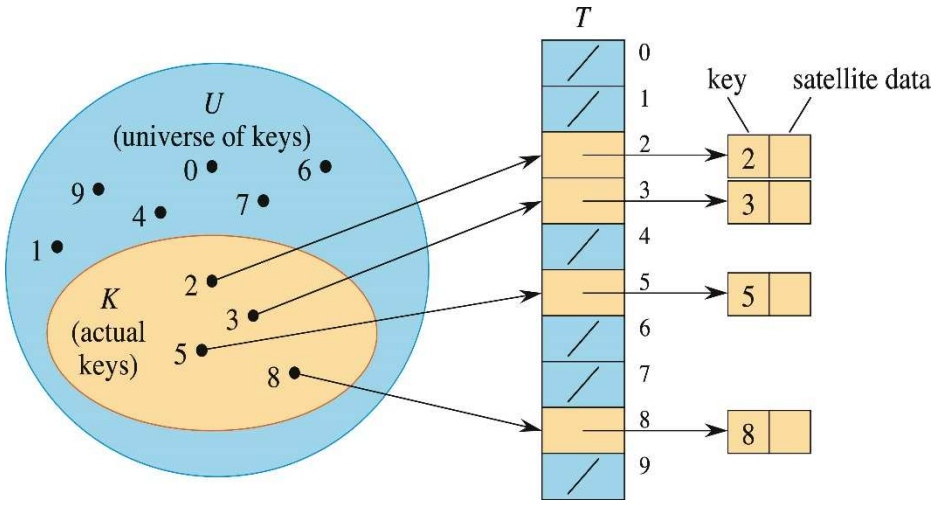
Part II

Data Structures

Four Types of Data Structures Considered:

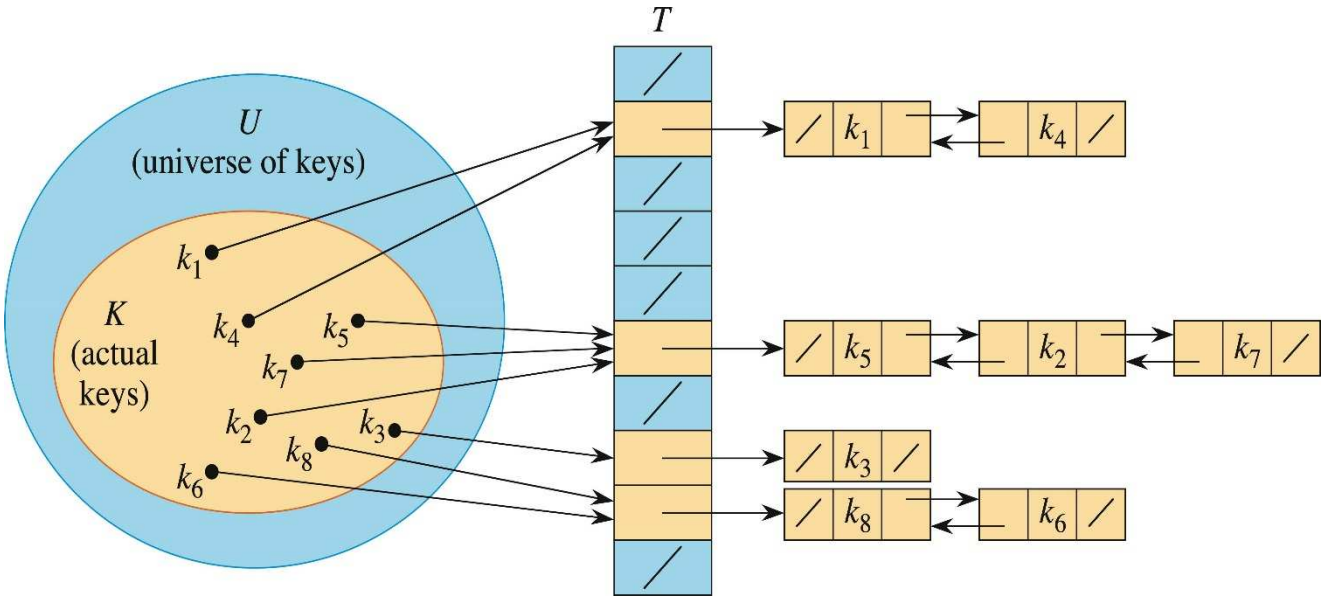
- Hash Tables
- Binary Search Trees
- Red-Black Trees
- Balanced Search Trees

- **Hash Tables**
  - suitable for large key space with small numbers of actual keys stored
  - average search time over the hash table equal to $O(1)$
  - collision-handling strategy required
  - keys must be stored to ensure right items retrieved
  - hash function maps a key to one table index



Hash Tables (continued)

- **Hash Table with Collision Resolution by Chaining**
  - multiple elements with different keys mapped to the same table entry (collisions)
  - they can be chained, with a lookup going through the chain
  - for load factor of $\alpha = n/m$, where n (or m) is the number of elements (or table entries)
  - under uniform distribution, one search (be successful or not) takes $\Theta(1+\alpha)$, according to Theorems 11.1 and 11.2
  - if hash table size (m) grows in proportion to n, then the mean search time is $\Theta(1)$
  - there are other more efficient ways for handling collisions



Hash Tables (continued)

- **Hash Functions**
  - division method
  - multiplication method

1. Division method: (more restrictive on the suitable m)

$h(k) = k \bmod m$, where the choice of $m = 2^p - 1$ is poor but a *prime* not close to an exact power of 2 is often good

For example, given n = 2000 character strings to be stored in a hash table with m = 701 entries, then an unsuccessful search takes some 3 accesses

Hash Tables (continued)

2. Multiplication method:

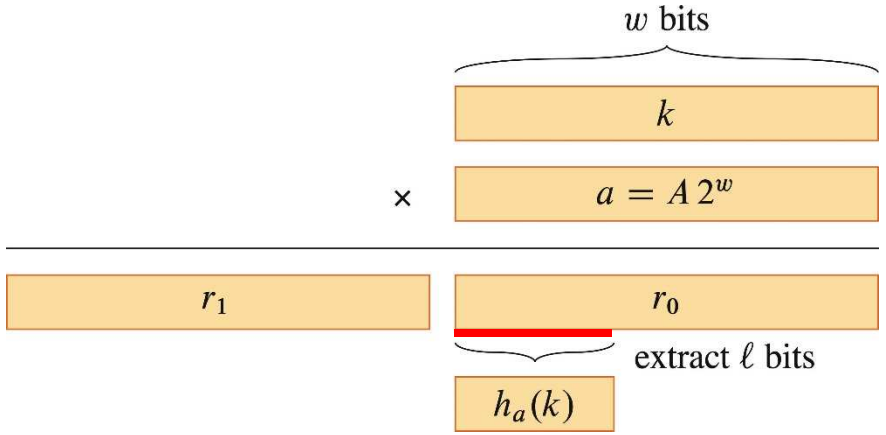
- Multiply k by a constant A, with $0 < A < 1$ and extract the fraction part of $k \cdot A$, e.g., $h(k) = \lfloor m\ (kA \mod 1) \rfloor$, where "$kA \mod 1$" equals $kA - \lfloor kA \rfloor$

  – table size (m) can be arbitrary, e.g., $m = 2^p$
  – let A be a fraction of the form $s/2^w$ as shown in the figure below
  – perform multiplication of w-bit k and w-bit a, to get a $2w$-bit product
  – product denoted by $r_1 \cdot 2^w + r_0$, then p-bit hash value is obtained from r_0

  – while arbitrary A works, $A \approx (\sqrt{5}\,-1)/2 = 0.6180339887\ \ldots$ recommended

Golden ratio: a root of x^2 + x -1 = 0.

+ Let $k = 123456$, $p = 14$, $m = 2^{14}$, $w = 32$ and $A = 2654435769/2^{32}$, we have $k \cdot a = (76300 \cdot 2^{32}) + 17612864$ to get $h(k) = 67$.



Hash Tables (continued)

- **Properties of Random Hashing**
  - family of hash functions, $\mathcal{H}$, with domain U and range $R = \{0, 1, \ldots, m-1\}$
  - $\mathcal{H}$ is *uniform* if probability of hashing a key k, $h(k)$, to q in R equals $1/m$
  - $\mathcal{H}$ is *universal* if probability of distinct keys, k_1 and k_2, with $h(k_1) = h(k_2)$ is $\leq 1/m$
  - $\mathcal{H}$ is ε-*universal* if probability of distinct keys, k_1 and k_2, with $h(k_1) = h(k_2)$ is $\leq 1/\varepsilon$
  - $\mathcal{H}$ is *d-independent* if distinct keys, $k_1, k_2, \ldots, k_d$, and slots of $q_1, q_2, \ldots, q_d$ in R, with $h(k_i) = h(q_i)$ for all $1 \leq i \leq d$, has the probability of $1/m^d$

Hash Tables (continued)

- **Universal Hashing**

  For the hash function of $\boldsymbol{h_{ab}(k) = ((a \cdot k + b) \bmod p) \bmod m}$, where p is a large prime number, with $p > m$, $a \in \{1, 2, \ldots p\text{-}1\}$ and $b \in \{0, 1, 2, p\text{-}1\}$, the collection of such hash functions is *universal*.

  In this case, m can be any number and does not have to be a prime.
  For example, for $p = 17$ and $m = 8$, we have $h_{34}(15) = 7$.

  ((3*15+4) mod 17) mod 8 = 7

Hash Tables (continued)

- **Open Addressing** (to deal with collision-chaining)
  - elements stored inside the table
  - calculating probe sequence of given key, instead of using pointers:
    $\langle h(k, 0), h(k, 1), h(k, 2), \ldots, h(k, m-1) \rangle$
  - if probe sequence is a permutation of $\langle 0, 1, 2, \ldots, m-1 \rangle$, every table entry is a candidate location for the element
  - the probe sequence is fixed for a given key
  - key has to be stored in the table

HASH-INSERT(T, k)

```
i = 0
repeat
    q = h(k, i)
    if T[q] == NIL
        T[q] = k
        return q
    else i = i + 1
until i == m
error "hash table overflow"
```

HASH-SEARCH(T, k)

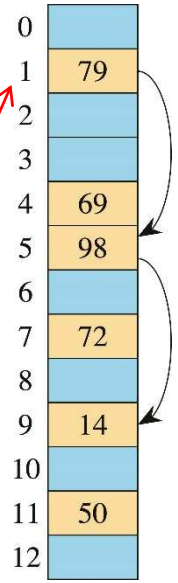
```
i = 0
repeat
    q = h(k, i)
    if T[q] == k
        return q
    i = i + 1
until T[j] == NIL or i = m
return NIL
```

Hash Tables (continued)

- **Uniform Hashing Analysis on Open Addressing**
  - key probe sequence equal likely in one of $m!$ permutations of $\langle 0, 1, 2, \ldots, m-1 \rangle$
  - probe sequences defined for open addressing
    - + linear probing: m different sequences
      - dictated by $h(k, i) = (h'(k) + i) \bmod m$
      - subject to *primary clustering*
    - + quadratic probing: m different sequences
      - dictated by $h(k, i) = (h'(k) + c_1 \cdot i + c_2 \cdot i^2) \bmod m$
      - subject to *secondary clustering*, i.e., $h(k_1, 0) = h(k_2, 0) \Rightarrow$ both keys follow the same probe sequence
    - + double hashing: m^2 different sequences
      - dictated by $h(k, i) = (h_1(k) + i \cdot h_2(k)) \bmod m$
      - many good choices: $h_2(k)$ relatively prime to m; m itself a prime; m a power of 2 and $h_2(k)$ always an odd number; or below:

        m a prime and m' slightly less than m (e.g., m-1 or m-2) with
        $h_1(k) = k \bmod m$
        $h_2(k) = 1 + (k \bmod m')$

e.g., for k =14 under m = 13, m' = 11;
upon inserting k = 14.



Hash Tables (continued)

- **Open-Address Hashing Analysis**

Theorem 1:

For an open-address hash table with load factor $\alpha = n/m < 1$, the expected number of probes in unsuccessful search under uniform hashing is at most $1/(1-\alpha)$.

Note: unsuccessful search ends at seeing an empty entry.

.

+ if α is a constant, an unsuccessful search runs in $O(1)$ time
+ inserting an element into an open-address hash table with load factor α takes at most $1/(1-\alpha)$ probes on an average under uniform hashing, because it has (1-α) for 1 probe, plus prob. α(1-α) to take 2 probes, plus prob. $\alpha^2(1-\alpha)$ to take 3 probes, etc., yielding $1+\alpha+\alpha^2+\alpha^3+ \ldots = 1/(1-\alpha)$

e.g., for α = 50/100, we have 2 probes.

Theorem 2:

For an open-address hash table with load factor α < 1, the expected number of probes in a successful search under uniform hashing is at most $\frac{1}{\alpha} \cdot \ln\frac{1}{1-\alpha}$.

► successful search for a key equals the sequence of inserting the key

2nd key insertion takes ≤ 1/(1-(1/*m*)) probes, on an average, when α is 1/*m*
3rd key takes ≤ 1/(1-(2/*m*)) probes, on an average, when α is 2/*m*
:
i+1th key takes ≤ 1/(1-(*i*/*m*)) probes

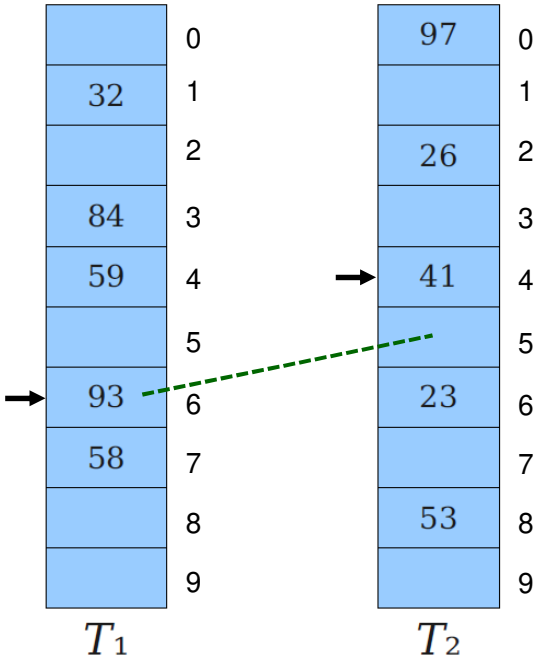
Mean no. of probes equals results over all *n* keys inserted

This is because **insertion and probes** follow the same hash function.
We have: $1/n \left(\sum_{i=0}^{n-1} 1/(1-i/m)\right)$

e.g., for α = 50/100, we have ≈ 1.39 probes, as ln(2) ~ 0.693. (base e ~ 2.718)

Hash Tables (continued)

- **Cuckoo Hashing**§
  - – move keys around upon insertion for worst probe case of *O*(1)
    - + quick search but possibly lengthy insertion
    - + multiple hash functions required for separate hash tables
  - – delete keys in the worst-case of *O*(1)
  - – example with two hash functions below:
    - + two hash tables, each with *m* elements
    - + two hash functions, h_1 and h_2, one for a table
    - + each key *x* at either $h_1(x)$ or $h_2(x)$
    - + new key '10' with $h_1(10) = 6$ and $h_2(10) = 4$
      - * move key '93' around, if $h_2(93)=5$, for insertion

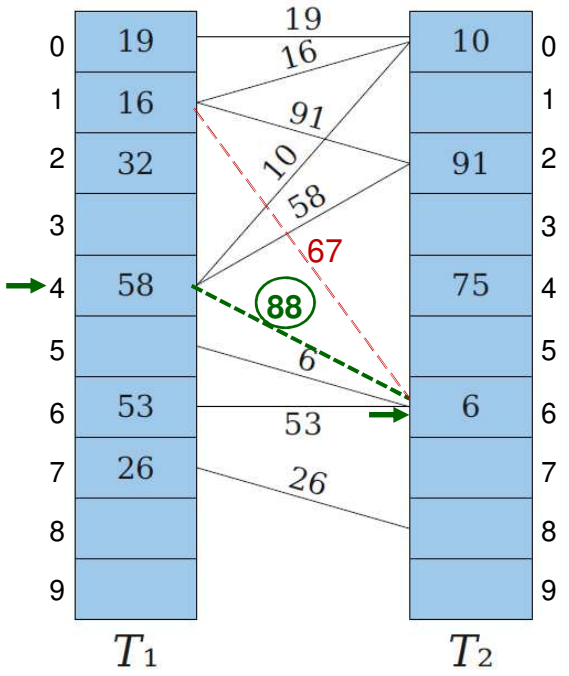


  - – Algorithm
    1. insert "new key" to T_1 located at h_1(new key), when available
    2. otherwise, "new key" displaces "existing key" in T_1; place "existing key" in T_2 located at h_2(existing key); repeat the displacement process, if needed.

