ESS101
Modelling and simulation

Exercises

Department of Electrical Engineering
Chalmers University of Technology

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Contents

Chapter 1

System dynamics and differential equations

Exercise 1.1 (From external to internal model [2])
Represent the following systems on state space form:

(a)
$$\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3u$$

(b)
$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 2\frac{d^2u}{dt^2} + \frac{du}{dt} + 2u$$

(c)
$$G(s) = \frac{5s+4}{s^2+5s+4}$$

(d)
$$G(s) = \frac{2s^2+15s+12}{s^2+5s+4}$$

Exercise 1.2 (Poles and zeros)

(a) Determine the poles and zeros for the system
$$G(s) = \frac{5s+4}{s^2+5s+4}$$

(b) Determine the poles and zeros for the system
$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) - 1/3u(t)\\ \dot{x}_2(t) &= -4x_2(t) + 16/3u(t)\\ y(t) &= x_1(t) + x_2(t)\end{aligned}$$

Exercise 1.3 (Linearization)
Consider the following system:

$$\begin{aligned}\dot{x}_1(t) &= x_1^2(t) + x_2(t)\\ \dot{x}_2(t) &= u(t)\\ y(t) &= x_1(t)\end{aligned}$$

Linearize the system around the point $x_1 = x_2 = u = 0$.

Exercise 1.4 (From state-space model to transfer function [2])
Determine the transfer function $G(s) = Y(s)/U(s)$ for the system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -x_1(t) - 2x_2(t) + u(t)\\ y(t) &= 0.5x_1(t) + 0.5x_2(t)\end{aligned}$$

Exercise 1.5 (Change of state variables)
The concentration y of a certain species in a mixing tank depends on the incoming flow rate x via a first order dynamics

$$Y(s) = \frac{4}{1+2s}X(s),$$

and the flow rate x is controlled by a valve with dynamics

$$X(s) = \frac{1.5}{1+0.5s}U(s),$$

where u is the valve control input voltage.

(a) Determine a state space model with input u, output y, and state variables with clear physical meaning.

(b) Determine a state space model on diagonal form. Can you see any physical interpretation of the state variables used?

Exercise 1.6 (Stability)
A dynamical system is described by the coupled differential equations

$$\begin{aligned}\dot{x}(t) + x(t) &= u(t)\\ \dot{y}(t) - y(t) + 2x(t) &= u(t)\end{aligned}$$

(a) Determine a state space model and the transfer function from u to y.

(b) Determine the stability properties, both from the transfer function and the state space model. How do you explain the different conclusions?

Exercise 1.7 (Steady-state gain)

(a) In an experiment, the impulse response of a linear system is determined as

$$g(t) = e^{-0.5t}\big(1 + \cos 0.5t\big)$$

What is the steady-state gain of the system?

(b) Determine the transfer function that asymptotically, as $\omega \to 0$, approximates the transfer function $G(s)$ below, and find out the frequency for which the approximate transfer function has the gain 0 dB.

$$G(s) = \frac{1}{s+1} - \frac{s+1}{s^2+2s}$$

Exercise 1.8 (Stability, discrete time)
In a mail server, the number of active tasks y can be influenced by the parameter *MaxUsers*, here denoted u. The dynamics can be described by the discrete-time transfer function

$$\frac{Y(z)}{U(z)} = \frac{0.47}{z - 0.43}$$

Feedback is applied using a proportional controller, with the aim to keep y reasonably constant.

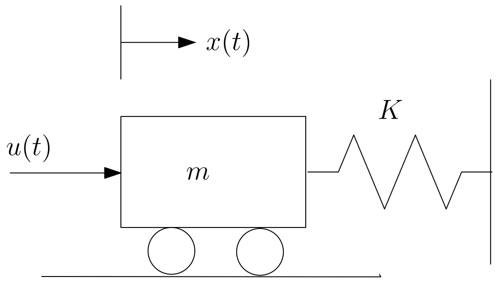
(a) Determine the gain K_p of the controller, so that the closed-loop system has a pole in the origin.

(b) What happens when the gain is gradually increased?

Chapter 2

Physical modelling

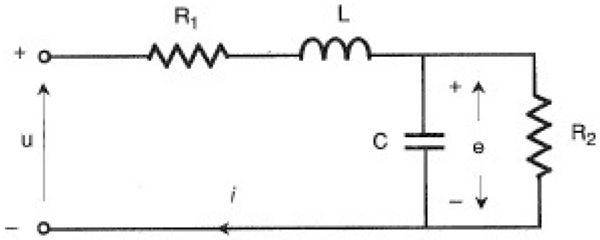
Exercise 2.1 (Simple cart)
Consider a cart with mass m, connected via a spring with spring constant K to a wall. Initially, the cart is at rest, i.e. $x(0) = 0$ and $\dot{x}(0) = 0$. At time $t = 0$, an impulse force $u(t) = \delta(t)$ is applied to the cart.



(a) Find a mathematical model for the cart and put into standard state-space form.

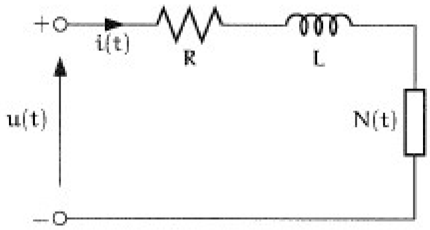
(b) Find the transfer function from force u to position x.

Exercise 2.2 (Electric circuit)
Consider the electric circuit:



Determine the differential equation for the circuit.

Exercise 2.3 (Electric circuit [3])
Consider the electric circuit

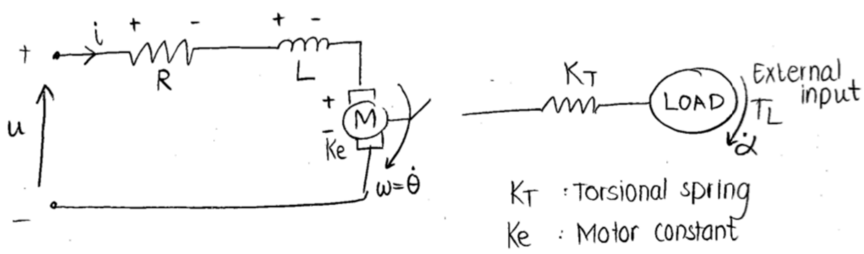


which includes the nonlinear component $N(t) = ki^2, k > 0$, for which voltage drop over the component is ki^2.

(a) Determine the differential equation for the circuit.

(b) Assume that the input signal is constant $u(t) = u_0$. Determine the corresponding stationary point and a linearized differential equation for small variations around this point.

Exercise 2.4 (DC motor with load)
Consider a DC motor with a flexible load according to the figure.



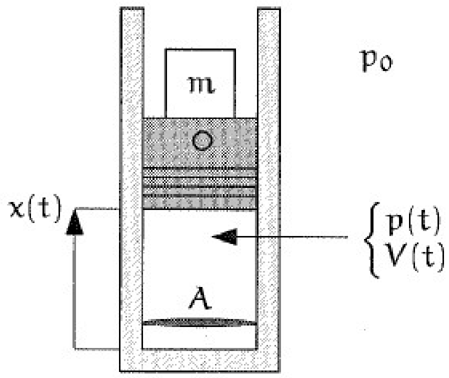
The input to the motor is the voltage u, and the motor has a motor constant K_e, a moment of inertia J_m, and damping b. The load comprises a moment of inertia J_L and a load torque T_L.

(a) Determine a state-space model of the motor alone, assuming the rotational speed $\omega = \dot{\theta}$ is the output.

(b) Determine a state-space model for motor and load with the rotational speed of the load, $\dot{\alpha}$, as output.

(c) Apply the Tikhonov theorem to neglect the electrical time constant.

Exercise 2.5 (Piston [3])
A weight with mass m is placed on a moving piston with cross section area A according to the figure below



For the air inside the cylinder the following relationship holds

$$p(t)V^{\gamma}(t) = C$$

where $p(t)$ is the pressure, $V(t)$ is the volume and γ and C are positive constants. Assume that the surrounding air pressure is constant, p_0.

(a) Describe the movement of the piston and the weight using a state space representation.

(b) Determine the systems stationary points and the linearized equations for small variations around these points.

Exercise 2.6 (Diver)
It is of extreme importance how one raises a diver from large depths. One has to consider the large pressure differences inside the divers body can cause injuries, decompression sickness, or the diver could even explode. We would like to determine a mathematical model for the diver, when the diver is lifted by an outer force, $F_{\text{lift}}(t)$, from a given depth.
Let $h(t)$ denote the diver's depth, m diver's mass, v diver's volume. Assume for simplicity the the density of the water ρ is constant.
The following relationships holds:

- The lifting force from the water is $g(\rho v - m)$.
- The friction force in water is proportional to the divers velocity.

An important physiological variable is the average internal pressure in the diver's tissues. Let $p(t)$ denote this pressure relative to the atmospheric pressure at sea level. The underwater pressure at depth h is ρh relative to atmospheric pressure. Then the following equation for the pressure $p(t)$ holds:

$$\frac{dp(t)}{dt} = k(\rho g h(t) - p(t))$$

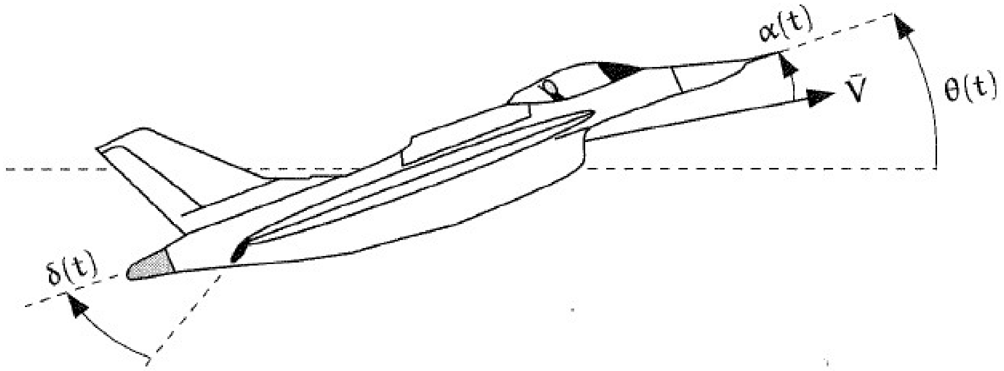
where k is a physiologic constant and g is the gravitational acceleration. Another important variable is the pressure difference between the bodies tissues and the surrounding pressure, i.e

$$q(t) = p(t) - \rho g h(t)$$

(a) Determine a mathematical model for the above stated problem, with $F_{\text{lift}}(t)$ as input signal and $q(t)$ as output signal.

(b) State a demand on the lifting force $F_{\text{lift}}(t)$ so that there exists a stationary pointy. Determine this stationary point.

Exercise 2.7 (Airplane [3])
An airplane's movement in pitch direction is described by the rudder angle $\delta(t)$ and by the angles $\alpha(t)$ and $\theta(t)$, the angles are defined in the figure below



where $\bar{V}$ is the velocity vector. For small angles $\alpha(t)$ and $\theta(t)$ the following approximative relationships holds

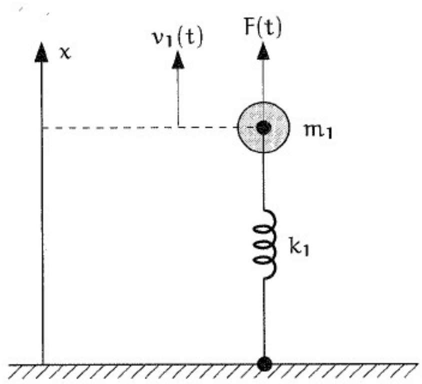
$$\frac{mV}{Sq}\dot{\alpha}(t) - C_1\alpha(t) - \frac{mV}{Sq}\dot{\theta}(t) = C_2\delta(t)$$

$$-\frac{c}{2V}C_3\dot{\alpha}(t) - C_4\alpha(t) + \frac{I_y}{Sqc}\ddot{\theta}(t) - \frac{c}{2V}C_5\dot{\theta}(t) = C_6\delta(t)$$

where V is the absolute value of $\bar{V}$, m is the mass of the airplane, S is the wing area, q is the dynamic pressure, I_y is the inertia of the airplane and c is the mean of the wing span. Furthermore is C_1, $C_2, \ldots, C_6$ so called aerodynamical derivatives (dimensionless). Introduce suitable state variables and make a state space representation of the system with $\delta(t)$ as input signal.

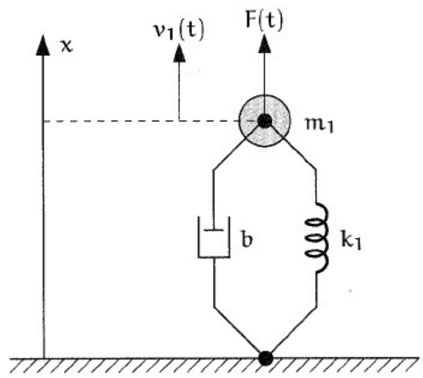
Exercise 2.8 (Car suspension [3])
We will now gradually build up a model of a car suspension system.

(a) A first simple approximation is given by the following mechanism.

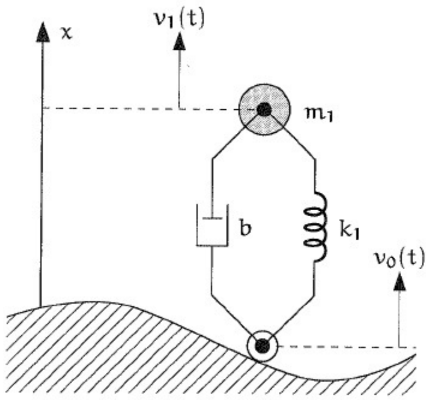


Here is m_1 the car's mass and the spring with spring constant k_1 represents the suspension. The mass-spring system is the lower end attached to the reference plane. Model the system with the force $F(t)$ is input signal and the velocity $v_1(t)$ as output signal. Take the gravitational force g in consideration. (Hint: Use the three phase method.)

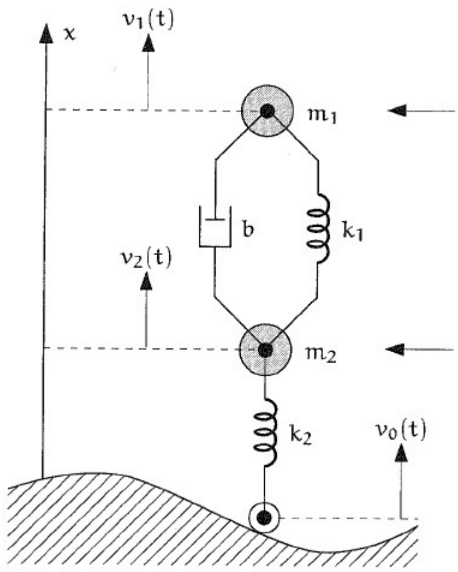
(b) The mechanism is now extended with a damper b. Start from a) and determine a model for the modified system.



(c) The next step is to add a friction less wheel to the lower end of the system from b), according to the figure below. By doing this it is possible to move the whole system horizontally over the surface (the roadway), the profile of the roadway is given by $v_0(t)$. Model the new system, let $v_0(t)$ be input signal and $v_1(t)$ be output signal.



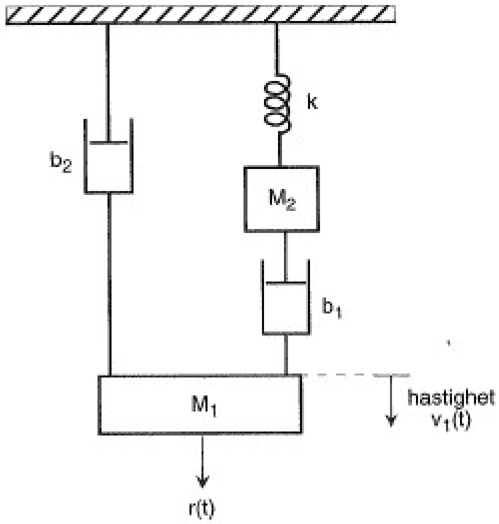
(d) Under the assumption that the wheel's dynamics is neglected, gives the model a good representation of a vehicles movement in vertical direction when travelling over a surface described by $v_0(t)$. The model can be even further improved by taking care of the wheel's vertical dynamics. This can be done by introducing another mass-spring system representing the wheel.



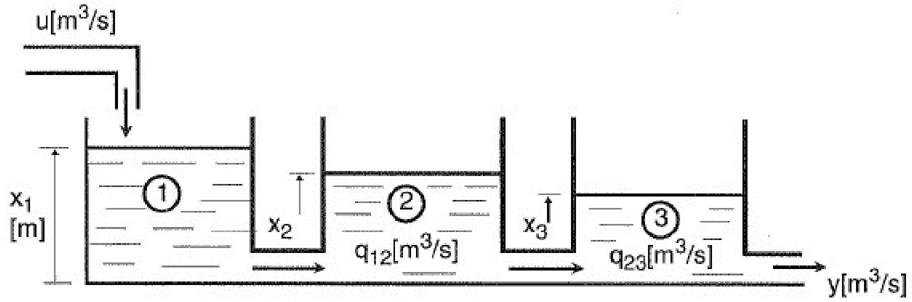
where m_1 is the mass of the car, m_2 is the mass of the wheel, k_1 is the suspension's spring constant, b is the suspensions damping and k_2 is the wheel's elasticity. Model the complete system, and again let $v_0(t)$ be input signal and $v_1(t)$ be output signal.

Exercise 2.9 (Two masses [3])
A mechanical system consists of two masses M_1 and M_2, which are connected together according the figure below with two dampers (damping constants b_1 and b_2) and a spring (with

spring constant k). Model the system on state space form, with the force $r(t)$ as input signal and the velocity $v_1(t)$ be output signal.



Exercise 2.10 (Water tanks [3])
Three water tanks with area $a = 0.1$ m^2 are connected together with pipes of the same dimensions. The inflow to the system u [m^3/s] is to tank 1 and the outflow from the system y [m^3/s] is from tank 3. The flows q_{12}, q_{23} and y are assumed to be proportional to the respectively pressure difference with proportional constant $K = 0.001$ m^3/s per meter water pillar.

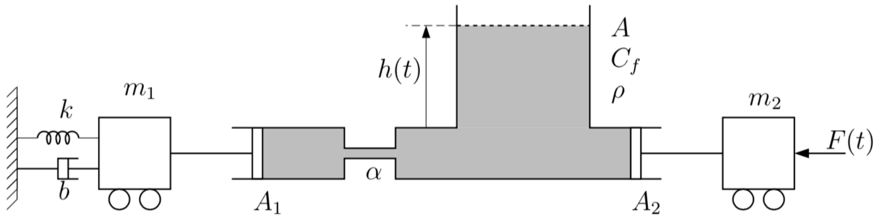


(a) Make a state space representation of the tank system, which has the tank level in number order as state variables and the outflow as output signal.

(b) Which are the stationary levels in the three tanks with a constant inflow of $u = 10^{-4}$ m^3/s?

(c) Determine the characteristic equation for the system, for determining the system's eigenvalues. You do not need to solve the equation!

Exercise 2.11 (Level control system)
Consider the level control system depicted below. The aim of the system is to control the tank level $h(t)$ using the force $F(t)$.



The left mass is connected to a wall via a spring constant k and a viscous damping b. The cross sectional areas of the the tank and pipes are A and A_1, A_2, respectively. The tank has the flow capacitance $C_f\,[m^4s^2/kg]$, and the density of the fluid is ρ. The pressure drop p_s over the pipe connecting the left cylinder with the tank is given by

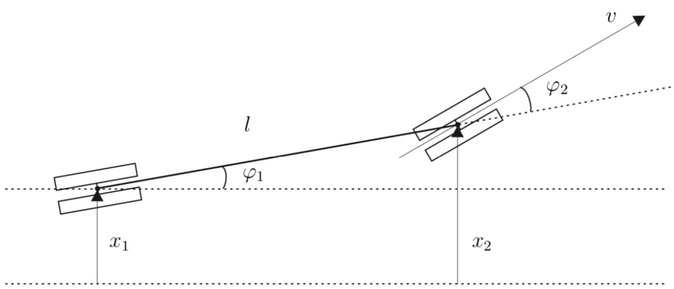
$$p_s(t) = \alpha Q_s(t),$$

where $Q_s(t)$ is the flow through the pipe.

