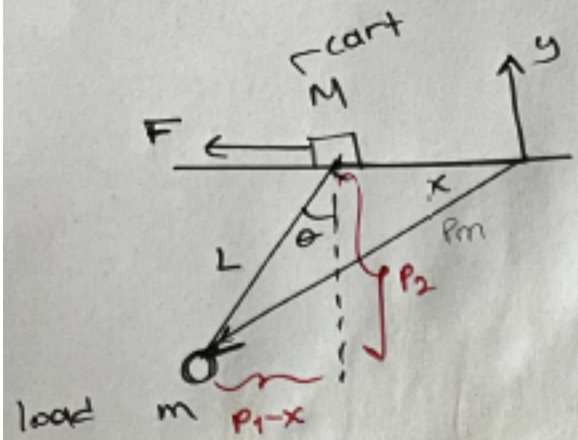
Example: Hanging crane with a constraint



$$q = \begin{bmatrix} x \\ p_m \end{bmatrix} = \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix}$$

Constraint: L is fixed

$$0 = c(q) = \frac{1}{2}\left[(p_1 - x)^2 + p_2^2 - L^2\right]$$

- $$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \dot{p}_m^T \dot{p}_m$$
  $$= \frac{1}{2} \dot{q}^T \underbrace{\begin{bmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}}_{W} \dot{q} = \frac{1}{2} \dot{q}^T W \dot{q}$$

- $$V = -mg \underbrace{[0 \ 0 \ 1]}_{e_3^T} q \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
  $$V = -mg\, e_3^T q$$

- $$L(q, \dot{q}, z) = T(\dot{q}) - V(q) - z^T \cdot c(q)$$
  $$= \frac{1}{2} \dot{q}^T W \dot{q} + mg\, e_3^T q - \frac{1}{2} z^T \cdot \overbrace{\left[q^T \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} q - L^2 \right]}^{c(q)}$$

E-L: $$W \ddot{q} + \nabla_q V + \nabla_q c \cdot z = Q = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix}$$

$$c(q) = 0$$

$$\begin{bmatrix} M & & \\ & m & \\ & & m \end{bmatrix} \ddot{q} - mg \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} q}_{\nabla_q c} \cdot z = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix}$$

$$c(q) = 0$$

$$\nabla_q c = \left(\frac{\partial c}{\partial q}\right)^T \qquad \frac{\partial c}{\partial q} = \frac{1}{2} \cdot 2 \cdot q^T \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(\frac{\partial c}{\partial q}\right)^T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot q$$

using $\alpha = x^T A x$, $\dfrac{\partial \alpha}{\partial x} = 2x^T A$

$$c(q) = \frac{1}{2}\left((p_1 - x)^2 + p_2^2 - L^2\right)$$
$$= \frac{1}{2} q^T \mathbf{C} q - L^2 \qquad \mathbf{C} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{c}(q) = \frac{\partial c}{\partial q} \cdot \dot{q} = q^T \mathbf{C} \cdot \dot{q}$$

$$\ddot{c}(q) = \frac{d}{dt}\left(q^T C \dot{q}\right)$$
$$= \underbrace{\frac{\partial}{\partial q}\left(q^T C \dot{q}\right)} \dot{q} + \frac{\partial}{\partial \dot{q}}\left(q^T C \dot{q}\right) \ddot{q}$$

$$\ddot{c}(q) = \dot{q}^T C \cdot \dot{q} + q^T C \ddot{q}$$

using the rule:
$$\left\{ \begin{array}{l} \alpha = y^T A X \\ \dfrac{\partial \alpha}{\partial y} = X^T A^T \end{array} \right\}$$
if A is symmetric
$A = A^T$

E-L:
equations:

$$W \ddot{q} - mg e_3 + C q \cdot z = Q = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix}$$
$$\dot{q}^T C \dot{q} + q^T C \ddot{q} = 0$$

$$\underbrace{\begin{bmatrix} W_{3\times3} & Cq_{3\times1} \\ q^T C_{1\times3} & 0 \end{bmatrix}}_{M(q)_{4\times4}} \begin{bmatrix} \ddot{q} \\ z \end{bmatrix}_{4\times1} = \begin{bmatrix} F \\ 0 \\ mg \\ -\dot{q}^T C \dot{q} \end{bmatrix}_{4\times1}$$

$M(q)$ is invertible if $p_m \neq \begin{bmatrix} x \\ 0 \end{bmatrix}$.

if $p_m = \begin{bmatrix} x \\ 0 \end{bmatrix}$, $M(q)$ is not full rank.
We do not allow the situation where load coincides with the cart

$q^T C$
$$\underset{1\times3}{\begin{bmatrix} x & x & 0 \end{bmatrix}} \underset{3\times3}{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$