§ R. Pagh and F. Rodler, "Cuckoo Hashing," *Journal of Algorithms*, Aug. 2001, pp. 121-133.

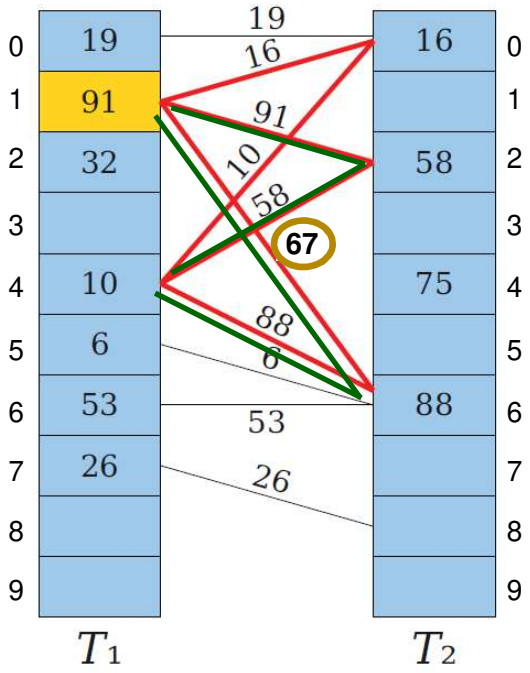
Hash Tables (continued)

- **Cuckoo Graph**
  - derived from cuckoo hash tables
    - + each table entry is a node
    - + each key is an edge, which links the entries that can hold the key
    - + an insertion adds a new edge to the graph
      - * let $h_1(88) = 4$ and $h_2(88) = 6$
      - * displace key '58' in T_1 to make room for key '88'
      - * displace key '91' in T_2 to make room for key '58'
      - * displace key '16' in T_1 to make room for key '91'
      - * displace key '10' in T_2 to make room for key '16'
      - * repeat ….
    - + insertion done by tracing a path over the graph
    - + what about inserting next key "67" with $h_1(67) = 1$ and $h_2(67) = 6$?



Hash Tables (continued)

- **Cuckoo Graph**
  – insertion succeeds if and only if the new edge (defined by the new key) is on at most one cycle



Proof sketch:

1. each edge denotes a key and requires a table entry (i.e., node) to hold it
2. for a cycle exists, the number of its edges equals the number of its nodes involved, so that all nodes (table entries) are taken
3. if a new edge (after added) is on two cycles, the edge must link two taken nodes AND all nodes on the two cycles have been used
4. the new key (edge) *cannot* be accommodated.

Side note: two cycles (one with 4 distinct nodes and the other with at least one distinct node) that share two edges, will include (at least) 5 distinct nodes but they contain (at least) 6 distinct edges; impossible.

Binary Search Trees

§ Binary Search Tree Property

+ Given tree node x, if node y is in the left (or right) subtree of x, we have $y.key \le x.key$ (or $y.key \ge x.key$).

Theorem 1

INORDER-TREE-WALK(x) across an n-node subtree rooted at x takes $\Theta(n)$.

(because the tree height may equal n)

Proof.

The tree walk has to visit every node, and hence, the time complexity $T(n)$ is lower bounded by $\Omega(n)$.

Next, considering that INORDER-TREE-WALK(x) is called on x whose left and right subtrees have k and $(n\text{-}k\text{-}1)$ nodes, respectively, we have

$T(n) \le T(k) + T(n\text{-}k\text{-}1) + d$ for some $d > 0$. For simplicity, we may let d equal 1 here.

By the substitution method, we prove that $T(n) \le (c+d)n + c$, for a constant c, below:

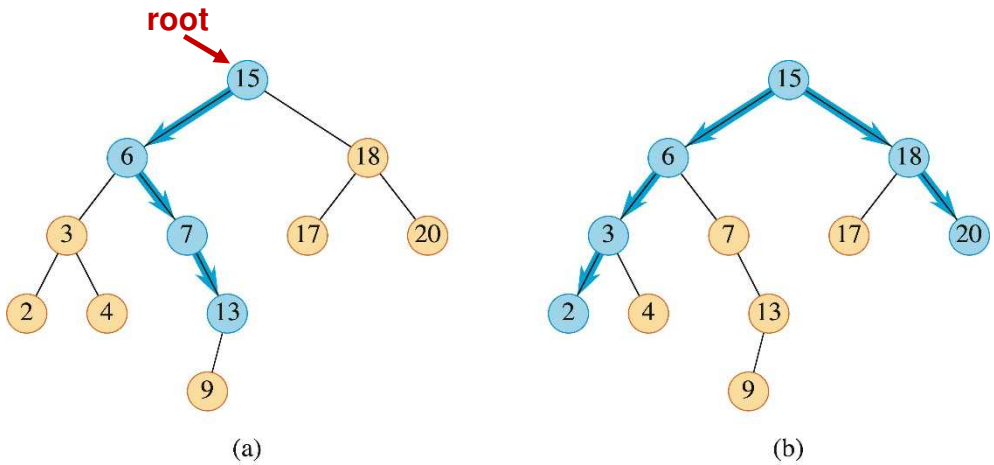
For simplicity, we may let c equal 1 here to have $T(n) \le 2 \cdot n + 1$.

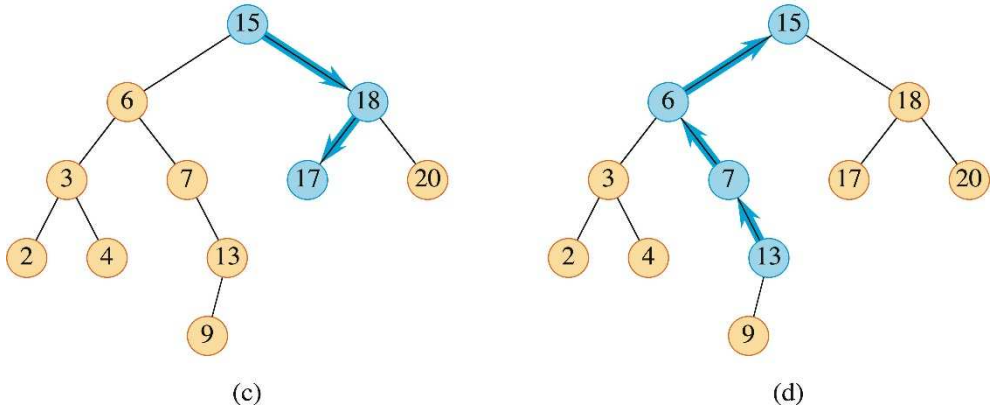
$$\begin{aligned} T(n) &\le T(k) + T(n\text{-}k\text{-}1) + d \\ &\le ((c+d)k + c) + ((c+d)(n\text{-}k\text{-}1) + c) + d \\ &= (c+d)n + c - (c+d) + c + d \\ &= (c+d)n + c \quad \text{(upper bounded)} \end{aligned}$$

Binary Search Trees (continued)

- **Querying binary search trees**
  - + searching
  - + successor and predecessor
  - + insertion and deletion

  - + Searching via TREE-SEARCH in time complexity $O(h)$






```
TREE-SEARCH(x, k)
if x == NIL or k == key[x]
      return x
if k < x.key
      return TREE-SEARCH(x.left, k)
else return TREE-SEARCH(x.right, k)
```

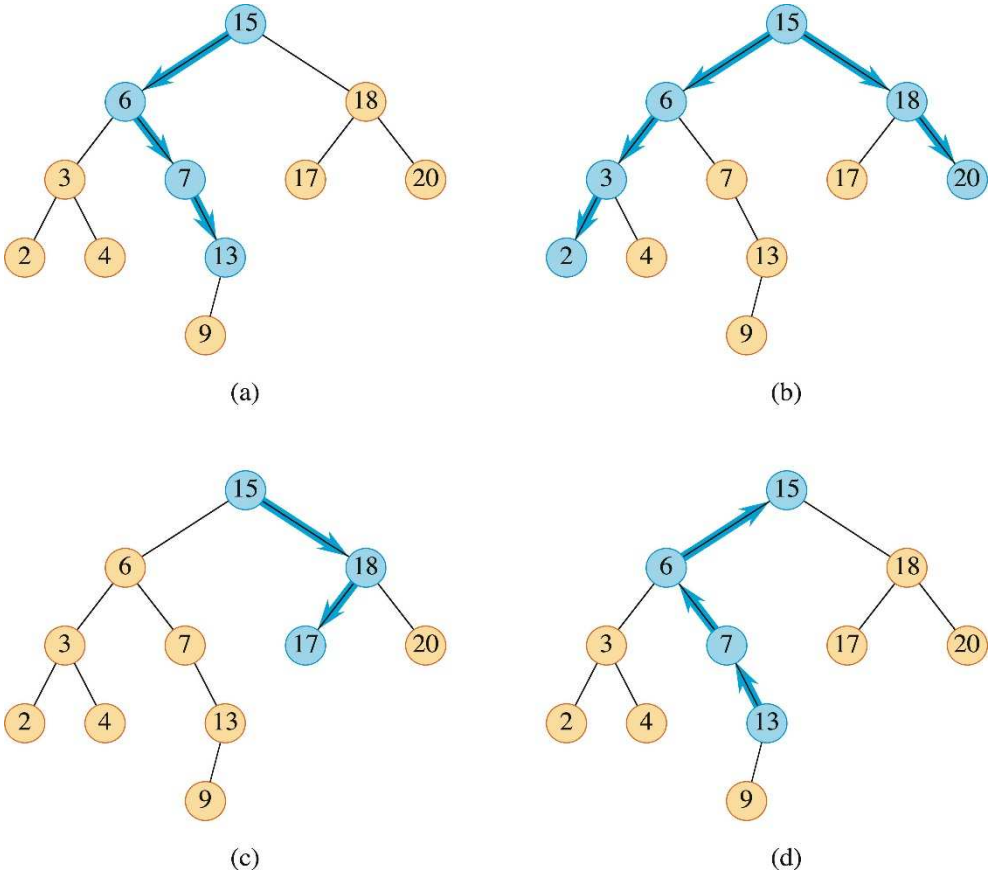
search path from root to *key* = 13 is
15 → 6 → 7 → 13

Binary Search Trees (continued)

- Successor and predecessor
  - + by inorder tree walk
  - + separate results for non-empty right subtree and for empty right subtree
  - + time complexity for a tree of height *h* is *O*(*h*)

```
TREE-SUCCESSOR(x)
  if x.right ≠ NIL
      return TREE-MINIMUM(x.right)
  y = x.p          y is the parent node of x.
  while y ≠ NIL and x == y.right
      x = y        climb up the tree.
      y = y.p
  return y
```

If node *x* has no right subtree, its successor is the lowest ancestor of *x* whose left subtree contains *x*.



successor of *key* = 15 is 17
successor of *key* = 13 is 15

Binary Search Trees (continued)

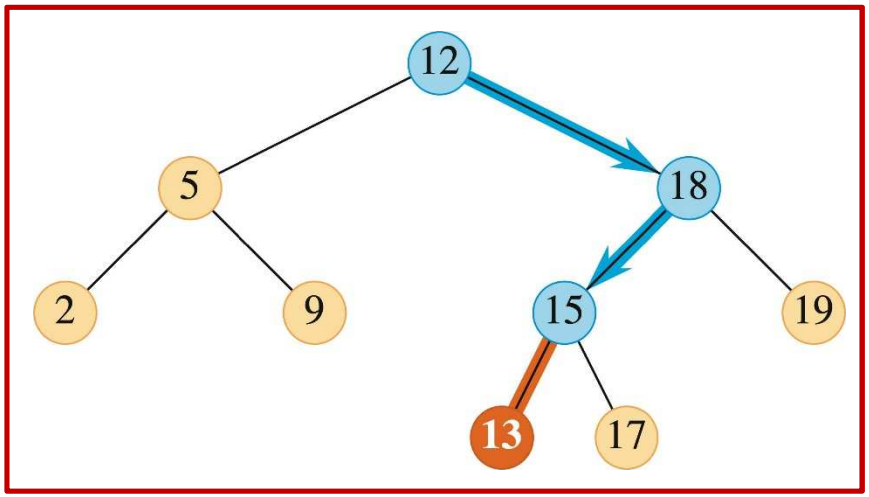
- Insertion
  - + tree node (say, *z*) to be added, with *z.key* = *v*, *z.left* = nil, *z.right* = nil
  - + *z* added to one appropriate node with an ***absent*** left or right child (never a full node)
  - + tree in-order walk, with the trailing pointer *y* maintained
  - + time complexity for a tree of height *h* is *O*(*h*)

TREE-INSERT(T, z)

```
y = NIL
x = T.root
while x ≠ NIL
      y = x
      if z.key < x.key
            x = x.left
      else x = x.right
z.p = y   // y is the parent node of z.
if y == NIL
      T.root = z          // tree T was empty
elseif z.key < y.key
      y.left = z
else y.right = z
```

y is the parent node of *x* after *x* is reassigned to move down

tree traversal downward until the insertion point, with the added node as a **leaf node**



Inserting *key* = 13 into binary search tree, with affected path marked.