(a) Specify the input and output of the system and suggest a choice of state variables.

(b) Determine a state space model of the system.

Exercise 2.12 (Bicycle model of a car)
The figure below illustrates a so called *bicycle model* of a car, based on the assumption that the track width is negligible.



The car has length l[m], and only the front wheels are possible to steer. The front wheels are here assumed to be driven with constant rotational speed, which results in a speed v[m/s] in a direction φ_2[rad] relative to the length axis of the car. The steering angle φ_2 is the input to the system.

(a) Find a nonlinear state space model for the car with x_1 and x_2, the lateral distances to the wheels measured in the "y-direction" in a fixed Cartesian coordinate system, as state variables.

(b) Assuming the angles φ_1 and φ_2 are small, determine a linear state model for the system and find the transfer function from φ_2 to x_2.

Exercise 2.13 (Submarine propulsion)
An unmanned underwater vessel is propelled by a an electric motor. The relation between the rotational speed N [rpm] and the motor voltage u [V] is given by the differential equation

$$J\frac{dN(t)}{dt} + M(N(t)) = \frac{K_T}{R}\big(u(t) - K_E N(t)\big), \quad M(N) = K_G N^2$$

where J is the total moment of inertia for the propeller and the rotor of the electric machine, R is the resistance in the motor circuit, and K_T and K_E are motor constants. The load torque is given by the function $M(N)$, which is quadratic in N.

(a) Which constant voltage u_0 is required to give the constant rotational speed N_0?

(b) Linearize the model around the operating point (u_0, N_0) in (a).

(c) What time constant does the linearized model have? How does it depend on the speed N_0?

Exercise 2.14 (Nonlinear circuit [4])
Consider an electric circuit with a voltage source u connected to a resistor and a capacitor C in series.

(a) Assume the resistor is linear with a resistance R_1. Derive a differential equation, describing the circuit.

(b) Assume now the resistor is nonlinear and is described by $u_R = R_1 i + R_2 i^5$, where u_R is the voltage over the resistor, and i is the current through the circuit. Formulate a model of the circuit.

Exercise 2.15 (Tikhonov theorem)
Consider the differential equation:

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \|\boldsymbol{x}\|^2 \\
\varepsilon \dot{\boldsymbol{z}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \boldsymbol{x}
\end{aligned}$$

Can we apply the Tikhonov theorem to approximate this model as a DAE? Justify!

Exercise 2.16 (Cart with pendulum)
Verify the statement in Example 2.9 in the Lecture Notes, namely that the model for a cart with pendulum, given by

$$\begin{bmatrix} M+m & -ml\cos\theta \\ -ml\cos\theta & J+mL^2 \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \ddot{\theta} \end{bmatrix}+\begin{bmatrix} ml\sin\theta\cdot\dot{\theta}^2 \\ -mgl\sin\theta \end{bmatrix}=\begin{bmatrix} F \\ 0 \end{bmatrix},$$

is transformed into the model

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}\tau^2}-\alpha\cos\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}+\alpha\sin\theta\Big(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\Big)^2 &= u, \qquad \alpha=\frac{m}{M+m} \\ -\beta\cos\theta\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2}+\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}-\sin\theta &= 0, \qquad \beta=\frac{ml^2}{J+ml^2} \end{aligned}$$

after introduction of the normalized (scaled) variables

$$\begin{aligned} \tau &= \omega_0 t \\ x &= \frac{1}{l}q \\ u &= \frac{1}{(M+m)l\omega_0^2}F \end{aligned}$$

Exercise 2.17 (Oscillator [1])
Consider a damped spring–mass system with dynamics

$$m\ddot{q}+c\dot{q}+kq=F.$$

Let $\omega_0=\sqrt{k/m}$ be the natural frequency and $\zeta=c/(2\sqrt{km})$ be the damping ratio.

(a) Show that by rescaling the equations, we can write the dynamics in the form

$$\ddot{q}+2\zeta\omega_0\dot{q}+\omega_0^2q=\omega_0^2u,$$

where $u=F/k$. This form of the dynamics is that of a linear oscillator with natural frequency ω_0 and damping ratio ζ.

(b) Show that the system can be further normalized and written in the form

$$\frac{dz_1}{d\tau}=z_2,\quad \frac{dz_2}{d\tau}=-z_1-2\zeta z_2+v.$$

The essential dynamics of the system are governed by a single damping parameter ζ. The *Q-value*, defined as $Q=1/2\zeta$, is sometimes used instead of ζ.

Exercise 2.18 (Modelica model)
Which subsystem/component is described by the Modelica code below?

```
model Component
  "Mechanical translational component"
  extends Interfaces.Compliant;
  parameter SI.Position s_rel0=0 "unstretched spring length";
  parameter Real c = 1 "spring constant";
  parameter Real d = 1 "damping constant";
  SI.Velocity v_rel "relative velocity between flange L and R";
equation
  v_rel = der(s_rel);
  f = c*(s_rel - s_rel0) + d*v_rel;
end Component;

partial model Compliant
  "Compliant connection of two translational 1D flanges"
  Flange_a flange_a
    "(left) driving flange (flange axis directed INTO cut plane)"
  Flange_b flange_b
    "(right) driven flange (flange axis directed OUT OF cut plane)";
  SI.Distance s_rel "relative distance (= flange_b.s - flange_a.s)";
  flow SI.Force f
    "force between flanges (positive in direction of flange axis R)";
equation
  s_rel = flange_b.s - flange_a.s;
  flange_b.f = f;
  flange_a.f = -f;
end Compliant;
```

Chapter 3

Lagrange modelling

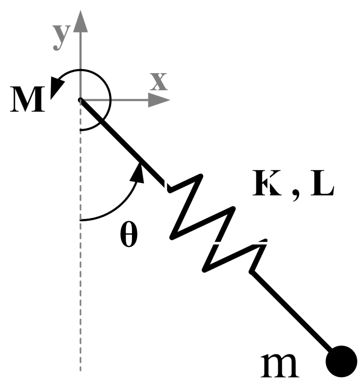
Exercise 3.1 (Flexible pendulum)

Derive the equations of motion of the system in the following figure, using

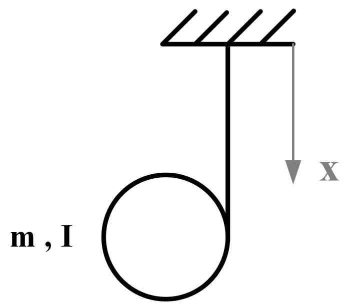
(a) the coordinate θ and L

(b) the coordinate x and y.

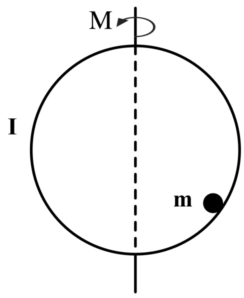
The external force is the torque M applied at the joint.



Exercise 3.2 (Drum on a cable)
Consider a drum with a cable of negligible mass around on it. The drum is a cylinder of mass m and intertia I. The cable end is attached to a fixed point to the ceiling, and the drum moves down while rolling out on cable. Write the model of this system using Lagrange's equations.



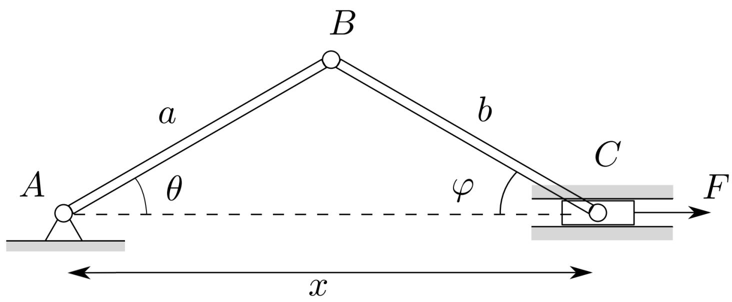
Exercise 3.3 (Ring and ball)
Consider the system sketched below. The ring has an inertia I and the ball is of mass m (sliding without friction). A torque M is applied to the vertical axis of the ring. Determine Lagrange's equations in minimum coordinates (two angles).



Exercise 3.4 (Crankshaft)
The crankshaft AB is in the arbitrary position which can be described using x or θ. The only external force is the force F applied at the piston C.

(a) Determine the generalized force Q_x if x is generalized coordinate.

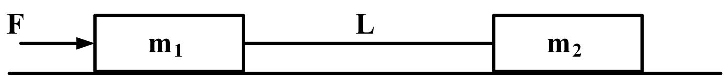
(b) Determine the generalized force Q_θ if θ is the generalized coordinate.



Exercise 3.5 (Two sliding masses)
Two masses m_1 and m_2 are positioned on a surface. They can slide freely, but they are connected with a rigid rod of length L.

(a) Derive the equation of the motion of the system using the Newton's Laws.

(b) Derive the equation of the motion of the system using the Lagrange's equations.

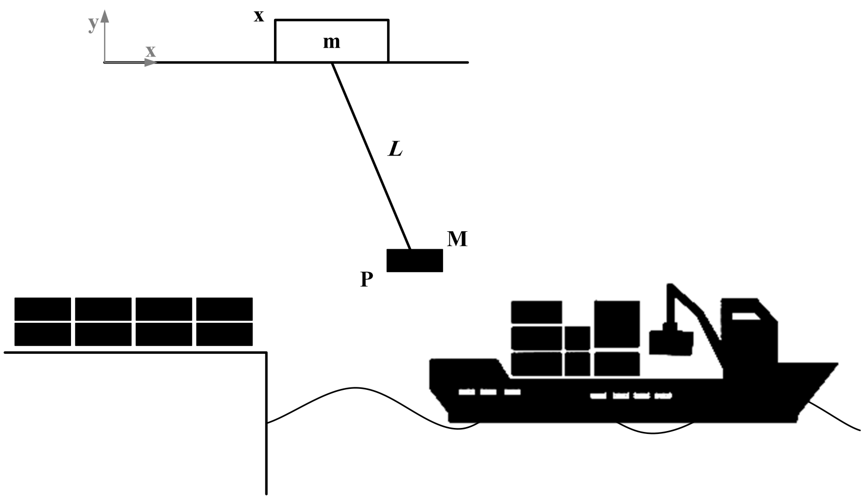


Exercise 3.6 (Harbor crane)
Harbor cranes are cranes with a rigid link of varying length $L(t)$. They operate in "swing" mode for speed.

(a) Write down the Lagrange function and model equation for the harbor crane.

(b) How is the cable length controlled?

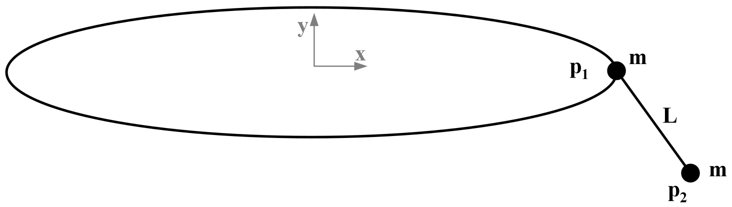
(c) What is the meaning of finding $z > 0$ and $z < 0$ when solving the model equation?



Exercise 3.7 (Strange crane)
Consider the “strange crane” showed in the following figure, where the rail is an ellipse, described by $\frac{1}{2}(\boldsymbol{p}_1^T A \boldsymbol{p}_1 - 1) = 0$ with A symmetric positive definite. The masses (both m) are linked by a rigid link of length L.

(a) Write the Lagrange function.

(b) Write the model equation for this system in terms of $\boldsymbol{q} = \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \end{bmatrix}$ (keep it simple).



Exercise 3.8 (Ball on a rail)
Consider the mechanical system depicted below. The ball A with mass m has the position (x_1, x_2), where x_1 is the horizontal and x_2 the vertical coordinate. The ball is gliding without friction along a rail that is described by the relation $x_2 = h(x_1)$. Further, the ball A is attached to one end of a spring, having the spring constant k. The other end of the spring (B) is gliding without friction along a vertical rail, so that the spring is always horizontal.
The forces acting on A are thus the spring force (assuming the neutral position of the force corresponds to $x_1 = 0$), gravity g, and the normal force $N = (N_1, N_2)$ from the rail.
In this problem, we will apply two different techniques to develop models for the system.

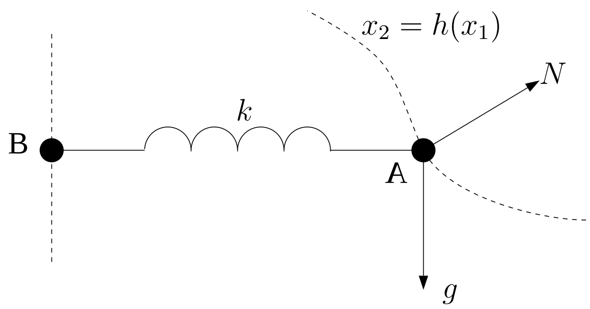
(a) Use the relation $x_2 = h(x_1)$ to find a relation between N_1 and N_2.

(b) Apply Newton's second law of motion to find a DAE in the variables $x_1, x_2, v_1, v_2, N_1, N_2$, describing the motion of A. Here v_1, v_2 are the horizontal and vertical velocities, respectively.

(c) What is the index of this DAE?

(d) In order to instead investigate the Euler-Lagrange approach, determine the Lagrange function for the system and derive a dynamic model in DAE form. Verify that only two independent initial conditions can be specified for the model.

(e) From the DAE derived in the previous step, derive a standard state-space (ODE) model of the system.
Hint: Use the constraint equation to make substitutions.



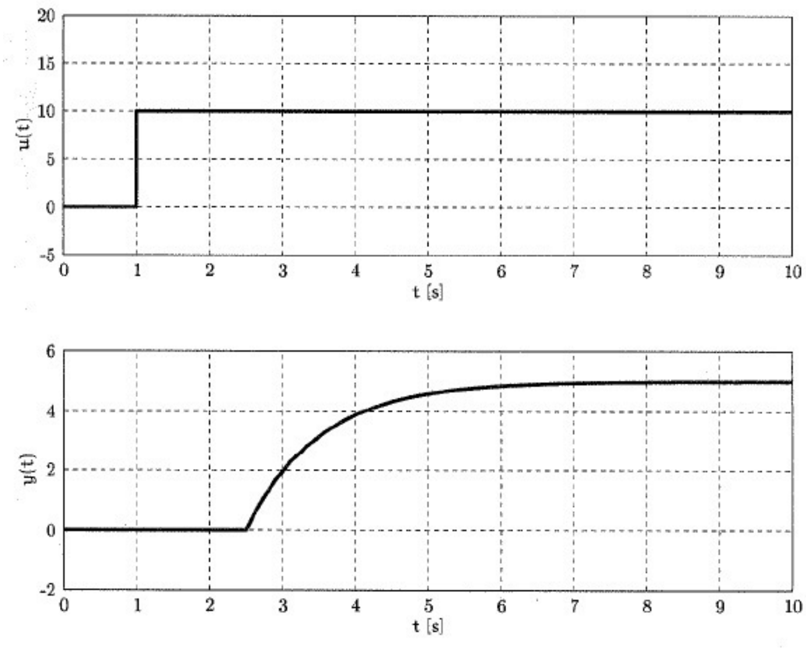
Chapter 4

System identification

Exercise 4.1 (Step response analysis [3])
The figures below show the input signal $u(t)$ and the output signal $y(t)$ for a system with transfer function

$$G(s) = A\frac{e^{-sT_1}}{1+sT_2}$$

Which values on T_1, T_2 and A can be determined from this experiment?



Exercise 4.2 (Frequency response analysis [3])
The input signal

$$u(t) = \sin(2t)$$

is applied to the linear system $G(s) = b/(s+a)$. After all transients have faded out, the output signal is

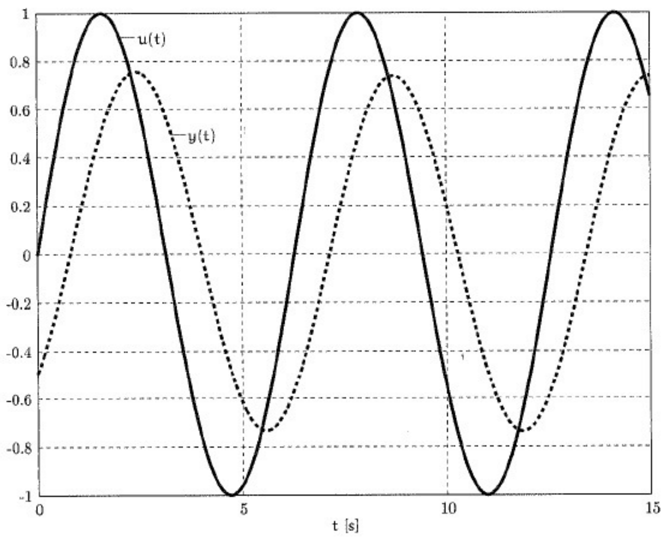
$$y(t) = 0.5\sin(2t-1)$$

Which values on a and b can be identified from this experiment?

Exercise 4.3 (Frequency response analysis [3])
The figure below shows the input and output signals for a model of the class

$$G(s) = \frac{b_1 s + b_2}{(s+a_1)(s+a_2)}$$

Is it possible to determine the parameters b_1, b_2, a_1 and a_2 from the information given in the figure? If not, give additional information which is needed in order to be able to determine the parameters.



Exercise 4.4 (Prediction [3])
A second order system given on discrete time form as

$$y(t) + a_1 y(t-1) + a_2 y(t-2) = b_1 u(t-1) + b_2 u(t-2)$$

During a step response experiment, i.e.

$$\begin{cases} u(t) = 0 & t < 0 \\ y(t) = 0 & t \le 0 \\ u(t) = 1 & t \ge 0 \end{cases}$$

the output signal sequence becomes

t	1	2	3	4
y(t)	1.0	3.0	5.3	7.35

Which values will the predictions $\hat{y}(5|4)$ and $\hat{y}(6|4)$ have?

Exercise 4.5 (From predictor to model [5])
Suppose a one-step-ahead predictor is given as

$$\hat{y}(t|t-1) = L_1(q)u(t-1) + L_2(q)y(t-1).$$

Calculate the system description of the type $y(t) = G(q)u(t) + H(q)e(t)$, from which this predictor was derived.

Exercise 4.6 (Parameter estimation [3])
A certain physical system can be described within the model structure

$$\dot{y}(t) = -\theta y^3(t) - y(t) + 5u(t) \tag{4.1}$$

The parameter θ is unknown and is to be determined from an experiment. Therefore, is the signal $\tilde{u}(t)$ used as input to the true system and the measurements $\tilde{y}(t_i)$, $i = 1,2,\ldots,N$ are given. After this the criteria

$$V_N(\theta) = \frac{1}{2N}\sum_{i=1}^{N}(\tilde{y}(t_i) - \hat{y}(t_i,\theta))^2$$

is calculated. In the criteria $\hat{y}(t_i,\theta)$ is the solution to (4.1) when $u(t) = \tilde{u}(t)$. $V_N(\theta)$ is calculated for a number of θ values within the interval $0 \le \theta \le 1$ (θ is known to exist in this interval) and it is found the the loss function can be approximated as

$$V_N(\theta) = \theta^3 + 1.50\theta^2 - 2.25\theta + 18.92$$

What value of θ should be chosen from this experiment? Motivate!

Exercise 4.7 (Least-squares estimation [3])
Using a least squares identification experiment the parameters in the model structure

$$y(t) + ay(t-1) = bu(t-1) \tag{4.2}$$

are to be identified. The following sums have been calculated:

$$\sum_{t=1}^{101} y^2(t) = 5.0 \qquad \sum_{t=1}^{101} y(t)u(t) = 1.0 \qquad \sum_{t=1}^{101} u^2(t) = 1.0$$
$$\sum_{t=2}^{102} y(t-1)y(t) = 4.5 \qquad \sum_{t=2}^{102} u(t-1)y(t) = 1.0$$

What values on $\theta^T = [a\ b]$ minimizes the quadratic criteria

$$V_N(\theta) = \frac{1}{2N}\sum_{i=1}^{N}(y(t_i) - \hat{y}(t_i,\theta))^2$$

where $\hat{y}(t,\theta)$ is the predictor given by (4.2)?

