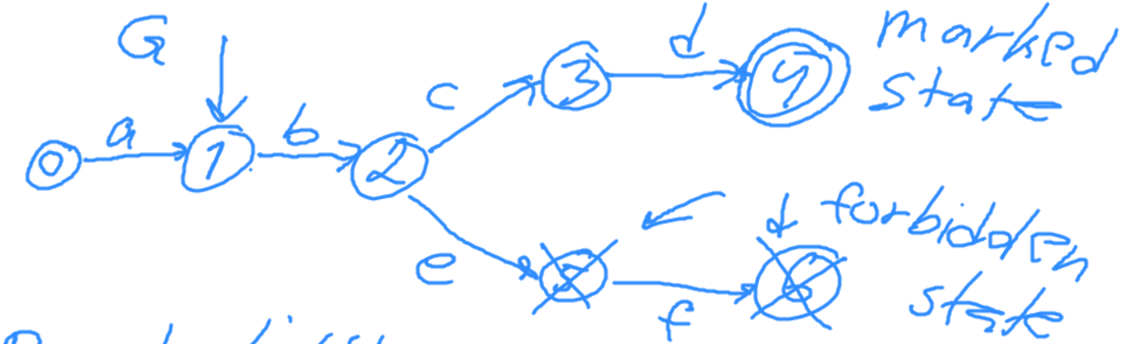
2 Discrete Mathematics

Predicate logic means that a propositional statement depends on one or more variables x, y, z. We assume that these variables take values from a domain $\Omega = \{a_1, a_2, \dots, a_n\}$ but often $\Omega = \mathbb{N} = \{0, 1, 2, \dots\}$

Predicates for automata

Ex



Reachability predicate

$P_r(x): x=1 \vee x=2 \vee \dots \vee x=6 \Leftrightarrow$

$1 \le x \le 6 \Leftrightarrow x \neq 0$

Coreachability predicate (going backward from marked states)

$P_c(x): 0 \le x \le 4 \Leftrightarrow x \neq 5 \wedge x \neq 6$

Forbidden state predicate

$$P_f(x) \overset{def}{=} P_r(x) \land \neg P_c(x) \Leftrightarrow$$

$$\Leftrightarrow 1 \leq x \leq 6 \land \underbrace{\neg (0 \leq x \leq 4)}_{(5 \leq x \leq 6)}$$

$$\Leftrightarrow 5 \leq x \leq 6 \Leftrightarrow x = 5 \lor x = 6$$

Acceptable state predicate

$$P_a(x) \overset{def}{=} P_r(x) \land \neg P_f(x) \Leftrightarrow$$

$$1 \leq x \leq 6 \land 0 \leq x \leq 4 \Leftrightarrow$$

$$1 \leq x \leq 4$$

Quantifiers in predicate logic

$\exists x$: There exists an x ...

$\forall x$: For all x

when Ω is finite

$$\exists x\, P(x) \overset{def}{=} P(a_1) \lor P(a_2) \lor \dots \lor P(a_n)$$

$$\forall x\, P(x) \overset{def}{=} P(a_1) \land P(a_2) \land \dots \land P(a_n)$$

Predicate equivalences

$$\neg \exists x [P(x)] \Leftrightarrow \forall x [\neg P(x)]$$

$$\neg (P(a_1) \lor \dots \lor P(a_n)) \Leftrightarrow (\neg P(a_1) \land \dots \land \neg P(a_n))$$

$$\Leftrightarrow \forall x [\neg P(x)]$$

Predicate Equivalences

Table 2.10 Equivalences and implications including predicates and quantifiers.

