VE281

Data Structures and Algorithms

Asymptotic Algorithm Analysis

Learning Objective:

- Understand best, worst, and average cases
- Understand Big-Oh, Big-Omega, Big-Theta notations
- Know how to analyze time complexity of a program

Outline

- Asymptotic Analysis: Big-Oh
- Relatives of Big-Oh
- Analyzing Time Complexity of Programs

How to Measure Efficiency?

- Empirical comparison: run programs
  - Use the wall-clock time to measure the runtime
  - Empirical comparison could be tricky. It depends on
    - Compiler
    - Machine (CPU speed, memory, etc.)
    - CPU load

- Asymptotic Algorithm Analysis
  - For most algorithms, running time depends on the "size" of the input.
  - Running time is expressed as $T(n)$ for some function T on input size n.

Input Dependency: Example

- Summing an array of n elements

```
// REQUIRES: a is an array of size n
// EFFECTS: return the sum
int sum(int a[], unsigned int n) {
  int result = 0;
  for(unsigned int i = 0; i < n; i++)
    result += a[i];
  return result;
}
```

- The runtime is roughly cn, where c is some constant.
- With n fixed, any array has roughly the the **same** runtime.

Best, Worst, Average Cases

- In the example of summing an array, all inputs of a given size take the same time to run.
- However, in some other cases, this is not true, i.e., not all inputs of a given size take the same time to run.
- Example: linear search

```
// REQUIRES: a is an array of size n
// EFFECTS: return the index of the element
// equals key. If no such element, return n.
int search(int a[], unsigned int n, int key) {
  for(unsigned int i = 0; i < n; i++)
    if(a[i] == key) return i;
  return n;
}
```

Which Statements Are True for Linear Search?

Select all the correct statements:

- **A.** The best case occurs when **`key`** is the first element in the array.
- **B.** In the worst case, we need to do **`n`** comparisons with **`key`**.
- **C.** The worst case in terms of the number of comparisons with **`key`** <u>**only**</u> occurs when **`key`** is not in the array.
- D. Suppose **`key`** is uniformly located in the array. Then, on average, the number of comparisons with **`key`** is **`n/2`**.

