PSS 4 - Lagrange Modelling 2

Exercises: 3.4, 3.5, 3.6, 3.7

Lectures Recap:

- $W(q)$?

Lagrange Function: $\mathcal{L}(q,\dot q) = T(q,\dot q) - U(q)$, $T(q,\dot q) = \frac{1}{2}\dot q^T W(q)\dot q$

$$= \frac{1}{2}\dot q^T W(q)\dot q - V(q)$$

(3.6)

where $W(q) = m\frac{\partial p^T}{\partial q}\frac{\partial p}{\partial q} = \frac{\partial^2 T}{\partial \dot q^2} = \frac{\partial^2 \mathcal{L}}{\partial \dot q^2}$

(3.61)

- $\nabla_{\dot q}\mathcal{L} = W(q)\dot q \Rightarrow \frac{d\nabla_{\dot q}\mathcal{L}}{dt} = W(q)\ddot q + \frac{\partial(W(q)\dot q)}{\partial q}\dot q$

- if $W(q)$ is independent of q, second term cancels and

$$W\ddot q + \nabla_q V = 0$$

- Constraints: $C(q) = 0$
- Changes in Lagrange procedure?

(ii) define $C(q) = 0$

(iv) $\mathcal{L}(q,\dot q,z) = T(q,\dot q) - U(q) - z^T C(q)$

$\to$ lagrange multiplier (It is a vector since we need a multiplier for each constraint.)

(V) E-L becomes

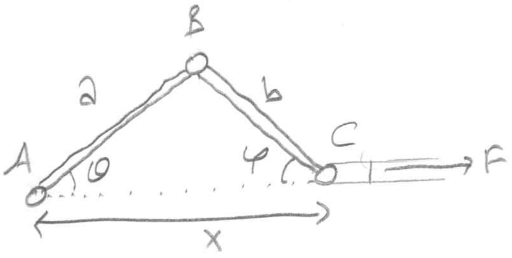
$$\frac{d}{dt}\nabla_{\dot q}\mathcal{L} - \nabla_q\mathcal{L} = Q, \quad \ddot C(q,\dot q,\ddot q) = 0 \Rightarrow \frac{\partial C}{\partial q}\ddot q + \frac{\partial}{\partial q}\left(\frac{\partial C}{\partial q}\dot q\right)\dot q = 0$$

$$\begin{bmatrix} W(q) & \frac{\partial C^T}{\partial q} \\ \frac{\partial C}{\partial q} & 0 \end{bmatrix}\begin{bmatrix} \ddot q \\ z \end{bmatrix} = \begin{bmatrix} Q - \frac{\partial}{\partial q}(W(q)\dot q)\dot q + \nabla_q T - \nabla_q U \\ -\frac{\partial}{\partial q}\left(\frac{\partial C}{\partial q}\dot q\right)\dot q \end{bmatrix} \quad (3.168)$$

* if W is constant, constrained E-L becomes

$$W\ddot q + \nabla_q V + \nabla_q C\, z = 0$$

3.4



a) Determine Q_x if $q = x$
b) Determine Q_θ if $q = \theta$

a) if $q = x$, then force F and generalized coordinates are in same reference frame, so $Q_x = F$.

b) if $q = \theta$, then $Q = \nabla q\, p\, F$ (3.114)

p? $\Rightarrow$ notice, $a \sin\theta = b \sin\varphi$

$x = a\cos\theta + b\cos\varphi$

- get rid of $\cos\varphi$

$\sin^2\varphi + \cos^2\varphi = 1 \Rightarrow \cos\varphi = \sqrt{1 - \sin^2\varphi} = \sqrt{1 - \frac{a^2}{b^2}\sin^2\theta}$

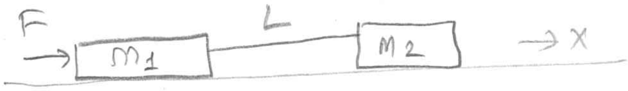
• $x = a\cos\theta + b\sqrt{1 - \frac{a^2}{b^2}\sin^2\theta} = a\cos\theta + \sqrt{b^2 - a^2\sin^2\theta}$

$$\frac{\partial x}{\partial \theta} = -a\sin\theta - \cancel{2}a^2\cos\theta\sin\theta \cdot \left(\frac{1}{\cancel{2}}\frac{1}{\sqrt{b^2 - a^2\sin^2\theta}}\right)$$

$$= -a\sin\theta - \frac{a^2\cos\theta\sin\theta}{\sqrt{b^2 - a^2\sin^2\theta}}$$

$$\underline{\underline{Q_\theta = \frac{\partial x}{\partial \theta} \times F}}$$

3.5



- rigid rod with length L.

a) Derive eq of motions using Newton's law.

b) Derive eq of motions using Lagrange

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a) Newton

$$m_1 \ddot{x}_1 = F - F_{rod}$$
$$m_2 \ddot{x}_2 = F_{rod}$$

b) Lagrange:

