - \nabla_{\mathbf{q}} V \\ -\frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} \end{bmatrix} \tag{29}$$

which delivers $\ddot{\mathbf{q}}$, $\mathbf{z}$ jointly if matrix $M(\mathbf{q})$ is full rank.

Constrained Euler-Lagrange, cont'd

$$\underbrace{\begin{bmatrix} W(\mathbf{q}) & \nabla_{\mathbf{q}}\mathbf{c} \\ \nabla_{\mathbf{q}}\mathbf{c}^{\top} & 0 \end{bmatrix}}_{:=M(\mathbf{q})} \begin{bmatrix} \ddot{\mathbf{q}} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} - \frac{\partial}{\partial \mathbf{q}}\left(W(\mathbf{q})\,\dot{\mathbf{q}}\right)\dot{\mathbf{q}} + \nabla_{\mathbf{q}} T - \nabla_{\mathbf{q}} V \\ -\frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} \end{bmatrix} \tag{30}$$

Writing the model in explicit form:

Constrained Euler-Lagrange, cont'd

$$\begin{bmatrix} W(\mathbf{q}) & \nabla_{\mathbf{q}}\mathbf{c} \\ (\nabla_{\mathbf{q}}\mathbf{c})^T & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} - \frac{\partial}{\partial \mathbf{q}}\left(W(\mathbf{q})\dot{\mathbf{q}}\right)\dot{\mathbf{q}} + \nabla_{\mathbf{q}} T - \nabla_{\mathbf{q}} V \\ -\frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} \end{bmatrix}$$

Writing the model in explicit form:

$$\left[\begin{array}{c} \ddot{\mathbf{q}} \\ \mathbf{z} \end{array}\right] = M\left(\mathbf{q}\right)^{-1} \left[\begin{array}{c} \mathbf{Q} - \frac{\partial}{\partial \mathbf{q}}\left(W\left(\mathbf{q}\right)\dot{\mathbf{q}}\right)\dot{\mathbf{q}} + \nabla_{\mathbf{q}} T - \nabla_{\mathbf{q}} V \\ -\frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} \end{array}\right] \tag{31}$$

Constrained Euler-Lagrange, cont'd

In order to be able to solve the Euler-Lagrange equations with constraints, we differentiate the constraint equation $\mathbf{c}(\mathbf{q}) = 0$ twice:

$$0 = \dot{\mathbf{c}}(\mathbf{q}) = \frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}$$

$$0 = \ddot{\mathbf{c}}(\mathbf{q}) = \frac{d}{dt}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right) = \frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} + \frac{\partial}{\partial \dot{\mathbf{q}}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)\ddot{\mathbf{q}} = \frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} + \frac{\partial \mathbf{c}}{\partial \mathbf{q}}\ddot{\mathbf{q}}$$

Replacing the constraint equation $\mathbf{c}(\mathbf{q}) = 0$ with $\ddot{\mathbf{c}}(\mathbf{q}) = 0$ now gives the Euler-Lagrange equation:

$$\begin{bmatrix} W(\mathbf{q}) & \nabla_{\mathbf{q}}\mathbf{c} \\ (\nabla_{\mathbf{q}}\mathbf{c})^T & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} - \frac{\partial}{\partial \mathbf{q}}\left(W(\mathbf{q})\dot{\mathbf{q}}\right)\dot{\mathbf{q}} + \nabla_{\mathbf{q}} T - \nabla_{\mathbf{q}} V \\ -\frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} \end{bmatrix}$$

Consistency Conditions

The original model that is describing the physical system reads as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^\top \tag{32a}$$

$$\mathbf{c}\left(\mathbf{q}\right) = 0 \tag{32b}$$

while the transformed model reads as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^\top \tag{33a}$$

$$\ddot{\mathbf{c}}\left(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\right) = 0 \tag{33b}$$

- Is the transformed model equivalent to the original one?
- The trajectories from the original model satisfy $\mathbf{c} = 0$ at all time, they also satisfy $\ddot{\mathbf{c}} = 0$ at all time. Hence, the trajectories of the original model are also trajectories of the transformed model.
- The trajectories of the transformed model satisfy $\ddot{\mathbf{c}} = 0$ at all time, which does not entail that they satisfy $\mathbf{c} = 0$ at all time.

