<u>Part III</u>

Advanced Design and Analysis Techniques

§ Two Types of Techniques:

- Dynamic Programming
- Greedy Algorithms

- **Dynamic Programming**
  - subproblems *overlap* (→ i.e., subproblems are repeated)
  - problem solved by *combining solutions* to its subproblems
  - *saving solutions* to common subproblems *in table(s)* for efficiency
  - commonly applicable to optimization problems

- **Greedy Algorithms**
  - following locally optimal choice (instead of exhaustive search)
  - simpler and more efficient approach
  - work for wide ranges of problems
  - yield optimal solutions to certain problems, e.g., combinatorial structures - *matroids*

(where a matroid possesses linear independence, i.e., part solutions are all independent, accelerating combinatorial optimization)

Dynamic Programming (continued)

- **Four Steps Involved**
  - characterize structure of optimal solution **(check applicability,** i.e., if optimal substructures exist**)**
  - recursively define expression of optimal solution (**establish recursive formulation**)
  - compute and keep value of optimal solution (**keep subproblem solutions in bottom-up way**)
  - construct optimal solution from kept information

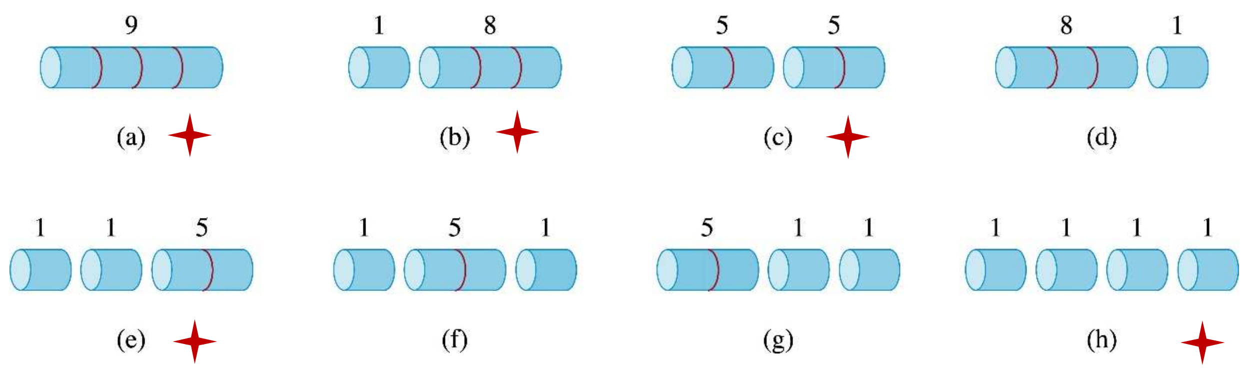
- **Example for Optimal Rod Cutting**
  - solve each subproblem *once*
  - *time-memory tradeoff*: additional memory involved for time complexity reduction
  - full solution in polynomial time if the number of *distinct subproblems* is polynomial
  - two approaches for dynamic programming implementation:
    - top-down with memorization †
    - bottom-up method, with *smallest* subproblems solved first and kept in table(s)

† Here, memoization is from "memo" and *not* from "memory."

<u>Dyanmic Programming</u> (continued)

- **Rod cutting problem**
  + rod with length n has $2^{n\text{-}1}$ ways (which equal leaf nodes of recursion tree) to cut
  + number of *distinct subproblems* far smaller, due to many identical subproblems
  + this problem exhibits *<u>optimal substructure</u>*, i.e., optimal solution consists of optimal solutions to related subproblems, which can be solved independently

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30



only five (5) distinct subproblems exist

Recursive top-down implementation

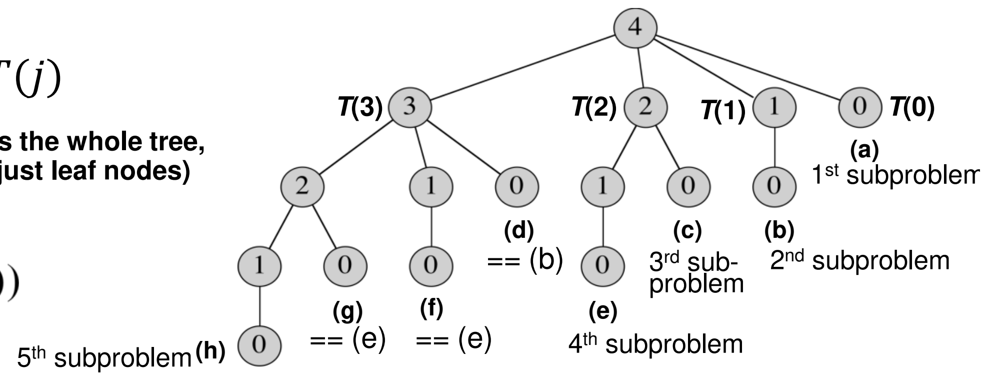
```
CUT-ROD(p, n)
  if n == 0
      return 0
  q = −∞
  for i = 1 to n
      q = max(q, p[i] + CUT-ROD(p, n − i))
  return q
```

root

$$T(n) = 1 + \sum_{j=0}^{n-1} T(j)$$

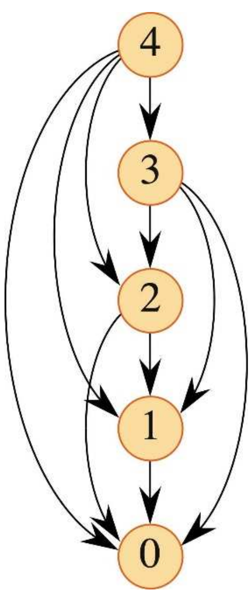
$$= \underline{2^n}$$ **(covers the whole tree, not just leaf nodes)**

Recursion tree showing recursive calls



Dynamic Programming (continued)

- **First three steps of typical dynamic programming**
  - – characterize structure of optimal solution (**applicability**)
  - – recursively define value of optimal solutions to subproblems (**recursion formulation**)
  - – compute & keep value of optimal solution to every subproblem (**keep subsolutions**)



Subproblem graph, being collapsed recursion tree

MEMOIZED-CUT-ROD(p, n)

```
let r[0..n] be a new array        // rely on this added array to keep earlier solutions
for i = 0 to n
    r[i] = −∞
return MEMOIZED-CUT-ROD-AUX(p, n, r)
```

