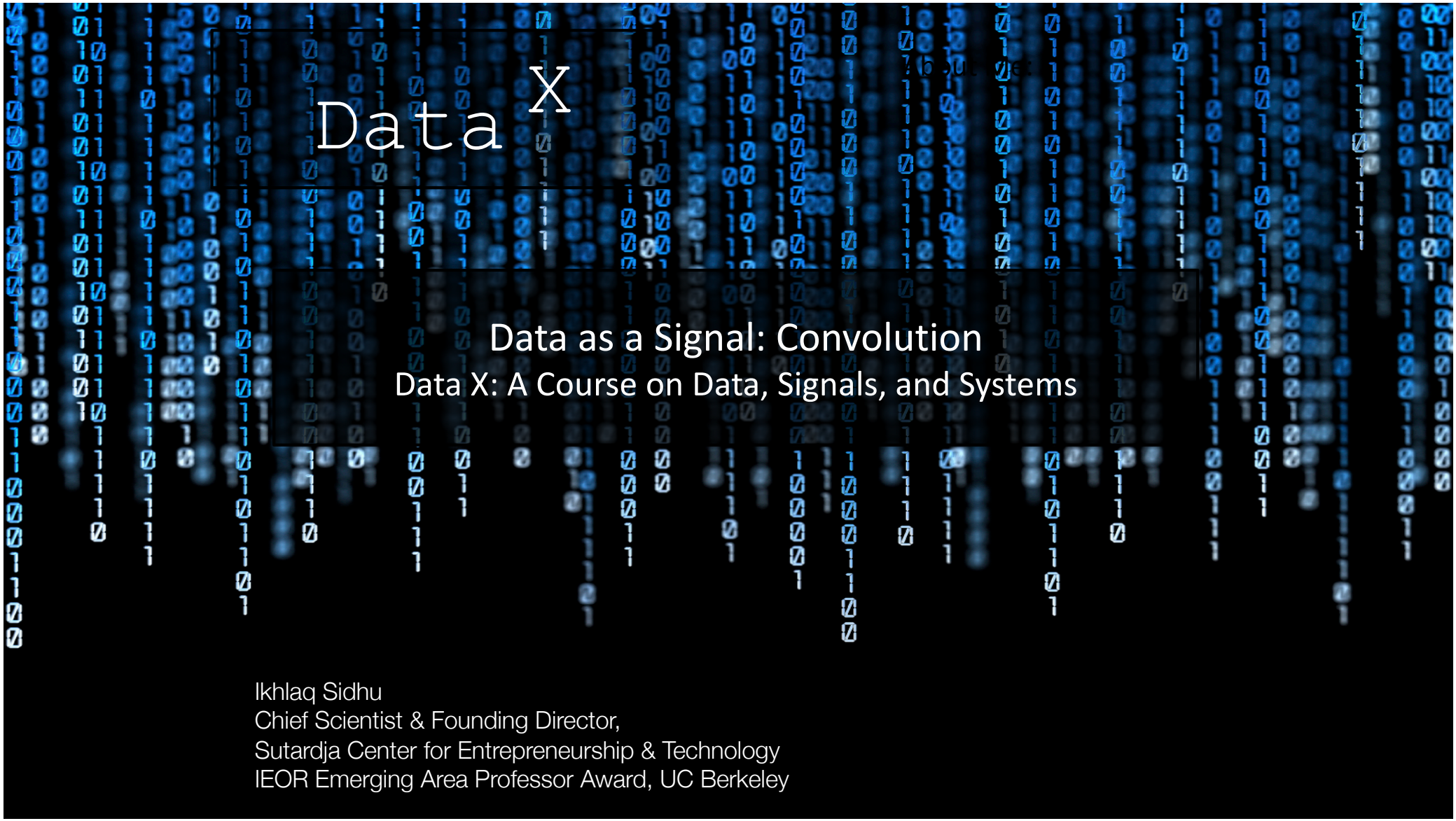

Data as a Signal: Convolution

Data X: A Course on Data, Signals, and Systems

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Sutardja Center for Entrepreneurship & Technology
IEOR Emerging Area Professor Award, UC Berkeley

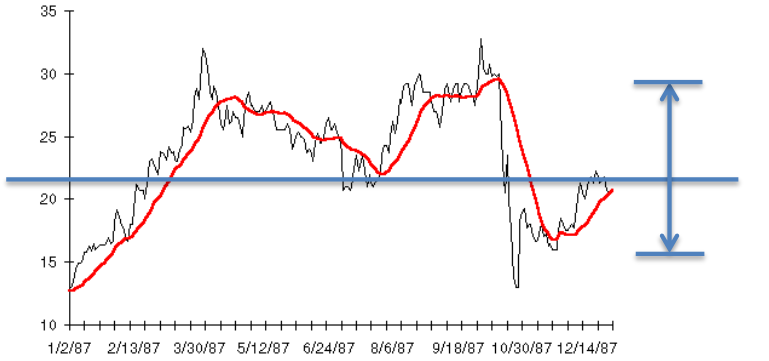
Before We Discuss Convolutional Neural Networks
Lets Discuss Convolution and LTI Systems First

Simple things we can do with a sequence: Mean, Variance

Discrete data

x_n = x1, x2, x3, …

Rec	Observed
1	60.323
2	61.122
3	60.171
4	61.187
5	63.221
6	63.639
7	64.989
8	63.761
9	66.019
10	67.857
11	68.169
12	66.513
13	68.655
14	69.564
15	69.331
16	70.551



variance

mean

Sample Mean	Population Mean
$\bar{x} = \frac{\Sigma x}{n}$	$\mu = \frac{\Sigma x}{N}$

where Σx is sum of all data values

N is number of data items in population

n is number of data items in sample

$$\sigma^2 = \frac{\sum (X-\mu)^2}{N}$$

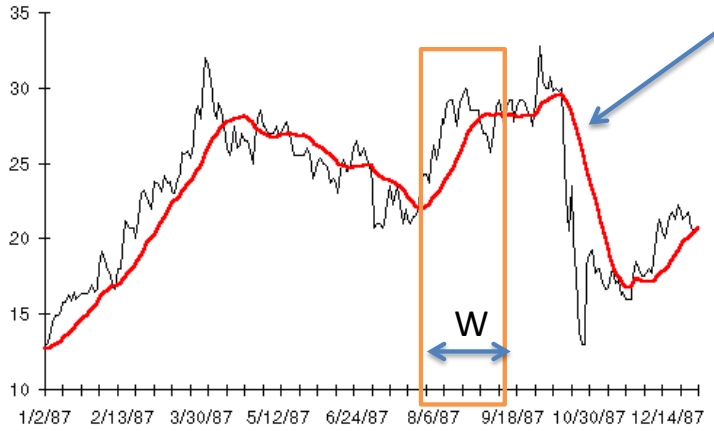
$$s^2 = \frac{\sum_{i=1}^{n} (X_i - X_{avg})^2}{n-1}$$

Moving Average of Sequence

(Simple) Moving Average

- Discrete data x_n = x1, x2, x3, …
- Moving Average

W =size of window
MA(n) is red line, at the right edge of the window



Note:

* It's a function of n (and maybe W)

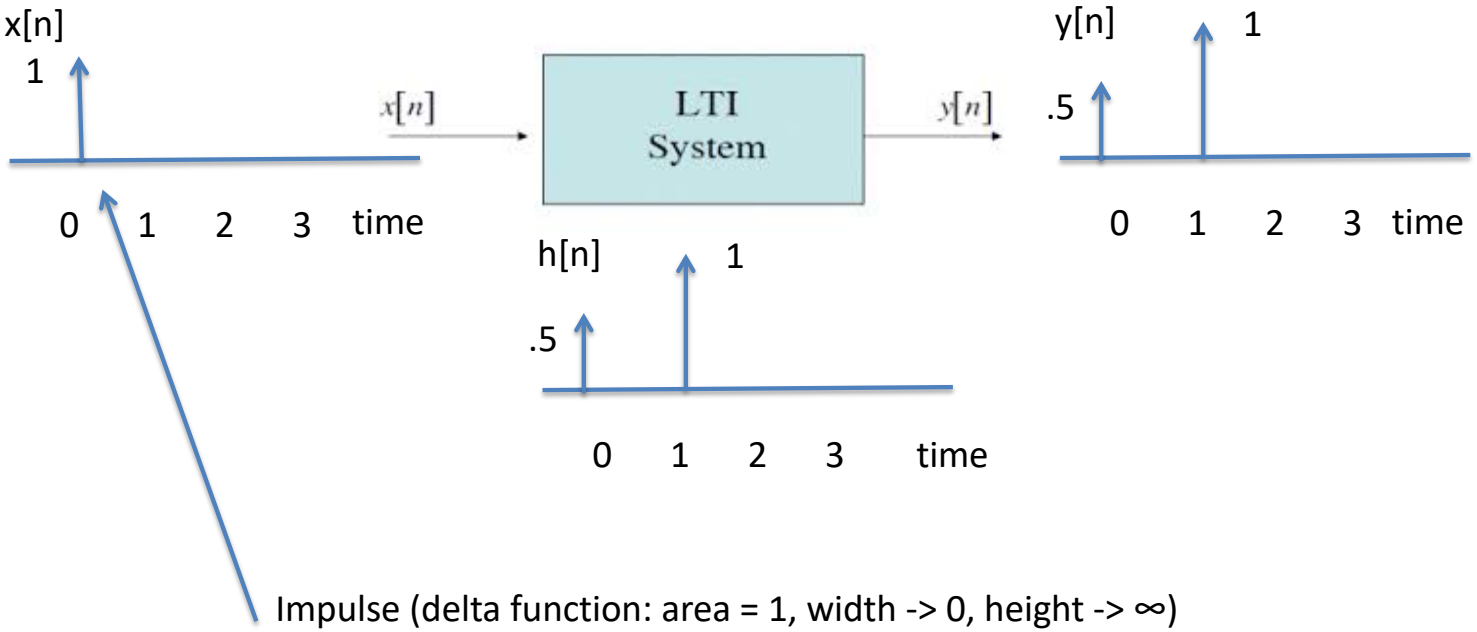
- Its smoother than the original (low pass filter)
- W = window size, can be in the summation

$$MA(n) = \frac{1}{W} \sum_{i=0.}^{W-1} x_{n-i}$$

$$MA(n) = \sum_{i=0.}^{W-1} \frac{x_{n-i}}{W}$$

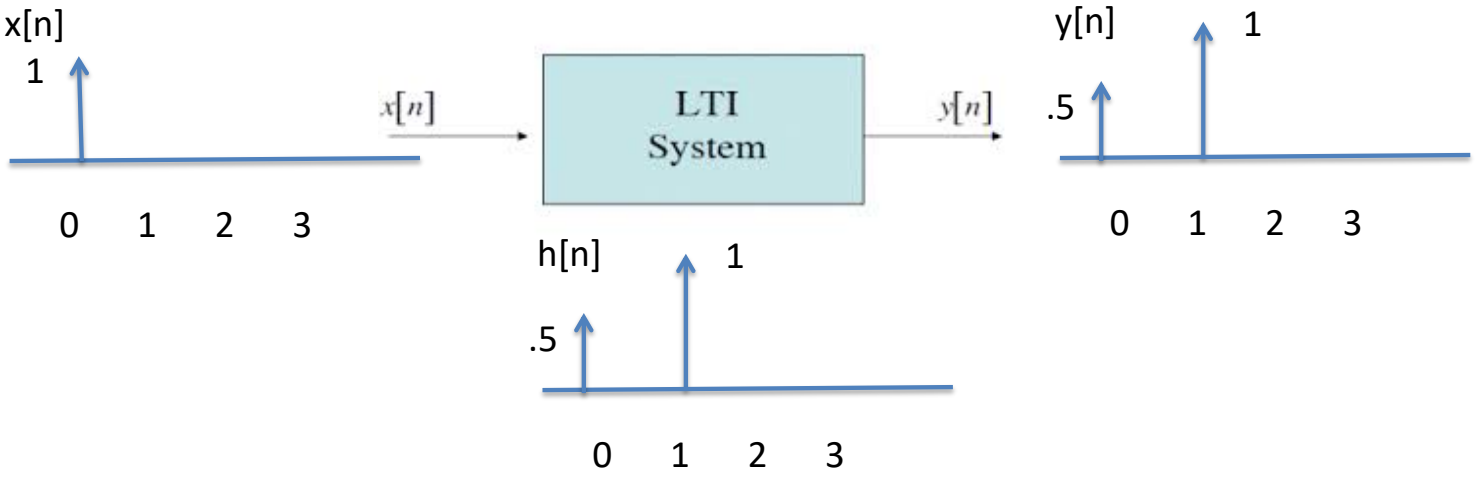
Linear Time Invariant System Approaches

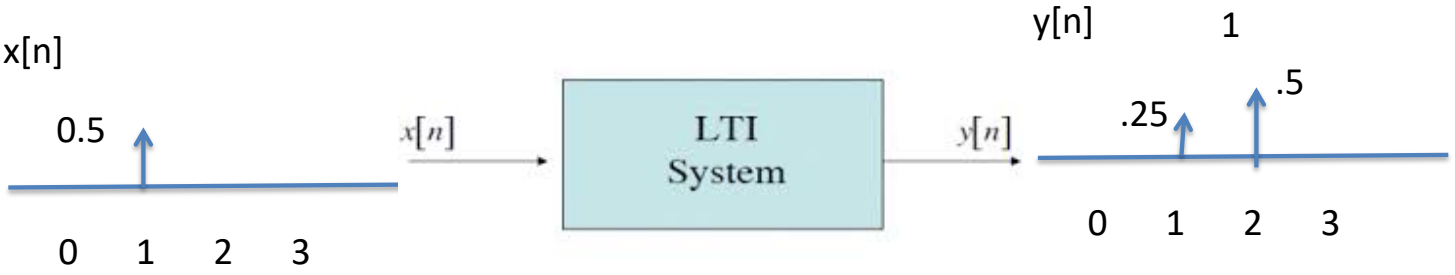
Linear Time Invariant System



Impulse (delta function: area = 1, width -> 0, height -> ∞)
In discrete systems, its just a value of “1”

