Lecture 2 - Linear algebra

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Agenda

Problem Set Review

Vector and Matrices

Systems of equations

Problem Set 1 Review

1. M&S pag 26, Exercises 1.a), 1.c), 1.e) and 1.g)
2. M&S pag 27, Exercise 5.
3. M&S pag 41 - 42, Exercises 1, 2, 3, 5, 7, 8, 9, 15, 16, 17, 19, 22, 25, 26, 28, 29, 31
4. M&S pag 78-79, Exercises 2, 3, 4, 7, 8, 9, 16, 17.
5. M&S pag 99, Exercise 5.

Exercise 1

1.8 EXERCISES

1.8.1 Constants and Variables and Levels of Measurement

1. Identify whether each of the following is a constant or a variable:

   a) Party identification of delegates at a political convention.
   b) War participation of the Great Powers.
   c) Voting record of members of Congress relative to the stated position of the president.
   d) Revolutions in France, Russia, China, Iran, and Nicaragua.
   e) An individual voter's vote choice in the 1992 presidential election.
   f) An individual voter's vote choice in the 1960–1992 presidential elections.
   g) Vote choice in the 1992 presidential election.

Exercise 2

5. Let $A = \{1, 5, 10\}$ and $B = \{1, 2, \ldots, 10\}$.
   a) Is $A \subset B$, $B \subset A$, both, or neither?
   b) What is $A \cup B$?
   c) What is $A \cap B$?
   d) Partition B into two sets, A and everything else. Call everything else C. What is C?
   e) What is $A \cup C$?
   f) What is $A \cap C$?

Exercise 3.1

2.4.1 Arithmetic Rules

Complete the following equations:

1. $x^1 =$ _____.
2. $-a \times (-b)^2 =$ _____.
3. $\sum_{i=1}^{4} x_i =$ _____.
4. $\prod_{m=6}^{9} x_m =$ _____.
5. $4! =$ _____.
6. $z^4 =$ _____.
7. $\sqrt[2]{9} =$ _____.
8. $\sqrt[3]{27} =$ _____.
9. $\left(\frac{3(2-4)}{2+3}\right)^3 =$ _____.

Exercise 3.2

2.4.3 Algebra Practice

15. Simplify into one term the following expressions:
    a) $xz + yz$.
    b) $mn + ln - pn$.
    c) $z \times y \times x - 2 \times y \times x$.
    d) $(z + x) \times y \times \frac{1}{z}$.
16. Simplify this expression as much as possible: $\frac{2x^2+20x+50}{2x^2-50}$.
17. Simplify this expression: $\frac{5+17x+4x+7}{42x}$.
18. Add these fractions: $\frac{2\sigma+13}{3\sigma} + \frac{4\sigma-5}{4\sigma}$.
19. Factor: $-7\theta^2 + 21\theta - 14$.
20. FOIL: $(2x - 3)(5x + 7)$.
21. Factor: $q^2 - 10q + 9$.
22. Factor and reduce: $\frac{\beta-\alpha}{\alpha^2-\beta^2}$.
23. Solve: $15\delta + 45 - 6\delta = 36$.
24. Solve: $.30\Omega + .05 = .25$.
25. Solve: $11 = (y + 1)2 + (6y - 12y)\frac{7}{2y}$.
26. Solve: $-4x^2 + 64 = 8x - 32$.
27. Complete the square and solve for x: $x^2 + 14x - 14 = 0$.
28. Complete the square and solve for y: $\frac{1}{3}y^2 + \frac{2}{3}y - 16 = 0$.
29. Solve using the quadratic formula: $2x^2 + 5x - 7$.

