ESS101
Modelling and Simulation, 2025

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Lecture 2 – Physical modelling

The process of going from characterizing a system **from its physical properties to determining a useful state space model**.

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t))\end{aligned}$$

Physical modelling

Many technical systems are governed by well-known physical phenomena, which can be used to build models.
Domain knowledge about the physical phenomena that characterize e.g. mechanical, electircal, fluid systems.
Sgeneral guidelines:

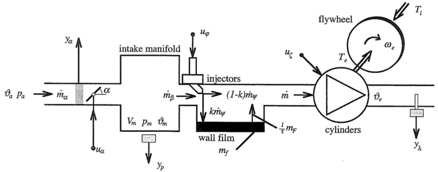
- **Physical modelling workflow:**
  - Structuring a model
  - Balance equations and constitutive relations
  - Forming a state-space model
- **Examples from different domains:**
  - Electric circuit
  - DC motor with load
  - Fluid system
- **Analogies between different domains**

Physical modelling work-flow

1. **Analyze the system's function and structure**
   - Subsystems? (how the system can be viewed as a connection of subsystems, interactions between)
   - Important variables? (which quantities/variables describe the mechanisms)
   - Qualitative relations
     (cause-effect, static-dynamic, fast-slow)?
2. **Determine basic relations/equations**
   - Balance equations
     (mass, energy, force,... e.g. Kirchoff)
   - Constitutive relations? Diff. equations and algebraic relations
     (e.g. Ohm's law, general gas law)
   - Dimensional check
3. **Formulate a model**
   - State space model
   - Choice of state variables?

Example: Structuring of an SI-engine model

From system sketch (a more systematic description of physical processes is obtained)...



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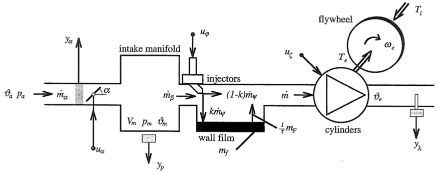
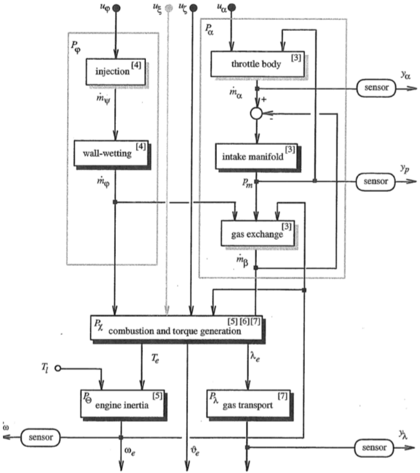


... **to block diagram:**



Determine basic relations

From qualitative description **to quantitative** description.
Determine equations describing the physical mechanisms relating variables involved.
Apply knowledge from physics, mechanics, electricity etc.

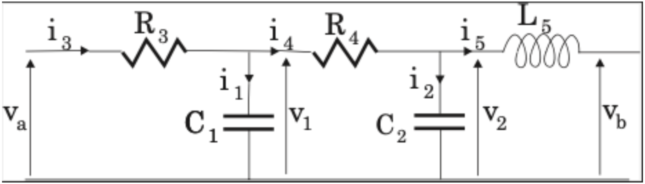
- **Balance equations:** encode principles of mass balance, energy balance, force balance etc. Relate several vaiables of the same kind, e.g. Kirchhoff's voltage and current laws.
- **Constitutive relations:** describe relations between different kinds of variables, e.g. Ohm's law, relations between voltage and current.

* Check dimensions, units.
* Result is a collection of equations, including derivatives.