Linear Time Invariant System





Convolution is actually really simple, but powerful

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- x[[n] = 1 2 3 2 4 0 0 0 …
- h[n] = 1 1 1
- y[n] = 1 1 1 ← h[n-0] x[0]

  2 2 2 ← h[n-1] x[1]

  3 3 3 ← h[n-2] x[2]

  2 2 2 ← h[n-3] x[3]

  4 4 4 ← h[n-4] x[4]

---

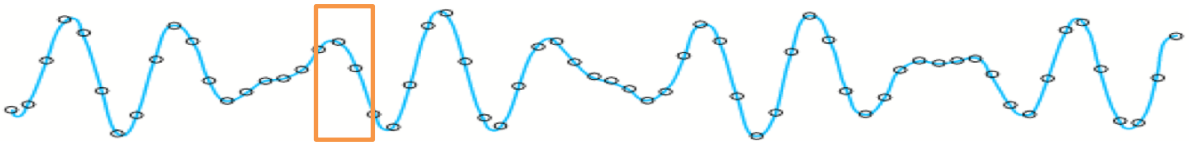
y[n] =1 3 6 7 9 6 4 …

Note: if h[n] = 1/3, 1/3, 1/3, then y[n] = 1/3, 1, 2, 7/3, 3, 2, 4/3

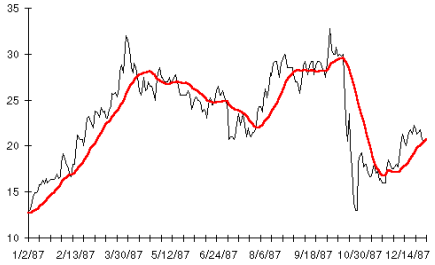
Note, MA with Window = 3 (if divided by 3)

Another Way to Think about Moving Average

x(t)



x(t) → LTI: h(n)=1,1,1,1 → y[n] is similar to moving average with window of 4



x(n) = array with sequence of numbers = [10, 3, 6, 12, …. … 43, 12, 1, 4]

h(n) = impulse response function from linear systems
= [1, 1, 1, 1]

MA(n, W) = y(n) = x(t) * h(t) (convolution)

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Continuous $$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

Circular $$(x \circledast y)_n \triangleq \sum_{m=0}^{N-1} x(m)y(n-m)$$

Another Convolution, different impulse response

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- x[[n] = 1 2 3 2 4 0 0 0 …
- h[n] = 3 2 1
- y[n] = 3 2 1 ← h[n-0] x[0]

  6 4 1 ← h[n-1] x[1]

  9 6 1 ← h[n-2] x[2]

  6 4 1 ← h[n-3] x[1]

  12 8 4 ← h[n-1] x[1]

---

y[n] =3 8 14 13 17 9 4 …

This is called filtering the data with an FIR filter with impulse response h[n] = 3 2 1

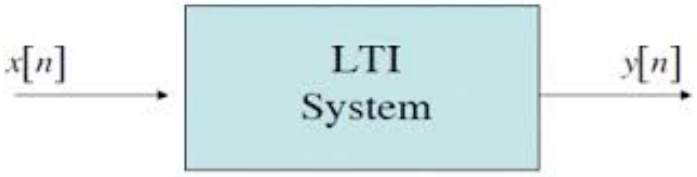
Code Example of Convolution

```
>>> np.convolve([1, 2, 3], [0, 1, 0.5])
array([ 0. ,  1. ,  2.5,  4. ,  1.5])
```

numpy.convolve(a, v, mode='full')[source]

- Returns the discrete, linear convolution of two one-dimensional sequences.
- The convolution operator is often seen in signal processing, where it models the effect of a linear time-invariant system on a signal [R17]. In probability theory, the sum of two independent random variables is distributed according to the convolution of their individual distributions.

Convolution: Why We Care



FIR: Finite Impulse Response Filter

Take Any Input Sequence x[n]
Filter for what you want with h[n]
Get result y[n]

h[n]	**Result**
1/W,1/W,1/W,1/W	Low pass filter or moving average, with 1/W cutoff
(a1,a2,…a10)	Any linear weighted sum
1, -1	Differential
Any sequence	Cross-correlation, allows you to look for similar pattern

Properties of Convolution

EQUATION 7-6
The commutative property of convolution. This states that the order in which signals are convolved can be exchanged.

$$a[n] * b[n] = b[n] * a[n]$$

EQUATION 7-8
The distributive property of convolution describes how parallel systems are analyzed.

$$a[n] * b[n] + a[n] * c[n] = a[n] * (b[n] + c[n])$$

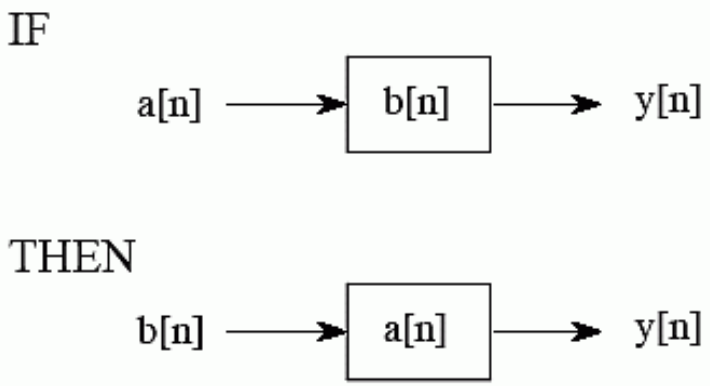


FIGURE 7-8
The commutative property in system theory. The commutative property of convolution allows the input signal and the impulse response of a system to be exchanged without changing the output. While interesting, this usually has no physical significance. (A signal appearing inside of a box, such as b[n] and a[n] in this figure, represent the *impulse response* of the system).

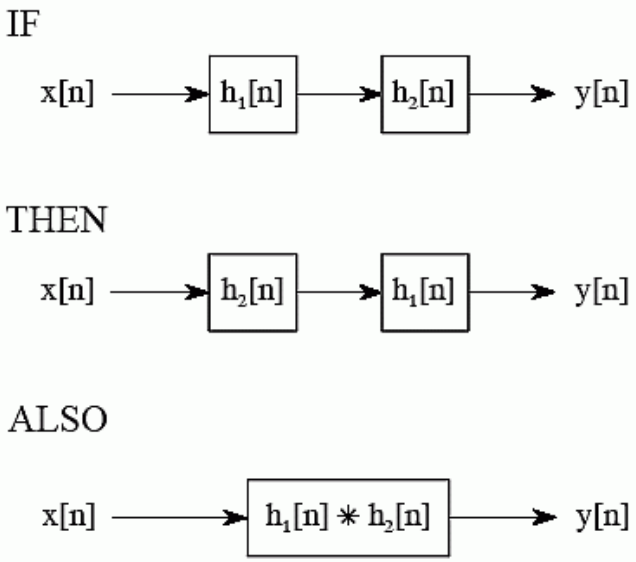


FIGURE 7-9
The associative property in system theory. The associative property provides two important characteristics of *cascaded* linear systems. First, the order of the systems can be rearranged without changing the overall operation of the cascade. Second, two or more systems in a cascade can be replaced by a single system. The impulse response of the replacement system is found by convolving the impulse responses of the stages being replaced.

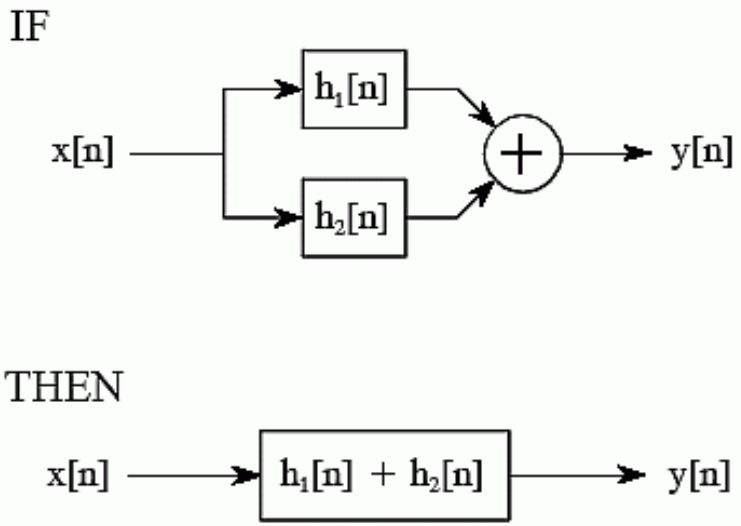
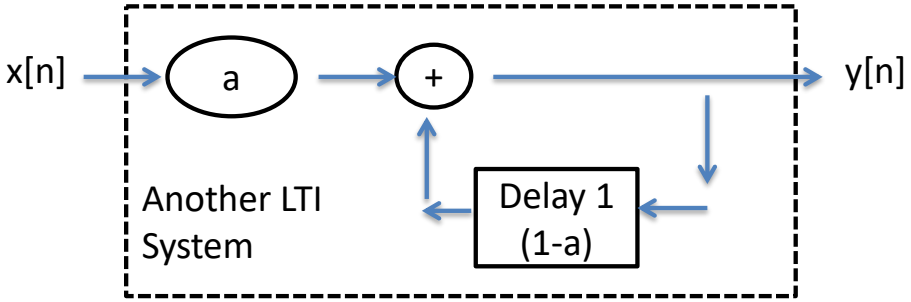


FIGURE 7-10
The distributive property in system theory. The distributive property shows that parallel systems with added outputs can be replaced with a single system. The impulse response of the replacement system is equal to the sum of the impulse responses of all the original systems.

Another LTI System: IIR: Infinite Impulse Response Filter

- Discrete data x_n = x1, x2, x3, …



$$y(n) = \alpha x(n) + (1-\alpha) y(n-1)$$

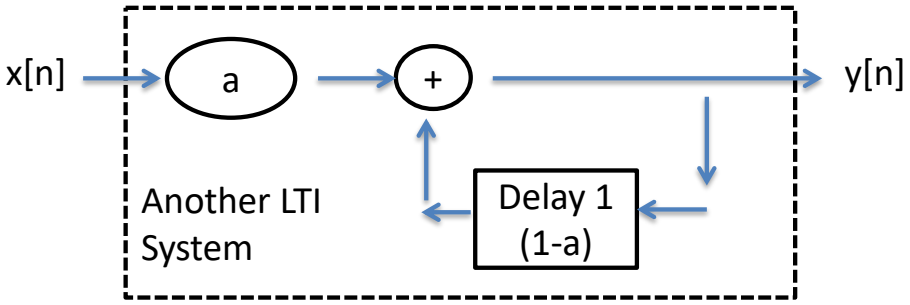
$$y(n) = \alpha x(n) + \alpha^2 x(n-1) + \alpha^3 x(n-2) \ldots$$

Infinitely back in x[n]

Another LTI System: IIR: Infinite Impulse Response Filter

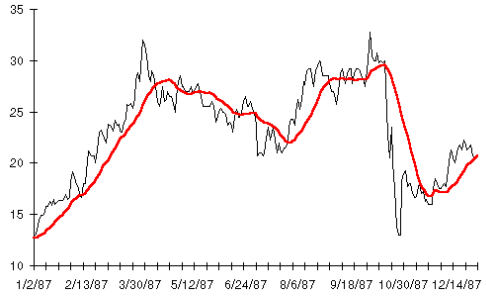
- Discrete data x_n = x1, x2, x3, …

$$y(n) = \alpha x(n) + (1-\alpha) y(n-1)$$



$$y(n) = \alpha x(n) + \alpha^2 x(n-1) + \alpha^3 x(n-2) \ldots$$

Exponential Moving Average



alpha = smoothing factor of 2/(W+1)

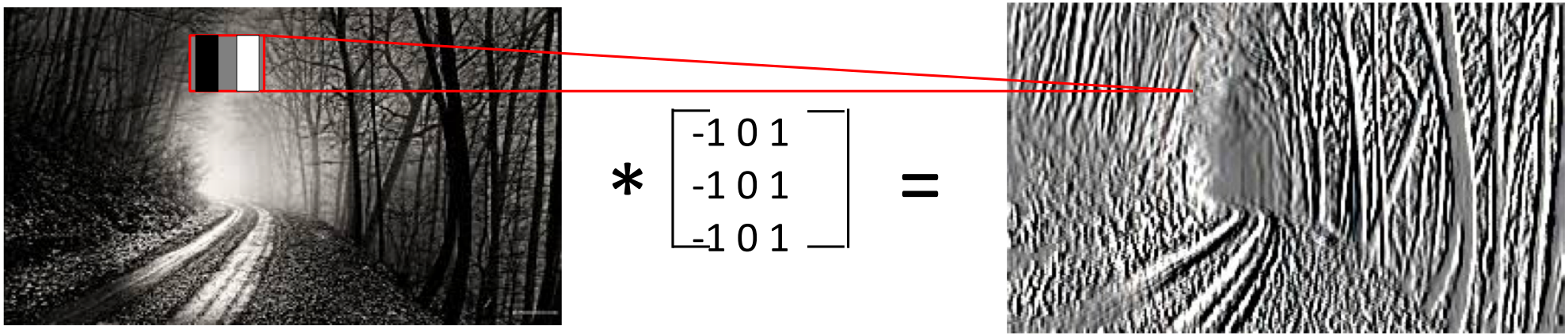
W is number of time periods

Infinitely back in x[n]

With Images
Convolutions are 2D

Examples in this section are from
Alyosha Efros, Slides in Guest Lecture

Convolutional of Two Signals



$$f[x,y] * g[x,y] = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} f[n_1,n_2] \cdot g[x-n_1, y-n_2]$$

elementwise multiplication and sum of a filter and the signal (image)

Image filtering

$h[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$f[\cdot,\cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[\cdot,\cdot]$

Image filtering

$h[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[.,.]$

	0	10							

Image filtering

$h[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$f[\cdot,\cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[\cdot,\cdot]$

	0	10	20						

Image filtering

$h[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$f[\cdot,\cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[\cdot,\cdot]$

	0	10	20	30					

Image filtering

$h[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$f[\cdot,\cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[\cdot,\cdot]$

	0	10	20	30	30				

Image filtering

$h[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[.,.]$

	0	10	20	30	30				
					?				