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  return n;
}
```

Best, Worst, Average Cases

- Best case: least number of steps required, corresponding to the ideal input

- Worst case: most number of steps required, corresponding to the most difficult input.

- Average case: average number of steps required, given any input.

A Common Misunderstanding

"The best case for my algorithm is $n = 1$ because that is the fastest."

- Wrong!
- Best case is a **special input** case of size n that is **cheapest** among all input cases of size n.

Which Case to Use?

- Average case or worst case is common.

- While average time appears to be the fairest measure, it may be difficult to determine.
  - Sometime, it requires domain knowledge, e.g., the distribution of inputs.

- Worst case is pessimistic, but it gives an upper bound.
  - Bonus: worst case usually easier to analyze.

How to Analyze Complexity of Algorithm?

- Guiding Principle #1: Ignore constant factors.
  - Justification:
  1. Way easier.
  2. Constants depend on architecture, compiler, etc.
  3. Lose very little predictive power (as we will see).

- Guiding Principle #2: Focus on running time for large input size n.
  - Justification: only big problem are interesting!
  - Thus, we will compare the runtime of two algorithms when n is very large.
    - E.g., $1000 \log_2 n$ is “better” than $0.001n$.

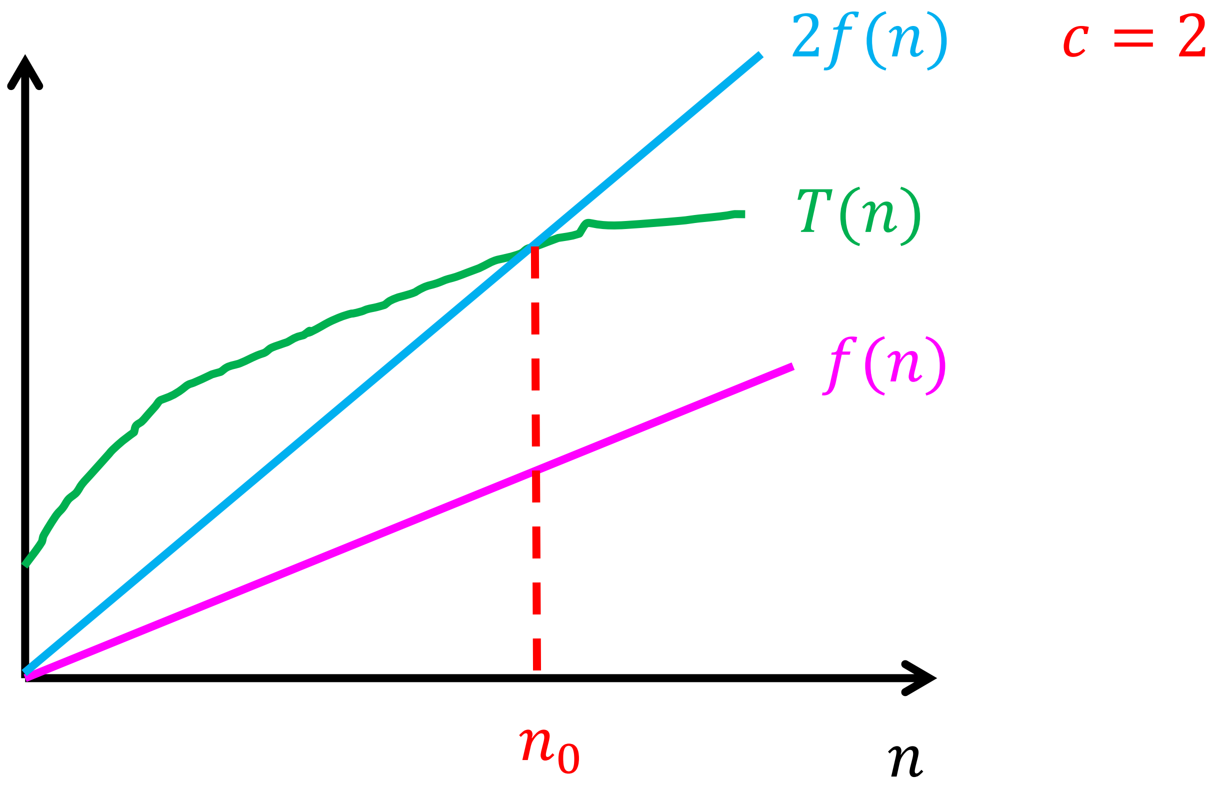
Asymptotic Analysis: Big-Oh

- Definition: A non-negatively valued function, $T(n)$, is in the **set** $O(f(n))$ if there **exist** two positive constants c and n_0 such that $T(n) \leq cf(n)$ for all $n > n_0$.

- Usage: The algorithm is in $O(n^2)$ in best/average/worst case.

- Meaning: For all data sets big enough (i.e.,$n > n_0$), the algorithm always executes in **less than** $cf(n)$ steps in best/average/worst case.

Graphic View of Big-Oh



Big-Oh Notation

- Strictly speaking, we say that $T(n)$ is **in** $O(f(n))$, i.e.,
  $T(n) \in O(f(n))$

- However, for convenience, people also write
  $T(n) = O(f(n))$

Big-Oh Example

- Claim: If $T(n) = a_k n^k + \cdots + a_1 n + a_0$, then
$$T(n) = O(n^k)$$

- Proof:
  - Need to pick constants c and n_0 so that for any $n > n_0$, $T(n) \leq c \cdot n^k$.
  - Choose $n_0 = 1$ and $c = |a_k| + \cdots + |a_1| + |a_0|$
  - Only need to show that for any $n > n_0, T(n) \leq cn^k$.

Big-Oh Example

- Claim: $2^{n+10} = O(2^n)$

- Proof:
  - Need to pick constants c and n_0 so that for any $n > n_0$,
    $$2^{n+10} \le c \cdot 2^n \qquad (*)$$
  - We note $2^{n+10} = 1024 \cdot 2^n$.
  - So if we choose $c = 1024$ and $n_0 = 1$, then (*) holds.