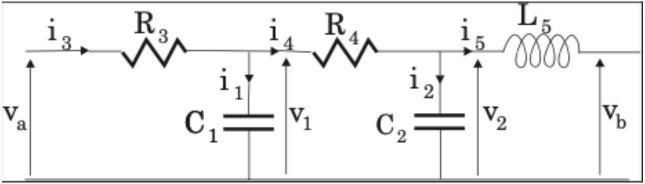
Formulate a model

- Arriving at a final model
- Simplify equations
- Obtain a state space model (if not possible, e.g. differential algebraic equations)

Example: Electric circuit

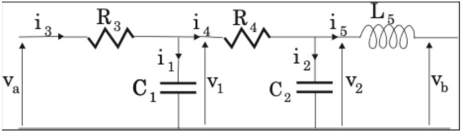


Example: Electric circuit



- The terminal voltages v_a and v_b are supplied as inputs.
- The capacitors C_1, C_2 and the inductor L_5 represent energy storages and thus give rise to dynamics.
- The resistors R_3 and R_4 are considered ideal and are hence static components.
- Temperature variations are considered small, and therefore component parameter values can be considered constant.

Example: Electric circuit



Describing dynamics

$$C_1 \frac{\mathrm{d}v_1}{\mathrm{d}t} = i_1$$
$$C_2 \frac{\mathrm{d}v_2}{\mathrm{d}t} = i_2$$
$$L_5 \frac{\mathrm{d}i_5}{\mathrm{d}t} = v_5$$

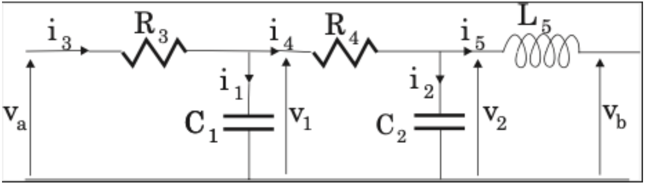
Static, constitutive relations

$$v_3 = R_3 i_3$$
$$v_4 = R_4 i_4.$$

Balance equations

$$v_1 = v_a - v_3$$
$$v_2 = v_1 - v_4$$
$$v_b = v_2 - v_5$$
$$i_3 = i_1 + i_4$$
$$i_4 = i_2 + i_5$$

Example: Electric circuit

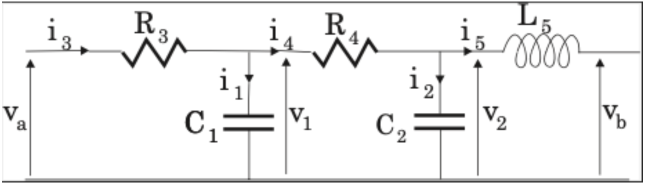


$$C_1 \frac{\mathrm{d}v_1}{\mathrm{d}t} = i_1 = i_3 - i_4 = \frac{1}{R_3} v_3 - \frac{1}{R_4} v_4 = \frac{1}{R_3}(v_a - v_1) - \frac{1}{R_4}(v_1 - v_2)$$

$$C_2 \frac{\mathrm{d}v_2}{\mathrm{d}t} = i_2 = i_4 - i_5 = \frac{1}{R_4}(v_1 - v_2) - i_5$$

$$L_5 \frac{\mathrm{d}i_5}{\mathrm{d}t} = v_5 = v_2 - v_b$$

Example: Electric circuit



introducing $\boldsymbol{x} = (v_1, v_2, i_5)$ and $\boldsymbol{u} = (v_a, v_b)$:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -\frac{1}{R_3C_1} - \frac{1}{R_4C_1} & \frac{1}{R_4C_1} & 0 \\ \frac{1}{R_4C_2} & -\frac{1}{R_4C_2} & -\frac{1}{C_2} \\ 0 & \frac{1}{L_5} & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} \frac{1}{R_3C_1} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{L_5} \end{bmatrix} \boldsymbol{u}(t)$$

Example: DC motor with load

Example 2.4 (DC motor with load)**.** Consider the DC motor illustrated in the figure below. The motor is supplied with a DC source with voltage u. The motor drives a rotating load, characterized by its moment of inertia J and friction coefficient b. We would like to derive a state-space model and a block diagram with transfer functions, describing the motor.

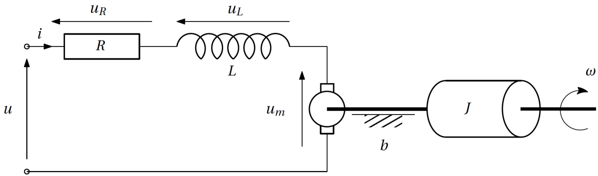
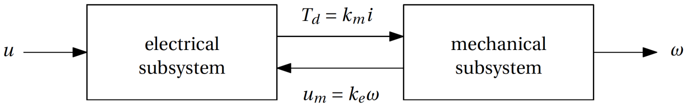


Figure 2.5: A simple sketch of a DC motor.