Exercise 4.1

3.4 EXERCISES

1. For each pair of ordered sets, state whether it represents a function or a correspondence:

   a) {5, -2, 7}, {0, 9, -8}
   b) {3, 1, 2, 6, -10}, {5, 7, 1, 4, 9}
   c) {3, 7, -4, 12, 7}, {8, -12, 15, -2, 17}

2. Simplify $h(x) = g(f(x))$, where $f(x) = x^2 + 2$ and $g(x) = \sqrt{x-4}$.
3. Simplify $h(x) = f(g(x))$ with the same f and g. Is it the same as your previous answer?
4. Find the inverse function of $f(x) = 5x - 2$.
5. Simplify $x^{-2} \times x^3$.
6. Simplify $(b \cdot b \cdot b) \times c^{-3}$.
7. Simplify $((qr)^\gamma)^\delta$.
8. Simplify $\sqrt{x} \times \sqrt[5]{x}$.
9. Simplify into one term $ln(3x) - 2ln(x+2)$.

Exercise 4.2

16. Rewrite the following by taking the log of both sides. Is the result a linear (affine) function?
$y = \alpha \times x_1^{\beta_1} \times x_2^{\beta_2} \times x_3^{\beta_3}$.

17. Rewrite the following by taking the log of both sides. Is the result a linear (affine) function?
$y = \alpha \times x_1^{\beta_1} \times \frac{x_2^{\beta_2}}{x_3^{\beta_3}}$.

Exercise 5.1

4.5 EXERCISES

1. Draw a graph to show that the sequence $\{1, -1, 1, -1, 1, -1\ldots\}$ is divergent.

2. Find the sum of the infinite series $\sum_{t=0}^{\infty}(\delta^t)^2$.

3. Show whether $f(x) = x + x^3$ has a limit at $x = 3$ and, if so, the value of the limit.

4. Show whether $f(x) = (x-3)(x+5)$ has a limit at $x = 4$ and, if so, the value of the limit.

5. Show whether $f(x) = \frac{3x^2-12}{x-2}$ has a limit at $x = 2$ and, if so, the value of the limit.

Morning challenge!

- Simplify $(-a)(-b)^3 - a^2$
- Simplify this expression assuming $x \neq -2$: $\frac{7+x^2-2x-15}{7x+14}$
- Solve: $5y + 5 - 2y = 11$
- Solve: $x^2 - 20 = 7x - 32$
- Simplify or evaluate the following expressions, using $f(x) = x^3 - 1$, $g(x) = ln(x)$ and $h(x) = \frac{1}{x^2}$
  - $f(g(x))$
  - $g(x)h(x)$
  - $f^{-1}(x)$
- Either find the following limits or show that they do not exist
  - $\lim_{x\to -2} \frac{x+2}{x^2-4}$
  - $\lim_{x\to 2} \frac{x-1}{x^3-8}$

Agenda

Vector and scalars

A *scalar* is a single element of a set

- e.g. x_i, 0, or 12.

A *vector* is an object with as many dimensions as the space in which it exists.

- e.g. $\vec{a} = (x_1, x_2)$, $\vec{b} = (0,0)$ in two dimensional space, or $C = (0,0,0)$ in 3-D space.

Dimension: the number of components in a vector.

Length of a vector: a measure of how big a vector is (do not confuse with the dimension of a vector).

$$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}$$

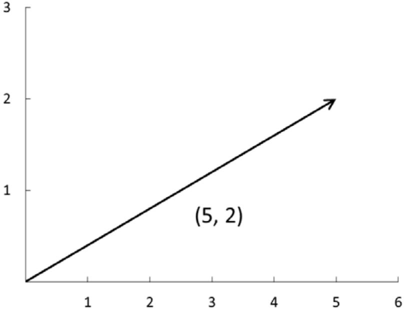

Figure: Vector (5,2)

Vector addition / subtraction

Vector sums

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}$$

Just add each corresponding element!

Length of a sum:

$$\|a+b\| = \sqrt{(a_1+b_1)^2+(a_2+b_2)^2}$$

Vector differences

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1-b_1 \\ a_2-b_2 \\ a_3-b_3 \end{bmatrix}$$

Subtract each corresponding element!