Running time complexity: $\Theta(n^2)$

(as it equals n–1, n-2, …,1 for i =1, 2 …, obtainable from recursion tree in preceding page, to visit leftmost path for calculation + the level just below the root for references)

MEMOIZED-CUT-ROD-AUX(p, n, r)

```
if r[n] ≥ 0
    return r[n]
if n == 0
    q = 0
else q = −∞
    for i = 1 to n
        q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n − i, r))
r[n] = q
return q
```

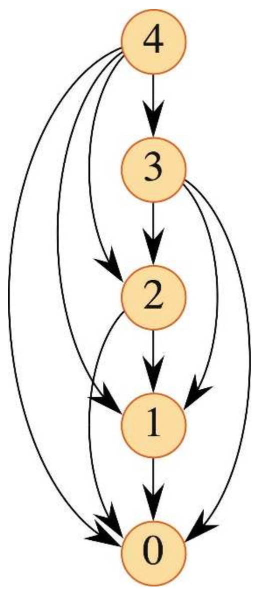
reference to subproblem solved before

top-down, same as divide-&-conquer method, but with memoization

Dynamic Programming (continued)

- Bottom-up method
  - same asymptotic complexity as the top-down method, with running time complexity of $\Theta(n^2)$
  - often with smaller constants, due to avoiding recursive calls (employed by top-down counterpart)

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30



Subproblem graph, being collapsed recursion tree

BOTTOM-UP-CUT-ROD(p, n)

```
let r[0..n] be a new array
r[0] = 0
for j = 1 to n
    q = −∞
    for i = 1 to j
        q = max(q, p[i] + r[j − i])
    r[j] = q
return r[n]
```

bottom-up, permitting table lookups

Dynamic Programming (continued)

- **Fourth step of typical dynamic programming**
  – reconstruct a solution to the problem
  – the solution of rod-cutting problem includes the choice of every cut (the cut listed in array ***s*** below), besides the optimal value (given in array ***r*** below)

EXTENDED-BOTTOM-UP-CUT-ROD(p, n)

```
let r[0..n] and s[0..n] be new arrays
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        if q < p[i] + r[j - i]
            q = p[i] + r[j - i]
            s[j] = i        // keep the cut location
    r[j] = q
return r and s
```

Solution of the problem printed by:

PRINT-CUT-ROD-SOLUTION(p, n)

```
(r, s) = EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
while n > 0
    print s[n]
    n = n - s[n]
```

‡ PRINT-CUT-ROD-SOLUTION(p, 3) prints "3" while
PRINT-CUT-ROD-SOLUTION(p, 8) prints "2" & "6"

EXTENDED-BOTTOM-UP-CUT-ROD(p, 8) returns ***r*** and ***s***, as follows:

i	0	1	2	3	4	5	6	7	8
$r[i]$	0	1	5	8	10	13	17	18	22
$s[i]$	0	1	2	3	2	2	6	1	2

Dynamic Programming (continued)

- **Elements of dynamic programming**
  - two key elements for optimization problems to apply dynamic programming:
    + optimal substructure
    + repeated subproblems
  - e.g., optimal parenthesization of matrix chain: $A_i \cdot A_{i+1} \cdot \ldots \cdot A_j$ consists of optimal solutions to split parenthesizing subproblems $A_i \cdot A_{i+1} \cdot \ldots \cdot A_k$ & $A_{k+1} \cdot A_{k+2} \cdot \ldots \cdot A_j$

Matrix-chain multiplication problem

Given a chain of n matrices $\langle A_1, A_2, \ldots, A_n \rangle$, find full parenthesization of the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$ that minimizes the number of scalar multiplications.

Let $\langle A_1, A_2, A_3 \rangle$ are of dimensions 10x100, 100x5, and 5x50.
The number of multiplications equals: $10 \cdot 100 \cdot 5 + 10 \cdot 5 \cdot 50 = 7500$ for $((A_1 \cdot A_2) \cdot A_3)$
and equals: $100 \cdot 5 \cdot 50 + 10 \cdot 100 \cdot 50 = 75000$ for $(A_1 \cdot (A_2 \cdot A_3))$

Dynamic Programming (continued)

- **Matrix-chain multiplication problem**
  - given <A_1, A_2, A_3, A_4>, we have ***five distinct ways*** for full parenthesization:

    $(A_1 \bullet (A_2 \bullet (A_3 \bullet A_4)))$
    $(A_1 \bullet ((A_2 \bullet A_3) \bullet A_4))$
    $((A_1 \bullet A_2) \bullet (A_3 \bullet A_4))$
    $((A_1 \bullet (A_2 \bullet A_3)) \bullet A_4)$
    $(((A_1 \bullet A_2) \bullet A_3) \bullet A_4)$

  - let $P(n)$ denote number of *alternative parenthesizations* of a sequence of n matrices and the two split subproducts be: $A_1 \bullet A_2 \bullet \ldots \bullet A_K$ and $A_{k+1} \bullet A_{k+2} \bullet \ldots \bullet A_n$, we have the recurrence below:

$$P(n) = \begin{cases} 1 & \text{if } n = 1 , \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2 . \end{cases}$$

    whose time complexity is $\Omega(2^n)$, exponential time complexity

    (proof by substitution, exercise problem on p. 381).

Dynamic Programming (continued)

- **Applying dynamic programming to matrix-chain multiplication**

  + four-step sequence:

    1. characterize structure of optimal parenthesization (**applicability**)
    2. establish recursive solution approach (**recursive formulation**)
    3. compute optimal costs (**memorization or table storage**) for subproblems
    4. construct an optimal solution for the problem

Recursive solution approach

– let $m[i, j]$ be the minimum number of scalar multiplications for matrix sequence $A_{i..j}$ with two split subsequences $A_i \bullet A_{i+1} \bullet \ldots \bullet A_K$ & $A_{k+1} \bullet A_{k+2} \bullet \ldots \bullet A_j$. We have:

$$m[i, j] = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j \quad \text{for a given } k$$

– consider every possible k, $i \leq k \leq j$-1, we have

$$m[i, j] = \begin{cases} 0 & \text{if } i = j\ , \\ \min\limits_{i \leq k < j} \{m[i, k] + m[k+1, j] + p_{i-1} p_k p_j\} & \text{if } i < j\ . \end{cases}$$

– finally, we define $s[i, j]$ to be a value of k at which we split the product $A_i A_{i+1} \cdots A_j$ in an optimal parenthesization.