(i) $q = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, (ii) $p = q$, $\dot{p} = \dot{q}$, $c(q) = x_2 - x_1 - L = 0$
$\ddot{c}(\ddot{q}) = \ddot{x}_2 - \ddot{x}_1 = 0$

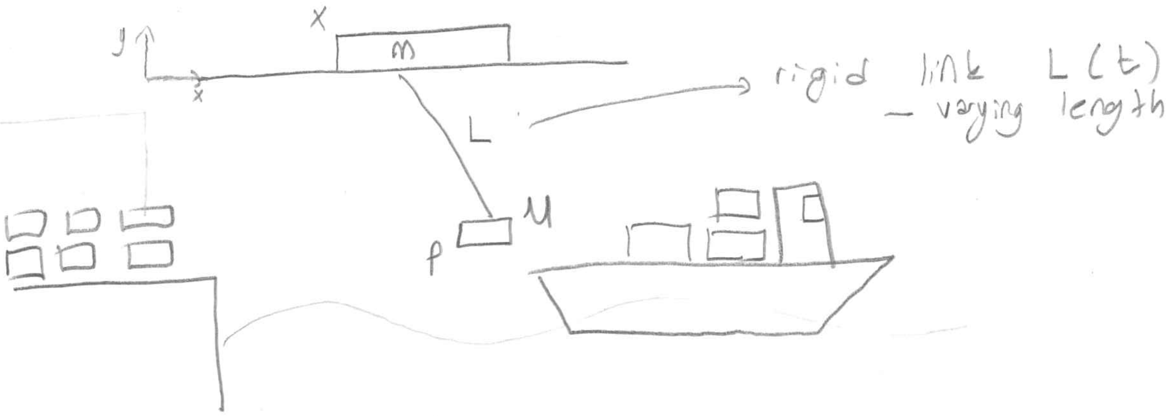
(iii) $V = 0$, $T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$

(iv) $\mathcal{L} = T(\dot{q}) - \lambda^T c(q) = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \lambda^T (x_2 - x_1 - L)$

$\hookrightarrow$ Lagrange Multiplier.

(v) E-L $\Rightarrow$
$$\begin{cases} m_1 \ddot{x}_1 - \lambda = F \\ m_2 \ddot{x}_2 + \lambda = 0 \\ \ddot{c} = 0 \quad \text{ie.} \quad \ddot{x}_2 - \ddot{x}_1 = 0 \end{cases}$$

3.6



a) Lagrange
b) Cable length control?
c) $z>0$ and $z<0$ meaning when solving model equation?

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a)
(i) Cartesian coordinates
$$q = \begin{bmatrix} x \\ p \end{bmatrix} \in \mathbb{R}^3 \quad \text{with} \quad p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

(ii) $$c(q,L) = \frac{1}{2}\Big(\underbrace{(x-p_x)^2 + p_y^2}_{\text{square distance between } m \text{ and } p.} - L^2\Big) = 0$$

$$\dot q = \begin{bmatrix} \dot x \\ \dot p \end{bmatrix}$$

(iii) $V = Mg\,p_y$, $T = \frac{1}{2} m \dot x^2 + \frac{1}{2} M \dot p^T \dot p = \frac{1}{2} \dot q^T \underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix}}_{W(q) = \text{constant}} \dot q$

(iv) $$\mathcal{L} = T - V - zC = \frac{1}{2}\dot q^T W \dot q - Mg\,p_y - z\frac{1}{2}\left((x-p_x)^2 + p_y^2 - L^2\right)$$

(V) E.L $\frac{d}{dt}\nabla_{\dot q}\mathcal{L} - \nabla_q \mathcal{L} = 0$

$$\ddot C(q,L) = \overset{(a)}{\frac{\partial C}{\partial q}\ddot q} + \overset{(b)}{\frac{\partial}{\partial q}\left(\frac{\partial C}{\partial q}\dot q\right)\dot q} + \overset{(c)}{\frac{\partial C}{\partial L}\ddot L} + \overset{(d)}{\frac{\partial}{\partial L}\left(\frac{\partial C}{\partial L}\dot L\right)\dot L} = 0$$

$$\frac{\partial C}{\partial q} = [x-p_x \quad p_x - x \quad p_y]$$

a) $[x-p_x \quad p_x-x \quad p_y]\,\ddot q$

b) $\frac{\partial}{\partial q}\left((x-p_x)\dot x + (p_x - x)\dot p_x + p_y \dot p_y\right)\dot q = [\dot x - \dot p_x \quad \dot p_x - \dot x \quad \dot p_y]\,\dot q = \dot q^T \dot q - 2\dot x \dot p_x$

c) $-L\ddot L$ $\quad \left(\frac{\partial C}{\partial L} = -L\right)$

d) $\frac{\partial}{\partial L}(-L\dot L)\dot L = -\dot L^2$

E.L
$(W\ddot q + \nabla_q C z = -\nabla_q V)$ from recap

$$\left.\begin{bmatrix} m & & \\ & M & \\ & & M \end{bmatrix}\ddot q + \begin{bmatrix} x-p_x \\ p_x - x \\ p_y \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ -Mg \end{bmatrix}\right\} \ddot q \text{ equations}$$

$$\left.(x-p_x)\ddot x + (p_x - x)\ddot p_x + p_y \ddot p_y + \dot q^T \dot q - 2\dot x \dot p_x - L\ddot L - \dot L^2 = 0\right\} \ddot C(q,L)$$

