Rigid body dynamics

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Chapter 1

Introduction: particles

Mechanics is part of physics and at the same time one of the cornerstones of many engineering educations. Mechanics can be defined as the science of material systems in rest or motion and the thereby appearing forces. In a general sense this encompasses all material systems, such as particles, rigid bodies, deformable bodies, and fluids. Usually, courses in mechanics only deal with particles and rigid bodies. Mechanics can be divided into statics and dynamics and dynamics can be divided into kinematics and kinetics. Kinematics is concerned with the motion of a system without regard to its causes (forces). One may be interested in the velocity of a part in a transmission system given the angular velocity of an axle. Kinetics is concerned with the interplay between forces and motion. One may be interested in the velocity of a part of a transmission system given the applied torque on an axle.

This chapter reviews the basic principles of mechanics as applied to particles. The kinematics of a single particle is treated and it is stressed that it is important to choose (generalized) coordinates in a sensible way. Newton's three laws are discussed and the various integrated versions of Newton's second law are derived. The important concepts of oscillations, equilibrium, and stability are introduced by simple examples. Finally, the laws for a system of particles are derived.

1.1 System of units

Some basic notions and quantities of mechanics are regarded as familiar and known. Thus mechanics takes place in three-dimensional space. Points in space specify positions. Physical quantities in space are, for example, length and volume. In space there is matter and to matter corresponds the quantity mass. A central notion in dynamics is of course time. Space, matter, and time have axiomatic character in mechanics.

To measure physical quantities there is a need for reference objects in terms of units. The length L of a particular distance can be given as $L = 2.8\,\mathrm{m}$, where 2.8 is the measure and m is the symbol for the unit, in this case meter. Here the SI system of units is used. It contains seven basic units, but in mechanics only three of these are needed, namely those of length, time, and mass. The corresponding units are meter (m), second (s), and kilogram (kg). Other units are derived units, e.g. the unit of velocity is meter per second (m/s) and the unit of force is Newton (N), where $1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2}$. The SI units for derived physical quantities can thus be traced to definitions (as for velocity) or physical laws (as for force).

It is here natural to recall the notion of dimension. Every physical quantity has a certain dimension. Thus a radius r has the dimension length, which is written $\dim r = \mathrm{L}$. The corresponding dimensions for time and mass are T and M. All other dimensions in mechanics can be expressed with the help of L, T, and M. For a velocity the dimension is $\dim v = \mathrm{LT}^{-1}$ and for force it is $\dim F = \mathrm{MLT}^{-2}$. A necessary condition for an equation to be correct is that all terms have the same dimension, and this dimensional control is often good to perform to eliminate many simple errors. It is noted that arguments of trigonometric and other elementary functions must be dimensionless. To be able to perform a check of dimensions all quantities must be denoted by algebraic symbols. Thus, this is one of several reasons to avoid the use of numerical values early in calculations.

One should be careful in the accuracy of the solution and in rounding off numbers when solving problems. Some engineering problems need an accuracy of one part in a thousand. So it is a good policy to retain at least four significant digits during the computations. A problem in this connection is that the value of the gravitational constant at the surface of the earth is not very accurate. Here the value $g = 9.81\,\mathrm{m/s^2}$ is used. Some problems in celestial mechanics need a much higher accuracy.

1.2 Particle kinematics

A particle is a body that only occupies a single point in space. It is thus specified by its position, i.e. three coordinates that are defined in some convenient manner. In practice a body may be regarded as a particle if it is small relative other relevant lengths. It is, for instance, possible to regard a satellite as a particle if its motion around the earth is considered, but not if the orientation of the satellite is of importance. For rigid bodies Newton's second law applies for the motion of the centre of mass and thus for many problems a rigid body may be regarded as a particle, e.g. for a box sliding down an inclined plane.

If $\boldsymbol{r}$ defines the position of the particle in a fixed reference frame (one where Newton's second law applies) then the velocity of the particle is defined as the time derivative of the position

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \dot{\boldsymbol{r}} \tag{1.1}$$

As is common in mechanics a dot over a letter denotes differentiation with respect to time. The acceleration is likewise defined as the time derivative of the velocity

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \dot{\boldsymbol{v}} \tag{1.2}$$

Both velocity and acceleration are vectors with magnitude and direction. The magnitude of the velocity is the speed v.

The simplest coordinates to describe the position of a particle are cartesian (rectangular) coordinates. These can depend on time

$$\boldsymbol{r} = x(t)\boldsymbol{e}_x + y(t)\boldsymbol{e}_y + z(t)\boldsymbol{e}_z \tag{1.3}$$

If the unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ are constant, i.e. if the reference frame is not accelerating, the velocity is obtained simply by differentiating the components

$$\boldsymbol{v} = v_x(t)\boldsymbol{e}_x + v_y(t)\boldsymbol{e}_y + v_z(t)\boldsymbol{e}_z \tag{1.4}$$

$$v_x = \dot{x}, \qquad v_y = \dot{y}, \qquad v_z = \dot{z} \tag{1.5}$$

Note that the motion in the three directions are uncoupled, i.e. none of the coordinates for a particular direction is included in the other directions. The motion is thus described as a superposition of three

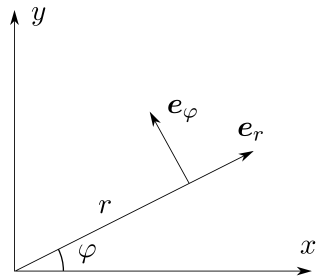


Figure 1.1: The definition of polar coordinates.

uncoupled rectilinear motions. The description in cartesian coordinates is thus very simple, but this very simplicity in fact also limits the usefulness. In many cases it is not natural with a description of motion along fixed directions. One case where it is natural to use cartesian coordinates is in the analysis of projectile motion.

Circular motion is an example where it is not useful to employ cartesian coordinates. Instead polar coordinates $r\varphi$ are defined

$$x = r\cos\varphi, \qquad\qquad y = r\sin\varphi \tag{1.6}$$

From the figure it is seen that the corresponding unit vectors are

$$\boldsymbol{e}_r = \cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y \tag{1.7}$$

$$\boldsymbol{e}_\varphi = -\sin\varphi\,\boldsymbol{e}_x + \cos\varphi\,\boldsymbol{e}_y \tag{1.8}$$

This can be inverted to give

$$\boldsymbol{e}_x = \cos\varphi\,\boldsymbol{e}_r - \sin\varphi\,\boldsymbol{e}_\varphi \tag{1.9}$$

$$\boldsymbol{e}_y = \sin\varphi\,\boldsymbol{e}_r + \cos\varphi\,\boldsymbol{e}_\varphi \tag{1.10}$$

The position in polar coordinates is simply

$$\boldsymbol{r} = r\boldsymbol{e}_r \tag{1.11}$$

When calculating the velocity in polar coordinates one must take into account that the unit vectors are not constant. Instead

$$\dot{\boldsymbol{e}}_r = -\dot\varphi\sin\varphi\,\boldsymbol{e}_x + \dot\varphi\cos\varphi\,\boldsymbol{e}_y = \dot\varphi\,\boldsymbol{e}_\varphi \tag{1.12}$$

$$\dot{\boldsymbol{e}}_\varphi = -\dot\varphi\cos\varphi\,\boldsymbol{e}_x - \dot\varphi\sin\varphi\,\boldsymbol{e}_y = -\dot\varphi\,\boldsymbol{e}_r \tag{1.13}$$

The velocity thus becomes

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot r\,\boldsymbol{e}_r + r\dot\varphi\,\boldsymbol{e}_\varphi \tag{1.14}$$

In the same way the acceleration becomes

$$\boldsymbol{a} = (\ddot r - r\dot\varphi^2)\boldsymbol{e}_r + (r\ddot\varphi + 2\dot r\dot\varphi)\boldsymbol{e}_\varphi \tag{1.15}$$

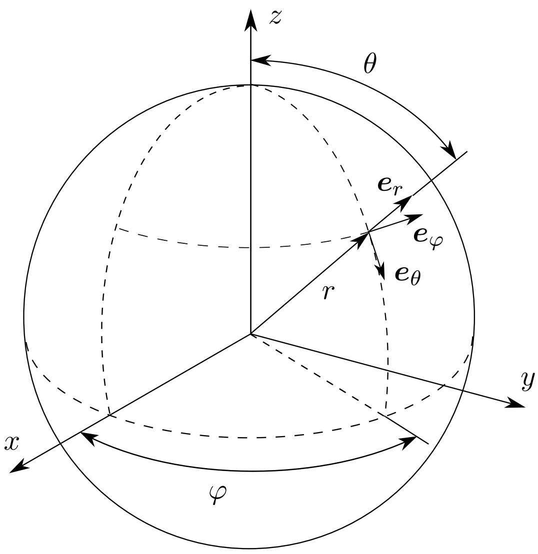


Figure 1.2: The definition of spherical coordinates.

These expressions are more complicated than in cartesian coordinates due to the varying unit vectors. Polar coordinates are particularly useful for circular motion when $\dot{r} = 0$ and $\ddot{r} = 0$ and the velocity and acceleration have the wellknown appearance

$$\boldsymbol{v} = r\dot{\varphi}\,\boldsymbol{e}_\varphi \tag{1.16}$$

$$\boldsymbol{a} = -r\dot{\varphi}^2\,\boldsymbol{e}_r + r\ddot{\varphi}\,\boldsymbol{e}_\varphi \tag{1.17}$$

Except for circular motion, polar coordinates are useful in many cases when a fixed point is referenced. This is the case for a central force and polar coordinates are thus useful when investigating planetary motion.

Polar coordinates describe motion in two dimensions. The corresponding three-dimensional system is cylindrical coordinates where a third cartesian coordinate, which is usually called z, is added. The position, velocity, and acceleration in cylindrical coordinates become

$$\boldsymbol{r} = r\,\boldsymbol{e}_r + z\,\boldsymbol{e}_z \tag{1.18}$$

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\varphi}\,\boldsymbol{e}_\varphi + \dot{z}\,\boldsymbol{e}_z \tag{1.19}$$

$$\boldsymbol{a} = (\ddot{r} - r\dot{\varphi}^2)\boldsymbol{e}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\boldsymbol{e}_\varphi + \ddot{z}\,\boldsymbol{e}_z \tag{1.20}$$

An example where cylindrical coordinates can be used is for a particle moving in a helix (a screw).

Spherical coordinates $r\theta\varphi$ are sometimes useful, e.g. for a spherical pendulum (a pendulum which can swing in all directions, not just in a plane). They are defined by

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta \tag{1.21}$$

The same letter r is usually used for the radius in both cylindrical and spherical coordinates although they have different geometrical meanings. In cylindrical coordinates r is the distance from a fixed axis (the z axis), but in spherical coordinates r is the distance from a fixed point (the origin). The angle φ,

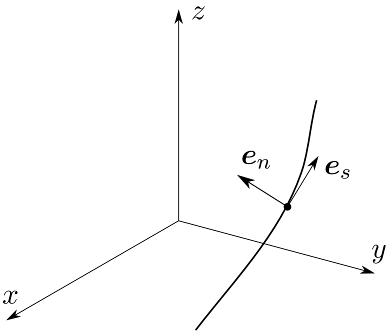


Figure 1.3: Path variables.

on the other hand, refers to exactly the same angle in the xy plane. By projections it is seen that the spherical unit vectors are

$$\boldsymbol{e}_r = \sin\theta\cos\varphi\,\boldsymbol{e}_x + \sin\theta\sin\varphi\,\boldsymbol{e}_y + \cos\theta\,\boldsymbol{e}_z \tag{1.22}$$

$$\boldsymbol{e}_\theta = \cos\theta\cos\varphi\,\boldsymbol{e}_x + \cos\theta\sin\varphi\,\boldsymbol{e}_y - \sin\theta\,\boldsymbol{e}_z \tag{1.23}$$

$$\boldsymbol{e}_\varphi = -\sin\varphi\,\boldsymbol{e}_x + \cos\varphi\,\boldsymbol{e}_y \tag{1.24}$$

Taking derivatives in the same way as for polar coordinates it is possible to obtain the expressions for the position, velocity, and acceleration

$$\boldsymbol{r} = r\,\boldsymbol{e}_r \tag{1.25}$$

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\,\boldsymbol{e}_\theta + r\dot{\varphi}\sin\theta\,\boldsymbol{e}_\varphi \tag{1.26}$$

$$\begin{aligned}\boldsymbol{a} = {} & (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\,\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)\,\boldsymbol{e}_\theta \\ & + (r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\,\boldsymbol{e}_\varphi\end{aligned} \tag{1.27}$$

As seen the expression for the acceleration is fairly complex.

The so far treated coordinate systems are sometimes called excentric, which means that they are defined independently of the particle path. Intrinsic coordinates, on the other hand, are defined in terms of the particle path. They are sometimes useful when the path is specified, and they also give some additional insight into the kinematics. The intrinsic coordinates are also called tangent and normal coordinates or path variables.

Consider a given particle path, i.e. a curve in three-dimensional space, and a point on the path. A natural reference direction at the point is the tangent to the curve. Let the unit vector $\boldsymbol{e}_s$ point along the path in the direction of travel of the particle. Locally at the point the curve and the tangent lies in the same plane (called the osculating plane), and the normal to the curve in this plane is taken as the second reference direction. The normal $\boldsymbol{e}_n$ is chosen so that it points towards the centre of curvature of the path. A third reference direction is most easily obtained as the cross product

$$\boldsymbol{e}_b = \boldsymbol{e}_s \times \boldsymbol{e}_n \tag{1.28}$$

where $\boldsymbol{e}_b$ is called the binormal. The three unit vectors $\boldsymbol{e}_s$, $\boldsymbol{e}_n$, $\boldsymbol{e}_b$ are thus chosen so that they form an orthonormal basis at every point along the path. Note that $\boldsymbol{e}_n$ (and thereby $\boldsymbol{e}_b$) shifts direction whenever the curvature of the path changes direction at a point of inflection.

A natural way to parametrise a curve is by the arc length s, i.e. the distance along the curve measured from some starting point. A point on the curve is then given by $\boldsymbol{r} = \boldsymbol{r}(s)$ and the tangent simply becomes

$$\boldsymbol{e}_s = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} \tag{1.29}$$

The normal points in the direction in which $\boldsymbol{e}_s$ changes and thus

$$\boldsymbol{e}_n = \rho\frac{\mathrm{d}\boldsymbol{e}_s}{\mathrm{d}s} \tag{1.30}$$

This is a way to define the radius of curvature ρ of the curve. It is possible to show that the derivatives of the other two unit vectors are

$$\frac{\mathrm{d}\boldsymbol{e}_n}{\mathrm{d}s} = -\frac{1}{\rho}\boldsymbol{e}_s + \frac{1}{\tau}\boldsymbol{e}_b \tag{1.31}$$

$$\frac{\mathrm{d}\boldsymbol{e}_b}{\mathrm{d}s} = -\frac{1}{\tau}\boldsymbol{e}_n \tag{1.32}$$

Here τ is called the torsion, which is usually defined by

$$\frac{1}{\tau} = \boldsymbol{e}_b \cdot \frac{\mathrm{d}\boldsymbol{e}_n}{\mathrm{d}s} \tag{1.33}$$

which follows directly from Eq. (1.31). The torsion is a measure of the rotation of the plane of the curve, i.e. the plane spanned by $\boldsymbol{e}_s$ and $\boldsymbol{e}_n$.

The velocity of the particle in tangent and normal coordinates is simply

$$\boldsymbol{v} = v\boldsymbol{e}_s \tag{1.34}$$

where $v = \dot{s}$ is the speed. The acceleration is obtained by differentiation and use of Eq. (1.30)

$$\boldsymbol{a} = \dot{v}\boldsymbol{e}_s + \frac{v^2}{\rho}\boldsymbol{e}_n \tag{1.35}$$

The first term is the tangential acceleration and the second is the centripetal acceleration directed towards the centre of curvature. Note the similarity between this formula and Eq. (1.17) for circular motion (with $v = r\dot{\varphi}$).

There are other coordinate systems that can be used, but this happens only in rather specialized situations. It is of more interest to use more than one system or combinations of systems. In fact any quantities, in practice lengths or angles, that can be used to fully specify the position of a particle can be used as coordinates. In the general case these are called generalized coordinates. They can even be more than three, in which case there exist constraints among them. This situation is more common for systems, see further Chapter 5.

Example 1.2.1

A particle is moving in a spiral with radius R and rising a distance a per turn. The position of the particle is thus

$$\boldsymbol{r} = R\cos\varphi\,\boldsymbol{e}_x + R\sin\varphi\,\boldsymbol{e}_y + \frac{a\varphi}{2\pi}\boldsymbol{e}_z$$

Here the position of the particle is given in terms of the angle $\varphi = \varphi(t)$. Determine the unit vectors in path variables and give the acceleration.

Solution. For an infinitesimal change in angle $\mathrm{d}\varphi$ the horizontal change in arc length is $R\mathrm{d}\varphi$ and the vertical change in arc length is $a\mathrm{d}\varphi/2\pi$, so the total infinitesimal arc length is

$$\mathrm{d}s = \sqrt{R^2 + \left(\frac{a}{2\pi}\right)^2}\,\mathrm{d}\varphi$$

Integration gives

$$s = b\varphi \qquad b = \sqrt{R^2 + \left(\frac{a}{2\pi}\right)^2}$$

The velocity of the particle is the time derivative of the position

$$\boldsymbol{v} = -R\dot{\varphi}\sin\varphi\,\boldsymbol{e}_x + R\dot{\varphi}\cos\varphi\,\boldsymbol{e}_y + \frac{a\dot{\varphi}}{2\pi}\boldsymbol{e}_z$$

But this should also be equal to $\boldsymbol{v} = \dot{s}\boldsymbol{e}_s = b\dot{\varphi}\boldsymbol{e}_s$, which gives the tangential unit vector

$$\boldsymbol{e}_s = \frac{1}{b}\left(-R\sin\varphi\,\boldsymbol{e}_x + R\cos\varphi\,\boldsymbol{e}_y + \frac{a}{2\pi}\boldsymbol{e}_z\right)$$

The normal unit vector can be obtained from

$$\frac{1}{\rho}\boldsymbol{e}_n = \frac{\mathrm{d}\boldsymbol{e}_s}{\mathrm{d}s} = \frac{\mathrm{d}\varphi}{\mathrm{d}s}\frac{\mathrm{d}\boldsymbol{e}_s}{\mathrm{d}\varphi} = \frac{1}{b^2}\left(-R\cos\varphi\,\boldsymbol{e}_x - R\sin\varphi\,\boldsymbol{e}_y\right)$$

As $\boldsymbol{e}_n$ is a unit vector it is

$$\boldsymbol{e}_n = -\cos\varphi\,\boldsymbol{e}_x - \sin\varphi\,\boldsymbol{e}_y$$

and the radius of curvature becomes

$$\rho = \frac{b^2}{R} = R + \frac{a^2}{4\pi^2 R}$$

Note that this radius is larger than R to which it reduces when a becomes small. The binormal is

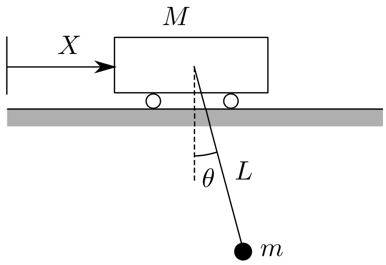
$$\boldsymbol{e}_b = \boldsymbol{e}_s \times \boldsymbol{e}_n = \frac{a}{2\pi b}\left(\sin\varphi\,\boldsymbol{e}_x - \cos\varphi\,\boldsymbol{e}_y\right) + \frac{R}{b}\boldsymbol{e}_z$$

Finally the acceleration becomes

$$\boldsymbol{a} = b\ddot{\varphi}\,\boldsymbol{e}_s + R\dot{\varphi}^2\boldsymbol{e}_n$$

Example 1.2.2

A pendulum of mass m and length L is suspended from a small cart of mass M that is free to move along a horizontal straight track. Introduce coordinates for the system and calculate the velocity and acceleration of the pendulum.



Solution. This system has two degrees of freedom which are naturally chosen as the horizontal displacement X of the cart and the angle from the vertical of the pendulum θ. The horizontal and vertical positions of the pendulum are then

$$x = X + L\sin\theta$$

$$y = -L\cos\theta$$

As these directions are fixed the velocity is obtained by differentiation

$$\dot{x} = \dot{X} + L\dot{\theta}\cos\theta$$

$$\dot{y} = L\dot{\theta}\sin\theta$$

Similarly the acceleration is obtained by further differentiation

$$\ddot{x} = \ddot{X} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta$$

$$\ddot{y} = L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta$$

1.3 Particle kinetics

As soon as the forces acting on a particle are essential in a problem, a kinematical description is not enough. This may happen because the force is known, but the path of the particle is unknown, as typified by planetary motion. Or the motion of the particle may be given, e.g. for circular motion, and the constraint forces are of interest. Intermediate problems are also common, e.g. the motion may be constrained to move along a curve or on a surface but the velocity along the curve or on the surface is unknown. To solve specific problems the coordinate systems from the previous section are often useful. Thus rectangular coordinates are convenient for projectile motion and spherical coordinates for a pendulum that can swing in all directions.

Newton's three laws lay the foundation of mechanics. The first law states that a body remains in equilibrium or moves with constant velocity if the resultant to all forces acting on the body is zero. This law can be seen as a special case of the second law, but can also be seen as the definition of an inertial frame, i.e. a system where Newton's second law is valid. The second law states that the change in momentum of the particle is equal to the resultant of all forces acting on the particle. The more usual way to formulate this is, of course, that the mass times the acceleration of the particle is equal to the force resultant. It is noted that this law is a vector equality and thus has three components in general. The third law states that the forces two particles exert on each other are equal in magnitude but opposite in direction.

Newton's second law is thus

$$m\boldsymbol{a} = \boldsymbol{F} \tag{1.36}$$

where m is the mass of the particle and $\boldsymbol{F}$ the sum of all forces on the particle. The acceleration is $\boldsymbol{a}$ and can be expressed in any of the ways of the previous section. Note that the equation is a vector equation with three components. Although the equation looks simple, this may not always be so when written out in components. Depending on the number of degrees of freedom of the particle the unknowns can be either (generalized) coordinates or constraint forces.

If Newton's second law is valid in one frame, it is also valid in all frames that translate with a constant velocity relative to the given frame, but not in a frame that is accelerating. A frame where Newton's second law is valid is called an inertial frame.

The notion of force is very central to mechanics, but it is also difficult to define. It is rather used in a phenomenological way. A force is a measure of how a material system is influenced by its surroundings. There must always be a source to a force. It is also assumed that a force is an objective quantity, i.e. that a force has the same value in all frames, also in accelerating frames. This means that fictitious forces, e.g. the "centrifugal force" is not a true or real force in the mechanical sense. This is not to say that fictitious forces may not be convenient and intuitive to use in some cases. Examples of forces are the gravitational force, various contact forces, and the elastic force in a spring. Additionally, a force may be distributed in different ways. The gravitational force is distributed over the whole body, while the force on the inside of a pressure tank is distributed over a surface. In many situations it is a good assumption that a distributed force is equivalent with a point force, thus the gravitational force may be substituted by a total gravitational force at the centre of gravity.

The magnitude of the gravitational force between two particles is

$$F = \frac{GmM}{r^2} \tag{1.37}$$

where m and M are the masses of the particles and r is the distance between them. The gravitational constant $G = 6.673 \cdot 10^{-11}$ Nm^2/kg^2, which gives all the known decimals to this constant. Furthermore, the force is directed along the line joining the two particles and is attractive. It may be shown that this formula is also valid for distributed masses provided the mass density is spherically symmetric. It is thus a very good model for planetary motion, e.g. for the attraction between the sun and the earth.

For engineering problems on the surface of the earth, one of the "particles" is the earth and the distance is the radius of the earth. This leads to the wellknown constant gravitational force

$$F = mg \tag{1.38}$$

where $g = 9.81$ m/s^2. The last decimal is here somewhat uncertain and in fact varies a little on the earth, primarily with the latitude.

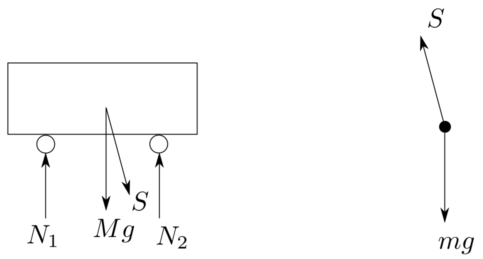
In all real systems there are damping forces. In many situations theses are small and are simply ignored, although it must be recognized that this is an approximation that usually breaks down after a sufficient time has elapsed. A simple pendulum is an example where the damping is very small, but given enough time the pendulum will of course come to a stop due to friction against the air and in the support. One common damping force is friction, which occurs when two bodies are sliding against each other. In so called dry friction the friction force F obeys

$$|F/N| \leq \mu \tag{1.39}$$

where N is the normal force between the two bodies in contact. This is an inequality, but if the two bodies are moving relative each other it becomes an equality, and the direction of F is always opposing the relative velocity. Usually μ is taken as a constant, but this is not completely true, it depends on the velocity to some extent.

At this point it is worth to mention the method with free-body diagrams that is very useful when solving problems in mechanics. In most cases it is good practice to draw a figure of each system (a single particle, a rigid body, or a system of more than one body) for which Newton's second law is going to be applied. The figure should contain all the forces on the system and maybe additional information such as lengths, angles, and accelerations.

Example 1.3.1
Put up the equations of motion for the system in **Example 1.2.2**.



Solution. Draw the free-body diagram for the cart and the pendulum. The force in the wire is S and must always be positive (a wire can support no compressive force). Newton's second law for the cart is

$$\rightarrow \quad M\ddot{X} = S\sin\theta$$

For the pendulum the accelerations were calculated in **Example 1.2.2**. The horizontal and vertical components of Newton's second law become

$$\rightarrow \quad m(\ddot{X} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) = -S\sin\theta$$

$$\uparrow \quad m(L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta) = S\cos\theta - mg$$

These are three equations for the unknowns X, θ, and S. The coordinates appear in differentiated form, but the force only enters as is and can thus easily be eliminated. Elimination of $S\sin\theta$ between the first two equations gives

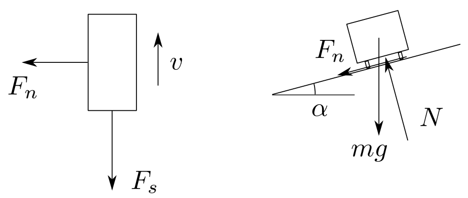
$$(M+m)\ddot{X} + m(L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) = 0$$

Multiply the first of Newton's second law for the pendulum by $\cos\theta$ and add to the second multiplied by $\sin\theta$

$$m\ddot{X}\cos\theta + mL\ddot{\theta} + mg\sin\theta = 0$$

These are the two equations of motion in terms of the coordinates X and θ. It is of course straightforward to eliminate $\ddot{X}$ and obtain one equation in θ.

Example 1.3.2
A car is driving through a curve with radius $50\,$m and $10°$ inclination. The car is driving with $50\,$km/h when it starts braking with $4.0\,$m/s^2. It is then just about to loose the grip. What is the largest constant speed the car can have through the curve?



Solution. This is a truly three-dimensional situation. Make free-body diagrams of the car projected vertically and from behind. The three components of Newton's second law in the vertical, normal, and tangential directions are

$$\uparrow \quad N\cos\alpha - mg - F_n\sin\alpha = 0$$

$$\leftarrow \quad F_n \cos\alpha + N \sin\alpha = m\frac{v^2}{R}$$

$$\otimes \quad -F_s = -ma$$

Here m is the mass of the car and $R = 50\,\mathrm{m}$, $\alpha = 10°$, $v = 50\,\mathrm{km/h}$, and $a = 4\,\mathrm{m/s^2}$. When decelerating the friction components in the normal and tangential directions, F_n and F_s, respectively, make up for the total resultant friction $F = \sqrt{F_n^2 + F_s^2}$. At the limit this friction is equal to the friction coefficient times the normal force, $F = \mu N$, where μ is the friction coefficient that is so far unknown. Solve for all the forces

$$N = mg\cos\alpha + \frac{mv^2}{R}\sin\alpha \qquad F_n = \frac{mv^2}{R}\cos\alpha - mg\sin\alpha \qquad F_s = ma$$

Put all the forces into the friction condition and solve for μ

$$\mu = \frac{\sqrt{\left(\frac{v^2}{R}\cos\alpha - g\sin\alpha\right)^2 + a^2}}{g\cos\alpha + \frac{v^2}{R}\sin\alpha} = 0.437$$

With the friction coefficient determined the highest possible speed is obtained from the same equations with $a = 0$

$$v = \sqrt{gR\frac{\mu\cos\alpha + \sin\alpha}{\cos\alpha - \mu\sin\alpha}} = 65\,\mathrm{km/h}$$

1.4 Derived laws

In many situations integrated versions of Newton's second law are useful. A first such law is obtained by noting that

$$\boldsymbol{a}\cdot \mathrm{d}\boldsymbol{r} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot \mathrm{d}\boldsymbol{r} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot \mathrm{d}\boldsymbol{v} = \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{v} = \frac{1}{2}\mathrm{d}(\boldsymbol{v}\cdot\boldsymbol{v}) = \frac{1}{2}\mathrm{d}(v^2) \tag{1.40}$$

Multiplying Newton's second law by $\mathrm{d}\boldsymbol{r}$ and integrating along the particle path then gives

$$\int \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \frac{1}{2}m\int \mathrm{d}(v^2) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \tag{1.41}$$

Here v_1 and v_2 are the speed of the particle at the beginning and end of the path. Define the kinetic energy of the particle

$$T = \frac{1}{2}mv^2 \tag{1.42}$$

and the work carried out by the forces

$$W = \int \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} \tag{1.43}$$

so that

$$W = T_2 - T_1 \tag{1.44}$$

Thus the work carried out by the forces equals the change in the kinetic energy. The line integral defining the work can alternatively be written

$$W = \int F_s \, \mathrm{d}s \tag{1.45}$$

where F_s is the component of the force tangent to the path and s is the arc length along the path. If this force component is constant the work is simply

$$W = F_s(s_2 - s_1) \tag{1.46}$$

In words, work is equal to force times distance.

In general the work is dependent on both the start and end points of the path as well as on the shape of the path. However, for some important forces the work is independent on the shape of the path and only depends on the start and end points. Such forces are called conservative and for such forces there exists a potential V such that

$$\boldsymbol{F} = -\nabla V \tag{1.47}$$

The line integral in Eq. (1.43) then becomes

$$W = V_1 - V_2 \tag{1.48}$$

so that

$$T_1 + V_1 = T_2 + V_2 = E \tag{1.49}$$

Here E is the total (mechanical) energy and the equation says that the energy is conserved if all acting forces are conservative.

Often several forces are acting on a particle, some conservative, some not. It is then useful to divide the forces into conservative and nonconservative and write

$$W^{(\mathrm{ik})} = T_2 + V_2 - T_1 - V_1 \tag{1.50}$$

where $W^{(\mathrm{ik})}$ is the work from the nonconservative forces and the potential energy only stems from the conservative forces. A common case when Eq. (1.50) is used is when all nonconservative forces are constraint forces, like normal forces and forces in cables, which do not do any work as the displacement is perpendicular to them and the work thus vanishes. In such cases the energy is conserved in spite of the presence of nonconservative forces.

Three common conservative forces are the constant gravitational force, the variable gravitational force, and the force in an elastic spring. The constant gravitational force is

$$F = mg \tag{1.51}$$

and is directed downward. The corresponding potential is

$$V = mgh \tag{1.52}$$

where h is the height above some convenient reference level. As is apparent from Eq. (1.47) the potential is defined apart from a constant. All constant forces admit a potential in the same manner.

The variable gravitational force is

$$F = -\frac{GmM}{r^2} \tag{1.53}$$

where r is the distance between the two particles with masses m and M and the minus sign is inserted because the force is opposite to the unit vector $\boldsymbol{e}_r$ if a polar (or spherical) coordinate system is introduced with the origin at one of the particles. The corresponding potential is

$$V = -\frac{GmM}{r} \tag{1.54}$$

This potential is always negative. The arbitrary constant in the potential is chosen so that the potential vanishes far away (large r).

For a linearly elastic spring the force is

$$F = -kx \tag{1.55}$$

where k is the spring constant and x is the spring elongation (or compression) from the unstretched position. The minus sign is inserted because the force is acting in the opposite direction to the elongation. Note that the spring is assumed to be massless and without damping. The corresponding potential is

$$V = \frac{1}{2}kx^2 \tag{1.56}$$

Here the arbitrary constant is naturally chosen so that the potential vanishes when the spring is unstretched. The spring may also be of torsional type. The restoring moment is then

$$M = -K\varphi \tag{1.57}$$

where K is the torsional spring constant (with dimension Nm/rad) and φ is the rotation of the spring from the unstretched angle. The corresponding potential is

$$V = \frac{1}{2}K\varphi^2 \tag{1.58}$$

If all forces on a particle are conservative and if a detailed description of the particle path is not requested, then it usually leads to simplifications to use energy conservation according to Eq. (1.49). Also in cases with nonconservative forces it can lead to simplifications to use energy considerations, now according to Eq. (1.50). This is particularly true for forces, like normal forces, that do not do any work, or when the work is easy to compute, e.g. for a constant friction force.

Another integrated version of Newton's second law is obtained by introducing the particle momentum

$$\boldsymbol{p} = m\boldsymbol{v} \tag{1.59}$$

so that Newton's second law can be written

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F} \tag{1.60}$$

Performing an integration in time over some convenient interval then gives

$$\boldsymbol{p}_2 - \boldsymbol{p}_1 = \int_{t_1}^{t_2} \boldsymbol{F}\,\mathrm{d}t \tag{1.61}$$

The time integral on the right-hand side is the impulse of the forces and the equation thus says that the change in the particle momentum is equal to the impulse of the acting forces. The impulse law can be useful when the time dependence of the forces is known. For a single particle, however, this law is rarely used.

For a particle the angular momentum with respect to a point O is defined as

$$\boldsymbol{L}_O = \boldsymbol{r} \times \boldsymbol{p} \tag{1.62}$$

where $\boldsymbol{r}$ is the position of the particle relative O. Taking the time derivative and using Newton's second law gives

$$\dot{\boldsymbol{L}}_O = \boldsymbol{M}_O \tag{1.63}$$

where it has been used that $\boldsymbol{v} \times \boldsymbol{v} = 0$ and where

$$\boldsymbol{M}_O = \boldsymbol{r} \times \boldsymbol{F} \tag{1.64}$$

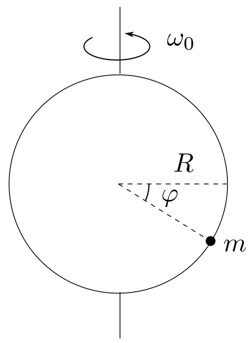
is the total moment of the forces with respect to O. For a particle Eq. (1.63) is seldom useful, but for a rigid body the corresponding equation determines the rotational motion. Integrating Eq. (1.63) with respect to time gives

$$\boldsymbol{L}_{O2} - \boldsymbol{L}_{O1} = \int_{t_1}^{t_2} \boldsymbol{M}_O \mathrm{d}t \tag{1.65}$$

The integral on the right-hand side defines the impulse moment of the forces. This equation is primarily useful in cases when (a component of) the impulse moment is zero and (the corresponding component of) the angular momentum is conserved.

Example 1.4.1

A particle of mass m is sliding without friction along a massless circular ring that can rotate freely around the vertical diameter. At an instant the particle is at rest at the middle of the ring while the ring has the angular velocity ω_0. What is the angular velocity of the particle at a later instant when the particle has dropped the angle φ?



Solution. Both energy and angular momentum about the vertical are conserved. There are also two unknowns, the angular velocity ω of the ring and the angular velocity $\dot{\varphi}$ of the particle along the ring. The velocity of the particle has two components, the tangential velocity $R\dot{\varphi}$ along the ring and the horizontal velocity $R\cos\varphi\,\omega$ in the circular motion around the vertical ring diameter. Taking the datum of the potential energy at the start, energy conservation gives

$$\frac{1}{2}m(R\omega_0)^2 = \frac{1}{2}m\left((R\dot{\varphi})^2 + (R\cos\varphi\,\omega)^2\right) - mgR\sin\varphi$$

The angular momentum only involves the horizontal velocity so conservation of angular momentum about the vertical gives

$$mR(R\omega_0) = m(R\cos\varphi)(R\cos\varphi\,\omega)$$

Solving gives the angular velocity of the particle along the ring

$$\dot\varphi = \sqrt{\frac{2g}{R}\sin\varphi - \left(\frac{\omega_0\sin\varphi}{\cos\varphi}\right)^2}$$

As the expression under the radical can not be negative this puts a limit on φ. Putting this expression to zero yields an equation for the maximum value for φ

$$\frac{2g}{R}\sin\varphi - \left(\frac{\omega_0\sin\varphi}{\cos\varphi}\right)^2 = 0$$

This can rewritten as a second order equation in $\sin\varphi$ which yields

$$\sin\varphi = -\alpha + \sqrt{\alpha^2+1} \qquad \alpha = \frac{R\omega_0^2}{4g}$$

It is seen that a large ω_0 gives a small maximum angle and that $\omega_0 = 0$ gives $\varphi = 90°$ as expected.

1.5 Oscillations, stability, and linearisation

In many cases the dynamic equations for a particle are not solvable by analytical methods. Important facts about the motion can, however, be obtained by linearizing the equations. This brings up the topics of oscillations and stability that are investigated shortly in this section and are later taken up again in Chapter 6.

As a first simple example consider a pendulum consisting of a particle of mass m suspended by a weightless string of length L. The pendulum is free to swing in a vertical plane under the action of gravity. The angle from the vertical of the string is denoted φ. Drawing the free-body diagram, the only forces on the particle are the gravitational force mg and the force in the string S. Newton's second law in the tangential direction gives

$$mL\ddot\varphi = -mg\sin\varphi \tag{1.66}$$

or

$$\ddot\varphi + \frac{g}{L}\sin\varphi = 0 \tag{1.67}$$

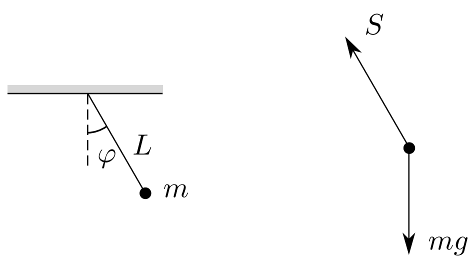


Figure 1.4: The pendulum.

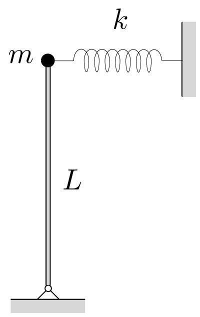


Figure 1.5: The inverted pendulum.