Consistency Conditions

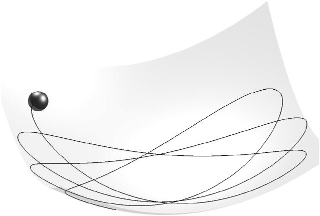
$\mathbf{c}(\mathbf{q}) = 0$ holds at all time if:

$$\boldsymbol{C}\left(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)\right) := \begin{bmatrix} \boldsymbol{c}\left(\boldsymbol{q}(0)\right) \\ \dot{\boldsymbol{c}}\left(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)\right) \end{bmatrix} = 0$$

- the transformed model delivers meaningful trajectories **if** the initial conditions $\mathbf{q}(0)$ and $\dot{\mathbf{q}}(0)$ satisfy $\mathbf{C}(\mathbf{q}(0), \dot{\mathbf{q}}(0)) = 0$.
- They are required in order to impose initial conditions in the model that are physically meaningful.

Consistency Conditions - Example

$$\boldsymbol{c}(\boldsymbol{p}) = \boldsymbol{p}_3 - \frac{1}{4}\left(\boldsymbol{p}_2^2 + \boldsymbol{p}_1^2\right) \in \mathbb{R}$$

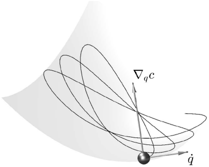


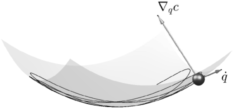
$$\boldsymbol{C}(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)) := \begin{bmatrix} \boldsymbol{p}_3 - \frac{1}{4}\boldsymbol{p}_1^2 - \frac{1}{4}\boldsymbol{p}_2^2 \\ \dot{\boldsymbol{p}}_3 - \frac{1}{2}\dot{\boldsymbol{p}}_1\boldsymbol{p}_1 - \frac{1}{2}\dot{\boldsymbol{p}}_2\boldsymbol{p}_2 \end{bmatrix}$$

- The first condition in $\mathbf{C}$ simply states that the initial condition $\mathbf{q}(0)$ must satisfy the constraints $\mathbf{c}(\mathbf{q})$.

Consistency Conditions - Example

$$\boldsymbol{C}\left(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)\right) := \begin{bmatrix} \boldsymbol{p}_3 - \frac{1}{4}\boldsymbol{p}_1^2 - \frac{1}{4}\boldsymbol{p}_2^2 \\ \dot{\boldsymbol{p}}_3 - \frac{1}{2}\dot{\boldsymbol{p}}_1\boldsymbol{p}_1 - \frac{1}{2}\dot{\boldsymbol{p}}_2\boldsymbol{p}_2 \end{bmatrix}$$



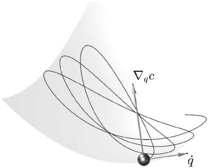


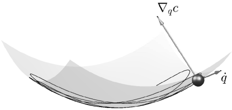
- The second condition states that: $\dot{\boldsymbol{c}}\left(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)\right) = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \nabla_{\boldsymbol{q}}\boldsymbol{c}^\top \dot{\boldsymbol{q}} = 0$

  i.e. it requires that the scalar product between the gradient $\nabla_{\mathbf{q}}\mathbf{c}$ and the velocities $\dot{\mathbf{q}}$ is zero.

Consistency Conditions - Example

$$
C(q(0), \dot{q}(0)) := \begin{bmatrix} p_3 - \frac{1}{4}p_1^2 - \frac{1}{4}p_2^2 \\ \dot{p}_3 - \frac{1}{2}\dot{p}_1 p_1 - \frac{1}{2}\dot{p}_2 p_2 \end{bmatrix}
$$



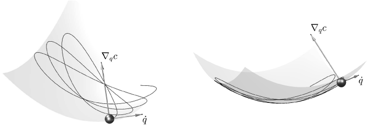


- The second condition states that:

  $$\dot{c}(q(0), \dot{q}(0)) = \frac{\partial c}{\partial q}\dot{q} = \nabla_q c^\top \dot{q} = 0$$ i.e. it requires that the scalar product between the gradient $\nabla_{\mathbf{q}}\mathbf{c}$ and the velocities $\dot{\mathbf{q}}$ is zero.
- The velocities $\dot{\mathbf{q}}$ to be orthogonal to the gradient $\nabla_{\mathbf{q}}\mathbf{c}$.
- The gradient $\nabla_{\mathbf{q}}\mathbf{c}$ is describing a normal to the surface described by the equation $\mathbf{c}(\mathbf{q}(0)) = 0$.