Dynamic Programming (continued)

- Recursive solution approach (without avoiding repeated subproblem computation)
  - from recurrence below directly leads to ***exponential complexity***:

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \,, \\ \min\limits_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1} p_k p_j\} & \text{if } i < j \,. \end{cases}$$

Time complexity of Line 5 (for i = 1 and j = n):

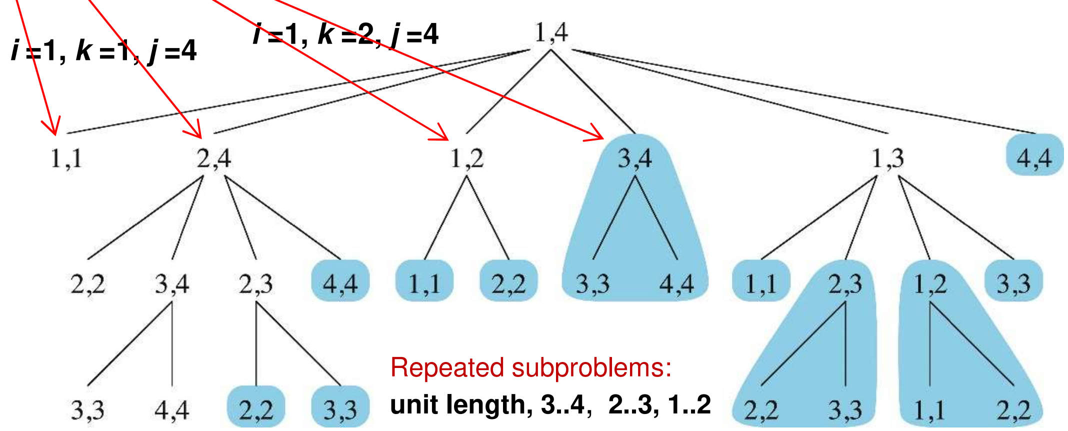
$$T(n) \ge 1 + \sum_{k=1}^{n-1} (T(k) + T(n-k) + 1)$$

$$\ge 2^{n-1}$$ (but only C(n, 2)+n distinct subproblems)

RECURSIVE-MATRIX-CHAIN(p, i, j)

1 **if** $i == j$
2   **return** 0
3 $m[i,j] = \infty$
4 **for** $k = i$ **to** $j-1$
5     $q =$ RECURSIVE-MATRIX-CHAIN(p, i, k)
        $+$ RECURSIVE-MATRIX-CHAIN$(p, k+1, j)$
        $+ p_{i-1} p_k p_j$
6   **if** $q < m[i,j]$
7         $m[i,j] = q$
8 **return** $m[i,j]$

// recursive calls, following divide-&-conquer alone
// without properly memorizing results in $m[i, j]$ from shortest chains upward



Dynamic Programming (continued)

- **Applying dynamic programming to matrix-chain multiplication**

Computing optimal costs

– relatively ***few distinct*** subproblems (i.e., sets of subproblems are overlapping)
– two auxiliary tables: *m*[1.. n, 1.. n] to store the *m*[*i*, *j*] cost and *s*[1.. n-1, 2 .. n] to store *k*, *i* ≤ *k* ≤ *j*-1, at which a split yields the lowest cost in computing *m*[*i*, *j*]

There are $\Theta(n^2)$ *distinct* subproblems, to yield time complexity of $\Theta(n^3)$.

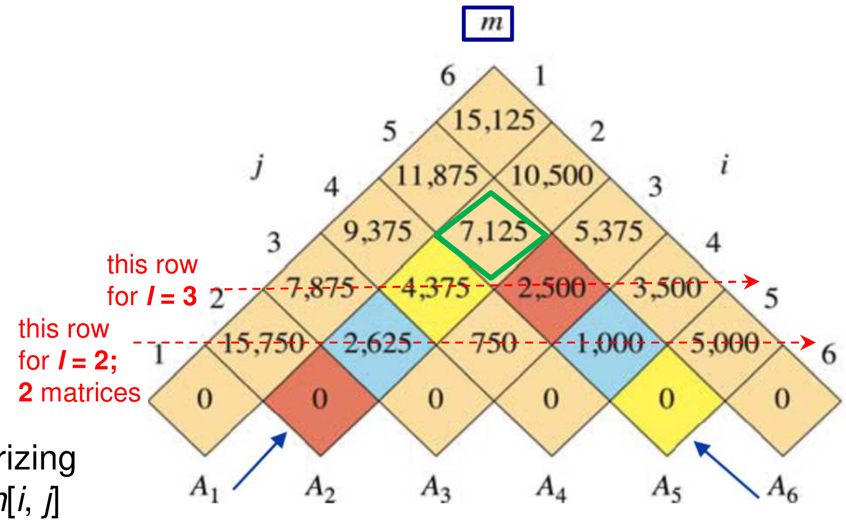
```
MATRIX-CHAIN-ORDER(p)
n = p.length − 1
let m[1..n, 1..n] and s[1..n − 1, 2..n] be new tables
for i = 1 to n
    m[i, i] = 0                       // nil initial cost for unit length stored in m[i, i]
for l = 2 to n                        // l is the chain length, for memorizing results in m[i, j] from shortest chains upward
    for i = 1 to n − l + 1
        j = i + l − 1                 // range from element i to element j for the chain length of l
        m[i, j] = ∞
        for k = i to j − 1
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q           // memorization
                s[i, j] = k
return m and s
```

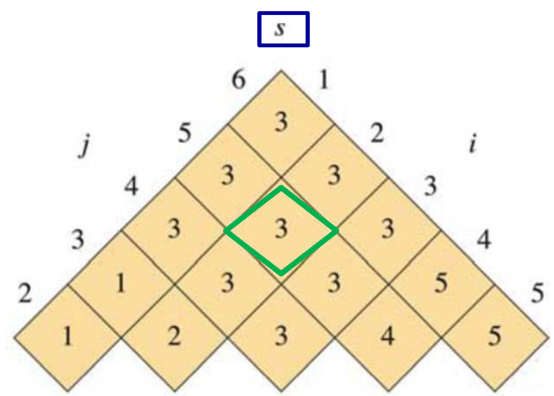
Dynamic Programming (continued)

Computing optimal costs

MATRIX-CHAIN-ORDER(p) Time complexity = $\Theta(n^3)$

```
n = p.length − 1
let m[1..n, 1..n] and s[1..n − 1, 2..n] be new tables
for i = 1 to n
    m[i, i] = 0
for l = 2 to n                    // l is the chain length, for memorizing results in m[i, j]
    for i = 1 to n − l + 1
        j = i + l − 1             // range from element i to element j
        m[i, j] = ∞
        for k = i to j − 1
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q
                s[i, j] = k
return m and s
```





matrix	A_1	A_2	A_3	A_4	A_5	A_6
dimension	30×35	35×15	15×5	5×10	10×20	20×25

m[1, 6] = 15,125

its computation relies on *m*[1, 1], *m*[2, 6]; *m*[1, 2], *m*[3, 6]; *m*[1, 3], *m*[4, 6]; *m*[1, 4], *m*[5, 6]; *m*[1, 5], *m*[6, 6].