Q_1	$\forall x[p(x)] \Rightarrow \exists x[p(x)]$
Q_2	$\neg \exists x[p(x)] \Leftrightarrow \forall x[\neg p(x)]$
Q_3	$\neg \forall x[p(x)] \Leftrightarrow \exists x[\neg p(x)]$
Q_4	$\exists x[p(x) \vee q(x)] \Leftrightarrow \exists x[p(x)] \vee \exists x[q(x)]$
Q_5	$\exists x[p(x) \wedge q(x)] \Rightarrow \exists x[p(x)] \wedge \exists x[q(x)]$
Q_6	$\forall x[p(x)] \vee \forall x[q(x)] \Rightarrow \forall x[p(x) \vee q(x)]$
Q_7	$\forall x[p(x) \wedge q(x)] \Leftrightarrow \forall x[p(x)] \wedge \forall x[q(x)]$

Set theory

$A = \{a_1, a_2, \ldots, a_n\}$

$a_i \in A \qquad b_i \notin A$

set of states in an automaton

$Q = \{q_1, q_2, \ldots, q_n\}$

$A = \{x \mid x \in \mathbb{N} \land x \leq 5\} =$

$= \{0, 1, 2, 3, 4, 5\}$

Generally a set A can be defined as

$A = \{x \mid p_A(x)\}$

A set is unordered without repeated elements

$\{a_1, a_2, a_2\} = \{a_1, a_2\} = \{a_2, a_1\}$

Basic operations on sets

$A \cup B = \{x \mid x \in A \lor x \in B\}$

$A \cap B = \{x \mid x \in A \land x \in B\}$

$A \setminus B = \{x \mid x \in A \land x \notin B\}$

$\sim A = \{x \mid x \notin A\}$

Ex $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$

$\Omega = \{x \mid 0 \le x \le 5\}$

$A \cup B = \{1, 2, 3, 4\}$

$A \cap B = \{2, 3\}$

$A \setminus B = \{1\}$

$\sim A = \{0, 4, 5\}$

Relations between set algebra and predicate logic

set algebra:	$\cup$ $\cap$ $\sim$ Ω $\emptyset$ $\subseteq$ $=$
predicate logic:	$\vee$ $\wedge$ $\neg$ T F $\Rightarrow$ $\Leftrightarrow$

Special sets.

Subset: $A \subseteq B \Leftrightarrow$

$\vDash \forall x [x \in A \rightarrow x \in B]$

Equal sets:

$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

Empty set:

$\emptyset = \{x \mid p(x) \wedge \neg p(x)\}$

$= \{\,\}$

Universal set

$\Omega = \{x \mid p(x) \vee \neg p(x)\}$

Power set: set of all possible subsets of a set A:

$$2^A = \{X \mid X \subseteq A\}$$

Ex: $A = \{1, 2, 3\}$

$$2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, A\}$$

Proper subsets:

$$A \subset B \iff A \subseteq B \land A \neq B$$

Proof of set expressions is performed by predicate logic

Ex $A \subseteq B \Rightarrow A \cap C \subseteq B \cap C \quad (*)$

$$A \subseteq B \iff \forall x\{x \in A \to x \in B\}$$

$$\forall x\{(x \in A \land x \in C) \to (x \in B \land x \in C)\} \iff$$

$$\forall x\{\underbrace{\neg(x \in A \land x \in C)}_{x \notin A \lor x \notin C} \lor (x \in B \land x \in C)\} \iff$$

$$\forall x\{(x \notin A \lor x \notin C \lor x \in B) \land (x \notin A \lor \underbrace{x \notin C \lor x \in C}_{T})\} \iff$$

$$\forall x\{\underbrace{\underbrace{x \notin A \lor x \in B}_{x \in A \to x \in B}}_{T \text{ since } A \subseteq B} \lor x \notin C)$$