This equation can not be solved by elementary functions (the solution can, however, be given as an elliptic integral). But as long as the angle φ is small, say less than $10°$, the sine function can be expanded with good accuracy as $\sin\varphi \simeq \varphi$ so the equation becomes

$$\ddot{\varphi} + \frac{g}{L}\varphi = 0 \tag{1.68}$$

This is a simple second order differential equation whose solution is

$$\varphi(t) = A\sin(\omega t + \delta) = C_1 \sin\omega t + C_2 \cos\omega t \tag{1.69}$$

Here $\omega = \sqrt{g/L}$ is the angular frequency (measured in rad/s) of the oscillations. The two integration constants can be taken as the amplitude A and phase δ or as C_1 and C_2. The frequency of the oscillations is $f = \omega/2\pi$ (measured in Hz) and the period is $T = 1/f$ (measured in seconds).

This method of linearizing the dynamic equations often leads to a qualitative understanding of at least part of the dynamics of the particle. In more complicated situations the linearization can involve trigonometric functions, roots, fractions, etc, and after these are expanded it may still be necessary to eliminate products of small quantities. Another important point is around which value the linearization is performed. Most often this is an equilibrium position, but this is by no means necessary. There may also exist ”dynamic equilibria” of the system as is exemplified later.

As a more complicated example consider an inverted pendulum that is stabilized by a horizontal spring. In this case the particle of mass m is connected to a massless rod of length L that can swing in a vertical plane under the action of gravity and a spring of stiffness k that is supposed to be unstretched when the rod is vertical upwards. In this case the tangential equation of motion is somewhat modified as compared to the pendulum (linearizing directly)

$$mL\ddot{\varphi} = mg\varphi - kL\varphi \tag{1.70}$$

This equation is written

$$\ddot{\varphi} + (k/m - g/L)\varphi = 0 \tag{1.71}$$

The solution to this equation has a different character depending on the sign of $(k/m - g/L)$. If this quantity is positive the solution is oscillatory as for the pendulum. But if this quantity is negative the solution of the differential equation is

$$\varphi(t) = C_1 e^{\lambda t} + C_2 e^{-\lambda t} \tag{1.72}$$

where $\lambda = \sqrt{g/L - k/m}$. Here the positive exponential will dominate after some time and the solution thus grows large quickly. It is then no longer valid, of course, as the assumptions of the linearization are no longer valid. The vertical equilibrium for the inverted pendulum is unstable in this case.

In a general case the dynamic equation for a particle with one degree of freedom may be stated as (after linearization)

$$\ddot{x} + \beta\dot{x} + \gamma x = 0 \tag{1.73}$$

The first derivative term usually originates from some damping in the motion of the particle, but it may have other sources too. For the solutions to Eq. (1.73) to be stable and oscillatory both constants must be positive, i.e. $\beta > 0$ and $\gamma > 0$. Otherwise the solution is unstable.

Example 1.5.1
Consider again the system with a cart and a pendulum from **Example 1.2.2** and **Example 1.3.1**. Linearize the equations of motion and determine the angular frequency of the system.

Solution. A linearization of the equations of motion from **Example 1.3.1** gives

$$(M + m)\ddot{X} + m(L\ddot{\theta}) = 0$$

$$m\ddot{X} + mL\ddot{\theta} + mg\theta = 0$$

Note that in this process also $\dot{\theta}$ is assumed small so that $\dot{\theta}^2$ can be neglected. It is a simple matter to eliminate $\ddot{X}$ and obtain

$$\frac{ML}{M + m}\ddot{\theta} + g\theta = 0$$

This equation is the standard equation for oscillations with the angular frequency

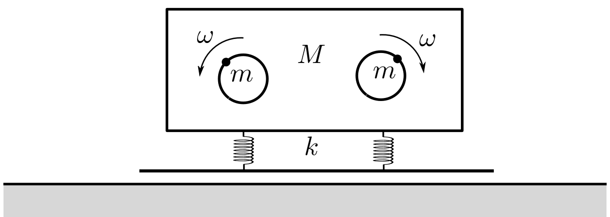
$$\omega = \sqrt{\frac{(M + m)g}{ML}}$$

If the mass of the cart is large, $M \to \infty$, the angular frequency for a simple pendulum is obtained again. With the mass of the cart finite the angular frequency is lower. The motion is such that the cart and the pendulum are moving in opposite directions so that the centre of mass is always fixed (in the horizontal direction). It is seen that there is a further solution to the linearized equations of motion, namely

$$X = v_0 t + X_0 \qquad \theta = 0$$

This is a pure translation with constant velocity of the system and corresponds to an angular frequency $\omega = 0$, see Chapter 6 for more on oscillations for systems with more than one degree of freedom.

Example 1.5.2
A vibrator consists of a frame with mass M and two counter-rotating excenter weights, each with mass m at the distance a from the corresponding rotation axle. The system is symmetrically attached to a bottom plate by springs with a total spring stiffness k. What is the force on the bottom plate if the weights are rotating with angular velocity ω?



Solution. The two rotating weights give a vertical force upwards on the frame that is

$$f = 2m(a\omega^2 \cos \omega t - g - \ddot{x})$$

where x is the vertical deflection of the frame and time is chosen so $t = 0$ corresponds to the weights being straight above the rotation axles. The equation for the vertical motion of the frame is

$$f - kx - Mg = M\ddot{x}$$

or

$$(M + 2m)\ddot{x} + kx = -(M + 2m)g + 2ma\omega^2 \cos \omega t$$

A particular solution can be assumed in the form

$$x = A + B \cos \omega t$$

It follows that

$$A = \frac{(M + 2m)g}{k}$$

$$B = \frac{2ma\omega^2}{k - (M + 2m)\omega^2}$$

The vertical force upwards on the bottom plate becomes

$$F = kx = -(M + 2m)g + \frac{2ma\omega^2}{k - (M + 2m)\omega^2} \cos \omega t$$

The first term is the influence of gravity and the second is due to the rotating masses. The force is strongly dependent on frequency and as no damping is included in the model the force becomes unbounded at the resonance, i.e. when

$$\omega^2 = \frac{k}{M + 2m}$$

To get a realistic model close to resonance damping must be included.

1.6 System of particles

The developments so far have essentially been for a single particle, although it is of course also useful for a few particles treated as individuals. It is then essential to remember Newton's third law that the forces between two particles (or any material systems) are equal (as vectors) except for the sign. When a system contains many particles it is often not possible, or even desirable, to keep track of each individual particle. This is the case for some problems with mass flow, but in particular also for rigid bodies, which can be regarded as systems of particles. It is for this reason that a system of particles is now studied.

Consider a system of N particles, each with mass m_i, $i = 1, 2, \ldots, N$, and position $\boldsymbol{r}_i$. The particles may be restrained in various ways, so the number of degrees of freedom is often less than $3N$.

The center of mass of the system is defined as

$$\overline{\boldsymbol{r}} = \frac{1}{m}\sum_{i=1}^{N} m_i \boldsymbol{r}_i \tag{1.74}$$

where m is the total mass of the system

$$m = \sum_{i=1}^{N} m_i \tag{1.75}$$

Note that an overbar, like in $\overline{\boldsymbol{r}}$, always denotes something for the centre of mass. The centre of mass is also commonly denoted the centre of gravity, in particular in everyday language, but strictly speaking the two concepts should be kept apart.

It is convenient to introduce the position $\boldsymbol{\rho}_i$ of particle i relative the centre of mass

$$\boldsymbol{r}_i = \overline{\boldsymbol{r}} + \boldsymbol{\rho}_i \tag{1.76}$$

The definition of the centre of mass then immediately leads to

$$\sum_{i=1}^{N} m_i \boldsymbol{\rho}_i = 0 \tag{1.77}$$

This relation is of course also valid for the relative velocities and accelerations.

Newton's second law for each particle is

$$m_i \boldsymbol{a}_i = \boldsymbol{F}_i + \boldsymbol{f}_i, \qquad i = 1, 2, \ldots, N \tag{1.78}$$

Here the forces are divided into two groups. The force $\boldsymbol{F}_i$ is the sum of all forces acting on the particle from outside the system, referred to as external forces. The force $\boldsymbol{f}_i$ is due to all the other particles of the system, referred to as internal or constraint forces. Summing Newton's second law for all the particles yields

$$\sum_{i=1}^{N} m_i \boldsymbol{a}_i = m\overline{\boldsymbol{a}} = \sum_{i=1}^{N} (\boldsymbol{F}_i + \boldsymbol{f}_i) = \sum_{i=1}^{N} \boldsymbol{F}_i = \boldsymbol{F} \tag{1.79}$$

where the definition of the centre of mass Eq. (1.74) is used. Newton's third law is also used and the last equality defines the total external force on the system.

Another way of writing the last equation is to introduce the linear momentum of the system

$$\boldsymbol{p} = \sum_{i=}^{N} \boldsymbol{p}_i = \sum_{i=1}^{N} m_i \boldsymbol{v}_i = m\overline{\boldsymbol{v}} \tag{1.80}$$

Again the definition of the centre of mass Eq. (1.74) is used. Newton's second law for the system can then be written

$$\dot{\boldsymbol{p}} = \boldsymbol{F} \tag{1.81}$$

As for a single particle this equation can be integrated

$$\boldsymbol{p}_2 - \boldsymbol{p}_1 = \int_{t_1}^{t_2} \boldsymbol{F}\,\mathrm{d}t \tag{1.82}$$

Here the right-hand side is called the impulse of the external forces. If it happens that this impulse vanishes, either because the sum of the external forces vanishes or because the integral becomes zero, then linear momentum is conserved. This conservation is useful in a number of situations, in particular in various types of collision problems. In other problems it may be more useful to rephrase this conservation into the statement that the centre of mass is moving at a constant speed.

The angular momentum of the system of particles about a fixed point O is the sum of the individual angular momenta of the particles

$$\boldsymbol{L}_O = \sum_{i=1}^{N} \boldsymbol{r}_i \times m_i \boldsymbol{v}_i \tag{1.83}$$

where $\boldsymbol{r}_i$ is the position of the i:th particle relative O. Write the position of the particles relative the centre of mass Eq. (1.76), so that the angular momentum of particle i is

$$\boldsymbol{L}_{Oi} = (\overline{\boldsymbol{r}} + \boldsymbol{\rho}_i) \times m_i(\overline{\boldsymbol{v}} + \dot{\boldsymbol{\rho}}_i) \tag{1.84}$$

Inserting this and using the definition of the centre of mass gives

$$\boldsymbol{L}_O = \overline{\boldsymbol{r}} \times m\overline{\boldsymbol{v}} + \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}} = \overline{\boldsymbol{r}} \times m\overline{\boldsymbol{v}} + \overline{\boldsymbol{L}} \tag{1.85}$$

This equation gives a division of the angular momentum as one term giving the angular momentum of the centre of mass plus one term giving the angular momentum of all the particles relative the centre of mass.

Taking the time derivative of Eq. (1.83) gives

$$\dot{\boldsymbol{L}}_O = \sum_{i=1}^{N} (\boldsymbol{v}_i \times m_i \boldsymbol{v}_i + \boldsymbol{r}_i \times m_i \dot{\boldsymbol{v}}_i) \tag{1.86}$$

In the second term Newton's second law Eq. (1.78) is used and terms of the form $\boldsymbol{r}_i \times \boldsymbol{f}_{ij}$ then appear, where $\boldsymbol{f}_{ij}$ is the force from particle j on particle i. When summed over all particles, pairs of terms can be grouped together

$$\boldsymbol{r}_i \times \boldsymbol{f}_{ij} + \boldsymbol{r}_j \times \boldsymbol{f}_{ji} = (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{f}_{ij} = 0 \tag{1.87}$$

Here Newton's third law is used and it is assumed that the force between the two particles is directed along the line joining the particles. Note that this assumption is a new postulate and, in particular, that this assumption will carry over also to rigid bodies. Collecting all the above then gives the law of the angular momentum for a system of particles about a fixed point O

$$\dot{\boldsymbol{L}}_O = \boldsymbol{M}_O \tag{1.88}$$

where $\boldsymbol{M}_O$ is the moment of all the external forces about O

$$\boldsymbol{M}_O = \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_i \tag{1.89}$$

This equation is a vector equation with three components, where the components can be taken in any convenient coordinate system.

Taking the time derivative of Eq. (1.85) gives

$$\dot{\boldsymbol{L}}_O = \overline{\boldsymbol{v}} \times m\overline{\boldsymbol{v}} + \overline{\boldsymbol{r}} \times m\overline{\boldsymbol{a}} + \dot{\overline{\boldsymbol{L}}} \tag{1.90}$$

With Eqs. (1.79) and (1.88) this becomes

$$\dot{\overline{\boldsymbol{L}}} + \overline{\boldsymbol{r}} \times \boldsymbol{F} = \boldsymbol{M}_O \tag{1.91}$$

But by the rule for connecting moments about different points this gives

$$\dot{\overline{\boldsymbol{L}}} = \overline{\boldsymbol{M}} \tag{1.92}$$

This is the law for the angular momentum for a system of particles about the centre of mass. The law for the angular momentum can be integrated in time, but this is most useful when the moment is zero so that the angular momentum, or a component, is conserved.

For a system of particles the kinetic energy is defined as the sum of the individual kinetic energies

$$T = \sum_i T_i = \sum_i \frac{1}{2} m_i v_i^2. \tag{1.93}$$

From Eq. (1.76) the velocity of particle i is

$$\boldsymbol{v}_i = \overline{\boldsymbol{v}} + \dot{\boldsymbol{\rho}}_i, \tag{1.94}$$

where $\overline{\boldsymbol{v}}$ is the velocity of the centre of mass. Thus

$$T = \frac{1}{2} \sum_i m_i (\overline{\boldsymbol{v}} + \dot{\boldsymbol{\rho}}_i) \cdot (\overline{\boldsymbol{v}} + \dot{\boldsymbol{\rho}}_i) = \frac{1}{2} \sum_i m_i \overline{\boldsymbol{v}} \cdot \overline{\boldsymbol{v}} + \sum_i m_i \overline{\boldsymbol{v}} \cdot \dot{\boldsymbol{\rho}}_i + \frac{1}{2} \sum_i m_i \dot{\boldsymbol{\rho}}_i \cdot \dot{\boldsymbol{\rho}}_i. \tag{1.95}$$

The second term vanishes due to Eq. (1.77). Thus

$$T = \frac{1}{2} m \overline{v}^2 + \sum_i \frac{m_i |\dot{\boldsymbol{\rho}}_i|^2}{2}. \tag{1.96}$$

This is known as König's theorem. The first term is the kinetic energy for a particle with the same mass as the whole system and the same velocity as the centre of mass. The second term is the contribution to the kinetic energy from the motion with respect to the centre of mass, i.e. the kinetic energy that the system would have had if the centre of mass had been fixed.

For particle i the change in kinetic energy is equal to the work done by all forces acting on the particle, external as well as internal:

$$W_i = \sum_i \Delta T_i. \tag{1.97}$$

Summing over i gives

$$W = T_2 - T_1, \tag{1.98}$$

where W is the work done by all forces, external and internal.

Example 1.6.1

Consider again the system in **Example 1.2.2**. If the pendulum is released from a horizontal position while the cart has no velocity, what is the speed of the cart when the pendulum reaches the vertical position?

Solution. There are no external forces on the system performing work except for the gravitational force. Thus energy is conserved. There are no horizontal forces at all on the system so horizontal linear momentum is conserved. Taking the datum for the potential energy at the starting position the energy at the start is zero. At the end position the pendulum has an angular velocity $\dot{\theta}$ and the cart has a velocity $\dot{X}$, both unknown. The velocity of the pendulum is $\dot{X} - L\dot{\theta}$, so energy conservation yields

$$\frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X} - L\dot{\theta})^2 - mgL = 0$$

Conservation of linear momentum gives

$$M\dot{X} + m(\dot{X} - L\dot{\theta}) = 0$$

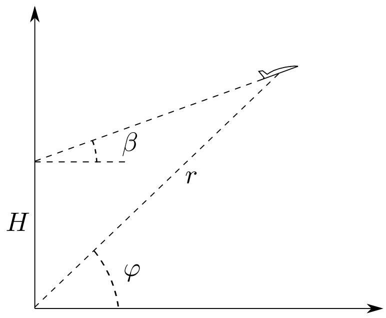
Eliminating $\dot{\theta}$ gives the speed of the cart

$$\dot{X} = \sqrt{\frac{2m^2gL}{M(M+m)}}$$

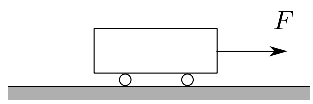
Problems

1.1 A car driving through a curve has speed $20\,\text{m/s}$ and acceleration $4\,\text{m/s}^2$ directed $45°$ to the direction of travel. Determine the radius of curvature for the path of the car at this instance.

1.2 An airplane is climbing with constant speed v and angle β. Determine the radial speed $\dot{r}$ and the angular speed $\dot{\varphi}$ as a function of the angle φ.



Prob. 1.2



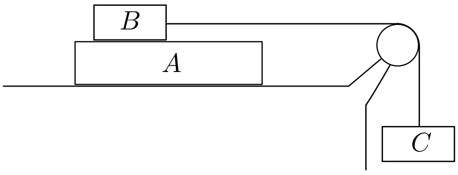
Prob. 1.3

1.3 A cart of mass m can move freely along a horizontal plane. The cart is at rest at time $t = 0$ when the force $F(t) = F_0\mathrm{e}^{-t/\tau}$ is applied. Determine the time it takes for the cart to reach half of its final velocity and how far it has gone then.

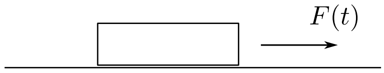
1.4 A body that falls freely through the air reaches after some time the velocity $60\,\text{m/s}$. How long time does it take the body to reach 90% of its final velocity if the air resistance is proportional to the square of the velocity?

1.5 A horizontal, plane plate is moving vertically in harmonic motion $z = h\cos\omega t$. A particle is thrown out at time $t = 0$ with the horizontal speed v_0 along the plate. The coefficient of friction between the particle and plate is μ. What is the condition for the particle staying in contact with the plate? If this condition is satisfied, give the equation that determines the time it takes for the horizontal motion to cease.

1.6 The three bodies A, B, and C in the figure have masses $2m$, m, and m, respectively. Between A and B the coefficient of friction is μ_1 and between A and the plane it is μ_2. The system is released from rest and the values of the coefficients of friction are such that B slides relative A and A slides relative the plane. Determine the acceleration of the three bodies.



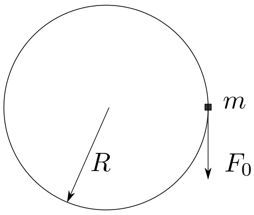
Prob. 1.6



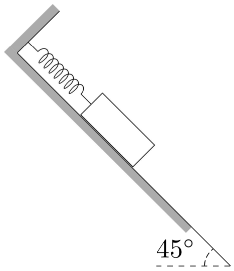
Prob. 1.7

1.7 A body with mass m can move on a horizontal surface. Between the body and the surface the coefficient of friction is 0.5. The body is at rest when a force $F(t) = F_0 \sin \omega t$, where $F_0 = mg$, is applied. The force is acting during half a period, i.e. during the time $0 < t < \pi/\omega$. How far does the body move during this time?

1.8 A particle with mass m is moving along a horizontal circular wire of radius R. Between the particle and the wire the coefficient of friction is μ. The particle starts from rest. A constant force F_0 that is all the time tangent to the circle is then applied. Determine the velocity of the particle after one complete lap.



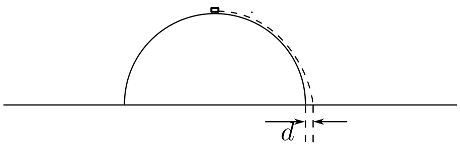
Prob. 1.8



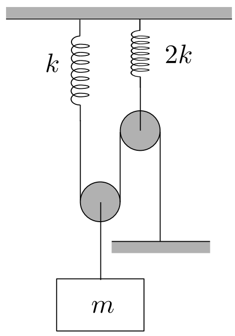
Prob. 1.9

1.9 A body with mass m lies on an inclined plane and is attached to a spring with stiffness k. There is friction between the body and the plane with the coefficient of friction μ. The system is released from rest with the spring compressed δ. What is the velocity of the body when it for the second time passes the position where the spring is unstretched?

1.10 A particle is released from rest at the top a smooth semi-cylinder and slides down along one side. After a while the particle loses contact with the cylinder and falls through the air. Where does the particle land, i.e. determine d in the figure. The radius of the cylinder is R.



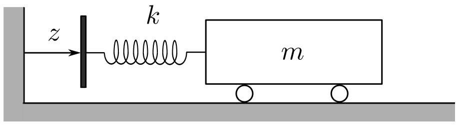
Prob. 1.10



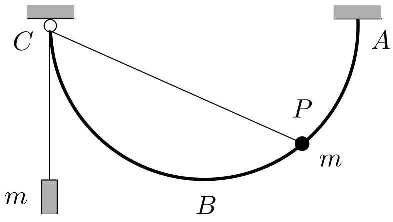
Prob. 1.11

1.11 A particle with mass m is part of the system in the figure. The pulleys are assumed to be massless. The system is released from rest with unstretched springs. Determine the motion of the particle.

1.12 A body of mass m is at rest when it is set into harmonic vibration due to an exiting motion $z = a \sin (2t\sqrt{k/m})$ of the left end of the spring. Here k is the spring constant. Determine the subsequent motion.

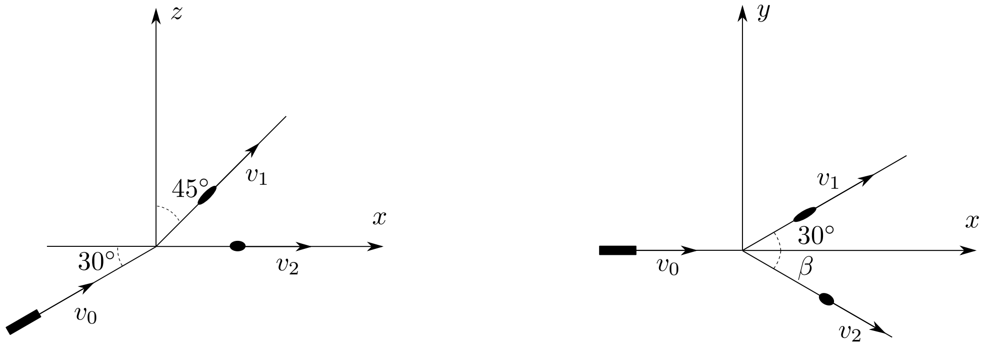


Prob. 1.12



Prob. 1.13

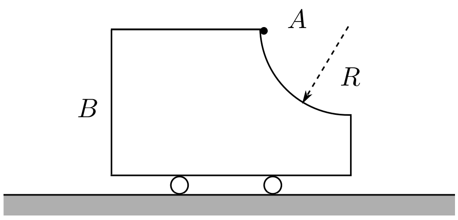
1.13 A small pearl P of mass m slides without friction along a semicircular, smooth wire of radius R in the vertical plane. The pearl is connected to another body, also of mass m, by a cord that runs through a smooth loop C. The system is released from rest in the position where the pearl is at point A. Determine the speed of the pearl when it passes the lowest position B.



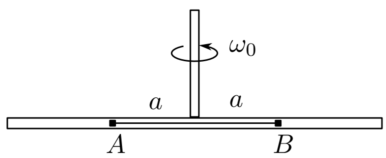
Prob. 1.14

1.14 A rocket is climbing with a speed v_0 and an angle $30°$ with the horizontal when is explodes into two pieces. One of them is continuing with the speed v_1 and an angle $45°$ to the vertical and with a change of direction $30°$ in the horizontal plane. The other continues horizontally with a change of direction β. The figures show the situation projected onto two planes, the vertical xz-plane and the horizontal xy-plane. How large is the angle β? The speeds v_0, v_1, and v_2 in the figures are there for the sake of clarity but must not be included in the answer.

1.15 A particle A with mass m is released from rest at the top and slides without friction along the path in the shape of a quarter of a circle with radius R according to the figure. The path is part of the cart B with mass M that moves freely on a horizontal plane. Determine the velocities of A and B when A leaves contact with B.



Prob. 1.15



Prob. 1.16

1.16 A narrow light pipe is connected to a vertical axis that can rotate freely. Two particles A and B, with mass m and $2m$, respectively, slide without friction inside the pipe and are connected by a cord of length $2a$. Initially the pipe is rotating with angular velocity ω_0 while the particles are held fixed relative the pipe. The particles are then released and B slides outwards. Determine the angular velocity and the velocity of the particles relative the pipe when A reaches the rotation axis.

Chapter 2

Relative motion

It is often convenient, or sometimes more or less necessary, to describe the motion of a body by describing its motion relative to a moving reference frame. Thus motion on the earth is almost always described relative to the rotating earth, not relative to a fixed system. And motion of parts in a car are of course described relative to the moving car. It is thus necessary to relate position, velocity, etc, in two frames that are moving relative to each other. It is then the rotations that cause most of the complications.

2.1 Rotating coordinate frames

As rotations are much simpler in two dimensions, this situation is first considered. Take a fixed XY system and a xy system that is rotated counterclockwise with the angle φ. From the figure it follows that

$$\boldsymbol{e}_x = \cos\varphi\,\boldsymbol{e}_X + \sin\varphi\,\boldsymbol{e}_Y \tag{2.1}$$

$$\boldsymbol{e}_y = -\sin\varphi\,\boldsymbol{e}_X + \cos\varphi\,\boldsymbol{e}_Y \tag{2.2}$$

This is conveniently written in matrix form

$$\begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_X \\ \boldsymbol{e}_Y \end{pmatrix} \tag{2.3}$$

Here

$$R = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \tag{2.4}$$

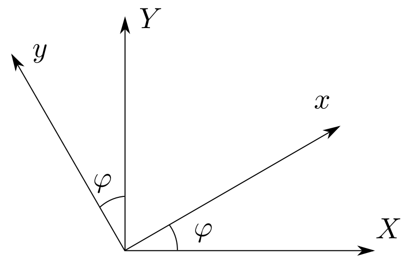


Figure 2.1: Coordinate rotation in 2D.

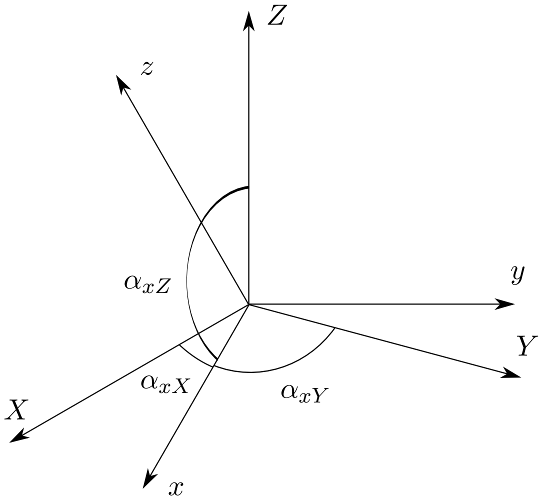


Figure 2.2: Coordinate rotation in 3D.

is the so called rotation matrix. This matrix is easily seen to satisfy

$$RR^{\mathrm{T}} = U \tag{2.5}$$

where T denotes the transpose and U is the unit matrix. This means that the rotation matrix is orthogonal and that

$$R^{-1} = R^{\mathrm{T}} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \tag{2.6}$$

It is thus a trivial matter to calculate the inverse of an orthogonal matrix.

Two consecutive rotations in two dimensions are around the same axis and therefore the order of the rotations does not matter. If the rotations are by the angles φ_1 and φ_2 the total rotation is by the angle $\varphi = \varphi_1 + \varphi_2$ and the rotation matrices commute: $R_\varphi = R_{\varphi_1} R_{\varphi_2} = R_{\varphi_2} R_{\varphi_1}$. However, this is particular to two dimensions, and in three dimensions the order of rotations is important.

Consider now three dimensions and two coordinate frames with coinciding origin, but with non-parallel axes. The frame XYZ with unit vectors $\boldsymbol{e}_X$, $\boldsymbol{e}_Y$, $\boldsymbol{e}_Z$ is regarded as fixed, whereas the frame xyz with unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ is regarded as rotating. Then the unit vectors can be expressed in terms of each other, for instance

$$\boldsymbol{e}_x = c_{xX}\boldsymbol{e}_X + c_{xY}\boldsymbol{e}_Y + c_{xZ}\boldsymbol{e}_Z \tag{2.7}$$

Here the quantities c_{xI}, $I = X, Y, Z$, can be expressed as the scalar product between the two unit vectors or as the direction cosine between the two unit vectors

$$c_{xI} = \boldsymbol{e}_x \cdot \boldsymbol{e}_I = \cos\alpha_{xI}, \qquad I = X, Y, Z \tag{2.8}$$

where α_{xX} is the angle between the x and X axes, etc. The three equations describing the rotation of the three unit vectors can conveniently be written in matrix form

$$\begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{pmatrix} = R \begin{pmatrix} \boldsymbol{e}_X \\ \boldsymbol{e}_Y \\ \boldsymbol{e}_Z \end{pmatrix} \tag{2.9}$$

where the matrix R is

$$R = \begin{pmatrix} c_{xX} & c_{xY} & c_{xZ} \\ c_{yX} & c_{yY} & c_{yZ} \\ c_{zX} & c_{zY} & c_{zZ} \end{pmatrix} \tag{2.10}$$

The matrix R is the rotation (transformation) matrix, and thus relates the unit vectors in two different frames.

The unit vectors are orthonormal and this leads to

$$c_{iX}c_{jX} + c_{iY}c_{jY} + c_{iZ}c_{jZ} = \delta_{ij}, \qquad i,\, j = x,\, y,\, z \tag{2.11}$$

Here the Kronecker delta is $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise. This equation amounts to six independent equations and thus only three out of the nine elements in the rotation matrix are independent. How these three are specified is investigated in the next chapter. It is apparent that the three elements in Eq. (2.7) cannot be chosen because $c_{xX}^2 + c_{xY}^2 + c_{xZ}^2 = 1$. In matrix form Eq. (2.11) is

$$RR^{\mathrm{T}} = U \tag{2.12}$$

where U is the unit matrix (now of size 3). The rotation matrix is still orthogonal. The inverse transformation to Eq. (2.9) is consequently

$$\begin{pmatrix} \boldsymbol{e}_X \\ \boldsymbol{e}_Y \\ \boldsymbol{e}_Z \end{pmatrix} = R' \begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{pmatrix} \tag{2.13}$$

with

$$R' = R^{-1} = R^{\mathrm{T}} \tag{2.14}$$

The property that the inverse of the rotation matrix is equal to the transpose is very convenient as it is much easier to calculate the transpose rather than the inverse.

An arbitrary vector $\boldsymbol{A}$ can be given components in the XYZ or the xyz system

$$\boldsymbol{A} = A_X\boldsymbol{e}_X + A_Y\boldsymbol{e}_Y + A_Z\boldsymbol{e}_Z = A_x\boldsymbol{e}_x + A_y\boldsymbol{e}_y + A_z\boldsymbol{e}_z \tag{2.15}$$

In matrix form this can be written

$$\begin{pmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_X & \boldsymbol{e}_Y & \boldsymbol{e}_Z \end{pmatrix} \begin{pmatrix} A_X \\ A_Y \\ A_Z \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \end{pmatrix} \left(R'\right)^{\mathrm{T}} \begin{pmatrix} A_X \\ A_Y \\ A_Z \end{pmatrix} \tag{2.16}$$

where Eq. (2.13) has been used. With Eq. (2.14) this gives

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = R \begin{pmatrix} A_X \\ A_Y \\ A_Z \end{pmatrix} \tag{2.17}$$

This equation describes how the components of a vector transform and it is noted that this transformation is identical to the one of the unit vectors Eq. (2.9). This is in particular also true of the vector (x, y, z) and when investigating transformations between frames in general this vector is used.

Rotation matrices become particularly simple when a rotation about a coordinate axis is taken. The situation is then in essence two-dimensional, so the result is

$$R_X(\theta_X) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & \sin\theta_X \\ 0 & -\sin\theta_X & \cos\theta_X \end{pmatrix} \tag{2.18}$$

$$R_Y(\theta_Y) = \begin{pmatrix} \cos\theta_Y & 0 & -\sin\theta_Y \\ 0 & 1 & 0 \\ \sin\theta_Y & 0 & \cos\theta_Y \end{pmatrix} \tag{2.19}$$

$$R_Z(\theta_Z) = \begin{pmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.20}$$

The index on R refers to the axis of rotation and the argument to the rotation angle. Sometimes one or both of these are omitted when it is clear from the context which rotation matrix that is referred to. The most important use of these simple rotation matrices is in the construction of more complicated rotation matrices as described in the next chapter.

Consider now consecutive rotations. One must then distinguish between body-fixed and space-fixed rotations. In the first case the consecutive rotations are with respect to body-fixed axes, which means that the orientation of the second axis of rotation depends on the first rotation. For space-fixed rotations, in contrast, the rotations are about fixed axes in space that do not affect each other. Here only body-fixed rotations are used as these are the most natural ones in most applications, see the next chapter.

As an example with body-fixed axes consider a first rotation with the angle φ about the X axis in the fixed XYZ system yielding the new $x_1y_1z_1$ system, followed by a rotation with the angle θ about the y_1 axis yielding the final xyz system. Using the notation in Eqs. (2.18) and (2.19) this is

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = R_X(\varphi) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{2.21}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_{y_1}(\theta) \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \tag{2.22}$$

Eliminating the intermediate $x_1y_1z_1$ system this gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{2.23}$$

where the total rotation matrix is

$$R = R_{y_1}(\theta) R_X(\varphi) \tag{2.24}$$

As matrices do not commute in general the order of the two rotation matrices is of course important. For body-fixed rotations the first rotation matrix is to the right, followed by the next rotation matrix directly to the left, and so on if there are more than two rotations. For space-fixed rotations the order

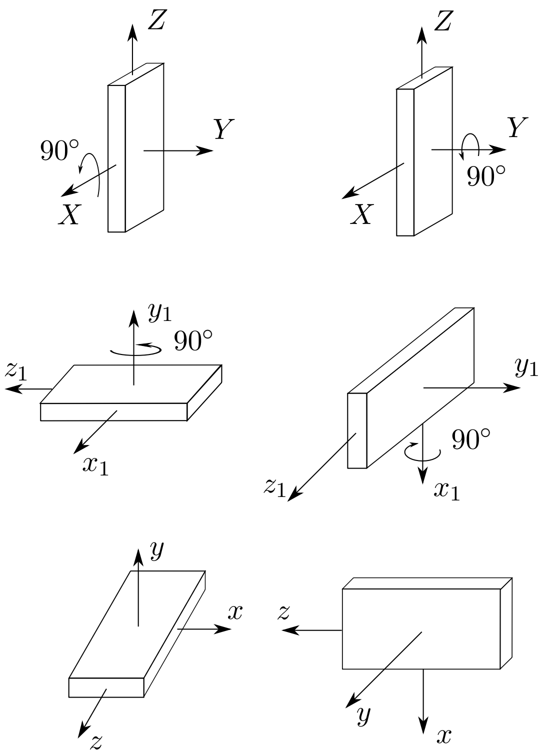


Figure 2.3: Coordinate rotations in 3D do not commute.

of the rotation matrices is reversed with the first matrix to the left followed by the next to the right, etc.

That the order of the rotation matrices is important corresponds to the fact that the order in which rotations are taken is important in general (the only exceptions are infinitesimal rotations, see the next section, and two or more rotations about the same axis). This also means that it is not possible to represent rotations by vectors (because vector addition is commutative). It is easy to illustrate that rotations are not commuting, by taking two $90°$ rotations about the X and y_1 axes and see that the resulting systems are in fact different.

Example 2.1.1

Consider a body-fixed rotation in two steps with a rotation by the angle φ about the fixed Z axis followed by a rotation with the angle θ about the rotated y_1 axis to the final xyz system. What are the coordinates in the XYZ system of the point that has coordinates $(x, y, z) = (a, 0, c)$ in the xyz system?

Solution. The total rotation matrix is

$$R = R_{y_1}(\theta)R_Z(\varphi)$$

which can be evaluated as

$$R = \begin{pmatrix} \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{pmatrix}$$

This gives the coordinates

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R^{\mathrm{T}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ 0 \\ c \end{pmatrix}$$

$$= \begin{pmatrix} a\cos\theta\cos\varphi + c\sin\theta\cos\varphi \\ a\cos\theta\sin\varphi + a\sin\theta\sin\varphi \\ -a\sin\theta + c\cos\theta \end{pmatrix}$$

This result can also be obtained by taking projections, but this is less systematic and the risk for errors is much larger. With three consecutive rotations projections become very difficult. It is noted that θ and φ are in fact spherical coordinates.

2.2 Angular velocity and angular acceleration

Although finite rotations do not commute, infinitesimal ones do. To see this, take three small rotations $\Delta\theta_X$, $\Delta\theta_Y$, $\Delta\theta_Z$ with respect to the respective axes. Use that $\cos\Delta\theta_X \simeq 1$ and $\sin\Delta\theta_X \simeq \Delta\theta_X$ for small rotations. The rotation matrices in Eqs. (2.18)–(2.20) then become unit matrices with two small elements symmetrically around the diagonal (with different signs). Multiplying together the three matrices and neglecting products of small quantities lead to the same matrix irrespective of the order of the matrices. This matrix becomes

$$R = \begin{pmatrix} 1 & \Delta\theta_Z & -\Delta\theta_Y \\ -\Delta\theta_Z & 1 & \Delta\theta_X \\ \Delta\theta_Y & -\Delta\theta_X & 1 \end{pmatrix} \tag{2.25}$$

It is thereby shown that infinitesimal rotations commute. It should be noted that $\Delta\theta_X$, $\Delta\theta_Y$, $\Delta\theta_Z$ are differential quantities in their own right, but that they are not differentials of finite expressions (like a $\mathrm{d}x$).

Now take a fixed point P in the moving xyz system which has the coordinates X_0, Y_0, Z_0. Before the rotation P then has the coordinates X_0, Y_0, Z_0 also in the fixed system X, Y, Z. After the infinitesimal rotation the corresponding coordinates are (invert, i.e. transpose, R in Eq. (2.25))

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & -\Delta\theta_Z & \Delta\theta_Y \\ \Delta\theta_Z & 1 & -\Delta\theta_X \\ -\Delta\theta_Y & \Delta\theta_X & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} \tag{2.26}$$

The change in the coordinates is

$$\begin{pmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{pmatrix} = \begin{pmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{pmatrix} = \begin{pmatrix} 0 & -\Delta\theta_Z & \Delta\theta_Y \\ \Delta\theta_Z & 0 & -\Delta\theta_X \\ -\Delta\theta_Y & \Delta\theta_X & 0 \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} \tag{2.27}$$

This can also be written as a cross product

$$\Delta\boldsymbol{r}_P = \boldsymbol{\Delta\theta} \times \boldsymbol{r}_P \tag{2.28}$$

where the vectors are

$$\Delta \boldsymbol{r}_P = \begin{pmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{pmatrix} \qquad \boldsymbol{r}_P = \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} \qquad \boldsymbol{\Delta\theta} = \begin{pmatrix} \Delta\theta_X \\ \Delta\theta_Y \\ \Delta\theta_Z \end{pmatrix} \tag{2.29}$$

with components in the fixed XYZ frame. As noted above the infinitesimal rotations do form the components of a vector, but there is no finite rotation vector from which the differential is formed. This is the reason for the notation $\boldsymbol{\Delta\theta}$ with also the $\boldsymbol{\Delta}$ bold-faced, as opposed to $\Delta\boldsymbol{r}$ where Δ is not bold-faced.