Image filtering

$h[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[.,.]$

	0	10	20	30	30				
						?			
					50				

Image filtering

$h[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

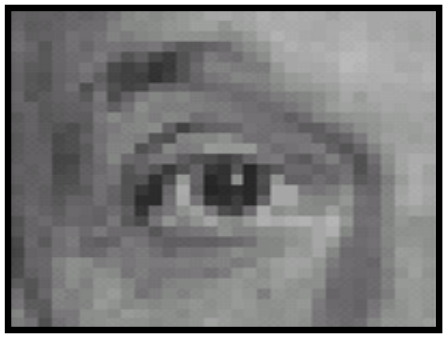
$f[\cdot,\cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[\cdot,\cdot]$

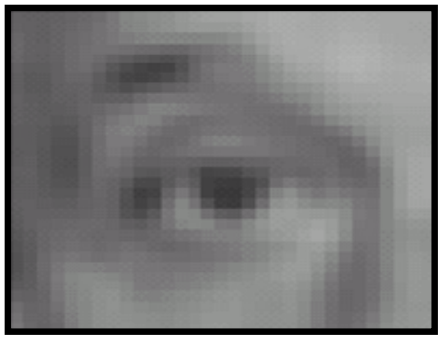
	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Linear filters: examples



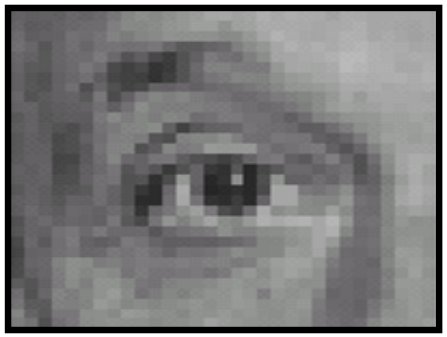
Original

$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} =$$



Blur (with a mean filter)

Practice with linear filters

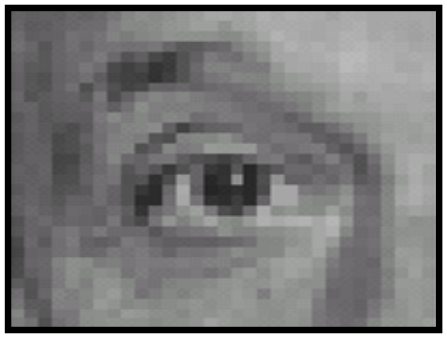


Original

0	0	0
0	1	0
0	0	0

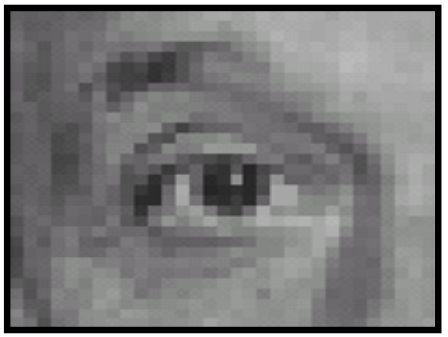
?

Practice with linear filters



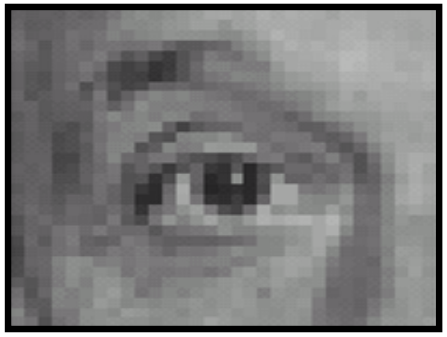
Original

0	0	0
0	1	0
0	0	0



Filtered
(no change)

Practice with linear filters

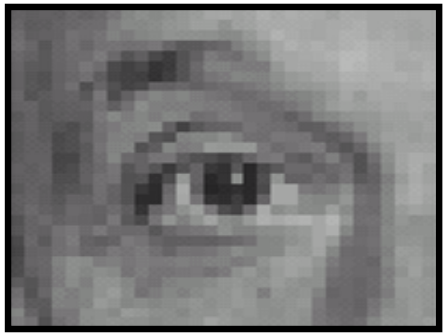


Original

0	0	0
0	0	1
0	0	0

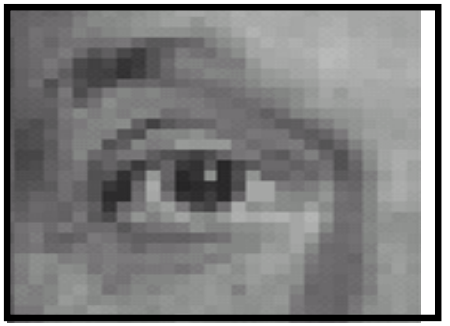
?