Binary Search Trees (continued)

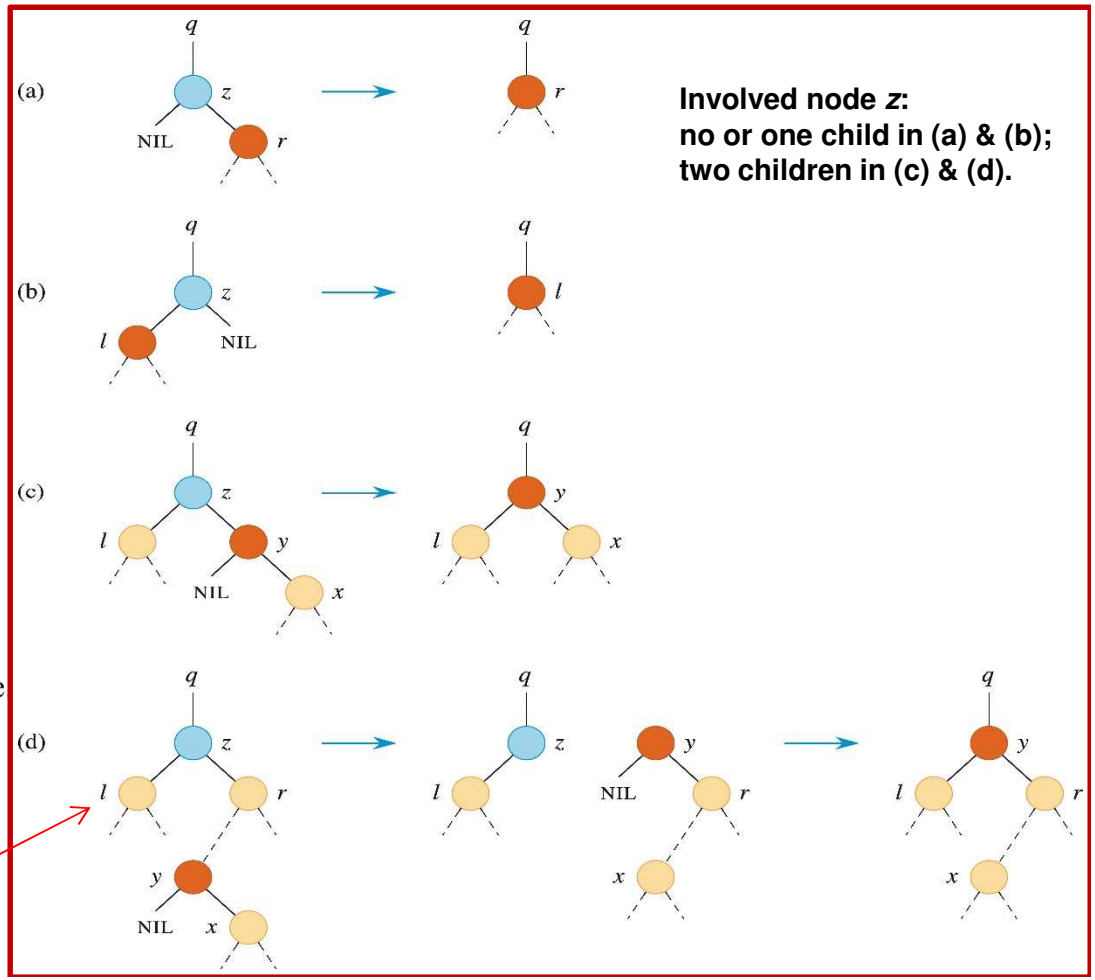
- Deletion (on *z*)
  - + three basic cases involved: *z* having no child, exactly one child, two children
  - + based on TRANSPLANT that replaces sibling subtree with another subtree
  - + time complexity for a tree of height *h* is *O*(*h*)

```
TRANSPLANT(T, u, v)
  if u.p == NIL
      T.root = v
  elseif u == u.p.left
      u.p.left = v
  else u.p.right = v
  if v ≠ NIL
      v.p = u.p
```

```
TREE-DELETE(T, z)
  if z.left == NIL
      TRANSPLANT(T, z, z.right)          // z has no left child
  elseif z.right == NIL
      TRANSPLANT(T, z, z.left)           // z has just a left child
  else // z has two children.
      y = TREE-MINIMUM(z.right)          // y is z's successor
      if y.p ≠ z   ← This is case (d) shown in the figure.
          // y lies within z's right subtree but is not the root of this subtree
          TRANSPLANT(T, y, y.right)
          y.right = z.right
          y.right.p = y
      // Replace z by y.
      TRANSPLANT(T, z, y)
      y.left = z.left
      y.left.p = y
```

Prepare a subtree rooted at ***y*** whose left subtree is empty.

Alternatively, one may cut the left-hand subtree of *z* and make it become the left-hand subtree of *y'*.



Involved node *z*:
no or one child in (a) & (b);
two children in (c) & (d).

Binary Search Trees (continued)

- Randomly build binary search trees
  - + worst-case tree height (h) being n-1
  - + can be shown that $h \geq \lfloor \lg n \rfloor$, which is the best case
  - + like quicksort, average case proven to be much closer to best case than worst case
  - + special case of creating binary search trees (via insertion alone) with random keys, we have following theorem:

Theorem 2. ← This theorem refers to the special case that the tree is created by insertion alone (and no deletion).

The expected height of a randomly built binary search tree on n distinct keys is $\boldsymbol{O(\lg n)}$.

Red-Black Trees

– Red-black tree is a binary search tree that satisfies:

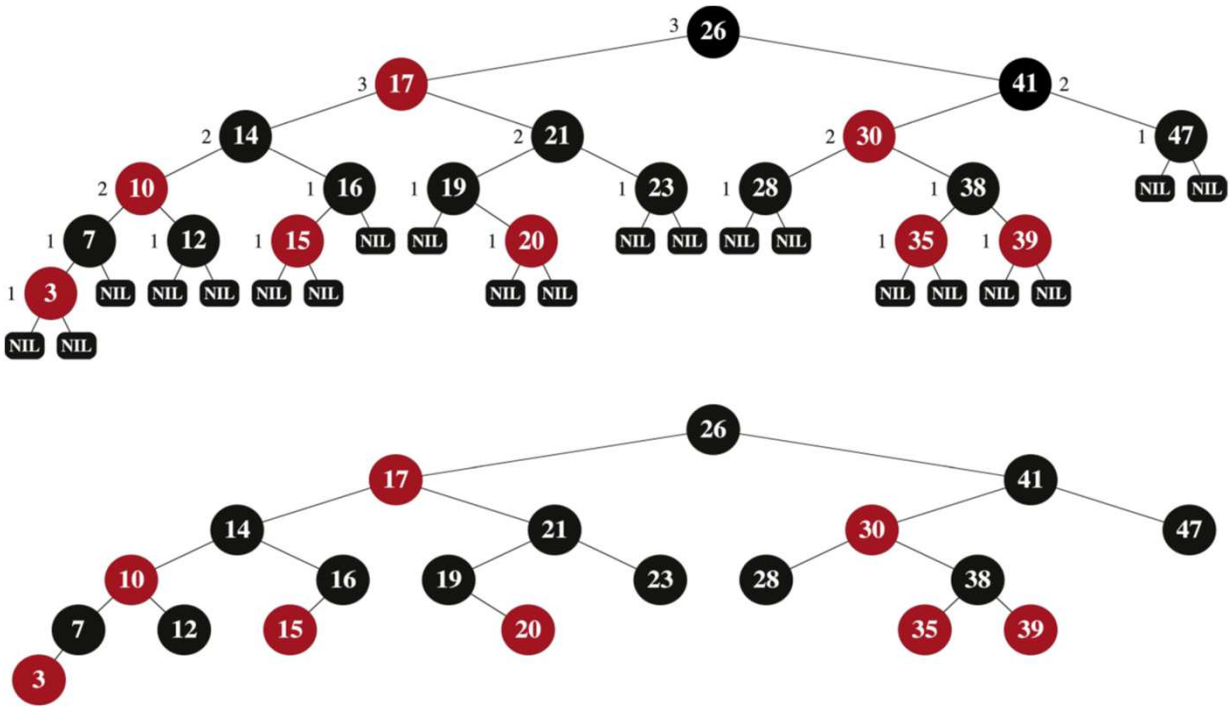
(1) every node is either red or black

(2) root is black

(3) every leaf (NIL) is black

(4) red node has its both children being black → CANNOT have two connected red nodes

(5) all simple paths from any node to its descendant leaves contains the same number of black nodes (called the node's black-height, *bh*)



Red-Black Trees (continued)

+ Red-black tree is a good search tree because of its bounded small height.

Lemma 1:

A red-black (RB) tree with *n* internal nodes has the height of $h \leq 2 \cdot \lg(n+1)$.

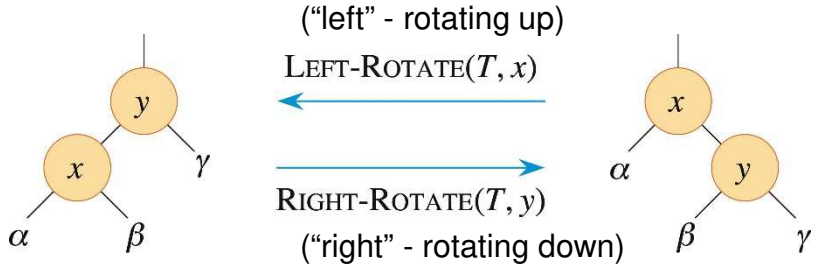
- proof by showing that any RB subtree rooted at Node *x* has at least $2^{bh(x)} - 1$ internal nodes via induction, as the bh of each *x*'s child is ≥ bh(x) -1 to contain ≥ $2^{bh(x)-1} - 1$ internal nodes
- from RB tree property (4), if the height of *x* is *h*, then bh(*x*) ≥ *h*/2, to yield $n \geq 2^{h/2} - 1$

– Lemma 1 signifies that all operations (SEARCH, MIN, MAX, SUCCESSOR, PREDECESSOR) on an RB tree with *n* internal nodes take $O(\lg n)$ each.

Red-Black Trees (continued)

Rotations

+ insertion to, and deletion from, an RB tree involves subtree rotations
+ left rotations and right rotations



LEFT-ROTATE(T, x)

```
1   y = x.right
2   x.right = y.left        // turn y's left subtree into x's right subtree
3   if y.left ≠ T.nil       // if y's left subtree is not empty ...
4       y.left.p = x        // ... then x becomes the parent of the subtree's root
5   y.p = x.p               // x's parent becomes y's parent
6   if x.p == T.nil         // if x was the root ...
7       T.root = y          // ... then y becomes the root
8   elseif x == x.p.left    // otherwise, if x was a left child ...
9       x.p.left = y        // ... then y becomes a left child
10  else x.p.right = y      // otherwise, x was a right child, and now y is
11  y.left = x              // make x become y's left child
12  x.p = y
```

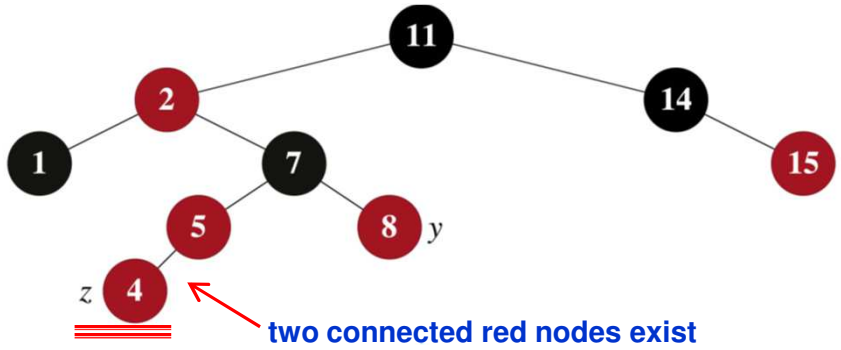
Red-Black Trees (continued)

Insertion

+ inserts a node to an RB tree, RB-INSERT(T, z), with the inserted node z in red and descending to a tree leaf node

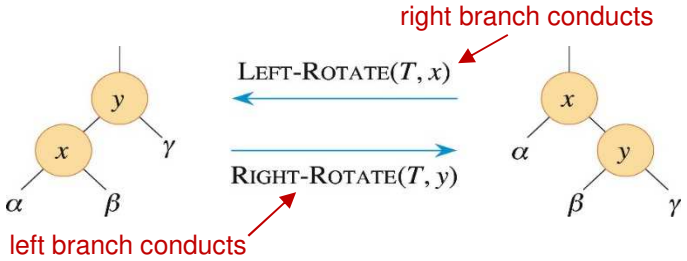
+ requires RB-INSERT-FIXUP(T, z) to ensure RB properties after insertion

```
RB-INSERT(T, z)
 1  x = T.root                  // node being compared with z
 2  y = T.nil                   // y will be parent of z
 3  while x ≠ T.nil             // descend until reaching the sentine
 4      y = x
 5      if z.key < x.key
 6          x = x.left
 7      else x = x.right
 8  z.p = y                     // found the location—insert z with parent y
 9  if y == T.nil
10      T.root = z              // tree T was empty
11  elseif z.key < y.key
12      y.left = z
13  else y.right = z
14  z.left = T.nil              // both of z's children are the sentinel
15  z.right = T.nil
16  z.color = RED               // the new node starts out red
17  RB-INSERT-FIXUP(T, z)       // correct any violations of red-black properties
```