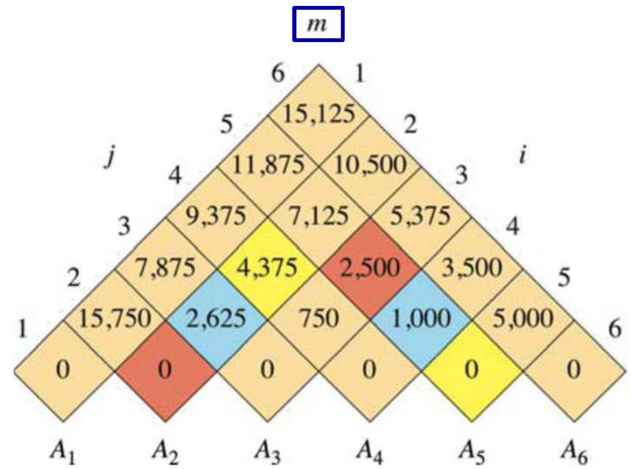
$$m[2,5] = \min \begin{cases} m[2,2] + \boxed{m[3,5]} + p_1 p_2 p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13{,}000\,, \\ m[2,3] + \boxed{m[4,5]} + p_1 p_3 p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125\,, \\ m[2,4] + \boxed{m[5,5]} + p_1 p_4 p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11{,}375 \end{cases}$$

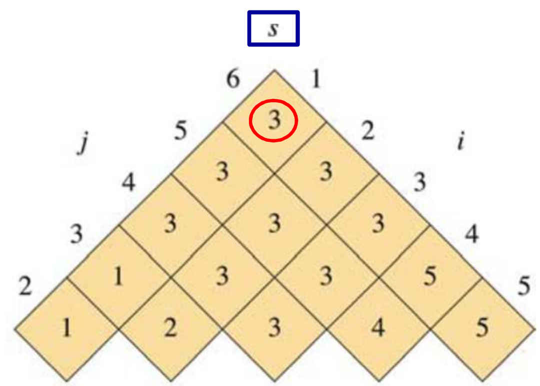
$$= 7125\,.$$

Dynamic Programming (continued)

Constructing an optimal solution

– Entry $s(i, j)$ keeps k where optimal parenthesization splits between A_k and A_{k+1}





PRINT-OPTIMAL-PARENS(s, i, j)

```
if i == j
    print "A"_i
else print "("
    PRINT-OPTIMAL-PARENS(s, i, s[i, j])
    PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
    print ")"
```

PRINT-OPTIMAL-PARENS$(s, 1, 6)$ prints the parenthesization $((A_1(A_2A_3))((A_4A_5)A_6))$.

PRINT-OPTIMAL-PARENS(s, 1, s[1, 6] = 3)

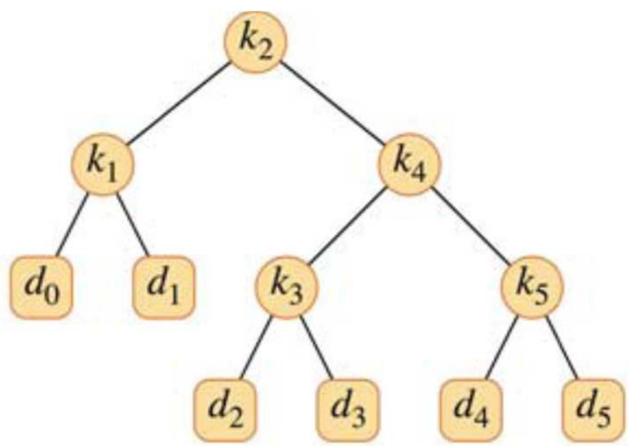
PRINT-OPTIMAL-PARENS(s, s[1, 6]+1=4, 6)

Dynamic Programming (continued)

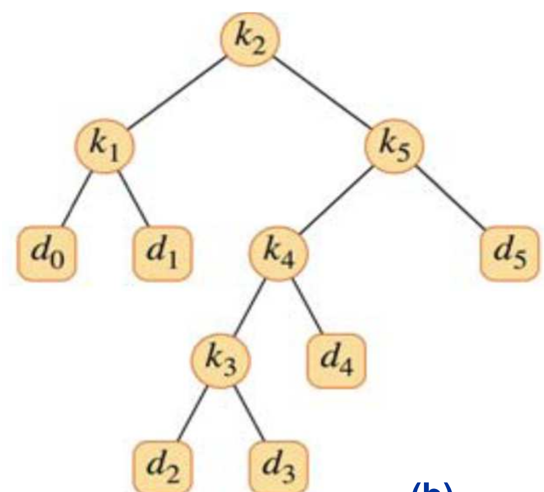
- **Optimal binary search trees (O-BSTs)**
  - given a set of sorted keys $K = <k_1, k_2, \ldots k_n>$, with p_i being search visit probability of k_i, build a binary search tree with the *mean search cost* minimized
  - $n+1$ "dummy keys (ranges)" (i.e., d_j) required to cover those value ranges outside K

contribution = $(\text{depth}_T(k_i)+1) \cdot p_i$

node	depth	probability	contribution
k_1	1	0.15	0.30
k_2	0	0.10	0.10
k_3	2	0.05	0.15
k_4	1	0.10	0.20
k_5	2	0.20	0.60
d_0	2	0.05	0.15
d_1	2	0.10	0.30
d_2	3	0.05	0.20
d_3	3	0.05	0.20
d_4	3	0.05	0.20
d_5	3	0.10	0.40
Total			2.80



(a)



(b)

Figure 15.9 Two binary search trees for a set of $n = 5$ keys with the following probabilities:

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

$$\sum_{i=1}^{n} p_i + \sum_{i=0}^{n} q_i = 1$$

(a) A binary search tree with expected search cost 2.80. **(b)** A binary search tree with expected search cost 2.75. This tree is optimal.

