8. Cont. Markov Processes

continuous-time stochastic process with the Markov conditional probability property is obtained when

$t_{k+1}-t_k=\Delta t$ and $\Delta t\to 0$.

Assume a given transition probability $p_{ij}=a_{ij}\Delta t$

Here a_{ij} = transition rate

The row sum in $\mathbb{P}$

$$\sum_{l=1}^{n} p_{jl} = \underbrace{p_{j1}+\dots+p_{jj-1}+p_{jj+1}+\dots+p_{jn}}_{\sum_{l\neq j}^{n} p_{jl}} + p_{jj} = 1$$

$$p_{jj} = 1-\sum_{l\neq j}^{n} p_{jl}$$

$$p_j(\underbrace{t+\Delta t}_{t_{k+1}}) = \underbrace{[p_1(t)\dots p_n(t)]}_{p(t_k)} \begin{bmatrix}p_{1j}\\ \vdots \\ p_{nj}\end{bmatrix} =$$

$$p_1(t)p_{1j}+\dots+p_j(t)p_{jj}+\dots+p_n(t)p_{nj}$$

$$= \sum_{i\neq j}^{n} \underbrace{p_{ij}}_{a_{ij}\Delta t}\, p_i(t) + p_{jj}\,p_j(t) =$$

$$= \sum_{i\neq j}^{n} a_{ij}\Delta t\, p_i(t) + \underbrace{\left(1-\sum_{l\neq j}^{n} \overset{a_{jl}\Delta t}{p_{jl}}\right)}_{p_{jj}} p_j(t)$$

$$\frac{dp_j(t)}{dt} = \lim_{\Delta t\to 0}\frac{p_j(t+\Delta t)-p_j(t)}{\Delta t} =$$

$$= \sum_{i\neq j}^{n} a_{ij}\,p_i(t) - \left(\sum_{l\neq j}^{n} a_{jl}\right)p_j(t)$$

stationary solution

$$\frac{dp_j}{dt} = 0 \Rightarrow$$

$$\sum_{i \neq j}^{n} a_{ij}\, p_i(t) = \left(\sum_{\ell \neq j}^{n} a_{j\ell}\right) p_j(t)$$

$$\begin{bmatrix} \text{Input flow} \\ \text{to state } q_j \end{bmatrix} = \begin{bmatrix} \text{Output flow} \\ \text{from state } q_j \end{bmatrix}$$

Ex Buffer

$$\begin{tikzcd} q_0 \arrow[r, bend left, "\lambda"] & q_1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & q_2 \arrow[l, bend left, "\mu"] \end{tikzcd}$$

$$q_0: \underbrace{\mu p_1}_{\text{input flow from } q_1} = \underbrace{\lambda p_0}_{\text{output flow from } q_0}$$

$$p_1 = \frac{\lambda}{\mu} p_0$$

$$q_1: \underbrace{\lambda p_0 + \mu p_2}_{\text{input flow to } q_1} = \underbrace{\mu p_1 + \lambda p_1}_{\text{output flow from } q_1}$$

