Lecture 3 - Calculus I

Theodore Landsman

Georgetown University

August 17th - 2022

Agenda

Differentiation

Optimization

Morning challenge!

- Find $C = \begin{bmatrix} 4 & 2 \\ -\frac{2}{3} & -1 \end{bmatrix} + \begin{bmatrix} 3 & 43 \\ -4 & 3 \end{bmatrix}$
- Suppose $A = \begin{bmatrix} 3 & 6 \\ -4 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} -10 & -4 \\ 5 & 2 \end{bmatrix}$. Find AB and BA. Is $AB = BA$?
- Find x, y and z: $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & x \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 2 & 7 & 1 \\ 8 & 2 & 8 \\ 1 & 8 & y \end{bmatrix} = \begin{bmatrix} z & 55 & 19 \\ 51 & 89 & 59 \\ 57 & 66 & 60 \end{bmatrix}$
- Suppose $A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$. Find A^{-1}
- The sum of the digits of a two-digit number is 7. When the digits are reversed, the number increases by 27. What is the number?

Agenda

Types of changes and derivatives

Types of changes:

- Discrete change: It is a measure of change in a variable across two *discrete* moments in time. For example, changes on the GDP of a country between two years.
- Instantaneous change: it is a measure of change in a variable at a specific moment in time. Using two moments, we want to identify a change in a variable when the interval between these two moments gets smaller and smaller, and for that we use limits.

A *derivative* is the *instantaneous rate of change* of a function.

Notation: the derivative of a function is represented as a $\frac{dy}{dx}$ or $f'(x)$

Definition of a derivative

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

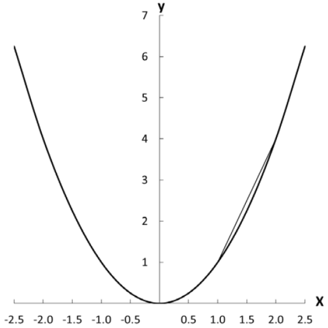


Figure 5.1: Graph of $y = x^2$ with Secant Line

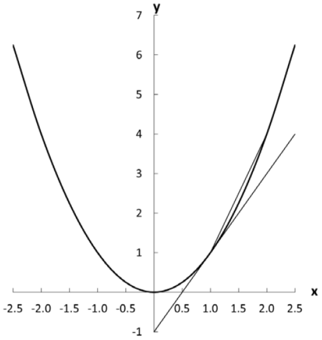


Figure 5.2: Graph of $y = x^2$ with Tangent Line

Example

Calculate $f'(x)$ and $f''(x)$ using the definition of the derivative:

$$f(x) = x^3 - 16x + 7$$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h\to 0} \frac{((x+h)^3 - 16(x+h) + 7) - (x^3 - 16x + 7)}{h}$$
$$f'(x) = \lim_{h\to 0} \frac{((x^3 + 3x^2h + 3xh^2 + h^3) - 16x - 16h + 7) - (x^3 - 16x + 7)}{h}$$
$$f'(x) = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 16h}{h}$$
$$f'(x) = \lim_{h\to 0} 3x^2 + 3xh + h^2 - 16$$
$$f'(x) = 3x^2 - 16$$
$$f''(x) = 6x$$

Five minutes practice

Calculate $f'(x)$ and $f''(x)$ using the definition of the derivative:

$$f(x) = x^2 - 4x + 3$$

Five minutes practice!

Calculate $f'(x)$ and $f''(x)$ using the definition of the derivative:

$$f(x) = x^2 - 4x + 3$$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{((x+h)^2 - 4(x+h) + 3) - (x^2 - 4x + 3)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{((x^2 + 2xh + h^2) - 4x - 4h + 3) - (x^2 - 4x + 3)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2xh + h^2 - 4h}{h}$$

$$f'(x) = \lim_{h\to 0} 2x + h - 4$$

$$f'(x) = 2x - 4$$

$$f''(x) = 2$$

Rules of differentiation

Table 6.1: List of Rules of Differentiation

Sum rule	$(f(x)+g(x))' = f'(x)+g'(x)$
Difference rule	$(f(x)-g(x))' = f'(x)-g'(x)$
Multiply by constant rule	$f'(ax) = af'(x)$
Product rule	$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$
Quotient rule	$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain rule	$(g(f(x))' = g'(f(x))f'(x)$
Inverse function rule	$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$
Constant rule	$(a)' = 0$
Power rule	$(x^n)' = nx^{n-1}$
Exponential rule 1	$(e^x)' = e^x$
Exponential rule 2	$(a^x)' = a^x(\ln(a))$
Logarithm rule 1	$(\ln(x))' = \frac{1}{x}$
Logarithm rule 2	$(\log_a(x))' = \frac{1}{x(\ln(a))}$
Trigonometric rules	$(\sin(x))' = \cos(x)$
	$(\cos(x))' = -\sin(x)$
	$(\tan(x))' = 1+\tan^2(x)$
Piecewise rules	Treat each piece separately

