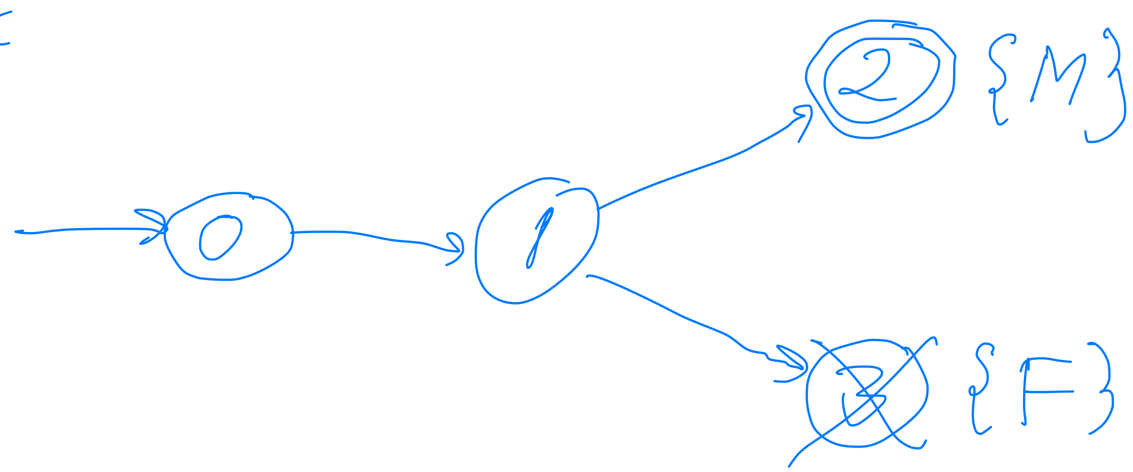
9 (cont) Temporal Logic

state labels in automata

An automaton including also state labels is called a transition system

Ex



AP = set of atomic propositions. In this example AP = {M, F}.

Labeling function $\lambda : \Sigma \to 2^{AP}$

set of states

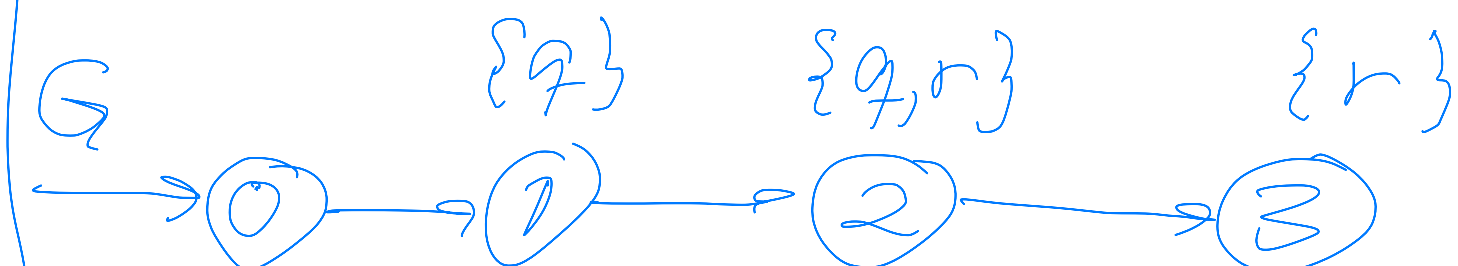
$\lambda(2) = \{M\}$ $\lambda(3) = \{F\}$

$\lambda(0) = \lambda(1) = \emptyset$ (default)

More generally

$AP = \{p, q, r, \ldots\}$

Ex



$[\![q]\!]$ = set of states including label q = $\{1, 2\}$

$[\![r]\!] = \{2, 3\}$

An algorithm will now be presented where

$[\![\exists \Diamond q]\!]$ is computed

Note that $[\![\exists \Diamond q]\!] = \{0, 1, 2\}$

μ-calculus

A tool to generate algorithms for analyzing in which states a temporal logic formula is valid. This includes both LTL, CTL and CTL*.

Ex Expansion law for

$$\exists \Diamond q \equiv q \vee \exists \bigcirc (\exists \Diamond q)$$

Introduce a logical variable $y = \exists \Diamond q \Rightarrow$
The expansion law can be expressed as

$$y = q \vee \exists \bigcirc y$$

This expression will be iterated until a fixed point is reached.

μ-calculus is a logic that includes fixed point operators.