Divide Eq. (2.28) by the time Δt during which the rotation takes place and go to the limit $\Delta t \to 0$. Then the velocity of the point P appears

$$\lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}_P}{\Delta t} = \frac{\mathrm{d}\boldsymbol{r}_P}{\mathrm{d}t} = \dot{\boldsymbol{r}}_P \tag{2.30}$$

The limit also defines the angular velocity of the coordinate system

$$\boldsymbol{\omega} = \lim_{\Delta t \to 0} \frac{\boldsymbol{\Delta\theta}}{\Delta t} = \dot{\theta}_X \boldsymbol{e}_X + \dot{\theta}_Y \boldsymbol{e}_Y + \dot{\theta}_Z \boldsymbol{e}_Z \tag{2.31}$$

Note that this equation is not written as the derivative of any vector, because as stressed above a rotation can not be given as vector. Therefore, the angular velocity vector is called nonholonomic, a term that is associated with expressions that can not be expressed as derivatives of another expression. The way angular velocity is derived is very different from the derivation of translational velocity as a rate of change of position. After division by Δt and taking the limit, Eq. (2.28) becomes

$$\dot{\boldsymbol{r}}_P = \boldsymbol{\omega} \times \boldsymbol{r}_P \tag{2.32}$$

It must be remembered that this equation is derived under the assumption that the components of $\boldsymbol{r}_P$ are fixed in the rotating frame.

It should be mentioned that there are other ways to define angular velocity. One way is to make the definition in terms of the unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ of the rotating frame and their rates of change

$$\boldsymbol{\omega} = (\dot{\boldsymbol{e}}_y \cdot \boldsymbol{e}_z)\boldsymbol{e}_x + (\dot{\boldsymbol{e}}_z \cdot \boldsymbol{e}_x)\boldsymbol{e}_y + (\dot{\boldsymbol{e}}_x \cdot \boldsymbol{e}_y)\boldsymbol{e}_z \tag{2.33}$$

From a mathematical point of view this definition can be regarded as more sound. Yet another way to define the angular velocity is by using the elements of the rotation matrix

$$\omega_X = \sum_{i=1}^{3} c_{3i}\dot{c}_{2i} \tag{2.34}$$

where the components are still given in the fixed XYZ frame. The other two components are obtained by cyclic permutation.

There is one exception to the statement that angular velocity is not the derivative of any vector. If the angular velocity is along a fixed direction, then it is called a simple angular velocity and can be written $\boldsymbol{\omega} = \omega\boldsymbol{e}$, where $\boldsymbol{e}$ specifies the direction and ω can be written as a derivative of an angle

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{2.35}$$

where θ is the angle around the axis.

Start now from a fixed XYZ frame and make a rotation φ around the Z axis to the $x'y'z'$ frame. The corresponding angular velocity is

$$\boldsymbol{\omega}_1 = \dot{\varphi}\boldsymbol{e}_Z = \dot{\varphi}\boldsymbol{e}_{z'} \tag{2.36}$$

Then make an additional rotation with the angle θ around the y' axis to the final xyz frame. The corresponding angular velocity is

$$\boldsymbol{\omega}_2 = \dot{\theta}\boldsymbol{e}_{y'} = \dot{\theta}\boldsymbol{e}_y \tag{2.37}$$

This rotation is ”simple” when considered by itself. Because angular velocities are vectors the total angular velocity is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = \dot{\varphi}\boldsymbol{e}_{z'} + \dot{\theta}\boldsymbol{e}_{y'} \tag{2.38}$$

with both terms expressed in the same frame. However, it is usually of most interest to express the angular velocity in the xyz frame, and this becomes

$$\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\,\boldsymbol{e}_x + \dot{\theta}\boldsymbol{e}_y + \dot{\varphi}\cos\theta\,\boldsymbol{e}_z \tag{2.39}$$

Alternatively, the angular velocity can be expressed in the fixed XYZ frame

$$\boldsymbol{\omega} = -\dot{\theta}\sin\varphi\,\boldsymbol{e}_X + \dot{\theta}\cos\varphi\,\boldsymbol{e}_Y + \dot{\varphi}\boldsymbol{e}_Z \tag{2.40}$$

Here the unit vectors are constant and it is clear that this vector can not be obtained as the derivative of any vector and thus this superposition of two simple rotations is not simple.

The way of obtaining the total angular velocity as a sum of simple parts is very typical and more or less all angular velocities are obtained in this way. And as in the case just illustrated the different parts are expressed in different coordinate frames and rotations to a common frame is thus necessary. This common frame is often the last frame, but occasionally other frames are useful.

Equation (2.32) for the derivative of the constant position vector in the rotating frame can directly be generalized to any vector $\boldsymbol{c}$ with constant components

$$\dot{\boldsymbol{c}} = \boldsymbol{\omega} \times \boldsymbol{c} \tag{2.41}$$

This equation can be used to generalize the equation to the derivative of any vector in a rotating frame

$$\begin{aligned}\dot{\boldsymbol{c}} &= \frac{\mathrm{d}}{\mathrm{d}t}(c_x\boldsymbol{e}_x + c_y\boldsymbol{e}_y + c_z\boldsymbol{e}_z) \\ &= \dot{c}_x\boldsymbol{e}_x + \dot{c}_y\boldsymbol{e}_y + \dot{c}_z\boldsymbol{e}_z + c_x\boldsymbol{\omega}\times\boldsymbol{e}_x + c_y\boldsymbol{\omega}\times\boldsymbol{e}_y + c_z\boldsymbol{\omega}\times\boldsymbol{e}_z \\ &= \frac{\delta\boldsymbol{c}}{\delta t} + \boldsymbol{\omega}\times\boldsymbol{c}\end{aligned} \tag{2.42}$$

Here a local time derivative operator is introduced

$$\frac{\delta\boldsymbol{c}}{\delta t} = \dot{c}_x\boldsymbol{e}_x + \dot{c}_y\boldsymbol{e}_y + \dot{c}_z\boldsymbol{e}_z \tag{2.43}$$

This time derivative thus gives the rate of change relative the rotating xyz frame. Equation (2.42) gives the time derivative of a vector given in a rotating frame as a superposition of the effects due to the time dependence of the components and the effects due to the rotation of the frame.

The angular acceleration $\boldsymbol{\alpha}$ of the rotating frame is defined as the time derivative of the angular velocity

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} \tag{2.44}$$

In the fixed XYZ frame this becomes

$$\boldsymbol{\alpha} = \dot{\omega}_X \boldsymbol{e}_X + \dot{\omega}_Y \boldsymbol{e}_Y + \dot{\omega}_Z \boldsymbol{e}_Z \tag{2.45}$$

Of course no time derivative of the unit vectors enters as these are constant in the fixed frame. In the frame xyz for which the angular acceleration is calculated the angular acceleration becomes

$$\boldsymbol{\alpha} = \frac{\delta \boldsymbol{\omega}}{\delta t} + \boldsymbol{\omega} \times \boldsymbol{\omega} = \frac{\delta \boldsymbol{\omega}}{\delta t} \tag{2.46}$$

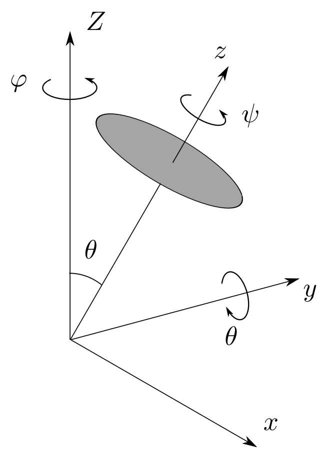
In this particular case no effects due to the rotation of the coordinate frame enter. If the angular velocity is instead expressed in another frame that is rotating with the angular velocity $\boldsymbol{\Omega}$ then the angular acceleration becomes

$$\boldsymbol{\alpha} = \frac{\delta \boldsymbol{\omega}}{\delta t} + \boldsymbol{\Omega} \times \boldsymbol{\omega} \tag{2.47}$$

Here the general rule that the rotation of the coordinate frame enters must be followed.

Example 2.2.1

The rotation of a spinning top is described by the three angles φ, θ, and ψ, where φ is the rotation about the fixed vertical direction, θ is the inclination of the spin axis with the vertical, and ψ is the spin angle about the symmetry axis of the top. Express the angular velocity and angular acceleration in the xyz coordinate system that is moving with the angles φ and θ, but which is not spinning, and is thus not fixed to the top.



Solution. The xyz coordinate system thus has the z axis as the symmetry axis of the top and the y axis horizontal. The inclination θ is about the y axis. The angular velocity is just the sum of the three simple rotations

$$\boldsymbol{\omega} = \dot{\varphi} \boldsymbol{e}_Z + \dot{\theta} \boldsymbol{e}_y + \dot{\psi} \boldsymbol{e}_z$$

But the angle θ is a rotation about the y axis so that

$$\boldsymbol{e}_Z = -\sin\theta\, \boldsymbol{e}_x + \cos\theta\, \boldsymbol{e}_z$$

The angular velocity thus becomes

$$\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\boldsymbol{e}_x + \dot{\theta}\boldsymbol{e}_y + (\dot{\psi} + \dot{\varphi}\cos\theta)\boldsymbol{e}_z$$

To calculate the angular acceleration it is first noted that the angular velocity of the xyz system is obtained by just deleting the spin term

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\,\boldsymbol{e}_x + \dot{\theta}\boldsymbol{e}_y + \dot{\varphi}\cos\theta\,\boldsymbol{e}_z$$

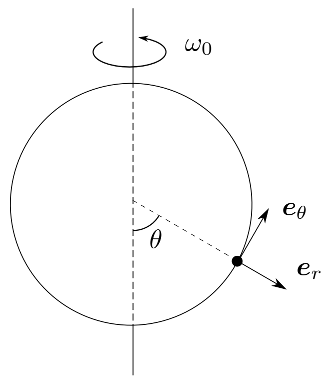
The angular acceleration then becomes

$$\begin{aligned}\boldsymbol{\alpha} &= \frac{\delta\boldsymbol{\omega}}{\delta t} + \boldsymbol{\Omega}\times\boldsymbol{\omega} \\ &= (\dot{\theta}\dot{\psi} - \ddot{\varphi}\sin\theta - \dot{\varphi}\dot{\theta}\cos\theta)\boldsymbol{e}_x + (\ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta)\boldsymbol{e}_y + (\ddot{\psi} + \ddot{\varphi}\cos\theta - \dot{\varphi}\dot{\theta}\sin\theta)\boldsymbol{e}_z\end{aligned}$$

where the terms $\dot{\theta}\dot{\psi}\boldsymbol{e}_x$ and $\dot{\varphi}\dot{\psi}\sin\theta\,\boldsymbol{e}_y$ comes from the cross product. If the angular velocity and acceleration are instead expressed in the fixed XYZ system or in a system fixed in the top, the expressions become lengthier.

Example 2.2.2

A particle of mass m is sliding without friction along a vertical ring of radius R that is rotating with the constant angular velocity ω_0. Determine the dynamical equilibria of the particle, i.e. positions along the ring for which the particle may be at rest relative the ring. Investigate also the stability of the equilibria.



Solution. Introduce the angle θ to give the position of the particle and a coordinate system $r\theta\varphi$ following the particle (this is in effect spherical coordinates). The angular velocity of this system is

$$\boldsymbol{\Omega} = -\omega_0\cos\theta\boldsymbol{e}_r + \omega_0\sin\theta\boldsymbol{e}_\theta + \dot{\theta}\boldsymbol{e}_\varphi$$

In this coordinate system the position of the particle is fixed

$$\boldsymbol{r} = R\boldsymbol{e}_r$$

so the velocity of the particle becomes

$$\boldsymbol{v} = \boldsymbol{\Omega}\times\boldsymbol{r} = R\dot{\theta}\boldsymbol{e}_\theta - R\omega_0\sin\theta\,\boldsymbol{e}_\varphi$$

Here the first term is the velocity in the circular motion along the ring and the second is the velocity in the circular motion with radius $R\sin\theta$ around the vertical. The acceleration becomes

$$\boldsymbol{a} = \frac{\delta\boldsymbol{v}}{\delta t} + \boldsymbol{\Omega}\times\boldsymbol{v}$$

$$= -R(\omega_0^2 \sin^2\theta + \dot{\theta}^2)\boldsymbol{e}_r + R(\ddot{\theta} - \omega_0^2 \sin\theta\cos\theta)\boldsymbol{e}_\theta - 2R\omega_0\dot{\theta}\cos\theta\,\boldsymbol{e}_\varphi$$

The two terms in ω_0^2 are the projected components of the centripetal acceleration due to the circular motion around the vertical. The term with $\dot{\theta}^2$ is likewise the centripetal acceleration due to the circular motion along the ring and the term with $\ddot{\theta}$ is the tangential acceleration in this motion. The last term is called the Coriolis acceleration and is due to the combined effects of two angular velocities (or an angular velocity and a velocity), see the next section.

In the free-body diagram of the particle two normal forces appear: N_1 in the radial direction and N_2 in the circumferential (φ) direction. Newton's second law in the r, θ, and φ directions gives

$$-mR(\omega_0^2 \sin^2\theta + \dot{\theta}^2) = mg\cos\theta - N_1$$

$$mR(\ddot{\theta} - \omega_0^2 \sin\theta\cos\theta) = -mg\sin\theta$$

$$-2mR\omega_0\dot{\theta}\cos\theta = -N_2$$

It is here the middle equation that determines θ and thereby the motion, whereas the other two equations can be used to determine the normal forces if these are of interest. The second equation can be rewritten as

$$\ddot{\theta} + \sin\theta\left(\frac{g}{R} - \omega_0^2\cos\theta\right) = 0$$

A dynamic equilibrium is characterized by a constant value of θ and this is possible if

$$\sin\theta\left(\frac{g}{R} - \omega_0^2\cos\theta\right) = 0$$

This gives three possibilities. Two simple solutions are $\theta = 0$ and $\theta = \pi$, corresponding to the particle being at rest at the bottom and top of the ring, respectively. A third possibility is

$$\cos\theta = \frac{g}{R\omega_0^2}$$

But this is only possible if $\omega_0 > \sqrt{g/R}$, i.e. the angular velocity of the ring must be above a certain threshold.

With the dynamic equilibria determined, what about the stability? For the dynamic equilibrium at $\theta = 0$ make a linearization, i.e. $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$

$$\ddot{\theta} + \left(\frac{g}{R} - \omega_0^2\right)\theta = 0$$

If

$$\omega_k^2 = \frac{g}{R} - \omega_0^2 > 0$$

then the solution is oscillatory with angular frequency ω_k and thus stable; otherwise the solution is exponential corresponding to an instability. A physical way to put this is that when the effect of gravity (through g) is stronger than the centrifugal effect (through ω_0) the motion is stable.

To determine the stability of the equilibrium at $\cos\theta^\star = g/R\omega_0^2 = \beta$, perform a linearization for $\xi = \theta - \theta^\star$ small

$$\sin(\xi + \theta^\star) = \sin\xi\cos\theta^\star + \cos\xi\sin\theta^\star \simeq \xi\beta + \sqrt{1-\beta^2}$$

$$\cos(\xi + \theta^\star) = \cos\xi\cos\theta^\star - \sin\xi\sin\theta^\star \simeq \beta - \xi\sqrt{1-\beta^2}$$

The equation for ξ becomes

$$\ddot{\xi} + \left(\xi\beta + \sqrt{1-\beta^2}\right)\left(\frac{g}{R} - \omega_0^2\left(\beta - \xi\sqrt{1-\beta^2}\right)\right) = 0$$

or

$$\ddot{\xi} + \xi\omega_0^2\left(1 - \beta^2\right) = 0$$

As $\beta < 1$ this equation always has stable solutions. It is noticed that when this equilibrium exists (and is stable) the equilibrium at $\theta = 0$ is unstable. Put in physical terms, for low angular velocity $\omega_0 < \sqrt{g/R}$ only the equilibrium around $\theta = 0$ exists and is stable and for higher angular velocity $\omega_0 > \sqrt{g/R}$ the equilibrium at $\theta = 0$ becomes unstable whereas a new equilibrium at $\cos\theta^\star = g/R\omega_0^2$ appears and is stable. This phenomenon where the qualitative behaviour of the system changes is called a bifurcation.

The equilibrium at $\theta = \pi$ can be shown to be always unstable, both gravity and the centrifugal effect are destabilizing.

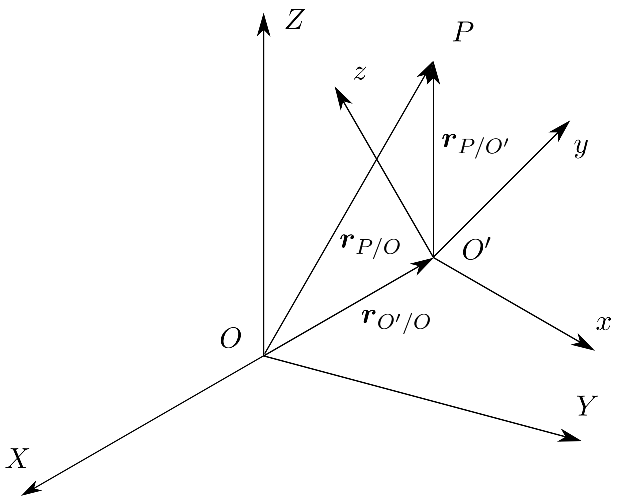


Figure 2.4: Coordinates in moving reference system.

2.3 Relative velocity and acceleration

Consider a point P in a moving coordinate frame xyz that is both translating and rotating. The absolute location of the point P can be given as the sum of P:s location in xyz and the location of this frame relative a fixed frame XYZ. Thus

$$\boldsymbol{r}_{P/O} = \boldsymbol{r}_{O'/O} + \boldsymbol{r}_{P/O'} \tag{2.48}$$

It is natural to give the coordinates of P in the moving frame

$$\boldsymbol{r}_{P/O'} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z \tag{2.49}$$

The absolute velocity of P then becomes (using Eq. (2.42))

$$\boldsymbol{v}_P = \boldsymbol{v}_{O'} + (\boldsymbol{v}_P)_{xyz} + \boldsymbol{\omega} \times \boldsymbol{r}_{P/O'} \tag{2.50}$$

where $\boldsymbol{v}_{O'}$ is the velocity of the xyz frame and $\boldsymbol{\omega}$ its angular velocity. The relative velocity of P in the moving frame is

$$(\boldsymbol{v}_P)_{xyz} = \frac{\delta}{\delta t}\boldsymbol{r}_{P/O'} = \dot{x}\boldsymbol{e}_x + \dot{y}\boldsymbol{e}_y + \dot{z}\boldsymbol{e}_z \tag{2.51}$$

The terms in Eq. (2.50) are as might be expected with the last term stemming from the rotation of the coordinate frame.

To obtain the absolute acceleration of the point P, Eq. (2.50) is differentiated with respect to time. For the derivative of the second term, Eq. (2.42) is used again

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_P)_{xyz} = (\boldsymbol{a}_P)_{xyz} + \boldsymbol{\omega} \times (\boldsymbol{v}_P)_{xyz} \tag{2.52}$$

where the relative acceleration is

$$(\boldsymbol{a}_P)_{xyz} = \frac{\delta}{\delta t}(\boldsymbol{v}_P)_{xyz} = \ddot{x}\boldsymbol{e}_x + \ddot{y}\boldsymbol{e}_y + \ddot{z}\boldsymbol{e}_z \tag{2.53}$$

The derivative of the third term in Eq. (2.50) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega} \times \boldsymbol{r}_{P/O'}) = \boldsymbol{\alpha} \times \boldsymbol{r}_{P/O'} + \boldsymbol{\omega} \times [(\boldsymbol{v}_P)_{xyz} + \boldsymbol{\omega} \times \boldsymbol{r}_{P/O'}] \tag{2.54}$$

where $\boldsymbol{\alpha}$ is the angular acceleration of the coordinate frame. The total acceleration becomes

$$\boldsymbol{a}_P = \boldsymbol{a}_{O'} + (\boldsymbol{a}_P)_{xyz} + \boldsymbol{\alpha} \times \boldsymbol{r}_{P/O'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{P/O'}) + 2\boldsymbol{\omega} \times (\boldsymbol{v}_P)_{xyz} \tag{2.55}$$

The first term on the right-hand side is the acceleration of the moving frame and the second is P:s relative acceleration in the moving frame. The third is due to the angular acceleration of the moving frame and is thus the tangential acceleration of P. The fourth term has the absolute value $\omega^2 r$, where r is the distance from P to the instantaneous axis of rotation. This is thus the centripetal acceleration.

The last term in Eq. (2.55) is called the Coriolis term and comes from two sources as seen in the derivation. This term is perpendicular to both $\boldsymbol{\omega}$ and $(\boldsymbol{v}_P)_{xyz}$ so that it always results in a change in the direction from $(\boldsymbol{v}_P)_{xyz}$. The magnitude of this term is often small, but can still be important because it causes a change in direction. This typically happens in problems with long time scales. Examples appear when the rotation of the earth must be taken into account, like in streams in the ocean or in air around low and high pressure centres.

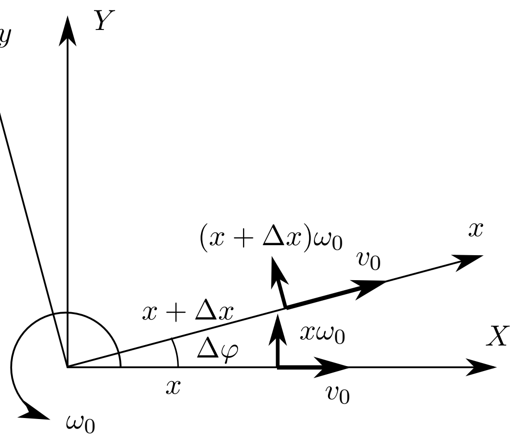


Figure 2.5: Illustration of the Coriolis acceleration.

To understand the origin of the Coriolis term it is instructive to study a simple example. Thus consider a fixed system XYZ and a system xyz which is coinciding with XYZ at time $t = 0$ and rotates about the Z axis with constant angular velocity ω_0. A point P is moving along the positive x

axis with constant speed v_0 and have the position x at time $t = 0$. At time $t = 0$ the point has the velocity

$$\boldsymbol{v}(0) = v_0\boldsymbol{e}_X + x\omega_0\boldsymbol{e}_Y \tag{2.56}$$

At a (small) later time Δt the velocity is

$$\boldsymbol{v}(\Delta t) = v_0\boldsymbol{e}_x + (x + \Delta x)\omega_0\boldsymbol{e}_y \tag{2.57}$$

where $\Delta x = v_0\Delta t$. During the time Δt the coordinate system xyz has rotated the angle $\Delta\varphi = \omega_0\Delta t$ so supposing a small angle the unit vectors are transformed according to

$$\boldsymbol{e}_x = \boldsymbol{e}_X \cos\Delta\varphi + \boldsymbol{e}_Y \sin\Delta\varphi = \boldsymbol{e}_X + \boldsymbol{e}_Y\Delta\varphi \tag{2.58}$$

$$\boldsymbol{e}_y = \boldsymbol{e}_Y \cos\Delta\varphi - \boldsymbol{e}_X \sin\Delta\varphi = \boldsymbol{e}_Y - \boldsymbol{e}_X\Delta\varphi \tag{2.59}$$

The definition of acceleration then gives

$$\boldsymbol{a} = \lim_{\Delta t\to 0} \frac{\boldsymbol{v}(\Delta t) - \boldsymbol{v}(0)}{\Delta t} = -x\omega_0^2\boldsymbol{e}_X + v_0\omega_0\boldsymbol{e}_Y + v_0\omega_0\boldsymbol{e}_Y \tag{2.60}$$

The first term is the centripetal acceleration, the second comes from the increased radius, and the last comes from change in direction of v_0 due to the rotation. The last two terms together is the Coriolis acceleration and as seen it has two causes.

An important application of the formula for the acceleration in a moving frame is when motion relative to the earth is considered. The rotation of the earth about its own axis is then the dominating effect. It can be shown that the effects due to the motion of the earth relative to the sun (and the sun's motion relative to the fix stars) are much smaller. The angular velocity of the earth is assumed constant both in magnitude and direction, although this is not absolutely true. The angular acceleration of the earth is then zero, so the term $\boldsymbol{\alpha} \times \boldsymbol{r}_{P/O'} = 0$. The centrifugal term depends on the latitude on the earth and is largest at the equator. It gives a contribution to g, which is thus a quantity depending on the latitude, although this is a minor effect.

The Coriolis term $2\boldsymbol{\omega} \times (\boldsymbol{v}_P)_{xyz}$ gives an effect when bodies are moving on earth. Consider a place on latitude λ (as measured from the equator) and introduce a coordinate system with the z axis vertical (along gravity including the centrifugal effect), the x axis directed to the east and the y axis to the north. The angular velocity of the earth is

$$\boldsymbol{\omega} = \Omega\cos\lambda\,\boldsymbol{e}_y + \Omega\sin\lambda\,\boldsymbol{e}_z \tag{2.61}$$

where

$$\Omega = \frac{2\pi}{24\cdot 60\cdot 60}\left(1 + \frac{1}{365.25}\right) = 7.29\times 10^{-5}\,\mathrm{rad/s} \tag{2.62}$$

where it is observed that it takes the earth 365.25 days to orbit the sun (and both rotations are counterclockwise). If the relative velocity of a particle relative to the earth is $(\boldsymbol{v}_P)_{xyz} = \dot{x}\boldsymbol{e}_x + \dot{y}\boldsymbol{e}_y + \dot{z}\boldsymbol{e}_z$ then the Coriolis acceleration becomes

$$\boldsymbol{a}_{cor} = 2\boldsymbol{\omega}\times(\boldsymbol{v}_P)_{xyz} = 2\Omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda)\boldsymbol{e}_x + 2\Omega\dot{x}\sin\lambda\,\boldsymbol{e}_y - 2\Omega\dot{x}\cos\lambda\,\boldsymbol{e}_z \tag{2.63}$$

In the earth-fixed frame Newton's second law for a particle acted on by only gravity then becomes

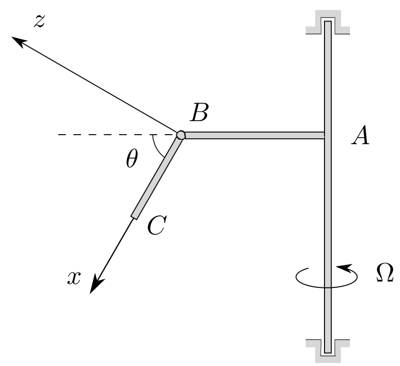
$$m(\ddot{x} + 2\Omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda)) = 0 \tag{2.64}$$

$$m(\ddot{y} + 2\Omega\dot{x}\sin\lambda) = 0 \tag{2.65}$$

$$m(\ddot{z} - 2\Omega\dot{x}\cos\lambda) = -mg \tag{2.66}$$

In the last equation the second term can be neglected for all reasonable velocities, because then $\Omega\dot{x} \ll g$. These equations can then be used in specific situations to investigate the effects of the Coriolis acceleration, see **Example 2.3.2** for an example.

Example 2.3.1
A rod BC of length b can rotate around the end B, which is attached to the rod AB of length a which rotates with constant angular velocity Ω according to the figure. The angle θ between AB and BC is an arbitrary function of time. Determine the velocity and acceleration of the point C expressed in the body-fixed coordinate system xyz.



Solution. The origin B of the coordinate system is moving in a circle with constant velocity. The tangential velocity is thus $a\Omega$ and the radial acceleration is $a\Omega^2$. Thus

$$\boldsymbol{v}_B = a\Omega\boldsymbol{e}_y$$

$$\boldsymbol{a}_B = -a\Omega^2\left(\cos\theta\,\boldsymbol{e}_x + \sin\theta\,\boldsymbol{e}_z\right)$$

The angular velocity of the coordinate system has two contributions due to $\dot{\theta}$ and Ω

$$\boldsymbol{\omega} = -\Omega\sin\theta\,\boldsymbol{e}_x + \dot{\theta}\boldsymbol{e}_y + \Omega\cos\theta\,\boldsymbol{e}_z$$

The angular acceleration of the coordinate system is according to Eq. (2.46)

$$\boldsymbol{\alpha} = -\Omega\dot{\theta}\cos\theta\,\boldsymbol{e}_x + \ddot{\theta}\boldsymbol{e}_y - \Omega\dot{\theta}\sin\theta\,\boldsymbol{e}_z$$

The point C is fixed in xyz so $\boldsymbol{r}_{C/B} = b\boldsymbol{e}_x$, $(\boldsymbol{v}_C)_{xyz} = 0$, and $(\boldsymbol{a}_C)_{xyz} = 0$. Equations (2.50) and (2.55) then give

$$\boldsymbol{v}_C = \boldsymbol{v}_B + \boldsymbol{\omega}\times\boldsymbol{r}_{B/C} = (a + b\cos\theta)\Omega\boldsymbol{e}_y - b\dot{\theta}\boldsymbol{e}_z$$

$$\begin{aligned}\boldsymbol{a}_C &= \boldsymbol{a}_B + \boldsymbol{\alpha}\times\boldsymbol{r}_{B/C} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_{B/C}) \\ &= -\left(b\dot{\theta}^2 + (a + b\cos\theta)\Omega^2\cos\theta\right)\boldsymbol{e}_x - 2b\Omega\dot{\theta}\sin\theta\,\boldsymbol{e}_y - \left(b\ddot{\theta} + (a + b\cos\theta)\Omega^2\sin\theta\right)\boldsymbol{e}_z\end{aligned}$$

Example 2.3.2
A particle is dropped at a height of $100\,\mathrm{m}$. How much is it displaced from the expected landing place due to the Coriolis effect? The latitude is $60°$.

Solution. The problem is here to find the solution to Eqs. (2.64)–(2.66) with the initial conditions $x = 0$, $y = 0$, $z = h = 100\,\mathrm{m}$ and $\dot{x} = \dot{y} = \dot{z} = 0$. The last equation immediately gives

$$z = h - \frac{1}{2}gt^2$$

As the motion will be predominantly vertical it is safe to neglect the term in $\dot{y}$ as compared to the term in $\dot{z}$ in the first equation that then gives

$$\ddot{x} = -2\Omega\dot{z}\cos\lambda = 2\Omega gt\cos\lambda$$

which is easily integrated to

$$x = \frac{1}{3}\Omega gt^3\cos\lambda$$

The particle reaches ground at the time $t = \sqrt{2h/g}$ which gives

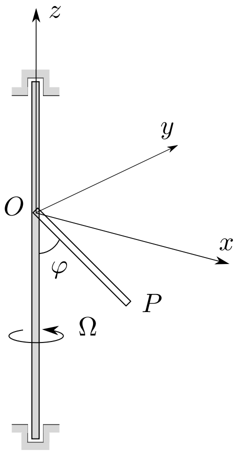
$$x = \frac{2}{3}\sqrt{2h^3}g\Omega\cos\lambda \simeq 11\,\mathrm{mm}$$

The displacement in the y direction can also be computed, but it will be much smaller. The particle is thus taking ground $11\,\mathrm{mm}$ to the east of the expected landing place. This distance is smaller than other disturbing effects, like wind, that can be anticipated in practice.

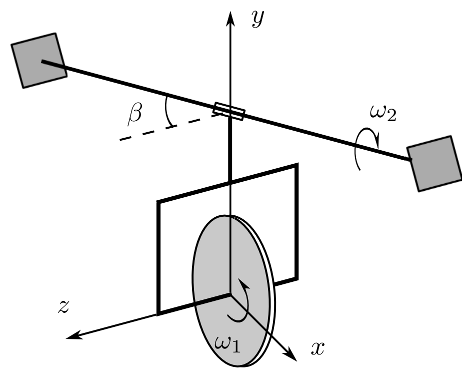
A physical explanation can be given as follows. For an observer that is not following the earth in its rotation both the dropping point P and the expected landing point A (vertically below P) are moving in circles with centres on the axis of the earth. Point P has the speed $(R+h)\Omega\cos\lambda$ whereas point A has the speed $R\Omega\cos\lambda$. The particle thus has a speed to the east that is larger than that of point A, and it thus moves more to the east than point A.

Problems

2.1 A bar OP is connected by a joint to a vertical axis which is rotating with the constant angular velocity Ω. The angle φ between OP and the vertical is varying with time according to $\varphi = \varphi_0 + \varphi_1 \sin qt$ where φ_0, φ_1, and q are constants. Determine the angular velocity and acceleration of OP. OP lies in the xz plane.



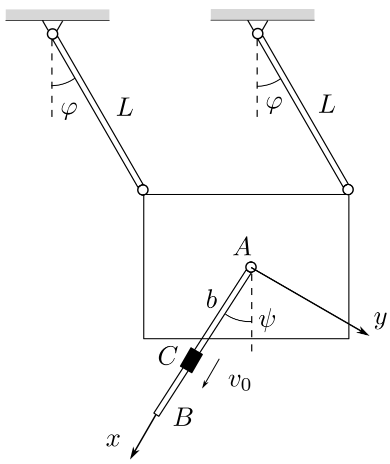
Prob. 2.1



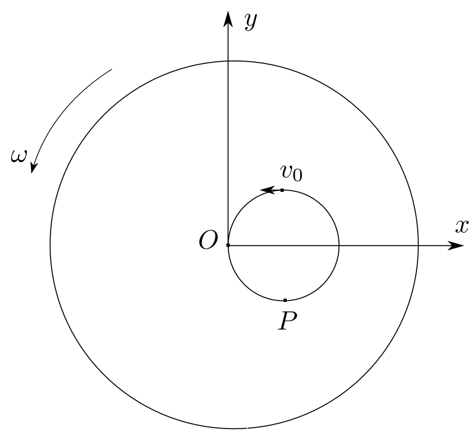
Prob. 2.2

2.2 A gyro consists of a disk which rotates with a constant angular velocity ω_1 about its axis. The whole gyro in turn rotates about the horizontal axis with variable angular velocity ω_2. The angle β between the gyro gimbal and the horizontal axis is an arbitrary function of time. Determine the angular acceleration of the gyro disk.

2.3 A disk is suspended in two equally long parallel bars, which are rotating with the constant angular velocity $\dot{\varphi} = \omega_1$. A third bar AB is rotating around A with constant angular velocity $\dot{\psi} = \omega_2$. A collar C is gliding along AB with the constant speed v_0. Determine the acceleration of C when $\varphi = \psi = 45°$.



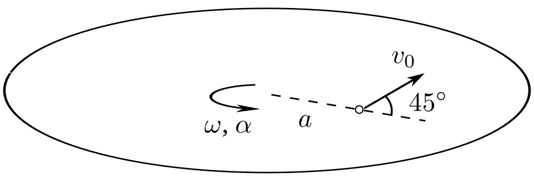
Prob. 2.3



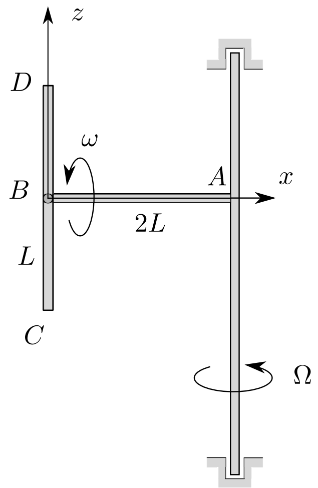
Prob. 2.4

2.4 A disk is rotating with the constant angular velocity ω around a fixed axis through O which is perpendicular to the disk. On the disk a particle is moving in a circle with radius r and constant relative speed v_0. Determine the velocity and acceleration of the particle at positions O and P.

2.5 At a particular instant a carousel has the angular velocity ω and the angular acceleration α. An ant is crawling on the carousel with constant relative velocity v_0 at the distance a from the carousel axis. The velocity makes the angle $45°$ with the radius in the outward direction and in the direction of the angular velocity. How large must the coefficient of friction be for the ant not to start sliding?



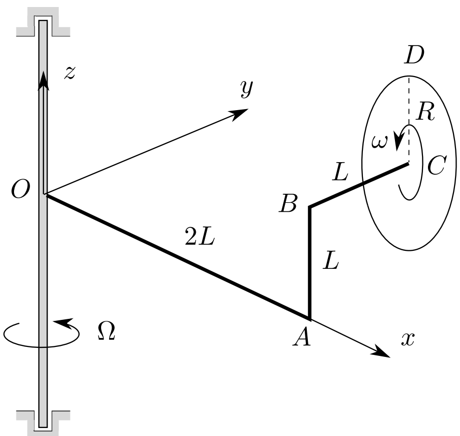
Prob. 2.5



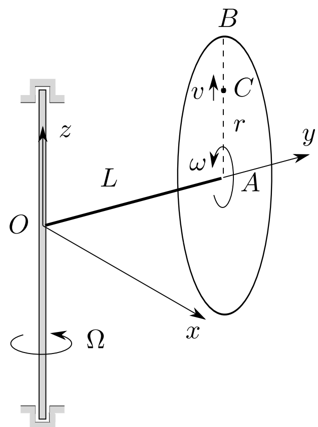
Prob. 2.6

2.6 A horizontal rod AB is rotating about a fixed vertical axis at a constant angular velocity Ω. Another rod CD is perpendicular to AB and rotates about AB with an angular velocity $\omega(t)$. Determine the velocity and acceleration of the point C at the position where CD is vertical. The xyz system is only rotating with the angular velocity Ω, not with ω.

2.7 A rod $OABC$ is composed of three parts such that they are all perpendicular to each other. The rod is rotating at a constant angular velocity Ω about a fixed vertical axis which is parallel to AB. A circular disk with radius R is perpendicular to BC and rotates at a constant angular velocity ω with respect to BC. Determine the velocity and acceleration of point D in the position shown.



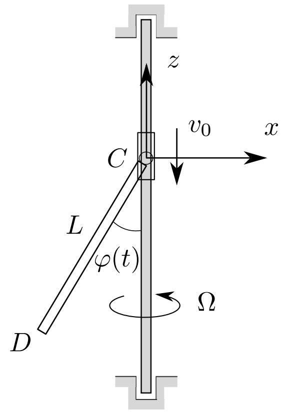
Prob. 2.7



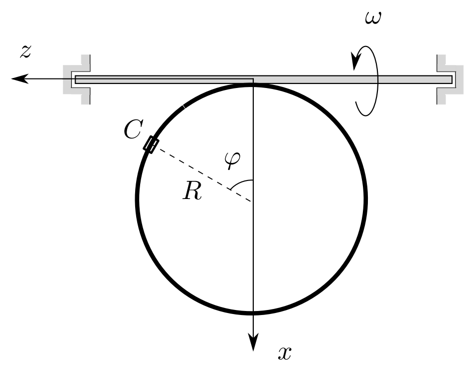
Prob. 2.8

2.8 A rod OA of length L is rotating at constant angular velocity Ω about a fixed vertical axis. A circular disk is mounted perpendicular to the rod and rotates at a constant angular velocity ω with respect to it. An insect C, which may be regarded as a particle, is crawling along a radius AB with constant velocity v with respect to the disk. At the particular instant when AB is vertical the insect is a distance r from A. Determine the velocity and acceleration of the insect at this instance.