Rules of differentiation Cont

Table 6.1: List of Rules of Differentiation

Sum rule	$(f(x)+g(x))' = f'(x)+g'(x)$
Difference rule	$(f(x)-g(x))' = f'(x)-g'(x)$
Multiply by constant rule	$f'(ax) = af'(x)$
Product rule	$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$
Quotient rule **Tricky Differentiation**	$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain rule	$(g(f(x))' = g'(f(x))f'(x)$
Inverse function rule	$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$
Constant rule	$(a)' = 0$
Power rule	$(x^n)' = nx^{n-1}$
Exponential rule 1	$(e^x)' = e^x$ **Easy Differentiation**
Exponential rule 2	$(a^x)' = a^x(\ln(a))$
Logarithm rule 1	$(\ln(x))' = \frac{1}{x}$
Logarithm rule 2	$(\log_a(x))' = \frac{1}{x(\ln(a))}$
Trigonometric rules	$(\sin(x))' = \cos(x)$
	$(\cos(x))' = -\sin(x)$
	$(\tan(x))' = 1+\tan^2(x)$
Piecewise rules	Treat each piece separately

Examples

- $f(x) = 2x^9 + x^2 + 8$
- $f(x) = \frac{x}{x^2+3}$
- $f(x) = (x+5)(x^3 + x^2 + 2)$
- $f(x) = \frac{5}{x^5}$
- $f(x) = \frac{1}{\sqrt{x}}$
- $f(x) = \frac{e^{2x}}{x^2}$
- $f(x) = ln(2x^4 - x^3 + 3x^2 - 3x)$

Agenda

What is optimization?

A method for finding all the extrema of a function.

In political science and economics, several approaches assume that (political) agents want to maximize/minimize an objective function. For example:

- Maximization of Utility
- Minimization of Risk
- Maximization of Welfare
- Maximization of the survival probability
- Minimization of errors

Extreme value theorem

A real-valued function that is continuous on a closed and bounded interval [a,b] must hit both its global maximum and minimum on that interval, at least once each.

Definitions:

- A high point is called a *maximum*.
- A low point is called a *minimum*.
- An extrema is local whenever it is the largest (or smallest) value of the function over some interval of values in the domain of a function (over some interval on the x-axis). Think of them as "valleys" and "peaks".
- A global extremum is the highest (or lowest) point on the function.

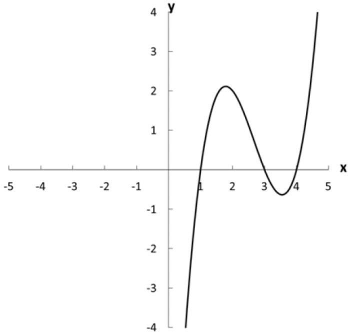
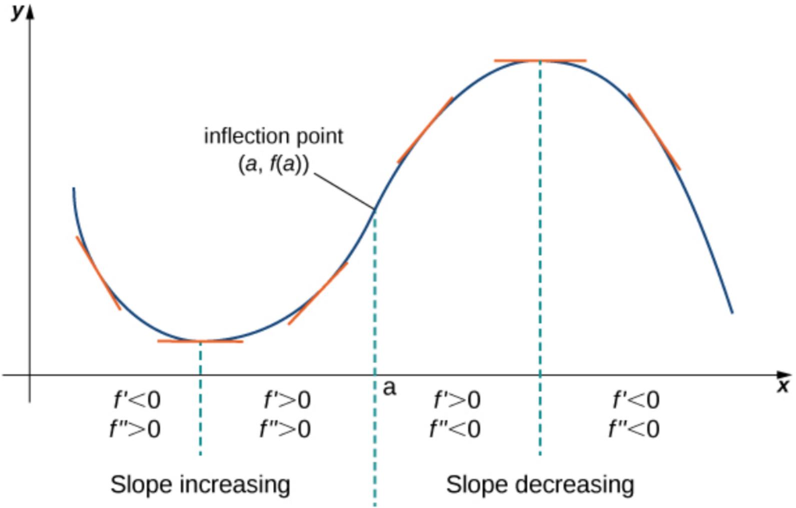


Figure 8.5: Graph of $f(x) = (x-1)(x-3)(x-4)$

Derivatives and the “shape” of a function



How to optimize!