Big-Oh Notation

- Big-oh notation indicates an **upper bound**.

- Example: If $T(n) = 3n^2$ then $T(n)$ is in $O(n^2)$.

- Look for the **tightest** upper bound:
  - While $T(n) = 3n^2$ is in $O(n^3)$, we prefer $O(n^2)$.

A Sufficient Condition of Big-Oh

$$\boxed{\text{If } \lim_{n\to\infty} \frac{f(n)}{g(n)} = c < \infty, \text{ then } f(n) \text{ is } O(g(n)).}$$

- With this theorem, we can easily prove that
  $T(n) = c_1 n^2 + c_2 n$ is $O(n^2)$
  - Proof: $\lim_{n\to\infty} \frac{c_1 n^2 + c_2 n}{n^2} = c_1 < \infty$

Rules of Big-Oh

- **Rule 1**: If $f(n) = O\big(g(n)\big)$, then $cf(n) = O(g(n))$.
  - Example: $3n^2 = O(n^2)$
- **Rule 2**: If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n) + f_2(n) = O(\max\{g_1(n), g_2(n)\})$
  - Example: $n^3 + 2n^2 = O(\max\{n^3, n^2\}) = O(n^3)$

Rules of Big-Oh

- **Rule 3**: If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$

- **Rule 4**: If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$

Common Functions and Their Growth Rates

constant: 1

logarithmic: $\log n$

refers to $\log_2 n$

square root: $\sqrt{n}$

linear: n

loglinear: $n \log n$

quadratic: n^2

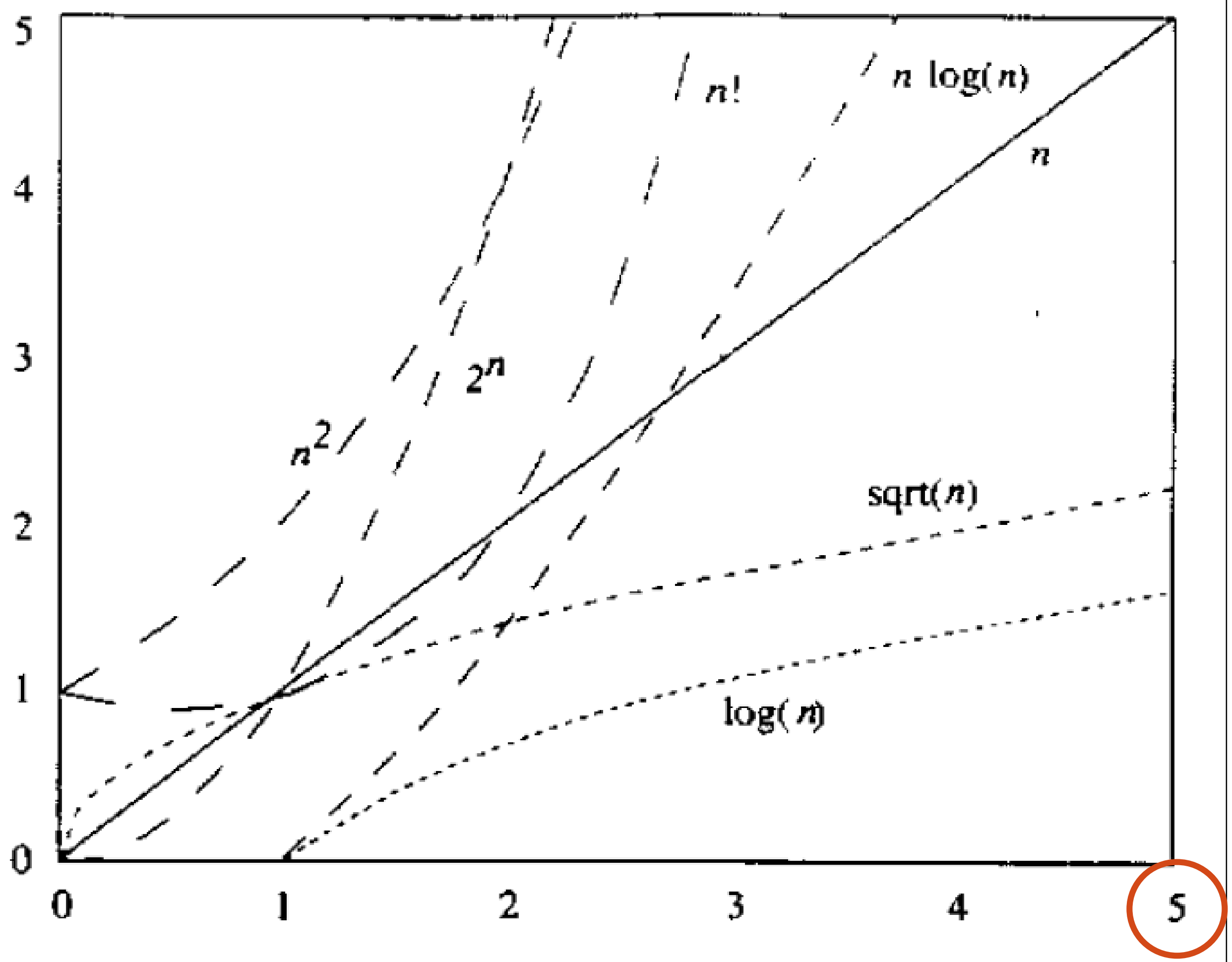
cubic: n^3

general polynomial: n^k

$k \geq 1$

exponential: $a^n, a > 1$

factorial: $n!$