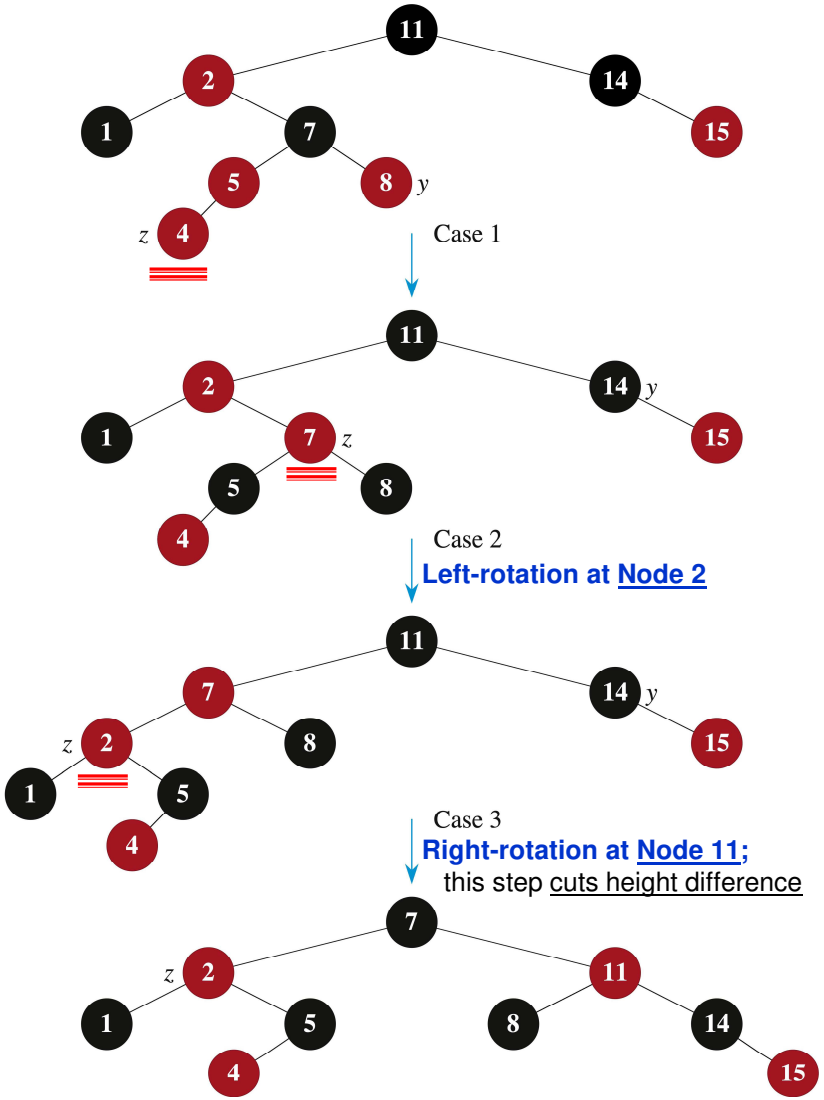
Red-Black Trees

+ RB-INSERT-FIXUP(T, z) to ensure RB properties



RB-INSERT-FIXUP(T, z)

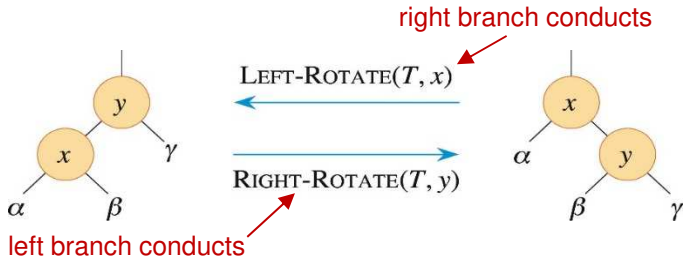
```
1  while z.p.color == RED
2      if z.p == z.p.p.left                  // is z's parent a left child?
3          y = z.p.p.right                   // y is z's uncle
4          if y.color == RED                 // are z's parent and uncle both red?
5                                                                          (a)
6              z.p.color = BLACK             }
7              y.color = BLACK               } case 1
8              z.p.p.color = RED             }
9              z = z.p.p                     }
10         else
11             if z == z.p.right             }
12                 z = z.p                   } case 2   rotation performed if two connected red nodes exist
13                 LEFT-ROTATE(T, z)         }                                                  (b)
14             z.p.color = BLACK             }
15             z.p.p.color = RED             } case 3
16             RIGHT-ROTATE(T, z.p.p)        }
17     else // same as lines 3–15, but with "right" and "left" exchanged
18         y = z.p.p.left
19         if y.color == RED
20             z.p.color = BLACK
21             y.color = BLACK
22             z.p.p.color = RED                                           (c)
23             z = z.p.p
24         else
25             if z == z.p.left
26                 z = z.p
27                 RIGHT-ROTATE(T, z)
28             z.p.color = BLACK
29             z.p.p.color = RED                                           (d)
30             LEFT-ROTATE(T, z.p.p)
31 T.root.color = BLACK
```

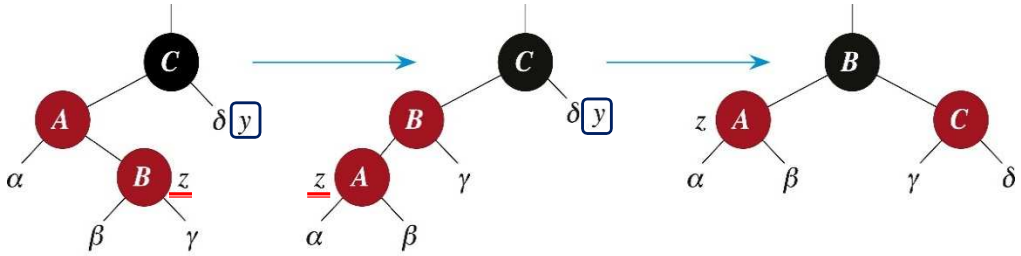


Red-Black Trees (continued)

+ RB-INSERT-FIXUP(T, z) to ensure RB properties

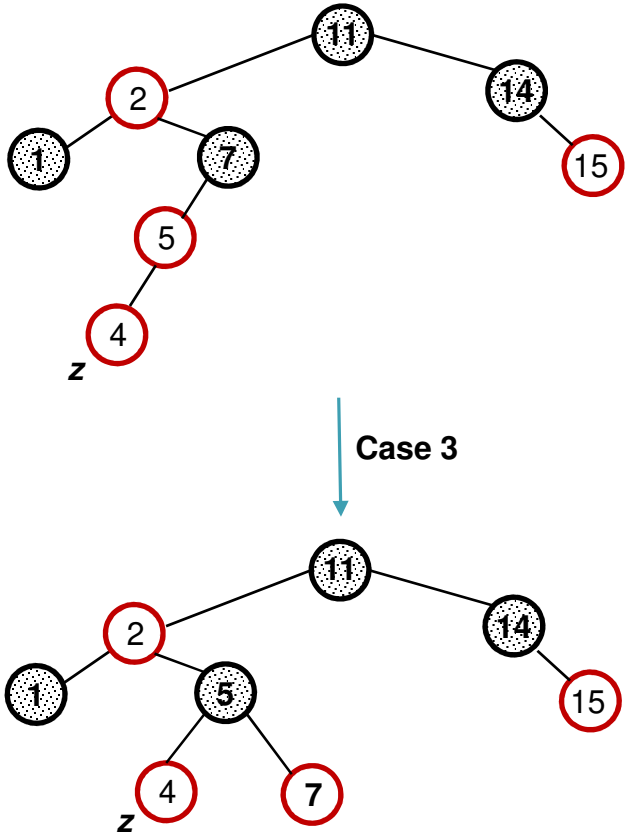
```
RB-INSERT-FIXUP(T, z)
 1  while z.p.color == RED
 2      if z.p == z.p.p.left                  // is z's parent a left child?
 3          y = z.p.p.right                   // y is z's uncle
 4          if y.color == RED                 // are z's parent and uncle both red?
 5              z.p.color = BLACK             }
 6              y.color = BLACK               }  case 1
 7              z.p.p.color = RED             }
 8              z = z.p.p                     }
 9          else
10              if z == z.p.right             }
11                  z = z.p                   }  case 2
12                  LEFT-ROTATE(T, z)         }
13              z.p.color = BLACK             }
14              z.p.p.color = RED             }  case 3
15              RIGHT-ROTATE(T, z.p.p)        }
16      else // same as lines 3–15, but with "right" and "left" exchanged
17          y = z.p.p.left
18          if y.color == RED
19              z.p.color = BLACK
20              y.color = BLACK
21              z.p.p.color = RED
22              z = z.p.p
23          else
24              if z == z.p.left
25                  z = z.p
26                  RIGHT-ROTATE(T, z)
27              z.p.color = BLACK
28              z.p.p.color = RED
29              LEFT-ROTATE(T, z.p.p)
30  T.root.color = BLACK
```



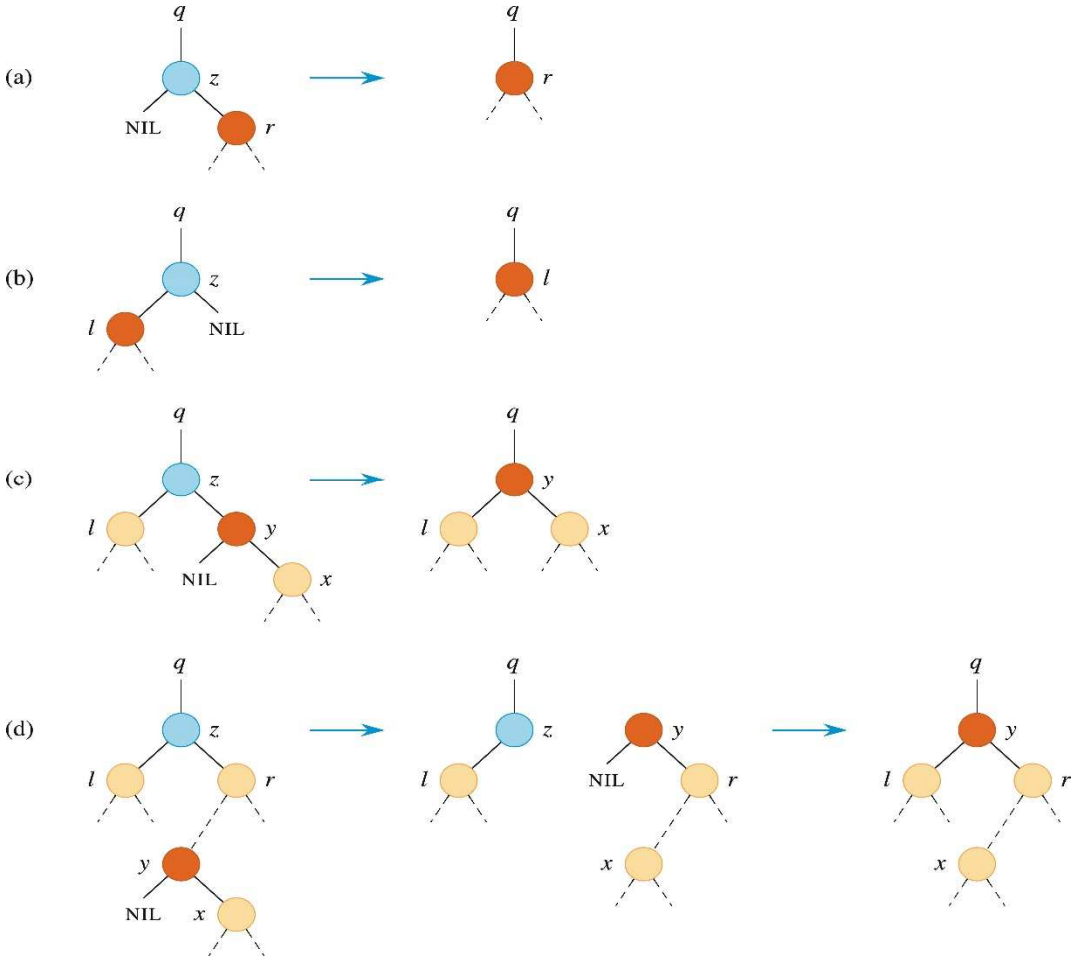


Case 2 — **Left-rotation at Node *A***

Case 3 — **Right-rotation at Node *C*, after making it red,** to cut tree height difference



Deletion Cases



Red-Black Trees (continued)

Deletion

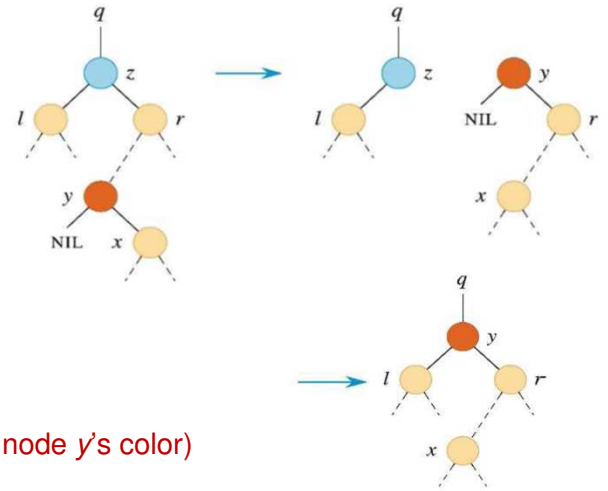
+ deletes an arbitrary node from the RB tree, RB-DELETE(T, z)
+ requires RB-DELETE-FIXUP(T, z) to ensure RB properties after deletion

```
RB-DELETE(T, z)
1   y = z
2   y-original-color = y.color
3   if z.left == T.nil
4       x = z.right
5       RB-TRANSPLANT(T, z, z.right)
6   elseif z.right == T.nil
7       x = z.left
8       RB-TRANSPLANT(T, z, z.left)
9   else y = TREE-MINIMUM(z.right)
10      y-original-color = y.color
11      x = y.right
12      if y ≠ z.right
13          RB-TRANSPLANT(T, y, y.right)
14          y.right = z.right
15          y.right.p = y
16      else x.p = y
17      RB-TRANSPLANT(T, z, y)
18      y.left = z.left
19      y.left.p = y
20      y.color = z.color
21  if y-original-color == BLACK
22      RB-DELETE-FIXUP(T, x)
```

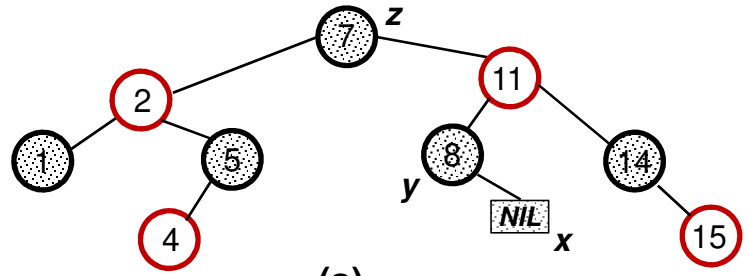
If the deleted node has just one child (must be black), x (must be red) denotes the node to replace node z

If the deleted node has two children, y denotes the node to replace deleted node z and x denotes the node to replace node y

```
// replace z by its right child

// replace z by its left child
// y is z's successor

// is y farther down the tree?
// replace y by its right child
// z's right child becomes
//       y's right child
// in case x is T.nil
// replace z by its successor y
// and give z's left child to y,
//       which had no left child

// if any red-black violations occurred,
//       correct them (based on replaced node y's color)
```

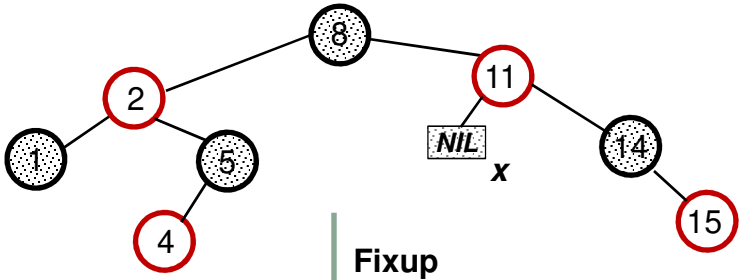



```
RB-TRANSPLANT(T, u, v)
1   if u.p == T.nil
2       T.root = v
3   elseif u == u.p.left
4       u.p.left = v
5   else u.p.right = v
6   v.p = u.p
```