Syntax for μ-calculus

$\psi ::= p \mid y \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid f(\psi) \mid \mu y.\psi$

$p \in AP$

f = function including the next modality + quantifier either $\exists\bigcirc$ or $\forall\bigcirc$,

$\mu y.\psi$ is the least fixed point

$\nu y.\psi$ is the greatest fixed point

Semantics of μ-calculus

Given a transition system

$G = \langle X, \Sigma, T, I, AP, \lambda \rangle$

the set of states $[\![\psi]\!]$ where the μ-calculus formula ψ holds is defined as follows:

$[\![p]\!] = \{x \mid p \in \lambda(x)\}$

$[\![y]\!] = Y \in 2^{X}$

$[\![\neg\psi]\!] = \sim[\![\psi]\!] = X \setminus [\![\psi]\!]$

$[\![\psi_1 \wedge \psi_2]\!] = [\![\psi_1]\!] \cap [\![\psi_2]\!]$

$[\![f(\psi)]\!] = \mathrm{Pre}^{f}([\![\psi]\!])$

$[\![\mu y.\psi]\!] = \mu Y.\, \underline{\Psi}(Y) =$ least fixed point for $Y =$

$= \bigcap \{Y \in 2^{X} \mid Y = \Psi(Y)\}$

$[\![\nu y . \psi]\!] = \nu Y . \psi(Y) =$

= greatest fixed point

for $Y = \bigcup \{ Y \in 2^{X} \mid Y = \psi(Y) \}$

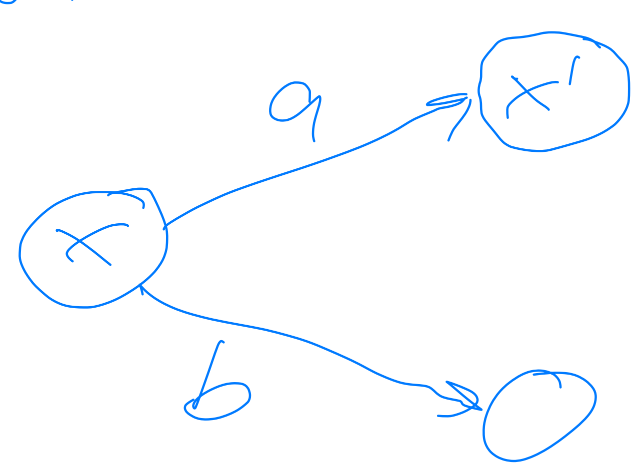
where $\psi(Y) = \psi([\![y]\!]) =$

$= [\![\psi(y)]\!]$

$\delta(x, a)$ = transition function

$\Sigma(x)$ = active event set

= possible events defined in state x



$x' = \delta(x, a)$

$\Sigma(x) = \{a, b\}$

$Pre^{\exists}(Y) = [\![\exists \bigcirc y]\!] =$

$= \{ x \mid \exists a \in \Sigma(x) : \delta(x, a) \subseteq Y \}$

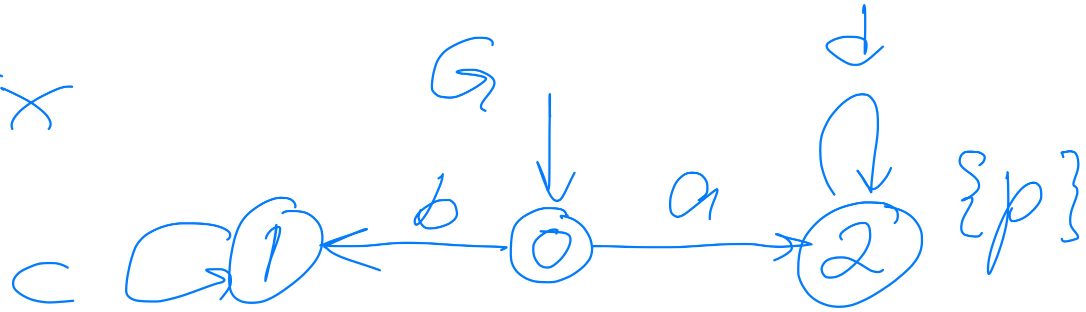
For at least one event a, $x' = \delta(x, a) \in Y \iff$ coreachability one step backward.