Exercise 4.8 (Weighted least-squares estimation)
Using notation from the Lecture Notes, the least squares method applied to a linear regression is based on the criterion

$$V_N(\theta) = \frac{1}{2}\|\boldsymbol{y} - \Phi\boldsymbol{\theta}\|^2 = \frac{1}{2}(\boldsymbol{y} - \Phi\boldsymbol{\theta})^\top(\boldsymbol{y} - \Phi\boldsymbol{\theta}) \tag{4.3}$$

and results in the LS estimate

$$\hat{\boldsymbol{\theta}}_N = (\Phi^\top\Phi)^{-1}\Phi^\top\boldsymbol{y}, \tag{4.4}$$

assuming the inverse exists.
The *weighted* least-squares is based on introducing a *weighted* criterion

$$V_N(\theta) = \frac{1}{2}\|\boldsymbol{y} - \Phi\boldsymbol{\theta}\|_W^2 = \frac{1}{2}(\boldsymbol{y} - \Phi\boldsymbol{\theta})^\top W(\boldsymbol{y} - \Phi\boldsymbol{\theta}), \tag{4.5}$$

where W is any symmetric, positive definite matrix.

(a) Determine the weighted least-squares estimate (WLSE).

(b) Let

$$\boldsymbol{y} = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \boldsymbol{\varphi}^\top(1) \\ \vdots \\ \boldsymbol{\varphi}^\top(N) \end{bmatrix}. \tag{4.6}$$

Determine the WLSE in the special case with $W = \mathrm{diag}(w_1,\ldots,w_n)$.

Exercise 4.9 (Linear regression with known parameter [5])
Consider the ARX model structure

$$y(t) + a_1 y(t-1) + \cdots + a_{n_a} y(t-n_a) = b_1 u(t-1) + \cdots + b_{n_b} u(t-n_b) + e(t),$$

where b_1 is known to be equal 0.5. Write the corresponding predictor in linear regression form.

Exercise 4.10 (Bias and variance of the sample mean)
Consider the following data samples

$$x[k] = A + w[k], \quad k = 1,\cdots,N \tag{4.7}$$

with $w[k]$ being white Gaussian noise, $w[k] \sim \mathcal{N}(0, \sigma^2)$. A possible estimator for the parameter A is the sample mean,

$$\hat{A} = \frac{1}{N} \sum_{k=1}^{N} x[k].$$

(a) Is $\hat{A}$ an unbiased estimate of A?

(b) How does the variance of $\hat{A}$ behave as $N \to \infty$?

(c) Use Matlab to generate data samples according to (4.7), with $A = 1$, $\sigma^2 = 0.1$. For $N = 10, 100, 200, 500, 1000$, plot the variance of the estimate, and compare it with the theoretical value given by the answer of question (b).

(d) Form an estimate of the variance σ^2 as

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{k=1}^{N} (x[k] - \hat{A})^2$$

Is this estimate unbiased? If not, suggest a modification to get an unbiased estimate.

Exercise 4.11 (Accuracy of FIR estimates)
Assume that a second order FIR model is fitted to data, coming from the system

$$y(t) = b_1 u(t-1) + b_2 u(t-2) + e(t),$$

where $e(t)$ is white noise.

(a) Verify that the covariance of the estimated parameters only depend on the input via $R_u(0)$ and $R_u(1)$, where $R_u(0) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u^2(t)$ and $R_u(1) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)u(t-1)$ (assuming these limits exist). What is the variance of $\hat{b}_1$ when $N \to \infty$?

(b) Suppose $R_u(0) = 1$, i.e. the variance of the input is fixed to 1. However, apart from this, the input can be chosen arbitrarily. How would you choose the input to minimize the variance of $\hat{b}_1$?

Exercise 4.12 (Parameter estimate convergence [3])
A system is given as

$$y(t) = u(t-1) + 0.5u(t-2) + e(t)$$

where $\{e(t)\}$ is white noise with variance 1. To describe the system, the following model structure is assigned

$$y(t) = bu(t-1) + e(t)$$

To what value will the least squares estimate of b converge, when the number of observations goes to infinity and

(a) $\{u(t)\}$ is white noise with variance 1?

(b) $\{u(t)\}$ has covariance function

$$
\begin{aligned}
&R_u(0) = 1,\\
&R_u(1) = 0.5,\\
&R_u(2) = 0.25,\\
&R_u(3) = 0.125,\\
&\text{and so on?}
\end{aligned}
$$

The input signal can be assumed to be independent of the disturbance.

Exercise 4.13 (Bias, variance and mean square error [6])
Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two different estimates of the same parameter θ. With N denoting the number of data points, the bias and variance of the estimates are

$$
\begin{aligned}
\text{Bias:} \quad & \mathbb{E}[\hat{\theta}_i] = \begin{cases} 1/N, & i = 1 \\ 0, & i = 2 \end{cases} \\
\text{Variance:} \quad & \mathbb{E}[(\hat{\theta}_i - \mathbb{E}[\hat{\theta}_i])^2] = \begin{cases} 1/N, & i = 1 \\ 3/N, & i = 2 \end{cases}
\end{aligned}
$$

Which of the two estimates is best in terms of the *mean square error (MSE)*, which is defined as $\mathbb{E}[(\hat{\theta} - \theta)^2]$? What are your comments to the result?

Exercise 4.14 (Estimation of ARX model)
Suppose you have had an ARX model estimated from data. The polynomial coefficients (first row) and their standard deviations (second row) are:

$$
B = \begin{bmatrix} 0 & 0 & 0 & 0.0664 & 0.0610 & 0.0209 \\ 0 & 0 & 0 & 0.0016 & 0.0034 & 0.0043 \end{bmatrix}
$$

$$
A = \begin{bmatrix} 1.0000 & -0.9855 & 0.0858 & 0.0439 & 0.0209 \\ 0 & 0.0452 & 0.0626 & 0.0057 & 0.0228 \end{bmatrix}
$$

Assume that other validation tests are satisfying. What is your next move? Why?

Exercise 4.15 (Variance of the variance)
A set of IID data samples is available, $\{x[1], \cdots, x[N]\}$, with $x[k] \sim \mathcal{N}(0, \sigma^2)$. An estimation of the variance σ^2 is required. A possible estimator is

$$
\hat{\sigma}^2 = \frac{1}{N} \sum_{k=1}^{N} x^2[k]
$$

(a) Is $\hat{\sigma}^2$ an unbiased estimate of σ^2?

(b) How does the variance of $\hat{\sigma}^2$ behave as $N \to \infty$?
Hint: For $x \sim \mathcal{N}(0, \sigma^2)$, $\mathbb{E}[x^4] = 3\sigma^4$.

Exercise 4.16 (Identifiability in closed-loop [3])
The parameters a and b in the system

$$y(t) + ay(t-1) = bu(t-1) + v(t)$$

are to be estimated. The system is controlled by a proportional controller $u(t) = -k_p y(t)$. Can the parameters a and b be uniquely determined by some identification method? In such case, which method? What happens if the controller is changed to $u(t) = -k_p y(t-1)$?

Exercise 4.17 (Bias of a least squares estimate)
Consider the model

$$\hat{y}(t) = \theta^{1/2} u(t), \quad t = 1, \dots, N \tag{4.8}$$

and the measurements generated using the true parameter θ_0

$$y(t) = \theta_0^{1/2} u(t) + e(t), \quad t = 1, \dots, N, \tag{4.9}$$

where $\{e(t)\}$ is a sequence of i.i.d. random variables with mean 0 and variance σ^2.

(a) Formulate and solve the least squares problem, providing the estimate $\hat{\theta}_N$ as a function of the data $\{u(t), y(t)\}, t = 1, \dots N$.

(b) Compute the bias of the estimate, assuming data is generated according to (4.9).

(c) Is the estimate consistent?

Exercise 4.18 (From state-space model to ARMA [5])
Consider the state-space model

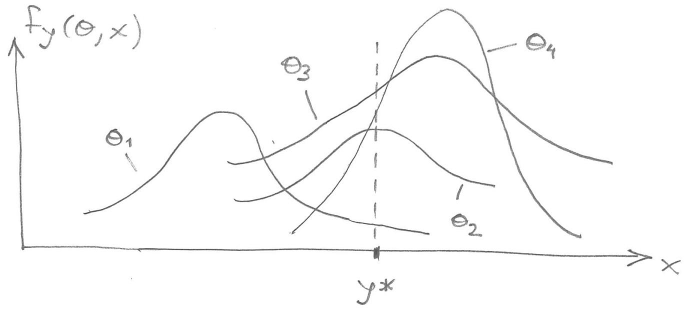
$$\begin{aligned} x_{k+1} &= f x_k + w_k \\ y_k &= h x_k + v_k, \end{aligned}$$

where x, f, h, w, and v are scalars. $\{w_k\}$ and $\{v_k\}$ are mutually independent white Gaussian noise noises with variances R_1 and R_2, respectively. Show that y_k can be represented as an ARMA process:

$$y_k + a_1 y_{k-1} + \cdots + a_n y_{k-n} = e_k + c_1 e_{k-1} + \cdots + c_n e_{k-n}$$

Determine n, a_i, c_i and the variances of e_k in terms of f, h, R_1, R_2. What is the relationship between e_k, w_k, and v_k?

Exercise 4.19 (Maximum likelihood estimation)
The likelihood function $f_y(\theta, x)$ is shown in the figure below for four different values of θ. Given the observation $y*$, which value of θ is an ML estimate?



Exercise 4.20 (ML estimation from single measurement)
Consider a single measurement $y \in \mathbb{R}$, used to estimate a single parameter $\theta \in \mathbb{R}$, based on the underlying model

$$y = \theta \cdot u + e, \tag{4.10}$$

where $u \in \mathbb{R}$ is a given (single) input, and the random variable e has a normal centred distribution, i.e. $e \sim \mathcal{N}(0, \sigma)$,
Determine the likelihood function and compute the ML estimate of θ.

Exercise 4.21 (ML estimation of a scalar quantity)
Consider the basic problem of taking N measurements $y(i)$, $i = 1, \ldots, N$ of a scalar quantity θ (e.g. the temperature in a room) in order to reduce the risk of having an erroneous, single measurement. Stacking the N measurements in the vector $\boldsymbol{y}$, the model can be written as

$$\boldsymbol{y} = \boldsymbol{\theta} + \boldsymbol{e}, \quad \boldsymbol{\theta} = \begin{bmatrix} \theta & \cdots & \theta \end{bmatrix}^\top, \tag{4.11}$$

where it is assumed that

$$\boldsymbol{e} \sim \mathcal{N}(0, \Sigma) \tag{4.12}$$

with a diagonal covariance matrix $\Sigma = \operatorname{diag}(\sigma_i^2)$, i.e. the noise is assumed to be uncorrelated. Determine the ML estimate of θ.

Exercise 4.22 (ML estimation of an input affine model)
Consider the following model, being *affine* in the input u:

$$y(t) = \theta_1 u(t) + \theta_0 + e(t), \quad t = 1, \ldots, N \tag{4.13}$$

The noise sequence $\{e(t)\}$ is assumed to be uncorrelated and Gaussian with mean 0 and variance σ_t^2.
Determine a system of linear equations that defines the ML estimates of θ_0 and θ_1.

Exercise 4.23 (Maximum likelihood of a Rayleigh distribution)
Consider the following data samples

$$y[k] = \theta u[k] + e[k], \quad k = 0, \cdots, N-1 \tag{4.14}$$

where $e[k]$ is white noise distribute according to the Rayleigh distribution,

$$f(e[k]) = \frac{1}{\sigma^2} e[k] \mathrm{e}^{-\frac{1}{2\sigma^2} e^2[k]}. \tag{4.15}$$

(a) Write the likelihood function of the outputs $y[k]$ as a function of the parameter θ.

(b) Using the likelihood function, write the estimation problem for finding θ. How does it relate with the least squares estimation?

(c) Intuitively, try to understand the role of the input $u[k]$ in the estimation of θ. Explain it.

Exercise 4.24 (From the penalty to the noise distribution)
A fitting problem is using the penalty

$$\min_{\theta} \quad \sum_{k=0}^{N-1} \frac{1}{2} \frac{e^2[k]}{1 + |e[k]|}, \tag{4.16}$$

where $e[k]$ is the error between the model and the measurements, $e[k] = y[k] - y^m[k]$.

(a) Assuming that the penalty is derived from a likelihood function, what noise distribution does the penalty correspond to? You can use Matlab to plot the penalty and the distribution.

(b) This penalty is called "fair" in the context of robust fitting. Can you give an intuition of what does it mean?

Exercise 4.25 (MLE for an exponential distribution)
Consider the system

$$y_k = -a y_k + b u_{k-1} + v_k,$$

where $\{v_k\}$ is a sequence of independent, identically distributed random variables, each with the probablioty density function $f(x) = \mu e^{-\mu x}$, $x \geq 0$. Design a maximum-likelihood method that permits estimation of a and b. What is the difference in algorithmic complexity for an unknown μ and for a known value of μ?

Chapter 5

Newton method

Exercise 5.1 (Computer code for basic Newton)
In this exercise, we will write a Matlab code that deploys a basic Newton iteration. You will reuse this code later in the course, so make it clean in the first place.

(a) Write a generic Matlab code where you declare a vector of variables $\boldsymbol{\theta}$ and a function $\boldsymbol{r}(\boldsymbol{\theta})$ symbolically. Compute the Jacobian $\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\theta}}$, and export the function and its Jacobian as a Matlab function. You can do that using a syntax of the type:

$$\text{matlabFunction}(\boldsymbol{r}, \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\theta}}, \text{"file"}, \text{"}\textit{choose your function name}\text{"}, \text{"vars"}, \{\boldsymbol{\theta}\});$$

Please consult "help matlabFunction" for help!!

(b) Write a Newton iteration that deploys:

Algorithm: Basic Newton iteration

Input: $\boldsymbol{\theta}$, tol, $\alpha \in]0,1]$
while $\|\boldsymbol{r}(\boldsymbol{\theta})\| \geq$ tol **do**
  Evaluate (using your exported function)

$$\boldsymbol{r}(\boldsymbol{\theta}) \quad \text{and} \quad \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\theta}}$$

  Compute the Newton direction $\Delta\boldsymbol{\theta}$ using:

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\theta}} \Delta\boldsymbol{\theta} = -\boldsymbol{r}(\boldsymbol{\theta})$$

  Newton step

$$\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha \Delta\boldsymbol{\theta}$$

return $\boldsymbol{\theta}$

Store $\|\boldsymbol{r}(\boldsymbol{\theta})\|$ over the iterations. At the end of the iterative procedure, plot $\|\boldsymbol{r}(\boldsymbol{\theta})\|$ in a semi-log plot. You can use the command:

semilogy([1 : niter],*store* $\|\boldsymbol{r}(\boldsymbol{\theta})\|$, "linestyle", "none", "marker", ".", "color", "k", "markersize", 15); gridon

where niter is the number of iterations your iteration took.

(c) Test your code first on the function $\boldsymbol{r}(\boldsymbol{\theta}) = 3\boldsymbol{\theta}$ where $\boldsymbol{\theta}, \boldsymbol{r} \in \mathbb{R}$. Use tol $= 10^{-6}$, the initial guess $\boldsymbol{\theta} = 1$ and full steps (i.e. $\alpha = 1$), What do you observe? Why?

(d) Test your code on the function $\boldsymbol{r}(\boldsymbol{\theta}) = \boldsymbol{\theta}^3 - 1$ where $\boldsymbol{\theta}, \boldsymbol{r} \in \mathbb{R}$.

1. Use tol $= 10^{-6}$, the initial guess $\boldsymbol{\theta} = 2$ and full steps (i.e. $\alpha = 1$), What do you observe?
2. Use tol $= 10^{-6}$, the initial guess $\boldsymbol{\theta} = 0$ and full steps (i.e. $\alpha = 1$), What do you observe? Why?

(e) Test your code on the function $\boldsymbol{r}(\boldsymbol{\theta}) = \tanh(\boldsymbol{\theta})$ where $\boldsymbol{\theta}, \boldsymbol{r} \in \mathbb{R}$.

1. Use tol $= 10^{-6}$, the initial guess $\boldsymbol{\theta} = 1$ and full steps (i.e. $\alpha = 1$), What do you observe?
2. Use tol $= 10^{-6}$, the initial guess $\boldsymbol{\theta} = 2$ and full steps (i.e. $\alpha = 1$), What do you observe? Why?
3. Use tol $= 10^{-6}$, the initial guess $\boldsymbol{\theta} = 2$ and reduced steps (e.g. $\alpha = 0.25$), What do you observe? Why?

Exercise 5.2 (Newton code for optimization)
Adapt your code from Exercise 5.1 to find the solution to the optimization problems:

$$\min_{\boldsymbol{\theta}} \quad \phi(\boldsymbol{\theta}) \tag{5.1}$$

What function $\boldsymbol{r}(\boldsymbol{\theta})$ do you need to choose?

(a) Test your code on the cost function

$$\phi = \frac{1}{2}(\boldsymbol{\theta} - 1)^2 \tag{5.2}$$

where $\boldsymbol{\theta} \in \mathbb{R}$. Use tol $= 10^{-6}$, the initial guess $\boldsymbol{\theta} = 2$ and full steps (i.e. $\alpha = 1$), What do you observe? Why?

(b) Test your code on the cost function

$$\phi = \boldsymbol{\theta}^2 + 5e^{-\boldsymbol{\theta}^2} + \boldsymbol{\theta} \tag{5.3}$$

where $\boldsymbol{\theta} \in \mathbb{R}$.

1. Use tol $= 10^{-6}$, the initial guess $\boldsymbol{\theta} = 2$ and full steps (i.e. $\alpha = 1$), What do you observe?
2. Use tol $= 10^{-6}$, the initial guess $\boldsymbol{\theta} = -1$ and full steps (i.e. $\alpha = 1$), What do you observe?

Why do you get a different solution? Plotting the function should help you here...

Exercise 5.3 (Newton for system identification)
You will now use your code from Exercise 5.1 for an application to system identification. The task is to fit the parameters of a model using a quadratic criterion (non-linear least-squares). The cost function is

$$J(\boldsymbol{\theta}, u, y) = \frac{1}{2}\sum_{k=1}^{N} e_k\left(\boldsymbol{\theta}, u, y_k\right)^2, \tag{5.4}$$

where $e_k\left(\boldsymbol{\theta}, u, y_k\right)$ is the error between the model and the data, i.e.

$$e_k\left(\boldsymbol{\theta}, u, y_k\right) = \hat{y}_k\left(\boldsymbol{\theta}, u\right) - y_k. \tag{5.5}$$

Let's use an output error model, where the prediction is obtained from a simulation:

$$\hat{y}_k\left(\boldsymbol{\theta}, u\right) = -a \cdot \hat{y}_{k-1}\left(\boldsymbol{\theta}, u\right) + b \cdot u_k \tag{5.6}$$

where $\theta = \begin{bmatrix} a & b \end{bmatrix}^\top$. We will assume the initial condition $\hat{y}_0 = 0$.
Note that:

$$\frac{\mathrm{d}J}{\mathrm{d}\boldsymbol{\theta}} = \sum_{k=1}^{N} e_k\left(\boldsymbol{\theta}, u, y_k\right) \frac{\mathrm{d}e_k\left(\boldsymbol{\theta}, u, y_k\right)}{\mathrm{d}\boldsymbol{\theta}} \tag{5.7}$$

and

$$\frac{\mathrm{d}e_k\left(\boldsymbol{\theta}, u, y_k\right)}{\mathrm{d}\boldsymbol{\theta}} = \frac{\mathrm{d}\hat{y}_k}{\mathrm{d}\boldsymbol{\theta}} \tag{5.8}$$

We will have to be careful here, and observe that a chain-rule applies:

$$\frac{\mathrm{d}\hat{y}_k}{\mathrm{d}\boldsymbol{\theta}} = \frac{\partial \hat{y}_k}{\partial \boldsymbol{\theta}} + \frac{\partial \hat{y}_k}{\partial \hat{y}_{k-1}} \frac{\mathrm{d}\hat{y}_{k-1}}{\mathrm{d}\boldsymbol{\theta}} \tag{5.9}$$

Moreover, we will use in the following the Hessian approximation:

$$\frac{\partial^2 J}{\partial \theta^2} \approx \frac{\mathrm{d}J}{\mathrm{d}\boldsymbol{\theta}}^\top \frac{\mathrm{d}J}{\mathrm{d}\boldsymbol{\theta}} = \sum_{k=1}^{N} \frac{\mathrm{d}\hat{y}_k}{\mathrm{d}\boldsymbol{\theta}}^\top \frac{\mathrm{d}\hat{y}_k}{\mathrm{d}\boldsymbol{\theta}} \tag{5.10}$$

and the Newton iteration for $i = 0, \ldots,$ will read as:

$$\theta_{i+1} = \theta_i - \alpha \left(\sum_{k=1}^{N} \frac{\mathrm{d}\hat{y}_k}{\mathrm{d}\boldsymbol{\theta}}^\top \frac{\mathrm{d}\hat{y}_k}{\mathrm{d}\boldsymbol{\theta}} \right)^{-1} \frac{\mathrm{d}J}{\mathrm{d}\boldsymbol{\theta}}^\top \tag{5.11}$$

We have prepared for you a partial code that execute these operations properly, see under Files in Canvas.