Practice with linear filters



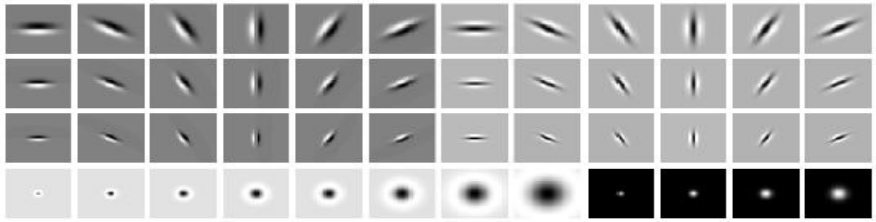
Original

0	0	0
0	0	1
0	0	0

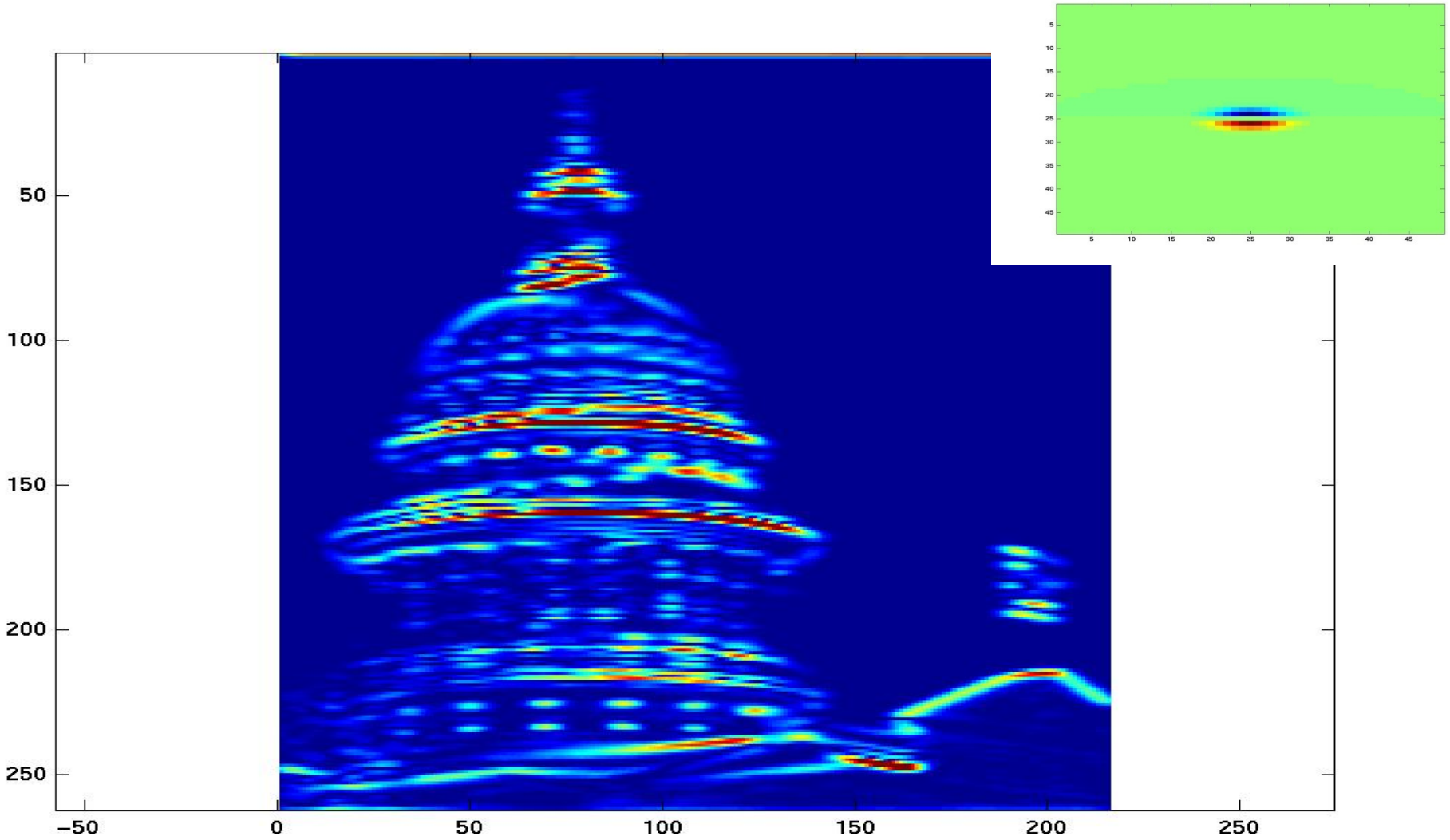


Shifted left
By 1 pixel

from http://www.texasexplorer.com/austincap2.jpg



Data



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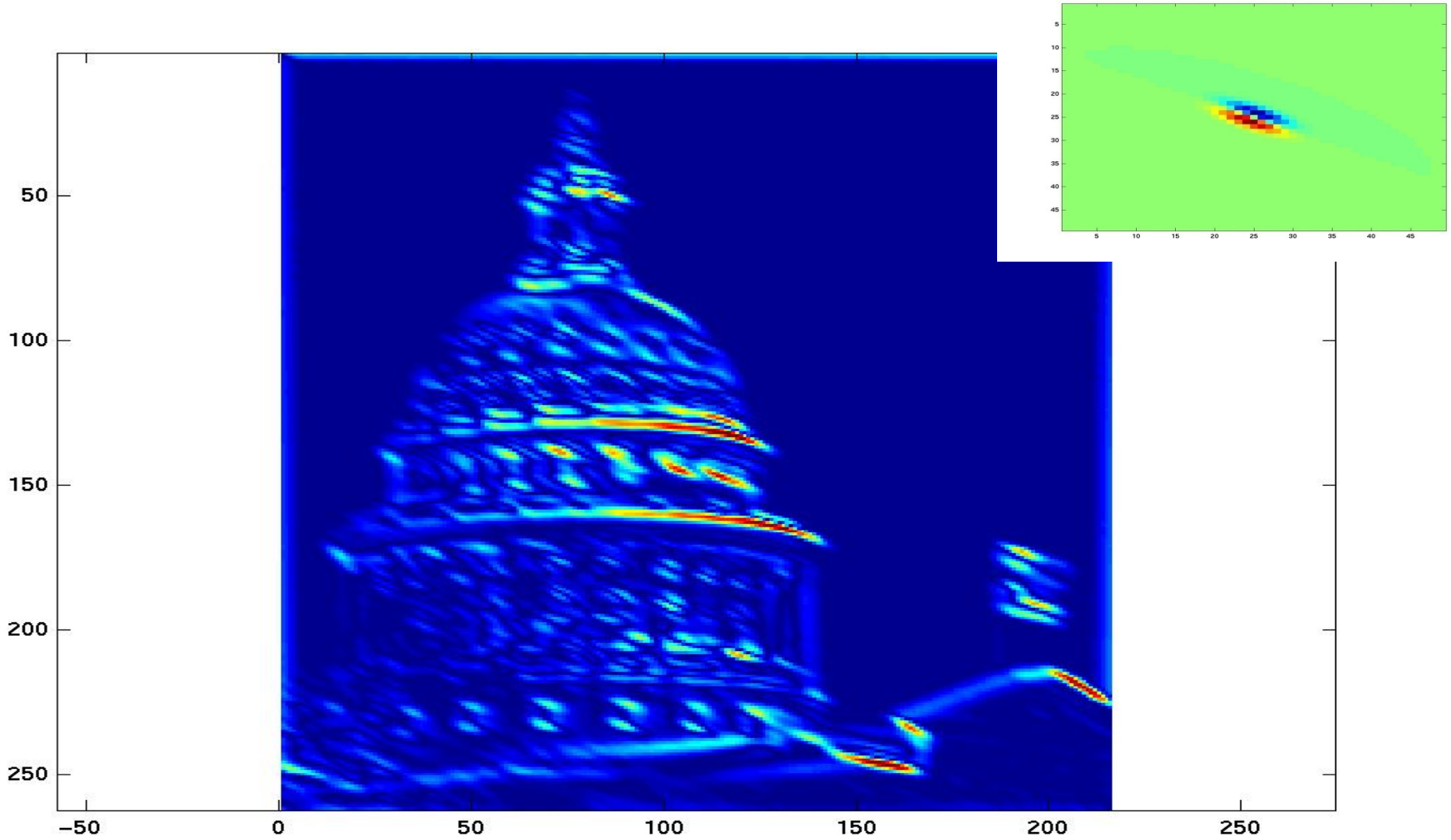
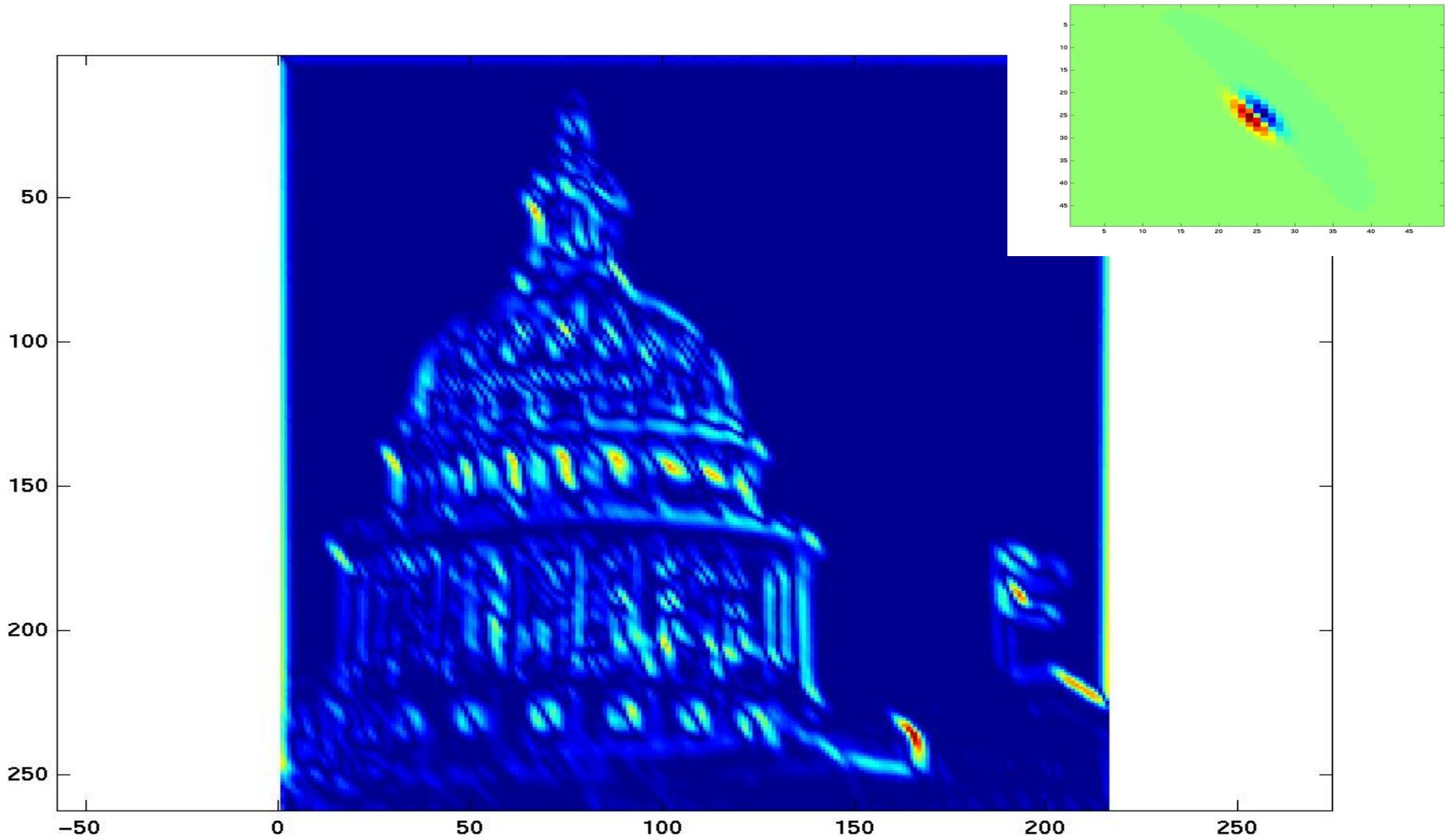
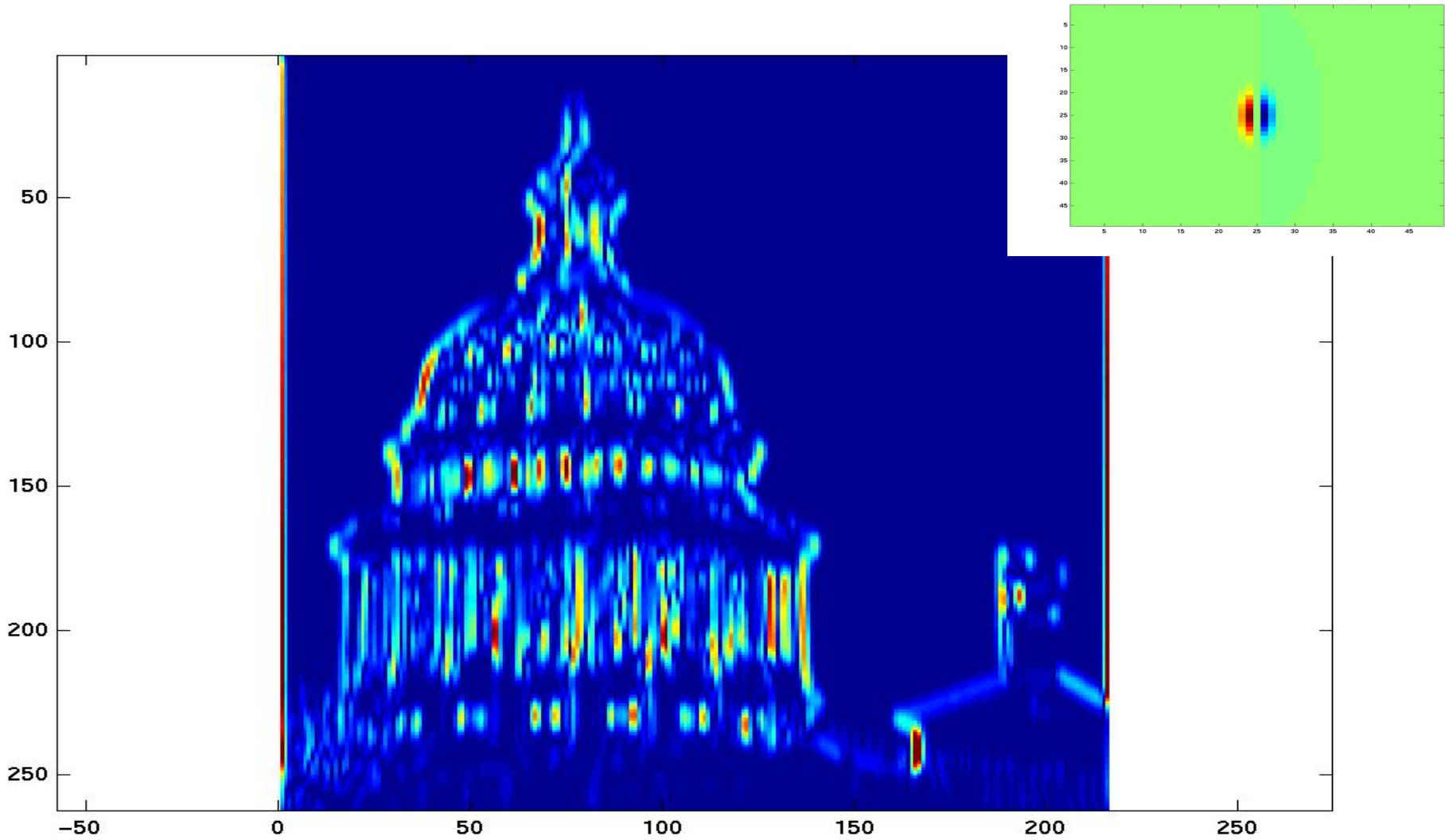