Common Functions and Their Growth Rates

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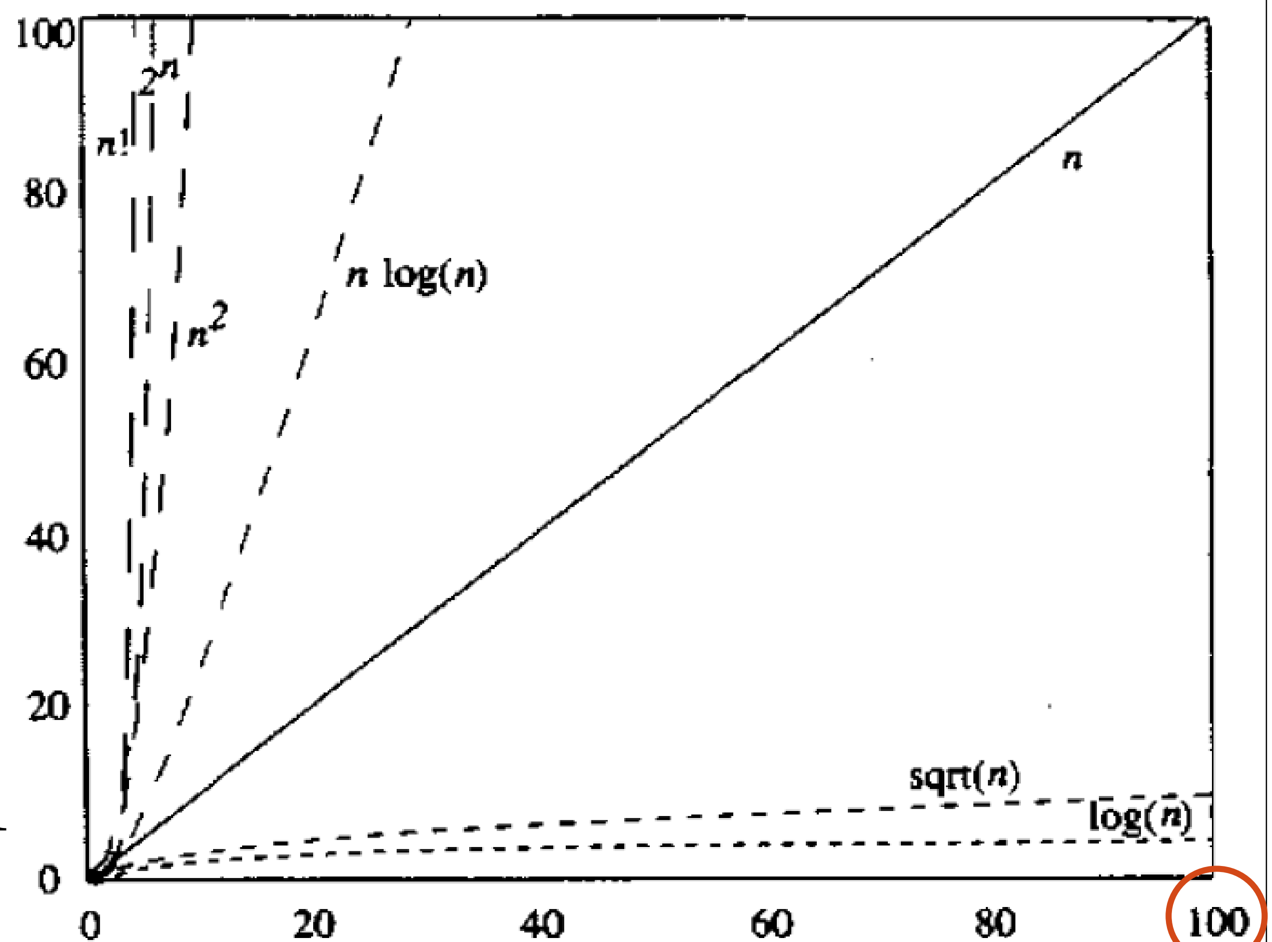


A Few Results about Common Functions

- For a polynomial in n of the form
$$f(n) = a_m n^m + a_{m-1} n^{m-1} + \cdots + a_1 n + a_0$$
where $a_m > 0$, we have $f(n) = O(n^m)$.

- For every integer $k \geq 1$, $log^k n = O(n)$.

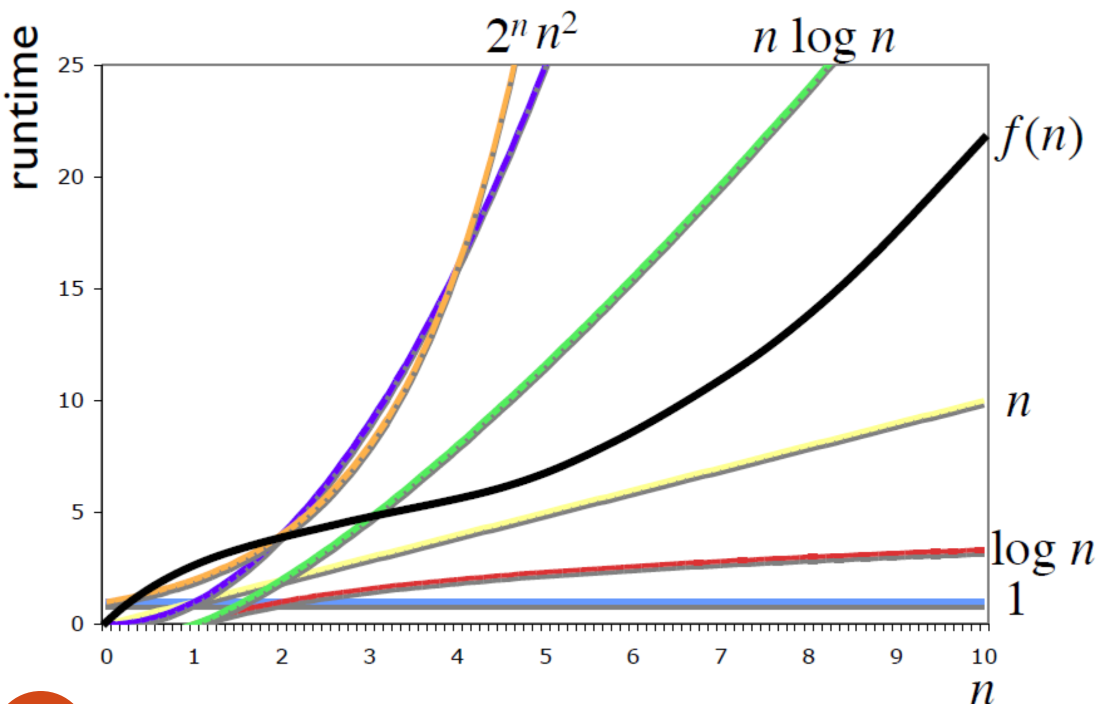
- For every integer $k \geq 1$, $n^k = O(2^n)$.

How Fast Is Your Code?

- Let $f(n)$ be the complexity of your code, how fast would you advertise it as? Choose one proper answer.

A. $\log n$ **B.** $n \log n$

C. n **D.** n^2



$f(n) = O(g(n))$; You want to pick a $g(n)$ that is as close to $f(n)$ as possible.

What Is a “Fast” Algorithm?

fast algorithm $\approx$ worst-case/average-case running time grows slowly with input size

- Usually as close to linear ($O(n)$) as possible.

Outline

- Asymptotic Analysis: Big-Oh
- Relatives of Big-Oh
- Analyzing Time Complexity of Programs

Relative of Big-Oh: Big-Omega

- Definition: For $T(n)$ a non-negatively valued function, $T(n)$ is in the **set** $\Omega(g(n))$ if there **exist** two positive constants c and n_0 such that $T(n) \geq cg(n)$ for all $n > n_0$.

- Meaning: For all data sets big enough (i.e., $n > n_0$), the algorithm always requires **more than** $cg(n)$ steps.