(a) Complete the missing lines in the code (denoted by "..." i.n the .m files).

(b) Try the code with $u_k = 0$, $e_k = 0$ for all k

(c) Try the code with the "true" parameter θ.true $= \begin{bmatrix} 0.95 & 1 \end{bmatrix}^\top$.

(d) Try the code with the initial guess $\theta_0 = \begin{bmatrix} 0.5 & 10 \end{bmatrix}^\top$.

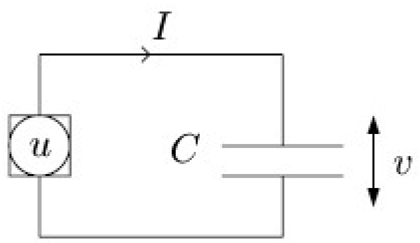
Chapter 6

Differential Algebraic Equations

Exercise 6.1 (DAE index)
What is the index for the following DAE?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} u$$

Exercise 6.2 (DAE index)
Consider the following electrical circuit, where an ideal voltage source is supplying a capacitor.



Let u be input signal and let $z = [v\ I]^T$ be generalized state vector. Determine a DAE on the form

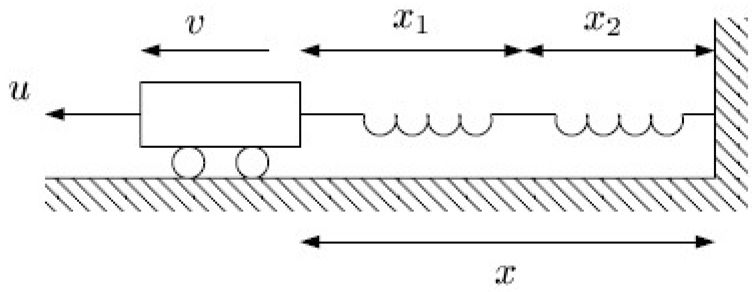
$$E\dot{z} + Fz = Gu$$

What is the index for the system?

Exercise 6.3 (Mass–spring system)
A frictionless mass with velocity v is affected by a force u as input signal; see figure below. If the forces in the springs with lengths x_1 and x_2, here denoted F_1 and F_2, then follows

$$F_1 = x_1 \qquad\qquad F_2 = \tan x_2$$

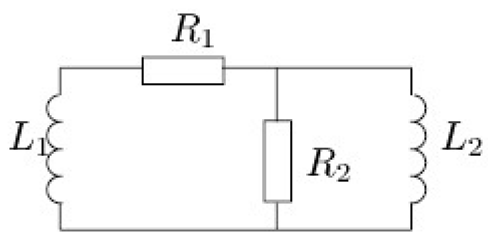
Determine a DAE in the variables x, x_1, x_2 and v.



Exercise 6.4 (Nonlinear electric circuit)
In the electrical circuit below, the resistors R_1 and R_2 are nonlinear with the relationships

$$i_1 = g_1(v_1) \qquad\qquad i_2 = g_2(v_2)$$

where i_1, i_2 are the currents through, and v_1, v_2 are the voltage over the components. Determine a DAE for the circuit. What is the index? Are there any demands on g_1 and g_2?



Exercise 6.5 (Index reduction)
Consider the DAE:

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_1 + \boldsymbol{x}_2 + z \tag{6.1a}$$

$$\dot{\boldsymbol{x}}_2 = z + u \tag{6.1b}$$

$$0 = \frac{1}{2}\left(\boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 - 1\right) \tag{6.1c}$$

(a) Why is it a DAE?

(b) What is the differential index of (6.1)?

(c) Perform an index-reduction of (6.1).

Exercise 6.6 (Index reduction)
Consider the differential equation:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \dot{\boldsymbol{x}} = \boldsymbol{x} \tag{6.2}$$

(a) Is this differential equation an implicit ODE or a DAE? Justify.

(b) Show that we can rewrite this equation in a semi-explicit form having 2 algebraic variables and one differential variable. *Hint: you need to do algebraic manipulations and time-differentiations.*

Exercise 6.7 (ODE or DAE?)
For the following differential equations, determine if they are ODE or DAEs, and if they are DAEs, specify (if possible) what are the algebraic and differential states.

(a)

$$\dot{\boldsymbol{x}}_1 + u + \boldsymbol{x}_1 + \boldsymbol{x}_2 = 0 \tag{6.3a}$$

$$u + \boldsymbol{x}_2 + \dot{\boldsymbol{x}}_2\dot{\boldsymbol{x}}_1 + \dot{\boldsymbol{x}}_2 u + \dot{\boldsymbol{x}}_2\boldsymbol{x}_1 + \dot{\boldsymbol{x}}_2\boldsymbol{x}_2 + u^2 = 0 \tag{6.3b}$$

(b)

$$u + \dot{\boldsymbol{x}}_1\boldsymbol{x}_1 + \dot{\boldsymbol{x}}_2\boldsymbol{x}_2 = 0 \tag{6.4a}$$

$$u\dot{\boldsymbol{x}}_1\boldsymbol{x}_1 + \dot{\boldsymbol{x}}_2 u\boldsymbol{x}_2 = 0 \tag{6.4b}$$

Exercise 6.8 (Fully implicit to semi-explicit?)
Consider the fully-implicit DAE:

$$\dot{x} + u + \tanh(\dot{x}) + xz = 0 \tag{6.5a}$$

$$\tanh(2u - z) = 0 \tag{6.5b}$$

where $x, z, u \in \mathbb{R}$ and tanh is the tangent hyperbolic function.

(a) Can you put (6.5) in the form of a semi-explicit DAE?

(b) Does (6.5) always provide a well-defined trajectory?

(c) What is the index of (6.5)?

Exercise 6.9 (Computer code for semi-explicit DAE)
We will write a computer code that let us treat a semi-explicit DAE

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \tag{6.6a}$$

$$0 = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \tag{6.6b}$$

like an ODE. To that end, we will need to solve (6.6b) in terms of $\boldsymbol{z}$, and then use the resulting solution in (6.6a). We can do that via the following function:

Algorithm: Make a semi-explicit DAE look like an ODE

Input: t, x, u, initial guess z, tolerance tol
while $\|\boldsymbol{g}(z, x, u)\| \geq$ tol **do**

Evaluate (using an exported function)

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \quad \text{and} \quad \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}}$$

Compute the Newton direction using:

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}} \Delta \boldsymbol{z} = \boldsymbol{g} \tag{6.7}$$

Take the Newton step

$$\boldsymbol{z} \leftarrow \boldsymbol{z} + \alpha \Delta \boldsymbol{z}$$

Evaluate $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u})$
return $\dot{\boldsymbol{x}}$

This function will then be submitted to the state-of-the-art ODE solver ode45 of Matlab (we will learn soon how this kind of solver works). We will feed in all cases the initial guess $z = 0$ (of adequate dimension).

(a) We have prepared a code template for you. A basic line-search is included to compute the step sizes α. Read the code carefully and complete the missing parts.

(b) Try the code on the semi-explicit DAE:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{z}_1^2 + \boldsymbol{x}_2 \\ \boldsymbol{z}_2 - \boldsymbol{x}_1 \end{bmatrix} \tag{6.8a}$$

$$0 = \begin{bmatrix} \boldsymbol{z}_1^3 + \boldsymbol{z}_1 + \boldsymbol{x}_2 + 1 \\ \boldsymbol{z}_2^3 + \boldsymbol{z}_2 - \boldsymbol{x}_1 - \tanh(\boldsymbol{z}_1) \end{bmatrix} \tag{6.8b}$$

Simulate from the initial conditions $\boldsymbol{x}_0 = \begin{bmatrix} 4 & 2 \end{bmatrix}^\top$ over the time interval $[0, 10]$.

(c) Try the code on the semi-explicit DAE:

$$\dot{\boldsymbol{x}}_1 = z\boldsymbol{x}_1 \tag{6.9a}$$
$$\dot{\boldsymbol{x}}_2 = z\boldsymbol{x}_2 - 1 \tag{6.9b}$$
$$0 = \boldsymbol{x}_1^2 + \boldsymbol{x}_2^3 - 1 \tag{6.9c}$$

Simulate from the initial conditions $\boldsymbol{x}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ over the time interval $[0, 10]$. What happens? Why? How can you fix this?

(d) Try the code on the semi-explicit DAE:

$$\dot{\boldsymbol{x}}_1 = 10\boldsymbol{x}_1\boldsymbol{z}_2 \tag{6.10a}$$
$$\dot{\boldsymbol{x}}_2 = -30\boldsymbol{z}_2\boldsymbol{x}_2^2 - \boldsymbol{z}_1^2 - 10 \tag{6.10b}$$
$$\dot{\boldsymbol{x}}_3 = \boldsymbol{z}_1 - 5\boldsymbol{z}_2 \tag{6.10c}$$
$$0 = \boldsymbol{x}_3 + \boldsymbol{z}_1 - 5\boldsymbol{z}_2 \tag{6.10d}$$
$$0 = 5\boldsymbol{z}_2 - \boldsymbol{z}_1 - \boldsymbol{x}_3 + 6\boldsymbol{x}_2^2\boldsymbol{z}_1^2 + 20\boldsymbol{x}_1^2\boldsymbol{z}_2 + 180\boldsymbol{x}_2^4\boldsymbol{z}_2 + \boldsymbol{x}_1^2 + 60\boldsymbol{x}_2^2 - 2\boldsymbol{x}_2^3 \tag{6.10e}$$

Simulate from the initial conditions $\boldsymbol{x}_0 = \begin{bmatrix} 4 & 2 & 0 \end{bmatrix}^\top$ over the time interval $[0, 10]$.

- What happens? Why?
- Can you draw a tentative conclusion regarding the index of DAEs?

Exercise 6.10 (Computer code for fully implicit DAE)
We will now write a computer code that allows us to treat fully implicit DAEs:

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = 0 \tag{6.11}$$

like an ODE. To that end, we will need to solve (6.11) for both $\dot{\boldsymbol{x}}$ and $\boldsymbol{z}$. We will then base our integration on the following function:

Algorithm: Make a fully implicit DAE look like an ODE

Input: t, x, u, initial guess $\dot{x}$, z, tolerance tol
while $\|\boldsymbol{F}(\dot{x}, z, x, u)\| \geq \mathrm{tol}$ **do**

Evaluate (using an exported function)

$$b = \boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{z}, \boldsymbol{x}, \boldsymbol{u}) \quad \text{and} \quad M = \begin{bmatrix} \frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}} & \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{z}} \end{bmatrix}$$

Compute the Newton direction using:

$$M \begin{bmatrix} \Delta \dot{\boldsymbol{x}} \\ \Delta \boldsymbol{z} \end{bmatrix} = -b$$

Take the Newton step

$$\dot{\boldsymbol{x}} \leftarrow \dot{\boldsymbol{x}} + \alpha \Delta \dot{\boldsymbol{x}}, \quad \text{and} \quad \boldsymbol{z} \leftarrow \boldsymbol{z} + \alpha \Delta \boldsymbol{z}$$

return $\dot{\boldsymbol{x}}$

This function will then be submitted to the state-of-the-art ODE solver ode45 of Matlab. We will feed the initial guess $\dot{x}, z = 0$ (of adequate dimension).

(a) We have prepared a code template for you. A basic line-search is included. Complete the missing bits.

(b) Try the code on the DAE:

$$F(\dot{x}, x, z, u) = \begin{bmatrix} \dot{x} + u + \tanh(\dot{x}) + xz \\ \tanh(2u - z) - z \end{bmatrix} = 0 \tag{6.12a}$$

Simulate from the initial conditions $\boldsymbol{x}_0 = 2$ over the time interval $[0, 10]$.

Chapter 7

Explicit integrators

Exercise 7.1 (Runge-Kutta 2)
Compute the one-step and global error order of RK2 with the following Butcher arrays:

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Exercise 7.2 (Runge-Kutta 2)
Write down the equation describing the region of stability of RK2.

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Exercise 7.3 (Runge-Kutta 1)
Show that the one-stage method RK1 (Butcher array shown below, left) is a special case of the two-stage RK2 (below, right) by choosing appropriate values of a, b_1 and b_2. What is the order of RK2 using the selected values of a, b_1 and b_2?

$$\begin{array}{c|c} 0 & 0 \\ \hline & d_1 \end{array} \qquad\qquad \begin{array}{c|cc} 0 & 0 & 0 \\ \mathrm{c} & \mathrm{a} & 0 \\ \hline & b_1 & b_2 \end{array}$$

Exercise 7.4 (Euler)
Consider the following initial value problem:

$$\dot{x}(t) = \frac{x(t)}{5} + 10e^{t/5}\cos(10t), \quad x(0) = 0$$

with the exact solution:

$$x(t) = e^{t/5}\sin(10t)$$

(a) Code an Euler scheme for the system.

(b) Compare the result with the exact solution.

(c) Report the global error as a function of Δt (loglog scale).

Chapter 8

Implicit integrators

Exercise 8.1 (IRK for linear dynamics)
Consider the Butcher Tableau:

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

(a) Write the equations matching the Tableau.

(b) Is it implicit or explicit?

(c) Write the pseudo code of this integrator.

(d) What becomes of the equation if the ODE is

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

(e) What will the Newton iteration do in this specific case?

(f) Can you draw a more general conclusion on using IRK method on linear dynamics?

Exercise 8.2 (IRK for semi-explicit DAE)
Consider a semi-explicit DAE

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \\ 0 &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \end{aligned}$$

where $\boldsymbol{x} \in \mathbb{R}^{n\times 1}$ and $\boldsymbol{z} \in \mathbb{R}^{m\times 1}$.

(a) Write an IRK scheme.

(b) Write r required for solving using Newton's method.

(c) Write the pseudo code.

(d) Consider the case with one stage only, i.e. $s = 1$. What is the condition to have a solution using Newton's method?

Exercise 8.3 (Integrator for Van der Pol oscillator)
Code the integrator of Exercise 8.1, and try it on Van der Pol oscillator with $u = 5$.

$$\dot{x} = y \tag{8.1a}$$

$$\dot{y} = u\left(1 - x^2\right) y - x \tag{8.1b}$$

Exercise 8.4 (Stability functions)
For the integration schemes with the given Butcher arrays, determine the stability functions and whether they are A-stable or not.

(a) Butcher array:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

(b) Butcher array:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

Exercise 8.5 (Collocation methods)
Code the procedure which builds the Butcher tableau of the IRK corresponding to the Gauss-Legendre method for any number of stages s. This procedure is detailed on page 152 of the lecture notes.

Appendix A

Solutions

A.1 System dynamics and differential equations

Solution to exercise 1.1
NB. There are many (in fact infinitely many) alternative state space forms, so the solutions below are only examples!

(a) With $\boldsymbol{x} = (y, \dot{y}, \ddot{y})$, the state space model becomes

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -2 & -5 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}$$

(b) With $(s^3 + a_1 s^2 + a_2 s + a_3) X_1(s) = U(s)$ and $X_2(s) = sX_1(s)$, $X_3(s) = sX_2(s)$, we get

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \boldsymbol{x}$$

(c) Similarly as in (b), we get

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 4 & 5 \end{bmatrix} \boldsymbol{x}$$

(d) This is a rare case with a direct (feed-through) term in the transfer function. Rewrite $G(s)$ as

$$G(s) = \frac{2s^2 + 15s + 12}{s^2 + 5s + 4} = 2 + \frac{5s + 4}{s^2 + 5s + 4},$$

from which follows that a state-space model with $D = 2$ and A, B, C as in (c) can be used.

Solution to exercise 1.2

(a) There are two poles, -1 and -4, and one zero in -0.8.

(b) The state space model is diagonal, so the eigenvalues (found from the characteristic equation $\det(sI - A) = 0$) can be seen directly: -1 and -4. Hence, the poles are the same, unless a cancellation appears. The transfer function can be calculated as

$$G(s) = \frac{-1/3}{s+1} + \frac{16/3}{s+4} = \frac{5s+4}{s^2+5s+4}$$

i.e. the same as in (a) and the poles and zero are the same.

Solution to exercise 1.3

$$\Delta\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Delta\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \Delta\boldsymbol{x}$$

Solution to exercise 1.4

$$G(s) = \frac{0.5}{s+1}$$

NB. Due to a cancellation of a pole/zero pair, the resulting transfer function is of first order, in spite of the fact that the state space model is of order 2.

Solution to exercise 1.5

(a) With $x_1 = y$ and $x_2 = x$, the model becomes

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -0.5 & 2 \\ 0 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}$$

(b) Writing $G(s) = Y(s)/U(s)$ as

$$G(s) = \frac{6}{(s+0.5)(s+2)} = \frac{4}{s+0.5} - \frac{4}{s+2}$$

suggests the diagonal form

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{bmatrix}-0.5 & 0\\ 0 & -2\end{bmatrix}\boldsymbol{x} + \begin{bmatrix}4\\ -4\end{bmatrix}u\\ y &= \begin{bmatrix}1 & 1\end{bmatrix}\boldsymbol{x}\end{aligned}$$

Since now the state variables are combinations of the original states x and y, it is not obvious how to interpret them in physical terms. This is a potential disadvantage of making state variable changes to get special realizations.

Solution to exercise 1.6

(a) With $x_1 = x$ and $x_2 = y$, the state space model becomes

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{bmatrix}-1 & 0\\ -2 & 1\end{bmatrix}\boldsymbol{x} + \begin{bmatrix}1\\ 1\end{bmatrix}u\\ y &= \begin{bmatrix}0 & 1\end{bmatrix}\boldsymbol{x}\end{aligned}$$

The transfer function is calculated as usual

$$G(s) = C(sI - A)^{-1}B = \frac{s-1}{(s+1)(s-1)} = \frac{1}{s+1}$$

(b) The characteristic polynomial is $\det(sI - A) = (s+1)(s-1)$, i.e. the eigenvalues are -1 and 1. Hence, the system is unstable. Yet, the transfer function looks stable with one pole in -1, but this is due to the pole/zero cancellation of the factor $(s-1)$.

Solution to exercise 1.7

(a) The impulse response $e^{-0.5t}\big(1 + \cos 0.5t\big)$ gives, after Laplace transformation (consult a table of Laplace transforms),

$$G(s) = \frac{1}{s+0.5} + \frac{s+0.5}{(s+0.5)^2 + 0.5^2},$$

which has the static gain $G(0) = 2 + 1 = 3$.

(b) Rewrite the transfer function with a common denominator, and approximate:

$$G(s) = \frac{1}{s+1} - \frac{s+1}{s^2+2s} = \frac{-1}{s(s+1)(s+2)} \approx \frac{-1}{2s},$$

where the approximation is valid for small ω. The gain is 1 (or 0 dB) for $\omega = 0.5$.

Solution to exercise 1.8
The closed-loop transfer function is given by

$$\frac{0.47K_p}{z - 0.43 + 0.47K_p}$$

(a) With $K_p = 0.43/0.47$, the closed-loop pole is in the origin.