$Pre^{\forall}(Y) = [\![\forall \bigcirc y]\!] =$

$= \{ x \mid \forall a \in \Sigma(x) : \delta(x, a) \subseteq Y \}$

All target states from x must belong to Y

Ex

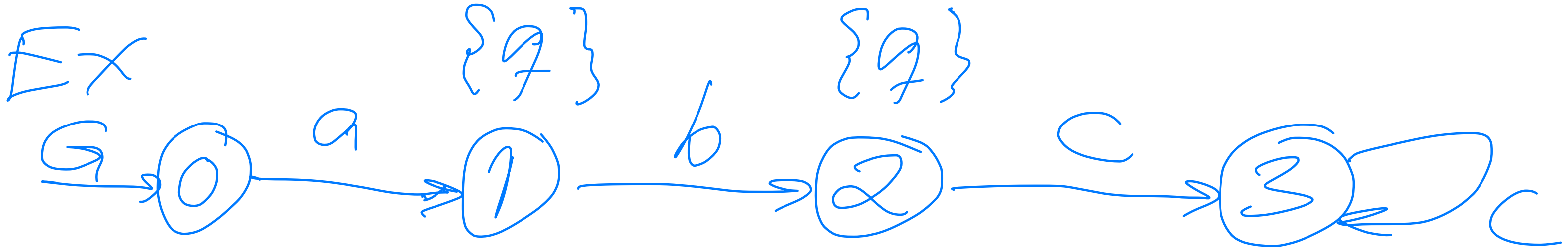


$\mathrm{Pre}^{\exists}(\llbracket p \rrbracket) = \mathrm{Pre}^{\exists}(\{2\}) = \{0,2\}$

$\mathrm{Pre}^{\forall}(\llbracket p \rrbracket) = \mathrm{Pre}^{\forall}(\{2\}) = \{2\}$

state 0 is not included since state 1 is missing in $\mathrm{Pre}^{\forall}(\{2\})$

Ex



$\llbracket \exists \Diamond q \rrbracket$?

$\exists \Diamond q \longleftrightarrow$ μ-calculus formula

$$\mu y.\ \underbrace{q \vee \exists \bigcirc y}_{\psi(y)}$$

$\llbracket \exists \Diamond q \rrbracket = \llbracket \mu y.\ \psi(y) \rrbracket =$

$= \mu Y.\ \underline{\Psi}(Y)$ where

$\underline{\Psi}(Y) = \llbracket \psi(y) \rrbracket = \llbracket q \vee \exists \bigcirc y \rrbracket$

$= \llbracket q \rrbracket \cup \llbracket \exists \bigcirc y \rrbracket = \llbracket q \rrbracket \cup \mathrm{Pre}^{\exists}(Y)$

where $Y = \llbracket y \rrbracket$

The least fixed point of $Y = \Psi(Y)$ is obtained by iterating $Y_{i+1} = \Psi(Y_i)$ for $i = 0, 1, 2, \ldots$ with $Y_0 = \emptyset$ until $Y_{i+1} = Y_i$