Example: DC motor with load

1. The structure of this electro-mechanical system can be illustrated by the simple diagram below. The two blocks represent the electrical and the mechanical subsystems, respectively. We also see how the subsystems are connected. The electrical subsystem delivers by induction the torque $T_d = k_m i$, where i is the current and k_m is the *torque constant*. Conversely, the rotation causes a *back-emf*, i.e. a voltage $u_m = k_e \omega$, where ω is the rotational speed and k_e is a constant.



Example: DC motor with load

2. Letting u_m denote the voltage over the motor and u_R, u_L be the component voltages, the constitutive relations and Kirchhoff's voltage law give the following equations for the electrical sybsystem:

$$u = u_R + u_L + u_m \tag{2.8a}$$
$$u_R = Ri \tag{2.8b}$$
$$u_L = L\frac{\mathrm{d}i}{\mathrm{d}t} \tag{2.8c}$$
$$u_m = k_e\omega \tag{2.8d}$$

For the mechanical subsystem, Newton's equation (a torque balance) and the constitutive relations give:

$$J\frac{\mathrm{d}\omega}{\mathrm{d}t} = T_d - T_f \tag{2.9a}$$
$$T_f = b\omega \tag{2.9b}$$
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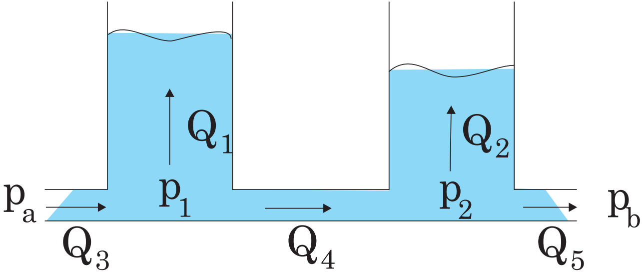
3. Choosing the differentiated variables, i and ω, as state variables, the following state-space model is readily derived:

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = -Ri - k_e\omega + u \tag{2.10a}$$

$$J\frac{\mathrm{d}\omega}{\mathrm{d}t} = k_m i - b\omega \tag{2.10b}$$

Example: Flow system



Example: Flow system

Potential energy (accumulation of mass), kinetic energy (due to flow), external pressures are inputs.

2. A mass balance for the tanks (with cross-sectional areas A_1 and A_2) and a force balance (Newton's equation) for the outflow pipe (with cross-sectional area A) give the following differential equations (ρ is the density of the liquid):

$$A_1 \frac{\mathrm{d}h_1}{\mathrm{d}t} = q_1 \tag{2.12a}$$

$$A_2 \frac{\mathrm{d}h_2}{\mathrm{d}t} = q_2 \tag{2.12b}$$

$$\rho l \frac{\mathrm{d}q_5}{\mathrm{d}t} = A(p_2 - p_b) \tag{2.12c}$$

In addition, we have two constitutive relations for the linear flow resistances R_3 and R_4, constitutive relations linking pressure and level in the tanks, and balance equations for the flows:

$$p_a - p_1 = R_3 q_3 \tag{2.13a}$$

$$p_1 - p_2 = R_4 q_4 \tag{2.13b}$$

$$p_1 = \rho g h_1 \tag{2.13c}$$

$$p_2 = \rho g h_2 \tag{2.13d}$$

$$q_1 = q_3 - q_4 \tag{2.13e}$$

$$q_2 = q_4 - q_5 \tag{2.13f}$$

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3. Using pressures p_1, p_2 and flow-rate q_5 as state variables, we can now form a state-space model by combining all equations in (2.12) and (2.13):

$$\frac{A_1}{\rho g} \frac{\mathrm{d}p_1}{\mathrm{d}t} = q_1 = q_3 - q_4 = \frac{1}{R_3}(p_a - p_1) - \frac{1}{R_4}(p_1 - p_2) \tag{2.14a}$$

$$\frac{A_2}{\rho g} \frac{\mathrm{d}p_2}{\mathrm{d}t} = q_2 = q_4 - q_5 = \frac{1}{R_4}(p_1 - p_2) - q_5 \tag{2.14b}$$

$$\frac{\rho l}{A} \frac{\mathrm{d}q_5}{\mathrm{d}t} = p_2 - p_b \tag{2.14c}$$

Similarities in different domains

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Analogies between different physical domains