(b) Larger values of K_p will give poles on the negative real axis. This will first result in an oscillatory response, and with larger values on K_p, the pole will migrate outside the unit circle, giving an unstable closed-loop.

A.2 Physical modelling

Solution to exercise 2.1

(a) Newton's second law of motion gives $m\ddot{x} = F = u - Kx$. With $x_1 = x$ and $x_2 = \dot{x}$ we get:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 \\ -K/m & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u = A\boldsymbol{x} + Bu \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x} = C\boldsymbol{x}\end{aligned}$$

(b)

$$G(s) = C(sI - A)^{-1}B = \frac{1}{ms^2 + K}$$

Solution to exercise 2.2

$$
\begin{aligned}
L\frac{di}{dt} &= -R_1 i - e + u \\
C\frac{de}{dt} &= i - e/R_2
\end{aligned}
$$

Solution to exercise 2.3

(a)

$$
\frac{di}{dt} = -\frac{R}{L}i - \frac{k}{L}i^2 + \frac{1}{L}u
$$

(b) Stationary point:

$$
i_0 = -\frac{R}{2k} + \sqrt{\frac{R^2}{4k^2} + \frac{u_0}{k}}
$$

Linearized system:

$$
\frac{d\Delta i}{dt} = -\left(\frac{R}{L} + \frac{2k}{L}i_0\right)\Delta i + \frac{1}{L}u_0
$$

Solution to exercise 2.4

(a) Kirchhoff's voltage law gives $u = Ri + L\frac{di}{dt} + K_e\omega$ and a force balance gives $J_m\ddot{\theta} = K_m i - b\omega$. With the states $x_1 = i$ and $x_2 = \omega = \dot{\theta}$ we get the state model

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{bmatrix} -R/L & -K_e/L \\ K_m/J_m & -b/J_m \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix}u \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix}\boldsymbol{x}
\end{aligned}
$$

(b) With the load added, the electric part stays the same, but the force balance is modified. In addition, a force balance for the load inertia has to be included. The mechanical equations become

$$
\begin{aligned}
J_m\ddot{\theta} &= K_m i - b\omega - K_T(\alpha - \theta) \\
J_L\ddot{\alpha} &= K_T(\alpha - \theta) - T_L
\end{aligned}
$$

and the new state space model with $x_1 = i$, $x_2 = \dot{\theta}$, $x_3 = \dot{\alpha}$, $x_4 = \theta$, and $x_5 = \alpha$ is

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -R/L & -K_e/L & 0 & 0 & 0 \\ K_m/J_m & -b/J_m & 0 & K_T/J_m & -K_T/J_M \\ 0 & 0 & 0 & -K_T/J_L & K_T/J_L \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1/L & 0 \\ 0 & 0 \\ 0 & -1/J_L \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ T_L \end{bmatrix}$$
$$y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}$$

(c) Defining $\varepsilon = L/R$, the first differentail equation in the state space model reads

$$\varepsilon \frac{dx_1}{dt} = -x_1 - \frac{K_e}{R} x_2 + \frac{1}{R} u.$$

This equation can readily be found to satisfy the conditions of Tikhonov's theorem, and by letting $\varepsilon \to 0$, we can substitute the d.e. with the algebraic equation $0 = -x_1 - \frac{K_e}{R} x_2 + \frac{1}{R} u$ or $x_1 = \frac{K_e}{R} x_2 + \frac{1}{R} u$, to be inserted in the other d.e.

Solution to exercise 2.5

(a)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{C}{mA^{\gamma-1} x_1^{\gamma}} - g - \frac{Ap_0}{m}$$

(b) Stationary point:

$$x_{20} = 0 \qquad x_{10} = \left(\frac{C}{mgA^{\gamma-1} + p_0 A^{\gamma}} \right)^{\frac{1}{\gamma}}$$

Linearized system:

$$\Delta \dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{\gamma C}{mA^{\gamma-1} x_{10}^{\gamma+1}} & 0 \end{bmatrix} \Delta \boldsymbol{x}$$

Solution to exercise 2.6

(a) With $x_1 = h, x_2 = \dot{h}, x_3 = p, u = F_{\text{lift}}$ and $y = q$, the following state-space model can be formulated:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t)\\
\dot{x}_2(t) &= -\tfrac{b}{m}x_2(t) - \tfrac{1}{m}u(t) - g(\tfrac{\rho v}{m} - 1)\\
\dot{x}_3(t) &= k\rho g x_1(t) - kx_3(t)\\
y(t) &= -\rho g x_1(t) + x_3(t).
\end{aligned}$$

(b) Stationary point: $x = \begin{bmatrix} h_0 & 0 & \rho g h_0 \end{bmatrix}^T$. Stationarity implies $u = g(m - \rho v)$.

Solution to exercise 2.7

$$\begin{aligned}
\dot{x}_1 &= \frac{C_1 Sq}{mV}x_1 + x_3 + \frac{C_2 Sq}{mV}u\\
\dot{x}_2 &= x3\\
\dot{x}_3 &= \frac{Sqc}{I_y}\left(\left(C_4 + \frac{Sqc}{2mV^2}C_1C_3\right)x_1 + \frac{c}{2V}(C_3 + C_5)x_3 + \left(C_6 + \frac{Sqc}{2mV^2}C_2C_3\right)u\right)
\end{aligned}$$

where $x_1 = \alpha$, $x_2 = \theta$, $x_3 = \dot{\theta}$ and $u = \delta$.

Solution to exercise 2.8

(a)

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= -\frac{k_1}{m_1}x_1 + \frac{1}{m_1}u - g\\
\dot{y} &= x_2
\end{aligned}$$

(b)

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= -\frac{k_1}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{1}{m_1}u - g\\
\dot{y} &= x_2
\end{aligned}$$

(c)

$$\begin{aligned}
\dot{x}_1 &= x_2 - u\\
\dot{x}_2 &= -\frac{k_1}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{b}{m_1}u - g\\
\dot{y} &= x_2
\end{aligned}$$

(d)

$$\begin{aligned}
\dot{x}_1 &= x_2 - x_4 \\
\dot{x}_2 &= -\frac{k_1}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{b}{m_1}x_4 - g \\
\dot{x}_3 &= x_4 - u \\
\dot{x}_4 &= \frac{k_1}{m_2}x_1 + \frac{b}{m_2}x_2 - \frac{k_2}{m_2}x_3 - \frac{b}{m_2}x_4 - g \\
\dot{y} &= x_2
\end{aligned}$$

Solution to exercise 2.9

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -\frac{b_1+b_2}{M_1} & \frac{b_1}{M_1} & 0 \\ \frac{b_1}{M_2} & -\frac{b_1}{M_2} & -\frac{k}{M_2} \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \frac{1}{M_1} \\ 0 \\ 0 \end{bmatrix} r$$

where $x_1 = v_1$, $x_2 = v_2$ and $x_3 = \int_0^t v_2 d\tau$.

Solution to exercise 2.10

(a)

$$\dot{\boldsymbol{x}} = \frac{K}{a}\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\boldsymbol{x} + \frac{1}{a}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \qquad y = K\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\boldsymbol{x}$$

(b) $x_1 = 0.1$m, $x_2 = 0.2$m and $x_3 = 0.3$m.

(c) Characteristic equation: $= \det(sI - A) = s^3 + 0.05s^2 + 0.0006s + 10^{-6} = 0$

Solution to exercise 2.11

(a) Input: $F(t)$; output: $h(t)$. Possible state variables are: v_2 (speed of mass m_2); h (tank level) or p (tank pressure); v_1 (speed of mass m_1); x (position of mass m_1) or F_k (spring force).

(b) With v_1, v_2, p and F_k as state variables, the model becomes

$$\begin{aligned}
\dot{v}_1(t) &= \frac{1}{m_1}\big(A_1(p(t) - \alpha A_1 v_1(t)) - F_k(t) - b v_1(t)\big) \\
\dot{v}_2(t) &= \frac{1}{m_2}(F(t) - A_2 p(t)) \\
\dot{p}(t) &= \frac{1}{C_f}\big(A_2 v_2(t) - A_1 v_1(t)\big) \\
\dot{F}_k(t) &= k v_1(t) \\
y(t) &= h(t)
\end{aligned}$$

Solution to exercise 2.12

(a) Decompose the velocity v of the front wheels along the fixed $x-$ and $y-$directions. The latter is the derivative of x_2, thus

$$\dot{x}_2(t) = v \sin(\varphi_1(t) + \varphi_2(t))$$

The vehicle speed along its length axis (which is the direction the rear wheels are moving) is $v_1(t) = v\cos\varphi_2(t)$, and the $y-$component of this is

$$\dot{x}_1(t) = v \cos(\varphi_2(t)) \sin(\varphi_1(t))$$

Finally, note that φ_1 depends on the states; assuming $\varphi_1 \in [-\pi, \pi]$ we have $\varphi_1(t) = \arcsin((x_2(t) - x_1(t))/l)$, so that the model becomes (with $u(t) = \varphi_2(t)$)

$$\begin{aligned}
\dot{x}_1(t) &= v \cos(u(t))(x_2(t) - x_1(t))/l \\
\dot{x}_2(t) &= v \sin(\arcsin((x_2(t) - x_1(t))/l) + u(t))
\end{aligned}$$

(b) For small angles φ we have $\sin(\varphi) \approx \varphi$ and $\cos(\varphi) \approx 1$. In matrix form, the model becomes

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{bmatrix} -v/l & v/l \\ -v/l & v/l \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ v \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}
\end{aligned}$$

Solution to exercise 2.13

(a) In stationarity we get

$$K_G N_0^2 = \frac{K_T}{R}\big(u_0 - K_E N_0\big) \quad \Rightarrow \quad u_0 = K_E N_0 + \frac{R K_G}{K_T} N_0^2$$

(b) With $\Delta N = N - N_0$, $\Delta u = u - u_0$, the linearization becomes

$$J\frac{d\Delta N(t)}{dt} + \big(2K_G N_0 + \frac{K_T K_E}{R}\big)\Delta N(t) = \frac{K_T}{R}\Delta u(t)$$

(c) The time constant is given by

$$T = \frac{J}{2K_G N_0 + \frac{K_T K_E}{R}}$$

i.e. it decreases with increasing speed.

Solution to exercise 2.14

(a) The resistor is described by $u_R = R_1 i$, the capacitor by $C\frac{d}{dt}u_C = i$ and Kirchhoff's voltage law gives $u = u_R + u_C$. After manipulation, the model becomes

$$\frac{d}{dt}u_C(t) = \frac{1}{R_1 C}\big(u(t) - u_C(t)\big)$$

(b) Unlike the linear case in (a), we cannot solve analytically for i, so we have to stay with a combination of a differential equation and an algebraic one, i.e. a DAE:

$$\begin{aligned} C\frac{d}{dt}u_C &= i \\ 0 &= u_C + R_1 i + R_2 i^5 - u \end{aligned}$$

Solution to exercise 2.15

No! This ODE fails the two assumptions of Tikhonov:

- The “fast dynamics" are not strictly stable because the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

  has the eigenvalues 0, 0.

- The function $g(\boldsymbol{x},\boldsymbol{z})$ in this case reads as:

$$g(\boldsymbol{x},\boldsymbol{z}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \boldsymbol{x}$$

  such that

$$\frac{\partial g}{\partial z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

  is rank-deficient.

Solution to exercise 2.16
First, use $q = lx$ to get

$$\begin{bmatrix} (M+m)l & -ml\cos\theta \\ -ml^2\cos\theta & J+mL^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} ml\sin\theta\cdot\dot{\theta}^2 \\ -mgl\sin\theta \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}.$$

Then, noting that

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \omega_0\frac{\mathrm{d}x}{\mathrm{d}\tau}$$
$$\ddot{x} = \omega_0^2\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2},$$

and similarly for $\dot{\theta}$ and $\ddot{\theta}$, we get the differential equations

$$(M+m)l\omega_0^2\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} - ml\omega_0^2\cos\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + ml\omega_0^2\sin\theta(\frac{\mathrm{d}\theta}{\mathrm{d}\tau})^2 = F$$
$$-ml^2\omega_0^2\cos\theta\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + (J+mL^2)\omega_0^2\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} - mgl\omega_0^2\sin\theta = 0$$

By defining

$$\alpha = \frac{m}{M+m} \qquad \beta = \frac{ml^2}{J+ml^2}$$

we finally arrive at the sought differential equations

$$\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} - \alpha\cos\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + \alpha\sin\theta\Big(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\Big)^2 = u$$
$$-\beta\cos\theta\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + \frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} - \sin\theta = 0$$

Solution to exercise 2.17

(a) With $\omega_0 = \sqrt{k/m}$ and $\zeta = c/(\sqrt{km})$, we have $\omega_0^2 = k/m$ and $2\zeta\omega_0 = c/m$. The differential equation can therefore be re-written as

$$\ddot{q} + \frac{c}{m}\dot{q} + \frac{k}{m}q = \frac{F}{m} \quad \Leftrightarrow$$
$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = \omega_0^2\frac{F}{m} = \omega_0^2 u$$

(b) Let $\tau = \omega_0 t$, so that $\frac{d\tau}{dt} = \omega_0$. Then $\dot{q} = \omega_0\frac{dq}{d\tau}$ and $\ddot{q} = \omega_0^2\frac{d^2q}{d\tau^2}$. The differential equation with τ as time variable then becomes

$$\omega_0^2\frac{d^2q}{d\tau^2} + 2\zeta\omega_0^2\frac{dq}{d\tau} + \omega_0^2 q = \omega_0^2 u \quad \Leftrightarrow$$
$$\frac{d^2}{d\tau^2} + 2\zeta\frac{dq}{d\tau} + q = u$$

With $z_1 = q$ and $z_2 = \frac{dq}{d\tau}$, the sought state model is obtained.

Solution to exercise 2.18
The model describes a component, consisting of a spring and a damper in parallell.

A.3 Lagrange modelling

Solution to exercise 3.1

(a) With $\boldsymbol{q} = (L, \theta)$, the Lagrange function is

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{L}^2 + L^2\dot{\theta}^2) - (-mgL\cos\theta + \frac{1}{2}kL^2)$$

The Lagrange equation $\frac{d}{dt}\nabla_{\dot{\boldsymbol{q}}}\mathcal{L} - \nabla_{\boldsymbol{q}}\mathcal{L} = F_q$ then becomes

$$m\ddot{L} - mL\dot{\theta}^2 - mg\cos\theta + kL = 0$$
$$mL^2\ddot{\theta} + 2mL\dot{L}\dot{\theta} + mgL\sin\theta = M$$

(b) With $\boldsymbol{q} = (x, y)$, the Lagrange function is

$$\mathscr{L} = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - (mgy + \frac{1}{2}k(x^2 + y^2)),$$

and the Lagrange equation becomes

$$m\ddot{x} + kx = F_x = \frac{M}{L}\cos\theta = -\frac{My}{x^2 + y^2}$$
$$m\ddot{y} + ky + mg = F_y = \frac{M}{L}\sin\theta = \frac{Mx}{x^2 + y^2}$$

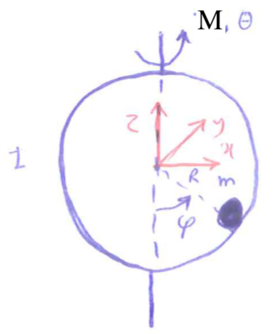
Solution to exercise 3.2
With the coordinate $q = x$, we have

$$\mathscr{L} = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 - (-mgx) = \frac{1}{2}\big(m + \frac{I}{R^2}\big)\dot{x}^2 + mgx$$

and the Lagrange equation $\frac{d}{dt}\nabla_{\dot{\boldsymbol{q}}}\mathscr{L} - \nabla_{\boldsymbol{q}}\mathscr{L} = F_q$ becomes

$$\big(m + \frac{I}{R^2}\big)\ddot{x} - mg = 0$$

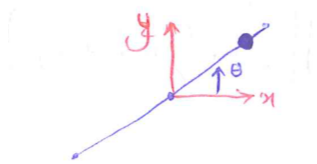
Solution to exercise 3.3



Introduce the two angles θ (rotation of the ring around the vertical axis) and φ (position of the ball in the ring), as seen in the figure to the right.
With $\boldsymbol{q} = (\varphi, \theta)$, the ball rotates in both φ and θ direction, and the ring in θ direction. Hence, the kinetic energy becomes

$$T = \frac{1}{2}m\big(R^2\dot{\varphi}^2 + (R\sin\varphi)^2\dot{\theta}^2\big) + \frac{1}{2}I\dot{\theta}^2$$

and the Lagrange function is



$$\mathscr{L} = T - V = \frac{1}{2}m\big(R^2\dot{\varphi}^2 + (R\sin\varphi)^2\dot{\theta}^2\big) + \frac{1}{2}I\dot{\theta}^2 - (-mgR\cos\varphi)$$

The Lagrange equation $\frac{d}{dt}\nabla_{\dot{\boldsymbol{q}}}\mathscr{L} - \nabla_{\boldsymbol{q}}\mathscr{L} = F_q$ therefore becomes

$$0 = mR^2\ddot{\varphi} - mR^2\dot{\theta}^2\sin\varphi\cos\varphi + mgR\sin\varphi$$
$$M = \frac{d}{dt}\big(mR^2(\sin\varphi)^2\dot{\theta} + I\dot{\theta}\big) = \big(mR^2(\sin\varphi)^2 + I\big)\ddot{\theta} + 2mR^2\dot{\varphi}\dot{\theta}\sin\varphi\cos\varphi$$

Solution to exercise 3.4

(a) Since F acts along the $x-$axis, we have $\delta W = F \cdot \delta x = Q_x \cdot \delta x$, i.e. $Q_x = F$.

(b) With $q = \theta$, the generalized force Q_θ can be found as follows:

$$\delta W = F \cdot \delta x = F\frac{dx}{d\theta}\delta\theta = Q_\theta \cdot \delta\theta \quad \Rightarrow \quad Q_\theta = F\frac{dx}{d\theta}$$

In order to express x in terms of θ, apply the law of sines:

$$\frac{a}{\sin\varphi} = \frac{b}{\sin\theta} \quad \Rightarrow \quad x = a\cos\theta + b\cos\varphi = a\cos\theta + b\sqrt{1 - \left(\frac{a}{b}\sin\theta\right)^2}$$

which implies

$$Q_\theta = F\frac{dx}{d\theta} = F\Big(-a\sin\theta - \frac{a^2\sin\theta\cos\theta}{b\sqrt{1-\left(\frac{a}{b}\sin\theta\right)^2}}\Big)$$

Solution to exercise 3.5

(a) Denoting by F_L the force of the rod, affecting the masses (in different directions), Newton's law applied to the masses gives:

$$\begin{aligned} m_1\ddot{x}_1 &= F - F_L \\ m_2\ddot{x}_2 &= F_L \end{aligned}$$

(b) For the Lagrange approach, use $\boldsymbol{q} = (x_1, x_2)$. The rod gives a constraint $c(\boldsymbol{q}) = x_2 - x_1 - L = 0$. The Lagrange function is (no potential energy)

$$\mathscr{L} = T - zc(\boldsymbol{q}) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - z(x_2 - x_1 - L)$$

The Lagrange equation $\frac{d}{dt}\nabla_{\dot{\boldsymbol{q}}}\mathscr{L} - \nabla_{\boldsymbol{q}}\mathscr{L} = F_q$ becomes

$$\begin{aligned} m_1\ddot{x}_1 - z &= F \\ m_2\ddot{x}_2 + z &= 0 \end{aligned}$$

The Lagrange multiplier z thus represents the force due to the rigid rod.