$\iff \forall x\, T$ $\therefore$ we have shown that (*) is valid

Set Table

S_1	$\sim\sim A$	$=$	A				
S_2	$\sim(A \cup B)$	$=$	$\sim A \cap \sim B$	S_3	$\sim(A \cap B)$	$=$	$\sim A \cup \sim B$
S_4	$A \cup B$	$=$	$B \cup A$	S_5	$A \cap B$	$=$	$B \cap A$
S_6	$A \cup (B \cup C)$	$=$	$(A \cup B) \cup C$	S_7	$A \cap (B \cap C)$	$=$	$(A \cap B) \cap C$
S_8	$A \cap (B \cup C)$	$=$	$(A \cap B) \cup (A \cap C)$	S_9	$A \cup (B \cap C)$	$=$	$(A \cup B) \cap (A \cup C)$
S_{10}	$A \cup A$	$=$	A	S_{11}	$A \cap A$	$=$	A
S_{12}	$A \cup \varnothing$	$=$	A	S_{13}	$A \cap \Omega$	$=$	A
S_{14}	$A \cup \Omega$	$=$	Ω	S_{15}	$A \cap \varnothing$	$=$	$\varnothing$
S_{16}	$A \cup \sim A$	$=$	Ω	S_{17}	$A \cap \sim A$	$=$	$\varnothing$
S_{18}	$A \cup (A \cap B)$	$=$	A	S_{19}	$A \cap (A \cup B)$	$=$	A
S_{20}	$A \cap B$	$\subseteq$	A	S_{21}	$A \cap B$	$\subseteq$	B
S_{22}	A	$\subseteq$	$A \cup B$	S_{23}	B	$\subseteq$	$A \cup B$

Equivalence Table

Table 2.6 Equivalence relations.

E_1	$\neg\neg p \Leftrightarrow p$		
E_2	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	E_3	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
E_4	$p \vee q \Leftrightarrow q \vee p$	E_5	$p \wedge q \Leftrightarrow q \wedge p$
E_6	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	E_7	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
E_8	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	E_9	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
E_{10}	$p \vee p \Leftrightarrow p$	E_{11}	$p \wedge p \Leftrightarrow p$
E_{12}	$p \vee \mathbf{F} \Leftrightarrow p$	E_{13}	$p \wedge \mathbf{T} \Leftrightarrow p$
E_{14}	$p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$	E_{15}	$p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$
E_{16}	$p \vee \neg p \Leftrightarrow \mathbf{T}$	E_{17}	$p \wedge \neg p \Leftrightarrow \mathbf{F}$
E_{18}	$p \vee (p \wedge q) \Leftrightarrow p$	E_{19}	$p \wedge (p \vee q) \Leftrightarrow p$
E_{20}	$p \rightarrow q \Leftrightarrow \neg p \vee q$	E_{21}	$\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$
E_{22}	$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$	E_{23}	$p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$
E_{24}	$\neg(p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \neg q$	E_{25}	$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
E_{26}	$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$		

ordered sets also called tuples

$\langle a_1, a_2, a_3 \rangle \neq \langle a_1, a_3, a_2 \rangle$

also ordinary parenthesis (a_1, a_2, a_3) can be used

cross product

$X = \{x_1, x_2, \dots, x_n\}$

$Y = \{y_1, y_2, \dots, y_m\}$

$$X \times Y = \{\langle x_1, y_1 \rangle, \langle x_1, y_2 \rangle, \dots, \langle x_1, y_m \rangle, \\ \langle x_2, y_1 \rangle, \dots \langle x_2, y_m \rangle, \\ \vdots \\ \langle x_n, y_1 \rangle, \dots \langle x_n, y_m \rangle\}$$

$X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$

Synchronous composition of automata G_1 and G_2 with state sets Q_1 and Q_2

$G_1 \| G_2$ generates the state set $Q_1 \times Q_2$

special rules for cross products e.g.

$(A \cap B) \times C = (A \times C) \cap (B \times C)$

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Note that $A \times B \neq B \times A$

Cross Product Examples

Example 2.19 — Show that $A \times B \neq B \times A$.

Assume that $A = \{1, 2, 3\}$ and $B = \{a, b\}$.

$$\begin{aligned} A \times B &= \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\} \\ B \times A &= \{\langle a, 1\rangle, \langle a, 2\rangle, \langle a, 3\rangle, \langle b, 1\rangle, \langle b, 2\rangle, \langle b, 3\rangle\} \end{aligned}$$

Hence, $A \times B \neq B \times A$ □

Example 2.20 — Show that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

$$\begin{aligned} (A \cup B) \times C &= \{\langle x, y\rangle | x \in (A \cup B) \wedge y \in C\} \\ &= \{\langle x, y\rangle | (x \in A \vee x \in B) \wedge y \in C\} \\ &= \{\langle x, y\rangle | (x \in A \wedge y \in C) \vee (x \in B \wedge y \in C)\} \\ &= (A \times C) \cup (B \times C) \end{aligned}$$