2.9 A small collar is sliding relative a vertical rod at a constant velocity v_0. At the same time the rod and collar are rotating at a constant angular velocity Ω. Another rod CD of length L is connected to the collar by a hinge and forms an angle $\varphi(t)$ with the vertical rod. Determine the velocity and acceleration of the centre of mass of CD.



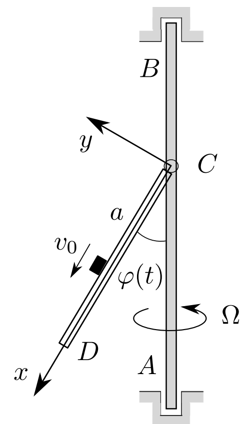
Prob. 2.9



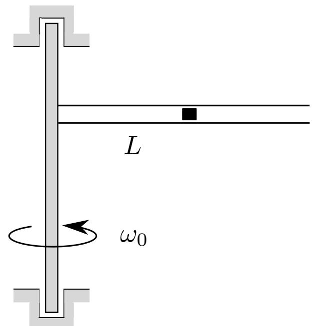
Prob. 2.10

2.10 A circular ring with radius R is welded to a fixed horizontal axis, so that the axis forms a tangent to the ring. The axis is rotating at a constant angular velocity ω. A small collar C is sliding along the ring. The relative position of C is given by the angle φ, where the angular velocity $\dot{\varphi}$ is constant. Determine the velocity and acceleration for C.

2.11 The vertical axis AB is rotating with constant angular velocity Ω. On the axis the bar CD is hinged and makes the angle $\varphi(t)$ with AB. A particle is moving along CD. At a particular instant the particle is at the distance a from C and has the constant velocity v_0 along the CD. Determine the acceleration of the particle at this instance.



Prob. 2.11



Prob. 2.12

2.12 A thin pipe is rotating in a horizontal plane with constant angular velocity ω_0 about a fixed axis. Inside the pipe a particle with mass m is sliding without friction. The particle is released from rest at the distance L from the rotation axis. Determine the position of the particle and the horizontal component of the normal force on the particle as functions of time.

2.13 A gun is fired horizontally with speed $450\,\text{m/s}$ towards a target $400\,\text{m}$ away at a latitude $30°$ south. How far sideways from the target is the bullet hitting?

Chapter 3

Rigid body kinematics

A rigid body can be viewed as a large (infinite) number of particles where all distances between the particles are fixed. No real bodies are of course absolutely rigid, but there exist numerous cases where a body can be regarded as rigid. In this chapter the kinematics of rigid bodies is considered, i.e. a description of position, velocity, and acceleration is given. A very important question in this connection is how to parametrize a general rotation in three dimensions. Also the different types of constraints that can exist between rigid bodies are described, and rolling is, in particular, investigated.

3.1 Velocity and acceleration relations

To describe the relation between the velocities and accelerations for arbitrary points in a rigid body it is convenient to attach a frame xyz to the body. This frame is thus body-fixed and moves completely with the body. The origin O and the orientation of the frame is arbitrary. For any other point P in the body the velocity and acceleration can be obtained from the equations of relative motion from the previous chapter. As P is fixed in the body and thus also in the frame the relative velocity $(\boldsymbol{v})_{xyz}$ and acceleration $(\boldsymbol{a})_{xyz}$ vanish, and Eqs. (2.50) and (2.55) give

$$\boldsymbol{v}_P = \boldsymbol{v}_O + \boldsymbol{\omega} \times \boldsymbol{r}_{P/O} \tag{3.1}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_O + \boldsymbol{\alpha} \times \boldsymbol{r}_{P/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{P/O}) \tag{3.2}$$

This is valid for every choice of O and P and this shows that the general motion of a rigid body can be described as a superposition of a translation and a rotation. In the translation all points in the body are moving in the same direction without a change in the orientation of the body and in the rotation the point O is fixed. If there is a point, that need not lie within the body, that is fixed during the motion, this is usually chosen as the point O. Otherwise the centre of mass is chosen, because the equations of motion for the rigid body are in general for the centre of mass, see the next chapter.

It should be stressed that Eqs. (3.1) and (3.2) are for the absolute velocity and acceleration. The equations can be expressed in any coordinate system although in many situations it is convenient to use the body-fixed system.

A natural question to put in this connection is whether the angular velocity is the same whatever point is used as the origin O of the body-fixed frame xyz. Consider two different origins, call them A and B. Assume for the moment that the corresponding angular velocities $\boldsymbol{\omega}_A$ and $\boldsymbol{\omega}_B$ may be

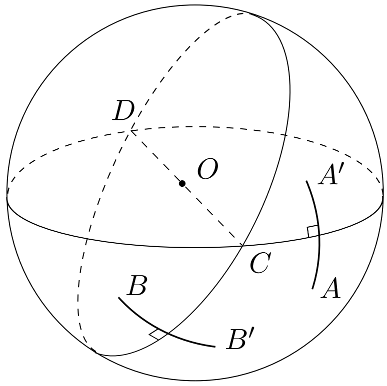


Figure 3.1: Euler's theorem: a rotation about O is a rotation about the line CD.

different. Take another arbitrary point P in the body and write the velocity of P using both A and B as the origin

$$\boldsymbol{v}_P = \boldsymbol{v}_A + \boldsymbol{\omega}_A \times \boldsymbol{r}_{P/A} \tag{3.3}$$

$$\boldsymbol{v}_P = \boldsymbol{v}_B + \boldsymbol{\omega}_B \times \boldsymbol{r}_{P/B} \tag{3.4}$$

But the velocity of B can also be written relative A

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega}_A \times \boldsymbol{r}_{B/A} \tag{3.5}$$

From these three equations there follows

$$\boldsymbol{\omega}_A \times \boldsymbol{r}_{P/A} = \boldsymbol{\omega}_A \times \boldsymbol{r}_{B/A} + \boldsymbol{\omega}_B \times \boldsymbol{r}_{P/B} \tag{3.6}$$

But as $\boldsymbol{r}_{P/B} = \boldsymbol{r}_{P/A} - \boldsymbol{r}_{B/A}$ it follows that

$$(\boldsymbol{\omega}_A - \boldsymbol{\omega}_B) \times \boldsymbol{r}_{P/B} = 0 \tag{3.7}$$

As P is arbitrary it is thereby shown that $\boldsymbol{\omega}_A = \boldsymbol{\omega}_B$, i.e. that the angular velocity is independent of the chosen reduction point in the body. As this is valid for all times and the angular acceleration is defined as the time derivative of the angular velocity, it follows that the angular acceleration is also independent of the reduction point.

Another way of looking upon the rotation of the body is the following. Consider a pure rotation where a point in the body is fixed and call this point O. Take two other points A and B that are arbitrary except that they are located at the same distance from O. After the rotation the same points are located at A' and B'. All the four points are thus located on a single sphere with origin at O. Now make the following geometrical construction. Draw the large circles between A and A' and between B and B'. (A large circle on a sphere is a circle with the same radius and centre as the sphere.) Through the middle points of these circles two additional large circles are drawn perpendicular to the two previous large circles. These new large circles cut each other at two diametrically opposite points C and D.

The point that before the rotation is located at C is assumed to move to C' due to the rotation. Due to the construction the distances between A and C and between A' and C are the same, thus $AC = A'C$. In the same way $BC = B'C$. As the body is rigid it is also true that $AC = A'C'$ and $BC = B'C'$. Thus $A'C = A'C'$ and $B'C = B'C'$ and as A and B are arbitrary, it follows that the

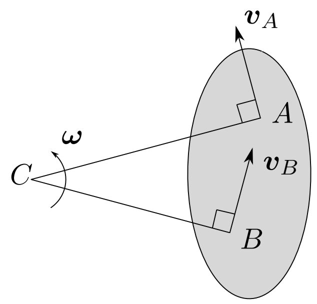


Figure 3.2: The instantaneous centre of rotation.

points C and C' coincide. The point C is thus at rest during the rotation and this is then true for all points along the line CD. The fact that an arbitrary rotation of a rigid body with one point fixed can be described as a rotation about a line through the point is called Euler's theorem.

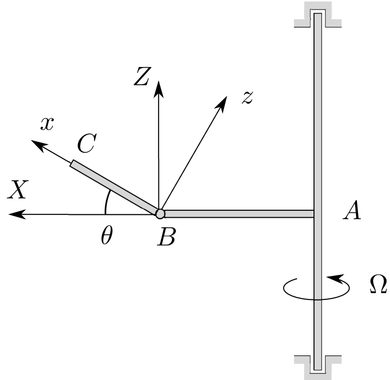
A corollary to Euler's theorem is Chasle's theorem which states that a general displacement of a rigid body can be viewed as a translation of an arbitrary point in the body plus a rotation about an axis through the point. Depending on the chosen point the axis of rotation is different.

Describing the translation of a point in the rigid body is the same as describing the motion of a particle and the same methods as used for a particle are employed. Thus three coordinates are in general needed to describe the translation of a rigid body. In addition three angles are in general needed to describe the rotation of the rigid body. In total six (generalized) coordinates are used to describe the general motion of a rigid body, i.e. the rigid body has six degrees of freedom. In most problems this number is reduced due to various constraints as discussed later in this chapter.

A useful concept for a rigid body in plane motion is the instantaneous centre of rotation. Let the point A in the body have velocity $\boldsymbol{v}_A$. Draw the normal to the velocity through A and locate the point C that lies the distance v_A/ω from A according to the figure, where ω is the angular velocity of the body. Due to the construction this point has the velocity zero and is therefore called the instantaneous centre of rotation (or simply the instant centre). Note that the point C need not lie within the body, it can be regarded as a thought point that is rigidly connected to the body. Note also that the point C is usually moving, both in space and with respect to the body, as the body is moving. The instant centre can be determined as soon as the directions (but not the size) of the velocity of two points in the body are known. If the directions of the velocities $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ are known, the instant centre is located where the normals to the two velocities intersect, see the figure. If the angular velocity of the body is zero, the body is purely translating and the instant centre is located at infinity (and is of no interest).

Example 3.1.1

The bar AB is rotating with the constant angular velocity Ω as shown in the figure. At B the bar BC is attached by a spherical joint and the bar is thus specified by the two spherical angles θ and φ according to the figure, where $\varphi = 0$ corresponds to the uppermost position of the bar. Determine the velocity of point C in the body-fixed system xyz.



Solution. To relate the unit vectors in the XYZ system, which is rotating with the angular velocity Ω, and the body-fixed system xyz, it is convenient to use the rotation matrices as in example **Example 2.1.1**. In this case the rotation from XYZ to xyz is first a rotation φ about the X axis followed by a rotation $-\theta$ about y

$$R = R_y(-\theta)R_X(\varphi)$$

This is evaluated as

$$R = \begin{pmatrix} \cos\theta & -\sin\theta\sin\varphi & \sin\theta\cos\varphi \\ 0 & \cos\varphi & \sin\varphi \\ -\sin\theta & -\cos\theta\sin\varphi & \cos\theta\cos\varphi \end{pmatrix}$$

From this matrix the relation between the unit vectors can be read off, and this is used when writing down the angular velocity of the bar BC

$$\boldsymbol{\omega} = \Omega\boldsymbol{e}_Z + \dot{\varphi}\boldsymbol{e}_X - \dot{\theta}\boldsymbol{e}_y = \Omega(\sin\theta\cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y + \cos\theta\cos\varphi\,\boldsymbol{e}_z) + \dot{\varphi}(\cos\theta\,\boldsymbol{e}_x - \sin\theta\,\boldsymbol{e}_z) - \dot{\theta}\boldsymbol{e}_y$$

The velocity of the point C is then

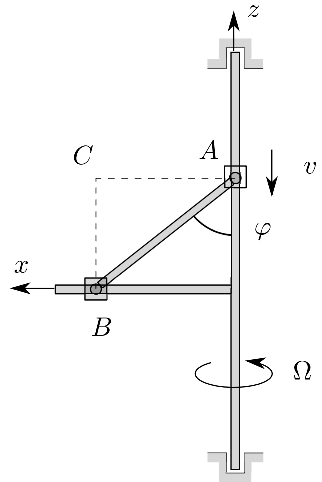
$$\boldsymbol{v}_C = a\Omega\boldsymbol{e}_Y + \boldsymbol{\omega}\times 2a\boldsymbol{e}_z$$

$$= a\Omega(-\sin\theta\sin\varphi\,\boldsymbol{e}_x + \cos\varphi\,\boldsymbol{e}_y - \cos\theta\sin\varphi\,\boldsymbol{e}_z) + 2a(\Omega\cos\theta\cos\varphi - \dot{\varphi}\sin\theta)\boldsymbol{e}_y + 2a(\dot{\theta} - \Omega\sin\theta)\boldsymbol{e}_z$$

It is of course also possible to derive the acceleration of C but the expression will be very lengthy.

Example 3.1.2

The bar AB of length $2L$ is attached to two collars at A and B that can slide along the vertical and horizontal guiding bars, respectively. At the moment shown, collar A has the downward constant velocity v. At the same time the whole system is rotating with the constant angular velocity Ω around the vertical. Determine the acceleration of the midpoint (the centre of mass) of AB.



Solution. Introduce the coordinate system according to the figure, where the z and x axes are along the vertical and horizontal guiding bars, respectively. The angular velocity of this system is $\boldsymbol{\Omega} = \Omega \boldsymbol{e}_z$. Although the motion is not planar, for the motion in the xz plane it is advantageous to use the instant centre. As the directions of the velocity at A and B are known, it follows that the instant centre is located at C. As the velocity of A is v the angular velocity of the bar for the motion in the zx plane is $v/(2L\sin\varphi)$. The total angular velocity of AB is then

$$\boldsymbol{\omega} = -\frac{v\boldsymbol{e}_y}{2L\sin\varphi} + \Omega\boldsymbol{e}_z$$

Using the instant centre the velocity of B in the x direction is $\omega 2L\cos\varphi$. As the velocity of the midpoint of AB is the mean of the velocities of A and B, this velocity becomes (the bar above $\boldsymbol{v}$ indicates that is the velocity of the centre of mass)

$$\overline{\boldsymbol{v}} = \frac{v\cos\varphi}{2\sin\varphi}\,\boldsymbol{e}_x + L\Omega\sin\varphi\,\boldsymbol{e}_y - \frac{v}{2}\,\boldsymbol{e}_z$$

To obtain the acceleration this expression is differentiated

$$\overline{\boldsymbol{a}} = \frac{\delta\overline{\boldsymbol{v}}}{\delta t} + \boldsymbol{\Omega}\times\overline{\boldsymbol{v}} = -\frac{v\dot{\varphi}}{2\sin^2\varphi}\,\boldsymbol{e}_x + L\Omega\dot{\varphi}\cos\varphi\,\boldsymbol{e}_y + \frac{v\Omega\cos\varphi}{2\sin\varphi}\,\boldsymbol{e}_y - L\omega^2\sin\varphi\,\boldsymbol{e}_x$$

But $\dot{\varphi}$ is the angular velocity in the xz plane of AB and inserting this the final result becomes

$$\overline{\boldsymbol{a}} = -\left(\frac{v^2}{4L\sin^3\varphi} + L\omega^2\sin\varphi\right)\boldsymbol{e}_x + \frac{v\Omega\cos\varphi}{\sin\varphi}\,\boldsymbol{e}_y$$

The first term in the x component is due to the decreasing outward velocity and the second term is the centripetal acceleration in the circular movement around the vertical. The y component is due to the Coriolis effect.

3.2 Euler and roll-pitch-yaw angles

The most common way to parametrize a general rotation in three dimensions is to use Euler angles. The idea is to use simple rotations about three consecutive coordinate axes to generate a general rotation. The simple rotation matrices about a coordinate axis are stated in Chap. 2 and are thus used for this purpose. Only body-fixed rotations are used here as these are almost exclusively used.

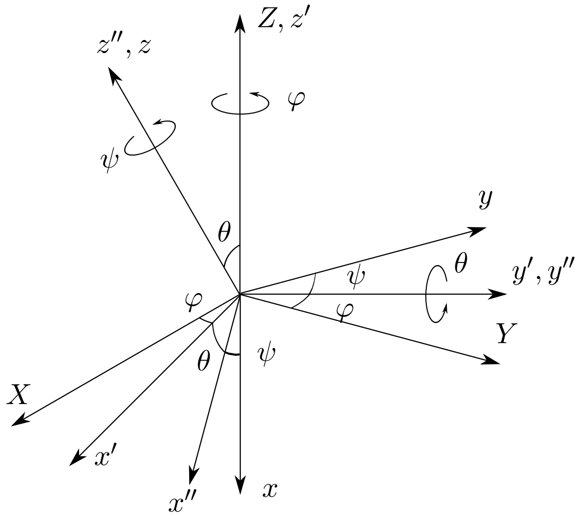


Figure 3.3: Euler angles.

The simple rotations can be taken about the three body-fixed coordinate axes in any order, with the only restriction that two consecutive rotations should never be about the same axis, as this is really only one rotation. The different possible sequences are denoted 3-2-1, 3-2-3, etc, where the first number denotes the first rotation, etc. There is a total of twelve different sequences. The rotation sequence starts from a fixed frame XYZ which coincides with the body-fixed xyz frame prior to the rotations. The 3-2-1 sequence thus starts with a rotation about the Z axis, giving a new orientation of the xyz frame which is denoted $x'y'z'$. The next rotation for the 3-2-1 sequence is about the y' axis (remember body-fixed rotations), leading to the frame $x''y''z''$, and the last rotation is about the x'' axis giving the final orientation of the xyz frame.

The twelve possible sequences divide into two groups, one group where three different axes are involved, like 3-2-1, and one where the first and last axes are the same, like 3-2-3 (but note that the Z and z'' axes are not the same physical axes except when the second rotation angle is zero). One sequence from each group is now presented in detail.

From the group with the first and last axes the same the sequence 3-2-3 is chosen, because this is probably the one mostly used (3-1-3 is also common). In this sequence the first rotation is with the angle φ about the fixed Z axis, which is often the vertical axis. The second rotation is with the angle θ about the y' axis and the last rotation is about the z'' axis with the angle ψ. The three angles are in turn called precession (φ), nutation (θ), and spin (ψ), a denomination that is natural e.g. for a three-axis gyro. Note that the first two angles φ and θ are the spherical angles of the z'' axis (and thereby also of the z (spin) axis), and that the names of the two angles also coincide (this is a reason to choose exactly this sequence).

The first rotation is thus

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_Z(\varphi) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{3.8}$$

where the rotation matrix is

$$R_Z(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.9}$$

The second rotation is

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = R_{y'}(\theta) \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \tag{3.10}$$

where the rotation matrix is

$$R_{y'}(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{3.11}$$

And the third and final rotation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_{z''}(\psi) \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} \tag{3.12}$$

where the rotation matrix is

$$R_{z''}(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.13}$$

The total rotation is the combination of all three rotations

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R(\varphi, \theta, \psi) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{3.14}$$

where the total rotation matrix is the product of the three simpler ones

$$R(\varphi, \theta, \psi) = R_{z''}(\psi) R_{y'}(\theta) R_Z(\varphi) \tag{3.15}$$

Note the order of the matrices with the first rotation to the right. Performing the matrix multiplications gives

$$R(\varphi, \theta, \psi) =$$

$$\begin{pmatrix} \cos\psi\cos\theta\cos\varphi - \sin\psi\sin\varphi & \cos\psi\cos\theta\sin\varphi + \sin\psi\cos\varphi & -\cos\psi\sin\theta \\ -\sin\psi\cos\theta\cos\varphi - \cos\psi\sin\varphi & -\sin\psi\cos\theta\sin\varphi + \cos\psi\cos\varphi & \sin\psi\sin\theta \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{pmatrix} \tag{3.16}$$

This matrix is quite complicated. If the θ angle is zero it is seen in the matrix that it is only the sum $\psi + \varphi$ that enters. This is a singularity of the 3-2-3 Euler angles, and corresponds to the fact that when $\theta = 0$ the precession and spin axes coincide and thus only the sum $\psi + \varphi$ has any significance. It is a fact that there exists a singularity of this type for all the twelve Euler sequences, and, in fact, for any parametrization of rotations that use only three parameters. The problem can be circumvented by employing four (or more) parameters, and such parametrizations are also in use, but are not discussed here.

The total angular velocity of the body in the 3-2-3 Euler rotation is

$$\boldsymbol{\omega} = \dot{\varphi}\, \boldsymbol{e}_Z + \dot{\theta}\, \boldsymbol{e}_{y'} + \dot{\psi}\, \boldsymbol{e}_{z''} \tag{3.17}$$

but it is noted that the three axes could equally well be denoted z', y'', and z, as the rotations are performed about these axes. This form of the angular velocity is not very useful as the unit vectors are in different frames. To transform the expression to the body-fixed frame xyz the following is used

$$\boldsymbol{e}_Z = -\cos\psi\sin\theta\,\boldsymbol{e}_x + \sin\psi\sin\theta\,\boldsymbol{e}_y + \cos\theta\,\boldsymbol{e}_z \tag{3.18}$$

$$\boldsymbol{e}_{y'} = \sin\psi\,\boldsymbol{e}_x + \cos\psi\,\boldsymbol{e}_y \tag{3.19}$$

The first equation follows from the last column of the total rotation matrix upon remembering that the unit vectors transform as the vector components and that the inverse of a rotation matrix is the transpose. The second equation likewise follows from the last simple rotation from $x''y''z''$ to xyz. The angular velocity in the body-fixed frame becomes

$$\boldsymbol{\omega} = (-\dot{\varphi}\sin\theta\cos\psi + \dot{\theta}\sin\psi)\boldsymbol{e}_x + (\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\boldsymbol{e}_y + (\dot{\varphi}\cos\theta + \dot{\psi})\boldsymbol{e}_z \tag{3.20}$$

In general, when stating the law of angular momentum for a rigid body, a body-fixed frame must be used, as is seen in the next chapter. Therefore, the angular velocity must usually be given in this frame as above. There is one important exception to this and that appears for the important case of rotationally symmetric bodies, like wheels of different kinds. Then it is advantageous to not include the spin $\dot{\psi}$ in the rotation of the frame, i.e. to use the $x''y''z''$ frame. In this frame the angular velocity becomes

$$\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\,\boldsymbol{e}_{x''} + \dot{\theta}\,\boldsymbol{e}_{y''} + (\dot{\varphi}\cos\theta + \dot{\psi})\boldsymbol{e}_{z''} \tag{3.21}$$

which is a simpler expression than Eq. (3.20).

If the angular acceleration is needed, it is simply obtained by taking the time derivatives of the components in Eq. (3.20), even though the frame is rotating. The reason is of course that it is exactly the angular velocity that is differentiated and the extra term in Eq. (2.42) becomes zero. If the expression Eq. (3.21) is used instead, then the extra term must be added as the angular velocities of the frame and the body are different.

When vehicles of different types, like cars, trains, boats, airplanes, etc., are modelled the Euler sequence 3-2-1 is usually employed, and in this connection the angles are called roll-pitch-yaw without any reference to Euler angles at all (many authors restrict the term ”Euler angles” to the sequences with two rotations about the same axis, like in the 3-2-3 case). The first rotation is then about the fixed Z axis (the vertical), which gives the yaw angle α. The second rotation is the pitch angle β about the y' axis (the lateral axis), and the third rotation is the roll about the x'' axis (the body-fixed forward axis in the vehicle). The total rotation matrix can be deduced as for the 3-2-3 sequence and the result is

$$R(\alpha,\beta,\gamma) = R_{x''}(\gamma)R_{y'}(\beta)R_Z(\alpha) =$$

$$\begin{pmatrix} \cos\beta\cos\alpha & \cos\beta\sin\alpha & -\sin\beta \\ \sin\gamma\sin\beta\cos\alpha - \cos\gamma\sin\alpha & \sin\gamma\sin\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\cos\beta \\ \cos\gamma\sin\beta\cos\alpha + \sin\gamma\sin\alpha & \cos\gamma\sin\beta\sin\alpha - \sin\gamma\cos\alpha & \cos\gamma\cos\beta \end{pmatrix} \tag{3.22}$$

This matrix is of the same complexity as the one for the 3-2-3 sequence. The angular velocity can be obtained in the same way. In this case it is usually the components in the body-fixed frame xyz that are of use

$$\boldsymbol{\omega} = (-\dot{\alpha}\sin\beta + \dot{\gamma})\boldsymbol{e}_x + (\dot{\alpha}\sin\gamma\cos\beta + \dot{\beta}\cos\gamma)\boldsymbol{e}_y + (\dot{\alpha}\cos\gamma\cos\beta - \dot{\beta}\sin\gamma)\boldsymbol{e}_z \tag{3.23}$$

The angular acceleration can be obtained by differentiating the components.

For the roll-pitch-yaw angles there is also a singularity, and in this case the singularity appears for $\beta = \pm 90°$, because then the yaw and roll axes coincide. This is rarely a problem for vehicles as this corresponds to the vehicle going vertically up or down (although this may happen for airplanes in extreme cases).

3.3 Constraints and interconnections

A rigid body in a mechanical system is rarely free to move with all six degrees of freedom. The body is usually connected to the surrounding or to other bodies in the system under consideration. This reduces the number of degrees of freedom and introduces constraint forces and moments. The body may be restrained in different ways by joints, restriction to a surface or curve, etc. Such geometrical constraints are discussed in this section, whereas rolling, which is a restriction on velocities, not displacements, is investigated in the next section.

A simple and common constraint is planar motion, where the number of degrees of freedom of a rigid body is reduced from six to three. All velocities lie in the plane and the angular velocity has only one component out of the plane. Also restriction to motion along a straight line or curve is common and then the body has only one translational degree of freedom.

Typically two bodies are connected so that rotations, translations, or both are possible between the bodies. As a special case the motion of one body may completely determine the motion of the other, a case that is common in various transmission systems. A mechanism is a construction with two or more bodies with the task of transmitting motion. The components in a mechanism are commonly called links.

The simplest rotational constraint is a revolute joint (pin connection), where there is only one rotational degree of freedom of one body relative the other. This means that any point on the line of the joint are common to the two bodies, which gives three translational constraints. In addition, two of the angular degrees of freedom of the two bodies are the same, which is often and more simply expressed as the constraint on the angular velocities that the two components perpendicular to the joint are the same. A simple way to treat the revolute joint is of course to introduce a single angular coordinate for one of the bodies, given the coordinates of the other body.

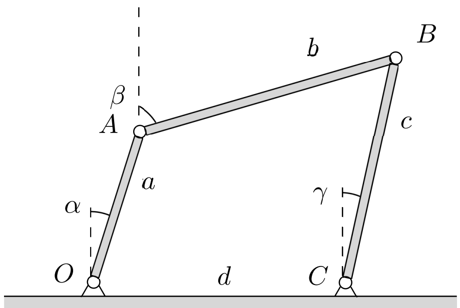
Another common rotational joint is the ball-and-socket joint, which is also called a spherical joint. This joint allows three degrees of rotational freedom between the bodies. The constraint is that the point of the joint be common to the two bodies. This can be realized by introducing three rotational coordinates (Euler angles) for one body relative the other.

A pure translational joint appears when two bodies can move relative each other without any relative angular freedoms. The simplest and most common case is a prismatic joint where there is only one translational degree of freedom between the bodies. The constraints are that the angular degrees of freedom are the same for the two bodies and that two of their translational degrees of freedom are also the same. This is simply realized by introducing a single translational degree of freedom of one of the bodies relative the other.

A common combination joint is a translational joint joined with a revolute or ball-and-socket joint, which is often called a collar connection. In the first case there are two relative degrees of freedom, in the second there are four.

When doing calculations by hand most constraints are taken care of by not introducing more coordinates than necessary, ideally not more than the number of degrees of freedom. This is very hard to implement in a general computer program for deducing the equations of motion and solving these for a general mechanical system. Instead six degrees of freedom are usually introduced for each rigid body and then the constraints are added. This requires a precise mathematical statement of all constraints, but this is a much easier task than trying to introduce only the necessary coordinates.

Example 3.3.1
Put up the constraints for the link mechanism in the figure if the positions of the three arms are described by the angles α, β, and γ.



Solution. The three angles α, β, and γ are thus the (generalized) coordinates in this case. Apparently these three coordinates are not independent, as giving one of them, say α, determines the other two. A simple way to put up the constraints is to write the distance between O and C in the horizontal and vertical directions

$$a\sin\alpha + b\sin\beta - c\sin\gamma = d$$

$$a\cos\alpha + b\cos\beta - c\cos\gamma = 0$$

These equations can be used to express, for instance, β and γ in terms of α, although two possible solutions are obtained. This can be seen by the fact, that fixing α, the point B may lie above or below a thought straight line between A and C. For this and other reasons it is probably better to keep all three angles as coordinates.

3.4 Rolling

Two bodies in contact that perform rotations are said to be in rolling contact. One of the bodies may be at rest, like the ground under a vehicle, or both may be moving, like in gears. For rolling to appear both bodies must be smooth (without corners) with a continuously varying radius of curvature. (Note that a flat surface has an infinite radius of curvature.) The contact between the two bodies can be a single point, like a sphere or thin disk rolling over a flat surface, or a whole line, like a cylinder rolling over a flat surface. The plane which is tangent to both bodies is called the plane of contact. If the two points in contact are denoted C_1 and C_2 the rolling introduces the constraint that the two points have the same velocity normal to the plane of contact

$$(\boldsymbol{v}_{C_1} - \boldsymbol{v}_{C_2}) \cdot \boldsymbol{e}_n = 0 \tag{3.24}$$

where $\boldsymbol{e}_n$ is the normal to the plane of contact. This is a single equation which means that a rolling body has five degrees of freedom in general. If the rolling is with slip there are no further restrictions,

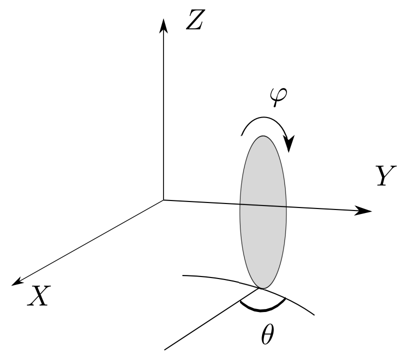


Figure 3.4: Rolling on a curved path on the XY-plane.

but if the rolling bodies do not slip relative each other then all three components of the velocities must be the same

$$\boldsymbol{v}_{C_1} = \boldsymbol{v}_{C_2} \tag{3.25}$$

A body that is rolling without slip has three degrees of freedom in general.

Consider first the case with a wheel rolling without slip along a straight line on a flat surface. If X is the position of the centre of the wheel and φ is the angle of the wheel, then the velocity of the contact point is $\dot{X} - R\dot{\varphi}$, where R is the radius of the wheel. But the surface under the wheel is at rest so the no-slip condition becomes

$$\dot{X} - R\dot{\varphi} = 0 \tag{3.26}$$

In this simple case the constraint can be integrated to

$$X - R\varphi = 0 \tag{3.27}$$

if $\varphi = 0$ at the same time as $X = 0$. Thus, in this case the constraint can be given as a condition between coordinates with no velocities involved. This case is not typical; in general the rolling constraint can not be integrated to yield a condition among coordinates.

Consider next a vertical wheel rolling without slip along a curved path on a flat surface. If X and Y are the coordinates of the wheel centre, φ is the rotation angle of the wheel, and θ is the angle between the wheel path and the positive X axis, then the no-slip conditions are

$$\dot{X} = R\dot{\varphi}\cos\theta \tag{3.28}$$

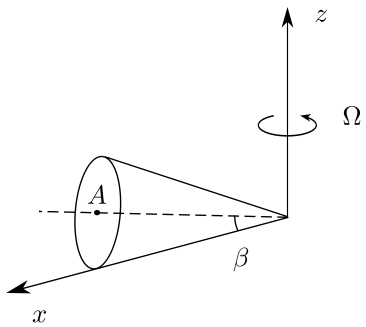
$$\dot{Y} = R\dot{\varphi}\sin\theta \tag{3.29}$$

But these constraints can not be integrated to yield conditions among the coordinates without any velocities involved. However, the vertical wheel has only two degrees of freedom and these are conveniently chosen as φ and θ. Namely, if φ and θ are determined, then Eqs. (3.28) and (3.29) determine $\dot{X}$ and $\dot{Y}$, and given initial conditions these can be used to determine X and Y so that a complete description of the motion is obtained.

A constraint that involves velocities and can not be integrated to a form without velocities is called nonholonomic. This designation is generally used for expressions that can not be integrated. Another example is the angular velocity vector that can not be integrated (which would result in a ”rotation vector”, something which does not exist). Constraints among the coordinates, but with no velocities, are called holonomic. In principle, a holonomic constraint can always be used to eliminate a coordinate, but this is not always so for nonholonomic constraints.

Example 3.4.1

A cone of height L and apex angle β is rolling without slip on a horizontal plane. The velocity of the centre A of the base of the cone is v. Determine the angular velocity and angular acceleration of the cone.



Solution. Introduce the coordinate system xyz with the z axis vertical and the x axis along the line of contact between the cone and the plane. The angular velocity of the coordinate system is simply

$$\boldsymbol{\Omega} = \Omega \boldsymbol{e}_z = \frac{v}{L\cos\beta}\boldsymbol{e}_z$$

because A follows a circular path with radius $L\cos\beta$. The angular velocity of the cone in addition has a spin component along the symmetry axis of the cone. Because of the rolling condition the line on the cone that is in contact with the plane has no velocity. Therefore, the total angular velocity of the cone is directed in this direction

$$\boldsymbol{\omega} = -\omega \boldsymbol{e}_x$$

where

$$\omega = \frac{v}{L\sin\beta}$$

where $L\sin\beta$ is the height of A above the plane. The spin part of the angular velocity of the cone is then

$$\boldsymbol{\omega}_s = \boldsymbol{\omega} - \boldsymbol{\Omega} = -\frac{\Omega}{\sin\beta}(\cos\beta\,\boldsymbol{e}_x + \sin\beta\,\boldsymbol{e}_z)$$

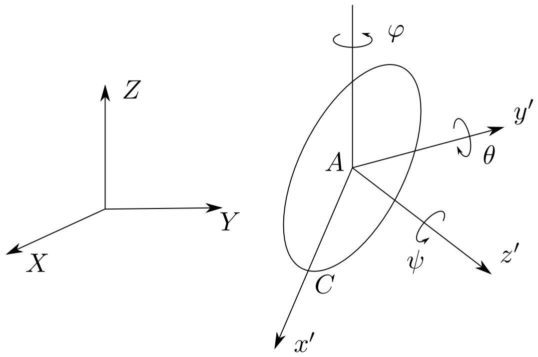
which is seen to be along the symmetry axis of the cone as expected.

The angular acceleration of the cone is

$$\boldsymbol{\alpha} = \frac{\delta\boldsymbol{\omega}}{\delta t} + \boldsymbol{\Omega}\times\boldsymbol{\omega} = \boldsymbol{\Omega}\times\boldsymbol{\omega} = -\frac{\Omega^2\cos\beta}{\sin\beta}\,\boldsymbol{e}_y = -\frac{v^2}{L^2\cos\beta\sin\beta}\,\boldsymbol{e}_y$$

In this example the angles can be identified as Euler angles with Ω as the precession rate, β as the constant nutation, and ω_s as the spin rate.

Example 3.4.2
A thin circular disk with radius R (a coin) is rolling on a horizontal plane. Introduce six coordinates for the disk and derive the constraints due to rolling.



Solution. First the three coordinates X_A, Y_A, and Z_A for the centre A of the disk are introduced to position the disk in space relative an inertial frame XYZ, where Z is the vertical coordinate and the rolling takes place on the XY-plane. Next the rotation of the disk is described by the three standard Euler angles, so that the starting position is with the disk in a horizontal position with the body-fixed system parallel with the inertial system. The first rotation is the precession φ about the Z axis, followed by the nutation θ about the intermediate y' axis, and the spin ψ about the symmetry axis of the disk. It is convenient to introduce the system $x'y'z'$ that is body-fixed except that it does not include the spin. The z' axis is the symmetry axis of the disk, the x' axis passes through the point of contact between the disk and the plane, and the y' axis is horizontal in the plane of the disk. Note that the nutation is about the y' axis and that the nutation angle is the tilt of the disk relative the horizontal position so that $\theta = 90^\circ$ gives an upright disk.