Dynamic Programming (continued)

- **Optimal binary search trees (O-BSTs)**
  - number of binary trees with n nodes: $\Omega(n^4/n^{3/2})$ // see Prob. 12-4, pp. 329-330
  - optimal solution to the problem obtainable from optimal solutions to its subproblems: root k_r has left subtree, k_i, k_{i+1}, … k_{r-1}, and right subtree, k_{r+1}, k_{r+2}, …, k_j
  - recursive solution: let $e[i, j]$ denote mean cost of searching O-BST with k_i, k_{i+1}, …, k_j

    then, $e[i,j] = p_r + (e[i, r-1] + \underbrace{w(i, r-1)}_{\text{due to tree height raised by 1}}) + (e[r+1, j] + \underbrace{w(r+1, j)}_{\text{due to tree height raised by 1}})$, with k_r the tree root,

    where $w(i, j)$ is *expected search prob.* over the tree, equal to $w(i,j) = \sum_{l=i}^{j} p_l + \sum_{l=i-1}^{j} q_l$

    We have final $e[i, j]$ as follows, due to $w(i,j) = w(i, r-1) + p_r + w(r+1, j)$:

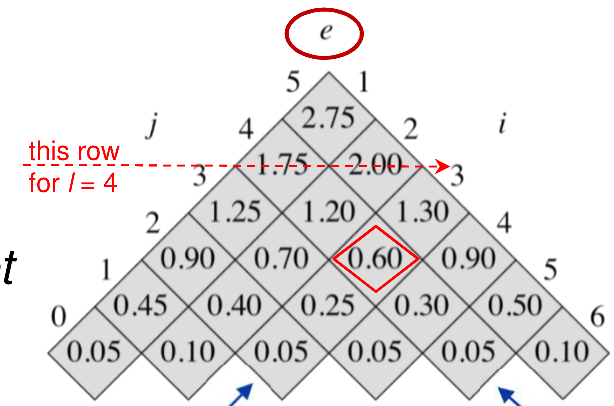
$$e[i,j] = \begin{cases} q_{i-1} & \text{if } j = i-1\,, \\ \min\limits_{i \le r \le j} \{e[i, r-1] + e[r+1, j] + \underbrace{w(i,j)}_{\text{due to tree height raised by 1}}\} & \text{if } i \le j\,. \end{cases}$$

There are relatively fewer distinct subproblems (i.e., O-BST involving k_i, k_{i+1}, … k_j):

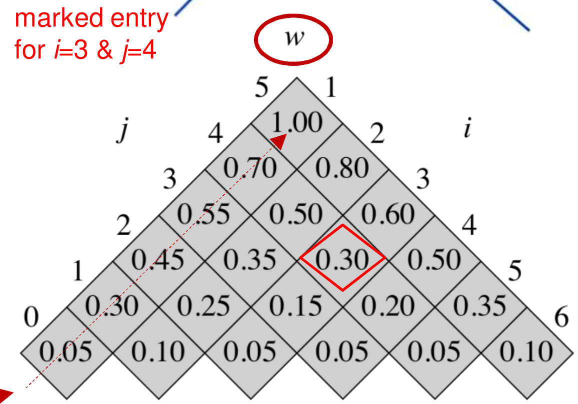
$$\underbrace{C(n, 2)}_{\text{selecting } i \,\&\, j \text{ out of } n \text{ values}} + \underbrace{n}_{\text{due to } i = j} = \Theta(n^2)$$

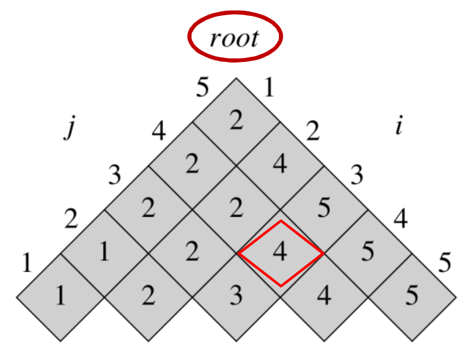
Dynamic Programming (continued)

- **Optimal binary search trees (O-BSTs)**
  - compute the expected search cost of an O-BST
  - we store the $e[i, j]$ values in a table $e[1..n+1, 0..n]$.
  - keep optimal BST for key subset of k_i, k_{i+1}, …, k_j in *root*



marked entry
for i=3 & j=4





OPTIMAL-BST(p, q, n)

```
let e[1..n+1, 0..n], w[1..n+1, 0..n],
        and root[1..n, 1..n] be new tables
for i = 1 to n+1
    e[i, i-1] = q_{i-1}
    w[i, i-1] = q_{i-1}
for l = 1 to n
    for i = 1 to n-l+1
        j = i+l-1
        e[i, j] = ∞
        w[i, j] = w[i, j-1] + p_j + q_j
        for r = i to j          // search subset of k_i .. k_j with l consecutive keys
            t = e[i, r-1] + e[r+1, j] + w[i, j]
            if t < e[i, j]
                e[i, j] = t
                root[i, j] = r
return e and root
```

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

Note. p_j & q_j

First, build w based on this equation, ~~one reverse~~ ~~diagonal~~ column then next

Resulting O-BST obtains from the *root* table.

Greedy Algorithms

- **Basics**
  - – make locally optimal choice, so only a single subproblem is solved in each step
  - – simpler and more efficient
  - – powerful and applicable to wide ranges of problems

An Example

Activity selection problem. To find a maximal subset of *compatible activities* from a set of activities, each with a starting time and an ending time.

Given the following set S of activities, **sorted according to their finish times**

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

Compatible activities, a_k and a_l, satisfy $f_k \le s_l$ or $s_k \ge f_l$.

Largest compatible sets: $\{a_1, a_4, a_8, a_{11}\}$ and $\{a_2, a_4, a_9, a_{11}\}$.

This cannot be identified via a greedy algorithm.

Greedy Algorithms (continued)

Given a set of activities, *S*, sorted according to their finish times, i.e.,

$f_1 \leq f_2 \leq f_3 \leq \cdots \leq f_{n-1} \leq f_n$

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

Let S_{ij} denote the set of activities that start after a_i finishes and that finish before a_j starts. Find out a maximum set of compatible activities in S_{ij}, with such a set represented by A_{ij}.