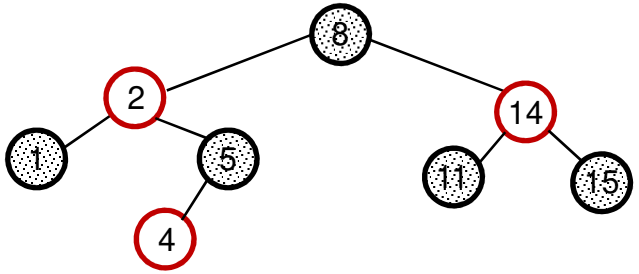


(a)

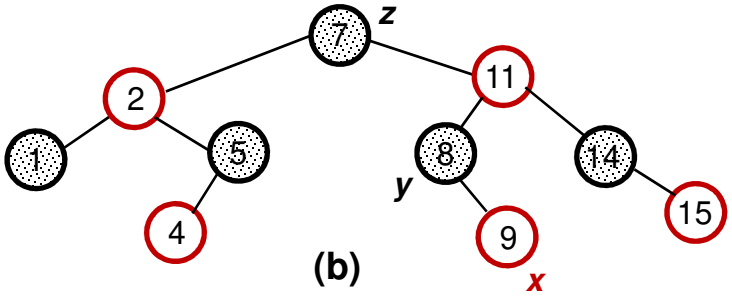
Delete '7'



Fixup

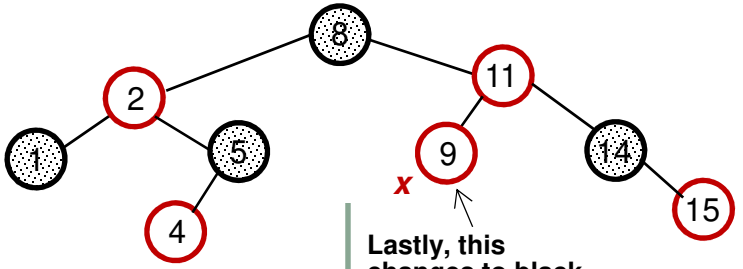


Deleting '2' then delete '11'

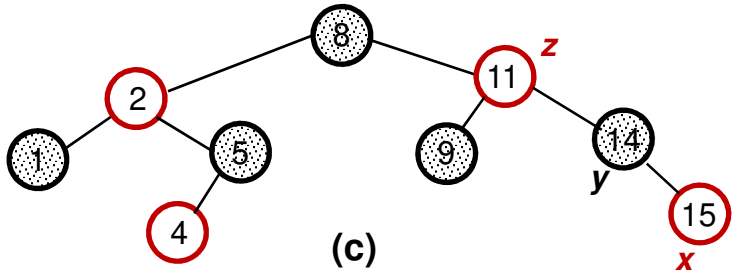


(b)

Delete '7'

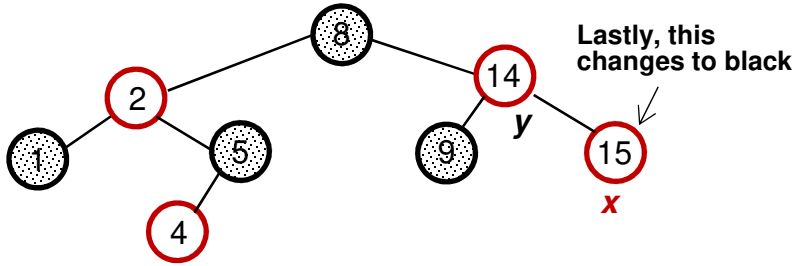


Lastly, this changes to black

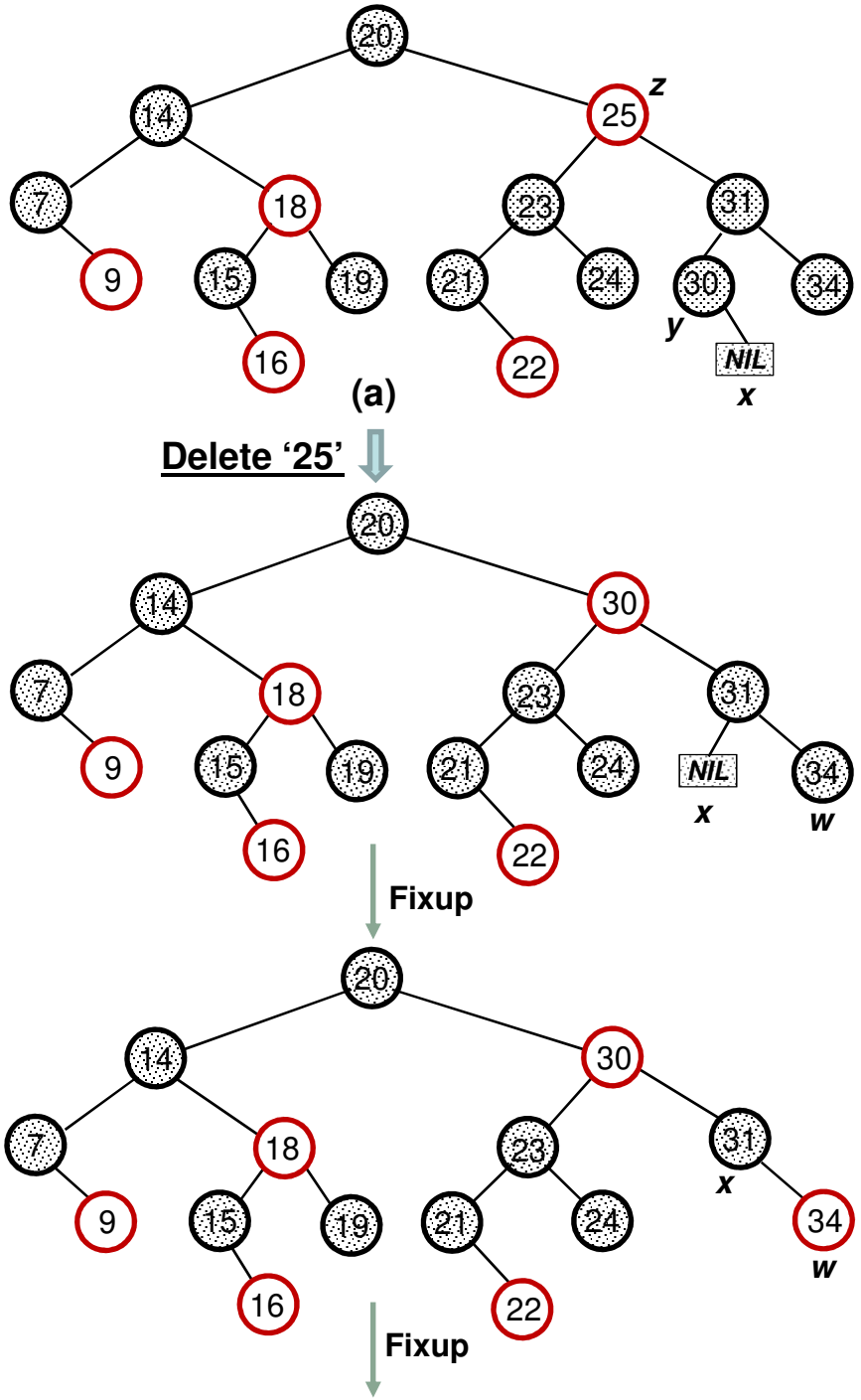


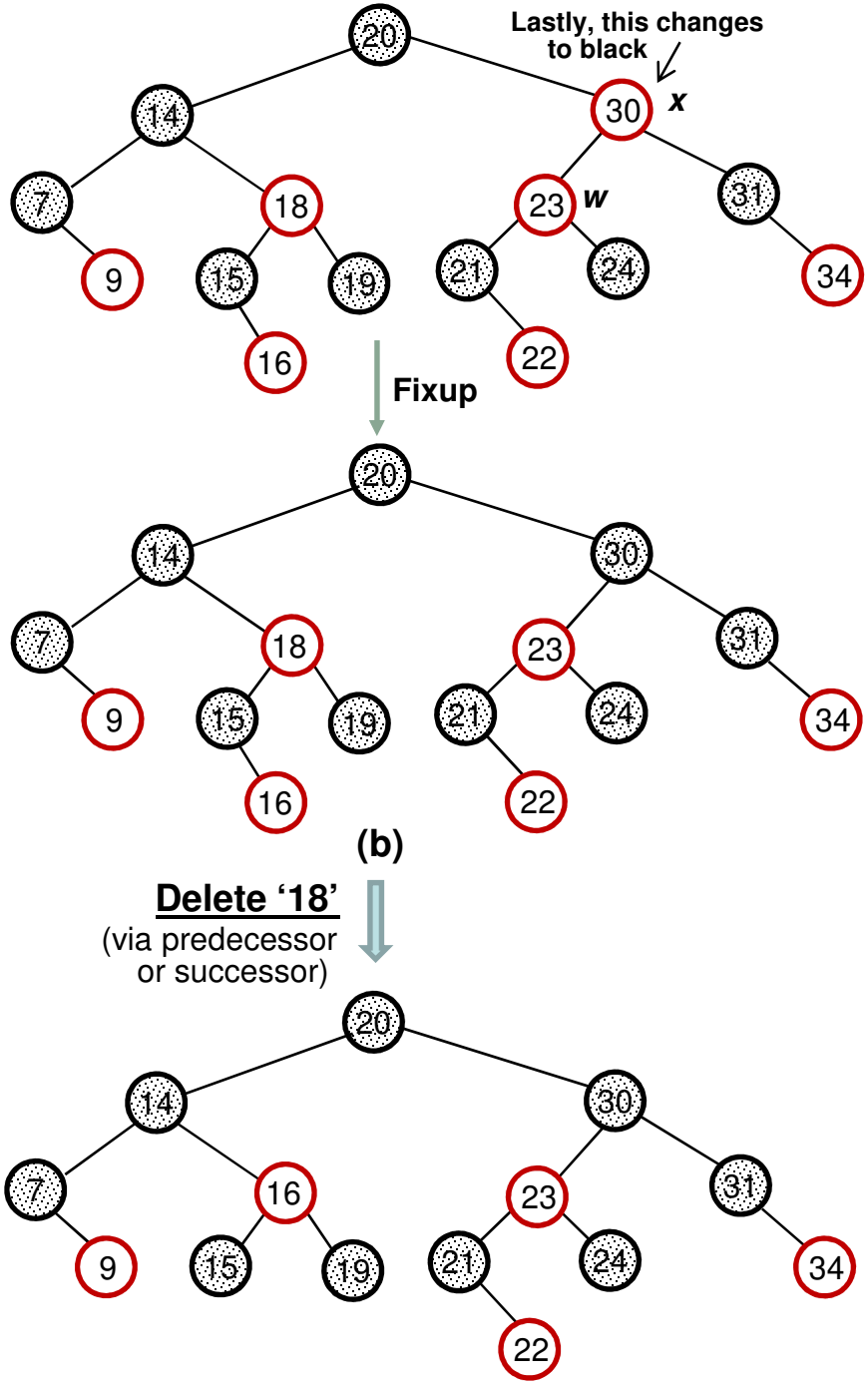
(c)

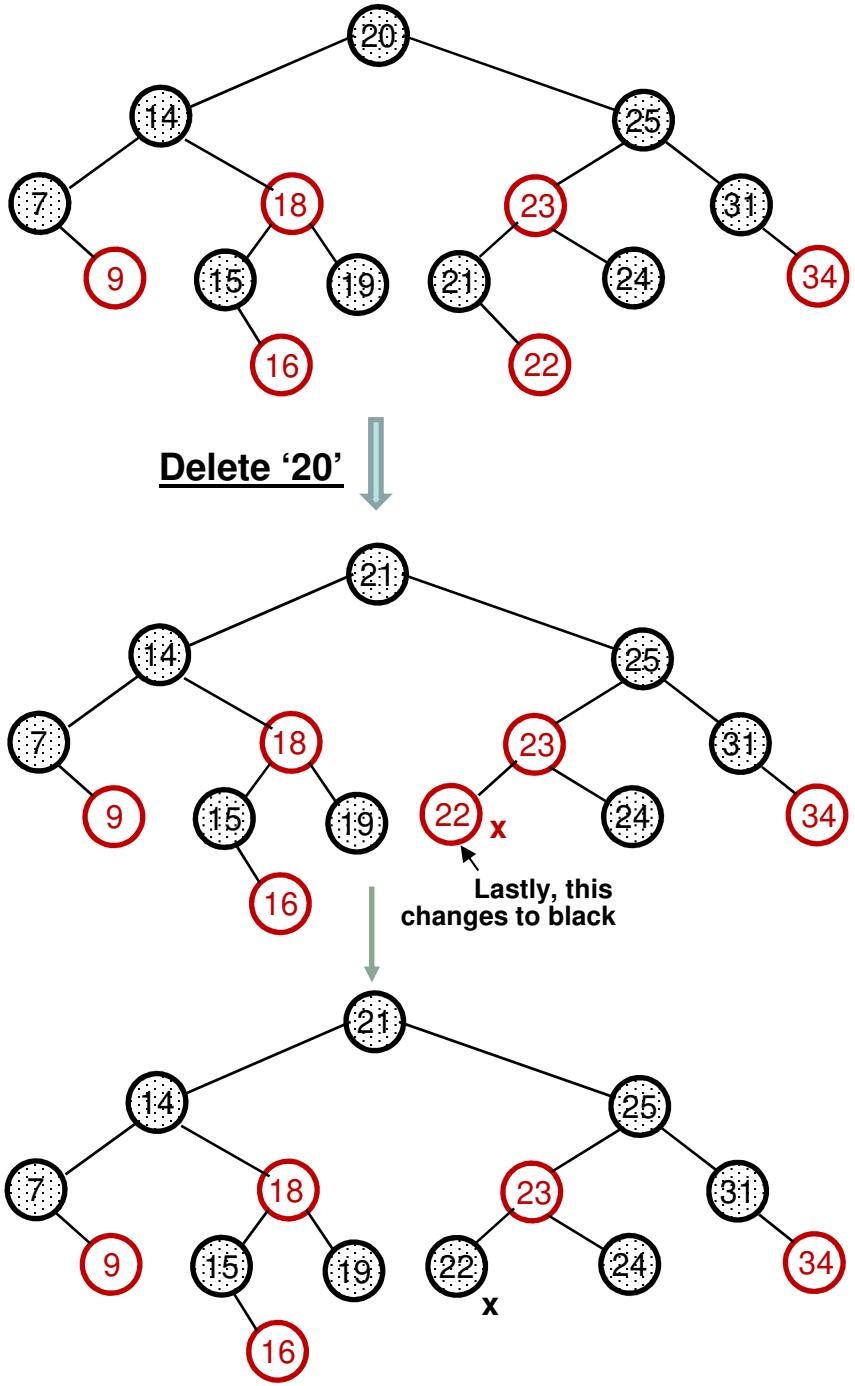
Deleting '11'

