Lecture 1 - Notation, functions and limits

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Today's Agenda

- Introductions
- Math Camp Logistics and schedule
- Notations, functions and limits!

Ovid/About Me

Schedule

	Monday 15th	Tuesday 16th	Wednesday 17th	Thursday 18th	Friday 19th
9:30 - 12:00	Oops!	Review Problem Sets + Lecture 2	Lecture 3	Lecture 4	Lecture 5
12:00 - 12:10	Break	Break	Break	Break	Break
12:10 - 1:00	Lecture 1	In-class Problem Set	In-class Problem Set	In-class Problem Set	Q and A with Prof Klasnja
1:00 - 2:00	Lecture 1	Lunch Break	Lunch Break	Lunch Break	Lunch Break
2:00 - 2:30	Break	Review Problem Sets	Review Problem Sets	Review Problem Sets	Q and A with Henry and Theo about Graduate Student Life
2:30 - 3:30	Computational bootcamp Software installation and Best Practices	Computational bootcamp R	Computational bootcamp R	Computational bootcamp Stata	Computational bootcamp Latex/Overleaf

Week's Agenda

- Lecture 1: Notation, functions and limits
- Lecture 2: Linear Algebra
- Lecture 3: Calculus 1 - Derivatives
- Lecture 4: Calculus 2 - Integrals and multivariate calculus
- Lecture 5: Probability

Goals of Math Camp

Do not worry - we do not expect you to master multivariate calculus!

- Familiarity
- Recognition
- Confidence

Agenda

Variables and measurements

Algebra

Functions

Series, sequences, and limits

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Variables and constants

Theory: A set of statements that involve concepts. The statements comprise assumptions, propositions, corollaries, and hypothesis.

Concepts must be measured so we use...

- **Variable:** A concept or measure[1] that takes different values in a given set.
- **Constant:** A concept or measure that has a single value for a given set.

[1]An operational indicator of a concept

Sets and sample spaces

A *set* is a collection of elements.

Common sets: Natural numbers ($\mathbb{N}$), Integers ($\mathbb{Z}$), Rational numbers ($\mathbb{Q}$), Real Numbers ($\mathbb{R}$), etc.

A set can be:

- **Finite or infinite:** $\mathbb{Z}$ is infinite, but all the integers from 1 to 10 is finite.
- **Countable or uncountable:** a countable set is one whose each of its element can be associated with a natural number (or an integer).
- **Bounded or unbounded:** A bounded set has finite size (but may have infinite elements).

Some important sets that we are going to use as political scientists:

- **Solution set:** a set that contains all solutions for an equation
- **Sample space:** a set that contains all values that a variable can take.

Unions and intersections

Much as sets contain elements, they also can contain, and be contained by, other sets.

Notation:

- $A \subset B$: "A is a **proper subset** of B" implies that set B contains all the elements in A, plus at least one more
- $A \subseteq B$: "A is a **subset** of B". In this case, it allows A and B to be the same.

Intersection: $A \cap B$. The set of elements common to two sets.

Union: $A \cup B$. The set that contains all elements in both sets.

Mutually exclusive sets: the intersection is the empty set.

Levels of measurement

Differences of kind: In some theories all we require of our concepts is that they distinguish one type from another.

- **Nominal**: No mathematical relationship. For example: Occupation (Grad Student, Researcher, Oncologist, Lawyer).

Differences of degree: At other times we are interested in differences in degree (whether one case *has more*, *is stronger*, etc.).

- **Ordinal:** There is a mathematical relationship ($<$, $\leq$, $>$, $\geq$, etc). For example, ideological scales (far left, moderate left, moderate, moderate right, far right).
- **Cardinal/Interval**: This requires that the distance between values be constant over the range of values. For example, some surveys ask you for your age (in years).
- **Ratio:** a cardinal variable that has a meaningful zero value. For example, Polity index goes from -10 to 10 in intervals of 1.