1. Take the derivative of $f(x)$ to get $f'(x)$
2. First derivative test: Set $f'(x) = 0$ and solve for x^* (critical points).
3. Take the derivative of $f'(x)$ to get $f''(x)$
4. Second derivative test: Calculate $f''(x^*)$
   - If $f''(x^*) > 0$, x^* is a local minimum.
   - If $f''(x^*) < 0$, x^* is a local maximum.
   - If $f''(x^*) = 0$, x^* may be an inflection point.
5. Substitute each x^* into $f(x)$ to get (x, y) for each point.
6. If the function is bounded, check the value of $f(x)$ at each bound
7. Compare the values of $f(x)$ and $f(bounds)$ to find global min/max.

Inflection point or extrema?

1. If $f'(x^*) = 0$ and $f''(x^*) = 0$, you should continue taking the derivative until $f^n(x^*) =$ a nonzero number.
   - If $n =$odd number then x^* is an inflection point, not and extremum.
   - If $n =$even number continue to step 2.

2. Calculate $f^n(x^*)$:
   - If $f^n(x^*) > 0$, the point is a local minimum.
   - If $f^n(x^*) < 0$, the point is a local maximum.

Example optimization!

Find the extrema of the following equation:

$$f(x) = x^3 - 3x^2 + 7, x \in [-4, 4]$$

Also, graph the function.

Example optimization!

1. Take the derivative of $f(x)$ to get $f'(x)$.

$$f(x) = x^3 - 3x^2 + 7$$
$$f'(x) = 3x^2 - 6x$$

2. First derivative test: Set $f'(x) = 0$ and solve for x^*:

$$f'(x) = 3x^2 - 6x = 0$$
$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$

Therefore, we will have $x_1^* = 0$ and $x_2^* = 2$

Example optimization!

3. Take the derivative of $f'(x)$ to get $f''(x)$.

$$f'(x) = 3x^2 - 6x$$
$$f''(x) = 6x - 6$$

4. Second derivative test: calculate $f''(x^*)$
   - For $x_1^* = 0$: $f''(0) = 6(0) - 6 = -6$.
     If $f''(x^*) < 0$, x^* is a local maximum.
   - For $x_2^* = 2$: $f''(2) = 6(2) - 6 = 6$.
     If $f''(x^*) > 0$, x^* is a local minimum.

Example optimization!

5. Substitute each x^* into $f(x)$ to get (x,y) for each point:

Remember that $f(x) = x^3 - 3x^2 + 7$

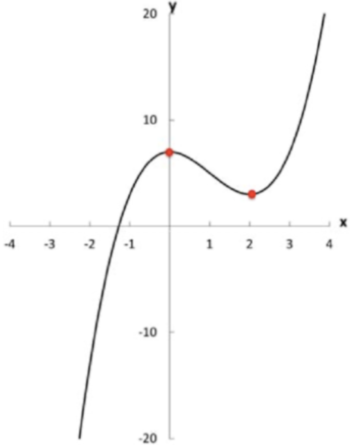
- For $x_1^* = 0$: $f(0) = 0^3 - 3(0)^2 + 7 = 7$.
  Local maximum at (0,7).
- For $x_2^* = 2$: $f(2) = 2^3 - 3(2)^2 + 7 = 3$.
  Local minimum at (2,3).

6. If the function is bounded, check the value of $f(x)$ at each bound:

Remember that $f(x) = x^3 - 3x^2 + 7, , x \in [-4, 4]$

- Lower bound:$f(-4) = (-4)^3 - 3(-4)^2 + 7 = -105$
  Global minimum at (-4, -105)
- Upper bound:$f(4) = (4)^3 - 3(4)^2 + 7 = 23$
  Global maximum at (4,23).

Example optimization!



7. Compare the values of $f(x^*)$ and $f(bounds)$ to find global min/max:
   - Global minimum at (-4,-105)
   - Local maximum at (0,7)
   - Local minimum at (2,3)
   - Global maximum at (4,23)