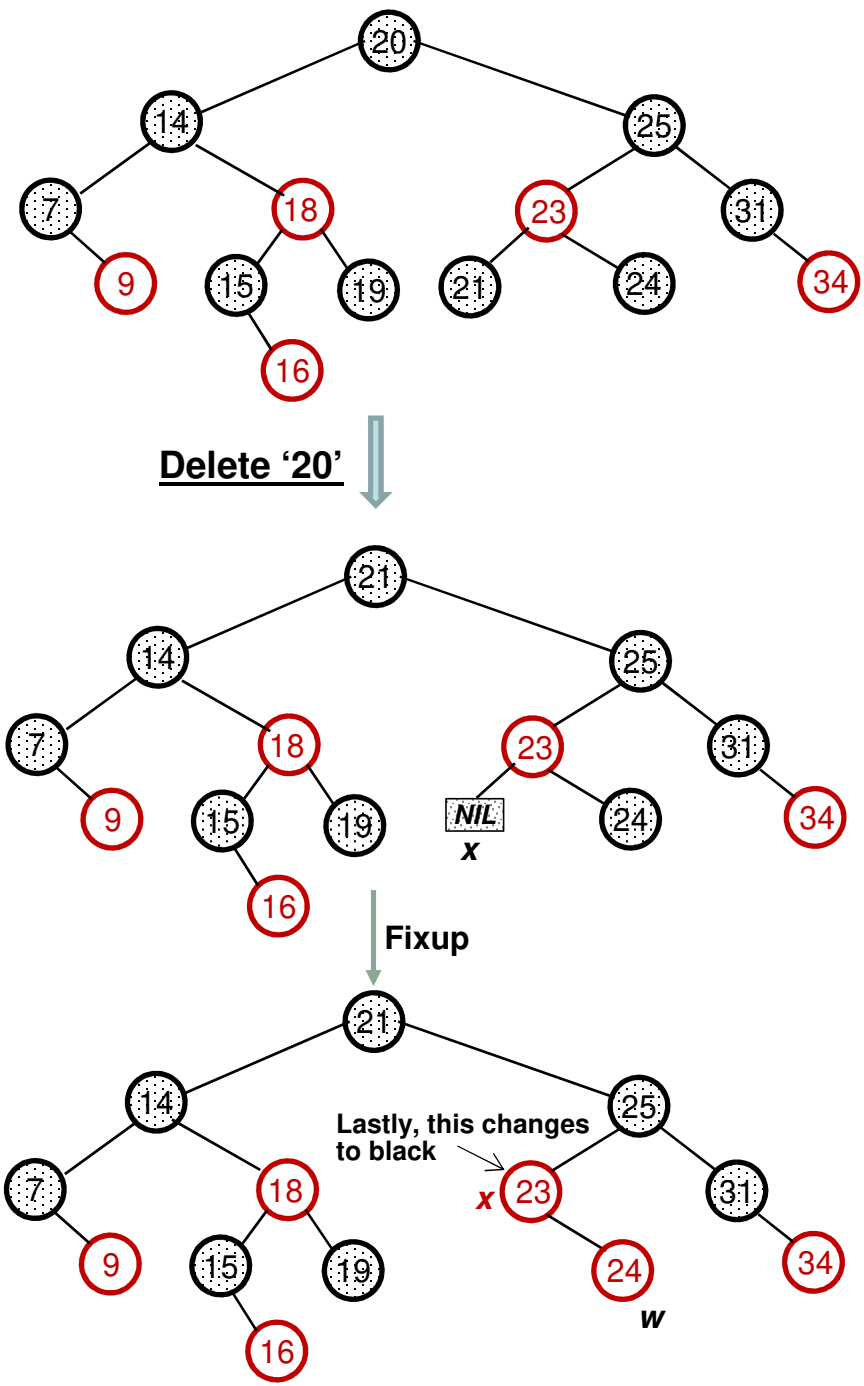


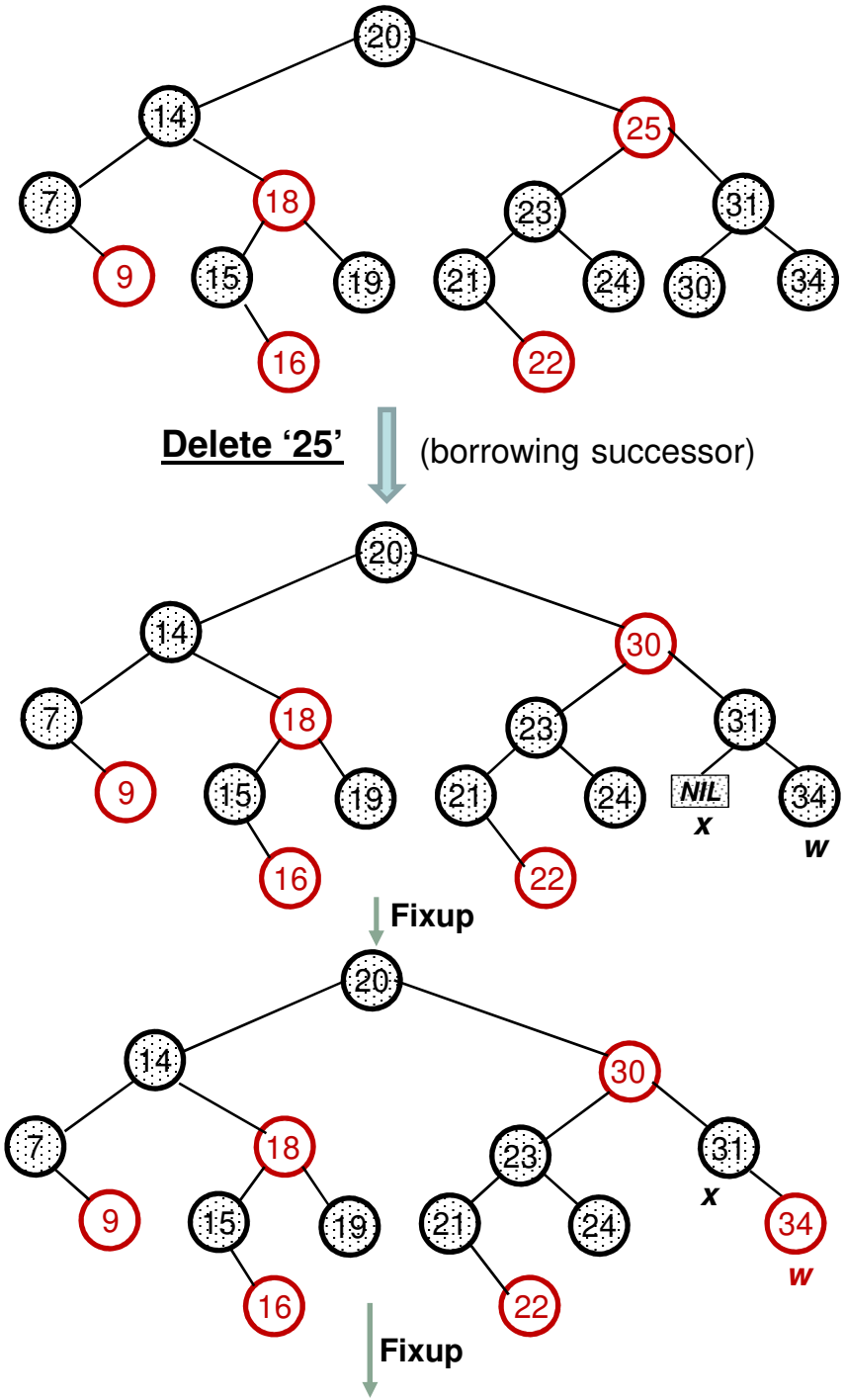
Lastly, this changes to black

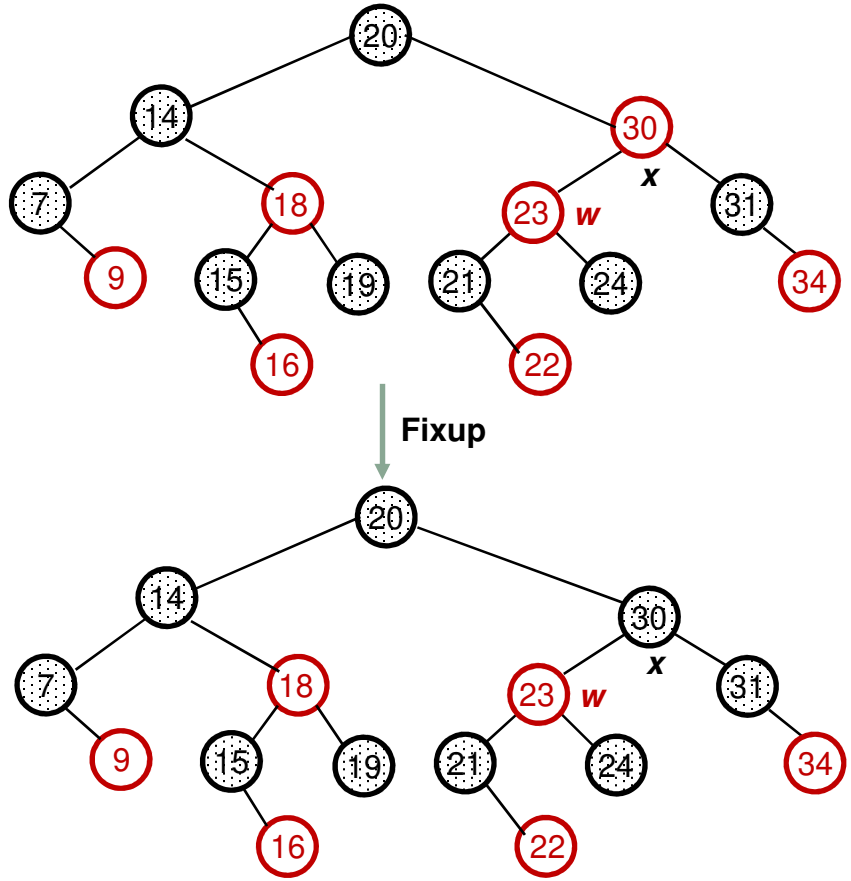

20
14
25
z
7
18
23
31
9
15
19
21
24
30
34
16
22
y
NIL
x
(a)
Delete '25'
20
14
30
7
18
23
31
9
15
19
21
24
NIL
34
x
w
16
22
Fixup
20
14
30
7
18
23
31
x
9
15
19
21
24
34
w
16
22
Fixup


Lastly, this changes
to black
20
14
30
x
7
18
23
w
31
9
15
19
21
24
34
16
22
Fixup
20
14
30
7
18
23
31
9
15
19
21
24
34
16
22
(b)
Delete '18'
(via predecessor
or successor)
20
14
30
7
16
23
31
9
15
19
21
24
34
22

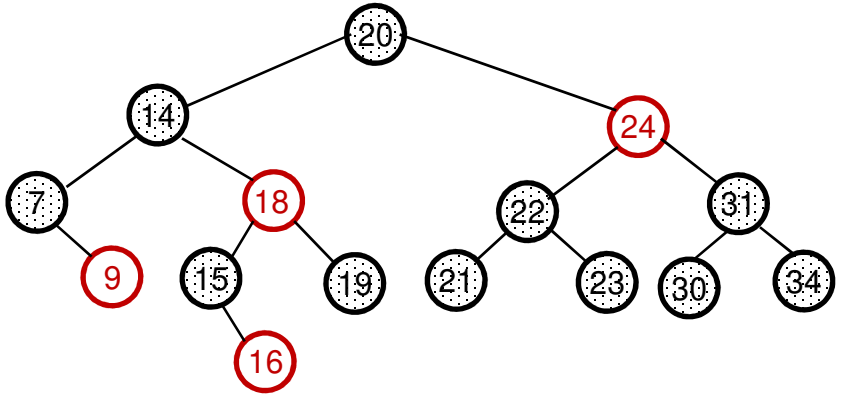

20
14
25
7
18
23
31
9
15
19
21
24
34
16
22
Delete ‘20’
21
14
25
7
18
23
31
9
15
19
22
x
24
34
16
Lastly, this
changes to black
21
14
25
7
18
23
31
9
15
19
24
34
16
x


20
14
25
7
18
23
31
9
15
19
21
24
34
16
Delete ‘20’
21
14
25
7
18
23
31
9
15
19
NIL
24
34
16
x
Fixup
21
14
25
Lastly, this changes
to black
7
18
x
23
31
9
15
19
24
34
16
w



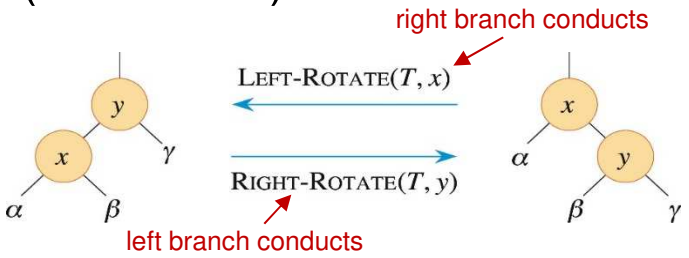


Another Solution (borrowing predecessor)



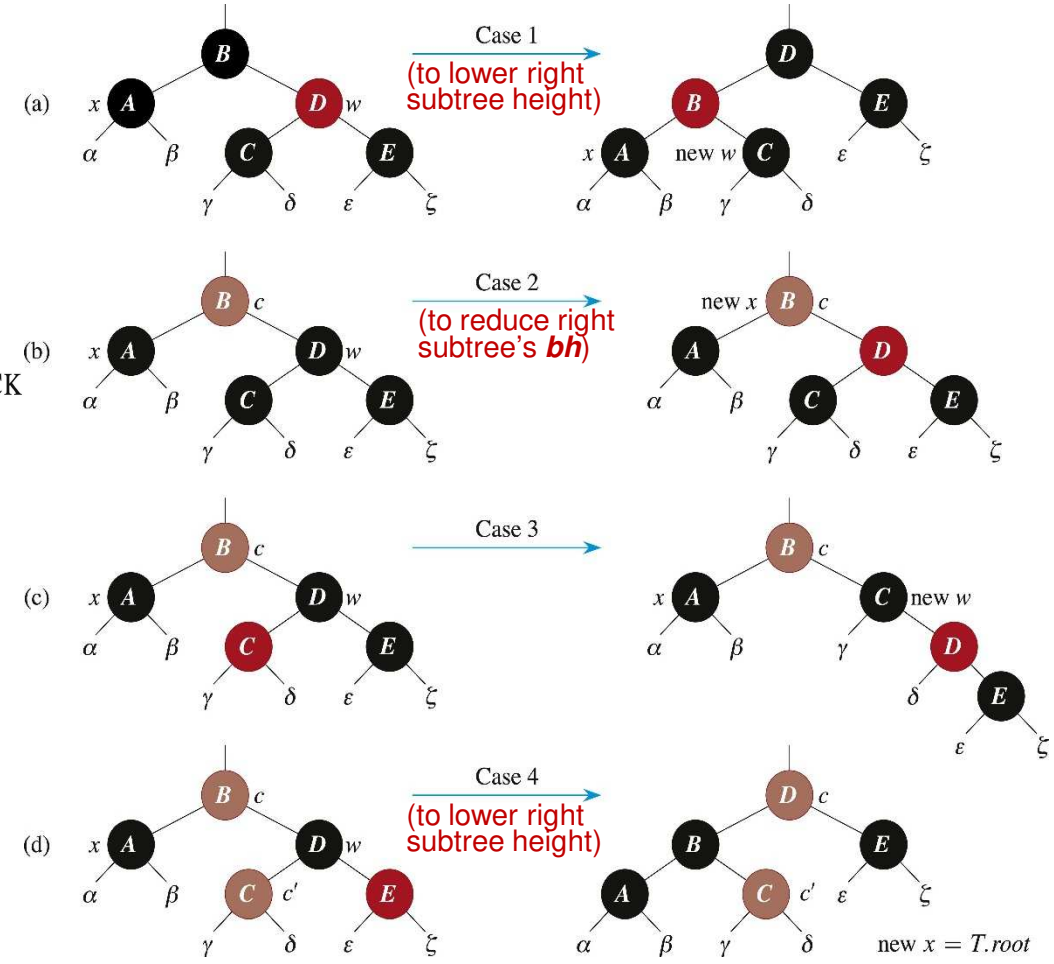
Red-Black Trees (continued)