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Basic properties of arithmetic

For variables that stand for real numbers or integers, these properties will always hold:

- **Associative properties:**
  - $(a+b)+c=a+(b+c)$
  - $(a\cdot b)\cdot c=a\cdot(b\cdot c)$
- **Commutative properties:**
  - $a+b=b+a$
  - $a\cdot b=b\cdot a$
- **Distributive properties:**
  - $a\cdot(b+c)=ab+ac$
- **Identity properties:**
  - $a+0=a$
  - $a\cdot 1=a$
- **Inverse properties (for real numbers not integers):**
  - $(-a)+a=0$
  - $a^{-1}\cdot a=1$

Order of operations and special products

Order of operations - PEMDAS:

- Parentheses ()
- Exponents $(^x)$
- Multiplication $(\cdot)$
- Division $(\div)$
- Addition $(+)$
- Subtraction $(-)$

Special products

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(x+a)(x+b) = x^2 + (a+b)x + ab$
4. $(a+b)(a-b) = a^2 - b^2$
5. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+ac+bc)$
6. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Solving equations

Steps:

- Isolate the variable you are looking for
- Combine like terms
- Factor and cancel
- Operate on both sides of the equation
- Check your answer

For quadratic equations:

- Try to create the following general form: $ax^2 + bx + c = 0$
- Then, the solutions will be given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

Example

Suppose:

$x^2 + 8x + 6 = 0$

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$x^2 + 8x + 6 = 0$

Option 1: using the quadratic solution formula:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

In this case, $a = 1$, $b = 8$ and $c = 6$:

$x = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 6}}{2}$

$x = -4 \pm \sqrt{\frac{64 - 24}{4}}$

$x = -4 \pm \sqrt{\frac{40}{4}}$

$x = -4 \pm \sqrt{10}$

Solution: $x_1 = -4 + \sqrt{10}$ and $x_2 = -4 - \sqrt{10}$

Example

Suppose:

$x^2 + 8x + 6 = 0$

Option 2: use the special products and isolate x:

$x^2 + 8x + 6 = 0$

$x^2 + 8x = -6$

(+16 in both sides)

$x^2 + 8x + 16 = 10$

$x^2 + 2 \cdot 4 \cdot x + 4^2 = 10$

$(x+4)^2 = 10$

$(x+4) = \pm\sqrt{10}$

$x = -4 \pm \sqrt{10}$

Solution: $x_1 = -4 + \sqrt{10}$ and $x_2 = -4 - \sqrt{10}$

Inequalities

All pairs of real numbers have exactly one of the following relations: $x = y$, $x > y$, or $x < y$.

Solving inequalities is similar to solving equations but there are a few extra properties:

- Adding any number to each side of these relations will not change them; this includes inequalities.
- Multiplication:
  - If a is positive and $x > y$, then $ax > ay$.
  - If a is negative and $x > y$, then $ax < ay$.
- Division:
  - If a is positive and $x > y$, then $\frac{x}{a} > \frac{y}{a}$.
  - If a is negative and $x > y$, then $\frac{x}{a} < \frac{y}{a}$.

Exponent, logarithms and root rules

Exponent rules

- $x^a \cdot x^b = x^{a+b}$
- $x^a \cdot z^a = (xz)^a$
- $(x^a)^b = x^{ab}$
- $\frac{x^a}{x^b} = x^{a-b}$
- $\frac{x^a}{z^a} = (\frac{x}{z})^a$

Logarithm rules

- $log(x_1 \cdot x_2) = log(x_1) + log(x_2)$
  for $x_1, x_2 > 0$
- $log(\frac{x_1}{x_2}) = log(x_1) - log(x_2)$
  for $x_1, x_2 > 0$
- $log(x^b) = b \cdot ln(x)$
  for $x > 0$

Exponent, logarithms and root rules

Root rules

- $\sqrt{x} = x^{\frac{1}{2}}$
- $\sqrt[n]{x} \cdot \sqrt[n]{z} = \sqrt[n]{xz}$
  for $n > 1$
- $\frac{\sqrt[n]{x}}{\sqrt[n]{z}} = \sqrt[n]{\frac{x}{z}}$
  for $n > 1$

Look at M&S page 70 for all the ways you cannot simplify roots!

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Functions and its characteristics

Functions provide a specific description of the association or relationship between two (or among several) concepts (in theoretical work) or variables (in empirical work).

- Functions assign one element of the range to an element of the domain (one x is assigned to one y)
- Noted as $f(x) : A \rightarrow B$ or "f maps A into B"
- A is the *domain*, or set of possible x values.
- B is the *codomain*, or set of possibly y values.
- Once A has gone through the function, the resulting values constitute the *range* or image of the function.