$Y_1 = \Psi(Y_0) = [\![q]\!] \cup Pre^{\exists}(Y_0)$

$= \{1,2\} \cup \underbrace{Pre^{\exists}(\emptyset)}_{\emptyset} = \{1,2\}$

$Y_2 = \Psi(Y_1) = [\![q]\!] \cup Pre^{\exists}(Y_1)$

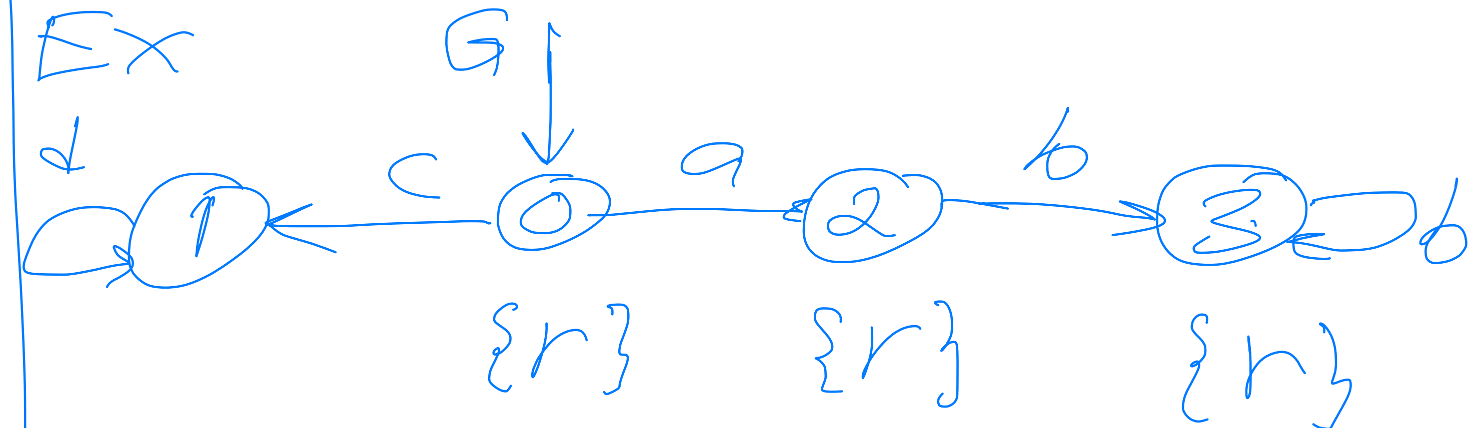
$= \{1,2\} \cup \{0,1\} = \{0,1,2\}$

$Y_3 = \Psi(Y_2) = \{1,2\} \cup \underbrace{Pre^{\exists}(\{0,1,2\})}_{\{0,1\}}$

$= \{0,1,2\} = Y_2 = Y_\omega$

$[\![\exists\Diamond q]\!] = \{0,1,2\}$

Ex



$\forall\Box r \leftrightarrow \nu y.\ \underbrace{r \wedge \forall\circ y}_{\Psi(y)}$

$[\![\forall\Box r]\!] = \nu Y.\ \Psi(Y)$ where

$\Psi(Y) = [\![r \wedge \forall\circ y]\!] = [\![r]\!] \cap Pre^{\forall}(Y)$

Greatest fixed point is obtained by iterating $Y_{i+1} = \Psi(Y_i)$ with $Y_0 = \Sigma = \{0,1,2,3\}$

$Y_1 = \Psi(Y_0) = \underbrace{\{0,2,3\}}_{[\![r]\!]} \cap \underbrace{Pre^{\forall}(\Sigma)}_{\Sigma} =$

$= \{0, 2, 3\}$

$Y_2 = \Psi(Y_1) = [\![r]\!] \cap Pre^{\forall}(\{0,2,3\}) = \{2,3\}$

state 0 not included since the target state 1 from state 0 is not included in Y in $Pre^{\forall}(Y)$ $Y = \{0, 2, 3\}$

$Y_3 = [\![r]\!] \cap \underbrace{Pre^{\forall}(\{2,3\})}_{\{2,3\}} = Y_2 = Y_\omega$

fixed point

$[\![\forall \Box r]\!] = Y_\omega = \{2, 3\}$

9. Temporal logic

For a transition system G with state set X and initial state set I, G satisfies a temporal logic formula ψ, written as $G \models \psi$, if ψ holds in all initial states of G, i.e. if

$$I \subseteq \Psi = [\![\psi]\!] = \{x \in X \mid x \in \psi\}$$

The set $[\![\psi]\!]$ can be determined by μ-calculus.

Ex

Evaluate the nonblocking condition $[\![\forall\Box\exists\Diamond p]\!]$, where the state label $\{p\}$ determines a marked state.

Since $\exists\Diamond p \equiv \mu Z.\, p \vee \exists\bigcirc Z$

$\forall\Box q \equiv \nu Y.\, q \wedge \forall\bigcirc Y$

$$\forall\Box\exists\Diamond p \equiv \nu Y.\,(\mu Z.\, p \vee \exists\bigcirc Z) \wedge \forall\bigcirc Y$$

$$[\![\mu Z.\, p \vee \exists\bigcirc Z]\!] = \mu Z.\, \Psi(Z)$$

where $\Psi(Z) = [\![p]\!] \cup Pre^{\exists}(Z) = Z$

Least fixed point:

$Z_{i+1} = \Psi(Z_i)$, $Z_0 = \emptyset$

$Z_1 = \Psi(Z_0) = [\![p]\!] \cup Pre^{\exists}(\emptyset) =$

$= \{2\} \cup \emptyset = \{2\}$

$Z_2 = [\![p]\!] \cup Pre^{\exists}(Z_1) =$

$\{2\} \cup \{0\} = \{0, 2\}$

$Z_3 = \{2\} \cup Pre^{\exists}(\{0,2\}) =$

$= \{2\} \cup \{0\} = \{0,2\} = Z_2 =$

$=$ fixed point Z^{ω}

$[\![\forall \Box \exists \Diamond p]\!] = \nu Y. \Psi(Y)$

where $\Psi(Y) = [\![\mu Z. p \vee \exists \bigcirc Z]\!]$

$\cap\, Pre^{\forall}(Y) = Z^{\omega} \cap Pre^{\forall}(Y) = Y$

Greatest fixed point:

$Y_{i+1} = \Psi(Y_i)$, $Y_0 = \Sigma$

$Y_1 = Z^{\omega} \cap Pre^{\forall}(\Sigma) = \{0,2\} \cap \{0,1,2\}$

$= \{0,2\}$

$Y_2 = Z^{\omega} \cap Pre^{\forall}(Y_1) =$

$= \{0,2\} \cap \{2\}$ (0 is excluded since its target state 1 is not included in $Y_1 = \{0,2\}$) $= \{2\}$

$Y_3 = Y_2 =$ fixed point Y^{ω}.