Solution to exercise 3.6

(a) Let

$$\boldsymbol{q} = \begin{bmatrix} x \\ \boldsymbol{p} \end{bmatrix}, \quad \boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

The kinetic and potential energies are

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} M \dot{\boldsymbol{p}}^T \dot{\boldsymbol{p}} = \frac{1}{2} \dot{\boldsymbol{q}}^T W \dot{\boldsymbol{q}}, \quad W = \mathrm{diag}\,(m, M, M)$$
$$V = -M \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T \boldsymbol{q} = -M \boldsymbol{g}^T \boldsymbol{q}$$

and the (scalar) constraint due to the link of length $L(t)$ can be written as

$$\boldsymbol{c}(\boldsymbol{q}, L) = \frac{1}{2}\Big(\Big(\begin{bmatrix} x \\ 0 \end{bmatrix} - \boldsymbol{p}\Big)^T \Big(\begin{bmatrix} x \\ 0 \end{bmatrix} - \boldsymbol{p}\Big) - L^2\Big) = \frac{1}{2}(\boldsymbol{q}^T D^T D \boldsymbol{q} - L^2) = 0, \quad D = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Hence, the Lagrange function is

$$\mathscr{L} = T - V - \boldsymbol{z}^T \boldsymbol{c}(\boldsymbol{q}) = \frac{1}{2} \dot{\boldsymbol{q}}^T W \dot{\boldsymbol{q}} + M \boldsymbol{g}^T \boldsymbol{q} - \frac{1}{2} \boldsymbol{z} \cdot (\boldsymbol{q}^T D^T D \boldsymbol{q} - L^2).$$

To form the model equations (see the Lecture Notes)

$$W(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \frac{\partial}{\partial \boldsymbol{q}} \left(W(\boldsymbol{q})\, \dot{\boldsymbol{q}} \right) \dot{\boldsymbol{q}} - \nabla_{\boldsymbol{q}} T + \nabla_{\boldsymbol{q}} V + \nabla_{\boldsymbol{q}} \boldsymbol{c} \cdot \boldsymbol{z} = Q$$
$$\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{q}} \ddot{\boldsymbol{q}} + \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{L}} \ddot{L} + \frac{\partial}{\partial \boldsymbol{q}} \left(\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} \right) \dot{\boldsymbol{q}} + \frac{\partial}{\partial L} \left(\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{L}} \dot{L} \right) \dot{L} = 0,$$

we need to work out the details:

$$\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{q}} = \boldsymbol{q}^T D^T D = \begin{bmatrix} x - p_x & p_x - x & p_y \end{bmatrix} \qquad \frac{\partial \boldsymbol{c}}{\partial L} = -L$$
$$\frac{\partial}{\partial \boldsymbol{q}} \Big(\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} \Big) = \dot{\boldsymbol{q}}^T D^T D = \begin{bmatrix} \dot{x} - \dot{p}_x & \dot{p}_x - \dot{x} & \dot{p}_y \end{bmatrix} \qquad \frac{\partial}{\partial L} \Big(\frac{\partial \boldsymbol{c}}{\partial L} \dot{L} \Big) = -\dot{L}$$

Thus, the model becomes

$$\begin{bmatrix} W & D^T D \boldsymbol{q} \\ \boldsymbol{q}^T D^T D & 0 \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} Q + M \boldsymbol{g} \\ -\dot{\boldsymbol{q}}^T D^T D \dot{\boldsymbol{q}} + L \ddot{L} + \dot{L}^2 \end{bmatrix}$$

(b) The cable length would typically be controlled by an electric machine, basically controlling the acceleration $\ddot{L}$, thus ensuring smoothness of the derivatives $\dot{L}$ and $\ddot{L}$.

(c) The force due to the constraint is

$$-\nabla_{\boldsymbol{q}} \boldsymbol{c} \cdot \boldsymbol{z} = -\begin{bmatrix} x - p_x \\ p_x - x \\ p_y \end{bmatrix} \boldsymbol{z},$$

i.e. $\boldsymbol{z} > 0$ is the normal case, when the cable conveys “pull” forces (load hanging down). The opposite case is of less interest and would need a rigid “cable”.

Solution to exercise 3.7

(a) With $\boldsymbol{q}^T = [\boldsymbol{p}_1^T\ \boldsymbol{p}_2^T]$, the kinetic and potential energies are

$$T = \frac{1}{2}m\dot{\boldsymbol{p}}_1^T\dot{\boldsymbol{p}}_1 + \frac{1}{2}m\dot{\boldsymbol{p}}_2^T\dot{\boldsymbol{p}}_2 = \frac{1}{2}m\dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T W\dot{\boldsymbol{q}}, \quad W = m\cdot I_4$$
$$V = -m\boldsymbol{g}^T\boldsymbol{p}_1 - m\boldsymbol{g}^T\boldsymbol{p}_2 = -m\begin{bmatrix}\boldsymbol{g}^T & \boldsymbol{g}^T\end{bmatrix}\boldsymbol{q}, \quad \boldsymbol{g}^T = \begin{bmatrix}0 & -g\end{bmatrix},$$

where I_4 is the 4×4 identity matrix. The constraints are

$$\boldsymbol{c}(\boldsymbol{q}) = \begin{bmatrix}\frac{1}{2}(\boldsymbol{p}_1^T A\boldsymbol{p}_1 - 1) \\ \frac{1}{2}\big((\boldsymbol{p}_1-\boldsymbol{p}_2)^T(\boldsymbol{p}_1-\boldsymbol{p}_2) - L^2\big)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\boldsymbol{q}^T C_1\boldsymbol{q} - 1 \\ \boldsymbol{q}^T C_2\boldsymbol{q} - L^2\end{bmatrix} = 0, \quad C_1 = \begin{bmatrix}A & 0 \\ 0 & 0\end{bmatrix}, C_2 = \begin{bmatrix}I & -I \\ -I & I\end{bmatrix}$$

The Lagrange function is

$$\mathscr{L} = T - V - \boldsymbol{z}^T\boldsymbol{c}(\boldsymbol{q}) = \frac{1}{2}\dot{\boldsymbol{q}}^T W\dot{\boldsymbol{q}} + m\begin{bmatrix}\boldsymbol{g}^T & \boldsymbol{g}^T\end{bmatrix}\boldsymbol{q} - \frac{1}{2}\boldsymbol{z}^T\begin{bmatrix}\boldsymbol{q}^T C_1\boldsymbol{q} - 1 \\ \boldsymbol{q}^T C_2\boldsymbol{q} - L^2\end{bmatrix}$$

(b) The model equations are (since W and T are independent of $\boldsymbol{q}$)

$$\begin{bmatrix}W & \nabla_{\boldsymbol{q}}\boldsymbol{c} \\ \nabla_{\boldsymbol{q}}\boldsymbol{c}^T & 0\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{q}} \\ \boldsymbol{z}\end{bmatrix} = \begin{bmatrix}Q - \nabla_{\boldsymbol{q}}V \\ -\frac{\partial}{\partial\boldsymbol{q}}\left(\frac{\partial\boldsymbol{c}}{\partial\boldsymbol{q}}\dot{\boldsymbol{q}}\right)\dot{\boldsymbol{q}}\end{bmatrix},$$

where

$$\nabla_{\boldsymbol{q}}V = -m\begin{bmatrix}\boldsymbol{g} \\ \boldsymbol{g}\end{bmatrix} = mg\begin{bmatrix}0 \\ 1 \\ 0 \\ 1\end{bmatrix}$$
$$\frac{\partial\boldsymbol{c}}{\partial\boldsymbol{q}} = \begin{bmatrix}\boldsymbol{q}^T C_1 \\ \boldsymbol{q}^T C_2\end{bmatrix}, \quad \frac{\partial}{\partial\boldsymbol{q}}\left(\frac{\partial\boldsymbol{c}}{\partial\boldsymbol{q}}\dot{\boldsymbol{q}}\right) = \begin{bmatrix}\dot{\boldsymbol{q}}^T C_1 \\ \dot{\boldsymbol{q}}^T C_2\end{bmatrix}$$

which finally gives

$$\begin{bmatrix}W & C_1\boldsymbol{q} & C_2\boldsymbol{q} \\ \boldsymbol{q}^T C_1^T & 0 & 0 \\ \boldsymbol{q}^T C_2^T & 0 & 0\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{q}} \\ \boldsymbol{z}\end{bmatrix} = \begin{bmatrix}Q - \nabla_{\boldsymbol{q}}V \\ -\dot{\boldsymbol{q}}^T C_1\dot{\boldsymbol{q}} \\ -\dot{\boldsymbol{q}}^T C_2\dot{\boldsymbol{q}}\end{bmatrix},$$

or

$$\begin{bmatrix}m\cdot I_2 & 0 & A\boldsymbol{p}_1 & \boldsymbol{p}_1-\boldsymbol{p}_2 \\ 0 & m\cdot I_2 & 0 & \boldsymbol{p}_2-\boldsymbol{p}_1 \\ \boldsymbol{p}_1^T A & 0 & 0 & 0 \\ \boldsymbol{p}_1^T-\boldsymbol{p}_2^T & \boldsymbol{p}_2^T-\boldsymbol{p}_1^T & 0 & 0\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{p}}_1 \\ \ddot{\boldsymbol{p}}_2 \\ z_1 \\ z_2\end{bmatrix} = \begin{bmatrix}Q_1 + m\boldsymbol{g} \\ Q_2 + m\boldsymbol{g} \\ -\dot{\boldsymbol{p}}_1^T A\dot{\boldsymbol{p}}_1 \\ -(\dot{\boldsymbol{p}}_1-\dot{\boldsymbol{p}}_2)^T(\dot{\boldsymbol{p}}_1-\dot{\boldsymbol{p}}_2)\end{bmatrix},$$

where I_2 is the 2×2 identity matrix.

Solution to exercise 3.8

(a) N is orthogonal to the rail, implying

$$(N_1, N_2) \perp (1, \frac{dx_2}{dx_1}) = (1, h'(x_1)) \quad \Rightarrow \quad N_1 = -N_2 h'(x_1)$$

(b) The DAE becomes

$$\begin{aligned}
\dot{x}_1 &= v_1 && \text{(A.1a)}\\
\dot{x}_2 &= v_2 && \text{(A.1b)}\\
m\dot{v}_1 &= N_1 - kx_1 && \text{(A.1c)}\\
m\dot{v}_2 &= N_2 - mg && \text{(A.1d)}\\
x_2 &= h(x_1) && \text{(A.1e)}\\
N_1 &= -N_2 h'(x_1) && \text{(A.1f)}
\end{aligned}$$

(c) To investigate the index, a first differentiation of the algebraic equations gives

$$\begin{aligned}
&\dot{x}_2 = h'(x_1)\dot{x}_1 \Leftrightarrow v_2 = h'(x_1)v_1 && \text{(A.2a)}\\
&\dot{N}_1 = -h'(x_1)\dot{N}_2 - N_2 h''(x_1)\dot{x}_1 = -h'(x_1)\dot{N}_2 - N_2 h''(x_1)v_1, && \text{(A.2b)}
\end{aligned}$$

delivering one more differential equation. The remaining algebraic equation is differentiated once more, giving

$$\dot{v}_2 = h'(x_1)\dot{v}_1 + h''(x_1)\dot{x}_1 v_1 = h'(x_1)\dot{v}_1 + h''(x_1)v_1^2, \tag{A.3}$$

which, using (A.1c),(A.1d), again leads to an algebraic equation:

$$N_2 - mg = h'(x_1)(N_1 - kx_1) + mh''(x_1)v_1^2. \tag{A.4}$$

A third differentiation now gives

$$\begin{aligned}
\dot{N}_2 &= h'(x_1)(\dot{N}_1 - kv_1) + h''(x_1)v_1(N_1 - kx_1) + 2mh''(x_1)v_1\dot{v}_1 + mv_1^2 h'''(x_1)v_1\\
&= h'(x_1)(\dot{N}_1 - kv_1) + h''(x_1)v_1(N_1 - kx_1) + 2h''(x_1)v_1(N_1 - kx_1) + mv_1^2 h'''(x_1)v_1, \quad \text{(A.5)}
\end{aligned}$$

thus finally giving an ODE by combining the original 4 differential equations with (A.2b) and (A.5). Hence, the original DAE has index 3.

(d) Using $\boldsymbol{q} = \boldsymbol{p} = (x_1, x_2)$, the kinetic and potential energies of the system can be written:

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2), \qquad V = mgx_2 + \frac{1}{2}kx_1^2 \tag{A.6}$$

With the constraint $c(\boldsymbol{q}) = x_2 - h(x_1) = 0$, the Lagrange function then reads as:

$$\mathscr{L} = T - V - zc = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - mgx_2 - \frac{1}{2}kx_1^2 - z(x_2 - h(x_1)) \tag{A.7}$$

The dynamics are constructed using:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\boldsymbol{q}}}^\top = m\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}, \qquad \frac{\partial \mathscr{L}}{\partial \boldsymbol{q}}^\top = -mg\begin{bmatrix} 0 \\ 1 \end{bmatrix} - kx_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} - z\begin{bmatrix} -h'(x_1) \\ 1 \end{bmatrix} \tag{A.8}$$

Adding the rail constraint, the model then follows from Euler-Lagrange's equation:

$$m\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + mg\begin{bmatrix} 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} -h'(x_1) \\ 1 \end{bmatrix} + kx_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \tag{A.9a}$$

$$x_2 - h(x_1) = 0 \tag{A.9b}$$

(The DAE model can be transformed into standard semi-explicit form, but that is not required in the problem formulation.)
The two 2nd order differential equations need 4 initial conditions, namely $x_1(0), x_2(0), \dot{x}_1(0)$, and $\dot{x}_2(0))$, but the constraint and its time derivative restricts these:

$$x_2(0) = h(x_1(0)) \tag{A.10}$$

$$\dot{x}_2(0) = h'(x_1(0))\dot{x}_1(0), \tag{A.11}$$

implying that only two independent initial conditions can be given.

(e) Differentiating the constraint equation gives

$$\dot{x}_2 = h'(x_1)\dot{x}_1 \tag{A.12a}$$

$$\ddot{x}_2 = h''(x_1)\dot{x}_1^2 + h'(x_1)\ddot{x}_1 \tag{A.12b}$$

Combining this with the 2nd row of (A.9a), we can solve for z:

$$z = -m\ddot{x}_2 - mg = -m\big(h''(x_1)\dot{x}_1^2 + h'(x_1)\ddot{x}_1\big) - mg \tag{A.13}$$

Inserting this expression into the first row of (A.9a) now gives a differential equation for x_1:

$$m\big(1 + h'(x_1)^2\big)\ddot{x}_1 + mh'(x_1)h''(x_1)\dot{x}_1^2 + mgh'(x_1) + kx_1 = 0. \tag{A.14}$$

Using the state-variables x_1 and $v_1 = \dot{x}_1$, the following state-space model is finally obtained:

$$\dot{x}_1 = v_1 \tag{A.15a}$$

$$\dot{v}_1 = -\frac{1}{1 + h'(x_1)^2}\Big(h'(x_1)h''(x_1)v_1^2 + gh'(x_1) + \frac{k}{m}x_1\Big) \tag{A.15b}$$

By adding an "output equation" for x_2, the model completely describes the system:

$$x_2 = h(x_1). \tag{A.16}$$

A.4 System identification

Solution to exercise 4.1
The delay in the response gives $T_1 = 1.5$. The steady-state gain is $A = 5/10 = 0.5$. The time constant T_2 can be found as the time when the response attains 63% of its final value, in this case 3.15, which gives $T_2 \approx 1$.

Solution to exercise 4.2
In stationarity we have

$$y(t) = |G(i\omega| \sin(\omega t + \arg G(i\omega)) = 0.5 \sin(2t - 1)$$

with $\omega = 2$. This gives

$$\begin{aligned} \arg G(i2) = -1 = -\arctan\left(\frac{2}{a}\right) \quad &\Rightarrow \quad a \approx 1.28 \\ |G(i2)| = 0.5 = \frac{b}{\sqrt{4+a^2}} \quad &\Rightarrow \quad b \approx 1.19. \end{aligned}$$

Solution to exercise 4.3
From the figure, we can determine the amplitude and the phase of $G(i\omega)$ for one frequency ω. This gives two equations, but we need 4 equations to solve for the 4 parameters. One way to obtain this is to repeat the experiment for another frequency.

Solution to exercise 4.4
The step response can be used to determine the system parameters from the following system of equations:

$$\begin{aligned} t = 1: &\quad 1.0 = b_1 \cdot 1 \\ t = 2: &\quad 3.0 = -a_1 \cdot 1.0 + b_1 \cdot 1 + b_2 \cdot 1 \\ t = 3: &\quad 5.3 = -a_1 \cdot 3.0 - a_2 \cdot 1.0 + b_1 \cdot 1 + b_2 \cdot 1 \\ t = 4: &\quad 7.35 = -a_1 \cdot 5.3 - a_2 \cdot 3.0 + b_1 \cdot 1 + b_2 \cdot 1 \end{aligned}$$

Solving this gives $b_1 = 1$, $b_2 = 0.5$, $a_1 = -1.5$, $a_2 = 0.7$. The predictions can then be computed from

$$\hat{y}(t|t-1) = -a_1 y(t-1) - a_2 y(t-2) + b_1 u(t-1) + b_2 u(t-2)$$

giving

$$\begin{aligned}
\hat{y}(5|4) &= 1.5y(4) - 0.7y(3) + u(4) + 0.5u(3) = 8.815 \\
\hat{y}(6|4) &= 1.5\hat{y}(5|4) - 0.7y(4) + u(5) + 0.5u(4) = 9.5775
\end{aligned}$$

Solution to exercise 4.5
We know that $e(t) = y(t) - \hat{y}(t|t-1)$, so that

$$y(t) = \hat{y}(t|t-1) + e(t) = L_1(q)u(t-1) + L_2(q)y(t-1) + e(t),$$

implying

$$(1 - q^{-1}L_2(q))y(t) = q^{-1}L_1(q)u(t) + e(t) \quad \Rightarrow \quad y(t) = \frac{q^{-1}L_1(q)}{1 - q^{-1}L_2(q)}u(t) + \frac{1}{1 - q^{-1}L_2(q)}e(t).$$

Solution to exercise 4.6
To find the value of θ that minimizes $V_N(\theta)$, differentiate $V_N(\theta)$:

$$\frac{dV_N(\theta)}{d\theta} = 3\theta^2 + 3\theta - 2.25 = 0,$$

which gives the candidates $\theta = 0.5$ and $\theta = -1.5$. The latter is discarded, since we know $0 \le \theta \le 1$. We have $V_N(0.5) = 18.295$, but we also need to check the values at the limits: $V_N(0) = 18.92$ and $V_N(1) = 19.17$, Hence, the best estimate is $\hat{\theta} = 0.5$.

Solution to exercise 4.7
With $\theta^T = [a \;\; b]$ and $\varphi^T(t) = [-y(t-1) \;\; u(t-1)]$, the model is $y(t) = \theta^T\varphi(t)$. The LS estimate is (see lecture slides for notation!)

$$\begin{aligned}
\hat{\theta}_N = R_N^{-1} f_N &= \left[\frac{1}{N}\sum_{t=2}^{N}\varphi(t)\varphi^T(t)\right]^{-1}\left[\frac{1}{N}\sum_{t=2}^{N}\varphi(t)y(t)\right] \\
&= \begin{bmatrix} \sum_{t=1}^{101} y^2(t) & -\sum_{t=1}^{101} y(t)u(t) \\ -\sum_{t=1}^{101} y(t)u(t) & \sum_{t=1}^{101} u^2(t) \end{bmatrix}^{-1} \begin{bmatrix} -\sum_{t=2}^{102} y(t-1)y(t) \\ \sum_{t=2}^{102} u(t-1)y(t) \end{bmatrix} \\
&= \begin{bmatrix} 5.0 & -1.0 \\ -1.0 & 1.0 \end{bmatrix}^{-1} \begin{bmatrix} -4.5 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -0.875 \\ 0.125 \end{bmatrix}
\end{aligned}$$

Solution to exercise 4.8

(a) The WLS solution is found by differentiating w.r.t. $\boldsymbol{\theta}$:

$$\frac{dV_N(\boldsymbol{\theta})}{d\boldsymbol{\theta}} = \boldsymbol{\theta}^\top \Phi^\top W \Phi - \boldsymbol{y}^\top W \Phi = 0,$$

giving

$$\hat{\boldsymbol{\theta}}_N = (\Phi^\top W \Phi)^{-1} \Phi^\top W \boldsymbol{y}.$$

(b) Inserting the expressions for $\Phi, \boldsymbol{y}$, and W in the above expression gives

$$\hat{\boldsymbol{\theta}}_N = \big(\sum_{i=1}^{N} w_i \boldsymbol{\varphi}(i) \boldsymbol{\varphi}^\top(i)\big)^{-1} \sum_{i=1}^{N} w_i \boldsymbol{\varphi}(i) y(i)$$

Solution to exercise 4.9
Note that the model can equivalently be written as

$$z(t) = y(t) - 0.5u(t-1) = -a_1 y(t-1) - \cdots - a_{n_a} y(t-n_a) + b_2 u(t-2) + \cdots + b_{n_b} u(t-n_b) + e(t) = \theta^T \varphi(t) + e(t),$$

which is a linear regression with predictor

$$\hat{z}(t|t-1,\theta) = \theta^T \varphi(t)$$

Solution to exercise 4.10

(a) Check bias:

$$\mathbb{E}[\hat{A}] = \mathbb{E}\big[\frac{1}{N}\sum_{k=1}^{N} x[k]\big] = \frac{1}{N}\sum_{k=1}^{N} \mathbb{E}[x[k]] = A$$

i.e. the estimator is unbiased.