Consistency Conditions - Example

$$C\left(q(0), \dot{q}(0)\right) := \begin{bmatrix} p_3 - \frac{1}{4}p_1^2 - \frac{1}{4}p_2^2 \\ \dot{p}_3 - \frac{1}{2}\dot{p}_1 p_1 - \frac{1}{2}\dot{p}_2 p_2 \end{bmatrix}$$

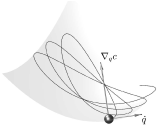


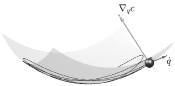
- The second condition states that: $\dot{c}\left(q(0), \dot{q}(0)\right) = \frac{\partial c}{\partial q}\dot{q} = \nabla_q c^\top \dot{q} = 0$

  i.e. it requires that the scalar product between the gradient $\nabla_{\mathbf{q}}\mathbf{c}$ and the velocities $\dot{\mathbf{q}}$ is zero.
- The velocities $\dot{\mathbf{q}}$ to be orthogonal to the gradient $\nabla_{\mathbf{q}}\mathbf{c}$.
- The gradient $\nabla_{\mathbf{q}}\mathbf{c}$ is describing a normal to the surface described by the equation $\mathbf{c}(\mathbf{q}(0)) = 0$.
- $\nabla_{\mathbf{q}}\mathbf{c}(\mathbf{q}(0))^\top \dot{\mathbf{q}}(0) = 0$ requires that the initial velocities $\dot{\mathbf{q}}(0)$ are tangent to the bowl surface.

Consistency Conditions - Example

$$C(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)) := \begin{bmatrix} \boldsymbol{p}_3 - \frac{1}{4}\boldsymbol{p}_1^2 - \frac{1}{4}\boldsymbol{p}_2^2 \\ \dot{\boldsymbol{p}}_3 - \frac{1}{2}\dot{\boldsymbol{p}}_1\boldsymbol{p}_1 - \frac{1}{2}\dot{\boldsymbol{p}}_2\boldsymbol{p}_2 \end{bmatrix}$$





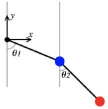
- The second condition states that: $\dot{\boldsymbol{c}}(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)) = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \nabla_{\boldsymbol{q}}\boldsymbol{c}^\top \dot{\boldsymbol{q}} = 0$ i.e. it requires that the scalar product between the gradient $\nabla_{\mathbf{q}}\mathbf{c}$ and the velocities $\dot{\mathbf{q}}$ is zero.
- The velocities $\dot{\mathbf{q}}$ to be orthogonal to the gradient $\nabla_{\mathbf{q}}\mathbf{c}$.
- The gradient $\nabla_{\mathbf{q}}\mathbf{c}$ is describing a normal to the surface described by the equation $\mathbf{c}(\mathbf{q}(0)) = 0$.
- $\nabla_{\mathbf{q}}\mathbf{c}(\mathbf{q}(0))^\top \dot{\mathbf{q}}(0) = 0$ requires that the initial velocities $\dot{\mathbf{q}}(0)$ are tangent to the bowl surface.
- This requirement is physically needed in order for the ball to slide on the surface of the bowl.

Physical modelling

The process of going from characterizing a system from its physical properties to determining a useful state space model.

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned}$$

Physical modelling - Example



$$
\begin{aligned}
x_1 &= l_1 \sin\theta_1 & \dot{x}_1 &= l_1\dot{\theta}_1 \cos\theta_1 \\
y_1 &= -l_1 \cos\theta_1 & \dot{y}_1 &= l_1\dot{\theta}_1 \sin\theta_1 \\
x_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2 & \dot{x}_2 &= l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_2 \cos\theta_2 \\
y_2 &= -l_1 \cos\theta_1 - l_2 \cos\theta_2 & \dot{y}_2 &= l_1\dot{\theta}_1 \sin\theta_1 + l_2\dot{\theta}_2 \sin\theta_2
\end{aligned}
$$

$$
\begin{aligned}
V &= m_1 g y_1 + m_2 g y_2 = -(m_1+m_2) l_1 g \cos\theta_1 - m_2 l_2 g \cos\theta_2 \\
T &= \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\
&= \tfrac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)]
\end{aligned}
$$

$$
\mathcal{L} = \tfrac{1}{2}(m_1+m_2) l_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1+m_2) l_1 g \cos\theta_1 + m_2 g l_2 \cos\theta_2
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \text{for } q_i = \theta_1, \theta_2.
$$

let $z_1 \equiv \dot{\theta}_1 \Rightarrow \ddot{\theta}_1 = \dot{z}_1$ and $z_2 \equiv \dot{\theta}_2 \Rightarrow \ddot{\theta}_2 = \dot{z}_2$.