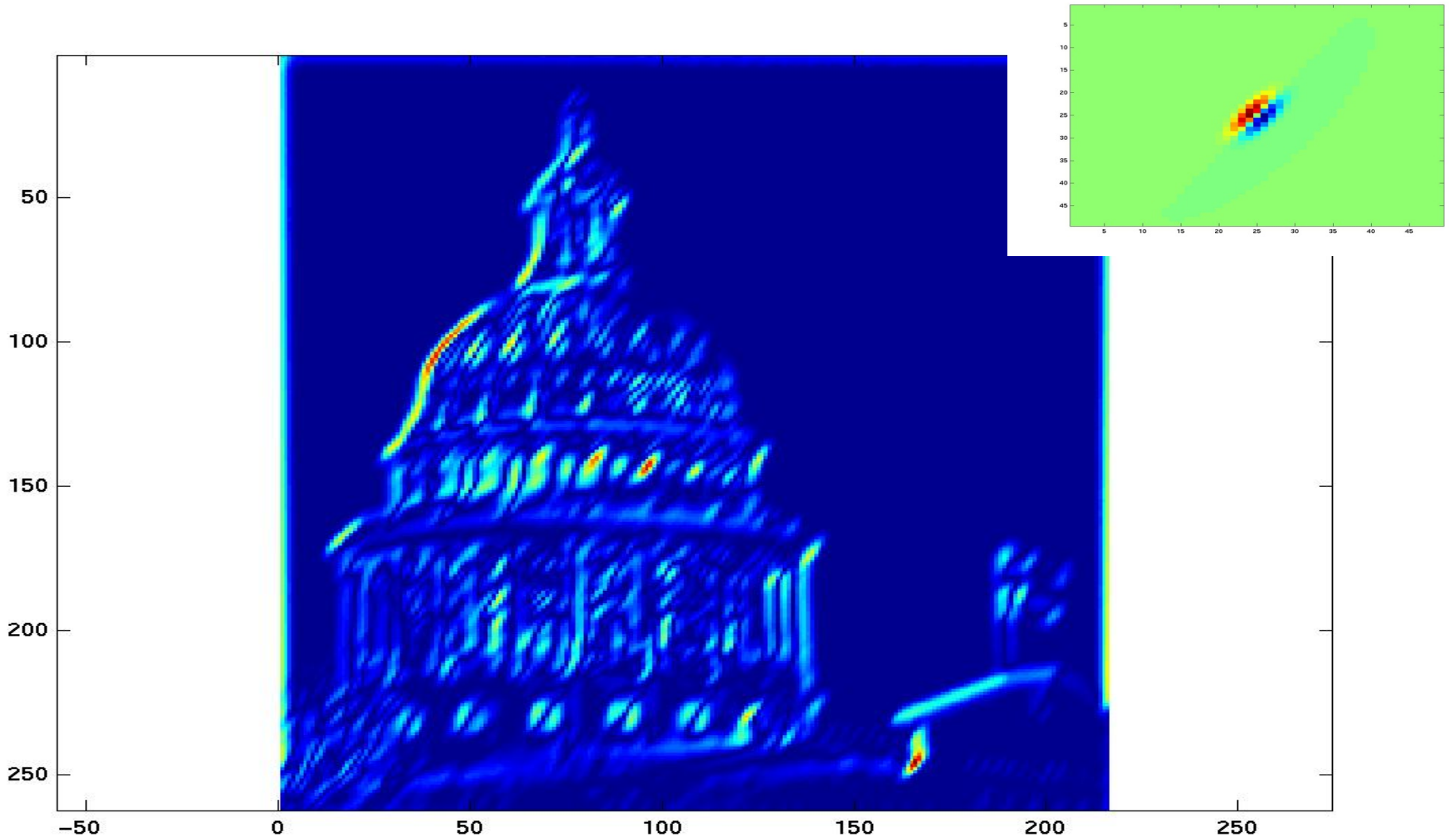
Kristen Grauman



Kristen Grauman



Kristen Grauman



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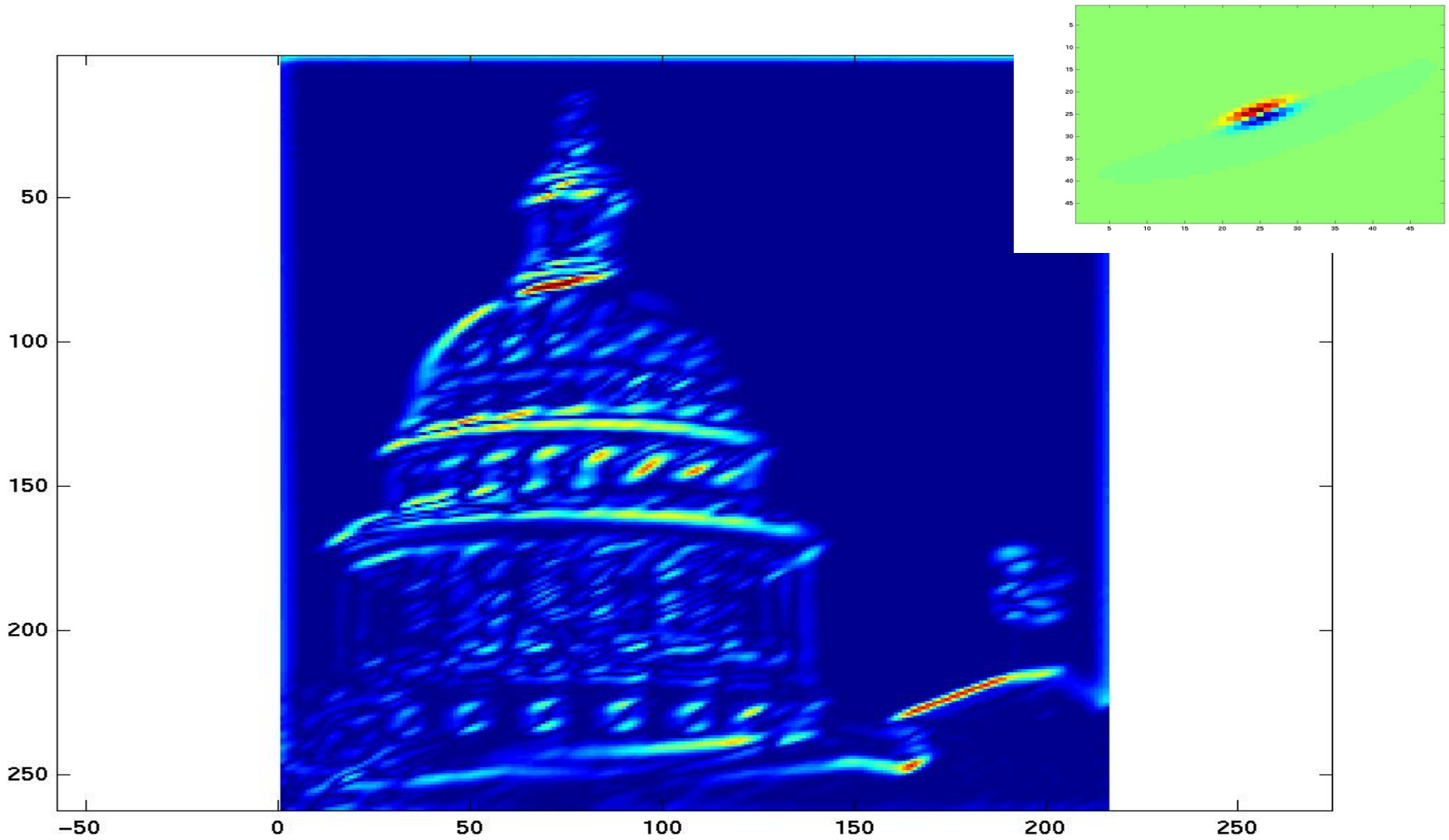

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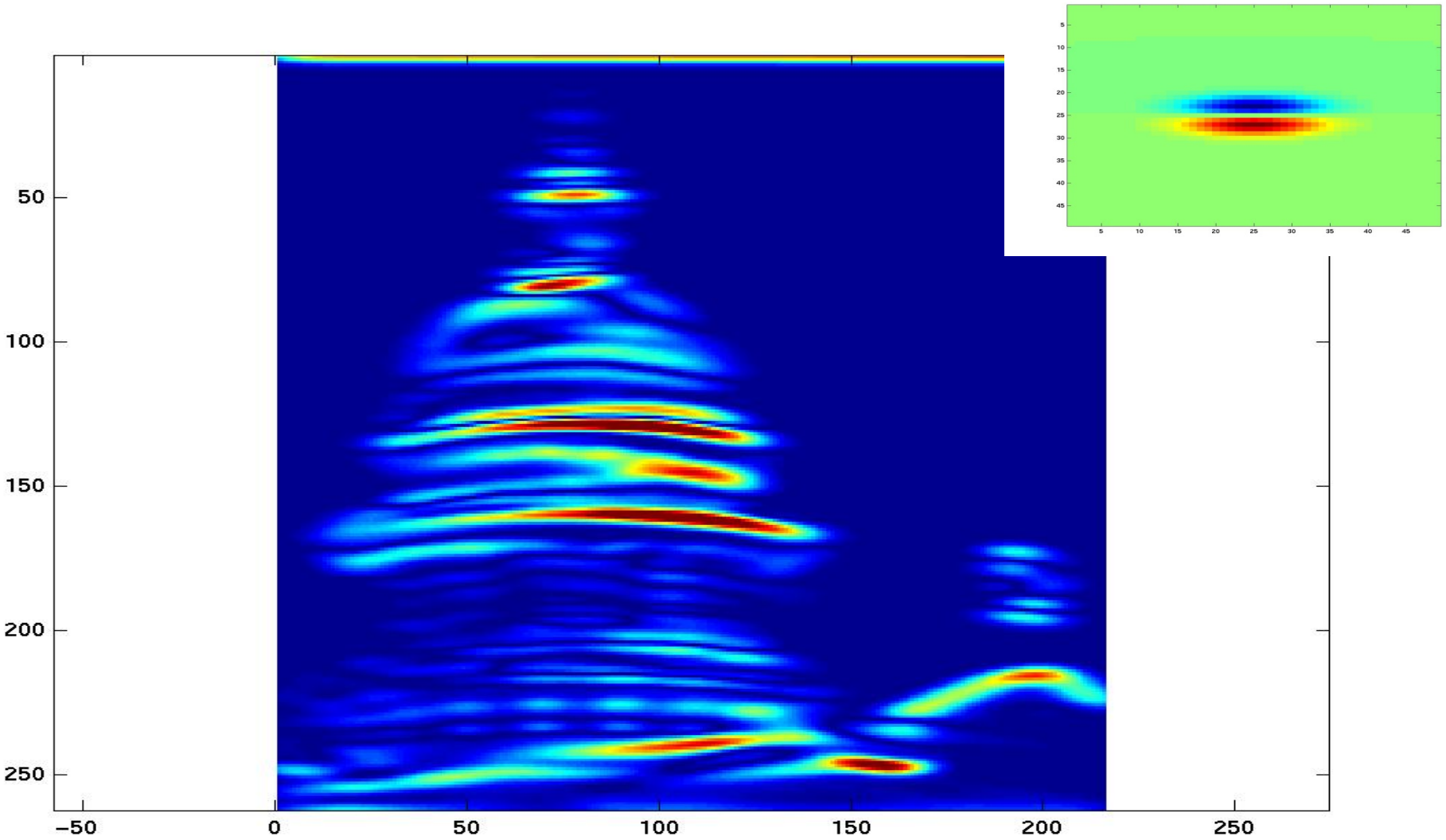

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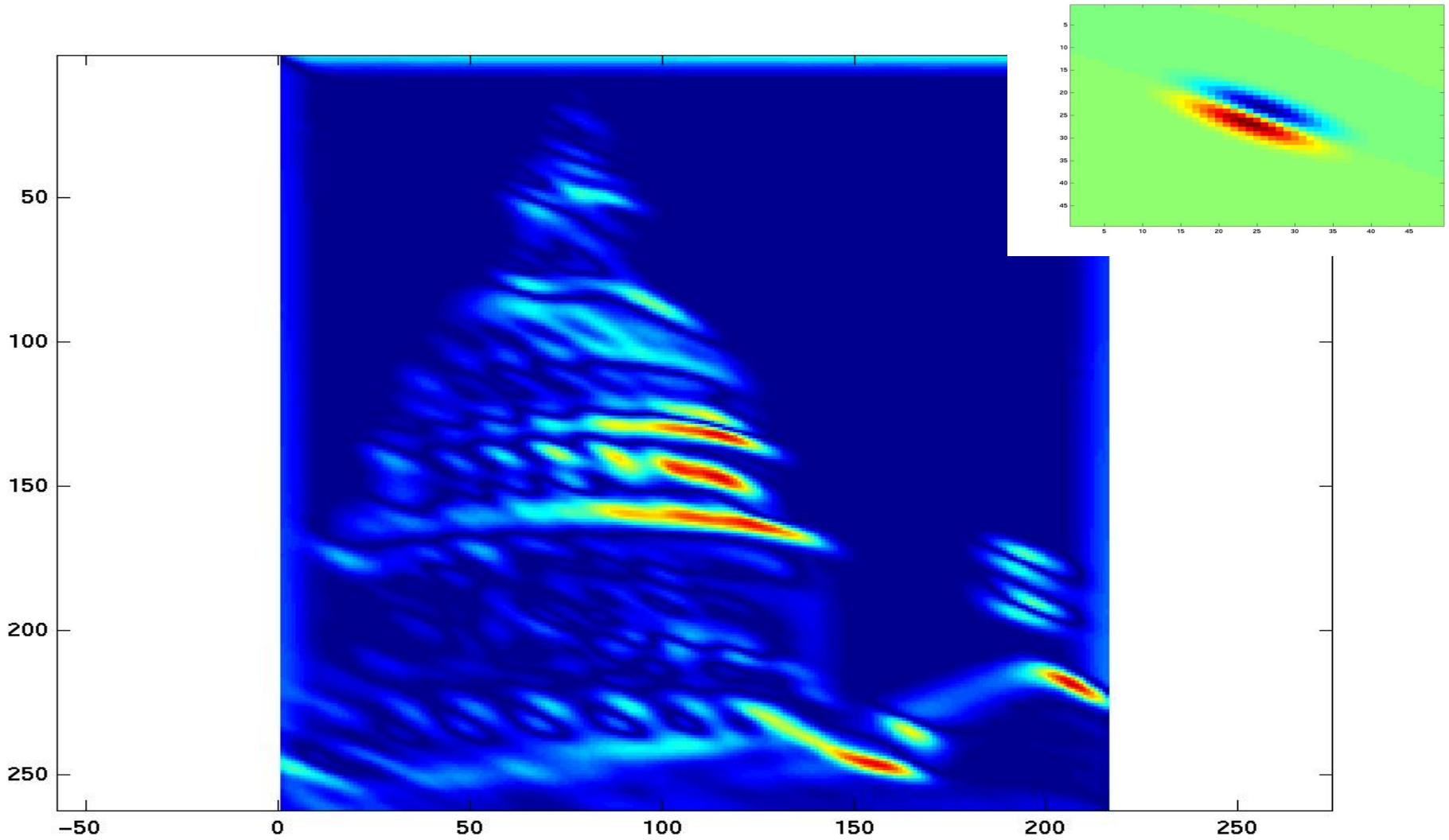