+ RB-DELETE-FIXUP(T, z) to ensure RB properties, i.e., identical black-height, *bh*, from any node



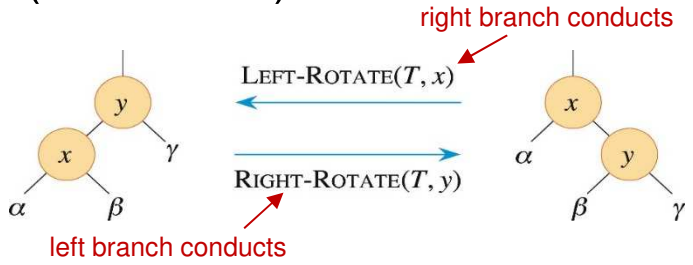
RB-DELETE-FIXUP(T, x) (based on the color of node x, which replaces y that replaces z)

```
1  while x ≠ T.root and x.color == BLACK
2      if x == x.p.left                    // is x a left child?
3          w = x.p.right                   // w is x's sibling
4          if w.color == RED
5              w.color = BLACK
6              x.p.color = RED
7              LEFT-ROTATE(T, x.p)                          } case 1
8              w = x.p.right
9          if w.left.color == BLACK and w.right.color == BLACK
10             w.color = RED                                } case 2
11             x = x.p
12         else
13             if w.right.color == BLACK
14                 w.left.color = BLACK
15                 w.color = RED
16                 RIGHT-ROTATE(T, w)                       } case 3
17                 w = x.p.right
18             w.color = x.p.color
19             x.p.color = BLACK
20             w.right.color = BLACK
21             LEFT-ROTATE(T, x.p)                          } case 4
22             x = T.root
```



Red-Black Trees (continued)

+ RB-DELETE-FIXUP(T, z) to ensure RB properties, i.e., identical black-height, *bh*, from any node



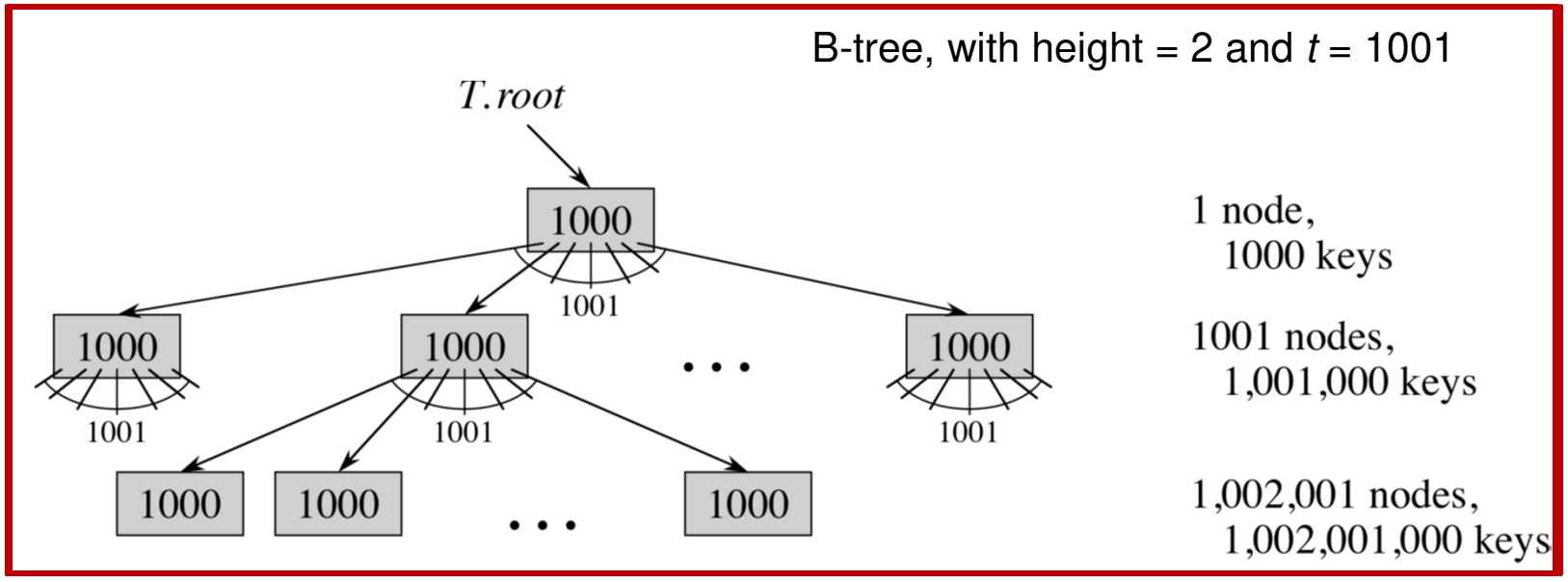
RB-DELETE-FIXUP(T, x) (based on the color of node *x*, which replaces y that replaces z)

```
1  while x ≠ T.root and x.color == BLACK
2      if x == x.p.left                    // is x a left child?
3          w = x.p.right                   // w is x's sibling
4          if w.color == RED
5              w.color = BLACK                          }
6              x.p.color = RED                          }
7              LEFT-ROTATE(T, x.p)                      }  case 1
8              w = x.p.right                            }
9          if w.left.color == BLACK and w.right.color == BLACK
10             w.color = RED                            }
11             x = x.p                                  }  case 2
12         else
13             if w.right.color == BLACK
14                 w.left.color = BLACK                 }
15                 w.color = RED                        }
16                 RIGHT-ROTATE(T, w)                   }  case 3
17                 w = x.p.right                        }
18             w.color = x.p.color                      }
19             x.p.color = BLACK                        }
20             w.right.color = BLACK                    }  case 4
21             LEFT-ROTATE(T, x.p)                      }
22             x = T.root
23     else // same as lines 3–22, but with "right" and "left" exchanged
24         w = x.p.left
25         if w.color == RED
26             w.color = BLACK
27             x.p.color = RED
28             RIGHT-ROTATE(T, x.p)
29             w = x.p.left
30         if w.right.color == BLACK and w.left.color == BLACK
31             w.color = RED
32             x = x.p
33         else
34             if w.left.color == BLACK
35                 w.right.color = BLACK
36                 w.color = RED
37                 LEFT-ROTATE(T, w)
38                 w = x.p.left
39             w.color = x.p.color
40             x.p.color = BLACK
41             w.left.color = BLACK
42             RIGHT-ROTATE(T, x.p)
43             x = T.root
44 x.color = BLACK
```

B-Trees

§ Balanced Search Trees (B-Trees) with t — non-root node has ≥ t & ≤ $2t$ children

+ balance achieved by keeping multiple (i.e., ≥ t-1) keys in each non-root node
+ node has at most $2t$-1 keys, called **full node** (with degree = $2t$)
+ keys stored in a node in non-decreasing order
+ node x (with $x.n$ keys) has $x.n$+1 children, pointed by $x.c_1$, $x.c_2$, …., $x.c_{x.n+1}$, then $k_1 \le x.key_1 \le k_2 \le x.key_2 \le \ldots \le x.key_{x.n} \le k_{x.n+1}$, where k_i is any key stored in subtree rooted at $x.c_i$ and $x.key_i$ is a key stored in node x
+ all leaves have the same height
+ simplest B-tree is for t = 2, called 2-3-4 tree (each *internal node* has 2, 3, or 4 children)

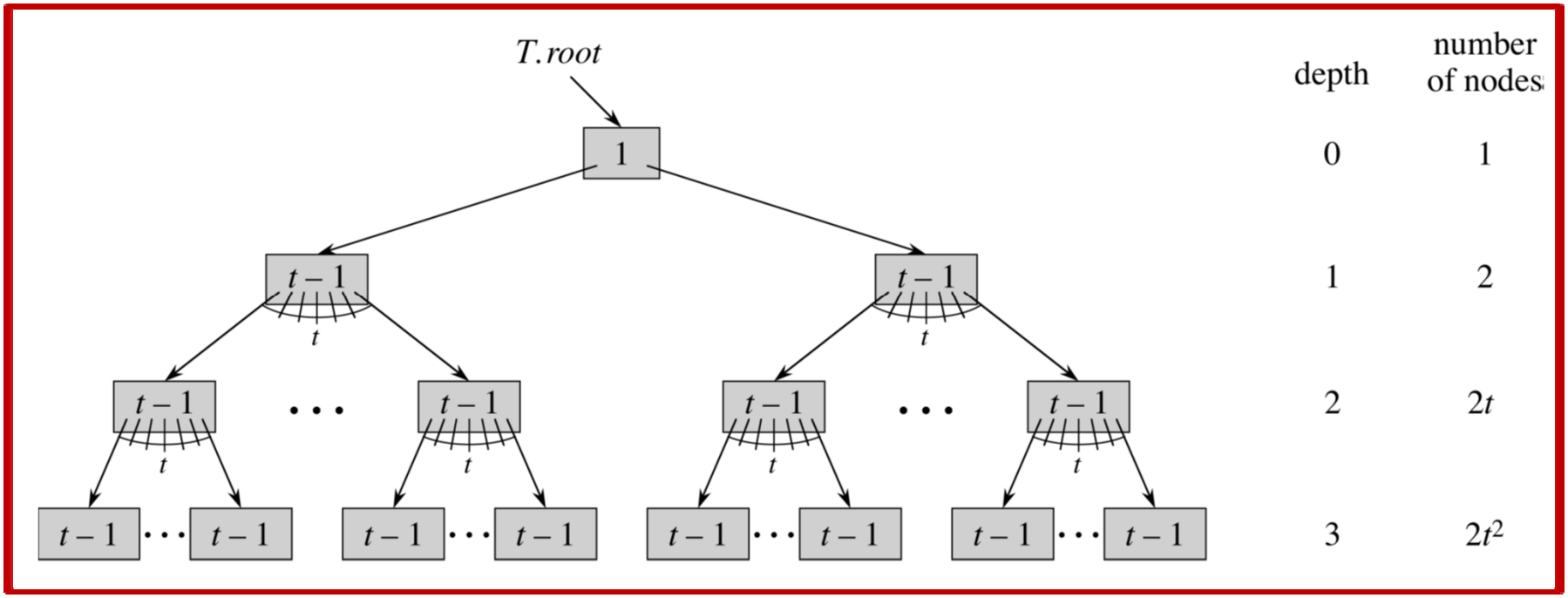


B-Trees (continued)

- **Theorem**

  For any n-key B-tree T of height h and minimum node degree $t \geq 2$ (i.e., number of keys in each non-root node $\geq t$-1), we have $h \leq \log_t \frac{n+1}{2}$.

  + Proof follows the fact of $n \geq 1 + (t-1) \cdot \sum_{i=1}^{h} 2 \cdot t^{i-1}$ illustrated in the figure below (to show the minimum possible number of keys in a B-tree with height = 3).



B-Trees (continued)

- **Basic operations**
  - + Searching for record with key = *k*: each internal node ***x*** makes an ($x.n$ +1)-way branching decision; returning (y, i) → node ***y*** and its index i with $y.key_i = k$

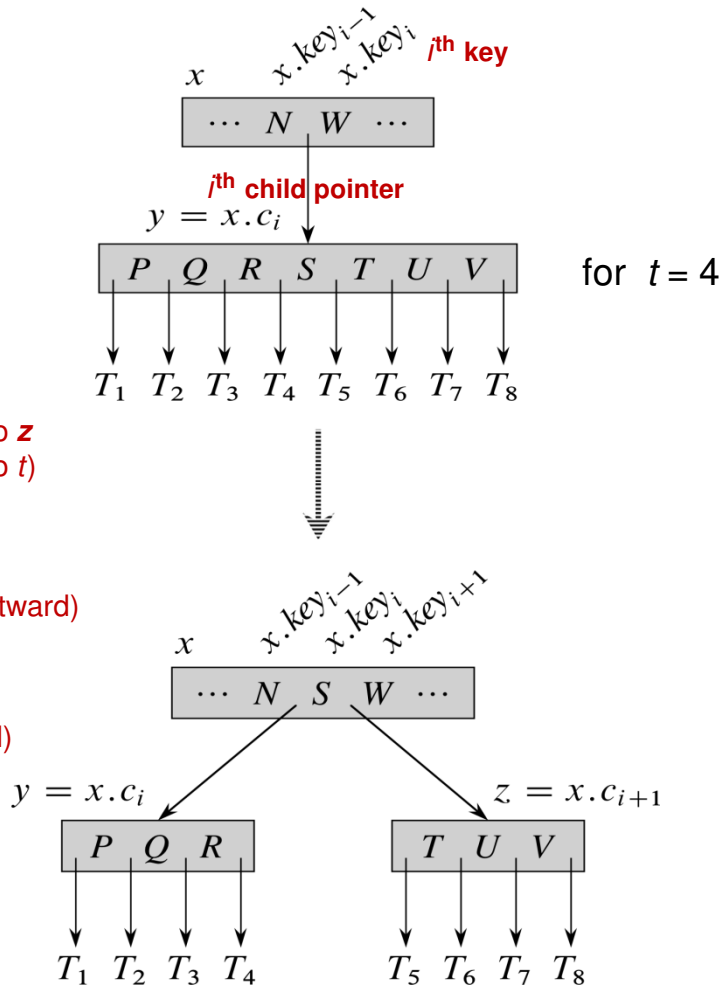
```
B-TREE-SEARCH(x, k)
1  i = 1
2  while i ≤ x.n and k > x.key_i
3       i = i + 1
4  if i ≤ x.n and k == x.key_i
5       return (x, i)
6  elseif x.leaf
7       return NIL
8  else DISK-READ(x.c_i)
9       return B-TREE-SEARCH(x.c_i, k)
```

x.leaf: Boolean variable to indicate if it is a leaf node

B-Trees (continued)

