Constrained Optimization

Suppose we want to minimize some function:

$$\min_x \; f(x)$$

subject to constraints:

$$g_i(x) = 0, \quad i = 1, \ldots, m$$

We build the **Lagrangian**:

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

If you used a **minus sign** instead:

$$\mathcal{L}(x, \lambda) = f(x) - \lambda g(x)$$

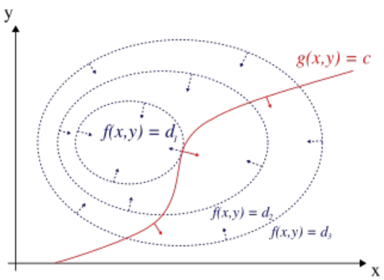
the solution is unchanged, since λ is a free multiplier.

- x = variables we optimize (like parameters β).
- λ_i = **Lagrange multipliers** (like "forces" that enforce constraints).
- Stationary point conditions:

$$\nabla_x \mathcal{L} = 0, \quad g_i(x) = 0$$

At the optimum, we impose the stationarity condition:

$$\nabla_x f(x) + \lambda \nabla_x g(x) = 0$$



- The point where the red constraint tangentially touches a blue contour is the maximum of $f(x, y)$ along the constraint g.
- The gradient of a function is perpendicular to the contour lines, the tangents to the contour lines of f and g are parallel if and only if the gradients of f and g are parallel

$$\nabla_{x,y} f = \lambda \, \nabla_{x,y} g.$$

Lagrangian Mechanics

When constraints are present:

$$\mathcal{L}(q, \dot{q}, \lambda) = L(q, \dot{q}, t) + \sum_{i=1}^{m} \lambda_i g_i(q, t)$$

Here:

- L = kinetic energy – potential energy.
- Constraints $g_i(q, t) = 0$ restrict motion.
- λ_i act as **constraint forces** (like normal forces or tension).