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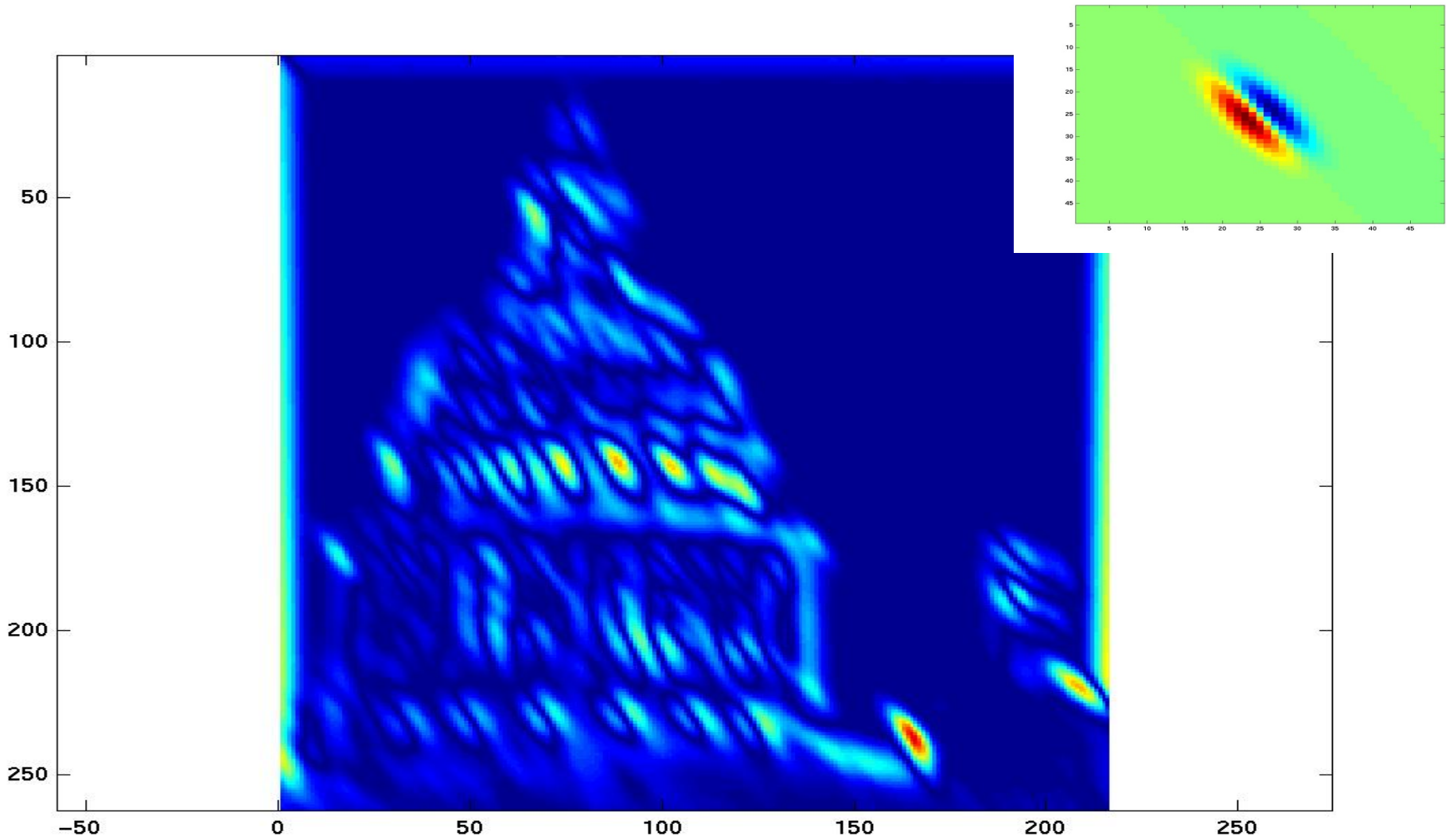
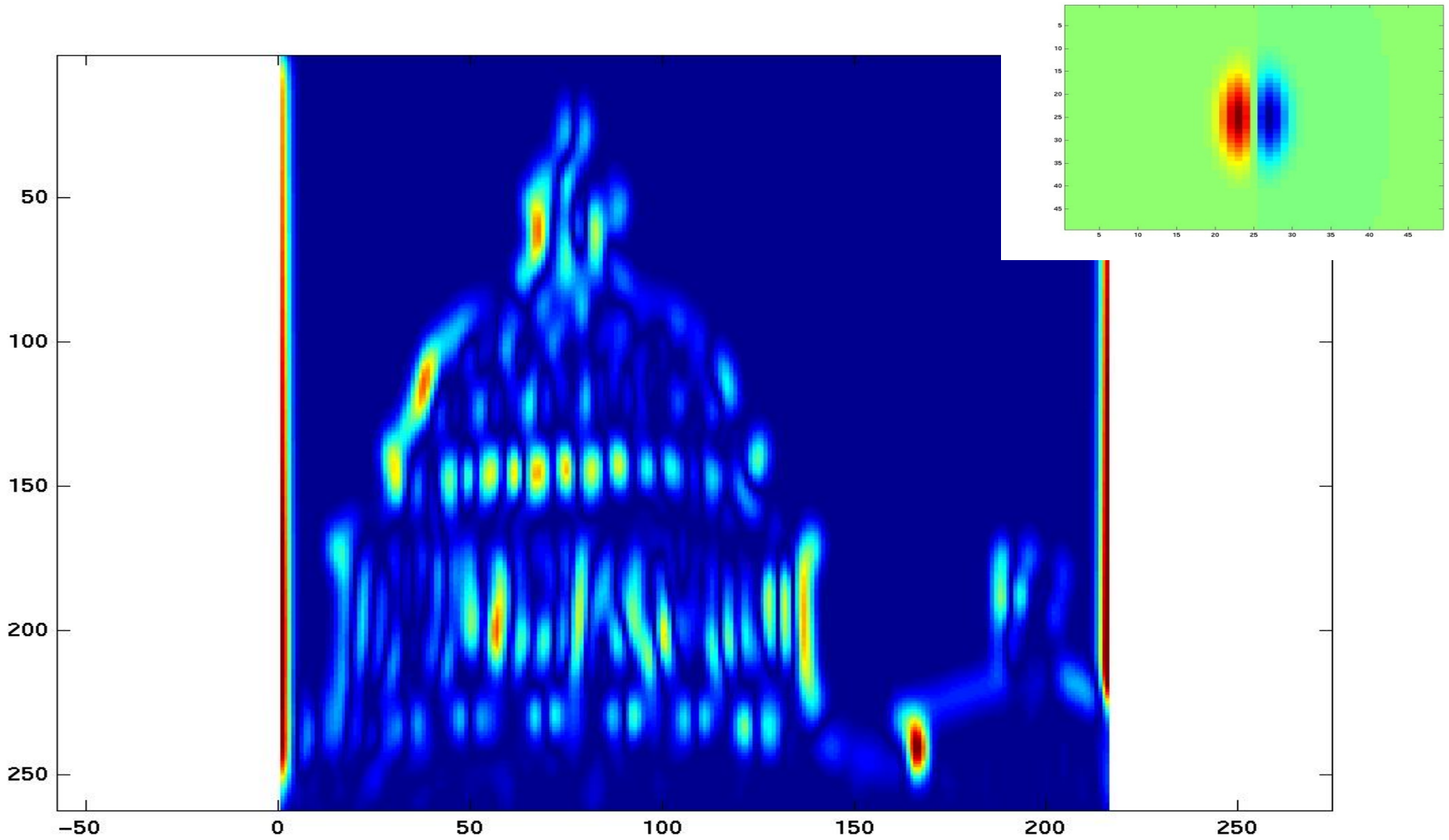
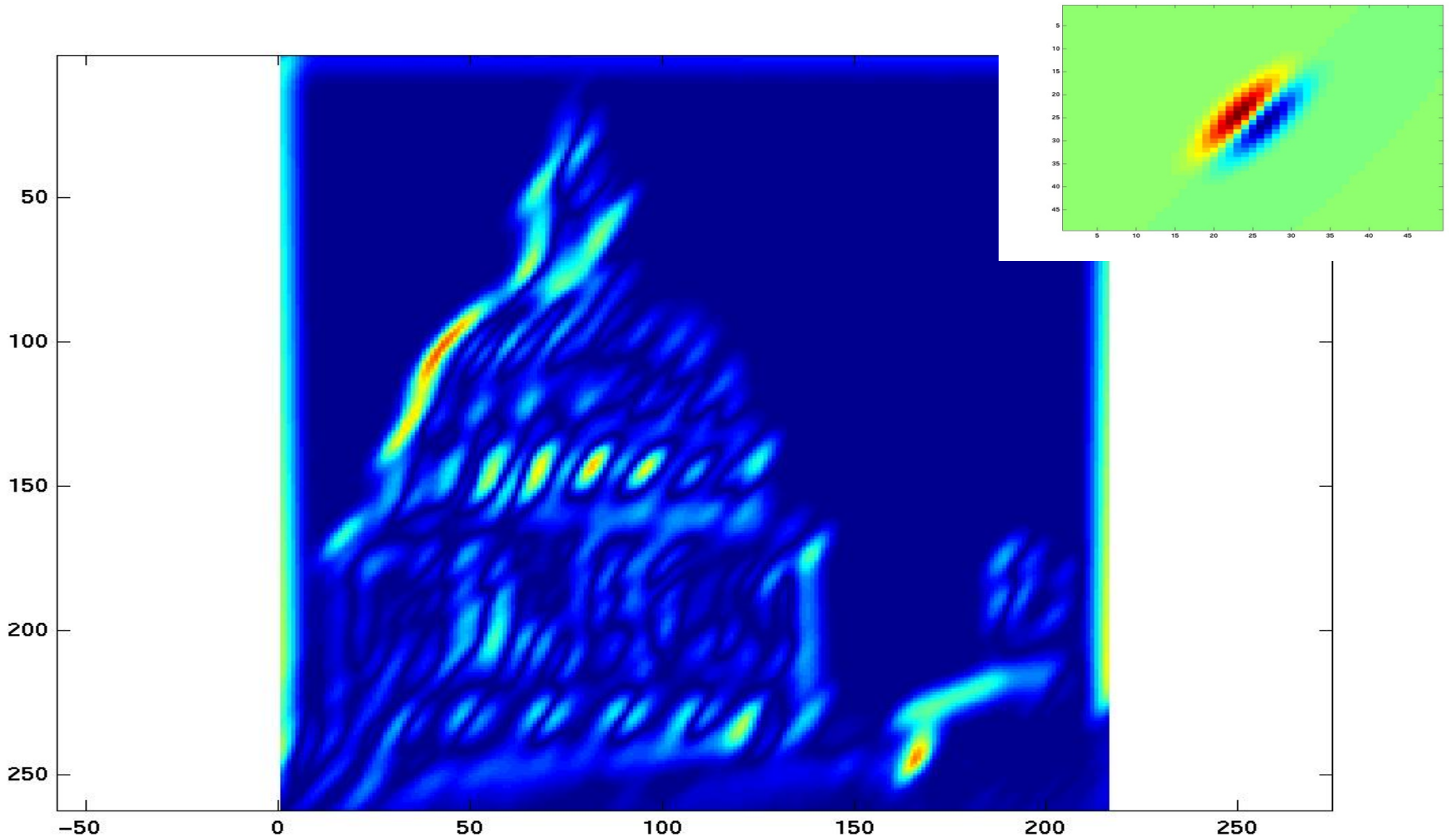


Kristen Grauman



Kristen Grauman



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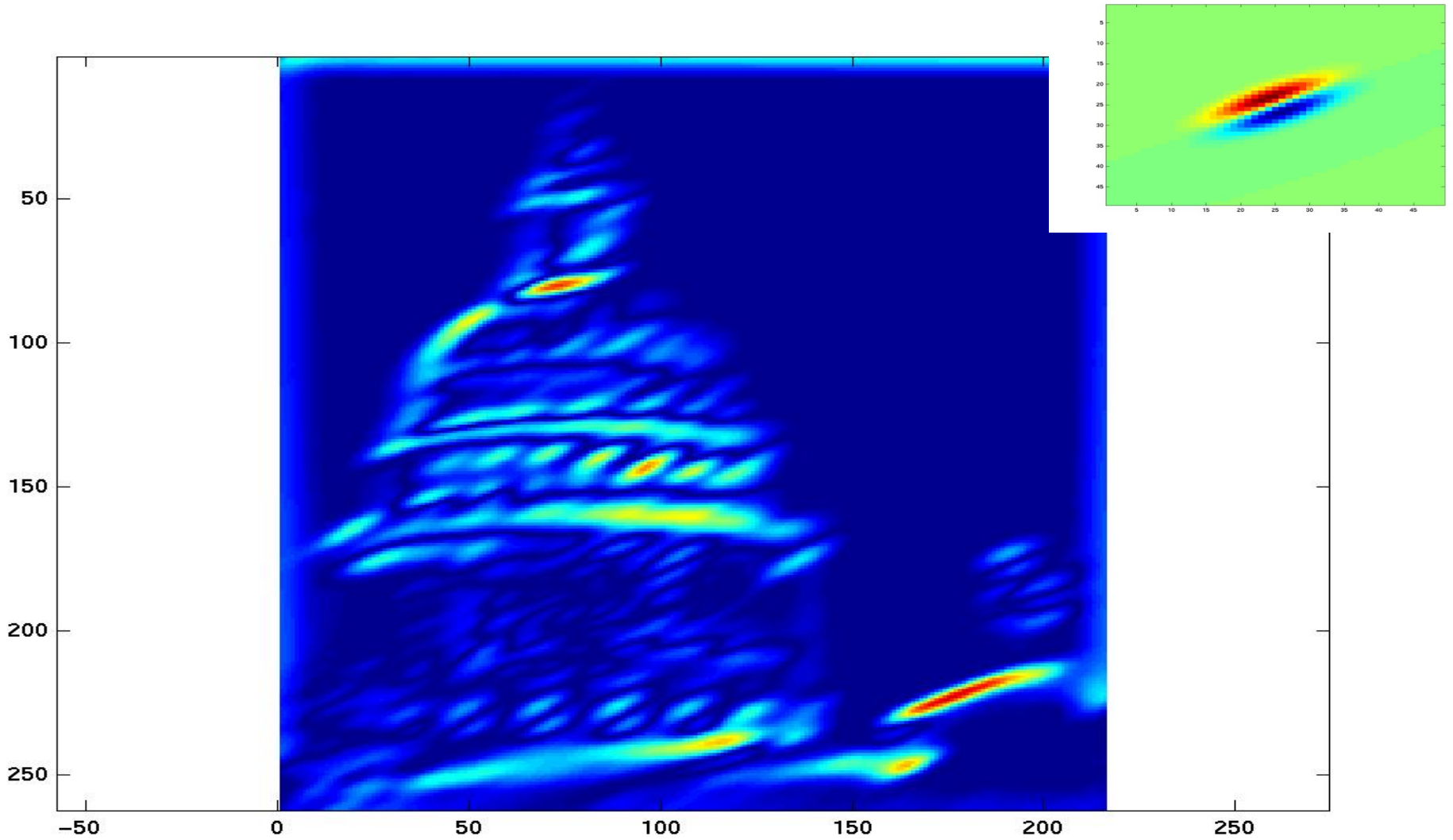