Graph examples

In which of these graphs can we observe a function?

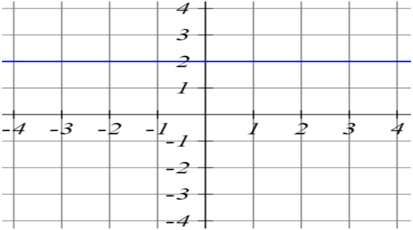


Figure: a)

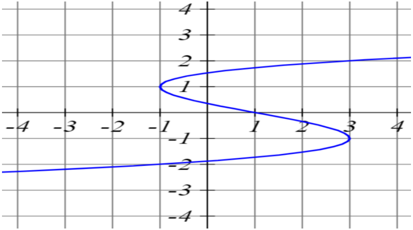


Figure: b)

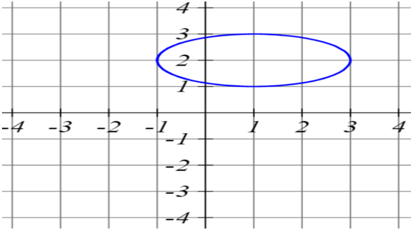


Figure: c)

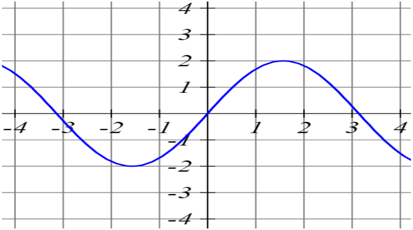


Figure: d)

Graph examples

In which of these graphs can we observe a function?

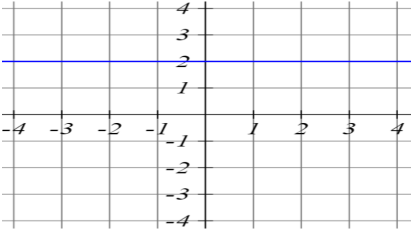


Figure: a)

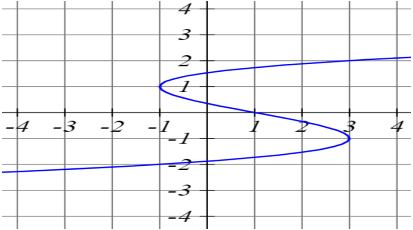


Figure: b)

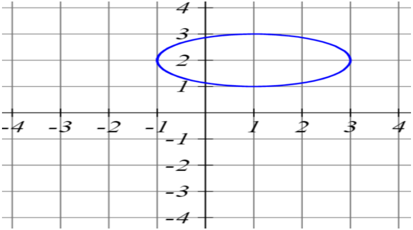


Figure: c)

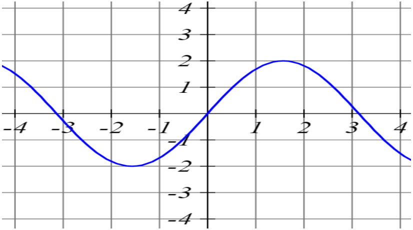


Figure: d)

Function composition and the inverse function

We can chain multiple functions using function composition.

- This is written either as $g \circ f(x)$ or $g(f(x))$.
- It is read as "g composed with f"
- Generally, $g \circ f(x) \neq f \circ g(x)$

Example: Suppose $f(x) = 2x$ and $g(x) = x^3$

- $g \circ f(x) = (2x)^3 = 8x^3$
- $f \circ g(x) = 2(x^3) = 2x^3$

The *inverse function* is the function that when composed with the original function returns the identity function: $f^{-1}(x) \circ f(x) = x$

How to find it? Just exchange $y(f(x))$ by x and isolate the "new" y.

Examples of functions of one variable - linear equation

- This is the classic linear equation $y = a + bx$ or $y = mx + n$
- a and b are constants and x is the variable.
- a is the intercept and b is the slope of the line, or the amount that y changes given a one-unit increase in x.