The condition for rolling is that the contact point C has zero velocity. To this end the angular velocity of the disk is needed

$$\boldsymbol{\omega} = \dot{\varphi}\boldsymbol{e}_Z + \dot{\theta}\boldsymbol{e}_{y'} + \dot{\psi}\boldsymbol{e}_{z'} = -\dot{\varphi}\sin\theta\,\boldsymbol{e}_{x'} + \dot{\theta}\boldsymbol{e}_{y'} + (\dot{\psi} + \dot{\varphi}\cos\theta)\boldsymbol{e}_{z'}$$

The velocity of the centre of the disk is simply

$$\boldsymbol{v}_A = \dot{X}_A\boldsymbol{e}_X + \dot{Y}_A\boldsymbol{e}_Y + \dot{Z}_A\boldsymbol{e}_Z$$

To express the constraint in the $x'y'z'$ system this velocity must be transformed to this system. This is accomplished with the rotation matrix in Eq. (3.16) with $\psi = 0$. The result is

$$\boldsymbol{v}_A = \left(\dot{X}_A\cos\theta\cos\varphi + \dot{Y}_A\cos\theta\sin\varphi - \dot{Z}_A\sin\theta\right)\boldsymbol{e}_{x'} + \left(\dot{X}_A\sin\varphi + \dot{Y}_A\cos\varphi\right)\boldsymbol{e}_{y'}$$

$$+ \left(\dot{X}_A\sin\theta\cos\varphi + \dot{Y}_A\sin\theta\sin\varphi + \dot{Z}_A\cos\theta\right)\boldsymbol{e}_{x'}$$

The velocity of the contact point C can be given by taking the velocity of A and add the relative velocity

$$\boldsymbol{v}_C = \boldsymbol{v}_A + \boldsymbol{\omega} \times R\boldsymbol{e}_{x'} = \boldsymbol{v}_A + R(\dot{\psi} + \dot{\varphi}\cos\theta)\boldsymbol{e}_{y'} - R\dot{\theta}\boldsymbol{e}_{z'}$$

The no-slip condition $\boldsymbol{v}_C = 0$ then gives

$$\dot{X}_A\cos\theta\cos\varphi + \dot{Y}_A\cos\theta\sin\varphi - \dot{Z}_A\sin\theta = 0$$

$$-\dot{X}_A\sin\varphi + \dot{Y}_A\cos\varphi + R(\dot{\psi} + \dot{\varphi}\cos\theta) = 0$$

$$\dot{X}_A\sin\theta\cos\varphi + \dot{Y}_A\sin\theta\sin\varphi + \dot{Z}_A\cos\theta - R\dot{\theta} = 0$$

For a dynamical analysis, the $x'y'z'$ system is definitely preferably, but the constraints are better expressed in the XYZ system. This is the same as solving the constraint equations for $\dot{X}_A$, $\dot{Y}_A$, and $\dot{Z}_A$ (or expressing the constraints directly in XYZ)

$$\dot{X}_A = R\dot{\theta}\sin\theta\cos\varphi + R\sin\varphi(\dot{\psi} + \dot{\varphi}\cos\theta)$$

$$\dot{Y}_A = R\dot{\theta}\sin\theta\sin\varphi - R\cos\varphi(\dot{\psi} + \dot{\varphi}\cos\theta)$$

$$\dot{Z}_A = R\dot{\theta}\cos\theta$$

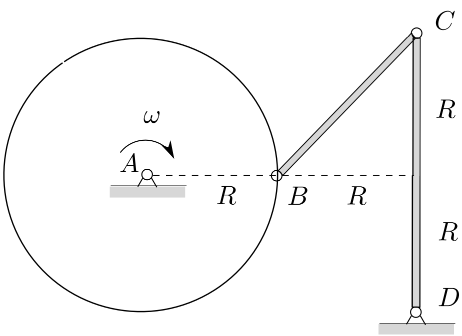
The last equation is easily integrated

$$Z_A = R\sin\theta$$

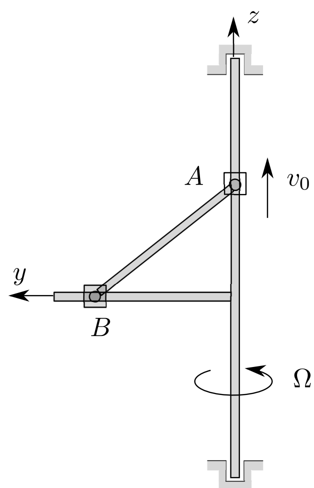
This simply gives the height of point A above the plane. The other two constraints can not, however, be integrated, so there are one holonomic constraint and two nonholonomic. In a dynamical analysis of the disk the constraints can still be used to eliminate X_A, Y_A, and Z_A and perform the calculations with only the Euler angles. Afterwards X_A, Y_A, and Z_A can be calculated from the constraints.

Problems

3.1 A system consists of a circular disk and two bars according to the figure. The disk is rotating around its centre A and CD is rotating around D. In the position shown the disk has the angular velocity ω. Determine the angular velocities of BC and CD.



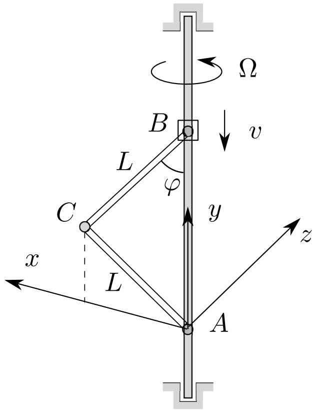
Prob. 3.1



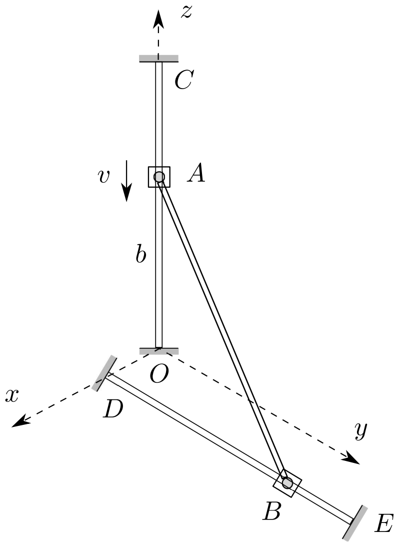
Prob. 3.2

3.2 The rod AB has length $125\,\text{mm}$ and is attached to two collars A and B that can slide along the vertical and horizontal guiding bars, respectively. In the moment shown collar A has the constant vertical velocity $v_0 = 0.2\,\text{m/s}$ upwards. The whole system is rotating with the constant angular velocity $\Omega = 2$ rad/s. Determine the angular acceleration of AB at the instant when collar A is $75\,\text{mm}$ above the horizontal bar.

3.3 Collar A is fixed on the vertical rotating axis. The angular velocity Ω of the axis and the velocity v of collar B are both constant. Determine the angular acceleration for BC.



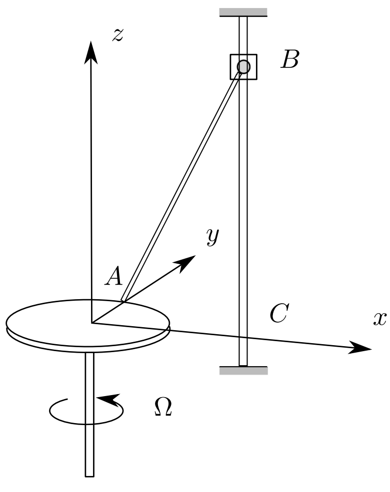
Prob. 3.3



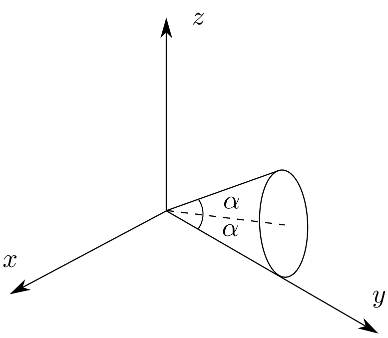
Prob. 3.4

3.4 Two collars A and B are connected by a rod of length $1.3L$. The collars slide along two fixed rods, OC and DE as shown. The distance between O and D is $0.3L$. Collar A is moving at constant velocity v. Determine the velocity and acceleration of collar B at the instant when $b = 0.4L$.

3.5 The bar AB with length $5R$ is connected to the rotating disk with radius R with a spherical joint at A. The disk is rotating with the constant angular velocity Ω. The end B of the bar is moving along a fixed vertical bar BC parallel to the axis of the disk. The bar BC is placed in the xz plane at a distance $2R$ from the origin. Determine the velocity of B in the position shown.



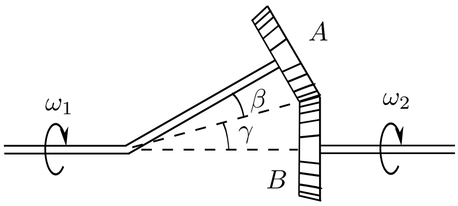
Prob. 3.5



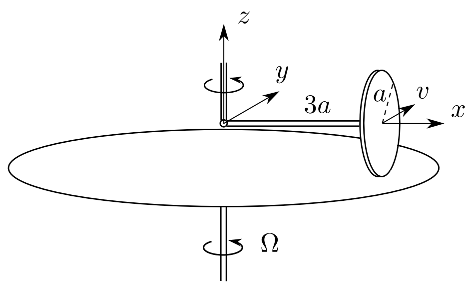
Prob. 3.6

3.6 A right circular cone has height h and top angle 2α. It is lying on a horizontal plane and is rotating with the middle point of the bottom P having a constant speed. The cone is rolling one turn on the plane in the time τ. Determine the angular velocity of the cone.

3.7 The gear A can rotate freely about its axis which in turn rotates with the variable angular velocity ω_1. The gear B is fixed on its axis and rotates with the variable angular velocity ω_2. Determine the angular velocity and angular acceleration for gear A.



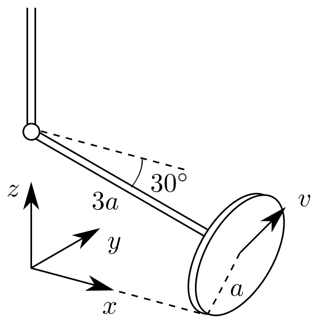
Prob. 3.7



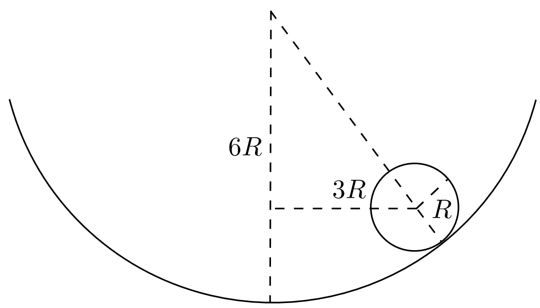
Prob. 3.8

3.8 A wheel of radius a is mounted on a horizontal axis of length $3a$. The axis is rotating freely about the vertical. The wheel rotates on a disk which in turn rotates with constant angular velocity Ω. The middle point of the wheel has the constant speed v. Determine the angular velocity and angular acceleration of the wheel and the acceleration of the contact point of the wheel.

3.9 A wheel of radius a is mounted on an axis of length $3a$ which makes the angle $30°$ with the horizontal. The other end of the axis is fixed in a spherical joint. The wheel is rolling without slip on a horizontal plane so that its centre has the constant speed v. Determine the angular velocity and angular acceleration of the wheel.



Prob. 3.9



Prob. 3.10

3.10 A sphere of radius R is rolling without slip with speed v in a circular path inside a half-sphere with radius $6R$. Determine the angular velocity and acceleration of the sphere and also the acceleration of the point in the sphere that is in contact with the half-sphere.

Chapter 4

Rigid body dynamics

With the kinematical description done, it is time to turn to kinetics, i.e. the interplay between motion and forces. First the two concepts of centre of mass and moments of inertia must be discussed as these are essential when the dynamics of rigid bodies is investigated. Then the general three-dimensional equations for a rigid body are derived and these consist of Newton's second law for the translational motion of the centre of mass and the law for the angular momentum for the rotational motion. In general the translation and rotation couple so that the equations must be solved simultaneously. Planar motion and rotation about a fixed axis are discussed as special cases. The free rotation of a rigid body is investigated and this has in particular applications for gyros.

4.1 Centre of mass

The centre of gravity for a body or system is the point where the resultant to the distributed gravitational force should act so as to give the same moment as the distributed gravitational force. The centre of mass is defined from the distribution of mass, and it is easily proved that the two centres coincide. Here this point is called the centre of mass, and the determination of this point is briefly discussed here.

From Chapter 1.6 the definition of the centre of mass for N particles, each with mass m_i and position $\boldsymbol{r}_i$ in some convenient frame, is

$$\overline{\boldsymbol{r}} = \frac{1}{m}\sum_{i=1}^{N} m_i \boldsymbol{r}_i \tag{4.1}$$

Here m is the total mass of the system of particles

$$m = \sum_{i=1}^{N} m_i \tag{4.2}$$

A rigid body can be viewed as an infinite number of particles rigidly connected and taking this limit the sum becomes an integral

$$\overline{\boldsymbol{r}} = \frac{1}{m}\int \boldsymbol{r}\,\mathrm{d}m \tag{4.3}$$

where the integration is over the whole body. In many cases the density of the body is constant and then the centre of mass becomes a purely geometrical concept

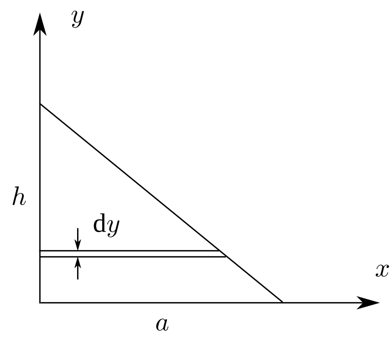
$$\overline{\boldsymbol{r}} = \frac{1}{V}\int \boldsymbol{r}\,\mathrm{d}V \tag{4.4}$$

where V is the volume of the body. This integral can easily be further specialized to cases where the body is a thin plate, slender bar, shell of revolution, etc.

In some cases the location of the centre of mass is self-evident, as simple examples like cubes, spheres, slender bars, etc. show. In other cases the location of the centre of mass is tabulated. Equation (4.1) is also useful as it can be used not only for particles but also for rigid bodies. Thus the centre of mass for bodies composed of simple parts with known centre of mass can easily be determined. It is noted that this equation can also be used for parts with negative mass, i.e. for a hole. An example is a square plate with an asymmetrical round hole. It is of course essential to attribute the correct masses to the parts in such cases.

Example 4.1.1

Determine the location of the centre of mass for a triangular plate with base a and height h.



Solution. The triangle can be taken as right-angled without loss of generality (because the answer does not depend on the base). The area of the triangle is

$$S = \frac{1}{2}ah$$

The area of the small surface element in the figure is

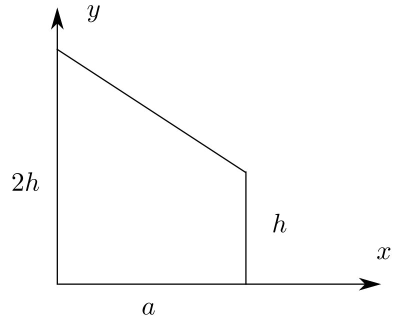
$$\mathrm{d}S = \left(a - \frac{ay}{h}\right)\mathrm{d}y$$

The height of the centre of mass then becomes

$$\overline{y} = \frac{1}{S}\int_0^h y\,\mathrm{d}S = \frac{2}{ah}\int_0^h y\left(a - \frac{ay}{h}\right)\mathrm{d}y = \frac{h}{3}$$

It follows that the centre of mass is located one third of the height from all three sides of a triangle.

Example 4.1.2
Determine the location of the centre of mass for the plate shown in the figure.



Solution. The plate can be seen as the difference between a larger triangle with base $2a$ and height $2h$ and a smaller triangle with base a and height h. For the larger triangle the area is $S_1 = 2ah$ and the centre of mass is located at $\overline{x}_1 = 2a/3$ and $\overline{y}_1 = 2h/3$ and for the smaller triangle $S_2 = ah/2$, $\overline{x}_2 = 4a/3$ and $\overline{y}_2 = h/3$. The centre of mass for the plate is then located at

$$\overline{x} = \frac{\overline{x}_1 S_1 - \overline{x}_2 S_2}{S_1 - S_2} = \frac{\frac{2a}{3} \cdot 2ah - \frac{4a}{3} \cdot \frac{ah}{2}}{2ah - \frac{ah}{2}} = \frac{4a}{9}$$

$$\overline{y} = \frac{\overline{y}_1 S_1 - \overline{y}_2 S_2}{S_1 - S_2} = \frac{\frac{2h}{3} \cdot 2ah - \frac{h}{3} \cdot \frac{ah}{2}}{2ah - \frac{ah}{2}} = \frac{7h}{9}$$

4.2 Moments of inertia

Moments of inertia describe the resistance of a body to rotation about a particular axis and in this way resemble mass which is the resistance to translation. The moments of inertia naturally appear when deriving the equations for the rotation of a rigid body. This topic is treated in the next section, but to not break the presentation there, moments of inertia are discussed in this section.

The moments of inertia for a rigid body with respect to the x, y, and z axes are defined by

$$I_{xx} = \int (y^2 + z^2)\mathrm{d}m \tag{4.5}$$

$$I_{yy} = \int (x^2 + z^2)\mathrm{d}m \tag{4.6}$$

$$I_{zz} = \int (x^2 + y^2)\mathrm{d}m \tag{4.7}$$

The integration is over the whole body. In a similar way the products of inertia are defined by

$$I_{xy} = \int xy\,\mathrm{d}m \tag{4.8}$$

$$I_{xz} = \int xz \, \mathrm{d}m \tag{4.9}$$

$$I_{yz} = \int yz \, \mathrm{d}m \tag{4.10}$$

Often the density of the body is constant and then $\mathrm{d}m = m \, \mathrm{d}V/V$, where m is the mass and V the volume of the body. Then, for example

$$I_{xx} = \frac{m}{V} \int (y^2 + z^2) \mathrm{d}V \tag{4.11}$$

The dimension of moment of inertia and product of inertia is kg m^2. Both quantities are additive (like a mass), i.e. the total moment of inertia for a body can be determined as a sum of moments of inertia for its parts.

A moment of inertia about an axis contains the distance from the axis squared times the mass element. This is exactly the quantity (multiplied by the angular velocity) that is contained in the angular momentum about the axis and this is why the moment of inertia appears when the rotation of a rigid body is investigated. The product of inertia I_{xy} is a measure (which can be both positive and negative) of how unsymmetrical the body is with respect to the xz and yz planes. If I_{xy} is nonzero an angular velocity about the x axis gives an angular momentum about the y axis and vice versa.

To give a value on the moment of inertia it is customary to instead specify the radius of gyration. It is defined as

$$k_x = \sqrt{I_{xx}/m} \tag{4.12}$$

and similarly for k_y and k_z. This is the distance from the axis which would give the same moment of inertia if all the mass was concentrated at this radius. The advantage with the radius of gyration is that a distance is a quantity for which one has an intuitive feeling, whereas a moment of inertia is more difficult.

For a body in the shape of a thin plate with the z axis normal to the plate it follows directly from the definition that

$$I_{zz} = I_{xx} + I_{yy} \tag{4.13}$$

because $z = 0$ for the plate.

To determine moments and products of inertia tables are used as much as possible. More complicated bodies can sometimes be obtained by adding contributions from simpler bodies. The products of inertia are often zero or very small. From the definition it is clear that if a coordinate plane is a plane of symmetry, then the products of inertia containing the third coordinate vanish.

Often the moments of inertia are needed for bodies that are tabulated, but with respect to other axes than the tabulated. Most often translations are needed but sometimes also rotations. Let the moments of inertia with respect to the xyz system be known and look for them in the $x'y'z'$ system with parallel axes to xyz. Let the origin of the xyz system have coordinates d_x, d_y, and d_z in the $x'y'z'$ system. Then

$$\begin{aligned} I_{x'x'} &= \int (y'^2 + z'^2) \mathrm{d}m = \int [(y + d_y)^2 + (z + d_z)^2] \mathrm{d}m \\ &= I_{xx} + 2d_y \int y \, \mathrm{d}m + 2d_z \int z \, \mathrm{d}m + m(d_y^2 + d_z^2) \end{aligned} \tag{4.14}$$

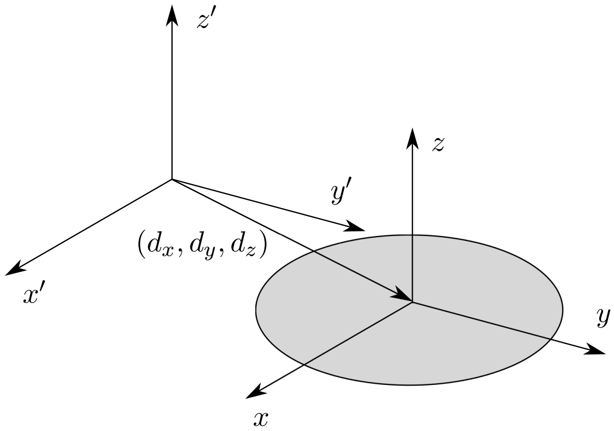


Figure 4.1: Steiner's theorem.

The two remaining integrals are the definition of the centre of mass in the xyz system so if the xyz system has the origin at the centre of mass of the body these integrals vanish and

$$I_{x'x'} = I_{xx} + m(d_y^2 + d_z^2) \tag{4.15}$$

Similar relations are of course valid for $I_{y'y'}$ and $I_{z'z'}$. For the products of inertia the corresponding relation is

$$I_{x'y'} = I_{xy} + m d_x d_y \tag{4.16}$$

and analogously for $I_{x'z'}$ and $I_{y'z'}$. From Eq. (4.15) it follows that the moment of inertia attains the smallest value in the centre of mass system. This is not true for the products of inertia, because the added term can be both positive and negative. Equations (4.15) and (4.16) are often called Steiner's theorem and are thus valid in parallel systems where one is the centre of mass system.

To discuss rotations of the coordinate system it is necessary to introduce the inertia matrix (the inertia tensor)

$$I = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix} \tag{4.17}$$

As is shown later in this chapter the rotational kinetic energy for a rigid body is

$$T = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}} I \boldsymbol{\omega} \tag{4.18}$$

where the angular velocity $\boldsymbol{\omega}$ is regarded as a column vector in this situation. (It is a slight abuse of notation to use the same letter both for the vector and the column matrix, but usually no confusion arises.) The important observation is that the kinetic energy is a scalar and is independent of the coordinate frame it is expressed in. Let xyz and $x'y'z'$ be two coordinate frames with the same origin but different orientation. Then

$$\boldsymbol{\omega}'^{\mathrm{T}} I' \boldsymbol{\omega}' = \boldsymbol{\omega}^{\mathrm{T}} I \boldsymbol{\omega} \tag{4.19}$$

The angular velocity $\boldsymbol{\omega}$ is a vector and a vector transforms according to Eq. (2.17)

$$\boldsymbol{\omega}' = R\boldsymbol{\omega} \tag{4.20}$$

Inserting this gives

$$\boldsymbol{\omega}^{\mathrm{T}} R^{\mathrm{T}} I' R \boldsymbol{\omega} = \boldsymbol{\omega}^{\mathrm{T}} I \boldsymbol{\omega} \tag{4.21}$$

As this must be true for any $\boldsymbol{\omega}$ it follows that the inertia matrix transforms according to

$$I' = R I R^{\mathrm{T}} \tag{4.22}$$

From this it is of course possible to express how the different elements transform.

The inertia matrix is a real symmetric matrix and can thus be transformed to a diagonal form in another coordinate system. The diagonal elements $\lambda_1 = I_1$, $\lambda_2 = I_2$, and $\lambda_3 = I_3$ and the rows in the rotation matrix $\boldsymbol{e}_1^{\mathrm{T}}$, $\boldsymbol{e}_2^{\mathrm{T}}$, $\boldsymbol{e}_3^{\mathrm{T}}$, are obtained as the solution to the eigenvalue problem

$$(I - \lambda U)\boldsymbol{e} = 0 \tag{4.23}$$

where U is the unit matrix. The rotation from the original xyz system to the principle system of inertia $\xi\eta\zeta$ system is

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4.24}$$

where the rotation matrix can be written

$$R = \begin{pmatrix} \boldsymbol{e}_1^{\mathrm{T}} \\ \boldsymbol{e}_2^{\mathrm{T}} \\ \boldsymbol{e}_3^{\mathrm{T}} \end{pmatrix} \tag{4.25}$$

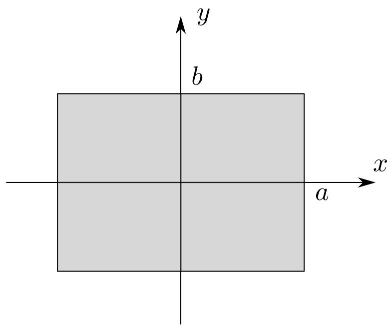
In the principle system the inertia matrix has the simple diagonal form

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{4.26}$$

It should be noted that the eigenvectors $\boldsymbol{e}_i$ must be normalized such that $\boldsymbol{e}_i \cdot \boldsymbol{e}_i = 1$. It is worth emphasizing that the principle system of inertia always exists irrespective of how unsymmetrical the body is. For more symmetric bodies it is possible to directly identify the principle axes of inertia. For a sphere or cube all coordinate systems with origin at the centre are principle systems. For a rotationally symmetric body the symmetry axis is a principle axis and the other two can be chosen arbitrarily.

Example 4.2.1

Determine the elements of the inertia matrix for a thin rectangular plate with mass m and sides $2a$ and $2b$.



Solution. Here the definition in terms of an integral is used, although some of the elements can be found directly in tables. Choose the coordinate system according to the figure. As $z = 0$ for the plate the definition gives

$$I_{xx} = \frac{m}{S}\int y^2\,\mathrm{d}S = \frac{m}{4ab}\int_{-b}^{b} y^2\,2a\,\mathrm{d}y = \frac{1}{3}mb^2$$

Similarly

$$I_{yy} = \frac{1}{3}ma^2$$

For a thin plate

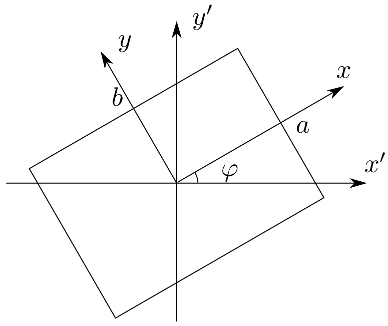
$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{3}m\left(a^2 + b^2\right)$$

By symmetry all products of inertia vanish

$$I_{xy} = I_{xz} = I_{yz} = 0$$

Example 4.2.2

Determine the inertia matrix for a thin rectangular plate with mass m and sides $2a$ and $2b$, which is rotated by an arbitrary angle φ.



Solution. The inertia matrix in the xyz system was determined in the previous example. The rotation to the $x'y'z'$ system is accomplished with the rotation matrix (the rotation is by the angle $-\varphi$)

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The inertia matrix in the rotated system is thus

$$\begin{aligned} I' &= RIR^{\mathrm{T}} \\ &= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3}mb^2 & 0 & 0 \\ 0 & \frac{1}{3}ma^2 & 0 \\ 0 & 0 & \frac{1}{3}m(a^2+b^2) \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \frac{m}{3}\begin{pmatrix} b^2\cos^2\varphi + a^2\sin^2\varphi & \left(a^2-b^2\right)\cos\varphi\sin\varphi & 0 \\ \left(a^2-b^2\right)\cos\varphi\sin\varphi & a^2\cos^2\varphi + b^2\sin^2\varphi & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \end{aligned}$$

For a square plate with $a = b$ the same inertia matrix as in the original xyz system is obtained; the inertia matrix is independent of φ in this case, maybe surprisingly.

4.3 General equations of motion

Starting from the equations for a system of particles, the equations of motion for a rigid body are now derived. In the general case a rigid body has six degrees of freedom so six equations are needed. Three of these are of course Newton's second law for the translational motion of the centre of mass. But three more equations are needed for the rotational motion and these are the law for the angular momentum. In contrast to the situation for a particle, Newton's second law and the law for the angular momentum are independent for a rigid body.

As a rigid body can be viewed as a collection of particles Newton's second law for a system of particles is directly valid also for a rigid body

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} \tag{4.27}$$

Here $\boldsymbol{F}$ is the sum of all forces acting on the rigid body. As no body is really a particle, this also shows that when a body is regarded as a particle it is the motion of the centre of mass that is studied.

When studying the rotational motion, the case with a rigid body rotating about a fixed point O is first investigated. In this case the starting point is the law of angular momentum for a system of particles about a fixed point (Eq. (1.88))

$$\dot{\boldsymbol{L}}_O = \boldsymbol{M}_O \tag{4.28}$$

where $\boldsymbol{M}_O$ is the sum of all moments about O acting on the rigid body. Dividing the rigid body into small elements with mass $\mathrm{d}m$ and position $\boldsymbol{r}$ relative O, the velocity of the element is $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$ and the angular momentum of the element is $\mathrm{d}\boldsymbol{L}_O = \boldsymbol{r} \times (\mathrm{d}m\,\boldsymbol{\omega} \times \boldsymbol{r})$. The total angular momentum is thus

$$\boldsymbol{L}_O = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})\mathrm{d}m \tag{4.29}$$

To proceed a coordinate system xyz with origin at O is introduced. Expanding the double cross product the x component of the angular momentum becomes

$$L_{Ox} = \int [(y^2 + z^2)\omega_x - xy\,\omega_y - xz\,\omega_z]\mathrm{d}m = I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \tag{4.30}$$

and similarly for the y and z components. Here the definition of the moments and products of inertia from the previous section are used. Written in matrix form the three components can be collected as

$$\boldsymbol{L}_O = I_O\boldsymbol{\omega} \tag{4.31}$$

where I_O is the inertia matrix with respect to O.

To take the time derivative it is in practice necessary to choose a coordinate frame where the moment of inertia matrix is constant. In the general case this must be a body-fixed frame, but in special cases it is more convenient to choose another frame. For rotationally symmetric bodies, typically wheels and gyros, it is easier to choose a coordinate frame that is not spinning about the symmetry axis.

If the angular velocity of the coordinate frame is $\boldsymbol{\Omega}$, which may thus be different from the angular velocity $\boldsymbol{\omega}$ of the body, then the time derivative of the angular momentum becomes

$$\dot{\boldsymbol{L}}_O = \frac{\delta \boldsymbol{L}_O}{\delta t} + \boldsymbol{\Omega} \times \boldsymbol{L}_O \tag{4.32}$$

where the local time derivative is

$$\frac{\delta \boldsymbol{L}_O}{\delta t} = \dot{L}_x \boldsymbol{e}_x + \dot{L}_y \boldsymbol{e}_y + \dot{L}_z \boldsymbol{e}_z \tag{4.33}$$

Here the time differentiation only acts on the components of the angular momentum as the moments of inertia are constant. For the x component

$$\left(\frac{\delta \boldsymbol{L}_O}{\delta t}\right)_x = I_{xx}\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z \tag{4.34}$$

and similarly for the y and z components.

Writing out the full equations for the angular momentum in terms of the moments of inertia and the components of the angular velocities gives rather lengthy expressions. It is usually better to first calculate the angular momentum and then take the time derivative according to the above.

In the case that the coordinate frame is chosen as the principal system of inertia all the products of inertia vanish and the equations simplify considerably. If the coordinate frame is body-fixed the components of the law of angular momentum become

$$I_1\dot{\omega}_x - (I_2 - I_3)\omega_y\omega_z = M_{Ox} \tag{4.35}$$

$$I_2\dot{\omega}_y - (I_3 - I_1)\omega_z\omega_x = M_{Oy} \tag{4.36}$$

$$I_3\dot{\omega}_z - (I_1 - I_2)\omega_x\omega_y = M_{Oz} \tag{4.37}$$

These equations are usually called Euler's equations.

Turning then to the general case when the rigid body has no fixed point, the starting point is the law for the angular momentum of a system of particles about the centre of mass. Repeating the argument above for the body divided into small elements, the only change is that the position $\boldsymbol{\rho}$ relative the centre of mass appears instead of the absolute position $\boldsymbol{r}$ relative the fixed point O. The result thus becomes

$$\overline{\boldsymbol{L}} = \overline{I}\boldsymbol{\omega} \tag{4.38}$$

where both the moment of inertia matrix and the angular momentum are about the centre of mass.

Euler's equations can also be written with respect to the centre of mass, the only difference is that the moments of inertia and the acting moments are now about the centre of mass.

The law for the angular momentum is formulated in a body-fixed (or almost body-fixed) frame, because it is necessary that the moment of inertia matrix is constant in the frame. However, it should be stressed that it is the absolute angular momentum and the absolute angular velocity that must be used, it is just the components that are taken in the body-fixed and moving system. As it is absolute quantities that are used no fictitious forces or the like appear.

With Newton's second law and the law for the angular momentum a complete set of equations for a rigid body is obtained. In the general case it is six coupled differential equations. In most cases the motion is restricted in some way, and exactly as for a particle unknown constraint forces then appear as unknowns instead of coordinates.

The law for the angular momentum is quite complicated if the moment of inertia matrix is full. But it should be noted that the equation has so far only been expressed for the components of the angular velocity. If the angular velocity is expressed by some angles, such as Euler angles, then the equations become truly formidable.

4.4 The kinetic energy

As for particles it is often useful to use derived laws. It is then first noted that the law of angular momentum is no longer a derived law in this sense. But the conservation of (a component of) the linear momentum or angular momentum may still be useful, although this more frequently happens for systems (of rigid bodies and particles).

The law for the kinetic energy or conservation of energy are often useful for rigid bodies, exactly as they are for particles. The kinetic energy for a rigid body is then needed. Consider again first the rotation about a fixed point O and divide the rigid body into small elements with mass $\mathrm{d}m$ and position $\boldsymbol{r}$. The square of the velocity that appears in the kinetic energy can be expressed as $v^2 = \boldsymbol{v}\cdot\boldsymbol{v} = (\boldsymbol{\omega}\times\boldsymbol{r})\cdot(\boldsymbol{\omega}\times\boldsymbol{r}) = \boldsymbol{\omega}\cdot(\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r}))$, where a permutation of the vectors in the scalar triple product is performed ($\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})$, with $\boldsymbol{a} = \boldsymbol{\omega}\times\boldsymbol{r}$). Thus

$$T = \int \frac{1}{2}v^2\,\mathrm{d}m = \frac{1}{2}\boldsymbol{\omega}\cdot\int \boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})\mathrm{d}m \tag{4.39}$$

Here the angular momentum is recognized so this can be written

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{L}_O = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}} I_O \boldsymbol{\omega} \tag{4.40}$$

where the last expression is in matrix notation.

For a general motion the kinetic energy can be written as a sum of the translational motion of the centre of mass plus the rotational motion about the centre of mass

$$T = \frac{1}{2}m\overline{v}^2 + \overline{T} \tag{4.41}$$

Similarly to the situation for the rotation about a fixed point, the rotational kinetic energy can be expressed as

$$\overline{T} = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\overline{I}\boldsymbol{\omega} \tag{4.42}$$

Potential energy and work are quantities that are deduced from the acting forces and has nothing to do with the mass distribution. The energy law is thus unchanged and is

$$W^{(\mathrm{ik})} = \Delta T + \Delta V \tag{4.43}$$

exactly as for a single particle or a system of particles. This relation is thus useful in many situations where position and velocity, but not time or acceleration, are involved.

4.5 Planar motion

Two common and important special cases are planar motion and rotation about a fixed axis. In planar motion all velocities are parallel to a certain plane, and thus all points in the rigid body along a line perpendicular to this plane have the same velocity. It is thus sufficient to consider the velocities in the plane of reference. It is noted that the forces acting on the body do not necessarily all lie in the reference plane, so a three-dimensional analysis may be necessary to determine the forces.

In planar motion the law for the centre of mass has only two components

$$m\ddot{\overline{x}} = F_x \tag{4.44}$$

$$m\ddot{\overline{y}} = F_y \tag{4.45}$$

where the right-hand sides include all the forces acting on the rigid body. It may be better to express the acceleration in some other coordinates like natural or polar coordinates.

The great simplification for planar motion appears in the law for the angular momentum about the centre of mass, which with only one component in the angular velocity becomes

$$\overline{I}\dot{\omega} = \overline{M} \tag{4.46}$$

Here $\overline{M}$ is the sum of all the moments acting about the centre of mass and the subscript z has been dropped for simplicity. In planar motion the rotation is always simple, so the angular velocity is the time derivative of an angle $\omega = \dot{\varphi}$.

A special case of planar motion of a rigid body is of course that one point (in the body or outside) is fixed during the motion. In such case it is advantageous to use the law for the angular momentum about a fixed point

$$I_O\dot{\omega} = M_O \tag{4.47}$$

Here I_O is the moment of inertia about O and M_O the corresponding moment. The advantage with using this version of the law for the angular momentum instead of the one for the centre of mass is that in the version about a fixed point the constraint forces do not enter. This means that the law for the angular momentum determines the rotation and afterwards the law for the centre of mass determines the constraint forces necessary to keep the rotation about the fixed axis.

In planar motion the expression for the kinetic energy simplifies considerably. The kinetic energy for rotation about a fixed point O is

$$T = \frac{1}{2}I_O\omega^2 \tag{4.48}$$

and for general planar motion it is

$$T = \frac{1}{2}m\overline{v}^2 + \frac{1}{2}\overline{I}\omega^2 \tag{4.49}$$

In planar motion there is a further way to write the kinetic energy and that is by using the instantaneous centre of rotation C. Let the distance between the centre of mass and the instantaneous centre of rotation be c. The speed of the centre of mass is then $c\omega$ and the kinetic energy becomes

$$T = \frac{1}{2}m(c\omega)^2 + \frac{1}{2}\overline{I}\omega^2 = \frac{1}{2}I_C\omega^2 \tag{4.50}$$

where Steiner's theorem is used in the last equality and I_C is the moment of inertia about the instantaneous centre of rotation.

When using energy principles it is sometimes necessary to calculate the work done by a moment acting about the axis O. Let the moment be due to a force $\boldsymbol{F}$ acting at the position $\boldsymbol{r}$ from some point on the fixed axis. Then the velocity of the point of application is $\boldsymbol{\omega} \times \boldsymbol{r}$ and the work becomes

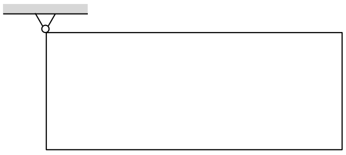
$$\mathrm{d}W = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \boldsymbol{F}\cdot\boldsymbol{v}\mathrm{d}t = \boldsymbol{F}\cdot(\boldsymbol{\omega}\times\boldsymbol{r})\mathrm{d}t = \boldsymbol{\omega}\cdot(\boldsymbol{r}\times\boldsymbol{F})\mathrm{d}t = \boldsymbol{\omega}\cdot\boldsymbol{M}_O\mathrm{d}t = M_O\mathrm{d}\varphi \tag{4.51}$$

If the body is rotating from the position $\varphi = \varphi_1$ to the position $\varphi = \varphi_2$ then the total work done is

$$W = \int_{\varphi_1}^{\varphi_2} M_O\,\mathrm{d}\varphi \tag{4.52}$$

The work is thus the moment times the angle and this carries over to an arbitrary collection of forces and is also true for a couple M. When the moment is constant the work is simply the moment times the total angular displacement.

Example 4.5.1
A rectangular plate with mass m and sides $2a$ and $2b$ can rotate freely about a horizontal axis. It is released from rest when one side is horizontal according to the figure. Which is the largest angular velocity in the resulting pendulum motion?



Solution. In the pendulum motion the angular velocity is the largest when the centre of mass is at it lowest point, i.e. when the centre of mass is below the axis of rotation.

The moment of inertia for the plate for an axis through the centre of mass perpendicular to the plate was calculated in **Example 4.2.1** as

$$\overline{I}_{zz} = \frac{1}{3}m\left(a^2+b^2\right)$$

Using Steiner's theorem this gives for the axis through the corner

$$I_{zz} = \frac{1}{3}m\left(a^2+b^2\right) + m\left(a^2+b^2\right) = \frac{4}{3}m\left(a^2+b^2\right)$$

Energy is conserved in this case

$$T_1 + V_1 = T_2 + V_2$$

As the plate is starting from rest $T_1 = 0$ and

$$T_2 = \frac{1}{2}I_{zz}\omega^2 = \frac{2}{3}m\left(a^2+b^2\right)\omega^2$$

where ω is the angular velocity to be determined.

Taking the datum for the potential energy at the axis

$$V_1 = -mgb$$

$$V_2 = -mg\sqrt{a^2+b^2}$$

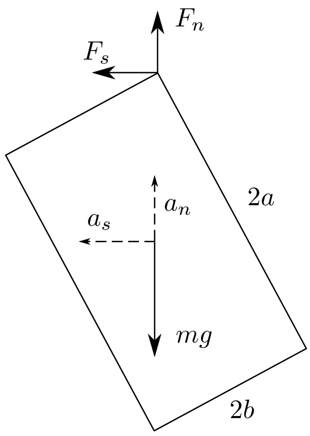
Energy conservation then gives

$$0 - mgb = \frac{2}{3}m\left(a^2+b^2\right)\omega^2 - mg\sqrt{a^2+b^2}$$

which is solved for the angular velocity

$$\omega = \sqrt{\frac{3g\left(\sqrt{a^2+b^2}-b\right)}{2\left(a^2+b^2\right)}}$$

Example 4.5.2
What are the reaction forces at the support for the plate of the previous example when the angular velocity is largest?