Length of differences:

$$\|a-b\| = \sqrt{(a_1-b_1)^2+(a_2-b_2)^2}$$

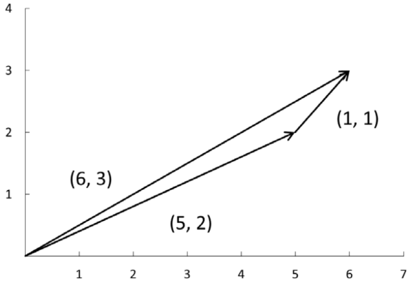

Figure: Vector addition: (5,2)+(1,1)

Scalar multiplication and dot product

Scalar multiplication

$$c \cdot \vec{a} = c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} c \cdot a_1 \\ c \cdot a_2 \\ c \cdot a_3 \end{bmatrix}$$

Multiply each element in the vector by the scalar!

Length of vector multiplied by a scalar:

$\|cx\| = |c|\|x\|$

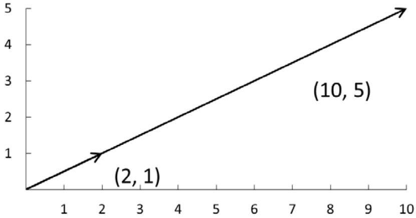

Figure: scalar multiplicaton: 5a where a=(2,1)

Dot product (or scalar product)

$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1 b_1 + a_2 b_2$$

This requires that the two vectors be of equal dimension.

Matrices

A *matrix* is a rectangular table of numbers or variables that are arranged in a specific order in rows and columns

- They can vary in size from a few columns and rows to hundreds of thousands of rows and columns. A dataset is a matrix.
- The size of a matrix is known as its dimensions and is expressed in terms of how many rows, n, and columns, m, it has, written as nxm (read "n by m").

Example:

$$A_{3x3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Important types of matrices

Zero matrix:

$$A_{3x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Diagonal matrix:

$$A_{3x3} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Identity matrix:

$$I_{1x1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric matrix:

$$A_{3x3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Lower triangular matrix:

$$A_{3x3} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Upper triangular matrix:

$$I_{1x1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Matrix transposition

The *transpose* switches the rows and columns of the matrix.

Example:

$$A_{2x3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T_{3x2} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Matrix addition, subtraction and scalar multiplication

Matrix addition and subtraction: simply add/subtract each corresponding element!

$$A_{3\times 3} \pm B_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$A_{3\times 3} \pm B_{3\times 3} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix}$$

Scalar multiplication:

$$5 \times A = \begin{bmatrix} 5 \times a_{11} & 5 \times a_{12} & 5 \times a_{13} \\ 5 \times a_{21} & 5 \times a_{22} & 5 \times a_{23} \\ 5 \times a_{31} & 5 \times a_{32} & 5 \times a_{33} \end{bmatrix}$$

Matrix multiplication

- In order to multiply two matrices, the number of columns in the first matrix must match the number of rows in the second matrix. e.g. $A_{n\times m}\cdot B_{m\times p}$
- This will result in a matrix of dimensions $n\times p$
- Therefore, $A\times B$ will not result in the same matris as $B\times A$
- Also, $A\times I=A$

$$A_{3x2}\times B_{2x3}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{bmatrix}\times\begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\end{bmatrix}$$

$$A_{3x2}\times B_{2x3}=\begin{bmatrix}a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} & a_{11}b_{13}+a_{12}b_{23}\\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} & a_{21}b_{13}+a_{22}b_{23}\\ a_{31}b_{11}+a_{32}b_{21} & a_{31}b_{12}+a_{32}b_{22} & a_{31}b_{13}+a_{32}b_{23}\end{bmatrix}$$

Matrix multiplication - example

Suppose $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$

$A \times B$?

