2 Discrete Mathematics

Atomic propositions:

p, q, r, s

Compound propositions:

$p \wedge q, \quad p \vee q, \quad p \to q, \quad p \leftrightarrow q$

Tautology = statement that is always true

$\models p \vee \neg p$

Contradiction

$\not\models p \wedge \neg p$

$\mathbb{T}$ and $\mathbb{F}$ are used to express tautology and contradiction

Equivalence $\Leftrightarrow$ is a biconditional statement that is always true i.e. a tautology

$\models p \leftrightarrow q$ if and only if (iff) $p \Leftrightarrow q$

Implication $\Rightarrow$ is a conditional statement that is a tautology

$\models p \to q$ iff $p \Rightarrow q$

Truth Table: Basic Connectives

Table 2.1 Truth table for the connectives negation $\neg$, conjunction $\wedge$, disjunction $\vee$, conditional $\rightarrow$, and biconditional $\leftrightarrow$.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Truth Table: Example

Table 2.2 Truth table for the propositions $(p \vee q) \wedge r$ and $p \vee (q \wedge r)$. Note the differences on line two and four.

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	F	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Truth Table: Conditional Statement Equivalence

Table 2.4 Truth table proving the equivalence $(p \to q) \Leftrightarrow (\neg q \to \neg p)$.

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Truth Table: Conditional and Biconditional Equivalences

Table 2.5 Truth table that proves the equivalences $p \rightarrow q \Leftrightarrow \neg p \vee q$ and $(p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow p \leftrightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
T	T	F	T	T	T	T	T
T	F	F	F	F	T	F	F
F	T	T	T	T	F	F	F
F	F	T	T	T	T	T	T

Equivalence Table

Table 2.6 Equivalence relations.

E_1	$\neg\neg p \Leftrightarrow p$		
E_2	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	E_3	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
E_4	$p \vee q \Leftrightarrow q \vee p$	E_5	$p \wedge q \Leftrightarrow q \wedge p$
E_6	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	E_7	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
E_8	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	E_9	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
E_{10}	$p \vee p \Leftrightarrow p$	E_{11}	$p \wedge p \Leftrightarrow p$
E_{12}	$p \vee \mathbf{F} \Leftrightarrow p$	E_{13}	$p \wedge \mathbf{T} \Leftrightarrow p$
E_{14}	$p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$	E_{15}	$p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$
E_{16}	$p \vee \neg p \Leftrightarrow \mathbf{T}$	E_{17}	$p \wedge \neg p \Leftrightarrow \mathbf{F}$
E_{18}	$p \vee (p \wedge q) \Leftrightarrow p$	E_{19}	$p \wedge (p \vee q) \Leftrightarrow p$
E_{20}	$p \rightarrow q \Leftrightarrow \neg p \vee q$	E_{21}	$\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$
E_{22}	$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$	E_{23}	$p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$
E_{24}	$\neg(p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \neg q$	E_{25}	$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
E_{26}	$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$		

Implication Table

Table 2.8 Implication relations.

I_1	$p \wedge q \Rightarrow p$	I_2	$p \wedge q \Rightarrow q$
I_3	$p \Rightarrow p \vee q$	I_4	$q \Rightarrow p \vee q$
I_5	$\neg p \Rightarrow p \to q$	I_6	$q \Rightarrow p \to q$
I_7	$\neg(p \to q) \Rightarrow p$	I_8	$\neg(p \to q) \Rightarrow \neg q$
I_9	$\neg p \wedge (p \vee q) \Rightarrow q$	I_{10}	$p \wedge (p \to q) \Rightarrow q$
I_{11}	$\neg q \wedge (p \to q) \Rightarrow \neg p$	I_{12}	$(p \to q) \wedge (q \to r) \Rightarrow p \to r$
I_{13}	$(p \vee q) \wedge (p \to r) \wedge (q \to r) \Rightarrow r$		

Ex Prove that the following is a tautology

$(p \to (q \to (r \to s))) \vee (p \wedge (\neg p \vee q) \wedge \neg(r \to s))$

$\overset{E_{23}, E_8}{\Longleftrightarrow} ((p \wedge q) \to (r \to s)) \vee$

$[(\underbrace{(p \wedge \neg p)}_{\mathbb{F}} \vee (p \wedge q)] \wedge \neg(r \to s))$

$\overset{E_{20}}{\Longleftrightarrow} (\neg(p \wedge q) \vee (r \to s)) \vee$

$\neg(\neg(p \wedge q) \vee (r \to s)) \Longleftrightarrow \mathbb{T}$

Ex show that $\neg p \wedge (p \vee q) \Rightarrow q$

$\underbrace{(\neg p \wedge p)}_{\mathbb{F}} \vee (\neg p \wedge q) \Rightarrow q$

Proof techniques

1) $p \Leftrightarrow q$ can be shown by proving that $p \Rightarrow q \wedge q \Rightarrow p$

2) Proof by contradiction

$p \Rightarrow q$ iff $\models p \to q$ iff

$\models \neg p \vee q$ iff $\models \neg(p \wedge \neg q)$

iff $\not\models p \wedge \neg q$

Ex show $\neg p \wedge (p \vee q) \Rightarrow q$ by contradiction

$(\neg p \wedge (p \vee q)) \wedge \neg q \Longleftrightarrow$

$(\neg p \wedge \neg q) \wedge (p \vee q) \Longleftrightarrow \neg(p \vee q) \wedge (p \vee q)$

$\Longleftrightarrow \mathbb{F}$

Predicate logic

$\Omega = \{a_1, a_2, \ldots, a_n\} =$

= universal set of elements

Predicate includes variables x, y, z

Predicate: $p(x)$

Existential operator

For all $\forall x[p(x)] \Longleftrightarrow p(a_1) \wedge p(a_2) \wedge \ldots \wedge p(a_n)$

Exist $\exists x[p(x)] \Longleftrightarrow p(a_1) \vee p(a_2) \vee \ldots \vee p(a_n)$

$\neg \forall x[p(x)] \Longleftrightarrow \exists x[\neg p(x)]$

Predicate Equivalences

Table 2.10 Equivalences and implications including predicates and quantifiers.

Q_1	$\forall x[p(x)] \Rightarrow \exists x[p(x)]$
Q_2	$\neg\exists x[p(x)] \Leftrightarrow \forall x[\neg p(x)]$
Q_3	$\neg\forall x[p(x)] \Leftrightarrow \exists x[\neg p(x)]$
Q_4	$\exists x[p(x) \vee q(x)] \Leftrightarrow \exists x[p(x)] \vee \exists x[q(x)]$
Q_5	$\exists x[p(x) \wedge q(x)] \Rightarrow \exists x[p(x)] \wedge \exists x[q(x)]$
Q_6	$\forall x[p(x)] \vee \forall x[q(x)] \Rightarrow \forall x[p(x) \vee q(x)]$
Q_7	$\forall x[p(x) \wedge q(x)] \Leftrightarrow \forall x[p(x)] \wedge \forall x[q(x)]$