Solution. The free-body diagram is shown in the figure. As the centre of mass is performing a circular motion Newton's second law for the centre of mass gives

$$\uparrow \quad F_n - mg = m\overline{a}_n = m\overline{r}\omega^2$$

$$\leftarrow F_s = m\overline{a}_s = m\overline{r}\dot{\omega}$$

where the distance to the centre of mass is $\overline{r} = \sqrt{a^2 + b^2}$. At this very instant there are no moments acting so the law of angular momentum gives

$$0 = I_{zz}\dot{\omega}$$

It follows that $\dot{\omega} = 0$ and inserting the angular velocity from the previous example gives

$$F_n = mg + \frac{3mg\left(\sqrt{a^2+b^2} - b\right)}{2\sqrt{a^2+b^2}}$$

$$F_s = 0$$

4.6 Rotation about a fixed axis

Rotation about a fixed axis can be viewed as a special case of planar motion, but this is only true in the kinematical sense. In the kinetic sense the rotation may be three-dimensional in that the system of forces and couples may include components not included in what is usually called a two-dimensional force system.

Choose a coordinate frame xyz so that the z axis is the rotation axis, i.e. $\boldsymbol{\omega} = \omega \boldsymbol{e}_z$. Let the frame be body-fixed so that the inertia matrix is constant. The angular momentum about the origin O on the axis then becomes

$$\boldsymbol{L}_O = -I_{xz}\omega \boldsymbol{e}_x - I_{yz}\omega \boldsymbol{e}_y + I_{zz}\omega \boldsymbol{e}_z \tag{4.53}$$

Calculating the time derivative and remembering that the angular velocity of the coordinate frame is also $\omega \boldsymbol{e}_z$, the six equations of motion become

$$m\ddot{\overline{x}} = F_x \tag{4.54}$$

$$m\ddot{\overline{y}} = F_y \tag{4.55}$$

$$0 = F_z \tag{4.56}$$

$$-I_{xz}\dot{\omega} + I_{yz}\omega^2 = M_x \tag{4.57}$$

$$-I_{yz}\dot{\omega} - I_{xz}\omega^2 = M_y \tag{4.58}$$

$$I_{zz}\dot{\omega} = M_z \tag{4.59}$$

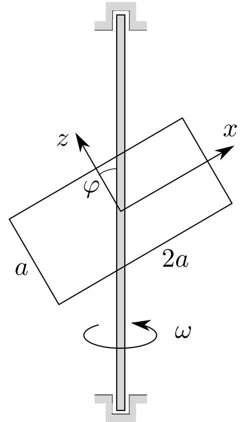
The two first and the last equations are here recognized as the equations for planar motion, but the two next to last equations also have important implications. If the products of inertia are nonzero it is realized that moments about the x and y axes are necessary to keep the motion planar. These moments must be produced by some forces keeping the axis in place. In addition the two first equations give that there are forces in the x and y directions to keep the body in place, if the centre of mass is not located on the rotation axis.

In many technical applications there are fast rotations of bodies about axes and the effects mentioned above are very important. Firstly, so as not obtain large net forces in the x and y directions it is necessary to have the centre of mass on the rotation axis. This is called static balancing. But even if this has been achieved there may still be large forces necessary to produce the large moments M_x and M_y that can be the result of a fast rotation. To avoid this it is also necessary to have the products of inertia I_{xz} and I_{yz} zero and this is called dynamic balancing. In this case the z axis is a principal axis of inertia and to have both static and dynamic balancing the centre of mass must be located on the axis of rotation that must, furthermore, be a principal axis.

This discussion also shows the physical significance of the products of inertia, quantities that are somewhat difficult to get a physical feeling for. For rotation about a fixed axis the products of inertia, if nonzero, give an angular momentum about perpendicular axes, which must be balanced by some moments, even in cases where the angular velocity is constant.

In the discussions it has been assumed that the rotation axis is the z axis. In some cases it may be better to choose the coordinate system so that the inertia matrix becomes as easy as possible to calculate and the angular velocity vector may then be oriented in a skew way in these coordinates.

Example 4.6.1
A rectangular plate with mass m and sides a and $2a$ is attached at a fixed angle φ to a vertical axis that rotates with the constant angular velocity ω. The axis is supported by bearings at the distance d above and below the centre of mass of the plate. What are the horizontal reaction forces at the bearings?



Solution. Here is an example where it is best to choose the coordinate system to fit the body, not the rotation. Introduce the xyz system according to the figure. The inertia (diagonal) matrix was calculated in **Example 4.2.1** with

$$I_{xx} = \frac{1}{12}ma^2, \qquad I_{yy} = \frac{1}{3}ma^2, \qquad I_{zz} = \frac{5}{12}ma^2$$

The angular velocity has components along the x and z axes

$$\boldsymbol{\omega} = \omega \sin\varphi\, \boldsymbol{e}_x + \omega \cos\varphi\, \boldsymbol{e}_z$$

The angular momentum of the plate becomes

$$\boldsymbol{L} = \frac{1}{12}ma^2\omega \sin\varphi\, \boldsymbol{e}_x + \frac{5}{12}ma^2\omega\cos\varphi\, \boldsymbol{e}_z$$

The coordinate system xyz rotates with the angular velocity $\boldsymbol{\omega}$ and there is no explicit time dependence in the components of $\boldsymbol{L}$ so

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{\omega} \times \boldsymbol{L} = -\frac{1}{3}ma^2\omega^2 \cos\varphi \sin\varphi\, \boldsymbol{e}_y$$

As the centre of mass is located on the rotation axis the sum of the horizontal forces on the bearings must be zero. Therefore, the forces above and below are of equal magnitude but opposite direction. Call the magnitude of these forces F. The law of the angular momentum then gives

$$2Fd = \frac{1}{3}ma^2\omega^2 \cos\varphi \sin\varphi$$

so that

$$F = \frac{ma^2\omega^2}{6d} \cos\varphi \sin\varphi$$

Note that these forces are thus in the rotating y direction.

4.7 Three-dimensional rotations

In this section the free rotations of a rigid body in three dimensions are investigated, in particular for rotationally symmetric bodies (or more generally for bodies having two principal moments of inertia equal).

Examples of bodies that are free to rotate in three dimensions are some gyros and free bodies in space or air if all resistance can be neglected. Then the motion of the centre of mass follows Newton's second law and is decoupled from the rotations. To start the studies of free rotations Euler's equations (4.35)–(4.37) with no acting moments are used

$$I_1\dot{\omega}_x - (I_2 - I_3)\omega_y\omega_z = 0 \tag{4.60}$$

$$I_2\dot{\omega}_y - (I_3 - I_1)\omega_z\omega_x = 0 \tag{4.61}$$

$$I_3\dot{\omega}_z - (I_1 - I_2)\omega_x\omega_y = 0 \tag{4.62}$$

Remember that these equations are formulated in a body-fixed principal system of the body and that they are formulated for the (absolute) angular velocity components in this coordinate system. The simplest type of solution to these is apparently that only one component of the angular velocity in nonzero

$$\omega_x = \omega_y = 0, \qquad \omega_z = \omega_3 = \text{constant} \tag{4.63}$$

For a body with three distinct values of the principal moments of inertia this is the only type of simple free rotation possible, i.e. a rotation about a fixed direction is only possible if this direction coincides with a principal axis.

To investigate the stability of the rotation about a principal axis a small perturbation of this solution is made. Thus ω_x and ω_y are assumed small (compared to ω_3) and ω_z is assumed to be almost ω_3. The last of Euler's equations then contains the product of two small angular velocities, and this product can thus be neglected altogether. It follows that $\omega_z = \omega_3$ is still constant to first order. Eliminating ω_y between the first two Euler equations then gives

$$\ddot{\omega}_x + \beta\omega_3^2\omega_x = 0 \tag{4.64}$$

where

$$\beta = \frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2} \tag{4.65}$$

For $\beta > 0$ this differential equation has solutions in the form of trigonometric functions, and as these remain limited for all times this gives stable solutions. If, on the other hand, $\beta < 0$, the solutions are exponentials and thus grow without limit as time goes on (and the approximate solution soon becomes invalid) and correspond to unstable solutions. From the expression for β it is seen that the solutions are stable if I_3 is the smallest or largest principal moment of inertia whereas the solutions are unstable if I_3 is the middle principal moment of inertia.

Assume from now on that $I_1 = I_2$, which is true, for example, for a rotationally symmetric body. If also $I_3 = I_1 = I_2$ then the general solution to Euler's equation is that all three components of the angular velocity are constant so the rotation of the body is about any direction fixed both in space and with respect to the body.

When $I_1 = I_2$ it follows from the third Euler equation that $\omega_z = \omega_3 =$ constant. Eliminating ω_y between the first two Euler equations then gives

$$\ddot{\omega}_x + \gamma^2 \omega_3^2 \omega_x = 0 \tag{4.66}$$

where

$$\gamma = \frac{I_3 - I_1}{I_1} \tag{4.67}$$

which has the general solution

$$\omega_x = \omega_{12} \cos(\gamma \omega_3 t + \alpha) \tag{4.68}$$

$$\omega_y = \omega_{12} \sin(\gamma \omega_3 t + \alpha) \tag{4.69}$$

Here ω_3, ω_{12}, and α are constants determined by initial conditions. The solution says that $\boldsymbol{\omega}$ rotates around the body-fixed z axis with the angular velocity $\gamma\omega_3$. Thus $\boldsymbol{\omega}$ describes a conical surface in the body, *the body cone*. The apex angle α_b of this cone is determined from

$$\tan \alpha_b = \frac{\omega_{12}}{\omega_3} \tag{4.70}$$

If $I_3 > I_1$ then $\boldsymbol{\omega}$ rotates in the positive direction around the z axis, otherwise in the negative direction.

What does this description of the rotation in the body-fixed system means in an inertial system? To investigate this it is observed that the angular momentum of the body with respect to the centre of mass $\overline{\boldsymbol{L}}$ is conserved. This direction is used as a space-fixed reference direction. The angular momentum is

$$\overline{\boldsymbol{L}} = I_1 \omega_x \boldsymbol{e}_x + I_1 \omega_y \boldsymbol{e}_y + I_3 \omega_3 \boldsymbol{e}_z \tag{4.71}$$

The component of $\overline{\boldsymbol{L}}$ in the xy plane is $I_1\omega_{12}$. The angle α_3 between $\overline{\boldsymbol{L}}$ and the z direction is given by

$$\tan \alpha_3 = \frac{I_1 \omega_{12}}{I_3 \omega_3} \tag{4.72}$$

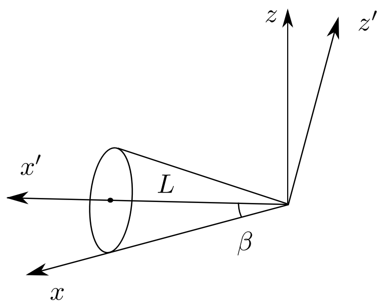
and is thus constant during the motion. The angle α_s between $\overline{\boldsymbol{L}}$ and $\boldsymbol{\omega}$ is given by

$$\cos \alpha_s = \frac{\boldsymbol{\omega} \cdot \overline{\boldsymbol{L}}}{\omega \overline{L}} = \frac{2T_{\text{rot}}}{\omega \overline{L}} \tag{4.73}$$

which is also constant during the motion because T_{rot} is also a conserved quantity. This means that $\boldsymbol{\omega}$ describes a cone around the fixed direction of $\overline{\boldsymbol{L}}$, *the space cone*. As the projections of $\boldsymbol{\omega}$ and $\overline{\boldsymbol{L}}$ in the xy plane are parallel, $\boldsymbol{\omega}$, $\overline{\boldsymbol{L}}$, and $\boldsymbol{e}_z$ lie in a single plane. The rotation can be visualized so that the body cone (fixed in the body) rolls on the space cone (fixed in space). Comparing the expressions for α_b and α_3 two cases can be distinguished: 1. $I_1 > I_3$ (a slender body) then $\alpha_3 > \alpha_b$ and the body cone rolls outside the space cone, 2. $I_1 < I_3$ (a flat body) then $\alpha_3 < \alpha_b$ and the body cone rolls inside the space cone. In both cases the resulting motion can be described as wobbling.

Example 4.7.1

A cone of mass m, height L, and apex angle β is rolling without slip on a horizontal plane. The velocity of the centre of the base of the cone is v. It is assumed that the cone has a slightly concave surface so that it is only supported at the apex and at the base. Determine the normal force from the plane at the base.



Solution. The cone was also considered in **Example 3.4.1**, where it was shown that the angular velocity of the cone is

$$\boldsymbol{\omega} = -\frac{\Omega\cos\beta}{\sin\beta}\boldsymbol{e}_x$$

where $\Omega = v/L\cos\beta$. To calculate the angular momentum of the cone it is best to use the system $x'y'z'$ which is a principal axis system of the cone. In this system the angular velocity is

$$\boldsymbol{\omega} = -\frac{\Omega\cos\beta}{\sin\beta}\left(\cos\beta\,\boldsymbol{e}_{x'} - \sin\beta\,\boldsymbol{e}_{z'}\right)$$

From tables and Steiner's theorem the moments of inertia of the cone are

$$I_{x'x'} = \frac{3}{10}mL^2 \qquad I_{z'z'} = \frac{3}{5}mL^2 + \frac{3}{20}\tan^2\beta\, mL^2$$

The angular momentum about the apex becomes

$$\boldsymbol{L}_O = mL^2\Omega\left[-\frac{3\cos^2\beta}{10\sin\beta}\boldsymbol{e}_{x'} + \cos\beta\left(\frac{3}{5} + \frac{3}{20}\tan^2\beta\right)\boldsymbol{e}_{z'}\right]$$

As the angular velocity of the coordinates system is

$$\boldsymbol{\Omega} = \Omega\boldsymbol{e}_z = \Omega\left(\cos\beta\,\boldsymbol{e}_{z'} + \sin\beta\,\boldsymbol{e}_{x'}\right)$$

the time derivative of the angular momentum becomes

$$\dot{\boldsymbol{L}}_O = \boldsymbol{\Omega}\times\boldsymbol{L} = -mL^2\Omega^2\left[\frac{3\cos^3\beta}{10\sin\beta} + \sin\beta\cos\beta\left(\frac{3}{5} + \frac{3}{20}\tan^2\beta\right)\right]\boldsymbol{e}_{y'}$$

But according to the law of angular momentum

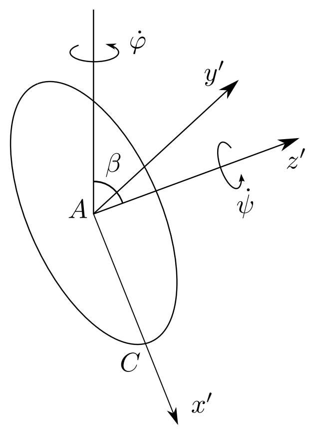
$$\dot{\boldsymbol{L}}_O = \boldsymbol{M}_O = \left(mg\cdot\frac{3}{4}L\cos\beta - N\cdot\frac{L}{\cos\beta}\right)\boldsymbol{e}_{y'}$$

This finally gives the normal force

$$N = \frac{3}{4}mg\cos^2\beta - \frac{mv^2}{L\cos\beta}\left[\frac{3\cos^3\beta}{10\sin\beta} + \sin\beta\cos\beta\left(\frac{3}{5} + \frac{3}{20}\tan^2\beta\right)\right]$$

For sufficiently great Ω the normal force becomes negative. This is of course not possible, what happens is that the cone loses contact with the plane and becomes a spinning top (in the shape of a cone) instead of a rolling cone.

Example 4.7.2
A thin circular disk with mass m and radius R is rolling on a horizontal plane so that the centre A of the disk follows a circular path with radius r. The normal to the disk is then tilted with the constant angle β to the normal to the plane. Determine the speed of A.



Solution. The disk was also considered in **Example 3.4.2**. A coordinate system $x'y'z'$ is introduced with the z' axis as the symmetry axis of the disk, the x' axis in the plane of the disk through the contact point and the y' axis horizontal in the plane of the disk. The three Euler angles φ, $\theta = \beta$, and ψ give the rotation about the vertical, the tilt of the disk from a horizontal position, and the spin of the disk, respectively. Note that this coordinate system is not body-fixed but is a principal system. The angular velocity of the disk is

$$\boldsymbol{\omega} = -\dot{\varphi}\sin\beta\,\boldsymbol{e}_{x'} + (\dot{\psi} + \dot{\varphi}\cos\beta)\boldsymbol{e}_{z'}$$

where it has been used that the nutation is constant so $\theta = \beta$ and $\dot{\theta} = 0$. The coordinate system has the same angular velocity if the spin $\dot{\psi}$ is put to zero. The velocity of the centre A can be written relative the contact point C

$$\boldsymbol{v}_A = \boldsymbol{\omega} \times \boldsymbol{r}_{A/C} = -R(\dot{\psi} + \dot{\varphi}\cos\beta)\boldsymbol{e}_{y'}$$

As the centre A goes in a circular path with radius r the precession velocity is

$$\dot{\varphi} = \frac{v}{r}$$

where v is the sought speed. It follows that

$$\dot{\psi} = -v\left(\frac{1}{R} + \frac{\cos\beta}{r}\right)$$

As the disk is in stationary motion $\ddot{\varphi} = \ddot{\psi} = 0$.

The moments of inertia are $I_{x'x'} = I_{y'y'} = mR^2/4$ and $I_{z'z'} = mR^2/2$. The angular momentum and its time derivative become

$$\boldsymbol{L}_A = mR^2\left(-\frac{1}{4}\dot{\varphi}\sin\beta\,\boldsymbol{e}_{x'} + \frac{1}{2}(\dot{\psi} + \dot{\varphi}\cos\beta)\boldsymbol{e}_{z'}\right)$$

$$\dot{\boldsymbol{L}}_A = \frac{\delta \boldsymbol{L}_A}{\delta t} + \boldsymbol{\Omega} \times \boldsymbol{L}_A = (-\dot{\varphi}\sin\beta\, \boldsymbol{e}_{x'} + \dot{\varphi}\cos\beta \boldsymbol{e}_{z'}) \times \boldsymbol{L}_A$$

$$= mR^2\dot{\varphi}\left(-\frac{1}{4}\dot{\varphi}\sin\beta\, \boldsymbol{e}_{x'} + \frac{1}{2}(\dot{\psi} + \dot{\varphi}\cos\beta)\boldsymbol{e}_{z'}\right)$$

Insert $\dot{\varphi}$ and $\dot{\psi}$ from above. Then the law for the angular momentum gives

$$-\frac{1}{4}mR^2\left(\frac{v}{r}\right)^2\left(\frac{2r}{R} + \cos\beta\right)\sin\beta\, \boldsymbol{e}_{y'} = (F_n r\sin\beta - Nr\cos\beta)\, \boldsymbol{e}_{y'}$$

To eliminate the forces Newton's second law for the centre of mass is used. The radial and vertical components are

$$-F_n = -\frac{mv^2}{r}$$

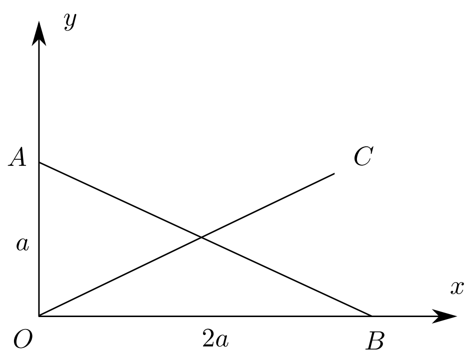
$$N - mg = 0$$

Solving for the velocity finally gives

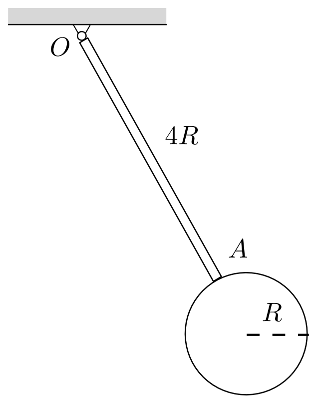
$$v = \frac{4gr^2\cos\beta}{(6r + R\cos\beta)\sin\beta}$$

Problems

4.1 The triangular plate OAB has sides a and $2a$ and mass m. Determine the matrix of inertia of the plate with respect to O. Determine also the moment of inertia about the axis OC, which cuts the mid-point of AB, and the principal moment of inertia and the principle axes of inertia with respect to O.



Prob. 4.1



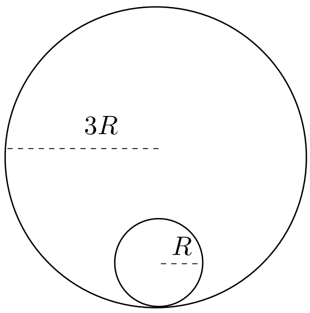
Prob. 4.2

4.2 A physical pendulum consists of a thin bar OA of length $4R$ and mass m and a circular disk of radius R and mass $2m$. The pendulum is swinging in a vertical plane. Determine the period of oscillations for small amplitudes.

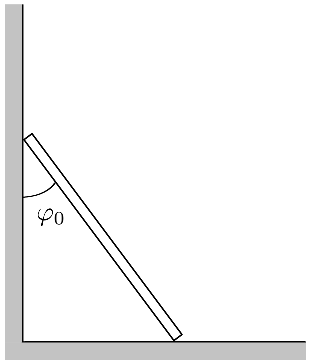
4.3 A car is standing still with a door wide open (90°). The car is starting with constant acceleration $2.0\,\text{m/s}^2$. With what velocity relative the car is the door hitting the frame at the back of the door. The door is approximated as a quadratic plate with side $0.80\,\text{m}$ and is pinned at the front.

4.4 A box, which can be regarded as homogeneous and twice as high as long, is standing on a lorry. The coefficient of friction between the box and the lorry is μ. How large can the acceleration of the lorry be without the box starting to move?

4.5 A homogeneous sphere of radius R can roll within a fixed cylindrical hole with radius $3R$ and horizontal axis. How large must the speed of the sphere be in the bottommost position so the sphere can complete a full turn within the hole?



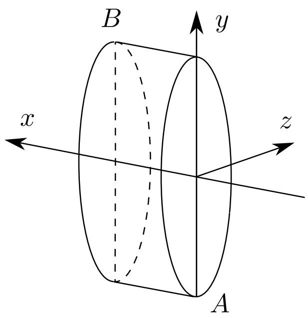
Prob. 4.5



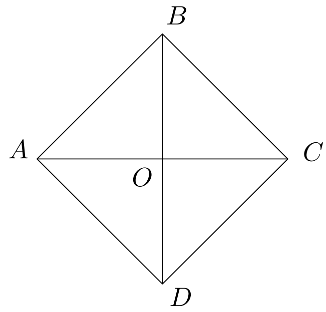
Prob. 4.6

4.6 A thin rod is leaning against a smooth wall and a smooth floor. It is released from rest in the position shown. Show that the rod loses contact with the wall before it reaches the floor, and determine the angle when this happens.

4.7 A cylindrical wheel with length a and radius a has mass m but is slightly inhomogeneous. It so happens that the centre of mass lies in the xy plane with y coordinate $-0.0060a$. The product of inertia $I_{xy} = -0.013ma^2$ and the other products of inertia are zero. The wheel is to balanced for rotation about the x axis by small weights at the positions A and B. What masses should these weights have?



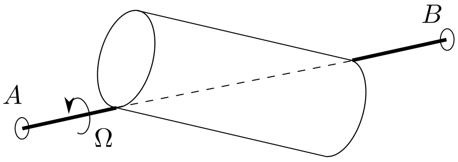
Prob. 4.7



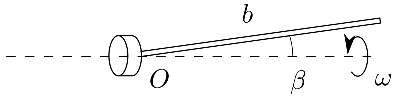
Prob. 4.8

4.8 The thin disk $ABCD$ consists of four triangles OAB, OBC, OCD, and ODA. The system is rotating about the diagonal AC but appears to be unbalanced. The reason for this is that the part OBC is thinner than the others, its mass is $4m$ while the others have mass $5m$ each. Balancing is to be performed by placing two small weights at the middle points of two of the sides of the quadrat. One of these is to be BC. Give the position of the other and the mass of the two weights.

4.9 A homogeneous cylinder with mass m, radius R, and height h is mounted on an axle according to the figure, so that its centre of mass is on the axle. It is mounted symmetrically between the two supports at A and B which are a distance L apart. Determine the moment M_0 and the constraint forces at A and B to obtain a given varying angular velocity Ω.



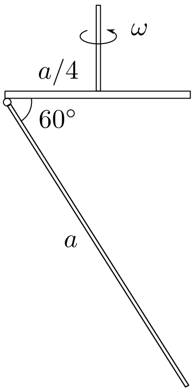
Prob. 4.9



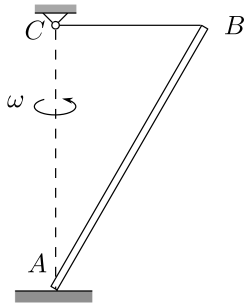
Prob. 4.10

4.10 A bore (a thin rod of mass m and length b) is mounted tilted with the angle β in the machine. The angular velocity ω is constant. Determine the bending moment and the forces at the mounting point O.

4.11 A thin rod of length a is joined to a rotating disk on the distance $a/4$ from the rotation axis. The rod can only swing in a vertical plane containing the rotation axis. How fast should the angular velocity ω of the disk be so that the rod can hang with the constant angle $60°$?



Prob. 4.11

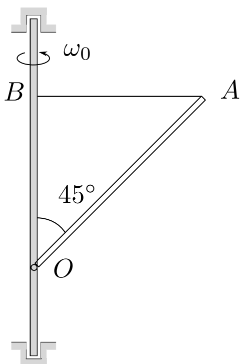


Prob. 4.12

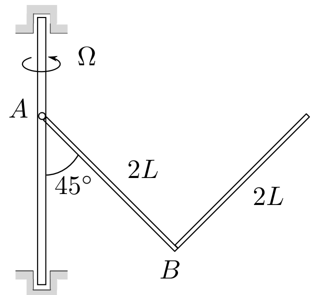
4.12 The rod AB is supported by a table at point A and has been put into rotation with constant angular velocity ω. The length of the rod is 380 mm. The horizontal cord BC has length 220 mm. At A the coefficient of friction is 0.40. What is the largest possible angular velocity?

4.13 A person wants to balance a thin rod of mass m and length a with a small weight of mass m attached on the top so that its top is at rest while the bottom point is taken in a circular path with radius $a/3$ with constant speed. At what angular velocity should the circular path be described?

4.14 A slender rod OA of length a is attached to a vertical axis OB with an angle $45°$ by a hinge at O. A cord AB is used to keep the rod OA in the position shown. Initially the rod and axis are rotating freely with an angular velocity ω_0. Suddenly the cord breaks. Determine the angular velocity about the vertical and about the hinge at O when the rod has fallen to a horizontal position.



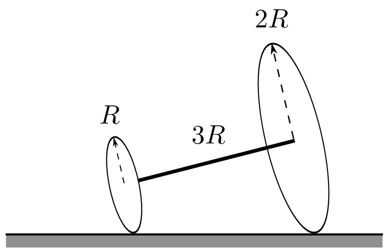
Prob. 4.14



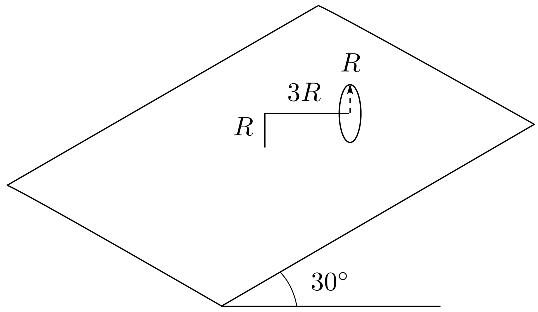
Prob. 4.15

4.15 A body is composed of two slender rods (each of mass m and length $2L$) that are welded together as shown. The body is attached to a rotating vertical axis by a hinge. Determine the angular velocity Ω of the axis so that rod AB makes an angle $45°$ with the vertical.

4.16 Two wheels with masses m and $3m$ and radii R and $2R$ are fixed on a light axis of length $3R$ according to the figure. The wheels can be regarded as thin circular disks. If the constant angular velocity of the wheels is ω about the common axis, how large are the normal forces under the wheels? What happens if the angular velocity becomes too large?



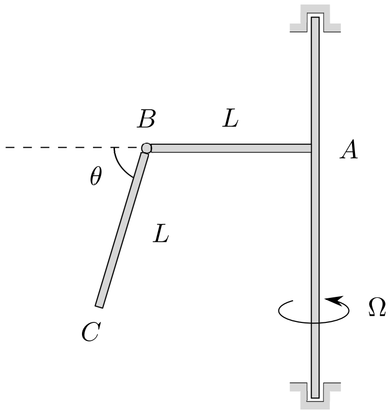
Prob. 4.16



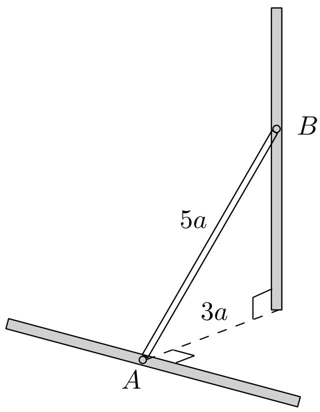
Prob. 4.17

4.17 A disk with mass m and radius R is mounted on an arm so that the wheel can move along a circular path of radius $3R$. The arm is attached to an inclined plane. If the wheel is released from rest at its uppermost position what is the speed at the lowest position?

4.18 The rod BC is attached to the rotating T-bar which rotates with constant angular velocity Ω. Derive the equation of motion of the rod.



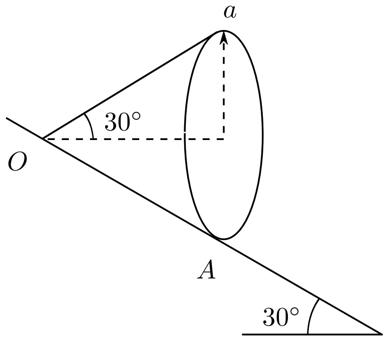
Prob. 4.18



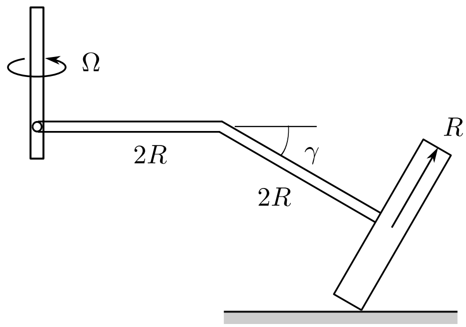
Prob. 4.19

4.19 A rod AB of length $5a$ is coupled to two collars, one following a horizontal and one a vertical bar. The rod is released from its uppermost position. Determine the speed of collar B when the rod is horizontal.

4.20 A cone of mass m, bottom radius a, and half top angle $30°$ is rolling on an inclined plane. The cone is supposed to make contact with the plane only at the points O and A. The cone is released from rest at the position where the contact line between the cone and the plane is horizontal. Determine the normal force at the point A when the cone is in its lowest position (the one in the figure).



Prob. 4.20



Prob. 4.21

4.21 A wheel with mass m is mounted on a curved axis according to the figure. The wheel is rolling on the horizontal plane. The constant angular velocity of the vertical axis is Ω. Determine the normal force on the wheel at the contact point.

Chapter 5

Lagrange's equations

In the formalism so far the constraints enter in two ways. On one hand the kinematical description must include the constraints, on the other the equations of motion must include the reaction forces that enforce the constraints. For systems containing more than one body the constraints may appreciably enlarge the complexity of putting up the equations. In the Lagrangian way of formulating the equations of motion the reaction forces are mostly absent, and the number of equations are correspondingly fewer.

5.1 Generalized coordinates and forces

A set of geometrical variables that uniquely defines the position of all parts of a mechanical system is called a set of generalized coordinates. In practice these variables are always lengths and angles. The smallest number of generalized coordinates that is necessary to describe the time development of a system is the number of degrees of freedom. Note ”the time development of a system” in this sentence. It is not always so that a set of coordinates corresponding to the number of degrees of freedom determines the position of every part of the system at every time instant. This situation appears for rolling where the rolling conditions involve velocities in a nonintegrable way, and thus only the velocities, and not the positions, are given at each instant.

If the number of coordinates is equal to the number of degrees of freedom the coordinates are called free. If the number of coordinates is greater than the number of degrees of freedom, there exist some constraints among the coordinates that in principle may be used to eliminate superfluous coordinates. This may be impracticable, or even impossible, particularly for nonholonomic constraints. However, in many cases, particularly simpler ones, one tries to select free generalized coordinates.

Suppose now that M generalized coordinates q_i, $i = 1, 2, \ldots, M$, are chosen for a system with N degrees of freedom. If $M > N$ there must exist $M - N$ constraints. These may be of the form

$$f_k(q_1, \ldots, q_M, t) = 0, \qquad k = 1, \ldots, M - N \tag{5.1}$$

Time can be included explicitly here, typically if a part of the system is moving in a prescribed way. Constraints of this form that only contain the generalized coordinates, but not any velocities, are called holonomic, as mentioned already in Chapter 3. In principle these constraints may be used to eliminate some coordinates to obtain a set of free coordinates. In practice this is often very difficult for larger systems.

Other forms of constraints also appear. Inequalities are not uncommon, a car rolling on the way or jumping at some irregularity is an example. This can also be expressed by saying that the number of degrees of freedom are changing during the motion, a situation that need some care in handling.

Nonholonomic constraints involve velocities in some way. A typical nonholonomic constraint appears in rolling as exemplified in Chapter 3. In some cases such constraints may be used to eliminate coordinates, in other cases this is impossible. A question in this connection is if a nonholonomic constraint can be integrated to yield a holonomic one. There is a general theorem stating under what conditions this is possible.

Virtual displacements play a central role in deriving Lagrange's equations. A virtual displacement of the generalized coordinates of a system means that they increase with an infinitesimal amount with time kept constant. The denomination "virtual" comes from the condition that time is kept fixed, i.e. that time dependent forces, moving supports, etc, are fixed.

Consider an arbitrary point P in a system. The position of P relative to some fixed frame is a function of the generalized coordinates and time

$$\boldsymbol{r} = \boldsymbol{r}(q_1, \ldots, q_M, t) \tag{5.2}$$

A virtual displacement of P is denoted $\delta\boldsymbol{r}$, and not $\mathrm{d}\boldsymbol{r}$, to distinguish it from a real displacement. Differentiation yields

$$\delta\boldsymbol{r} = \sum_{i=1}^{M} \frac{\partial \boldsymbol{r}}{\partial q_i}\delta q_i \tag{5.3}$$

As the displacement is virtual there is no term from an explicit time dependence of $\boldsymbol{r}$. For a real displacement there would in general also be a term containing $\mathrm{d}t$.

To calculate a virtual displacement Eq. (5.3) can be used. To do this the position of P must be calculated as a function of the generalized coordinates, which is then differentiated. Often there are simpler ways to proceed where the smallness of $\delta\boldsymbol{r}$ is employed. This is most easily seen in examples and is not so easily described in general terms.

When a particle in a system is moving in a virtual displacement the forces on the particle do virtual work. As the virtual displacement is infinitesimal the acting forces can be regarded as constant, so the virtual work is

$$\delta W = \boldsymbol{F}\cdot\delta\boldsymbol{r} = \sum_{i=1}^{M} \boldsymbol{F}\cdot\frac{\partial \boldsymbol{r}}{\partial q_i}\delta q_i \tag{5.4}$$

where Eq. (5.3) is used in the last step. The definition of a generalized force Q_i is that it is the coefficient in front of the corresponding infinitesimal virtual displacement δq_i in the expression for the virtual work

$$\delta W = \sum_{i=1}^{M} Q_i \delta q_i \tag{5.5}$$

For a single particle the generalized force is thus

$$Q_i = \boldsymbol{F}\cdot\frac{\partial \boldsymbol{r}}{\partial q_i} \tag{5.6}$$

It should be remembered that $\boldsymbol{F}$ is the sum of all the forces acting on the particle. If the position of the particle is described by three cartesian coordinates then the generalized forces are simply the cartesian components of the sum of all the forces acting on a particle. If a generalized coordinate is an angle then the corresponding generalized force is a moment as is seen from a dimensional analysis.

To obtain the total generalized forces for a whole system a summation over all particles is performed. When constraint forces are acting on the system, they do not do any virtual work if the virtual displacement does not break the constraint. If the system is holonomic and the coordinates are free then the constraint forces do not perform any virtual work, and this is one good reason to choose such coordinates.

If any of the forces acting on the system is conservative, then the corresponding virtual work is minus the change in potential energy

$$\delta W = -\delta V = -\sum_{i=1}^{M} \frac{\partial V}{\partial q_i}\delta q_i \tag{5.7}$$

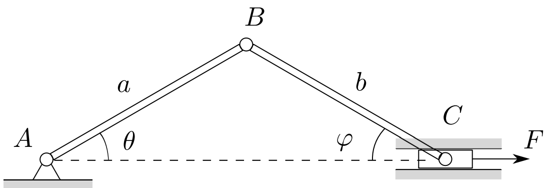
This means that the total generalized force can be written

$$Q_i^{\text{tot}} = -\frac{\partial V}{\partial q_i} + Q_i \tag{5.8}$$

where the potential energy contains all the conservative forces and Q_i all the rest. In the following the notation Q_i is used for the generalized force corresponding to forces that are not described by a potential energy.

Example 5.1.1

The crankshaft AB is in the arbitrary position given by the angle θ. Determine the generalized force Q_θ for the force F applied at the piston C.



Solution. The angle φ of the upper link is determined from the law of sines

$$\sin\varphi = \frac{a}{b}\sin\theta$$

The position of the piston is

$$x = a\cos\theta + b\cos\varphi = a\cos\theta + \sqrt{b^2 - a^2\sin^2\theta}$$

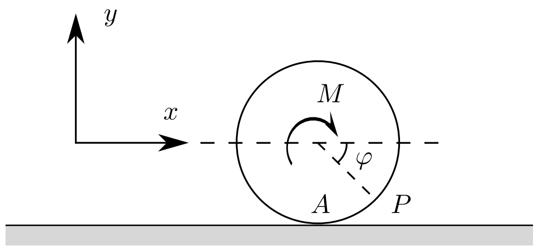
This gives the virtual displacement

$$\delta x = \frac{\mathrm{d}x}{\mathrm{d}\theta}\delta\theta = -\left(a\sin\theta + \frac{a^2\sin\theta\cos\theta}{\sqrt{b^2 - a^2\sin^2\theta}}\right)\delta\theta$$

and the generalized force

$$Q_\theta = F\frac{\mathrm{d}x}{\mathrm{d}\theta} = -Fa\sin\theta\left(1 - \frac{a\cos\theta}{\sqrt{b^2 - a^2\sin^2\theta}}\right)$$

Example 5.1.2
A wheel with radius R is rolling with slip on a horizontal surface. A moment M is applied at the hub. Determine the generalized forces.