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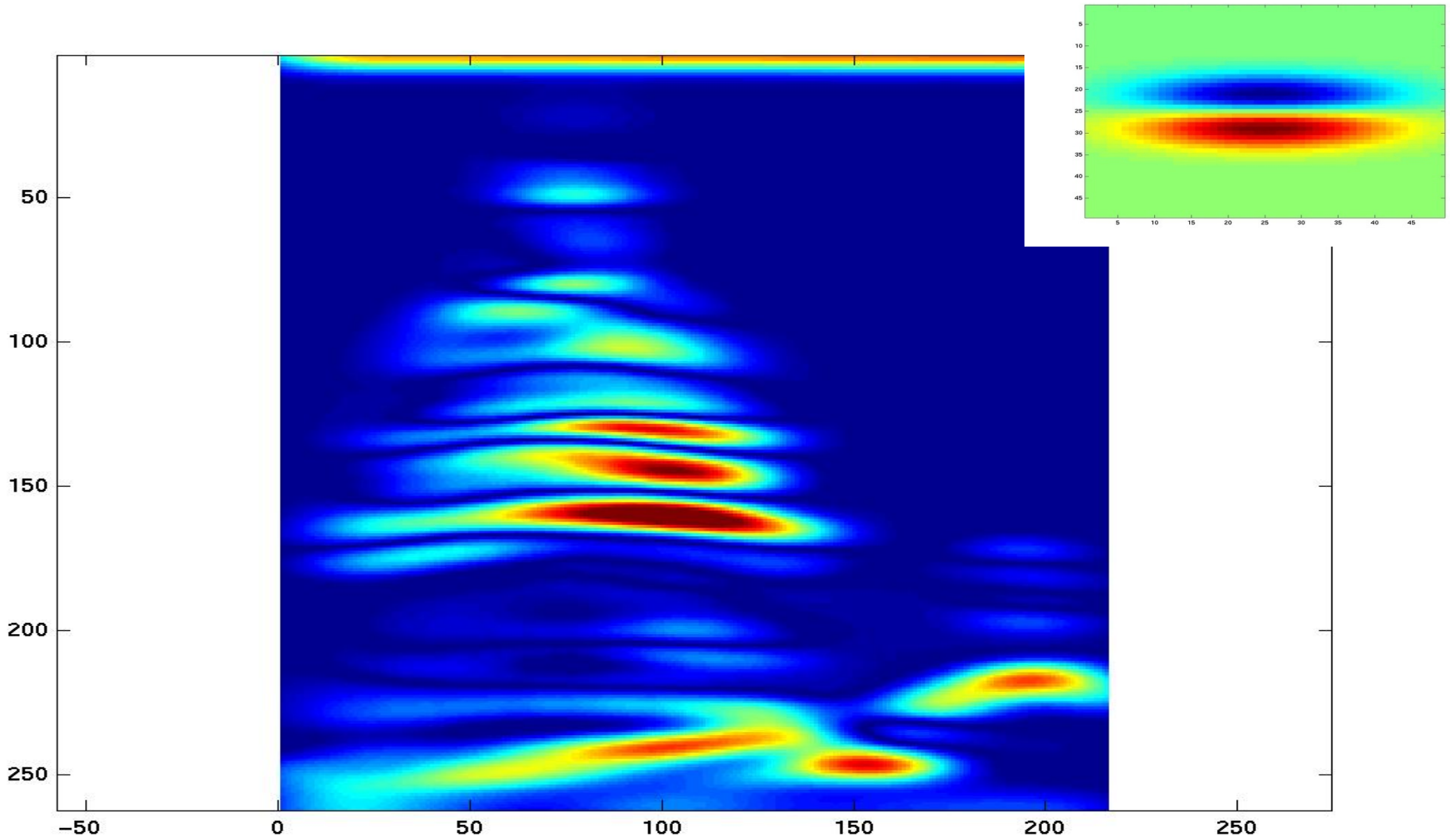
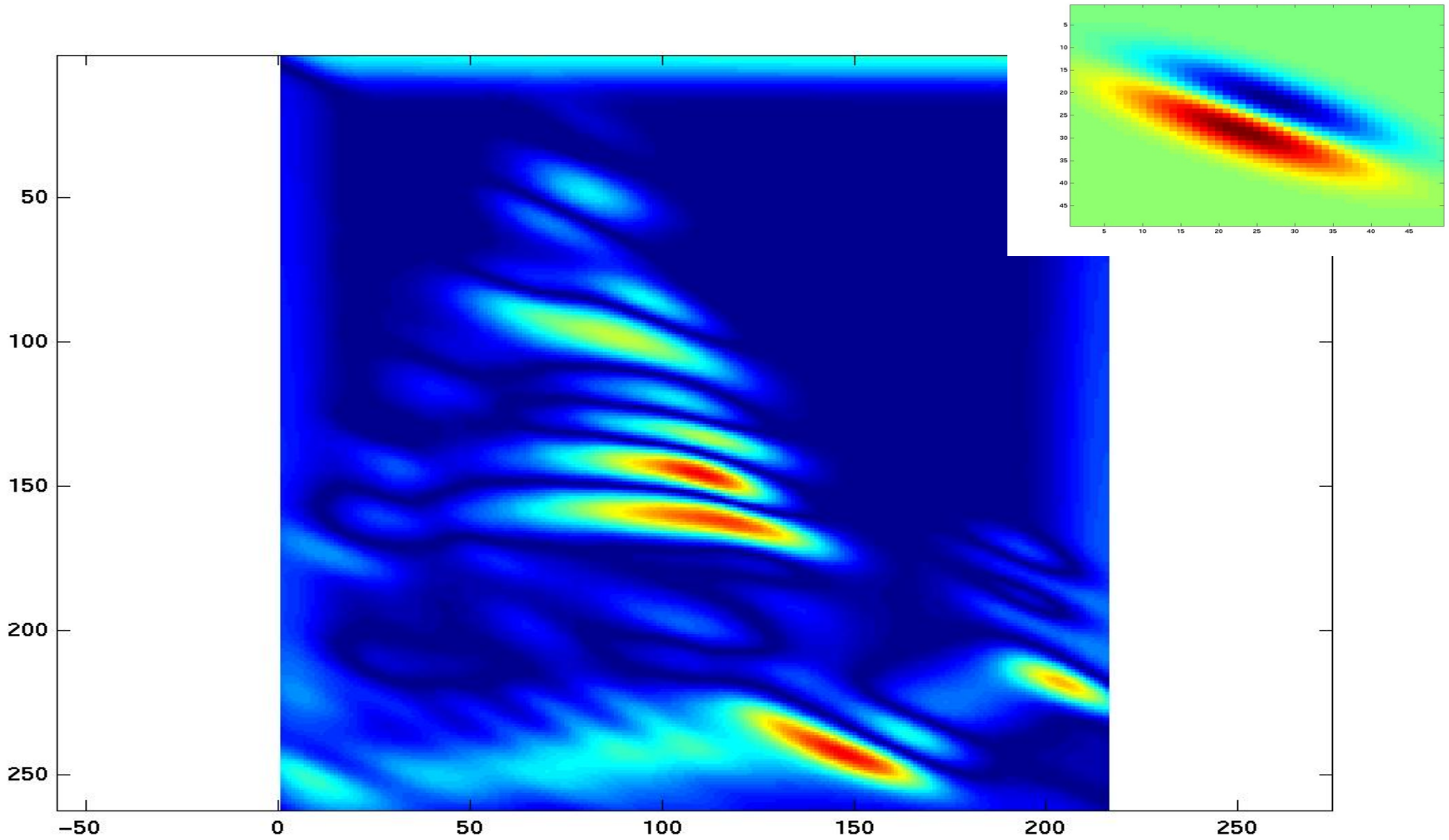
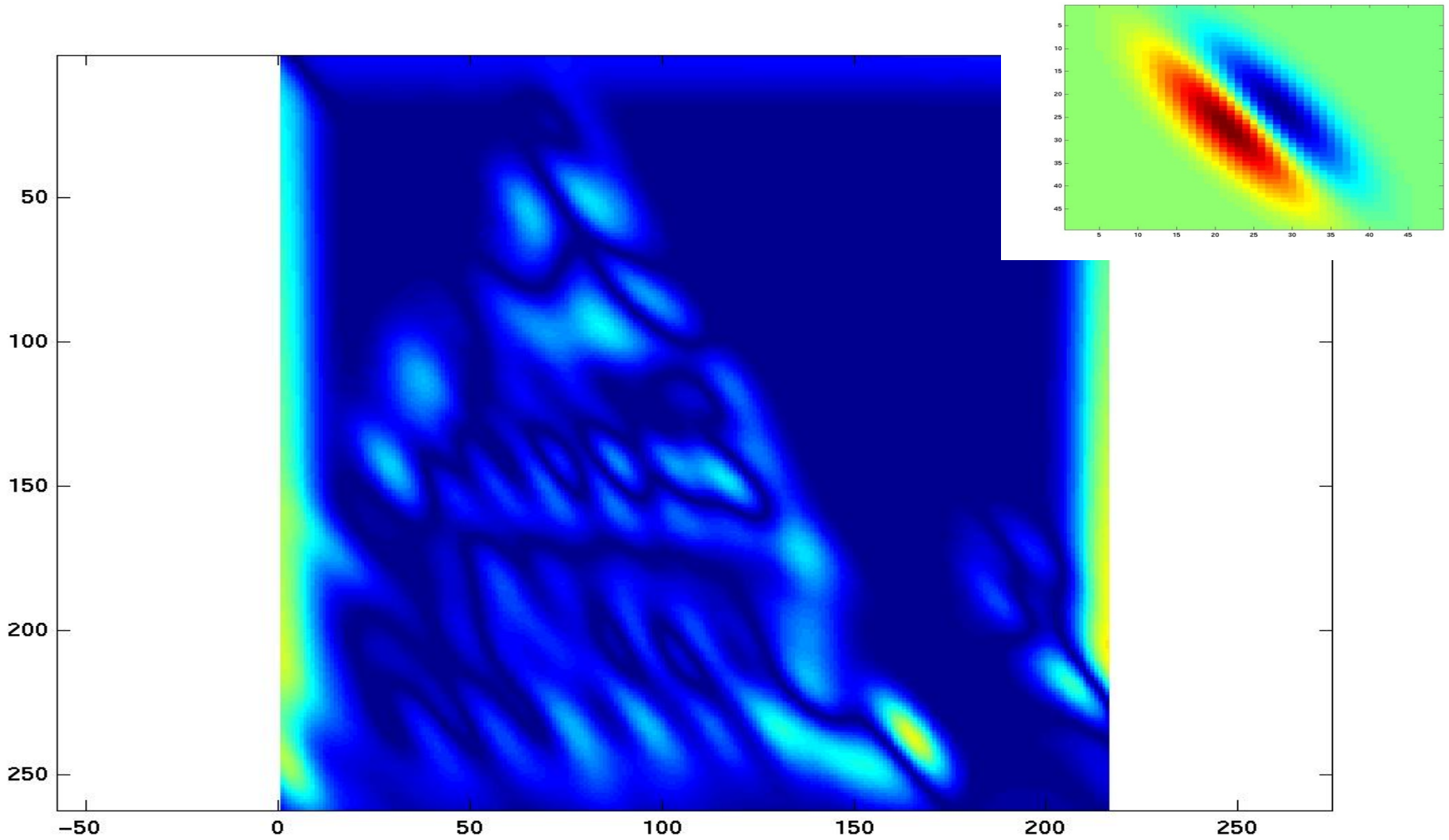