If A_{ij} contains a_k, then the solution is left with two subproblems: maximum set of compatible activities in S_{ik} and the maximum set of compatible activities in S_{kj}, namely, $A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$.

Let the size of an optimal solution for S_{ij} be denoted by $c[i, j]$, with a_k in the optimal solution set. We have the recurrence of

$$c[i, j] = c[i, k] + c[k, j] + 1.$$

Consider every possible a_k, we have:

$$c[i, j] = \begin{cases} 0 & \text{if } S_{ij} = \emptyset , \\ \max\limits_{a_k \in S_{ij}} \{c[i, k] + c[k, j] + 1\} & \text{if } S_{ij} \neq \emptyset . \end{cases}$$

Note: S_i denotes the subset of activities $\{a_{i+1}, a_{i+2}, \ldots , a_n\}$ that start after a_i finishes.

One may adopt dynamic programming to solve the above recurrence (either top-down with memoization or bottom-up with auxiliary table(s)), or preferably to solve *just one* desirable subproblem (instead of *all*) in each step via a greedy choice, called the greedy approach.

Greedy Algorithms (continued)

- **Greedy Choice**
  - activity selection problem chooses the activity which ***finishes the earliest***
  - typically, one follows top-down manner to choose an activity for adding to the optimal solution, then finds maximal compatible activities for left subproblem
  - the greedily chosen activity always part of some optimal solution, as follows:

Theorem 1. This refers to the greedy choice property.

If S_k is nonempty and a_m has the earliest finish time in S_k, then a_m is included in some optimal solution.

Unlike S_{kv}, S_k denotes the subset of activities $\{a_{k+1}, a_{k+2}, \ldots, a_n\}$ that start **after a_k finishes.**

Recursive greedy algorithm

With greedy selection of a_m, only one subproblem (S_{m+1}) is left after selecting a_m.

REC-ACTIVITY-SELECTOR(s, f, k, n)

```
m = k + 1
while m ≤ n and s[m] < f[k]      // find the first activity in S_k to finish
    m = m + 1                     after a_k is added to the solution (i.e., compatible with a_k).
if m ≤ n
    return {a_m} ∪ REC-ACTIVITY-SELECTOR(s, f, m, n)
else return Ø
```

Greedy Algorithms (continued)

- **Recursive greedy algorithm**

Idea

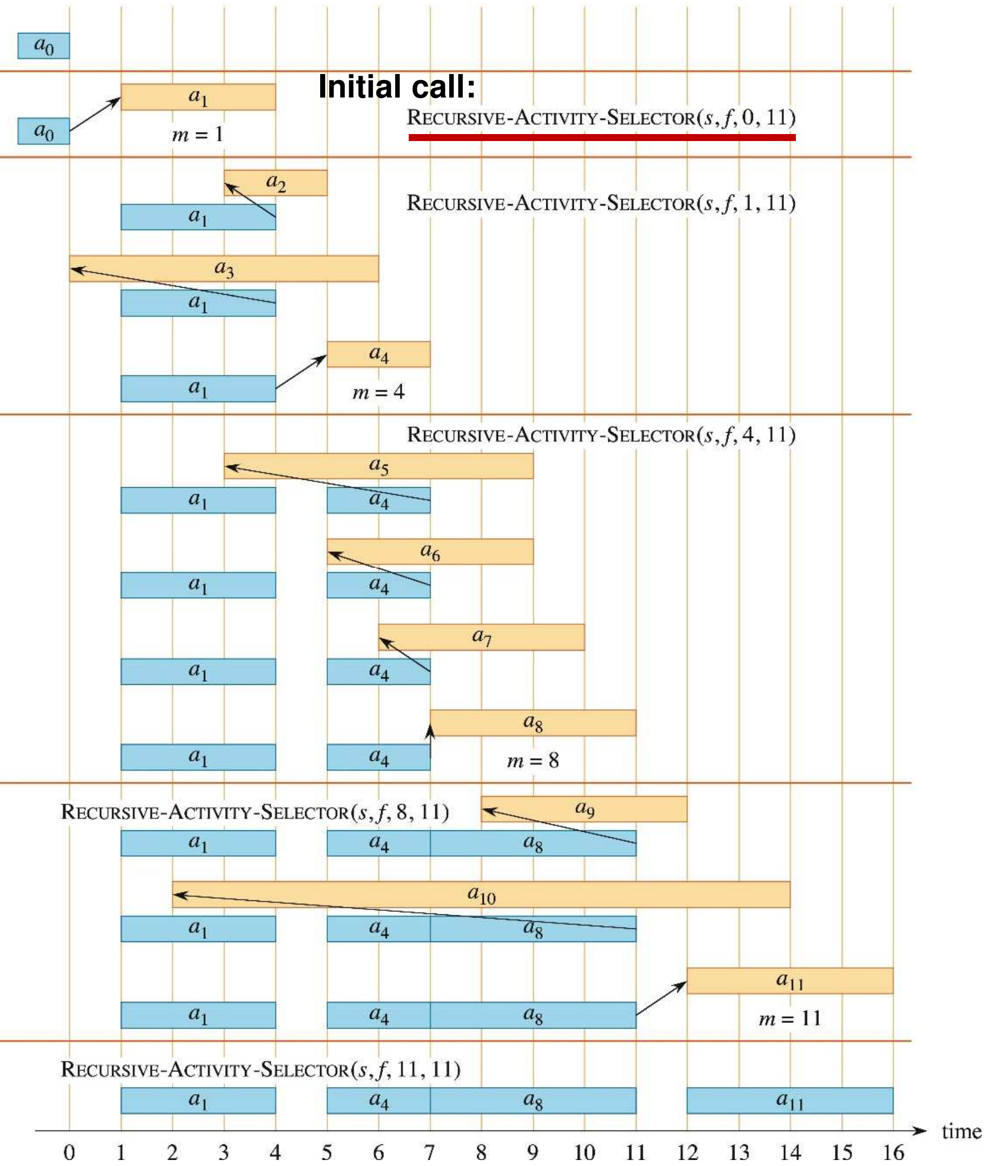
The **while** loop checks $a_{k+1}, a_{k+2}, \ldots, a_n$ until it finds an activity a_m that is compatible with a_k (need $s_m \geq f_k$).

- If the loop terminates because a_m is found ($m \leq n$), then recursively solve S_m,
- If the loop never finds a compatible a_m ($m > n$), then just return empty set.