- $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} = (2 \times 2) + (1 \times 5) = 9 = \begin{bmatrix} 9 & - \\ - & - \end{bmatrix}$
- $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} = (2 \times 4) + (1 \times 3) = 11 = \begin{bmatrix} 9 & 11 \\ - & - \end{bmatrix}$
- $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} = (3 \times 2) + (2 \times 5) = 16 = \begin{bmatrix} 9 & 11 \\ 16 & - \end{bmatrix}$
- $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} = (3 \times 4) + (2 \times 3) = 18 = \begin{bmatrix} 9 & 11 \\ 16 & 18 \end{bmatrix}$

$$A \times B = \begin{bmatrix} 9 & 11 \\ 16 & 18 \end{bmatrix}$$

Trace and determinant

The *trace* of an $n \times n$ square matrix is the sum of its diagonal.

$Tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ... + a_{nn}$

The *determinant* of a matrix is a commonly used function that converts the matrix into a scalar. Determinant for a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Is given by:

$|A| = a_{11}a_{22} - a_{12}a_{21}$

Example:

$$B = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$$

$|B| = (4 \cdot 2) - (3 \cdot 2) = 8 - 6 = 2$

Ovid ♡

Trace and determinant

- For 3×3 (or more dimensions) matrices, we can use the *Laplace expansion*.
- Consider the following matrix: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
- Define the *minor* of element a_{12} as the determinant of the submatrix a_{12}. The submatrix a_{12} is the matrix remaining when we eliminate the elements in the row and column of which a_{12} is the intersection.
- Thus, the minor of a_{12} is given by:

  $$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$
- Finally, define a cofactor as a minor with a prescribed sign, with the following structure:

  $$\begin{bmatrix} a_{11}(+) & a_{12}(-) & a_{13}(+) \\ a_{21}(-) & a_{22}(+) & a_{23}(-) \\ a_{31}(+) & a_{32}(-) & a_{33}(+) \end{bmatrix}$$

Trace and determinant

- Thus, the determinant of a three-by-three matrix is the sum of the products of elements in any given row or column, alternating in sign, and the determinants of specific 2×2 submatrices.
- For the first row, we would have:

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- Example: Suppose $B = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$
- In this case, this determinant is given by:

$$|B| = 3 \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} - 2 \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} + 5 \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 3 \times (-2) - 2 \times (5) + 5 \times (4) = 4$$

Inverse Matrix

- Square matrices are invertible if the determinant is non-zero.
- If the determinant of the matrix is zero, then it is *singular* and cannot be inverted.
- A matrix multiplied by its inverse returns the identity matrix: $A \times A^{-1} = A^{-1} \times A = I$
- For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
- The inverse (A^{-1}) is given by the following expression: $A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$
- And generally, by this expression: $A^{-1} = \frac{1}{|A|} C^T$
- Where C^T is the transpose of the matrix of cofactors of A. It is called also the adjoint Matrix of A: adj(A).

Inverse Matrix

Example 2×2 matrix:

- Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
- $A^{-1} = \frac{1}{(4-6)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{(-2)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$

Example 3×3 matrix:

- Suppose $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ -6 & -2 & 2 \end{bmatrix}$
- First, we have to calculate Matrix C. Therefore, we have to calculate each minor:

  $M_{11} = \begin{vmatrix} 4 & 3 \\ -2 & 2 \end{vmatrix} = (4 \times 2) - (-2 \times 3) = 8 + 6 = 14$

Inverse Matrix

Example 2×2 matrix:

- Repeating this process we get: $M_{11} = 14$; $M_{12} = 18$; $M_{13} = 24$; $M_{21} = 6$; $M_{22} = 8$; $M_{23} = 10$; $M_{31} = 2$; $M_{32} = 3$; $M_{33} = 4$
- With this we can calculate C:
$$C = \begin{bmatrix} 14 & -18 & 24 \\ -6 & 8 & -10 \\ 2 & -3 & 4 \end{bmatrix}$$
- Then $C^T = \begin{bmatrix} 14 & -6 & 2 \\ -18 & 8 & -3 \\ 24 & -10 & 4 \end{bmatrix}$
- The next step is to calculate the determinant of A: $|A| = 1 \times M_{11} - 2 \times M_{12} + 1 \times M_{13} = 14 - 36 + 24 = 2$
- Finally, $A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{2} \begin{bmatrix} 14 & -6 & 2 \\ -18 & 8 & -3 \\ 24 & -10 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -3 & 1 \\ -9 & 3 & \frac{-3}{2} \\ 12 & -5 & 2 \end{bmatrix}$