$I = \{0\} \not\subseteq [\![\forall \Box \exists \Diamond p]\!] = Y^{\omega} = \{2\}$

$\Rightarrow G \not\models \forall \Box \exists \Diamond p$

Remove state $\Rightarrow$

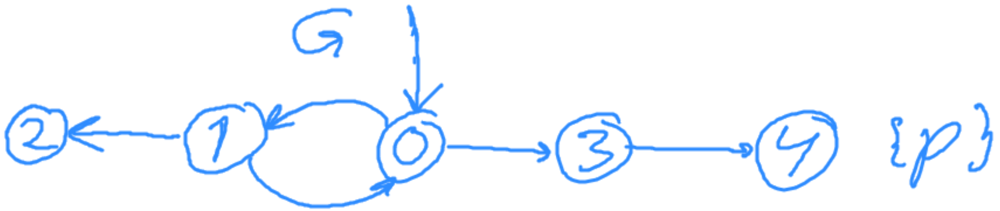
$Z^{\omega} = \{0,2\}$ $\quad Y_1 = Y_2 = Y^{\omega} = \{0,2\}$

$\Rightarrow I \subseteq Y^{\omega}$ and $G \models \forall \Box \exists \Diamond p$.

Correct since state 2 is a blocking state.

Removing this blocking state
$\Rightarrow$ G is nonblocking and satisfies
the nonblocking condition $\forall\Box\exists\Diamond p$

Ex



Evaluate again the nonblocking
condition $\forall\Box\exists\Diamond p$

$[\![\exists\Diamond p]\!] = \mu Z.\,\Psi(Z)$ where
$\Psi(Z) = [\![p]\!] \cup Pre^{\exists}(Z) = Z$

Least fixed point iteration:

$Z_0 = \emptyset$, $Z_1 = [\![p]\!] \cup Pre^{\exists}(\emptyset) = \{4\} \cup \emptyset$

$Z_2 = [\![p]\!] \cup Pre^{\exists}(\{4\}) =$
$= \{4\} \cup \{3\} = \{3,4\}$

$Z_3 = [\![p]\!] \cup Pre^{\exists}\{3,4\} = \{4\} \cup \{0,3\} =$
$= \{0,3,4\}$

$Z_4 = \{4\} \cup \{1,0,3\} = \{0,1,3,4\}$

$Z_5 = Z_4$ = fixed point $Z^{\omega} = \{0,1,3,4\}$

$[\![\forall\Box\exists\Diamond p]\!] = \nu Y.\ Z^{\omega} \cap Pre^{\forall}(Y)$

Greatest fixed point iteration:

$Y_0 = \mathbb{X}$, $Y_1 = Z^{\omega} \cap Pre^{\forall}(\mathbb{X}) =$
$= Z^{\omega} \cap \mathbb{X} = Z^{\omega} = \{0,1,3,4\}$

$Y_2 = Z^{\omega} \cap Pre^{\forall}(\underbrace{\{0,1,3,4\}}_{Y_1}) =$

$= Z^w \cap \{0,3,4\} = ($state 1 excluded since its target state is not included in Y_1 in $Pre^{\forall}(Y_1)) = \{0,3,4\}$

$Y_3 = Z^w \cap Pre^{\forall}(\underbrace{\{0,3,4\}}_{Y_2}) = \{3,4\}$

since the target state 1 of state 0 not included in Y_2

$Y_4 = Y_3 =$ fixed point $Y^w = \{3,4\}$

$I = \{0\} \not\subseteq Y^w = [\![\forall\Box\exists\Diamond p]\!] \Rightarrow$

$G \not\models \forall\Box\exists\Diamond p$

10. Reinforcement learning (RL)

RL = optimization method where actions are sent to the plant and resulting rewards are evaluated such that optimal actions are selected after an initial learning phase.

Currently a very popular method within modern AI and machine learning.

AlphaGo is a popular program where its success is based on RL.