- Big-omega gives a lower bound.
- We usually want the greatest lower bound.

Big-Omega Example

- Consider $T(n) = c_1 n^2 + c_2 n$, where c_1 and c_2 are positive.
- What is the big-omega notation for $T(n)$?

- Solution:
  - $c_1 n^2 + c_2 n \geq c_1 n^2$ for all $n > 1$.
  - $T(n) \geq cn^2$ for $c = c_1$ and $n_0 = 1$.
  - Therefore, $T(n)$ is in $\Omega(n^2)$ by the definition.

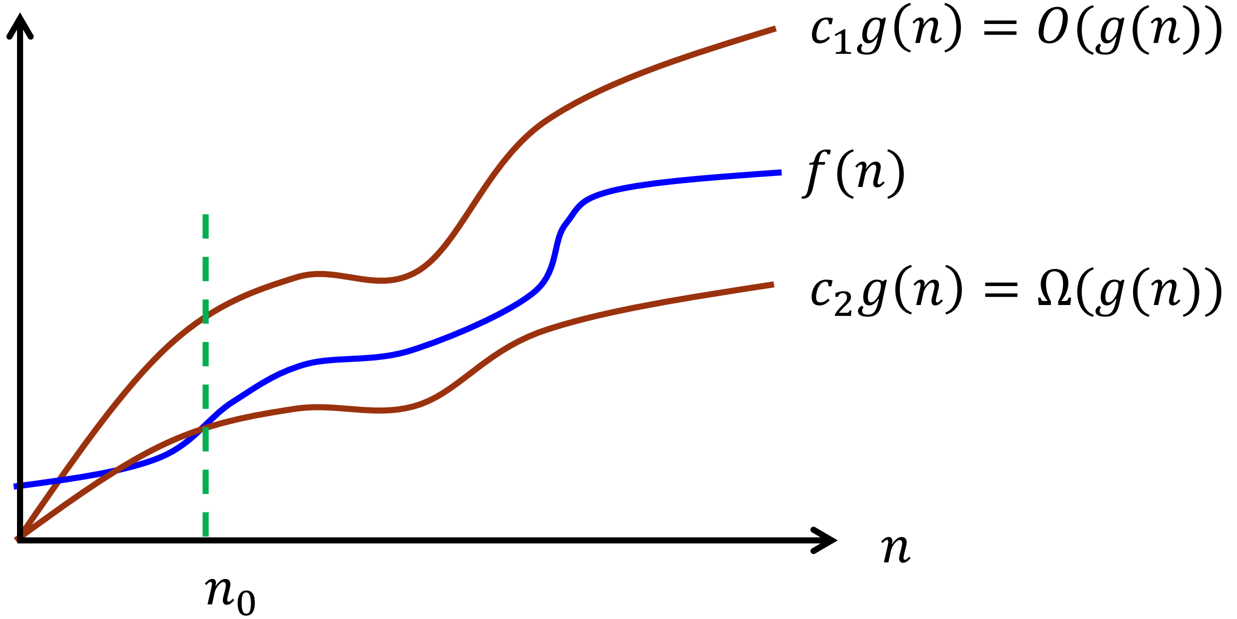
Theta Notation

- When big-oh and big-omega coincide, we indicate this by using big-theta (Θ) notation.

- Definition: $T(n)$ is said to be in the set $\Theta(g(n))$ if it is in $O(g(n))$ and it is in $\Omega(g(n))$.
  - In other words, there **exist** three positive constants c_1, c_2, and n_0 such that $c_1 g(n) \leq T(n) \leq c_2 g(n)$ for all $n > n_0$.

Theta Notation

$$f(n) = \Theta(g(n))$$



- Question: Does $f(n) = \Theta\big(g(n)\big)$ indicate $g(n) = \Theta(f(n))$?

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Analyzing Time Complexity of Programs

- For atomic statement, such as assignment, its complexity is $\Theta(1)$.

- For branch statement, such as if-else statement and switch statement, its complexity is that of the most expensive Boolean expression plus that of the most expensive branch.