$$\mu p_2 = \underbrace{\mu p_1 - \lambda p_0}_{=0} + \lambda p_1 =$$

$$= \lambda p_1$$

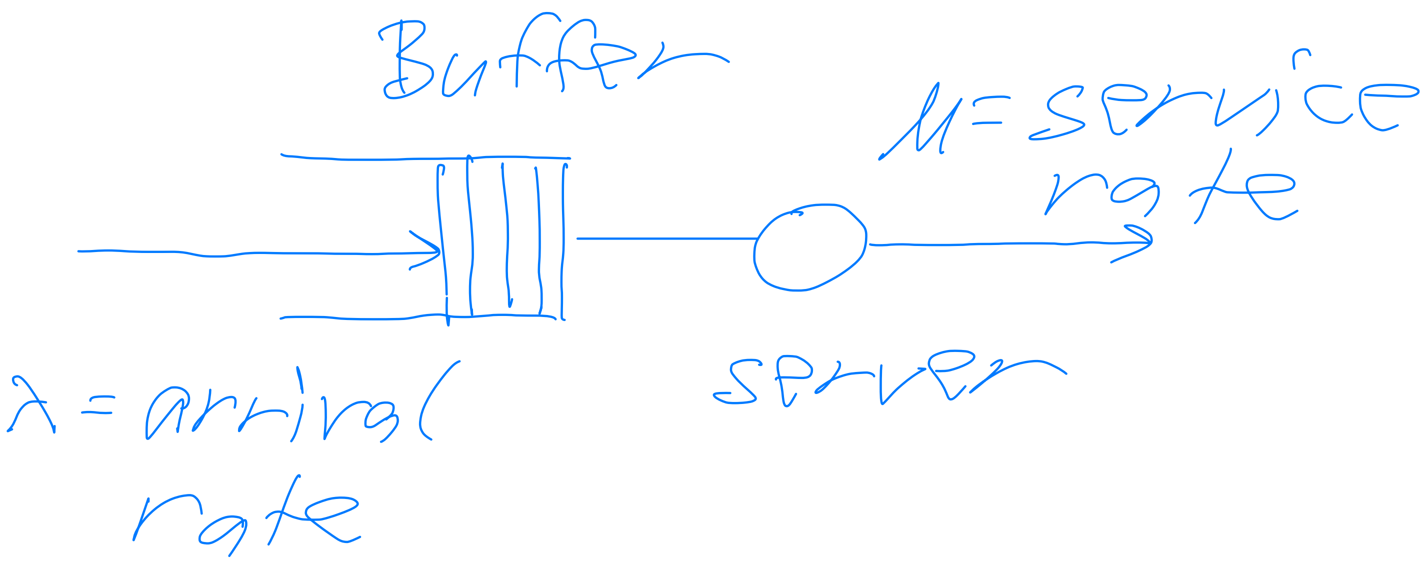
$$p_2 = \frac{\lambda}{\mu} p_1 = \frac{\lambda^2}{\mu^2} p_0$$

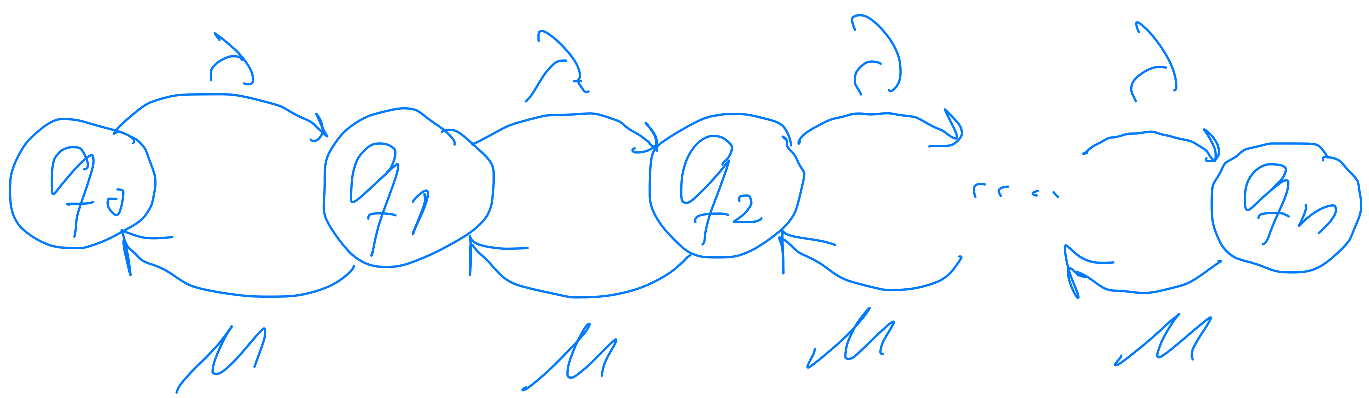
$$p_0 + p_1 + p_2 = 1 \Rightarrow$$

$$p_0 + \frac{\lambda}{\mu} p_0 + \frac{\lambda^2}{\mu^2} p_0 = 1$$

$$p_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}}$$

General buffer example





$$\begin{bmatrix} \text{Input flow} \\ \text{to state } q_i \end{bmatrix} = \begin{bmatrix} \text{Output flow} \\ \text{from state } q_i \end{bmatrix}$$