Matrix and vector properties

Table 12.2: Matrix and Vector Transpose Properties

Inverse	$(A^T)^T = A$
Additive property	$(A+B)^T = A^T + B^T$
Multiplicative property	$(AB)^T = B^T A^T$
Scalar multiplication	$(cA)^T = cA^T$
Inverse transpose	$(A^{-1})^T = (A^T)^{-1}$
If A is symmetric	$A^T = A$

Table 12.3: Matrix Determinant Properties

Transpose property	$\det(A) = \det(A^T)$
Identity matrix	$\det(I) = 1$
Multiplicative property	$\det(AB) = \det(A)\det(B)$
Inverse property	$\det(A^{-1}) = \frac{1}{\det(A)}$
Scalar multiplication $(n \times n)$	$\det(cA) = c^n \det(A)$
If A is triangular or diagonal	$\det(A) = \prod_{i=1}^{n} a_{ii}$

Table 12.4: Matrix Inverse Properties

Inverse	$(A^{-1})^{-1} = A$
Multiplicative property	$(AB)^{-1} = B^{-1}A^{-1}$
Scalar multiplication $(n \times n)$	$(cA)^{-1} = c^{-1}A^{-1}$ if $c \neq 0$

Agenda

Problem Set Review

Vector and Matrices

Systems of equations

Linear combinations

Add or subtract scalar multiples (a_n) of vectors to get a new vector.

$$a_1x_1 + a_2x_2 + ... + a_nx_n$$

Example: Imagine three vectors of GOVT-701 grades of the incoming PhD students in Fall 2021.

- A corresponds to a mid-term (30% of the final grade)
- B to Problem Sets (20% of the final grade)
- C to the final exam (50% of the final grade)

With this information you can calculate a new vector with the final grade of each student:

$$0.3 \times \begin{bmatrix} 85 \\ 100 \\ ... \\ 95 \end{bmatrix} + 0.2 \times \begin{bmatrix} 95 \\ 100 \\ ... \\ 100 \end{bmatrix} + 0.5 \times \begin{bmatrix} 95 \\ 80 \\ ... \\ 80 \end{bmatrix} = \begin{bmatrix} 92 \\ 90 \\ ... \\ 88.5 \end{bmatrix}$$

Linear Independence

A set of vectors is *linearly independent* if we cannot write any vector in the set as a combination of other vectors in the set.

So the only way for $a_1x_1 + a_2x_2 + ... + a_nx_n = 0$, is if every scalar multiplier is zero.

Examples:

- Suppose $v_1 = (1,3)$ and $v_2 = (3,9)$. These vectors are not linearly independent because $3v_1 - v_2 = 0$
- Suppose $v_1 = (1,3)$ and $v_2 = (2,9)$. These vectors are linearly independent because the only a_i that allow $a_iv_1 - a_iv_2 = 0$ are $a_i = 0$

Matrix rank

The **rank** of this matrix is the maximum number of linearly independent rows (or columns).

The main question here is: **How many rows (or columns) of the matrix give us new information?**

We can test linear independence by taking the determinant of a matrix:

- If the determinant is non-zero, the vectors are linearly independent.
- If the determinant is zero, they are dependent.

Systems of equations

How determined is your system of equations?

- Uniquely determined
  - Same number of equations and variables to solve for.
  - Yields one unique solution
- Overdetermined
  - More equations than unknowns
  - Equations may be contradictory
- Undetermined
  - More unknown than equations
  - Can occur if equations are not linearly independent - each equation must give us new information if we want to solve the system.
  - Infinite number of possible solutions

Solving systems of equations - Substitution

1. Pick and equation and solve for one of the variables.
2. Plug this into the other equations.
3. Pick another variable and solve for that one, then plug it in. Repeat process until only one variable is left and you can solve for the numerical solution.
4. Working backwards, plug the value for the variable that you solved for into one of the equations for another variable.
5. Repeat until you have a numerical solution for every variable.