Time

$\Theta(n)$—each activity examined exactly once, assuming that activities are already sorted according to finish times.

i	s_i	f_i
0	–	0
1	1	4
2	3	5
3	0	6
4	5	7
5	3	9
6	5	9
7	6	10
8	7	11
9	8	12
10	2	14
11	12	16



Greedy Algorithms (continued)

- **Iterative version of greedy algorithm**

GREEDY-ACTIVITY-SELECTOR(s, f)

```
n = s.length
A = {a_1}
k = 1
for m = 2 to n
    if s[m] ≥ f[k]
        A = A ∪ {a_m}        // Activity a_m being most recently added to A.
        k = m
return A
```

Greedy Algorithms (continued)

- **Elements of greedy strategy to yield *global optimum***
  - a sequence of choices, each of which is best at the moment
  - key elements: greedy-choice property and optimal substructure
  - greedy algorithm with three steps listed below, to yield optimal solutions (for matriods, i.e., linear independence)

1. Cast the optimization problem as one in which we make a choice and are left with one subproblem to solve.
2. Prove that there's always an optimal solution that makes the greedy choice, so that the greedy choice is always safe.
3. Demonstrate optimal substructure by showing that, having made the greedy choice, combining an optimal solution to the remaining subproblem with the greedy choice gives an optimal solution to the original problem (with **linear independence**)

Greedy strategy typically will

- Make a choice at each step.
- Make the choice *before* solving the subproblems.
- Solve *top-down*.

Greedy Algorithms (continued)

- **Huffman codes**
  - for efficient data compression using variable code length
  - fixed-length code is simple but larger in its encoded footprint
  - prefix codes: no codeword is a prefix of any other codeword

Example

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

← totally, 100K

Fixed length: 3 bits per symbol type or 300,000 bits for 100K symbols

Variable length: $(45 \cdot 1 + 13 \cdot 3 + 12 \cdot 3 + 16 \cdot 3 + 9 \cdot 4 + 5 \cdot 4) \cdot 1{,}000 = 224{,}000$ bits

savings of ~25%

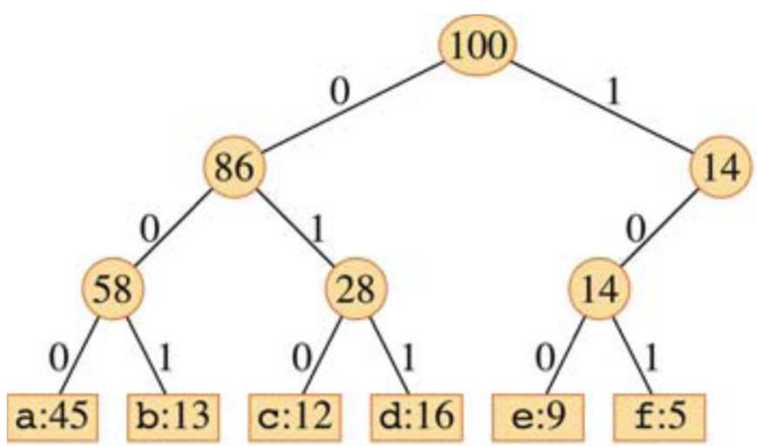
Greedy Algorithms (continued)

- **Prefix codes**
  - no codeword is a prefix of any other codeword
  - simplifying decoding
  - achieving *optimal* data compression for character coding

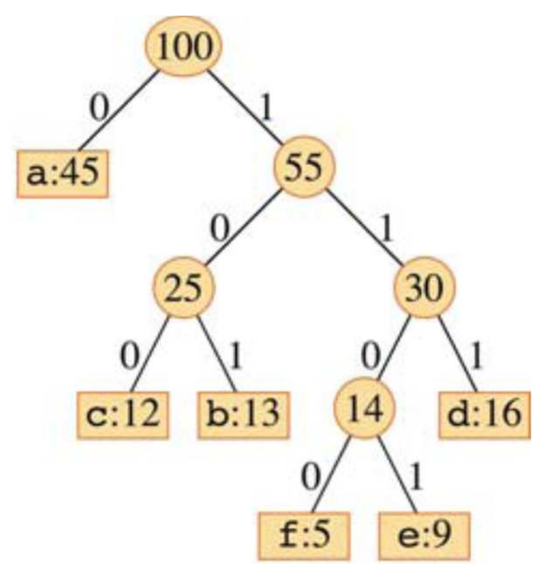
Tree representations of character coding

Number of bits (cost) required of ***T***: $B(T) = \sum_{c \in C} c.freq \cdot d_T(c)$, where $d_T(c)$ is codeword length.

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100



(a) Fixed-length coding, with the same height, but not a full binary tree, for all codewords.



(b) Optimal coding, always denoted by full binary tree with variable heights.