(b) Check variance:

$$\mathrm{Var}(\hat{A}) = \mathbb{E}[(\hat{A}-A)^2] = \mathbb{E}\big[\frac{1}{N^2}\big(\sum_{k=1}^{N}(x[k]-A)\big)^2\big] = \frac{1}{N^2} N\sigma^2 = \frac{\sigma^2}{N}$$

(c) Matlab!

(d) Check bias:

$$\begin{aligned}
\mathbb{E}[\hat{\sigma}^2] &= \mathbb{E}[\frac{1}{N}\sum_{k=1}^{N}(x[k]-\hat{A})^2] = \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}[((x[k]-A)-(\hat{A}-A))^2] \\
&= \frac{1}{N}\sum_{k=1}^{N}\Big(\mathbb{E}[(x[k]-A)^2] - 2\mathbb{E}[(x[k]-A)(\hat{A}-A)] + \mathbb{E}[(\hat{A}-A)^2]\Big) \\
&= \frac{1}{N}\sum_{k=1}^{N}(\sigma^2 - \frac{2}{N}\sigma^2 + \frac{1}{N}\sigma^2) = \frac{N-1}{N}\sigma^2,
\end{aligned}$$

where the independence was used in the second to last step. The conclusion is that the estimate is (slightly) biased, but this can be circumvented by dividing by $N-1$ instead of N in the expression for $\hat{\sigma}^2$.

Solution to exercise 4.11

(a) Writing the model as $y(t) = \theta^T\varphi(t)$ gives the covariance of the estimates as

$$\mathbb{E}(\hat{\theta}-\theta_0)(\hat{\theta}-\theta_0)^T = \begin{bmatrix} \mathbb{E}(\hat{b}_1-b_1)^2 & \mathbb{E}(\hat{b}_1-b_1)(\hat{b}_2-b_2) \\ \mathbb{E}(\hat{b}_2-b_2)(\hat{b}_1-b_1) & \mathbb{E}(\hat{b}_2-b_2)^2 \end{bmatrix} \approx \frac{\sigma_e^2}{N}\bar{R}^{-1}$$

with

$$\bar{R} = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\varphi(t)\varphi^T(t) = \begin{bmatrix} R_u(0) & R_u(1) \\ R_u(1) & R_u(0) \end{bmatrix} \quad\Rightarrow\quad \bar{R}^{-1} = \frac{1}{R_u^2(0)-R_u^2(1)}\begin{bmatrix} R_u(0) & -R_u(1) \\ -R_u(1) & R_u(0) \end{bmatrix}$$

This shows that the accuracy depends only on $R_u(0)$ and $R_u(1)$ (and on σ_e and N of course). Specifically, the variance of $\hat{b}_1$ is given by $\frac{\sigma_e^2}{N}\frac{R_u(0)}{R_u^2(0)-R_u^2(1)}$.

(b) With $R_u(0) = 1$, the variance of $\hat{b}_1$ is obtained if $R_u(1) = 0$; choosing the input as white noise would fulfil this condition.

Solution to exercise 4.12

We will use the general result that the prediction error estimate converges to the value that minimizes the variance of the prediction errors. Let

$$\varepsilon(t,b) = y(t) - \hat{y}(t|b) = y(t) - bu(t-1) = (1-b)u(t-1) + 0.5u(t-2) + e(t),$$

giving (using the fact that the input is independent of $e(t)$)

$$\mathbb{E}\,\varepsilon^2(t) = (1-b)^2\,\mathbb{E}\,u^2(t-1) + (1-b)\,\mathbb{E}\,u(t-1)u(t-2) + 0.25\,\mathbb{E}\,u^2(t-2) + \mathbb{E}\,e^2(t)$$

(a) If $u(t)$ is white noise, then $\mathbb{E}\,u(t-1)u(t-2) = 0$ and $\mathbb{E}\,\varepsilon^2(t) = (1-b)^2+0.25+1$, which attains its minimal value for $\hat{b} = 1$.

(b) In this case, we have $R_u(1) = 0.5$, giving $\mathbb{E}\,\varepsilon^2(t) = (1-b)^2+0.5(1-b)+0.25+1 = b^2-2.5b+2.75$. By differentiating, we find the minimum $\hat{b} = 1.25$.

Solution to exercise 4.13
Note that for the MSE we have

$$\mathbb{E}[(\hat{\theta}-\theta)^2] = \mathbb{E}\left[((\hat{\theta}-\mathbb{E}[\theta])+(\mathbb{E}[\theta]-\theta))^2\right] = \mathrm{Var}\,\hat{\theta}+(\mathrm{Bias}\,\hat{\theta})^2$$

Hence, we have for the two estimates

$$MSE = \begin{cases} \frac{1}{N}+\frac{1}{N^2} = \frac{N+1}{N^2}, & i=1 \\ \frac{3}{N}+0 = \frac{3}{N}, & i=2 \end{cases}$$

Hence, if bias and variance are of the same order, then the variance is dominating the MSE. Bottom line: unbiasedness is negotiable!

Solution to exercise 4.14
It seems a good idea to reduce the number of A-parameters, since the estimated standard deviation of e.g. a_4 is of the same order as the coefficient itself.

Solution to exercise 4.15

(a) Check bias:
$$\mathbb{E}[\hat{\sigma}^2] = \mathbb{E}\left[\frac{1}{N}\sum_{k=1}^{N} x^2[k]\right] = \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}[x^2[k]] = \sigma^2$$
i.e. the estimate is unbiased.

(b) Check variance:

$$\begin{aligned}\mathrm{Var}(\hat{\sigma}^2) &= \mathbb{E}\left[(\frac{1}{N}\sum_{k=1}^{N} x^2[k]-\sigma^2)^2\right] = \frac{1}{N^2}\mathbb{E}\left[(\sum_{k=1}^{N}(x^2[k]-\sigma^2))^2\right] \\ &= \frac{1}{N}\mathbb{E}\left[(x^2-\sigma^2)^2\right] = \frac{1}{N}(\mathbb{E}[x^4]-\sigma^4) = \frac{1}{N}(3\sigma^4-\sigma^4) = \frac{2\sigma^4}{N},\end{aligned}$$

where we have used the independence in step 3. Hence, the variance decays as $1/N$.

Solution to exercise 4.16
With the control $u(t) = -k_p y(t)$, the closed-loop system becomes

$$y(t) = -(a + k_p b)u(t-1) + v(t).$$

Since all parameter combinations (a, b), giving the same value for $(a + k_p b)$, give identical descriptions of the closed-loop system, the parameters a and b cannot be uniquely determined. With the controller $u(t) = -k_p y(t-1)$, the system instead becomes

$$y(t) = -ay(t-1) - k_p by(t-2) + v(t),$$

so that parameters a and $k_p b$ can be determined using e.g. an ARX method. With knowledge of k_p, a and b can both be determined.

Solution to exercise 4.17

(a) The LS criterion can be written

$$\begin{aligned} V_N(\theta) &= \frac{1}{N}\sum_{t=1}^{N}\big(y(t) - \hat{y}(t)\big)^2 = \frac{1}{N}\sum\big(y(t) - \theta^{1/2}u(t)\big)^2 \\ &= \frac{1}{N}\sum y^2(t) - 2\theta^{1/2}\frac{1}{N}\sum y(t)u(t) + \theta\frac{1}{N}\sum u^2(t). \end{aligned}$$

To find the minimum, differentiate w.r.t. θ and set the derivative equal to zero, giving

$$\frac{\mathrm{d}}{\mathrm{d}\theta}V_N(\theta) = -\theta^{-1/2}\frac{1}{N}\sum y(t)u(t) + \frac{1}{N}\sum u^2(t) = 0,$$

implying that the LS estimate is

$$\hat{\theta}_N = \frac{\big(\frac{1}{N}\sum y(t)u(t)\big)^2}{\big(\frac{1}{N}\sum u^2(t)\big)^2}$$

(b) Inserting the expression for $y(t)$ gives

$$\hat{\theta}_N = \frac{\big(\theta_0^{1/2}\frac{1}{N}\sum u^2(t) + \frac{1}{N}\sum e(t)u(t)\big)^2}{\big(\frac{1}{N}\sum u^2(t)\big)^2} = \theta_0 + 2\theta_0^{1/2}\frac{\frac{1}{N}\sum e(t)u(t)}{\frac{1}{N}\sum u^2(t)} + \frac{\big(\frac{1}{N}\sum e(t)u(t)\big)^2}{\big(\frac{1}{N}\sum u^2(t)\big)^2},$$

so that

$$\mathbb{E}\hat{\theta}_N = \theta_0 + \frac{\mathbb{E}\big(\frac{1}{N}\sum e(t)u(t)\big)^2}{\big(\frac{1}{N}\sum u^2(t)\big)^2} = \theta_0 + \frac{\sigma^2}{\big(\frac{1}{N}\sum u^2(t)\big)^2}$$

where we have used the i.i.d. property of the noise in the last step. Hence, the LS estimate is biased in this case.

(c) With standard assumptions (see Lecture Notes), the sum $\frac{1}{N}\sum e(t)u(t)$ converges to 0 as $N \to \infty$, implying that $\hat{\theta}_N$ is a consistent estimate.

Solution to exercise 4.18
TBD

Solution to exercise 4.19
The ML estimate is θ_3, which gives the largest value of the likelihood function. Note that it is *not* a maximum of $f_y(\theta_3, y*)$.

Solution to exercise 4.20
Since $e \sim \mathcal{N}(0, \sigma)$, its PDF is given by

$$f_e(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}x^2}$$

and the likelihood function becomes

$$L(\theta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-\theta\cdot u)^2}$$

which, assuming $u \neq 0$, is maximized when the argument of the exponential is zero, i.e.

$$\hat{\theta} = \frac{y}{u}$$

Solution to exercise 4.21
Using the independence assumption, the PDF for the noise sequence is

$$f_{\boldsymbol{e}}(\boldsymbol{x}) = \frac{1}{\det(2\pi\Sigma)^{\frac{1}{2}}} e^{-\frac{1}{2}\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}} = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2\sigma_i^2}x_i^2},$$

so that the log-likelihood function becomes

$$\log L(\theta) = \text{const} - \frac{1}{2}\sum_{i=1}^{N} \frac{(y_i - \theta)^2}{\sigma_i^2}$$

The maximum is found by setting the derivative w.r.t. θ to zero:

$$\sum_{i=1}^{N} \frac{y_i - \theta}{\sigma_i^2} = 0,$$

giving the ML estimate

$$\hat{\theta} = \frac{\sum y_i / \sigma_i^2}{\sum 1/\sigma_i^2}.$$

Hence, the measurements are weighted by the inverse of their respective noise variance. If the noise is i.i.d., then the estimate becomes simply the mean of the measurements.

Solution to exercise 4.22

The noise sequence is Gaussian and uncorrelated, hence independent. The log-likelihood function then reads as:

$$\log L(\boldsymbol{\theta}) = \log \prod_{t=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_t} e^{-\frac{1}{2\sigma_t^2}(y(t)-\theta_1 u(t)-\theta_0)^2} = -\sum_{t=1}^{N} \frac{1}{2\sigma_t^2} \left(y(t) - \boldsymbol{\theta}^\top \boldsymbol{\varphi}(t)\right)^2 + \text{const},$$

where $\boldsymbol{\theta} = [\theta_0 \;\; \theta_1]^\top$ and $\boldsymbol{\varphi}(t) = [1 \;\; u(t)]^\top$. To find the maximum, we set the gradient to zero:

$$\nabla_{\boldsymbol{\theta}} \log L(\boldsymbol{\theta}) = \sum_{t=1}^{N} \frac{y(t) - \boldsymbol{\theta}^\top \boldsymbol{\varphi}(t)}{\sigma_t^2} \boldsymbol{\varphi}(t) = 0,$$

leading to the following system of linear equations:

$$\left[\sum_{t=1}^{N} \frac{1}{\sigma_t^2} \boldsymbol{\varphi}(t) \boldsymbol{\varphi}^\top(t)\right] \cdot \boldsymbol{\theta} = \sum_{t=1}^{N} \frac{y(t)}{\sigma_t^2} \boldsymbol{\varphi}(t)$$

Solution to exercise 4.23

(a)

(b)

(c)

Solution to exercise 4.24

TBD

Solution to exercise 4.25

TBD

A.5 Differential Algebraic Equations

Solution to exercise 6.1
The DAE has x_1, x_2, x_3 as differential variables and x_4 as an algebraic variable. It can easily be converted into a semi-explicit DAE with the algebraic equation $x_3 = 7u$, which clearly is not solvable for x_4, i.e. the index is higher than 1. Differentiating this equation once gives $\dot{x}_3 = 7\dot{u}$, which combined with the fourth equation $3\dot{x}_3 + x_4 = 8u$ gives the new algebraic equation $x_4 = 8u - 21\dot{u}$ that provides a solution for x_4. Thus, we arrived at an index-1 DAE by applying one differentiation, i.e. the original DAE has index 2.

Solution to exercise 6.2
The capacitor obeys the equation $I = C\frac{dv}{dt}$ and the voltage source gives $v = u$, giving the DAE

$$\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{I} \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The current I cannot be solved for from the algebraic equation, but differentiating gives

$$\dot{v} = \dot{u} \quad \Rightarrow \quad I = c\dot{u}$$

so that the original DAE has index 2.

Solution to exercise 6.3
The forces in the two springs are equal, giving the DAE

$$\begin{aligned} m(\ddot{x}_1 + \ddot{x}_2) + x_1 &= u \\ x_1 - \tan x_2 &= 0 \end{aligned}$$

Hint: If you are uncertain, apply Lagrange modelling, noting that the potential energy of the nonlinear spring is given by $V_2 = -\log(|\cos x_2|)$, where log is the natural logarithm.

Solution to exercise 6.4

The DAE becomes

$$
\begin{aligned}
L_1 \frac{d}{dt} i_1 + v_1 + v_2 &= 0 \\
L_2 \frac{d}{dt} (i_1 - i_2) - v_2 &= 0 \\
i_1 - g_1(v_1) &= 0 \\
i_2 - g_2(v_2) &= 0
\end{aligned}
$$

and since the matrix (using the notation $x = (i_1, i_2)$, $z = (v_1, v_2)$)

$$
\begin{bmatrix} \frac{\partial F}{\partial \dot{x}} & \frac{\partial F}{\partial v} \end{bmatrix} = \begin{bmatrix} L_1 & 0 & 1 & 1 \\ L_2 & -L_2 & 0 & -1 \\ 0 & 0 & -g_1' & 0 \\ 0 & 0 & 0 & -g_2' \end{bmatrix}
$$

is full rank if the derivatives of g_1 and g_2 are nonzero, the index is 1.

Solution to exercise 6.5

(a) We can answer in a simple or formal way:

- The simple way relies on observing that variable z does not enter as time-differentiated in (6.1), it is an algebraic variable and therefore the model constitutes a DAE.
- For the complex and formal way, we observe that (6.1) is given by the fully implicit differential equation

$$
F(\dot{s}, s, u) = \begin{bmatrix} \dot{\boldsymbol{x}}_1 - \boldsymbol{x}_1 - \boldsymbol{x}_2 - z \\ \dot{\boldsymbol{x}}_2 - z - u \\ \frac{1}{2}\left(\boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 - 1\right) \end{bmatrix}, \tag{A.17}
$$

where $\boldsymbol{s} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & z \end{bmatrix}^\top$. Since the matrix

$$
\frac{\partial F}{\partial \dot{\boldsymbol{s}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A.18}
$$

is rank-deficient, we conclude the model is a DAE.

(b) We observe that (6.1) is a semi-explicit DAE with

$$
g(\boldsymbol{x}, z, u) = \frac{1}{2}\left(\boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 - 1\right) \tag{A.19}
$$

and $\frac{\partial g}{\partial z} = 0$, hence it is of index larger than 1. In order to compute precisely the differential index, we need to perform time-differentiations on (6.1) until it is transformed in an ODE. Because (6.1a)-(6.1b) are already ODEs (functions of z), we can leave them alone, and focus on (6.1c). We then observe that:

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\boldsymbol{x}, z, u) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(\boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 - 1\right)\right) = \boldsymbol{x}_1 \dot{\boldsymbol{x}}_1 + \boldsymbol{x}_2 \dot{\boldsymbol{x}}_2 \tag{A.20}$$

Replacing $\dot{\boldsymbol{x}}_1$, $\dot{\boldsymbol{x}}_2$ by there expressions from (6.1a)-(6.1b), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\boldsymbol{x}, z, u) = \boldsymbol{x}_1 (\boldsymbol{x}_1 + \boldsymbol{x}_2 + z) + \boldsymbol{x}_2 (z + u) \tag{A.21}$$

which is not yet a differential equations. A second time-derivative delivers:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(\boldsymbol{x}, z, u) = \dot{\boldsymbol{x}}_1 (\boldsymbol{x}_1 + \boldsymbol{x}_2 + z) + \dot{\boldsymbol{x}}_2 (z + u) + \boldsymbol{x}_1 (\dot{\boldsymbol{x}}_1 + \dot{\boldsymbol{x}}_2 + \dot{z}) + \boldsymbol{x}_2 (\dot{z} + \dot{u}) \tag{A.22}$$

we then can solve $\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(\boldsymbol{x}, z, u) = 0$ for $\dot{z}$:

$$\dot{z} = \frac{-\dot{\boldsymbol{x}}_1 (\boldsymbol{x}_1 + \boldsymbol{x}_2 + z) - \dot{\boldsymbol{x}}_2 (z + u) - \boldsymbol{x}_1 \dot{\boldsymbol{x}}_1 - \boldsymbol{x}_1 \dot{\boldsymbol{x}}_2 - \boldsymbol{x}_2 \dot{u}}{\boldsymbol{x}_1 + \boldsymbol{x}_2} \tag{A.23}$$

which is an ODE as long as $\boldsymbol{x}_1 + \boldsymbol{x}_2 \neq 0$.

(c) We have already performed this task in the previous question. The index-reduced DAE is the one occurring “one step before getting an ODE", i.e. we can write:

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_1 + \boldsymbol{x}_2 + z \tag{A.24a}$$

$$\dot{\boldsymbol{x}}_2 = z + u \tag{A.24b}$$

$$0 = \boldsymbol{x}_1 (\boldsymbol{x}_1 + \boldsymbol{x}_2 + z) + \boldsymbol{x}_2 (z + u) \tag{A.24c}$$

Here as well we need $\boldsymbol{x}_1 + \boldsymbol{x}_2 \neq 0$ to be able to solve (A.24c).

Solution to exercise 6.6

(a) Since matrix E is rank deficient (3rd line is the sum of the 1st and 2nd lines), it is a DAE.

