Lagrange Equations:

$$L(q,\dot q) = T(q,\dot q) - V(q) \qquad q \in \mathbb{R}^{n_q}$$

L: Lagrange function

$$L(q,\dot q) = \frac{1}{2}\dot q^T w(q)\,\dot q - V(q)$$

Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0$$

$$\frac{d}{dt}\nabla_{\dot q} L - \nabla_q L = 0$$

Gradient of L:

$$\begin{cases} \nabla_{\dot q} L = \left(\dfrac{\partial L}{\partial \dot q}\right)^T \\ \nabla_q L = \left(\dfrac{\partial L}{\partial q}\right)^T \end{cases}$$

$\alpha = x^T A x$ $\quad x: \dot q$

$\frac{\partial \alpha}{\partial x} = 2x^T A$ $\quad A = w$

$$\nabla_{\dot q} L = \nabla_{\dot q}\left(\frac{1}{2}\dot q^T w(q)\,\dot q - \cancel{V(q)}^{\,0}\right)$$

$$\nabla_{\dot q} L = \nabla_{\dot q}\left(\frac{1}{2}\dot q^T w(q)\,\dot q\right)$$

$$\frac{\partial}{\partial \dot q}\left(\dot q^T w\,\dot q\right) = 2\dot q\, w(q)$$

apply another transpose to get gradient:

$$2\, w_{(q)}^T \dot q$$

$$\nabla_{\dot q} L = w(q)\cdot \dot q$$

$$\frac{d}{dt}\nabla_{\dot q} L = \frac{d}{dt}\left(w(q)\,\dot q\right)$$

$$= \frac{\partial}{\partial \dot q}\left(w(q)\cdot \dot q\right)\ddot q + \frac{\partial}{\partial q}\left(w(q)\cdot \dot q\right)\dot q$$

$$= w(q)\cdot \ddot q + \frac{\partial}{\partial q}\left(w(q)\,\dot q\right)\dot q$$

$w(q)$ = symmetric positive-definite (the inertia tensor) describes the mass distribution

$w(q) = w(q)^T$

Euler-Lagrange Eq:

$$\frac{d}{dt}\nabla_{\dot q}L - \nabla_q L = 0$$

$$w(q)\,\ddot q + \frac{\partial}{\partial q}\left(w(q)\,\dot q\right)\dot q - \nabla_q L = 0 \qquad (\nabla_q L = \nabla_q T - \nabla_q V)$$

$$w(q)\,\ddot q + \frac{\partial}{\partial q}\left(w(q)\,\dot q\right)\dot q - \nabla_q T + \nabla_q V = 0$$

if w is constant:

$$w\cdot\ddot q + \cancelto{0}{\frac{\partial}{\partial q}\left(w\dot q\right)\dot q} - \cancelto{0}{\left(\frac{\partial}{\partial q}\frac{1}{2}\dot q^T w\,\dot q\right)^T} + \nabla_q V = 0$$

$$\ddot q = -W^{-1}\cdot\nabla_q V$$