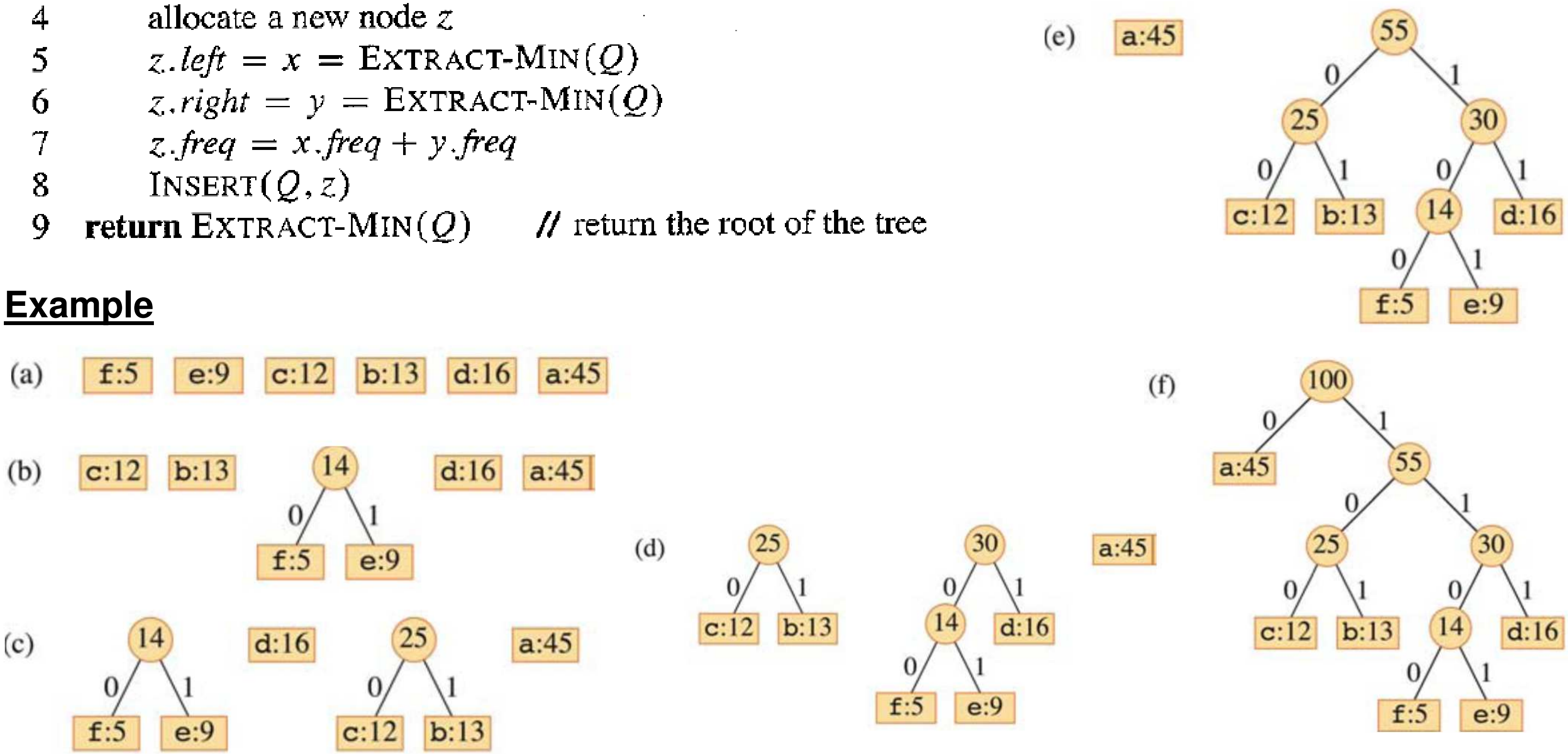
Greedy Algorithms (continued)

- **Huffman code construction**

```
HUFFMAN(C)
n = |C|
Q = C
for i = 1 to n − 1
    allocate a new node z
    z.left = x = EXTRACT-MIN(Q)
    z.right = y = EXTRACT-MIN(Q)
    z.freq = x.freq + y.freq
    INSERT(Q, z)
return EXTRACT-MIN(Q)      // return the root of the tree
```

C is a set of n characters and each character $c \in C$ is an object with an attribute $c.freq$

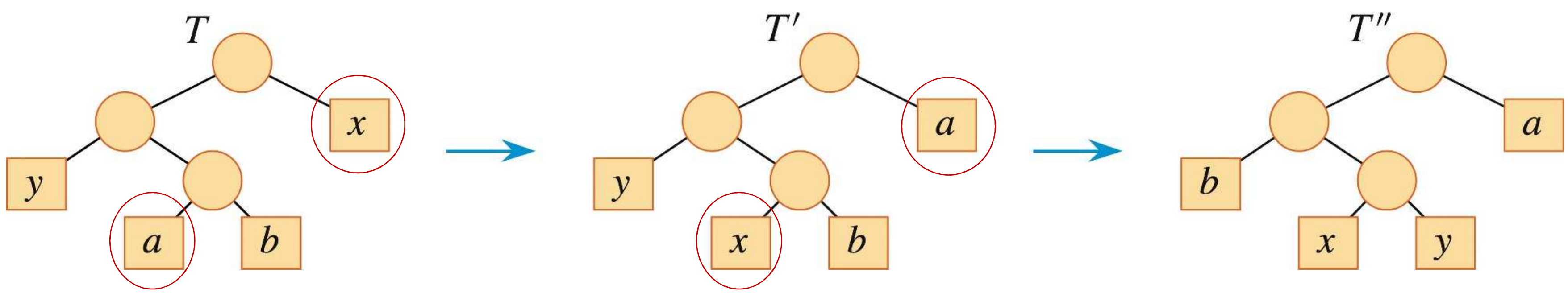
Example



<u>Greedy Algorithms</u> (continued)

<u>Lemma 1</u>

Let C be an alphabet in which each character $c \in C$ has frequency $c.freq$. Let x and y be two characters in C having the lowest frequencies. Then there exists an <u>optimal prefix code for C</u> in which the codewords for x and y have the same length and differ only in the last bit.



T denotes <u>arbitrary optimal prefix code</u>

$$
\begin{aligned}
B(T) - B(T') &= \sum_{c \in C} c.freq \cdot d_T(c) - \sum_{c \in C} c.freq \cdot d_{T'}(c) \\
&= x.freq \cdot d_T(x) + a.freq \cdot d_T(a) - x.freq \cdot d_{T'}(x) - a.freq \cdot d_{T'}(a) \\
&= x.freq \cdot d_T(x) + a.freq \cdot d_T(a) - x.freq \cdot d_T(a) - a.freq \cdot d_T(x) \\
&= (a.freq - x.freq)(d_T(a) - d_T(x)) \\
&\geq 0,
\end{aligned}
$$

therefore, <u>$B(T) \geq B(T')$</u>

But, <u>$B(T') \geq B(T)$</u> since T is optimal, so we have <u>$B(T') \equiv B(T)$</u>.

Similarly, $B(T') \geq B(T'')$ to yield <u>$B(T) \geq B(T'')$</u>, since $B(T') \equiv B(T)$.

Given that T is optimal, we have $B(T'') \equiv B(T)$, implying that <u>***T''***</u> is <u>another optimal prefix code</u> with x and y differing only in the last bit.

Greedy Algorithms (continued)

Lemma 2

Let C be a given alphabet with frequency $c.freq$ defined for each character $c \in C$. Let x and y be two characters in C with **smallest frequencies.** Let C' be the alphabet C with the characters x and y removed and a new character z added, so that $C' = C - \{x, y\} \cup \{z\}$. Define *freq* for C' as for C, except that $z.freq = x.freq + y.freq$. Let T' be any tree representing an optimal prefix code for the alphabet C'. Then the tree T, obtained from T' by replacing the leaf node for z with an internal node having x and y as children, represents an optimal prefix code for the alphabet C.

Theorem

Procedure HUFFMAN produces an optimal prefix code.