(b) We observe that the DAE reads as:

$$\dot{\boldsymbol{x}}_1 + \dot{\boldsymbol{x}}_2 = \boldsymbol{x}_1 \tag{A.25a}$$

$$\dot{\boldsymbol{x}}_3 = \boldsymbol{x}_2 \tag{A.25b}$$

$$\dot{\boldsymbol{x}}_1 + \dot{\boldsymbol{x}}_2 + \dot{\boldsymbol{x}}_3 = \boldsymbol{x}_3 \tag{A.25c}$$

Subtracting (A.25a) and (A.25b) from (A.25c), we get:

$$\dot{\boldsymbol{x}}_1 + \dot{\boldsymbol{x}}_2 = \boldsymbol{x}_1 \tag{A.26a}$$
$$\dot{\boldsymbol{x}}_3 = \boldsymbol{x}_2 \tag{A.26b}$$
$$0 = \boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{x}_3 \tag{A.26c}$$

A time-differentiation of (A.26c) yields:

$$\dot{\boldsymbol{x}}_1 + \dot{\boldsymbol{x}}_2 = \boldsymbol{x}_1 \tag{A.27a}$$
$$\dot{\boldsymbol{x}}_3 = \boldsymbol{x}_2 \tag{A.27b}$$
$$0 = \boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{x}_3 \tag{A.27c}$$
$$\dot{\boldsymbol{x}}_1 + \dot{\boldsymbol{x}}_2 - \dot{\boldsymbol{x}}_3 = 0 \tag{A.27d}$$

We then do (A.27d) - (A.27a) + (A.27b) to get:

$$\dot{\boldsymbol{x}}_1 + \dot{\boldsymbol{x}}_2 = \boldsymbol{x}_1 \tag{A.28a}$$
$$\dot{\boldsymbol{x}}_3 = \boldsymbol{x}_2 \tag{A.28b}$$
$$0 = \boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{x}_3 \tag{A.28c}$$
$$0 = \boldsymbol{x}_1 - \boldsymbol{x}_2 \tag{A.28d}$$

A time-differentiation of (A.28d) yields:

$$\dot{\boldsymbol{x}}_1 + \dot{\boldsymbol{x}}_2 = \boldsymbol{x}_1 \tag{A.29a}$$
$$\dot{\boldsymbol{x}}_3 = \boldsymbol{x}_2 \tag{A.29b}$$
$$0 = \boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{x}_3 \tag{A.29c}$$
$$0 = \boldsymbol{x}_1 - \boldsymbol{x}_2 \tag{A.29d}$$
$$\dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_2 = 0 \tag{A.29e}$$

We then do (A.29e)+(A.29a) to get the semi-explicit DAE:

$$\dot{\boldsymbol{x}}_1 = \frac{1}{2}\boldsymbol{x}_1 \tag{A.30a}$$
$$0 = \boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{x}_3 \tag{A.30b}$$
$$0 = \boldsymbol{x}_1 - \boldsymbol{x}_2 \tag{A.30c}$$

Solution to exercise 6.7

(a) Differential equation (6.3) can be put in the fully implicit form:

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, u) = \begin{bmatrix} \dot{\boldsymbol{x}}_1 + u + \boldsymbol{x}_1 + \boldsymbol{x}_2 \\ u + \boldsymbol{x}_2 + \dot{\boldsymbol{x}}_2 (\dot{\boldsymbol{x}}_1 + u + \boldsymbol{x}_1 + \boldsymbol{x}_2) - u^2 \end{bmatrix} = 0 \tag{A.31}$$

and we observe that:

$$\frac{\partial \boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, u)}{\partial \dot{\boldsymbol{x}}} = \begin{bmatrix} 1 & 0 \\ \dot{\boldsymbol{x}}_2 & \dot{\boldsymbol{x}}_1 + u + \boldsymbol{x}_1 + \boldsymbol{x}_2 \end{bmatrix} \tag{A.32}$$

We should note here that the lower-right element of $\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}$ is zero as it is the first equation in (A.31). It has then a column of zero and is therefore rank-deficient. This is a DAE. We observe here that (6.3) can be written as:

$$\dot{\boldsymbol{x}}_1 + u + \boldsymbol{x}_1 + \boldsymbol{x}_2 = 0 \tag{A.33a}$$
$$\boldsymbol{x}_2 + u + u^2 = 0 \tag{A.33b}$$

hence $\boldsymbol{x}_2$ here plays the role of an algebraic variable. We would tend to rename it z to stress that observation.

(b) We perform the same tasks, i.e.:

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, u) = \begin{bmatrix} u + \dot{\boldsymbol{x}}_1 \boldsymbol{x}_1 + \dot{\boldsymbol{x}}_2 \boldsymbol{x}_2 \\ u \dot{\boldsymbol{x}}_1 \boldsymbol{x}_1 + \dot{\boldsymbol{x}}_2 u \boldsymbol{x}_2 \end{bmatrix} = 0 \tag{A.34}$$

and

$$\frac{\partial \boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, u)}{\partial \dot{\boldsymbol{x}}} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \\ u\boldsymbol{x}_1 & u\boldsymbol{x}_2 \end{bmatrix} \tag{A.35}$$

We observe that $\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}$ is rank deficient as the second row is the first one multiplied by u. In case of doubt, one can also compute $\det\left(\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}\right) = 0$. This is a DAE as per the formal definition. We observe that it can be rewritten as:

$$u + \dot{\boldsymbol{x}}_1 \boldsymbol{x}_1 + \dot{\boldsymbol{x}}_2 \boldsymbol{x}_2 = 0 \tag{A.36a}$$
$$u(\dot{\boldsymbol{x}}_1 \boldsymbol{x}_1 + \dot{\boldsymbol{x}}_2 \boldsymbol{x}_2) = 0 \tag{A.36b}$$

such that $u = -\dot{\boldsymbol{x}}_1 \boldsymbol{x}_1 + \dot{\boldsymbol{x}}_2 \boldsymbol{x}_2$ and

$$-u^2 = 0 \tag{A.37}$$

should hold. The system reduces to

$$\dot{\boldsymbol{x}}_1 \boldsymbol{x}_1 + \dot{\boldsymbol{x}}_2 \boldsymbol{x}_2 = 0 \tag{A.38}$$

This DAE is degenerate in the sense that it does not provide $\dot{\boldsymbol{x}}_{1,2}$.

Solution to exercise 6.8

(a) In theory we can (c.f. Lectures), but we cannot provide explicit expressions here providing $\dot{\boldsymbol{x}}$ as an explicit function of $\boldsymbol{x}, z, u$ from (6.5). We would have to solve it numerically.

(b) We use the Implicit Function Theorem here to decide if (6.5) can be solved for $\dot{\boldsymbol{x}}$, z. We write:

$$F(\dot{x}, x, z, u) = \begin{bmatrix} \dot{x} + u + \tanh(\dot{x}) + xz \\ \tanh(2u - z) \end{bmatrix} = 0 \tag{A.39}$$

and compute:

$$M = \begin{bmatrix} \frac{\partial F}{\partial \dot{x}} & \frac{\partial F}{\partial z} \end{bmatrix} = \begin{bmatrix} 2 - \tanh(\dot{x})^2 & x \\ 0 & \tanh(2u - z)^2 - 1 \end{bmatrix} \tag{A.40}$$

and

$$\det(M) = -(\tanh(2u - z)^2 - 1)(\tanh(\dot{x})^2 - 2) \tag{A.41}$$

We then observe that

$$\tanh(2u - z) \neq 1 \tag{A.42}$$

$$\tanh(\dot{x}) \neq 2^{\frac{1}{2}} \tag{A.43}$$

for all finite $u, z, \dot{x}$, which implies that we can solve for $\dot{\boldsymbol{x}}$, z, so that the DAE gives a well-defined trajectory.

(c) From the above, it follows that the index is 1.

Solution to exercise 6.9
Solution will be posted at the course homepage.

Solution to exercise 6.10
Solution will be posted at the course homepage.

A.6 Explicit integrators

Solution to exercise 7.1
The Butcher array gives

$$\begin{aligned} K_1 &= f(x_k, u(t_k)) \\ K_2 &= f(x_k + \Delta t K_1, u(t_k + \Delta t)) \\ x_{k+1} &= x_k + \frac{\Delta t}{2}(K_1 + K_2) \end{aligned}$$

Taylor expansion of K_2 in the last equation then gives for x_{k+1}:

$$\begin{aligned} x_{k+1} = x_k + \frac{\Delta t}{2}\big(f(x_k, u_k) + f(x_k, u_k) + \Delta t \frac{\partial f}{\partial x} K_1 + \Delta t \frac{\partial f}{\partial u} \dot{u}(t_k) + \mathcal{O}(\Delta t^2)\big) \\ = x_k + \Delta t f(x_k, u_k) + \frac{1}{2}\Delta t^2 \big(\frac{\partial f}{\partial x} f(x_k, u_k) + \frac{\partial f}{\partial u} \dot{u}_k\big) + \mathcal{O}(\Delta t^3) \end{aligned}$$

The first three terms are the same as obtained when Taylor-expanding $x(t_{k+1})$ with $x(t_k) = x_k$, and hence the 1-step error is of order 3, giving a global error of order 2.

Solution to exercise 7.2
Since the order of the RK2 scheme is 2, the stability region is determined by the equation (see Section 6.4 in the Lecture Notes)

$$\left|1 + \lambda \Delta t + \frac{1}{2}\lambda^2 \Delta t^2\right| < 1$$

Solution to exercise 7.3
For RK1, we have

$$x_{k+1} = x_k + \Delta t d_1 K_1 = x_k + \Delta t d_1 f(x_k, u_k)$$

which is of order 1 if $d_1 = 1$. For RK2 we have

$$\begin{aligned} K_1 &= f(x_k, u_k) \\ K_2 &= f(x_k + \Delta t a K_1, u(t_k + c\Delta t)) \\ x_{k+1} &= x_k + \Delta t(b_1 K_1 + b_2 K_2) \end{aligned}$$

With $a = c = 0$ and $b_1 = b_2 = 0.5$, the algorithms are identical, and thus the RK2 method is also of order 1. Determining the coefficients in an appropriate way is thus important to achieve errors of desired order.

Solution to exercise 7.4

Solution will be posted at the course homepage.

A.7 Implicit integrators

Solution to exercise 8.1

(a) From the Butcher array, it follows that $s = 1$ and $a = b = c = 1$. The equations are

$$\begin{aligned} K_1 &= \boldsymbol{f}\big(\boldsymbol{x}_k + \Delta t \cdot K_1, \boldsymbol{u}(t_k + \Delta t)\big) \\ \boldsymbol{x}_{k+1} &= \boldsymbol{x}_k + \Delta t \cdot K_1 \end{aligned} \quad \Rightarrow \quad \boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta t \cdot \boldsymbol{f}\big(\boldsymbol{x}_{k+1}, \boldsymbol{u}_{k+1}\big)$$

(b) The equations describe the implicit Euler scheme.

(c) Defining

$$\boldsymbol{r}\left(\boldsymbol{x}_{k+1}, \boldsymbol{x}_k, \boldsymbol{u}_{k+1}\right) := \boldsymbol{x}_k + \Delta t \cdot \boldsymbol{f}\left(\boldsymbol{x}_{k+1}, \boldsymbol{u}_{k+1}\right) - \boldsymbol{x}_{k+1} = 0, \tag{A.44}$$

the pseudo code for the implicit Euler method is given below:

Algorithm: Implicit Euler method

Input: Initial conditions $\boldsymbol{x}_0$, input profile $\boldsymbol{u}(.)$, step size Δt
for $k = 0, \ldots, N-1$ **do**
  Guess $\boldsymbol{x}_{k+1}$, one can e.g. use $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k$
  while $\|\boldsymbol{r}\left(\boldsymbol{x}_{k+1}, \boldsymbol{x}_k, \boldsymbol{u}_{k+1}\right)\| > \text{Tol}$ **do**
    Compute the solution $\Delta \boldsymbol{x}_{k+1}$ to the linear system:

$$\frac{\partial \boldsymbol{r}\left(\boldsymbol{x}_{k+1}, \boldsymbol{x}_k, \boldsymbol{u}_{k+1}\right)}{\partial \boldsymbol{x}_{k+1}} \Delta \boldsymbol{x}_{k+1} + \boldsymbol{r}\left(\boldsymbol{x}_{k+1}, \boldsymbol{x}_k, \boldsymbol{u}_{k+1}\right) = 0 \tag{A.45}$$

    where $\boldsymbol{r}$ is given by (A.44). Update:

$$\boldsymbol{x}_{k+1} \leftarrow \boldsymbol{x}_{k+1} + \alpha \Delta \boldsymbol{x}_{k+1} \tag{A.46}$$

    for some step size $\alpha \in]0,1]$ (a full step $\alpha = 1$ generally works for implicit integrators)

return $\boldsymbol{x}_{1,\ldots,N}$

(d) With $\boldsymbol{f}(x, u) = A\boldsymbol{x} + B\boldsymbol{u}$, the update equation becomes

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta t \cdot (A\boldsymbol{x}_{k+1} + B\boldsymbol{u}_{k+1}) \quad \Rightarrow \quad (I - \Delta t \cdot A)\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta t \cdot B\boldsymbol{u}_{k+1},$$

so that $\boldsymbol{x}_{k+1}$ is obtained as the solution to a system of linear equations.

(e) The residual function r in this case becomes

$$\boldsymbol{r}\left(\boldsymbol{x}_{k+1}, \boldsymbol{x}_k, \boldsymbol{u}_{k+1}\right) := \boldsymbol{x}_k + \Delta t \cdot \left(A\boldsymbol{x}_{k+1} + B\boldsymbol{u}_{k+1}\right) - \boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta t \cdot B\boldsymbol{u}_{k+1} - \left(I - \Delta t \cdot A\right)\boldsymbol{x}_{k+1}$$

and since this is linear in $\boldsymbol{x}_{k+1}$, the Newton method solves it in one step (making it equivalent to the answer in (d)).

(f) For the general case with s stages, the linear dynamics will lead to all equations becoming linear in $\{K_i\}$, so that it is enough to solve a system of $s \cdot n$ linear equations, where n is the dimension of the state vector.

Solution to exercise 8.2

(a) An IRK scheme for a semi-explicit DAE is similar to the ODE case, but each stage's equation need to be augmented with the corresponding algebraic equation ($i = 1, \dots, s$):

$$\begin{aligned} K_i &= \boldsymbol{f}\big(\boldsymbol{x}_k + \Delta t \cdot \sum_{j=1}^{s} a_{ij} K_j,\ \boldsymbol{z}_i,\ \boldsymbol{u}(t_k + c_i \Delta t)\big) \\ 0 &= \boldsymbol{g}\big(\boldsymbol{x}_k + \Delta t \cdot \sum_{j=1}^{s} a_{ij} K_j,\ \boldsymbol{z}_i,\ \boldsymbol{u}(t_k + c_i \Delta t)\big) \end{aligned}$$

and the update becomes as before

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta t \cdot \sum_{i=1}^{s} b_i K_i$$

(b) Defining

$$\boldsymbol{w} = \big(\boldsymbol{K}_1, \dots, \boldsymbol{K}_s, \boldsymbol{z}_1, \dots, \boldsymbol{z}_s\big),$$

the residual function $\boldsymbol{r}(\boldsymbol{w}, \boldsymbol{x}_k, \boldsymbol{u}(\cdot))$ has the i:th ($i = 1, \dots, s$) block row

$$\boldsymbol{r}_i(\boldsymbol{w}, \boldsymbol{x}_k, \boldsymbol{u}(\cdot)) := \begin{bmatrix} \boldsymbol{f}\left(\boldsymbol{z}_i,\ \boldsymbol{x}_k + \Delta t \sum_{j=1}^{s} a_{ij} \boldsymbol{K}_j,\ \boldsymbol{u}(t_k + c_i \Delta t)\right) - \boldsymbol{K}_i \\ \boldsymbol{g}\left(\boldsymbol{z}_i,\ \boldsymbol{x}_k + \Delta t \sum_{j=1}^{s} a_{ij} \boldsymbol{K}_j,\ \boldsymbol{u}(t_k + c_i \Delta t)\right) \end{bmatrix} \tag{A.47}$$

(c) The pseudo code follows below:

Algorithm: IRK for semi-explicit DAEs

Input: Initial conditions $\boldsymbol{x}_0$, input profile $\boldsymbol{u}(.)$, Butcher tableau, step size Δt

for $k = 0, \ldots, N-1$ **do**

Guess for $\boldsymbol{w}$ (one can e.g. use $\boldsymbol{K}_i = \boldsymbol{x}_k$, $\boldsymbol{z}_i = 0$)

while $\|\boldsymbol{r}(\boldsymbol{w}, \boldsymbol{x}_k, \boldsymbol{u}(\cdot))\| > \text{Tol}$ **do**

Compute $\boldsymbol{r}(\boldsymbol{w}, \boldsymbol{x}_k, \boldsymbol{u}(\cdot))$ and $\frac{\partial \boldsymbol{r}(\boldsymbol{w}, \boldsymbol{x}_k, \boldsymbol{u}(\cdot))}{\partial \boldsymbol{w}}$ with $\boldsymbol{r}$ given by (A.47).

Compute the solution $\Delta \boldsymbol{w}$ to the linear system:

$$\frac{\partial \boldsymbol{r}(\boldsymbol{w}, \boldsymbol{x}_k, \boldsymbol{u}(\cdot))}{\partial \boldsymbol{w}} \Delta \boldsymbol{w} + \boldsymbol{r}(\boldsymbol{w}, \boldsymbol{x}_k, \boldsymbol{u}(\cdot)) = 0 \tag{A.48}$$

Update:

$$\boldsymbol{w} \leftarrow \boldsymbol{w} + \alpha \Delta \boldsymbol{w} \tag{A.49}$$

for some step size $\alpha \in]0,1]$ (a full step $\alpha = 1$ generally works for implicit integrators)

Take RK step:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta t \sum_{j=1}^{s} b_i \boldsymbol{K}_i \tag{A.50}$$

return $\boldsymbol{x}_{1,\ldots,N}$

(d) For $s = 1$, we have $\boldsymbol{w} = (\boldsymbol{K}_1, \boldsymbol{z}_1)$ and

$$\boldsymbol{r}(\boldsymbol{w}, \boldsymbol{x}_k, \boldsymbol{u}(\cdot)) := \begin{bmatrix} \boldsymbol{f}(\boldsymbol{z}_1,\ \boldsymbol{x}_k + \Delta t a \boldsymbol{K}_1,\ \boldsymbol{u}(t_k + c\Delta t)) - \boldsymbol{K}_1 \\ \boldsymbol{g}(\boldsymbol{z}_1,\ \boldsymbol{x}_k + \Delta t a \boldsymbol{K}_1,\ \boldsymbol{u}(t_k + c\Delta t)) \end{bmatrix}$$

Solving using Newton's method requires the following matrix being of full rank:

$$\frac{\partial \boldsymbol{r}(\boldsymbol{w}, \boldsymbol{x}_k, \boldsymbol{u}(\cdot))}{\partial \boldsymbol{w}} = \begin{bmatrix} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} a \Delta t - I & \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}_1} \\ \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} a \Delta t & \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}_1} \end{bmatrix} = \begin{bmatrix} -I & \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}_1} \\ 0 & \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}_1} \end{bmatrix} + a \Delta t \cdot \begin{bmatrix} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} & 0 \\ \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} & 0 \end{bmatrix}$$

Since Δt is usually chosen small, the first matrix in this expression should be full rank. This happens exactly when $\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}$ is full rank, i.e. when the DAE has index 1.

Solution to exercise 8.3

Solution will be posted at the course homepage.

Solution to exercise 8.4

Denoting the Butcher array as

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

the stability function is given by $R(\mu) = 1 + \mu b^T (I - \mu A)^{-1}\mathbf{1}$, where $\mu = \lambda \Delta t$ and $\mathbf{1}$ is a column vector with all entries equal to 1.

(a)

$$R(\mu) = 1 + \mu \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\mu & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 + \mu + \frac{1}{2}\mu^2$$

From the stability function, it is clear that A-stability does not hold, since $|R(\mu)| \to \infty$, $|\mu| \to \infty$. This also follows from the fact that the scheme is explicit, as can be seen from the Butcher array.

(b)

$$R(\mu) = 1 + \mu \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\mu/2 & 1 - \mu/2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1 + \mu/2}{1 - \mu/2}$$

Since $|1 + \mu/2| \le |1 - \mu/2|$ for all μ in the left half-plane (geometrical interpretation?), $|R(\mu)| \le 1$ for the same μ, i.e. the scheme is A-stable.

Solution to exercise 8.5
Solutions will be posted at the course homepage.

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