3.6b

b) Cable length could be controlled through $\ddot{l}$ to guarante smoothness of $\dot{l}$ and $\ddot{l}$.

c) The force due to constraint is

$$-\nabla_q C\, z = -\begin{bmatrix} x-p_x \\ p_x-x \\ p_y \end{bmatrix} z \quad \left(W\ddot{q} = -\nabla_q C z - \nabla_q V \right)$$

if $z > 0$ is the normal case when the cable conveys pull force.

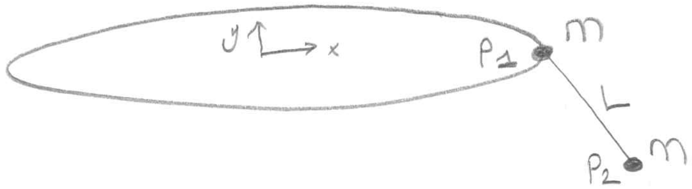
(Assume system is stable and $\ddot{q} = 0$,;

$$0 = \begin{bmatrix} x-p_x \\ p_x-x \\ p_y \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ Mg \end{bmatrix} \Rightarrow \begin{array}{l} x = p_x \\ p_y z = -Mg \end{array}$$

$$\Downarrow$$

$$Mg\downarrow ,\; P_y\uparrow \Rightarrow z\uparrow$$

3.7



- rail equation

$$\frac{1}{2}(p_1^T A p_1 - 1) = 0 \quad , A \text{ symmetric positive definite.}$$

- rigid link L.

a) Lagrange Function

b) $q = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, write model (keep it simple)

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a)

(i) $q = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$

(ii) constraints: $c(q) = \begin{bmatrix} \frac{1}{2}(p_1^T A p_1 - 1) \\ \frac{1}{2}((p_1 - p_2)^T (p_1 - p_2) - L^2) \end{bmatrix} = 0$

rail constraint

length constraint.

(iii) Energies:

$$T = \frac{1}{2} m \dot{p}_1^T \dot{p}_1 + \frac{1}{2} m \dot{p}_2^T \dot{p}_2 = \frac{1}{2} \dot{q}^T W \dot{q} \quad , W = m I_4$$

notice! constant.

$$V = mg [0 \ 1] (p_1 + p_2)$$

(iv) $\mathcal{L} = T - V - z^T C(q) = \frac{1}{2} \dot{q}^T W \dot{q} - mg [0 \ 1](p_1 + p_2) - z^T \begin{bmatrix} \frac{1}{2}(p_1^T A p_1 - 1) \\ \frac{1}{2}((p_1 - p_2)^T (p_1 - p_2) - L^2) \end{bmatrix}$

b) Model Equations:

Recall: $\ddot{C} = \frac{\partial}{\partial q}\left(\frac{\partial c}{\partial q} \dot{q}\right) \dot{q} + \frac{\partial c}{\partial q} \ddot{q}$

$$\frac{\partial c}{\partial q} = \begin{bmatrix} p_1^T A & 0 \\ p_1^T - p_2^T & p_2^T - p_1^T \end{bmatrix} \qquad \frac{\partial c}{\partial q} \dot{q} = \begin{bmatrix} p_1^T A \dot{p}_1 \\ (p_1 - p_2)^T (\dot{p}_1 - \dot{p}_2) \end{bmatrix}$$

$$\frac{\partial}{\partial q}\left(\frac{\partial c}{\partial q} \dot{q}\right) = \begin{bmatrix} \dot{p}_1^T A & 0 \\ (\dot{p}_1 - \dot{p}_2)^T & (\dot{p}_2 - \dot{p}_1)^T \end{bmatrix}$$

General Form

$$\begin{bmatrix} W & \left(\frac{\partial c}{\partial q}\right)^T \\ \frac{\partial c}{\partial q} & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ z \end{bmatrix} = \begin{bmatrix} Q - \frac{\partial V}{\partial q} \\ -\frac{\partial}{\partial q}\left(\frac{\partial c}{\partial q} \dot{q}\right) \dot{q} \end{bmatrix}$$

when W is constant

$$\begin{bmatrix} mI_2 & 0 & Ap_1 & p_1 - p_2 \\ 0 & mI_2 & 0 & p_2 - p_1 \\ p_1^T A & 0 & 0 & 0 \\ p_1^T - p_2^T & p_2^T - p_1^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \\ -mg \\ -\dot{p}_1^T A \dot{p}_1 \\ -(\dot{p}_1 - \dot{p}_2)^T (\dot{p}_1 - \dot{p}_2) \end{bmatrix}$$