Solving systems of equations - Substitution

Example:

Suppose:

$$x+y=5 \tag{1}$$
$$3x-2y=5 \tag{2}$$

From (1), you can get: $x=5-y$. Replacing x in (2):

$$\begin{aligned} 3(5-y)-2y &= 5 \\ 15-3y-2y &= 5 \\ -5y &= -10 \\ y &= 2 \end{aligned}$$

Finally, replacing in (1):

$$\begin{aligned} x+2 &= 5 \\ x &= 3 \end{aligned}$$

Solving systems of equations - Elimination

Assumptions:

- We can reorder equations however we like.
- We can manipulate equations by applying operations to both the left and right hand.

Suppose the following systems of equation:

$$ax + by + cz = d \quad (1)$$
$$ex + fy + gz = h \quad (2)$$
$$jx + ky + mz = n \quad (3)$$

This system of equations can be expressed with matrices and vectors:

$$\begin{bmatrix} a & b & c \\ e & f & g \\ j & k & m \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ h \\ n \end{bmatrix}$$

Solving systems of equations - Elimination

Your goal is to create an upper triangular matrix:

$$\begin{bmatrix} a & b & c \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ s \\ t \end{bmatrix}$$

Steps:

1. Keep one equation with all three variables
2. Use combinations of the equations to systematically eliminate variables.
3. Organize equations in upper triangular matrix form.
4. Back substitute to solve.

Solving systems of equations - Elimination

Example:

$$2x - y + 3z = 9 \tag{1}$$
$$x + 4y - 5z = -6 \tag{2}$$
$$x - y + z = 2 \tag{3}$$

- Equation (1) remain as is.
- Eliminate x from (2), creating (2)': -2×(2)+(1):

$$-2(x+4y-5z)+(2x-y+3z) = -2(-6)+9$$
$$-9y+13z = 21$$

- Eliminate x from (3), creating (3)': -2×(3)+(1):

$$-2(x-y+z)+(2x-y+3z) = -2(2)+9$$
$$y+z = 5$$

Solving systems of equations - Elimination

- Eliminate y from (3)': 9×(3)'+(2)':

$$\begin{aligned} 9(y+z)+(-9y+13z) &= 9(5)+21 \\ 22z &= 66 \end{aligned}$$

- $z = 3$
- Plug into equation (2)':

$$\begin{aligned} -9y+13(3) &= 21 \\ -9y &= -18 \end{aligned}$$

- $y = 2$
- Substitute z and y into equation (1):

$$\begin{aligned} 2x-(2)+3(3) &= 9 \\ 2x &= 2 \end{aligned}$$

- $x = 1$

Ovid ♡

Solving systems of equations - Matrix inversion

Consider $Ax = c$

1. Check the determinant of A. If it is non-zero, can be inverted.
2. Calculate A^{-1}
3. The multiplication of A^{-1} by the vector of constants (c) will provide the answer

Example:

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix}$$

- $|A| = 2 \times (4-5) - (-1) \times (1+5) + 3 \times (-1-4) = 2 \times (-1) + 1 \times (6) + 3 \times (-5) = -11$
- $M_{11} = -1$; $M_{12} = 6$; $M_{13} = -5$; $M_{21} = 2$; $M_{22} = -1$; $M_{23} = -1$; $M_{31} = -7$; $M_{32} = -13$; $M_{33} = 9$

Solving systems of equations - Matrix inversion

- Thus, the matrix C^T (or adj(A)) is:

$$C^T = \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix}$$

- Then, A^{-1}:

$$A^{-1} = \frac{1}{-11} \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix}$$

- Finally, the results are given by the following expression:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix} \times \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$