```
if(Boolean_Expression_1) {Statement_1}
else if (Boolean_Expression_2) {Statement_2}
…
else if (Boolean_Expression_n) {Statement _n}
else {Statement_For_All_Other_Possibilities}
```

Analyzing Time Complexity of Programs

- For subroutine call, its complexity is that of the subroutine.
- For loops, such as while and for loop, its complexity is related the number of operations required in the loop.

Time Complexity Example One

- What is the time complexity of the following code?

```
sum = 0;
for(i = 1; i <= n; i++)
  sum += i;
```

- The entire time complexity is $\Theta(n)$.

Time Complexity Example Two

- What is the time complexity of the following code?

```
sum = 0;
for(i = 1; i <= n; i++)
   for(j = 1; j <= i; j++)
      sum++;
```

- Note that the statements

```
j <= i;
j++;
sum++;
```

all occur (roughly) $1 + 2 + \cdots + n = n(n+1)/2$ times.

- The time complexity is $\Theta(n^2)$.

Time Complexity Example Three

- What is the time complexity of the following code?

```
sum = 0;
for(i = 1; i <= n; i *= 2)
  for(j = 1; j <= n; j++)
    sum++;
```

- The outer loop occurs $\log n$ times.
- The statements **`sum++ / j<=n / j++`** occur $n \log n$ times.
- The time complexity is $\Theta(n \log n)$.

What Is the Time Complexity of the Following Code?

- Choose the correct answer.

```
sum = 0;
for(i = 1; i <= n; i *= 2)
  for(j = 1; j <= i; j++)
    sum++;
```

A. $\Theta(\log n)$ **B.** $\Theta(n \log n)$
C. $\Theta(n)$ **D.** $\Theta(n^2)$

Multiple Parameters

- Example: Compute the rank ordering for all C (i.e., 256) pixel values in a picture of P (i.e., 64×64) pixels.

```
for(i=0; i<C; i++)    // Initialize count
    count[i] = 0;
```

$\Theta(C)$

```
for(i=0; i<P; i++)    // Look at all pixels
    count[value[i]]++; // Increment count
```

$\Theta(P)$

```
sort(count);           // Sort pixel counts
```

$\Theta(C \log C)$

- The time complexity is $\Theta(P + C \log C)$.
- One general application is to analyze graph algorithm

Space/Time Trade-off Principle

- One can often reduce time if one is willing to sacrifice space, or vice versa.

- Example: factorial
  - Iterative method: Get "n!" using a for-loop.
  - This requires $\Theta(1)$ memory space and $\Theta(n)$ runtime.

  - Table lookup method: Pre-compute the factorials for $1,2,\cdots,N$ and store all the results in an array.
  - This requires $\Theta(n)$ memory space and $\Theta(1)$ runtime (fetching from an array).