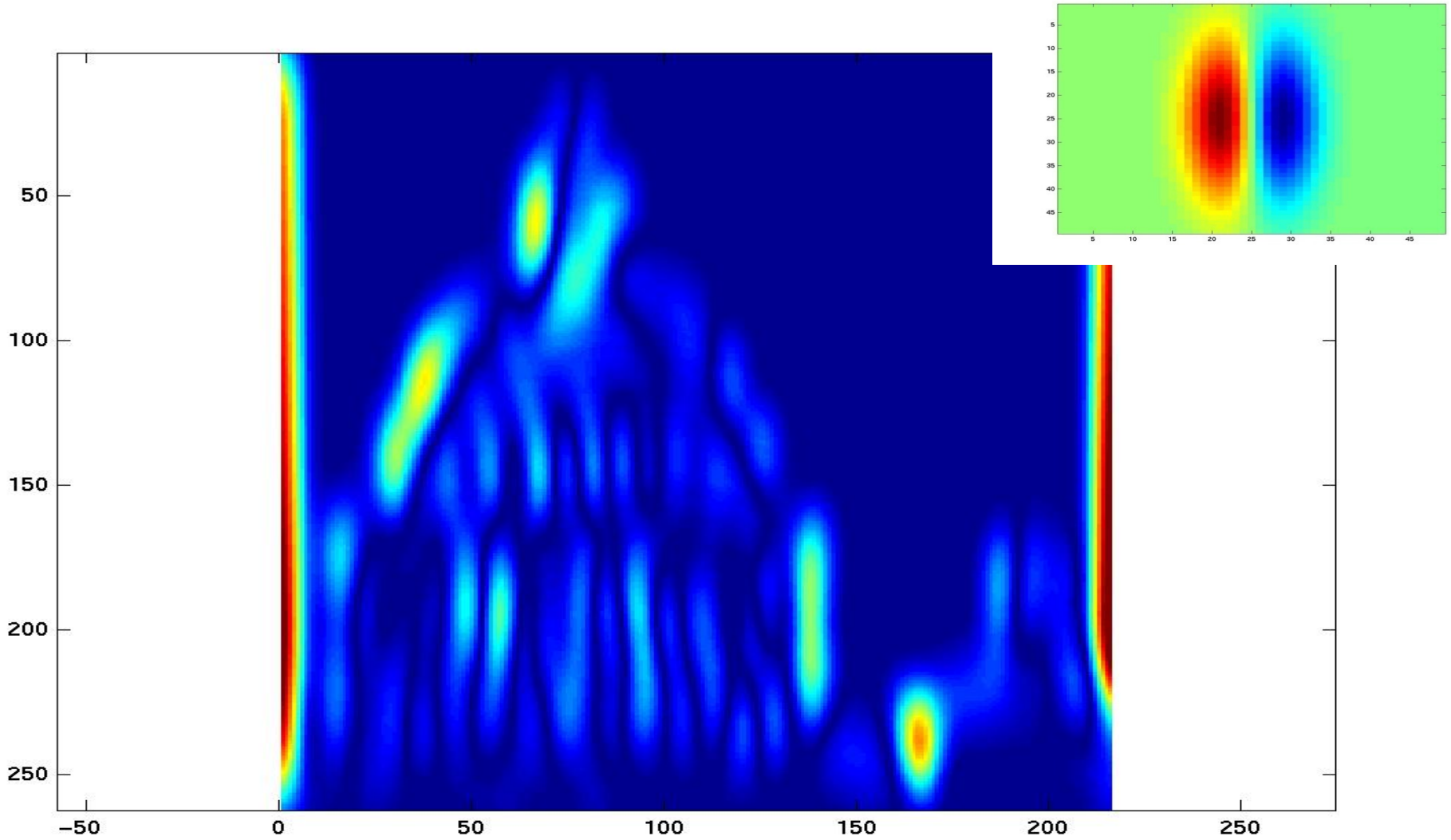
Kristen Grauman



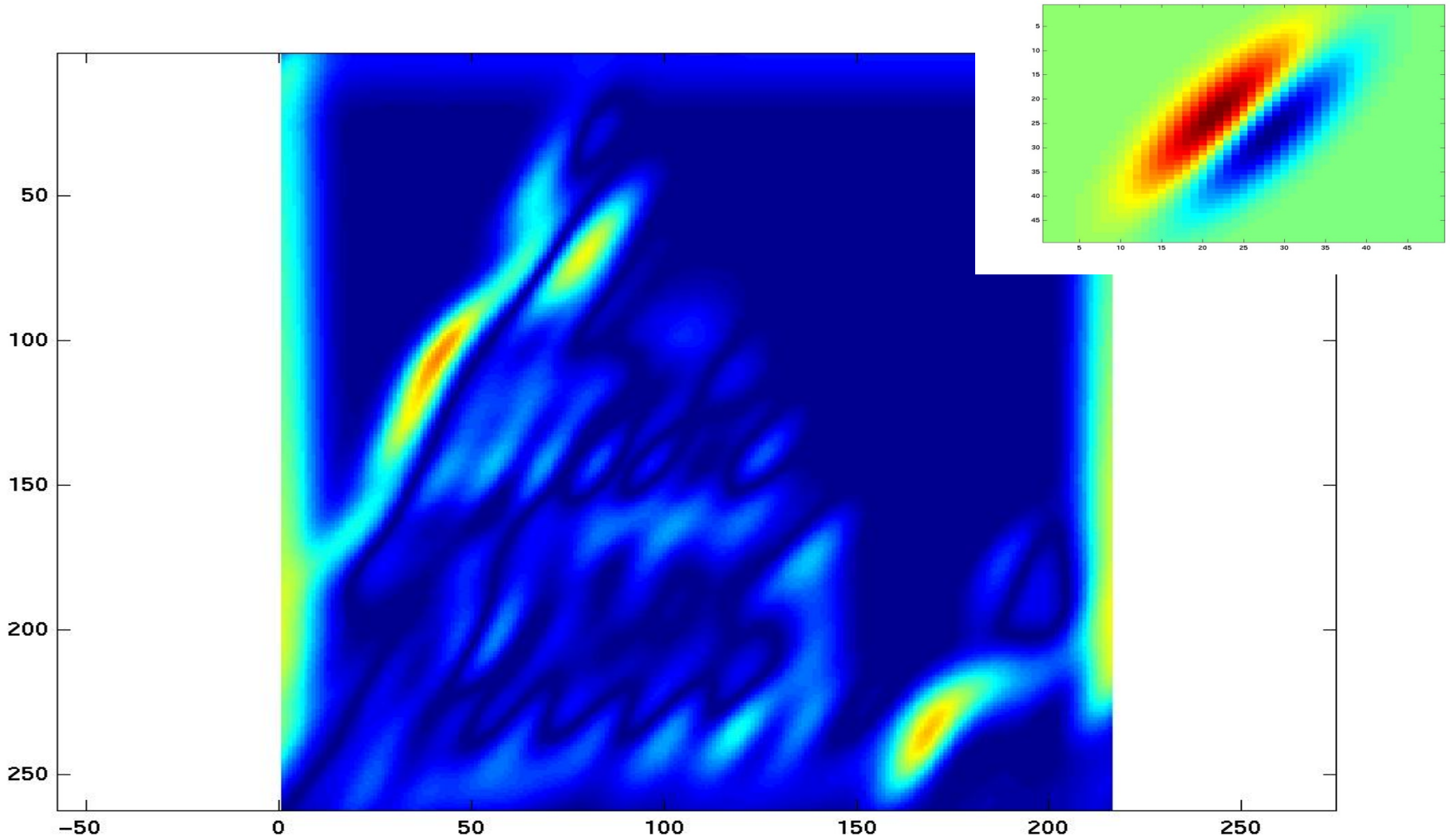
Kristen Grauman



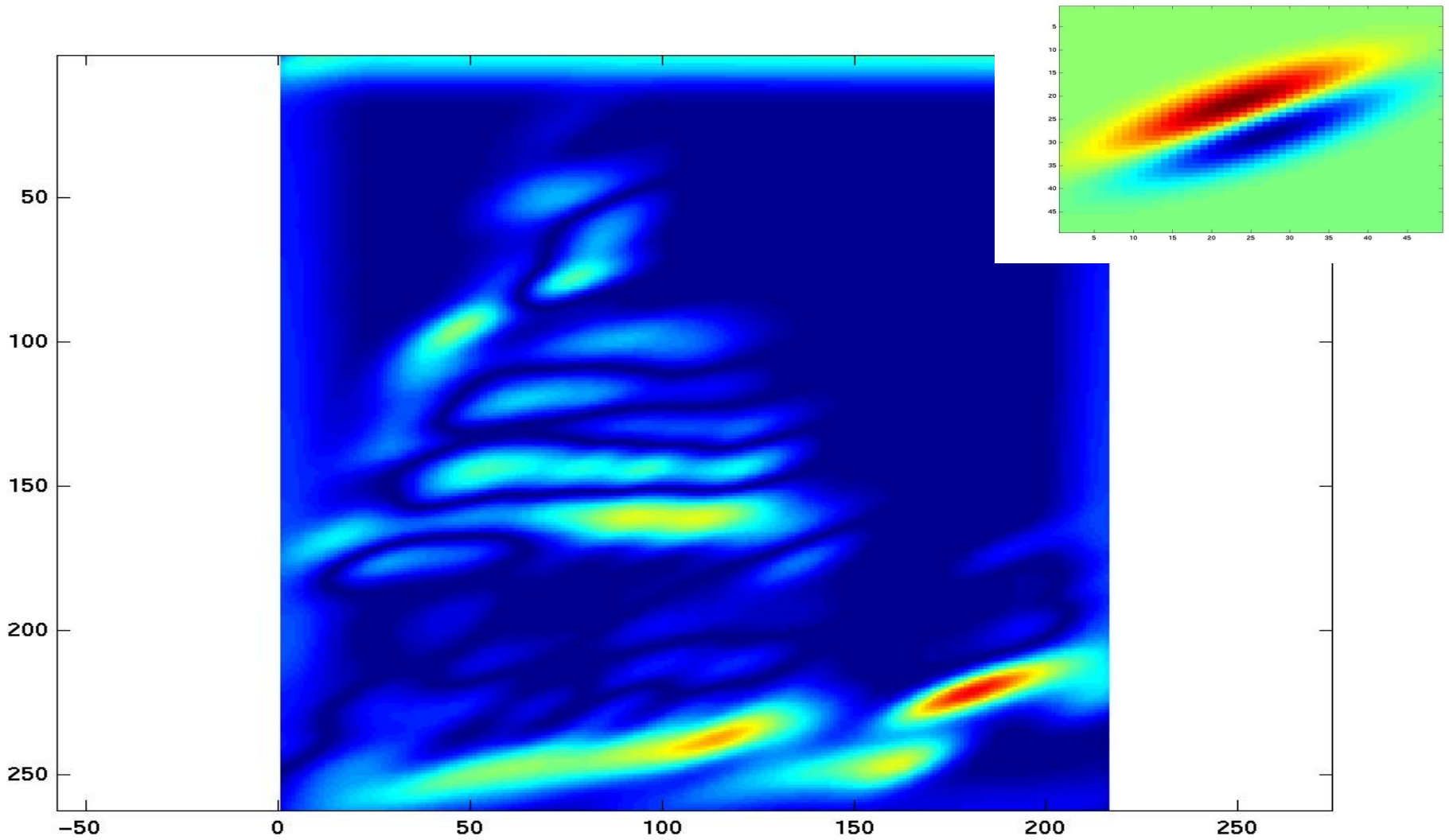
Kristen Grauman



Kristen Grauman



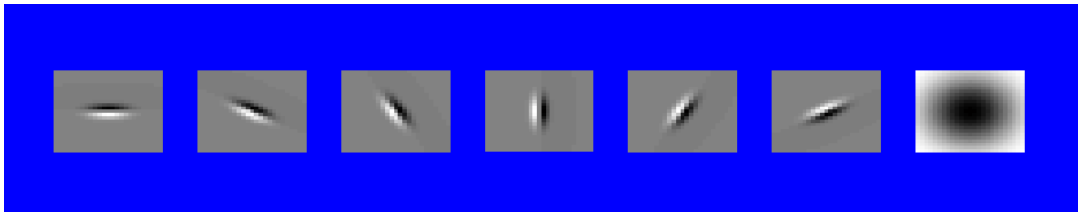
Kristen Grauman



Kristen Grauman

Can you match the texture to the response?

Filters



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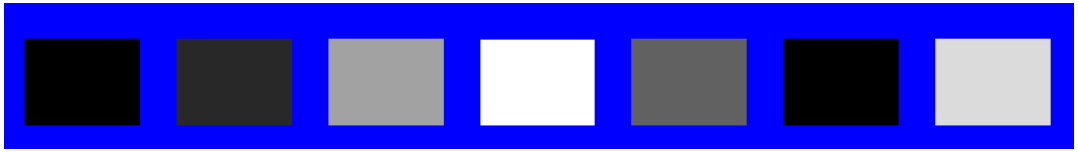
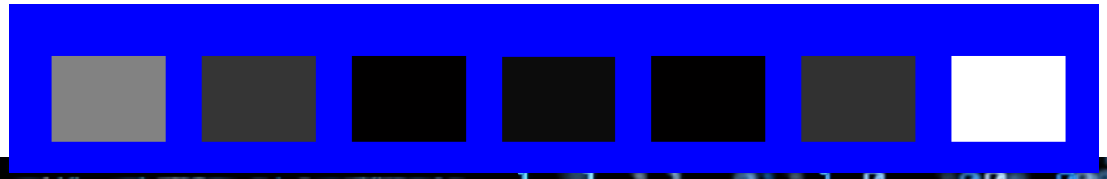
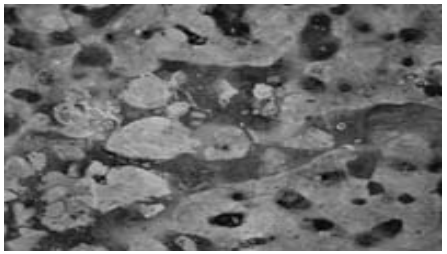


2



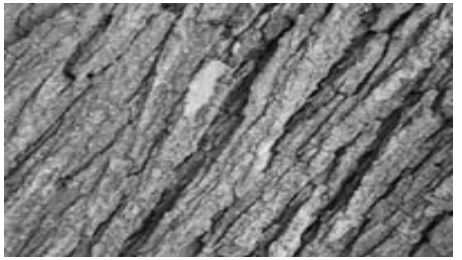
3

A



B

C



Data X

End of Section