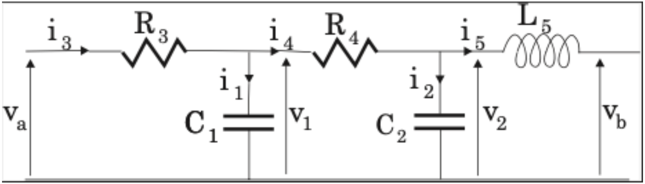
Same model equations can describe systems *in different domains*.

	General	Electrical	Flow	Mechanical
Intensity	e	u	p	F
Flow	f	i	Q	v
Resistance	$e = \gamma f$	$u = Ri$	$p = R_f q$	$F = dv$
Inductance	$e = \alpha \frac{d}{dt} f$	$u = L \frac{d}{dt} i$	$p = L_f \frac{d}{dt} q$	$F = m \frac{d}{dt} v$
Capacitance	$f = \beta \frac{d}{dt} e$	$i = C \frac{d}{dt} u$	$q = C_f \frac{d}{dt} p$	$v = 1/k \frac{d}{dt} F$
Stored energy:				
Inductance	$\frac{1}{2} \alpha f^2$	$\frac{1}{2} L i^2$	$\frac{1}{2} L_f q^2$	$\frac{1}{2} m v^2$
Capacitance	$\frac{1}{2} \beta e^2$	$\frac{1}{2} C u^2$	$\frac{1}{2} C_f p^2$	$\frac{1}{2} \frac{1}{k} F^2$

Physical modelling - Summary

The process of going from *characterizing a system from its physical properties* to determining a useful state space model.

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$



introducing $\boldsymbol{x} = (v_1, v_2, i_5)$ and $\boldsymbol{u} = (v_a, v_b)$:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -\frac{1}{R_3C_1} - \frac{1}{R_4C_1} & \frac{1}{R_4C_1} & 0 \\ \frac{1}{R_4C_2} & -\frac{1}{R_4C_2} & -\frac{1}{C_2} \\ 0 & \frac{1}{L_5} & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} \frac{1}{R_3C_1} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{L_5} \end{bmatrix} \boldsymbol{u}(t)$$

Some remarks on physical modelling

- **Domain knowledge and experience** is essential and help in making judgment on what is important and not. Analogies can help.

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- There are **standard choices of state variables**, e.g. positions and velocities of masses, charge of capacitor, current of inductor, accumulated mass or volume, and temperature.

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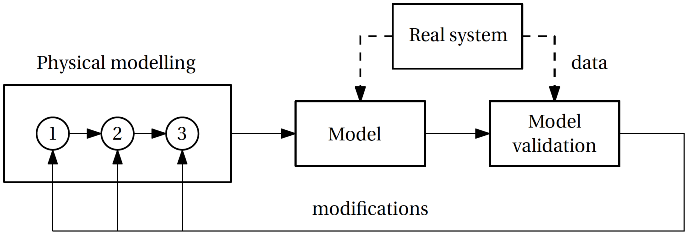
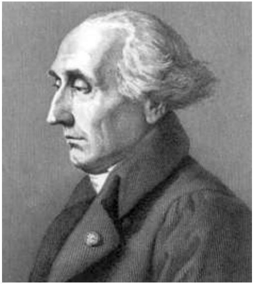
- Another good habit: check dimensions/units!
- There are standard choices of state variables, e.g. positions and velocities of masses, charge of capacitor, current of inductor, accumulated mass or volume, and temperature.
- Note that **step 1** in the 3-step work-flow becomes **significant** for realistic modelling tasks!

Model validation



Lagrange modelling

- Generalized coordinates
- Kinetic and potential energy
- Lagrange function
- Euler-Lagrange's equation

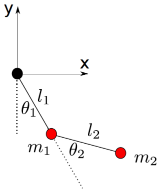


Learning objective:

- Use methods and tools to **develop mathematical models of dynamical systems by using basic physical laws**. The emphasis will be on complex mechanical systems.