Solution. Introduce the position x of the wheel and the rotation angle φ. The generalized forces to these two are thus desired. The moment performs the virtual work

$$\delta W_M = M\delta\varphi$$

To get the virtual work corresponding to the friction force F_f the virtual displacement of the contact point A is needed. Determine first the virtual displacement for an arbitrary point P on the rim. The position of P is

$$\boldsymbol{r}_P = (x + R\cos\varphi)\,\boldsymbol{e}_x - R\sin\varphi\,\boldsymbol{e}_y$$

The corresponding virtual displacement is

$$\delta\boldsymbol{r}_P = \frac{\partial\boldsymbol{r}_P}{\partial x}\delta x + \frac{\partial\boldsymbol{r}_P}{\partial\varphi}\delta\varphi = \boldsymbol{e}_x\delta x - \boldsymbol{e}_x R\sin\varphi\,\delta\varphi - \boldsymbol{e}_y R\cos\varphi\,\delta\varphi$$

For the contact point A $\varphi = \pi/2$ so

$$\delta\boldsymbol{r}_A = \boldsymbol{e}_x\delta x - \boldsymbol{e}_x R\delta\varphi$$

Note that it is not possible to put $\boldsymbol{r}_A = x\boldsymbol{e}_x - R\boldsymbol{e}_y$ and take the derivatives because the variation of A with φ is not captured in this way. The friction force is $F_f\boldsymbol{e}_x$ so the virtual work becomes

$$\delta W_f = -F_f\delta x + RF_f\delta\varphi$$

The total virtual work is $\delta W = \delta W_M + \delta W_f$ so the generalized forces become

$$Q_x = F_f$$

$$Q_\varphi = -RF_f + M$$

5.2 Lagrange's equations

Lagrange's equations are a system of differential equations in time with the generalized coordinates as dependent variables. These equations are now derived and it is then convenient to regard the mechanical system to be composed of particles that are numbered with the letter p. If the system contains rigid bodies the number of particles is infinite and the summation over p should be replaced by an integral, but this does not affect the derivation in any way.

Suppose that M generalized coordinates are used to describe the system. To begin with the coordinates are supposed to be free, the case when constraints exist is discussed afterwards. For particle p with mass m_p the position is a function of the coordinates and possibly time

$$\boldsymbol{r}_p = \boldsymbol{r}_p(q_1, \ldots, q_M, t) \tag{5.9}$$

The velocity of the particle is

$$\dot{\boldsymbol{r}}_p = \frac{\mathrm{d}\boldsymbol{r}_p}{\mathrm{d}t} = \sum_{i=1}^{M} \frac{\partial \boldsymbol{r}_p}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial \boldsymbol{r}_p}{\partial t} = \sum_{i=1}^{M} \frac{\partial \boldsymbol{r}_p}{\partial q_i}\dot{q}_i + \frac{\partial \boldsymbol{r}_p}{\partial t} \tag{5.10}$$

where the generalized velocity

$$\dot{q}_i = \frac{\mathrm{d}q_i}{\mathrm{d}t} \tag{5.11}$$

is introduced. Such a velocity is in practice always an "ordinary" velocity or an angular velocity. From Eq. (5.10) it is seen that

$$\frac{\partial \dot{\boldsymbol{r}}_p}{\partial \dot{q}_i} = \frac{\partial \boldsymbol{r}_p}{\partial q_i} \tag{5.12}$$

a result that is soon needed. Furthermore

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{r}_p}{\partial q_i} = \sum_{j=1}^{M}\left(\frac{\partial}{\partial q_j}\frac{\partial \boldsymbol{r}_p}{\partial q_i}\right)\dot{q}_j + \frac{\partial}{\partial t}\frac{\partial \boldsymbol{r}_p}{\partial q_i} = \frac{\partial}{\partial q_i}\left(\sum_{j=1}^{M}\frac{\partial \boldsymbol{r}_p}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r}_p}{\partial t}\right) = \frac{\partial \dot{\boldsymbol{r}}_p}{\partial q_i} \tag{5.13}$$

a result that is also needed soon.

Start from Newton's second law for particle p

$$m_p \ddot{\boldsymbol{r}}_p = \boldsymbol{F}_p \tag{5.14}$$

To introduce the generalized forces this equation is scalar multiplied by the virtual displacement $\delta \boldsymbol{r}_p$ of particle p and the result is summed over all particles

$$\sum_p m_p \ddot{\boldsymbol{r}}_p \cdot \delta \boldsymbol{r}_p = \sum_p \boldsymbol{F}_p \cdot \delta \boldsymbol{r}_p = \delta W \tag{5.15}$$

according to the definition of the total virtual work on the system. The generalized forces are introduced by Eqs. (5.5) and (5.8) and at the same time Eq. (5.3) is used to express the virtual displacement in the generalized coordinates

$$\sum_{i=1}^{M}\sum_p m_p \ddot{\boldsymbol{r}}_p \cdot \frac{\partial \boldsymbol{r}_p}{\partial q_i}\delta q_i = \sum_{i=1}^{M}\left(-\frac{\partial V}{\partial q_i} + Q_i\right)\delta q_i \tag{5.16}$$

The scalar product on the left-hand side is manipulated with the help of Eqs. (5.12) and (5.13)

$$\begin{aligned}
\ddot{\boldsymbol{r}}_p \cdot \frac{\partial \boldsymbol{r}_p}{\partial q_i} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\boldsymbol{r}}_p \cdot \frac{\partial \boldsymbol{r}_p}{\partial q_i}\right) - \dot{\boldsymbol{r}}_p \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{r}_p}{\partial q_i} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\boldsymbol{r}}_p \cdot \frac{\partial \dot{\boldsymbol{r}}_p}{\partial \dot{q}_i}\right) - \dot{\boldsymbol{r}}_p \cdot \frac{\partial \dot{\boldsymbol{r}}_p}{\partial q_i}
\end{aligned}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}\dot{\boldsymbol{r}}_p^2\right) - \frac{\partial}{\partial q_i}\left(\frac{1}{2}\dot{\boldsymbol{r}}_p^2\right) \tag{5.17}$$

The summation over p in Eq. (5.16) introduces the total kinetic energy of the system

$$T = \sum_p \frac{1}{2} m_p \dot{\boldsymbol{r}}_p^2 \tag{5.18}$$

The kinetic energy is a function of the generalized coordinates and velocities and possibly time. Equation (5.16) can be written

$$\sum_{i=1}^{M}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} - Q_i\right)\delta q_i = 0 \tag{5.19}$$

As this must be true for all possible virtual displacements, which have been chosen as free, each term in the sum must vanish individually

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i, \qquad i = 1, \ldots, M \tag{5.20}$$

These are the M Lagrange's equations for the system. Usually the Lagrangian is introduced

$$L = T - V \tag{5.21}$$

As the potential energy is independent of the generalized velocities the equations can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i, \qquad i = 1, \ldots, M \tag{5.22}$$

The two equations are of course equivalent, and the introduction of the Lagrangian primarily has theoretical interest for further developments in analytical mechanics.

When developing Langrange's equations the assumption was made that the coordinates are free. However, this is not necessary, the equations can be used also with constraints. But as these must be regarded as added on afterwards it is essential that the constraints are not used during the derivation of Lagrange's equations. The constraint forces naturally enter in the generalized forces as they perform virtual work when the constraints are not satisfied. Langrange's equations in the present form can thus be used with constraints of any type, also nonholonomic.

Lagrange's equations are particularly suited for situations where the motion is investigated with no regard to the constraint forces. This is particularly true for larger systems with many constraint forces. If a constraint force is needed a convenient Newtonian equation can be used or else generalized coordinates that are not free, and that thus introduce the wanted constraint force among the generalized forces, are used.

Lagrange's equations are a set of second order differential equations in the generalized coordinates. If the coordinates are not free some constraints and constraint forces are also present as unknowns. There is one not uncommon case when one of Lagrange's equations can be reduced to a first order equation. If the kinetic and potential energies are independent of a generalized coordinate, say q_k, and the corresponding generalized force Q_k vanishes, then that Langrange's equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} = 0 \tag{5.23}$$

The coordinate q_k is called cyclic or ignorable. Equation (5.23) can apparently be integrated to

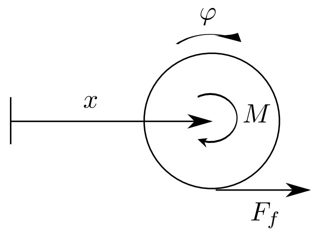
$$\frac{\partial L}{\partial \dot{q}_k} = C \tag{5.24}$$

where C is a constant depending on the initial conditions. This is then a first order differential equation. A common case when a coordinate is cyclic is the angular coordinate for a rotationally symmetric body, like a wheel or gyro, when there is no moment applied about the symmetry axis. Equation (5.24) is in fact a conservation principle and contain conservation of linear momentum (when q_k is a length) and angular momentum (when q_k is an angle) as special cases.

In many problems energy is also conserved and this yields a further equation that can be useful to simplify calculations, although it should be remembered that energy is quadratic in the generalized velocities. When Lagrange's equations are used, the kinetic and potential energies are anyway calculated, so putting up energy conservation entails no appreciable extra effort.

Example 5.2.1

Put up Lagrange's equations for the wheel of **Example 5.1.2**. The wheel has mass m and moment of inertia with respect to the hub $\overline{I}$.



Solution. The generalized coordinates are x and φ. The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\overline{I}\dot{\varphi}^2$$

and the potential energy is simply $V = 0$. The generalized forces were calculated in **Example 5.1.2**

$$Q_x = F_f$$

$$Q_\varphi = -RF_f + M$$

Assume now that the wheel rolls with slip and that $\dot{x} > R\dot{\varphi}$. Then the friction force is $F_f = -\mu mg$. Differentiations easily give Lagrange's equations

$$m\ddot{x} = -\mu mg$$

$$\overline{I}\ddot{\varphi} = \mu mgR + M$$

These are seen to be the same as those obtained by a Newtonian approach.

Assume next rolling without slip, so that $\dot{x} = R\dot{\varphi}$. It is possible to still use x and φ as generalized coordinates, although they are no longer free. The friction force F_f is then unknown, so Lagrange's equations are

$$m\ddot{x} = F_f$$

$$\overline{I}\ddot{\varphi} = -F_fR + M$$

Together with the rolling condition $\dot{x} = R\dot{\varphi}$, these form a system of three equations for x, φ, and F_f. Note that it is not possible to use the constraint to simplify the kinetic energy to

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\overline{I}\dot{\varphi}^2 = \frac{1}{2}\left(mR^2 + \overline{I}\right)\dot{\varphi}^2$$

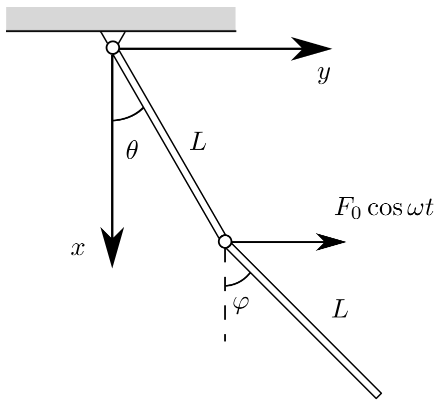
and then derive Lagrange's equations for x andφ. However, it is possible to eliminate x altogether and use φ as the single generalized coordinate. Then

$$Q_\varphi = M$$

because the friction force F_f performs no virtual work when the constraint is satisfied. The single Lagrange's equation becomes

$$\left(mR^2 + \overline{I}\right)\ddot{\varphi} = M$$

Example 5.2.2
A double pendulum consists of two equal bars, each with mass m and length L, which can swing in a vertical plane. A horizontal force is applied at the pin connecting the two bars. Derive the equations of motion in the two angles θ and φ by using Lagrange's equations.



Solution. It is noticed that the system is conservative and holonomic with free generalized coordinates θ and φ. Denote the upper bar with index 1, the lower with index 2. In the coordinate system in the figure the position of the centre of mass of the bars is

$$\boldsymbol{r}_1 = \frac{1}{2}L\sin\theta\,\boldsymbol{e}_x + \frac{1}{2}L\cos\theta\,\boldsymbol{e}_y$$

$$\boldsymbol{r}_2 = L\left(\sin\theta + \frac{1}{2}\sin\varphi\right)\boldsymbol{e}_x + L\left(\cos\theta + \frac{1}{2}\cos\varphi\right)\boldsymbol{e}_y$$

The corresponding velocities are obtained by direct differentiations

$$\dot{\boldsymbol{r}}_1 = \frac{1}{2}L\dot{\theta}\cos\theta\,\boldsymbol{e}_x - \frac{1}{2}L\dot{\theta}\sin\theta\,\boldsymbol{e}_y$$

$$\dot{\boldsymbol{r}}_2 = L\left(\dot{\theta}\cos\theta + \frac{1}{2}\dot{\varphi}\cos\varphi\right)\boldsymbol{e}_x - L\left(\dot{\theta}\sin\theta + \frac{1}{2}\dot{\varphi}\sin\varphi\right)\boldsymbol{e}_y$$

The kinetic energy is

$$T = \frac{1}{2}I_{zz}\dot{\theta}^2 + \frac{1}{2}m\dot{\boldsymbol{r}}_2 + \frac{1}{2}\overline{I}_{zz}\dot{\varphi}^2$$

$$= \frac{1}{6}mL^2\dot{\theta}^2 + mL^2\left[\dot{\theta}^2 + \frac{1}{4}\dot{\varphi}^2 + \dot{\theta}\dot{\varphi}\cos(\varphi-\theta)\right] + \frac{1}{24}mL^2\dot{\varphi}^2$$

$$= mL^2\left[\frac{2}{3}\dot{\theta}^2 + \frac{1}{6}\dot{\varphi}^2 + \frac{1}{2}\dot{\theta}\dot{\varphi}\cos(\varphi-\theta)\right]$$

The potential energy is

$$V = mg\left[-\frac{1}{2}L\cos\theta - L\left(\cos\theta + \frac{1}{2}\cos\varphi\right)\right]$$

The force is applied at the point

$$\boldsymbol{r} = L\sin\theta\boldsymbol{e}_x + L\cos\theta\boldsymbol{e}_y$$

so the generalized forces can be calculated as

$$Q_\theta = \boldsymbol{F}\cdot\frac{\partial\boldsymbol{r}}{\partial\theta} = F_0\cos\omega t L\cos\theta$$

$$Q_\varphi = \boldsymbol{F}\cdot\frac{\partial\boldsymbol{r}}{\partial\varphi} = 0$$

The derivatives with respect to the generalized velocities become

$$\frac{\partial T}{\partial\dot{\theta}} = mL^2\left[\frac{4}{3}\dot{\theta} + \frac{1}{2}\dot{\varphi}\cos(\varphi-\theta)\right]$$

$$\frac{\partial T}{\partial\dot{\varphi}} = mL^2\left[\frac{1}{3}\dot{\varphi} + \frac{1}{2}\dot{\theta}\cos(\varphi-\theta)\right]$$

Further differentiations give

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{\theta}} = mL^2\left[\frac{4}{3}\ddot{\theta} + \frac{1}{2}\ddot{\varphi}\cos(\varphi-\theta) - \frac{1}{2}\dot{\varphi}(\dot{\varphi}-\dot{\theta})\sin(\varphi-\theta)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{\varphi}} = mL^2\left[\frac{1}{3}\ddot{\varphi} + \frac{1}{2}\ddot{\theta}\cos(\varphi-\theta) - \frac{1}{2}\dot{\theta}(\dot{\varphi}-\dot{\theta})\sin(\varphi-\theta)\right]$$

$$\frac{\partial T}{\partial\theta} = \frac{1}{2}mL^2\dot{\varphi}\dot{\theta}\sin(\varphi-\theta)$$

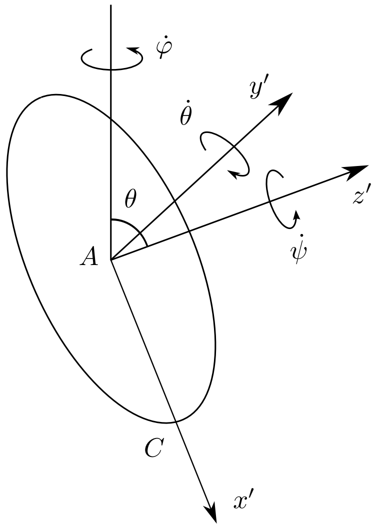
$$\frac{\partial T}{\partial\varphi} = -\frac{1}{2}mL^2\dot{\theta}\dot{\varphi}\sin(\varphi-\theta)$$

Lagrange's equations become (note that there are cancellations)

$$mL^2\left[\frac{4}{3}\ddot{\theta} + \frac{1}{2}\ddot{\varphi}\cos(\varphi-\theta) - \frac{1}{2}\dot{\varphi}^2\sin(\varphi-\theta)\right] + \frac{3}{2}mgL\sin\theta = F_0\cos\omega t L\cos\theta$$

$$mL^2\left[\frac{1}{3}\ddot{\varphi} + \frac{1}{2}\ddot{\theta}\cos(\varphi-\theta) - \frac{1}{2}\dot{\theta}^2)\sin(\varphi-\theta)\right] + \frac{1}{2}mgL\sin\varphi = 0$$

Example 5.2.3
Put up Lagrange's equations for the general motion of the rolling disk of **Example 3.4.2** and **Example 4.7.2**.



Solution. A coordinate system $x'y'z'$is introduced with the z' axis as the symmetry axis of the disk, the x' axis in the plane of the disk through the contact point and the y' axis horizontal in the plane of the disk. Note that this coordinate system is not body-fixed but is a principal system. The three Euler angles φ, θ, and ψ give the rotation about the vertical, the tilt of the disk from a horizontal position, and the spin of the disk, respectively. By using the constraints it is enough to use the Euler angles φ, θ, and ψ as generalized coordinates. This minimizes the number of equations.

From **Example 3.4.2** the angular velocity of the disk is

$$\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\, \boldsymbol{e}_{x'} + \dot{\theta}\boldsymbol{e}_{y'} + (\dot{\psi} + \dot{\varphi}\cos\theta)\boldsymbol{e}_{z'}$$

The velocity of the centre A can be written relative the contact point C

$$\boldsymbol{v}_A = \boldsymbol{\omega} \times \boldsymbol{r}_{A/C} = -R(\dot{\psi} + \dot{\varphi}\cos\theta)\boldsymbol{e}_{y'} + R\dot{\theta}\boldsymbol{e}_{z'}$$

The inertia matrix is diagonal with nonzero entries

$$\overline{I}_{x'x'} = \overline{I}_{y'y'} = \frac{1}{4}mR^2 \qquad \overline{I}_{z'z'} = \frac{1}{2}mR^2$$

The kinetic energy of the disk becomes

$$\begin{aligned} T &= \frac{1}{2}mv_A^2 + \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\overline{I}\boldsymbol{\omega} \\ &= \frac{1}{2}mR^2\left[(\dot{\psi} + \dot{\varphi}\cos\theta)^2 + \dot{\theta}^2\right] + \frac{1}{2}\cdot\frac{1}{4}\left[\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2 + 2(\dot{\psi} + \dot{\varphi}\cos\theta)^2\right] \\ &= mR^2\left[\frac{5}{8}\dot{\theta}^2 + \frac{3}{4}\dot{\psi}^2 + \frac{1}{8}\dot{\varphi}^2(1 + 5\cos^2\theta) + \frac{3}{2}\dot{\varphi}\dot{\psi}\cos\theta\right] \end{aligned}$$

The potential energy due to gravity is

$$V = mgR\sin\theta$$

With only three generalized coordinates the constraints are always satisfied and there are therefore no generalized forces.

Only the generalized velocities $\dot{\varphi}$ and $\dot{\psi}$, and not the angles φ and ψ, are present in T and V and φ and ψ are thus cyclic coordinates. Therefore the quantities

$$\frac{\partial T}{\partial \dot{\varphi}} = mR^2\left[\frac{1}{4}\dot{\varphi}(1+5\cos^2\theta) + \frac{3}{2}\dot{\psi}\cos\theta\right]$$

$$\frac{\partial T}{\partial \dot{\psi}} = \frac{3}{2}mR^2(\dot{\psi} + \dot{\varphi}\cos\theta)$$

are conserved. The following derivatives are also needed

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{5}{4}mR^2\dot{\theta}$$

$$\frac{\partial T}{\partial \theta} = \frac{1}{4}mR^2(5\dot{\varphi}^2\cos\theta\sin\theta - 6\dot{\varphi}\dot{\psi}\sin\theta)$$

Some further differentiations then give Lagrange's equations

$$\ddot{\varphi}(1+5\cos^2\theta) - 10\dot{\varphi}\dot{\theta}\cos\theta\sin\theta + 6\ddot{\psi}\cos\theta - 6\dot{\theta}\dot{\psi}\sin\theta = 0$$

$$\ddot{\psi} + \ddot{\varphi}\cos\theta - \dot{\varphi}\dot{\theta}\sin\theta = 0$$

$$5\ddot{\theta} + 5\dot{\varphi}^2\cos\theta\sin\theta + 6\dot{\varphi}\dot{\psi}\sin\theta + 4\frac{4g}{R}\cos\theta = 0$$

These equations are quite complicated and can not be solved analytically. In the next section a linearization is performed to investigate stability.

5.3 Linearization

The equations of motion for most systems become complicated and nonlinear and are not possible to solve analytically. Numerical solutions are always possible and often necessary, but approximate analytical solutions are also very valuable as they provide important qualitative and quantitative information which is not possible to obtain in such generality from a numerical solution. As an example, an analytical solution will give an oscillation frequency as a function of masses, stiffnesses, and lengths not obtainable by a numerical solution.

The simplest and most common way to obtain approximate solutions is by linearizing the equations of motion about some reference state. This is often a static equilibrium, but it can also a dynamic "equilibrium", e.g. a constant rotation rate about an axis or a vehicle going straight ahead with constant velocity.

To perform a linearisation the reference state is first determined. Let this state be described by the generalized coordinates $q_i^\star$. Introduce the relative generalized coordinates ξ_i as

$$\xi_i = q_i - q_i^\star \tag{5.25}$$

At the linearisation the coordinates ξ_i and velocities $\dot{\xi}_i$ are assumed small. To be really proper this "smallness" must be introduced for dimensionless variables, but this is often skipped. This is necessary, however, if the range of validity of the linearisation is investigated. If the equations of motion are already derived, Eq. (5.25) is inserted and a series expansion is performed keeping only linear terms in ξ_i, $\dot{\xi}_i$, and $\ddot{\xi}_i$ (also cross terms, like $\xi_i\dot{\xi}_i$, are of course skipped). If Lagrange's equations are used this method is unnecessarily complicated. Instead the approximations can be performed in the kinetic and potential energies where only quadratic terms are kept. Lagrange's equations then become linear.

Assume now that the system under study is holonomic with time independent constraints. From the general derivation of the kinetic energy from the previous section it then follows that the kinetic energy has the form

$$T = \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} M_{ij}\dot{\xi}_i\dot{\xi}_j \tag{5.26}$$

Here M_{ij} is call the mass matrix (compare the expression for a particle) and can be assumed to be symmetric. The mass matrix is generally a function of the generalized coordinates, but if T is to be expanded to second order, then the mass matrix needs only be calculated in the equilibrium position $\xi_i = 0$. This leads to the conclusion that the kinematic analysis can be performed in the equilibrium position, a fact that can simplify the calculations considerably.

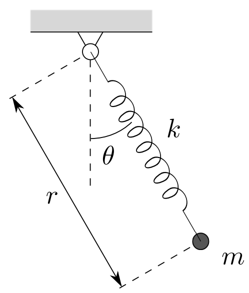
The potential energy only depends on the generalized coordinates and thus has a Taylor series expansion about the equilibrium position

$$V = V^\star + \sum_{i=1}^{M}\left(\frac{\partial V}{\partial q_i}\right)^\star \xi_i + \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} K_{ij}\xi_i\xi_j \tag{5.27}$$

where the coefficients are calculated in the equilibrium position. The value of the constant $V^\star$ is immaterial and the constants before the linear terms vanish if the system is conservative because this defines the equilibrium position. The matrix K_{ij} is called the stiffness matrix (compare the expression for a spring) and is symmetric. To determine the stiffness matrix deviations to second order in the generalized coordinates must be determined. This means, for example, that the elongation of a spring must be determined to second order, except when the equilibrium position is when the spring is unstretched, in which case it is enough to determine the elongation to first order.

Example 5.3.1

Derive the linearized equations of motion for a particle suspended in a spring that can swing in a vertical plane. The particle has mass m and the spring has stiffness k and unstretched length L.



Solution. Introduce the generalized coordinates r and θ (polar coordinates) and put up Lagrange's equations. There are no generalized forces and

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$V = -mgr\cos\theta + \frac{1}{2}k(r-L)^2$$

The exact Lagrange's equations become

$$m\ddot{r} - mr\dot{\theta}^2 + k(r-L) - mg\cos\theta = 0$$

$$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} + mg\sin\theta = 0$$

The equilibrium position is obtained by putting all time derivatives to zero which gives

$$r^\star = L + \frac{mg}{k} \qquad \theta^\star = 0$$

This can of course be seen directly in this simple case. Introduce the relative coordinates

$$\xi_1 = r - r^\star \qquad \xi_2 = \theta - \theta^\star$$

Using $\cos\xi_2 \simeq 1$ and $\sin\xi_2 \simeq \xi_2$ and deleting all nonlinear terms gives

$$m\ddot{\xi}_1 + k\xi_1 = 0$$

$$r^\star\ddot{\xi}_2 + g\xi_2 = 0$$

Two uncoupled oscillations are thus obtained, one vertical with angular frequency $\omega_1 = \sqrt{k/m}$ due to the spring and one horizontal (pendulum) with $\omega_2 = \sqrt{g/r^\star}$ due to gravity.

If it should happen that the two frequencies become equal, $\omega_1 = \omega_2$, then the two oscillations couple. This is a nonlinear phenomenon and the result is that energy is gradually transferred between the two types of oscillation.

Example 5.3.2
Derive the linearized equations of motion for the double pendulum of **Example 5.2.2**.

Solution. The simplest possibility is of course to directly linearize the final equations in **Example 5.2.2**. However, to demonstrate the advantages with an earlier linearization, this approach is followed. The kinematical analysis can then be performed in the equilibrium position, which means that the velocity of the lower bar is $\overline{v}_2 = L\dot{\theta} + L\dot{\varphi}/2$, so the kinetic energy becomes

$$\begin{aligned} T &= \frac{1}{2}I_{zz}\dot{\theta}^2 + \frac{1}{2}m\overline{v}_2^2 + \frac{1}{2}\overline{I}_{zz}\dot{\varphi}^2 \\ &= \frac{1}{6}mL^2\dot{\theta}^2 + \frac{1}{2}m(L\dot{\theta} + L\dot{\varphi}/2)^2 + \frac{1}{24}mL^2\dot{\varphi}^2 \\ &= mL^2\left[\frac{2}{3}\dot{\theta}^2 + \frac{1}{6}\dot{\varphi}^2 + \frac{1}{2}\dot{\theta}\dot{\varphi}\right] \end{aligned}$$

Here the linear approximation in velocity of course gives quadratic terms in T. In the potential energy the trigonometric functions must be expanded to second order

$$\begin{aligned} V &= mg\left[-\frac{1}{2}L\cos\theta - L\left(\cos\theta + \frac{1}{2}\cos\varphi\right)\right] \\ &= mg\left[-\frac{1}{2}L\left(1 - \frac{1}{2}\theta^2\right) - L\left(1 - \frac{1}{2}\theta^2\right) - \frac{1}{2}L\left(1 - \frac{1}{2}\varphi^2\right)\right] \end{aligned}$$

$$= mgL\left[\frac{3}{4}\theta^2 + \frac{1}{4}\varphi^2 - 2\right]$$

It is then an easy matter to derive Lagrange's equations

$$\frac{4}{3}\ddot{\theta} + \frac{1}{2}\ddot{\varphi} + \frac{3g}{2L}\theta = F_0 L\cos\omega t$$

$$\frac{1}{3}\ddot{\varphi} + \frac{1}{2}\ddot{\theta} + \frac{g}{2L}\varphi = 0$$

These equations are of course identical to those obtained by a direct linearization of the final equations in **Example 5.2.2**. If only the linear equations are needed then the present method, starting with linearizations already in the kinematical analysis, is simpler. The two linear equations in θ and φ are expected to result in oscillatory motion. How these equations are solved is discussed in the next chapter.

Example 5.3.3

Derive the equations of motion for the rolling disk, when only small deviations from rolling straight ahead is considered. Investigate the stability.

Solution. The Lagrangian equations for the rolling disk were derived in **Example 5.2.2**. Rolling in a straight line is characterized by

$$\varphi = 0 \qquad \theta = \frac{\pi}{2} \qquad \dot{\psi} = \frac{v}{R}$$

where v is the rolling speed. Now introduce

$$\theta = \frac{\pi}{2} + \xi \qquad \dot{\psi} = \frac{v}{R} + \omega$$

It is now assumed that φ, ξ, and ω and derivatives of these are small. Using also

$$\cos\theta = -\sin\xi \simeq -\xi \qquad \sin\theta = \cos\xi \simeq 1$$

it is straightforward to linearize and obtain

$$\ddot{\varphi} - \frac{6v}{R}\dot{\xi} = 0$$

$$\dot{\omega} = 0$$

$$5\ddot{\xi} + \frac{6v}{R}\dot{\varphi} - \frac{4g}{R} = 0$$

Integrating the first once gives

$$\dot{\varphi} = \frac{6v}{R}\xi + C$$

where C is a constant. Inserting in the last equation gives

$$5\ddot{\xi} + \left[36\left(\frac{v}{R}\right)^2 - \frac{4g}{R}\right]\xi = -\frac{6vC}{R}$$

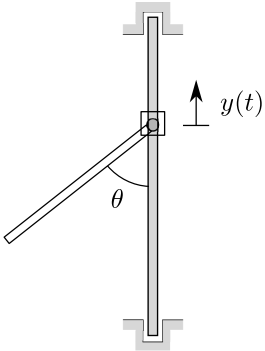
If the factor in front of the ξ term is positive the solution is stable, otherwise unstable. Thus increasing v has a stabilizing effect. For stability

$$v > \frac{1}{3}\sqrt{gR}$$

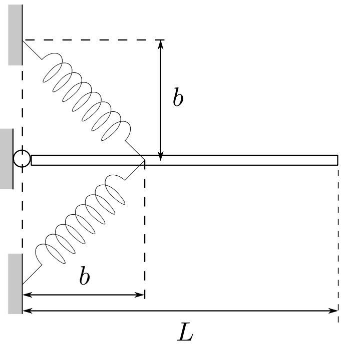
For a coin with radius $R = 14\,\text{mm}$ this gives $v > 0.12\,\text{m/s}$.

Problems

5.1 A collar has a given vertical motion $y(t)$. A rod of mass m and length L is attached to the collar and can swing in a vertical plane. Derive Lagrange's equation in the angle θ for the motion of the rod.



Prob. 5.1

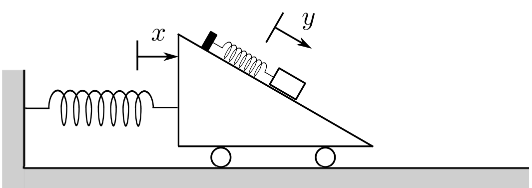


Prob. 5.2

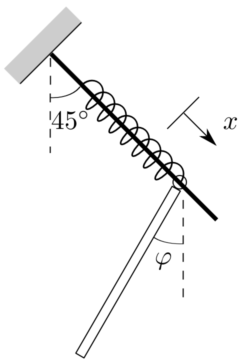
5.2 A bar with mass m and length L is connected by a pin to a fixed support. The bar is supported by two equal springs with stiffness k each. The springs are unstretched when the bar is horizontal. Derive Lagrange's equation in the angle θ which is zero in the shown position and positive downwards.

5.3 A cylinder with mass m and radius R is rolling without slip on a horizontal surface. The centre of mass is located at the distance d from the cylinder axis. The radius of gyration with respect to the centre of mass is k. Derive Lagrange's equation in the angle θ for the rolling of the cylinder. Here θ is defined to be zero in the equilibrium position.

5.4 A cart with mass $5m$ is attached to a support by a spring with stiffness $8k$. A box with mass m is gliding without friction on an inclined plane (angle β) on the cart and is attached by a spring with stiffness $3k$. Derive Lagrange's equations in the variables x and y measured from unstretched springs.



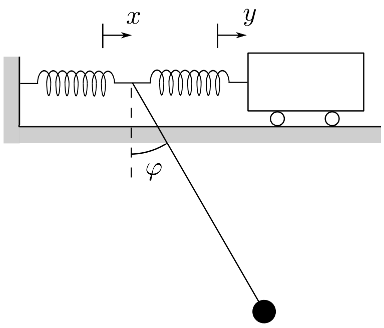
Prob. 5.4



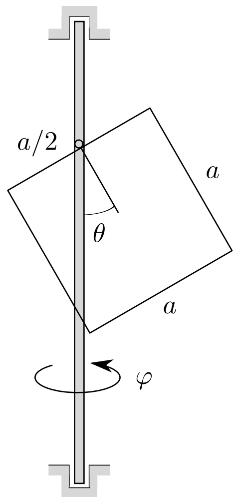
Prob. 5.5

5.5 A rod with mass m and length a is attached to a spring with stiffness k which has the fixed direction $45°$ to the vertical. Derive Lagrange's equations in the variables x (measured from unstretched spring) and φ.

5.6 Two equal springs, each with stiffness k, a pendulum with mass m and length d, and a body with mass m constitutes a mechanical system according to the figure. Derive Lagrange's equations in the three coordinates x, y, and φ.



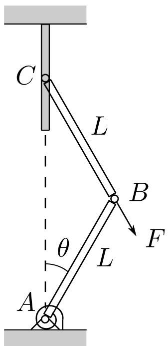
Prob. 5.6



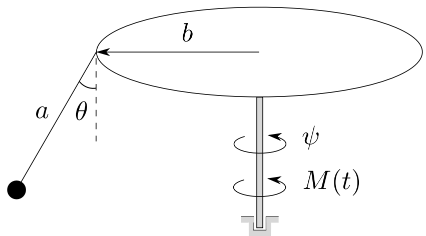
Prob. 5.7

5.7 A quadratic plate with mass m and side a is connected by a pin to a vertical, freely rotating axis. Derive Lagrange's equations in the angles θ and φ.

5.8 Two equal arms AB and BC, each with mass m and length L are connected according to the figure. The applied force F is always parallel to the arm BC. The torsional spring at the support A has stiffness k and is unstretched when $\theta = 60°$. Derive Lagrange's equation.



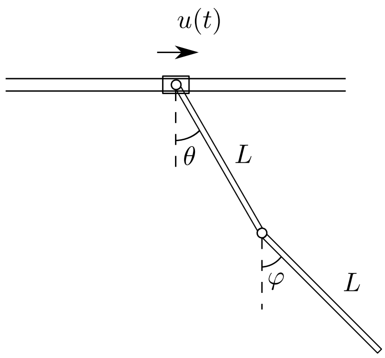
Prob. 5.8



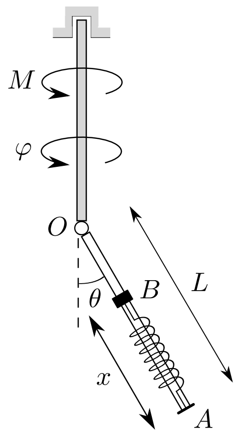
Prob. 5.9

5.9 A pendulum with mass m and length a is suspended on a circular disk at the distance b from the rotation axis. The pendulum can only swing in a vertical plane through the axis. The disk has moment of inertia I about its axis and rotates under the action of an applied moment $M(t)$. Derive Lagrange's equations in the angles θ (the pendulum angle) and ψ (the angle of the axis).

5.10 A double pendulum consists of two equal bars of mass m and length L each. The pendulum can only swing in one plane and its point of suspension is moving with a given velocity $u(t)$ along a horizontal rod in the swing plane. Derive Lagrange's equations.



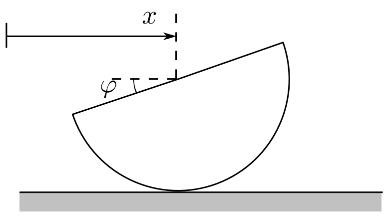
Prob. 5.10



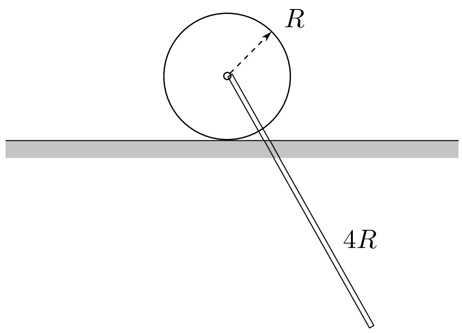
Prob. 5.11

5.11 A rod OA of negligible mass and length L is attached to a rotating vertical axis so that OA can rotate about a pin at O. A small collar B with mass m can slide along OA with negligible friction. The collar is attached to an elastic spring AB with stiffness k and unstretched length b. A torque $M(t)$ is applied to the vertical axis. Derive Lagrange's equations in x, φ, and θ.

5.12 A homogeneous semicylinder with mass m and radius R is moving on a horizontal surface without friction. Derive Lagrange's equations in the variables x and φ.



Prob. 5.12



Prob. 5.13

5.13 A homogeneous circular cylinder with mass $4m$ and radius R rolls without slip on a horizontal surface. A bar with mass m and length $4R$ is pinned to the axis of the cylinder and can swing freely. If the system is released from rest with the bar horizontal, what is the angular velocity of the bar when it has rotated $90°$?

Chapter 6

Coupled oscillations

In the previous chapters oscillations have appeared in many cases. Oscillations often appear in technical applications, often as an unwanted phenomenon that may lead to large forces in the system or annoyance for people. So far, however, oscillations have only been treated in systems that can be described by a single degree of freedom. But oscillations are of course equally important in systems with more than one degree of freedom. Such cases are treated in this chapter.

6.1 Oscillations with several degrees of freedom

Before turning to a general treatment of oscillations with more than one degree of freedom, it is instructive to look at a simple two degree of freedom system. Consider the two bodies connected by two springs in the figure. The two bodies each has mass m and the springs each has stiffness k. Introducing the deviations from the equilibrium position x_1 and x_2, Newton's second law becomes

$$m\ddot{x}_1 + kx_1 + k(x_1 - x_2) = 0 \tag{6.1}$$

$$m\ddot{x}_2 + k(x_2 - x_1) = 0 \tag{6.2}$$

For the moment only free oscillations are of interest so no exciting forces are included. Damping is neglected.