□

3 Automata, Formal Languages and Petri Nets

Def. of an automaton

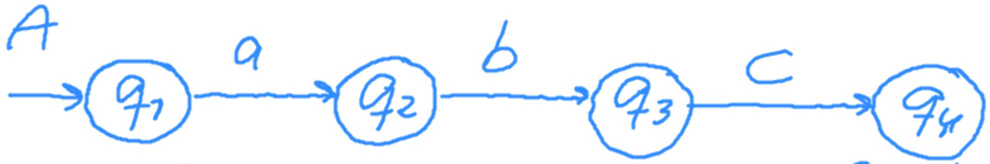
$A = \langle Q, \Sigma, \delta, q_i \rangle$

$Q = \{q_1, q_2, \dots, q_n\}$ = set of states

$\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_m\}$ = set of events (the alphabet)

$\delta : Q \times \Sigma \to Q$ = transition function

q_i = initial state



$Q = \{q_1, q_2, q_3, q_4\}$, $\Sigma = \{a, b, c\}$

$q_2 = \delta(q_1, a)$, $q_3 = \delta(q_2, b)$, $q_4 = \delta(q_3, c)$

Note that $\delta(q_2, a)$ is not defined $\Rightarrow$ $\delta(q, \sigma)$ is a partial function. A total function is obtained by introducing self-loops for those events that are not defined e.g. $\delta(q_2, a) = \delta(q_2, c) = q_2$

Generally a transition function is written

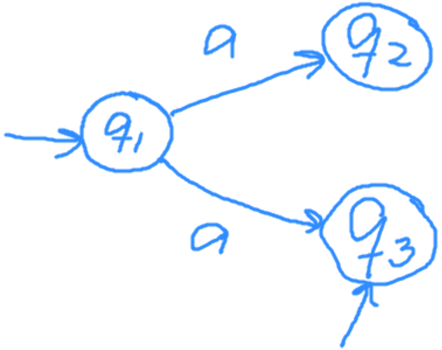
$$q' = \delta(q, a)$$

q = source state, q' = target state

Alternatively a transition can also be defined as a transition relation

$$\langle q, a, q' \rangle \in Q \times \Sigma \times Q$$

This is common for non-deterministic automata



$$T = \{\langle q_1, a, q_2 \rangle, \langle q_1, a, q_3 \rangle\}$$

$$A = \{Q, \Sigma, T, Q_i\}$$

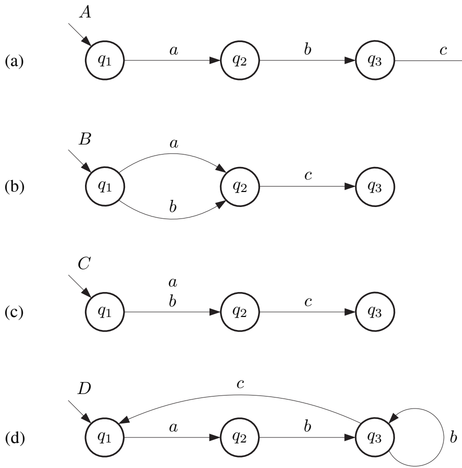
$$Q_i = \{q_1, q_3\}$$

Alternatively this non-deterministic choice can also be defined by the following transition function

$$\delta(q_1, a) = \{q_2, q_3\}$$

$$\delta : Q \times \Sigma \longrightarrow 2^Q$$

Automata Examples



Formal languages

Automaton A generates the string

ε, a, ab, abc

empty string

The language for A is denoted

$$L(A) = \{\varepsilon, a, ab, abc\}$$

The set of all strings for a given set of events/alphabet $\Sigma = \{a, b, c\}$ is denoted

$$\Sigma^* = \{\varepsilon, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, aab, aac, \ldots\}$$

Any language L for an automaton is a subset of Σ^*

$$L(A) \subseteq \Sigma^*$$