Lagrange Mechanics

- Physical modelling complemented with **domain specific knowledge**.
- We will look into how domain specific techniques can facilitate physical modelling process, based on ***Lagrange Mechanics***.
- Lagrange Mechanics is a tool to **build mathematical models for mechanical systems**.
  - allows to describe arbitrarily complex mechanical systems, e.g. relative moving parts, accelerated frames.
  - allows to build simple models.



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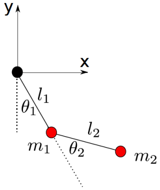
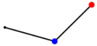




Generalized coordinates of a mechanical system

Definition (Generalized coordinates)

A vector $\mathbf{q}(t) \in \mathsf{R}^{n_q}$ that describes the " configuration" (position of all parts) of a mechanical system is called *generalized coordinates*.

- does not tell how the system will evolve, but can tell **in what configuration the system is at a given time**.
- If n_q is equal to the number of degrees-of-freedom (DOF) of the system, the generalized coordinates are independent or *free*.
- In practice, the generalized coordinates are **usually positions, lengths or angles**.

Lagrange Mechanics

- Lagrange mechanics is based on a description of the mechanical system in terms of **energy**.
- To build models using Lagrange equations, we need to compute **Kinetic** and **Potential energy** functions of the system, denoted as T and V.
- Leading to **dynamical description, equation of motion** - describe the **behavior of a physical system** as a set of mathematical functions in terms of dynamic variables.

Kinetic energy

Consider a mechanical system with N particles, having masses $\{m_i\}$ and positions $\{\mathbf{p}_i\} \in \mathrm{R}^D$ with $D = 1, 2$ or 3.

The *kinetic energy T* of the system is defined as

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\mathbf{p}}_i^T \dot{\mathbf{p}}_i, \quad \mathbf{p}_i = \mathbf{p}_i(t)$$

Using generalized coordinates $\mathbf{q}$,

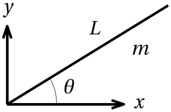
$$\mathbf{p}(t) = \mathbf{p}(\mathbf{q}(t)) \quad \Rightarrow \quad \dot{\mathbf{p}} = \frac{\partial \mathbf{p}}{\partial \mathbf{q}} \dot{\mathbf{q}},$$

$$T = \frac{1}{2} \dot{\mathbf{q}}^T \Big(\sum_{i=1}^{N} m_i \frac{\partial \mathbf{p}_i}{\partial \mathbf{q}}^T \frac{\partial \mathbf{p}_i}{\partial \mathbf{q}} \Big) \dot{\mathbf{q}} = \frac{1}{2} \dot{\mathbf{q}}^T W(\mathbf{q}) \dot{\mathbf{q}}$$

Potential energy

Example:

- The potential energy **due to gravity** in most mechanical applications: $V = mgz$ where z is the height of the mass.
- The mass m is concentrated at the end of a rigid rod, the vertical position is given by: $p_z = Lsin\theta$, its potential energy is given by $V = mgLsin\theta$.



Euler-Lagrange's equation – summary

Kinetic, potential energies and the Lagrangian, expressed in generalized coordinates $\mathbf{q}$:

$$T = T(\mathbf{q}, \dot{\mathbf{q}}), \quad V = V(\mathbf{q}), \quad \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})$$

The Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0,$$

$$\nabla_{\mathbf{q}}\mathcal{L} = \big(\frac{\partial \mathcal{L}}{\partial \mathbf{q}}\big)^T, \quad \nabla_{\dot{\mathbf{q}}}\mathcal{L} = \big(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\big)^T, \quad T = \frac{1}{2}\dot{\mathbf{q}}^T W(\mathbf{q})\dot{\mathbf{q}},$$

the Euler-Lagrange equation reads

$$W(\mathbf{q})\ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}\big(W(\mathbf{q})\dot{\mathbf{q}}\big)\dot{\mathbf{q}} - \nabla_{\mathbf{q}} T + \nabla_{\mathbf{q}} V = 0$$