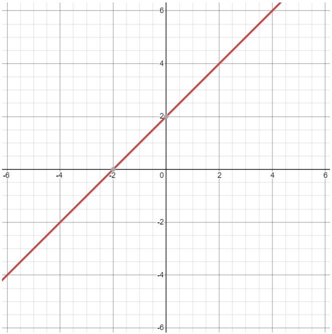


Figure: y=2+x

Examples of functions of one variable - Quadratic function

- This is the classical quadratic function: $f(x) = ax^2 + bx + c$ or $f(x) = \alpha + \beta_1 x + \beta_2 x^2$
- If we set $a > 0$ ($\beta_2 > 0$) we get a curve shaped like an U (a convex parabola).
- If we set $a < 0$ ($\beta_2 < 0$) we get a curve shaped like an inverse U (a concave parabola).

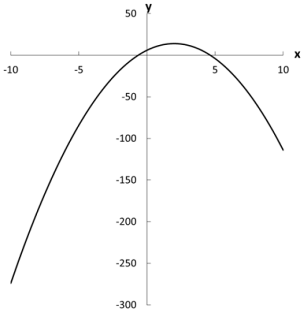
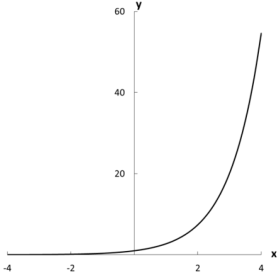


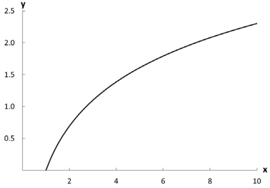
Figure: $y = -2x^2 + 8x + 6$

Exponent, logarithms and roots - Graphs

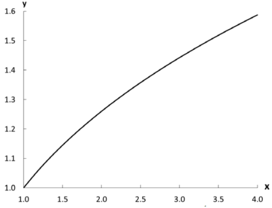
Exponential function



Logarithmic funtion



Root function



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Sequences and series

A *sequence* is an ordered list of numbers

- A sequence can be infinite, such as 1,2,3,4...
- Or a sequence can be finite, such as 5,10,15,20,25

A *series* is the sum of a sequence.

- Typically noted as $\sum_{i=1}^{N} x_i$ which means add the terms in the sequence beginning at x_1 and stopping at x_n.
- For an infinite sequence, $N = \infty$

Limits

Limits help us describe the behavior of a sequence, series, or function as it approaches a given value.

- A sequence/series/function *converges* if it has a finit limit.
- A sequence/series/function *diverges* if it has no limit or the limit is $\pm\infty$

The limit of a sequence is the number L such that as we approach infinity, x_i gets arbitrarily close to L. Noted as: $\lim_{i\to\infty} x_i = L$

Example of the limit of a sequence:

- The limit of the sequence $\{i\}_{i=1}^{\infty}$ does not have an "endpoint" and approaches infinity, so it diverges.
- The limit of the sequence $\{\frac{3}{10^i}\}_{i=1}^{\infty}$ approaches zero as $i \to \infty$, so it converges.

Limits

The limit of a series is similar, but you are not looking for an "endpoint". In this case, you are looking for the sum of all elements in an infinite sequence. For example:

- $\lim_{n\to\infty}\sum_{i=1}^{n} i = \infty$ So, this series is divergent.
- $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{1}{2^i}$. This series converges. Where?

For a function $y = f(x)$, the limit is the value of y that the function tend towards as small steps are taken towards a value $x = c$

If you are looking at a piecewise function, remember that you can approach c from above or below - the limits may differ!

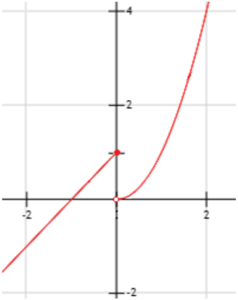
Limits

Example: Estimate the value of the following limits: $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ for the following function:

$$f(x) = \begin{cases} x+1 & x \le 0 \\ x^2 & x > 0 \end{cases}$$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x+1 = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$



Math Camp Exercises - Day 1

1. M&S pag 26, Exercises 1.a), 1.c), 1.e) and 1.g)
2. M&S pag 27, Exercise 5.
3. M&S pag 41 - 42, Exercises 1, 2, 3, 5, 7, 8, 9, 15, 16, 17, 19, 22, 25, 26, 28, 29, 31
4. M&S pag 78-79, Exercises 2, 3, 4, 7, 8, 9, 16, 17.
5. M&S pag 99, Exercise 5.