- **Basic operations**
  - + Splitting a full node: for *non-full internal node* ***x*** and a *full child* of ***x*** (say, ***y*** = $x.c_i$), ***y*** is split at median key *S*, which moves up into ***x***, with those > *S* forming a new subtree

```
B-TREE-SPLIT-CHILD(x, i)                    (split i-th child of x)
1   z = ALLOCATE-NODE()
2   y = x.c_i                               (Node x's i-th child is full)
3   z.leaf = y.leaf                         (status – leaf?)
4   z.n = t - 1
5   for j = 1 to t - 1
6       z.key_j = y.key_{j+t}
7   if not y.leaf
8       for j = 1 to t                      (move child pointers from y to z
9           z.c_j = y.c_{j+t}                with pointer # from 1 to t)
10  y.n = t - 1
11  for j = x.n + 1 downto i + 1
12      x.c_{j+1} = x.c_j                   (shift child pointers in x rightward)
13  x.c_{i+1} = z
14  for j = x.n downto i
15      x.key_{j+1} = x.key_j               (shift keys in x rightward)
16  x.key_i = y.key_t                       (move middle key in y up)
17  x.n = x.n + 1
18  DISK-WRITE(y)
19  DISK-WRITE(z)
20  DISK-WRITE(x)
```



B-Trees (continued)

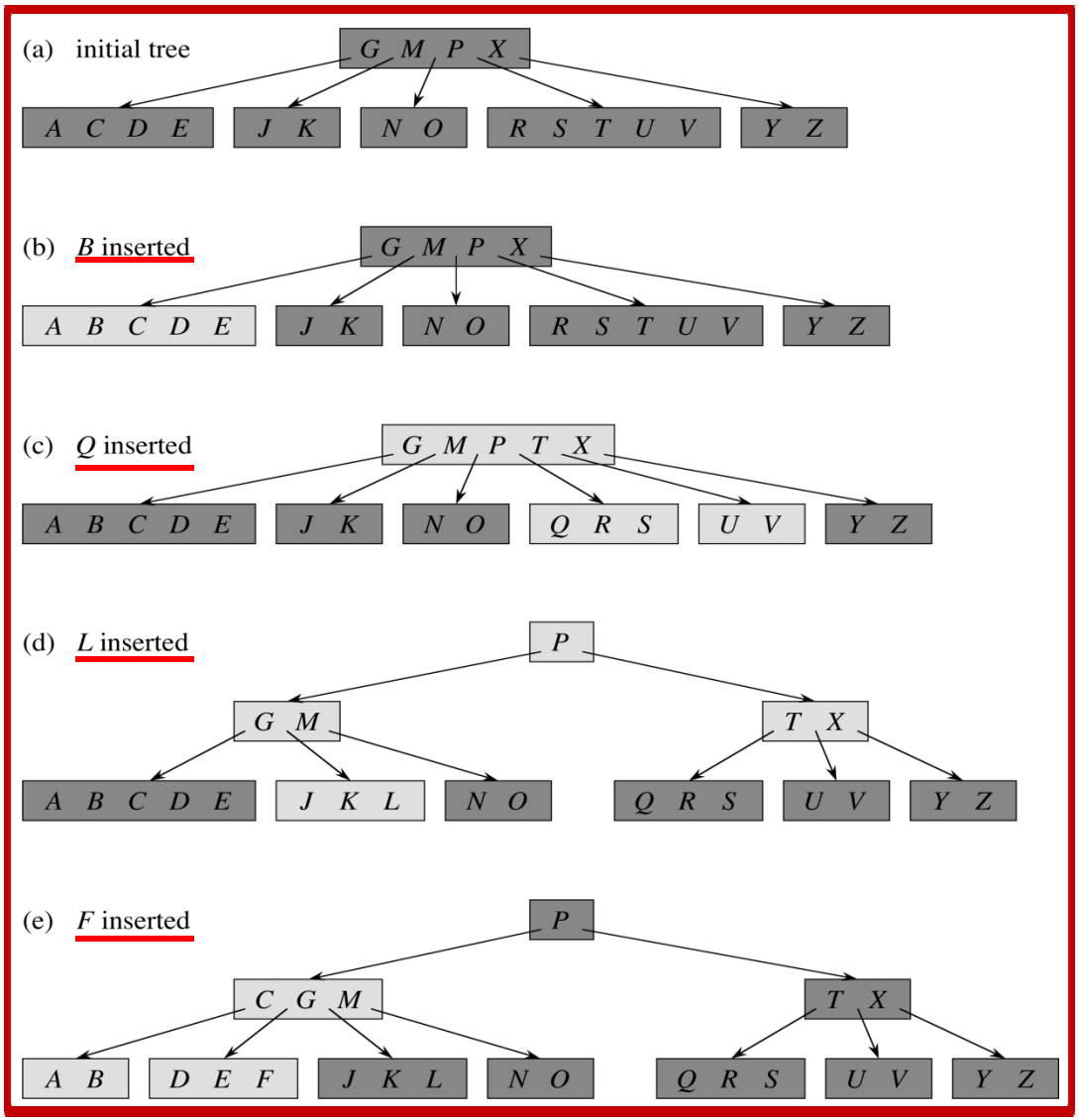
- **Basic operations**
  - + Inserting a key: B-tree ***T*** of height *h*, takes *O*(*h*) disk accesses (nodes kept as disk pages)
    - – *can't* insert the key into a newly created node *nor* into an internal node directly
      ⟶ so, insert it only to a **leaf node**
    - – **never** descend through a **full node**, achieved by B-TREE-SPLIT-CHILD, to avoid back-tracking altogether
    - – root split increases the height by 1

```
B-TREE-INSERT(T, k)
 1  r = T.root
 2  if r.n == 2t − 1          // splitting the root?
 3      s = ALLOCATE-NODE()
 4      T.root = s
 5      s.leaf = FALSE
 6      s.n = 0
 7      s.c_1 = r
 8      B-TREE-SPLIT-CHILD(s, 1)        (split 1st child of S)
 9      B-TREE-INSERT-NONFULL(s, k)
10  else B-TREE-INSERT-NONFULL(r, k)
```

```
B-TREE-INSERT-NONFULL(x, k)
 1  i = x.n
 2  if x.leaf                  (upon a leaf, key is inserted therein)
 3      while i ≥ 1 and k < x.key_i
 4          x.key_{i+1} = x.key_i      (shift key in x rightwar
 5          i = i − 1
 6      x.key_{i+1} = k
 7      x.n = x.n + 1
 8      DISK-WRITE(x)
 9  else while i ≥ 1 and k < x.key_i
10          i = i − 1
11      i = i + 1
12      DISK-READ(x.c_i)           (descend one level)
13      if x.c_i.n == 2t − 1       // full child?
14          B-TREE-SPLIT-CHILD(x, i)
15          if k > x.key_i
16              i = i + 1
17      B-TREE-INSERT-NONFULL(x.c_i, k)
```

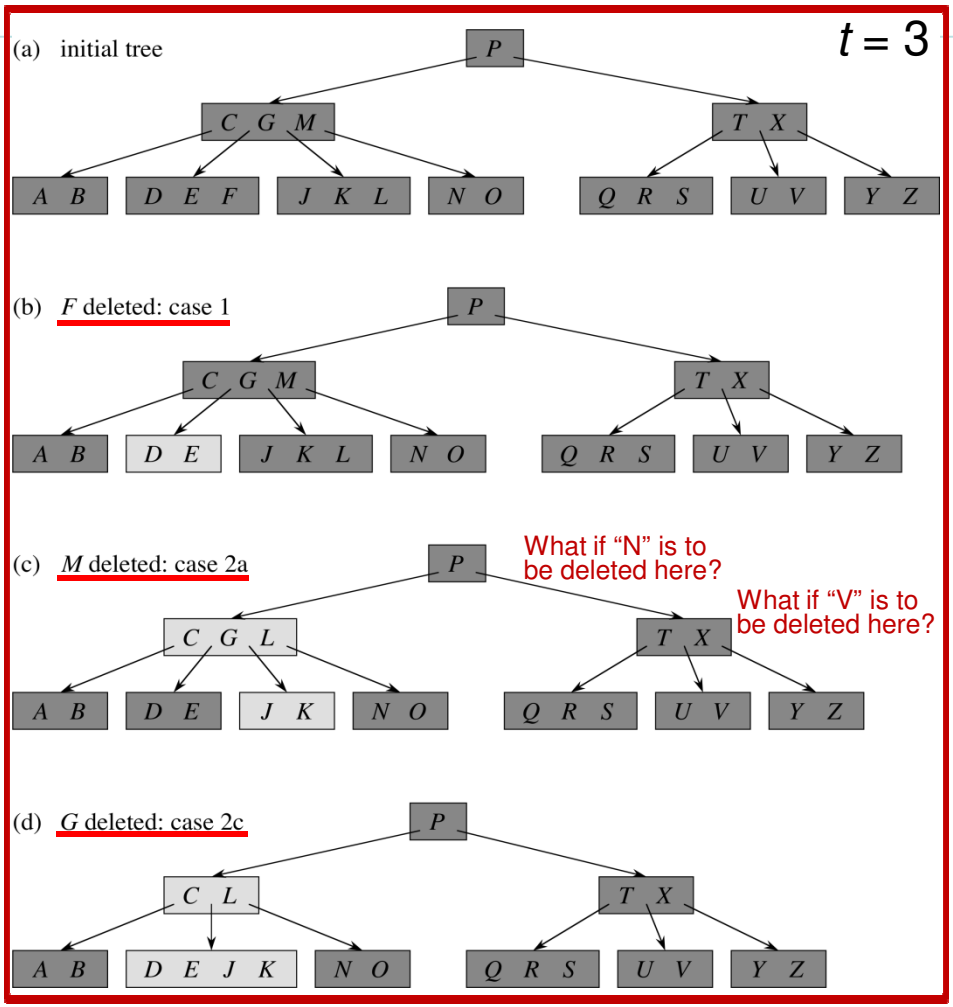
B-Trees (continued)

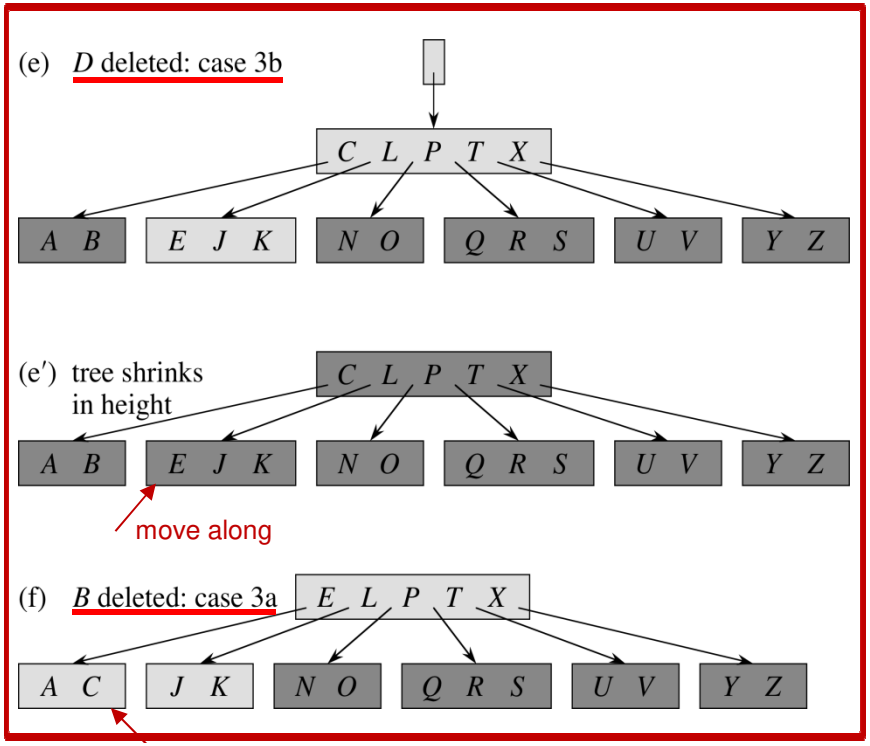
+ Inserting a key: B-tree ***T*** with $t = 3$ and modified nodes lightly shaded



B-Trees (continued)

- **Basic operations**
  - + Deleting a key: delete key *k* from the subtree rooted at ***x*** in B-tree ***T*** with the minimum degree of *t* (where the key may be in any node, not just at a leaf node)
    - – number of keys kept in any **internal node** upon descending through would be **at least *t***





B-Trees (continued)

+ Deleting key *k* from the subtree rooted at ***x*** in B-tree ***T*** with the minimum degree of *t*
  – **Deletion Procedure** sketched below:

1. If the key k is in node x and x is a leaf, delete the key k from x.
2. If the key k is in node x and x is an internal node, do the following:
   a. If the child y that precedes k in node x has at least t keys, then find the predecessor k' of k in the subtree rooted at y. Recursively delete k', and replace k by k' in x. (We can find k' and delete it in a single downward pass.)
   b. If y has fewer than t keys, then, symmetrically, examine the child z that follows k in node x. If z has at least t keys, then find the successor k' of k in the subtree rooted at z. Recursively delete k', and replace k by k' in x. (We can find k' and delete it in a single downward pass.)
   c. Otherwise, if both y and z have only $t-1$ keys, merge k and all of z into y, so that x loses both k and the pointer to z, and y now contains $2t-1$ keys. Then free z and recursively delete k from y.
3. If the key k is not present in internal node x, determine the root $x.c_i$ of the appropriate subtree that must contain k, if k is in the tree at all. If $x.c_i$ has only $t-1$ keys, execute step 3a or 3b as necessary to guarantee that we descend to a node containing at least t keys. Then finish by recursing on the appropriate child of x.
   a. If $x.c_i$ has only $t-1$ keys but has an immediate sibling with at least t keys, give $x.c_i$ an extra key by moving a key from x down into $x.c_i$, moving a key from $x.c_i$'s immediate left or right sibling up into x, and moving the appropriate child pointer from the sibling into $x.c_i$.
   b. If $x.c_i$ and both of $x.c_i$'s immediate siblings have $t-1$ keys, merge $x.c_i$ with one sibling, which involves moving a key from x down into the new merged node to become the median key for that node.

Targeted node with the key, but it may be non-leaf

This is Rule 1 - Key *k* present:
At internal node *x* which contains the key (say, *k*) to be deleted, it checks if predecessor or successor of *k* can be borrowed to replace *k* (and in this case, only the borrowed key is moved; and its associated pointer stays).

Prepare to walk down the tree

visited

This is Rule 2 - Key *k* not present:
While at internal node *x*, it checks if the root of the target child has at least *t* key
If not, it prepares that root node by borrowing a key (plus associated pointer, i.e., a subtree) from its left or right sibling, if possible.