q_0: $\mu p_1 = \lambda p_0 \Rightarrow p_1 = \frac{\lambda}{\mu} p_0$

q_1: $\underbrace{\lambda p_0}_{\mu p_1} + \mu p_2 = \mu p_1 + \lambda p_1$

$\mu p_2 = \lambda p_1 \Rightarrow p_2 = \frac{\lambda}{\mu} p_1 = \frac{\lambda^2}{\mu^2} p_0$

q_2: $\underbrace{\lambda p_1}_{\mu p_2} + \mu p_3 = \mu p_2 + \lambda p_2$

$\mu p_3 = \lambda p_2 \Rightarrow p_3 = \frac{\lambda}{\mu} p_2 = \left(\frac{\lambda}{\mu}\right)^3 p_0$

Let $\rho = \frac{\lambda}{\mu} \Rightarrow$

$$p_j = \left(\frac{\lambda}{\mu}\right)^j p_0 = \rho^j p_0$$

$p_0 + p_1 + \ldots + p_n = 1$

$p_0 + \rho p_0 + \ldots + \rho^n p_0 = 1$

$(1 + \rho + \rho^2 + \ldots + \rho^n) p_0 = 1$

$$p_0 = \frac{1}{1 + \rho + \ldots + \rho^n} = \frac{1}{\sum_{j=0}^{n} \rho^j}$$

Introduce an infinite buffer by letting $n \to \infty$

$$\sum_{j=0}^{\infty} \rho^j = \frac{1}{1-\rho} \quad \Rightarrow$$

$$\boxed{\begin{aligned} p_0 &= \frac{1}{1/(1-\rho)} = 1-\rho \\ p_j &= \rho^j p_0 = \rho^j (1-\rho) \end{aligned}}$$

Queueing theory

Performance measures:

$\bar{N}$ = average total number of jobs in the system (work in progress)

$\bar{N}_Q$ = — " — waiting in the queue before the server

$\bar{N}_S$ = — " — in the server (utilization factor)

$\bar{T}$ = average time in the system

$\bar{T}_Q$ = — " — queue

$\bar{T}_S$ = — " — server

$$\overline{N} = \sum_{j=0}^{\infty} j p_j = 0 p_0 + 1 \cdot p_1 + 2 \cdot p_2 \ldots = (1-\rho) \sum_{j=0}^{\infty} j \rho^j =$$

$$= (1-\rho)(0 + 1\rho + 2\rho^2 + \ldots) = (1-\rho) \frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho}$$

$$\overline{N}_s = \sum_{j=1}^{\infty} p_j = 1 - p_0 =$$

$= (p_0 =$ prob of no element in the system $) =$

$$= 1 - (1-\rho) = \rho$$

$$\overline{N}_Q = \overline{N} - \overline{N}_s = \frac{\rho}{1-\rho} - \rho = \ldots = \frac{\rho^2}{1-\rho} = \rho \overline{N}$$

$$\overline{T}_s = \frac{1}{\mu}$$

$\overline{T} = \overline{N}/\lambda$ because of Little's law $\overline{N} = \lambda \overline{T}$

$$\overline{T} = \frac{\overset{\lambda/\mu}{\rho}}{(1-\rho)\lambda} = \frac{1}{\mu(1-\rho)} = \frac{1}{1-\rho} \overline{T}_s$$

$$\overline{T}_Q = \overline{T} - \overline{T}_s = \left(\frac{1}{1-\rho} - 1\right) \overline{T}_s$$

$$= \frac{\rho}{1-\rho} \overline{T}_s = \rho \overline{T}$$

ρ	$\bar{N} = \frac{\rho}{1-\rho}$	$T/T_s = \frac{1}{1-\rho}$
0.5	1	2
0.9	9	10
0.99	99	100
0.999	999	1000