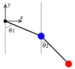
$$
\dot{z}_1 = \frac{m_2 g \sin\theta_2 \cos(\theta_1 - \theta_2) - m_2 \sin(\theta_1 - \theta_2)[l_1 z_1^2 \cos(\theta_1 - \theta_2) + l_2 z_2^2] - (m_1+m_2) g \sin\theta_1}{l_1 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]},
$$

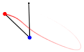
$$
\dot{z}_2 = \frac{(m_1+m_2)[l_1 z_1^2 \sin(\theta_1 - \theta_2) - g \sin\theta_2 + g \sin\theta_1 \cos(\theta_1 - \theta_2)] + m_2 l_2 z_2^2 \sin(\theta_1 - \theta_2) \cos(\theta_1 - \theta_2)}{l_2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]}
$$

Double pendulum example

```
def deriv(y, t, L1, L2, m1, m2):
    """Return the first derivatives of y = theta1, z1, theta2, z2."""
    theta1, z1, theta2, z2 = y

    c, s = np.cos(theta1-theta2), np.sin(theta1-theta2)

    theta1dot = z1
    z1dot = (m2*g*np.sin(theta2)*c - m2*s*(L1*z1**2*c + L2*z2**2) -
             (m1+m2)*g*np.sin(theta1)) / L1 / (m1 + m2*s**2)
    theta2dot = z2
    z2dot = ((m1+m2)*(L1*z1**2*s - g*np.sin(theta2) + g*np.sin(theta1)*c) +
             m2*L2*z2**2*s*c) / L2 / (m1 + m2*s**2)
    return theta1dot, z1dot, theta2dot, z2dot

def calc_E(y):
    """Return the total energy of the system."""

    th1, th1d, th2, th2d = y.T
    V = -(m1+m2)*L1*g*np.cos(th1) - m2*L2*g*np.cos(th2)
    T = 0.5*m1*(L1*th1d)**2 + 0.5*m2*((L1*th1d)**2 + (L2*th2d)**2 +
            2*L1*L2*th1d*th2d*np.cos(th1-th2))
    return T + V

# Maximum time, time point spacings and the time grid (all in s).
tmax, dt = 30, 0.01
t = np.arange(0, tmax+dt, dt)
# Initial conditions: theta1, dtheta1/dt, theta2, dtheta2/dt.
y0 = np.array([3*np.pi/7, 0, 3*np.pi/4, 0])

# Do the numerical integration of the equations of motion
y = odeint(deriv, y0, t, args=(L1, L2, m1, m2))
```

Physical modelling - Example





$$
\begin{aligned}
x_1 &= l_1 \sin\theta_1 & \dot{x}_1 &= l_1\dot{\theta}_1 \cos\theta_1 \\
y_1 &= -l_1 \cos\theta_1 & \dot{y}_1 &= l_1\dot{\theta}_1 \sin\theta_1 \\
x_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2 & \dot{x}_2 &= l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_2 \cos\theta_2 \\
y_2 &= -l_1 \cos\theta_1 - l_2 \cos\theta_2 & \dot{y}_2 &= l_1\dot{\theta}_1 \sin\theta_1 + l_2\dot{\theta}_2 \sin\theta_2
\end{aligned}
$$

$$\dot{z}_1 = \frac{m_2 g \sin\theta_2 \cos(\theta_1-\theta_2) - m_2 \sin(\theta_1-\theta_2)[l_1 z_1^2 \cos(\theta_1-\theta_2) + l_2 z_2^2] - (m_1+m_2) g \sin\theta_1}{l_1[m_1 + m_2 \sin^2(\theta_1-\theta_2)]},$$

$$\dot{z}_2 = \frac{(m_1+m_2)[l_1 z_1^2 \sin(\theta_1-\theta_2) - g\sin\theta_2 + g\sin\theta_1\cos(\theta_1-\theta_2)] + m_2 l_2 z_2^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)}{l_2[m_1 + m_2 \sin^2(\theta_1-\theta_2)]}$$