For free oscillations it is possible to assume

$$x_1 = u_1 \cos\omega t \tag{6.3}$$

$$x_2 = u_2 \cos\omega t \tag{6.4}$$

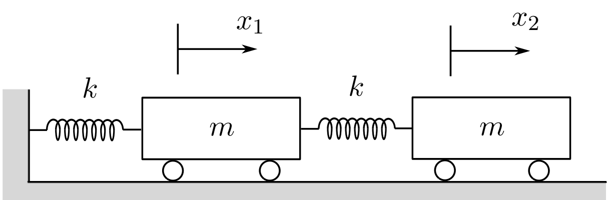


Figure 6.1: Two-degree of freedom oscillating system.

where the amplitudes u_1 and u_2 and the angular frequency ω are to be determined. Insertion gives

$$-m\omega^2 u_1 + 2ku_1 - ku_2 = 0 \tag{6.5}$$

$$-m\omega^2 u_2 + ku_2 - ku_1 = 0 \tag{6.6}$$

For nontrivial solutions to exist the determinant of the system must vanish

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} = 0 \tag{6.7}$$

This is a second order equation in ω^2 with solutions

$$\omega^2 = \frac{k}{2m}(3 \pm \sqrt{5}) \tag{6.8}$$

This gives two positive real angular frequencies

$$\omega_1 = 0.62\sqrt{k/m} \tag{6.9}$$

$$\omega_2 = 1.62\sqrt{k/m} \tag{6.10}$$

There are also two negative roots but they have no physical significance. The two angular frequencies ω_1 and ω_2 are called eigenfrequencies (or natural frequencies) and are the ones the system is oscillating with in the absence of excitations. That the number of eigenfrequencies coincides with the number of degrees of freedom is a general rule as will be shown shortly.

When the eigenfrequencies are determined the quotient between the amplitudes u_1^j and u_2^j for eigenmode j are given by Eqs. (6.5) or (6.6)

$$\frac{u_1^j}{u_2^j} = \frac{k}{2k - m\omega_j^2} = \frac{k - m\omega_j^2}{k} \tag{6.11}$$

This gives the numerical values

$$\frac{u_1^1}{u_2^1} = 0.62 \tag{6.12}$$

$$\frac{u_1^2}{u_2^2} = -1.62 \tag{6.13}$$

In mode 1 the two bodies are oscillating in phase with a little larger amplitude on the outer body, while in mode 2 they are oscillating in antiphase with a larger amplitude on the inner body. The general solution for free oscillations of the two bodies is a superposition of the two eigenmodes

$$x_1 = A_1 u_1^1 \cos(\omega_1 t + \alpha_1) + A_2 u_1^2 \cos(\omega_2 t + \alpha_2) \tag{6.14}$$

$$x_2 = A_1 u_2^1 \cos(\omega_1 t + \alpha_1) + A_2 u_2^2 \cos(\omega_2 t + \alpha_2) \tag{6.15}$$

The amplitudes A_1 and A_2 and the phases α_1 and α_2 are determined by initial conditions.

When the oscillations of a general system with N degrees of freedom are investigated there is a further complication that did not arise for the simple example just studied. Most mechanical systems are governed by nonlinear differential equations which are not solvable analytically. To be able to proceed the equations are therefore linearized around some equilibrium or simple dynamical ”equilibrium”, according to the discussion in the previous chapter. Assume the system to be conservative with free generalized coordinates ξ_i defined relative the equilibrium state. The Lagrangian is then

$$L = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(M_{ij}\dot{\xi}_i\dot{\xi}_j - K_{ij}\xi_i\xi_j\right) \tag{6.16}$$

The mass matrix M_{ij} and the stiffness matrix K_{ij} are both constant and symmetric after the linearization. In Lagrange’s equations the derivative of the Lagrangian with respect to the generalized velocity $\dot{\xi}_k$ is needed

$$\begin{aligned}\frac{\partial L}{\partial \dot{\xi}_k} &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}\left(\frac{\partial \dot{\xi}_i}{\partial \dot{\xi}_k}\dot{\xi}_j + \dot{\xi}_i\frac{\partial \dot{\xi}_j}{\partial \dot{\xi}_k}\right) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}\left(\delta_{ik}\dot{\xi}_j + \dot{\xi}_i\delta_{jk}\right)\\ &= \frac{1}{2}\sum_{j=1}^{N} M_{kj}\dot{\xi}_j + \frac{1}{2}\sum_{i=1}^{N} M_{ik}\dot{\xi}_i = \sum_{i=1}^{N} M_{ki}\dot{\xi}_i \end{aligned}\tag{6.17}$$

Here the symmetry of the mass matrix is used in the last equality. In the same way one obtains

$$\frac{\partial L}{\partial \xi_k} = -\sum_{i=1}^{N} K_{ki}\xi_i \tag{6.18}$$

Lagrange’s equations thus become

$$\sum_{i=1}^{N}\left(M_{ki}\ddot{\xi}_i + K_{ki}\xi_i\right) = 0 \qquad k = 1, 2, \ldots, N \tag{6.19}$$

This constitutes a system of N coupled, linear, and homogeneous ordinary differential equations which are conveniently written on matrix form

$$M\ddot{\boldsymbol{\xi}} + K\boldsymbol{\xi} = 0 \tag{6.20}$$

To determine the natural frequencies the following anzats can be made

$$\boldsymbol{\xi} = \boldsymbol{u}\cos\omega t \tag{6.21}$$

where $\boldsymbol{u}$ is a constant amplitude vector with N components and ω is the angular frequency to be determined. Insertion gives

$$(K - \omega^2 M)\boldsymbol{u} = 0 \tag{6.22}$$

which is a linear and homogeneous system of equations. To be solvable the determinant of the system must vanish

$$\det(K - \omega^2 M) = 0 \tag{6.23}$$

This is a polynomial equation of degree N in ω^2, and as M and K are symmetric it follows that the N solutions ω_i^2 become real. If all the ω_i^2 are positive the resulting motion becomes harmonic and the equilibrium is thus stable. On the other hand, if any ω_i^2 is negative, the motion becomes exponentially growing and the equilibrium is unstable. In this case the assumptions of the linearization soon break down. If $\omega_i = 0$ is a solution this corresponds to a rigid body motion (translation or rotation) of the whole system.

The eigenfrequencies ω_i are chosen as positive. The corresponding eigenvectors $\boldsymbol{u}^i$ can be determined from Eq. (6.22). The eigenfrequencies and eigenvectors are often called modal frequencies and modal vectors. The modal vectors are usually normalized so that

$$\left(\boldsymbol{u}^i\right)^{\mathrm{T}} \cdot \boldsymbol{u}^i = 1 \tag{6.24}$$

The modal vectors are satisfying an orthogonality relation that can be obtained by taking Eq. (6.22) for modes i and j and multiply from the left by $\boldsymbol{u}^j$ and $\boldsymbol{u}^i$, respectively, and taking the difference

$$\left(\boldsymbol{u}^j\right)^{\mathrm{T}} \left(K - \omega_i^2 M\right) \boldsymbol{u}^i - \left(\boldsymbol{u}^i\right)^{\mathrm{T}} \left(K - \omega_j^2 M\right) \boldsymbol{u}^j = 0 \tag{6.25}$$

The symmetry of M and K gives

$$\left(\omega_j^2 - \omega_i^2\right) \left(\boldsymbol{u}^i\right)^{\mathrm{T}} M \boldsymbol{u}^j = 0 \tag{6.26}$$

which can also be written

$$\left(\boldsymbol{u}^i\right)^{\mathrm{T}} M \boldsymbol{u}^j = m_i \delta_{ij} \tag{6.27}$$

This is the orthogonality relation between the modal vectors and it also defines the modal mass m_i.

The general solution for free oscillations can now be written

$$\boldsymbol{\xi} = \sum_{i=1}^{N} A_i \boldsymbol{u}^i \cos\left(\omega_i t + \alpha_i\right) \tag{6.28}$$

The amplitude A_i and phases α_i are determined from the initial conditions.

Example 6.1.1
Calculate the eigenfrequencies and modal vectors for the double pendulum considered in **Example 5.2.2** and **Example 5.3.2**.

Solution. The linearized equations for the double pendulum were derived in **Example 5.3.2** and are here given again

$$\frac{4}{3}\ddot{\theta} + \frac{1}{2}\ddot{\varphi} + \frac{3g}{2L}\theta = 0$$

$$\frac{1}{3}\ddot{\varphi} + \frac{1}{2}\ddot{\theta} + \frac{g}{2L}\varphi = 0$$

Putting

$$\theta = u_1 \cos\omega t \qquad \varphi = u_2 \cos\omega t$$

gives

$$-\frac{4}{3}\omega^2 u_1 - \frac{1}{2}\omega^2 u_2 + \frac{3g}{2L}u_1 = 0$$

$$-\frac{1}{3}\omega^2 u_2 - \frac{1}{2}\omega^2 u_1 + \frac{g}{2L}u_2 = 0$$

To get a nontrivial solution the determinant must vanish

$$\left[-\frac{4}{3}\omega^2 + \frac{3g}{2L}\right]\left[-\frac{1}{3}\omega^2 + \frac{g}{2L}\right] - \frac{1}{2}\omega^2 \cdot \frac{1}{2}\omega^2 = 0$$

Solving this equation gives the two eigenfrequencies

$$\omega^2 = \frac{g}{L}\left(3 \pm \frac{6}{\sqrt{7}}\right)$$

with numerical values

$$\omega_1 = 0.86\sqrt{\frac{g}{L}}$$

$$\omega_2 = 2.30\sqrt{\frac{g}{L}}$$

The quotient between the two components in each modal vector are

$$\frac{u_2}{u_1} = \frac{-\frac{4}{3}\omega^2 + \frac{3g}{2L}}{\frac{1}{2}\omega^2}$$

For mode 1 this quotient becomes 1.43 and for mode 2 it is -2.10. The normalized mode vectors are then

$$\begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} = \begin{pmatrix} 0.57 \\ 0.82 \end{pmatrix} \qquad \begin{pmatrix} u_1^2 \\ u_2^2 \end{pmatrix} = \begin{pmatrix} 0.43 \\ -0.90 \end{pmatrix}$$

In the first mode the two bars in the double pendulum swing together, in the second mode they are out of phase. This is typical, in the first mode the mass in the system tends to move more or less in phase.

6.2 Modal analysis

In the previous section the free oscillations for a conservative system with several degrees of freedom were investigated. If there are applied forces on the system a similar approach can be used. With applied forces but still without damping the linearized equations of motion become

$$M\ddot{\boldsymbol{\xi}} + K\boldsymbol{\xi} = \boldsymbol{Q} \tag{6.29}$$

where the force vector $\boldsymbol{Q}$ contains the generalized forces on the system. Assume that the eigenfrequencies ω_i and modal vectors $\boldsymbol{u}^i$ to the corresponding homogeneous system have been determined. It is then useful to make an expansion in the modal vectors

$$\boldsymbol{\xi} = \sum_{i=1}^{N} Y_i(t)\boldsymbol{u}^i \tag{6.30}$$

Insertion then gives

$$\sum_{i=1}^{N}\left(\ddot{Y}_i M + Y_i K\right)\boldsymbol{u}^i = \boldsymbol{Q} \tag{6.31}$$

Use Eq. (6.22) for the modal vectors to obtain

$$\sum_{i=1}^{N} \left(\ddot{Y}_i + \omega_i^2 Y_i \right) M \boldsymbol{u}^i = \boldsymbol{Q} \tag{6.32}$$

Multiply by $\boldsymbol{u}^k$ and use Eq. (6.27) to obtain

$$m_k \left(\ddot{Y}_k + \omega_k^2 Y_k \right) = Q_k \qquad k = 1, 2, \dots, N \tag{6.33}$$

where m_k is the modal mass and

$$Q_k = \boldsymbol{u}^k \cdot \boldsymbol{Q} \tag{6.34}$$

is the modal force. Thus N uncoupled ordinary differential equations are obtained.

If the forcing is time harmonic

$$Q_k = f_k \cos \omega t \tag{6.35}$$

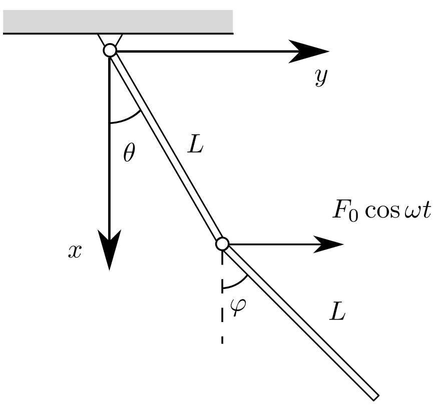
it is easy to see that a particular solution to Eq. (6.33) is

$$Y_k = \frac{f_k}{m_k(\omega_k^2 - \omega^2)} \cos \omega t \tag{6.36}$$

When the forcing frequency ω is close to one of the eigenfrequencies ω_k of the system the response becomes large, in principle infinite at resonance. In most cases this is an unwanted phenomenon and in some cases it may even lead to catastrophic failure of systems.

In the presentation of several degrees of freedom damping has been neglected. Damping can be included, but this leads to complex eigenfrequencies and other complications that will not be treated here.

Example 6.2.1
Consider the double pendulum with an applied force at the pin connecting the two rods, which has been treated in **Examples 5.2.2, 5.3.2, 6.1.1**. Assume that the pendulum starts from rest in the equilibrium position and determine the subsequent motion due to the force.



Solution. The eigenfrequencies and mode vectors were calculated in **Example 6.1.1** and are

$$\omega_1 = 0.86\sqrt{\frac{g}{L}}$$

$$\omega_2 = 2.30\sqrt{\frac{g}{L}}$$

$$\begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} = \begin{pmatrix} 0.57 \\ 0.82 \end{pmatrix} \qquad \begin{pmatrix} u_1^2 \\ u_2^2 \end{pmatrix} = \begin{pmatrix} 0.43 \\ -0.90 \end{pmatrix}$$

Also the modal masses are needed

$$m_k = \begin{pmatrix} u_1^k & u_2^k \end{pmatrix} \begin{pmatrix} \frac{4}{3}mL^2 & \frac{1}{2}mL^2 \\ \frac{1}{2}mL^2 & \frac{1}{3}mL^2 \end{pmatrix} \begin{pmatrix} u_1^k \\ u_2^k \end{pmatrix} = \frac{4}{3}mL^2\left(u_1^k\right)^2 + mL^2 u_i^k u_2^k + \frac{1}{3}mL^2\left(u_2^k\right)^2$$

Inserting the numerical values for the modal vectors gives

$$m_1 = 1.12mL^2 \qquad m_2 = 0.13mL^2$$

Because the generalized coordinates are angles the modal masses do not have the dimension of a mass.

The generalized forces were determined in **Example 5.2.2**

$$Q_\theta = F_0 L\cos\omega t \qquad Q_\varphi = 0,$$

and this gives the modal forces

$$Q_k = \left(u_1^k, u_2^k\right)\cdot\left(LF_0\cos\omega t, 0\right) = u_1^k\, LF_0\cos\omega t$$

The solution is expanded in the modal vectors

$$\begin{pmatrix} \theta \\ \varphi \end{pmatrix} = Y_1(t)\boldsymbol{u}^1 + Y_2(t)\boldsymbol{u}^2$$

which gives an uncoupled equation for each modal amplitude

$$m_k\left(\ddot{Y}_k + \omega_k^2 Y_k\right) = f_k\cos\omega t$$

where $f_k = LF_0 u_1^k$. The general solution to this differential equation is

$$Y_k(t) = A_k\cos\omega_k t + B_k\sin\omega_k t + \frac{f_k\cos\omega t}{m_k\left(\omega_k^2 - \omega^2\right)}$$

The initial conditions in this case are homogeneous so $\theta = \varphi = 0$ and $\dot{\theta} = \dot{\varphi} = 0$ at time $t = 0$. This leads to homogeneous initial conditions also for the modal amplitudes, i.e. $Y_k = 0$ and $\dot{Y}_k = 0$, $k = 1, 2$. This determines the constants

$$A_k = -\frac{LF_0 u_1^k}{m_k\left(\omega_k^2 - \omega^2\right)} \qquad B_k = 0$$

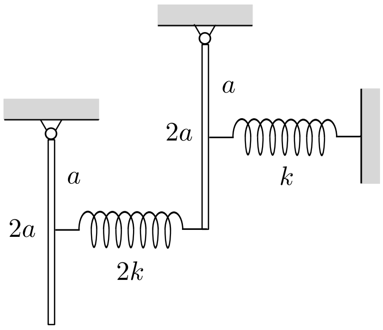
The solution for the modal amplitudes is thus

$$Y_k(t) = \frac{LF_0 u_1^k}{m_k\left(\omega_k^2 - \omega^2\right)}\left(\cos\omega t - \cos\omega_k t\right)$$

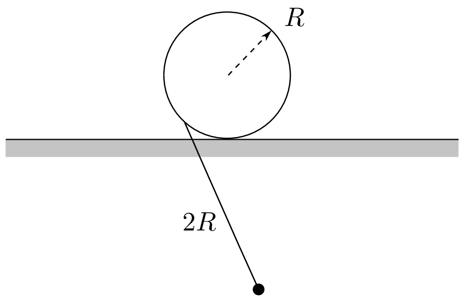
The solution is thereby complete.

Problems

6.1 Two equal rods, each with mass m and length $2a$, are hanging and are connected by two springs according to the figure. The springs have stiffness k and $2k$ and are unstretched when the rods are vertical. Determine the eigenfrequencies.



Prob. 6.1



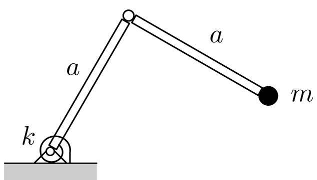
Prob. 6.2

6.2 A thin-walled cylindrical shell with mass m and radius R has a pendulum with mass m and length $2R$ attached to the periphery. Determine the eigenfrequencies.

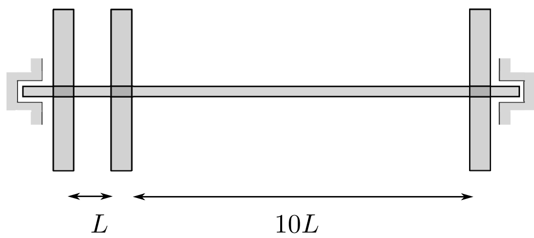
6.3 A thin rod with mass m is hanging in two vertical springs with stiffness k and $2k$, one attached at each end. The rod is horizontal in the equilibrium position. Determine the eigenfrequencies.

6.4 Determine the eigenfrequencies and eigenvectors for a system of three particles that are gliding without friction on a horizontal circular ring and are pairwise connected with three springs. The particles have mass m, m, and $2m$ and the springs are all equal with stiffness k and are unstretched in the equilibrium position.

6.5 A pendulum is constructed with two equal light rods, each with length a, that are connected by a pin. One of the rods is connected with a pin and a torsional spring with stiffness k to a fixed support. The other rod has a particle with mass m at the outer end. The pendulum can swing in a vertical plane and is in equilibrium when the rod connected to the torsional spring makes an upward angle $30°$ with the horizontal. Determine the eigenfrequencies.



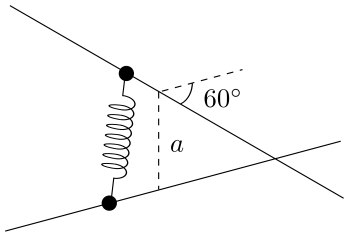
Prob. 6.5



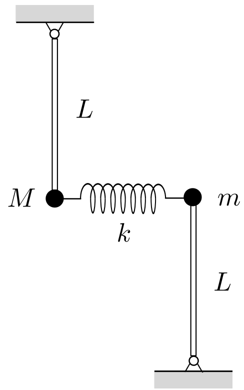
Prob. 6.6

6.6 An axle is freely rotating in its bearings. Three equal wheels, each with moment of inertia I, are mounted on the axle such that the length of the part between two of the wheels is ten times that of the other length. The torsional stiffness of the longer part is k. Determine the eigenfrequencies for torsional vibrations.

6.7 Two particles, each with mass m, can glide without friction along its horizontal bar. The two bars are making the angle $60°$ with each other. The smallest (vertical) distance between the bars is a. The particles are connected by a spring with stiffness k and unstretched length $a/2$. Determine the eigenfrequencies and eigenvectors.



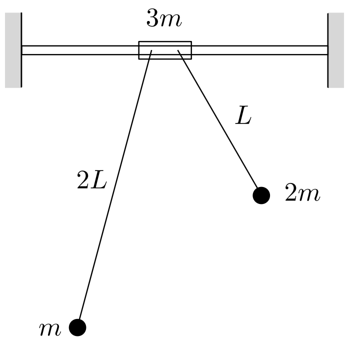
Prob. 6.7



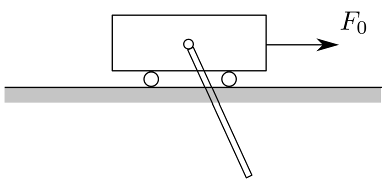
Prob. 6.8

6.8 One hanging and one standing (inverted) pendulum are connected by a spring of stiffness k. The pendulums consist of a light rod with a particle at the end according to the figure. When the pendulums are vertical the spring is horizontal and unstretched. What are the conditions for the pendulums to perform small oscillations around this state?

6.9 A body with mass $3m$ can move freely along a fixed horizontal rod. Two simple pendulums are attached to the body so that they swing in a vertical plane containing the rod. Determine the eigenfrequencies for the system for small oscillations.



Prob. 6.9



Prob. 6.10

6.10 A homogeneous rod with mass m and length a is suspended on a cart of mass m. Use modal analysis to determine the motion if the cart is pulled with a constant force F_0 and the cart and rod start from rest in the equilibrium position.

6.11 Two rods, each of mass m and length a, are hanging vertically from pins at the same height. The lower ends of the rods are connected by a spring of stiffness k which is unstretched when the rods are vertical. The system is at rest when a horizontal force $F_0 \cos \omega t$ is applied at the end of one of the rods. Use modal analysis to determine the subsequent motion.

Answers to problems

Chapter 1

1.1 141 m
1.2 $\dot{r} = v\cos(\varphi - \beta), \quad \dot{\varphi} = -\frac{v\sin^2(\varphi-\beta)}{H\cos\beta}$
1.3 $\tau\ln 2, \quad \frac{F_0\tau^2}{m}(\ln 2 - 0.5)$
1.4 9.0 s
1.5 $h\omega^2 < g, \quad v_0 - \mu g t + \mu h\omega\sin\omega t = 0$
1.6 $a_A = \frac{1}{2}(\mu_1 - 3\mu_2)g, \quad a_B = a_C = \frac{1}{2}(1-\mu_1)g$
1.7 $0.37g/\omega^2$
1.8 $\sqrt{\frac{F_0R}{\mu m}(1 - \mathrm{e}^{-4\pi\mu})}$
1.9 $\sqrt{\frac{k}{m}\delta^2 + \sqrt{2}\delta g(1-3\mu) - \frac{4\mu m g^2}{k}(1-\mu)}$
1.10 $0.12R$
1.11 $x(t) = \frac{3mg}{4k}(1-\cos\omega t), \quad \omega = \sqrt{4k/3m}$
1.12 $x(t) = \frac{a}{3}(2\sin\omega t - \sin 2\omega t)$
1.13 $v_B = \sqrt{4gR(1-\sqrt{2}/3)}$
1.14 30°
1.15 $v_A = \sqrt{\frac{2gR}{1+m/M}}, \quad v_B = \frac{m}{M}v_A$
1.16 $\frac{3}{8}\omega_0, \quad a\omega_0\sqrt{\frac{5}{8}}$

Chapter 2

2.1 $(0, -q\varphi_1\cos qt, \Omega), \quad (\Omega q\varphi_1\cos qt, q^2\varphi_1\sin qt, 0)$
2.2 $\left(-\dot{\omega}_2\sin\beta - \omega_2\dot{\beta}\cos\beta + \omega_1\dot{\beta}, \ddot{\beta} + \omega_1\omega_2\sin\beta, \dot{\omega}_2\cos\beta - \omega_2\dot{\beta}\sin\beta\right)$
2.3 $\left(-b\omega_2^2, -L\omega_1^2, -2\omega_2 v_0\right)$
2.4 $\boldsymbol{v}_O = -v_0\boldsymbol{e}_y, \quad \boldsymbol{a}_O = \left(2\omega v_0 + \frac{v_0^2}{r}\right)\boldsymbol{e}_x$

$\boldsymbol{v}_P = (v_0 + r\omega)\,\boldsymbol{e}_x + r\omega\boldsymbol{e}_y, \quad \boldsymbol{a}_P = -r\omega^2\boldsymbol{e}_x + \left(r\omega^2 + 2\omega v_0 + v_0^2/r\right)\boldsymbol{e}_y$

2.5 $\mu = \frac{1}{g}\sqrt{\left(a\omega^2 + \sqrt{2}\omega v_0\right)^2 + \left(a\alpha + \sqrt{2}\omega v_0\right)^2}$
2.6 $\boldsymbol{v}_C = -L(2\Omega+\omega)\boldsymbol{e}_y, \quad \boldsymbol{a}_C = 2L\Omega(\Omega+\omega)\boldsymbol{e}_x - L\dot{\omega}\boldsymbol{e}_y + L\omega^2\boldsymbol{e}_z$
2.7 $\boldsymbol{v}_D = (-L\Omega - R\omega, 2L\Omega, 0) \quad \boldsymbol{a}_D = (-2L\Omega^2, -L\Omega^2 - 2R\Omega\omega, -R\omega^2)$
2.8 $\boldsymbol{v}_C = -(L\Omega + r\omega)\boldsymbol{e}_x + v\boldsymbol{e}_z, \quad \boldsymbol{a}_C = -2\omega v\boldsymbol{e}_x - (L\Omega^2 + 2r\Omega\omega)\boldsymbol{e}_y - r\omega^2\boldsymbol{e}_z$
2.9 $\bar{\boldsymbol{v}} = -\frac{L}{2}\dot{\varphi}\cos\varphi\boldsymbol{e}_x - \frac{L}{2}\Omega\sin\varphi\boldsymbol{e}_y + \left(\frac{L}{2}\dot{\varphi}\sin\varphi - v_0\right)\boldsymbol{e}_z,$

$\bar{\boldsymbol{a}} = \frac{L}{2}\left(\dot{\varphi}^2\sin\varphi - \ddot{\varphi}\cos\varphi + \Omega^2\sin\varphi\right)\boldsymbol{e}_x - L\Omega\dot{\varphi}\cos\varphi\boldsymbol{e}_y + \frac{L}{2}(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi)\boldsymbol{e}_z$

2.10 $\boldsymbol{v} = R\dot{\varphi}\sin\varphi\boldsymbol{e}_x + R\omega(1-\cos\varphi)\boldsymbol{e}_y + R\dot{\varphi}\cos\varphi\boldsymbol{e}_z$

$\boldsymbol{a} = (R\dot{\varphi}^2\cos\varphi - R\omega^2(1-\cos\varphi))\boldsymbol{e}_x + 2R\omega\dot{\varphi}\sin\varphi\boldsymbol{e}_y - R\dot{\varphi}^2\sin\varphi\boldsymbol{e}_z$

2.11 $\boldsymbol{a} = -a(\dot{\varphi}^2 + \Omega^2\sin^2\varphi)\boldsymbol{e}_x + (a\ddot{\varphi} - a\Omega^2\cos\varphi\sin\varphi + 2v_0\dot{\varphi})\boldsymbol{e}_y - (2a\Omega\dot{\varphi}\cos\varphi + 2v_0\Omega\sin\varphi)\boldsymbol{e}_z$
2.12 $x = L\cosh\omega_0 t, \quad N = 2mL\omega_0^2\sinh\omega_0 t$
2.13 13 mm

Chapter 3

3.1 $\omega_{BC} = \omega, \quad \omega_{CD} = \frac{1}{2}\omega$
3.2 $(-3, -4, 0)$ rad/s^2
3.3 $-\frac{v\Omega}{2L\sin^2\varphi}\boldsymbol{e}_x - \frac{v^2\cos\varphi}{4L^2\sin^3\varphi}\boldsymbol{e}_z$
3.4 $\frac{1}{3}v\boldsymbol{e}_y, \quad -\frac{25v^2}{27L}\boldsymbol{e}_y$
3.5 $R\Omega/\sqrt{5}$ downwards
3.6 $\frac{2\pi}{\tau}\cot\alpha\boldsymbol{e}_y$
3.7 $\frac{1}{\sin\beta}(\sin\gamma\,\dot{\omega}_2 - \cos\beta\sin(\beta+\gamma)\dot{\omega}_1)$
3.8 $(3\Omega - \frac{v}{a}, 0, \frac{v}{3a}), \quad (3\Omega - \frac{v}{a})\frac{v}{3a}\boldsymbol{e}_y \quad (\frac{v^2}{3a} - 2v\Omega, 0, 9a\Omega^2 - 6v\Omega + \frac{v^2}{a})$
3.9 $(-3\sqrt{3}+1, 0, 3+\sqrt{3})\frac{v}{6a}, \quad -\frac{v^2}{9\sqrt{3}a^2}(3\sqrt{3}-1)\boldsymbol{e}_y$
3.10 $\frac{v}{5R}(-4, 0, -3), \quad \frac{v^2}{25R^2}(12, 0, 16)$

Chapter 4

4.1 $\frac{ma^2}{6}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad \frac{2}{15}ma^2$
$I_1 = \frac{5-\sqrt{13}}{12}ma^2, \quad \boldsymbol{e}_1 = (0.96, 0.28, 0)$
$I_2 = \frac{5+\sqrt{13}}{12}ma^2, \quad \boldsymbol{e}_2 = (-0.28, 0.96, 0)$
$I_3 = \frac{5}{6}ma^2, \quad \boldsymbol{e}_3 = (0, 0, 1)$
4.2 $\frac{13\pi}{3}\sqrt{\frac{R}{g}}$
4.3 2.19 m/s
4.4 $a = \mu g$ if $\mu \leq 0.5$, $a = 0.5g$ if $\mu > 0.5$
4.5 $\sqrt{\frac{54gR}{7}}$
4.6 $\arccos\left(\frac{2}{3}\cos\varphi_0\right)$
4.7 $m_A = 0.007m, \quad m_B = 0.013m$
4.8 $BC : \frac{1}{2}m, \quad AB : \frac{1}{6}m$
4.9 $M_0 = \frac{1}{12}m\dot{\Omega}\left(6R^2 + (h^2 - 3R^2)\sin^2\beta\right), \quad \sin\beta = \frac{2R}{\sqrt{4R^2+h^2}}$
$A_1 = -B_1 = \frac{m\Omega^2}{24L}\left(h^2 - 3R^2\right)\sin 2\beta$
$A_2 = -B_2 = \frac{m\dot{\Omega}}{24L}\left(h^2 - 3R^2\right)\sin 2\beta$
4.10 $F = \frac{1}{2}mb\omega^2\sin\beta, \quad M = \frac{1}{6}mb^2\omega^2\sin 2\beta$
4.11 $2\sqrt{\frac{\sqrt{3}g}{a}}$
4.12 14.2 rad/s
4.13 $\frac{3}{2}\sqrt{\frac{3\sqrt{2}g}{a}}$
4.14 $\frac{1}{2}\omega_0, \quad \sqrt{\omega_0^2/4 + 3g/\sqrt{2}a}$
4.15 $\sqrt{\frac{\sqrt{2}g}{a}}$
4.16 $N_1 = \frac{17}{10}mg - \frac{429}{80\sqrt{10}}mR\omega^2$ small wheel
$N_2 = \frac{23}{10}mg + \frac{429}{80\sqrt{10}}mR\omega^2$ large wheel
4.17 $\sqrt{\frac{216gR}{55}}$
4.18 $\frac{1}{3}L\ddot{\theta} + L\Omega^2\sin\theta(\frac{1}{2} + \frac{1}{3}\cos\theta) = \frac{1}{2}g\cos\theta$

4.19 $\sqrt{12ga}$

4.20 $\frac{9\sqrt{3}}{16}mg$

4.21 $mg\frac{2(1+\cos\gamma)}{2+2\cos\gamma-\sin\gamma}+mR\Omega^2\frac{4\cos\gamma+4\cos^2\gamma-16\sin\gamma-17\sin\gamma\cos\gamma}{4(2+2\cos\gamma-\sin\gamma)}$

Chapter 5

5.1 $\frac{1}{3}L\ddot{\theta}+\frac{1}{2}(\ddot{y}+g)\sin\theta=0$

5.2 $\frac{1}{3}mL^2\ddot{\theta}+2kb^2\sin(\theta/2)-\frac{1}{2}mgL\cos\theta=0$

5.3 $(R^2+k^2+d^2+2Rd\cos\theta)\ddot{\theta}-Rd\sin\theta\dot{\theta}^2+gd\sin\theta=0$

5.4 $6m\ddot{x}+m\ddot{y}\cos\beta+8kx=0$
$m\ddot{y}+m\ddot{x}\cos\beta+3ky-mg\sin\beta=0$

5.5 $\frac{1}{3}a\ddot{\varphi}-\frac{\sqrt{2}}{4}\ddot{x}(\cos\varphi+\sin\varphi)-\frac{\sqrt{2}}{4}\dot{\varphi}\dot{x}(\cos\varphi-\sin\varphi)+\frac{1}{2}g\sin\varphi=0$
$\ddot{x}-\frac{\sqrt{2}}{4}a\ddot{\varphi}(\cos\varphi+\sin\varphi)-\frac{\sqrt{2}}{4}a\dot{\varphi}^2(\cos\varphi-\sin\varphi)-\frac{1}{\sqrt{2}}g+\frac{k}{m}x=0$

5.6 $md\ddot{\varphi}\cos\varphi-md\dot{\varphi}^2\sin\varphi+m\ddot{x}+kx-k(y-x)=0$
$m\ddot{y}+k(y-x)=0$
$md\ddot{\varphi}+m\ddot{x}\cos\varphi+mg\sin\varphi=0$

5.7 $\left(\frac{1}{3}\sin^2\theta+\frac{1}{12}\cos^2\theta\right)\ddot{\varphi}+\frac{1}{2}\dot{\varphi}\dot{\theta}\sin\theta\cos\theta=0$
$\frac{5}{12}\ddot{\theta}-\frac{1}{4}\dot{\varphi}^2\sin\theta\cos\theta+\frac{g}{2a}\sin\theta=0$

5.8 $mL^2\left[\ddot{\theta}\left(\frac{2}{3}+2\sin^2\theta\right)+\dot{\theta}^2\sin 2\theta\right]+k\left(\theta-\frac{\pi}{3}\right)-2mgL\sin\theta=FL\sin 2\theta$

5.9 $ma\ddot{\theta}-m\dot{\psi}^2\cos\theta(b+\sin\theta)+mg\sin\theta=0$
$\left(I+m(b+a\sin\theta)^2\right)\ddot{\psi}+2ma\dot{\psi}\dot{\theta}\cos\theta(b+a\sin\theta)=M$

5.10 $8L\ddot{\theta}+9\dot{u}\cos\theta+3L\ddot{\varphi}\cos(\theta-\varphi)+3L\dot{\varphi}^2\sin(\theta-\varphi)+9g\sin\theta=0$
$2L\ddot{\varphi}+3L\ddot{\theta}\cos(\theta-\varphi)-3L\dot{\theta}^2\sin(\theta-\varphi)+3\dot{u}\cos\varphi+3g\sin\theta=0$

5.11 $-2m(L-x)\dot{x}\dot{\varphi}\sin^2\theta+m(L-x)^2\ddot{\varphi}\sin^2\theta+2m(L-x)^2\dot{\varphi}\dot{\theta}\sin\theta\cos\theta=M$
$-2m(L-x)\dot{x}\dot{\theta}+m(L-x)^2\ddot{\theta}+m(L-x)^2\dot{\varphi}^2\cos\theta+mg(L-x)\sin\theta=0$
$m\ddot{x}-m(L-x)\dot{\varphi}^2\sin^2\theta-m(L-x)\dot{\theta}^2+mg\cos\theta+k(x-b)=0$

5.12 $\ddot{x}+\frac{4}{3\pi}R\ddot{\varphi}\cos\varphi-\frac{4}{3\pi}R\dot{\varphi}^2\sin\varphi=0$
$\frac{4}{3\pi}\ddot{x}\cos\varphi+R\ddot{\varphi}+\frac{4}{3\pi}g\sin\varphi=0$

5.13 $\sqrt{\frac{21g}{25R}}$

Chapter 6

6.1 $\sqrt{\frac{3}{8}\left(\frac{2g}{a}+\frac{k}{m}(11\pm\sqrt{113})\right)}$

6.2 $\sqrt{\frac{g}{2R}}$ double root

6.3 $\omega_{1,2}=\sqrt{\frac{k}{m}(6\pm\sqrt{12})}\qquad\begin{pmatrix}x\\a\varphi\end{pmatrix}=\begin{pmatrix}1\\-6\mp4\sqrt{3}\end{pmatrix}$
x is displacement of centre of mass and φ is rotation of rod

6.4 $\omega_1=0\quad \boldsymbol{u}^1=\begin{pmatrix}1\\1\\1\end{pmatrix}\qquad \omega_2=\sqrt{\frac{k}{2m}}\quad \boldsymbol{u}^2=\begin{pmatrix}1\\0\\-1\end{pmatrix}\qquad \omega_3=\sqrt{\frac{3k}{4m}}\quad \boldsymbol{u}^3=\begin{pmatrix}1\\-1\\1\end{pmatrix}$

6.5 $\left(\frac{g}{3a}\left[1+\frac{2k}{mga}\pm\sqrt{\left(1+\frac{2k}{mga}\right)^2+6\left(1-\frac{2k}{mga}\right)}\right]\right)^{1/2}$

6.6 $0, \quad 1.209\sqrt{I/k}, \quad 4.532\sqrt{I/k}$

6.7 $\omega_1 = \sqrt{k/4m} \quad \boldsymbol{u}^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \omega_2 = \sqrt{3k/4m} \quad \boldsymbol{u}^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

6.8 $M > m$ and $k > \frac{Mmg}{2L(M-m)}$

6.9 $0 \quad \sqrt{\frac{(7\pm\sqrt{13})g}{6L}}$

6.10 $\begin{pmatrix} x \\ a\varphi \end{pmatrix} = \frac{F_0 t^2}{4m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{F_0}{8m}\begin{pmatrix} 1 \\ -4 \end{pmatrix}(1 - \cos(t\sqrt{12g/5a}))$

6.11 $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \frac{3F_0}{2ma}\left[\frac{\cos\omega t - cos\omega_1 t}{\omega_1^2 - \omega^2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\cos\omega t - cos\omega_2 t}{\omega_2^2 - \omega^2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$

$\omega_1 = \sqrt{\frac{3g}{2a}} \quad \omega_2 = \sqrt{\frac{3g}{2a} + \frac{6k}{m}}$