VE281

Data Structures and Algorithms

Priority Queues and Heaps

Learning Objectives:

- Know what a priority queue is
- Know what a min heap is
- Know how a min heap performs enqueue and extractMin operations
- Know how to efficiently initialize a min heap

Outline

- Priority Queue
- Min Heap and Its Operations
- Min Heap Initialization and Application

Priority Queues

- Two kinds of priority queues:
  - Min priority queue.
  - Max priority queue.

- We will focus on **min priority queue**.
  - The max priority queue is similar.

What Is Min Priority Queue?

- A collection of items.
- Each item has a key (or "**priority**").

- Support the following operations:
  - **`isEmpty`**
  - **`size`**
  - **`enqueue`**: put an item into the priority queue.
  - **`dequeueMin`**: remove element with **min** key.
  - **`getMin`**: get item with **min** key.

Applications of Priority Queue

- Banking services
  - VIP customer who arrives later gets served first.

- Network bandwidth management
  - The prioritized traffic, such as real-time data, is forwarded with the least delay once it reaches the network router.

- Discrete event simulation
  - One event happening triggers a few others, which are put into a queue.
  - Simulating in the order of the **beginning time** of the events.

Min Priority Queue: Implementation

- A collection of items.
- Each item has a key (or "**priority**").

- Support the following operations:
  - **`isEmpty`**
  - **`size`**
  - **`enqueue`**: put an item into the priority queue.
  - **`dequeueMin`**: remove element with **min** key.
  - **`getMin`**: get item with **min** key.

What's the time complexity for an unsorted array-based implementation?

Priority Queue Implemented with Heap

- Priority queues are most commonly implemented using **Binary Heaps** (will be shown soon).

- Complexity of the operation using heap implementation:
  - **`isEmpty`**, **`size`**, and **`getMin`** are $O(1)$ time complexity in the worst case.
  - **`enqueue`** and **`dequeueMin`** are $O(\log n)$ time complexity in the worst case, where n is the size of the priority queue.

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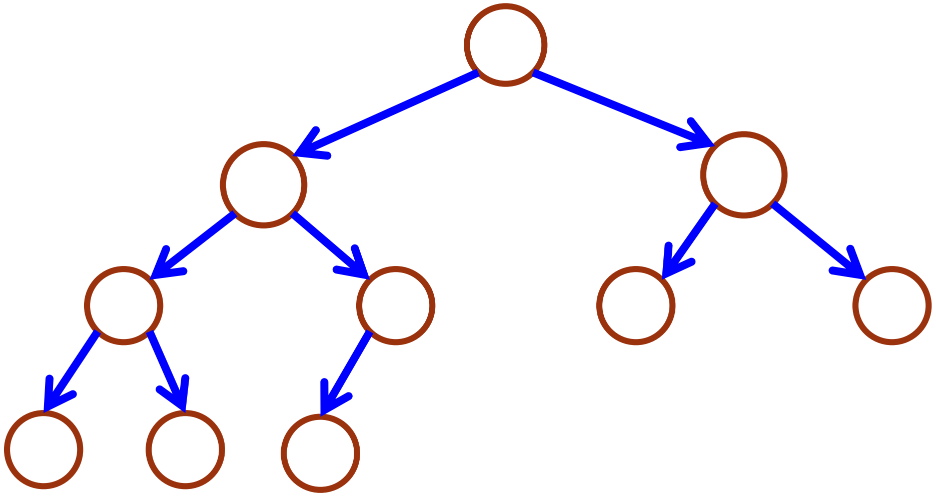
Binary Heap

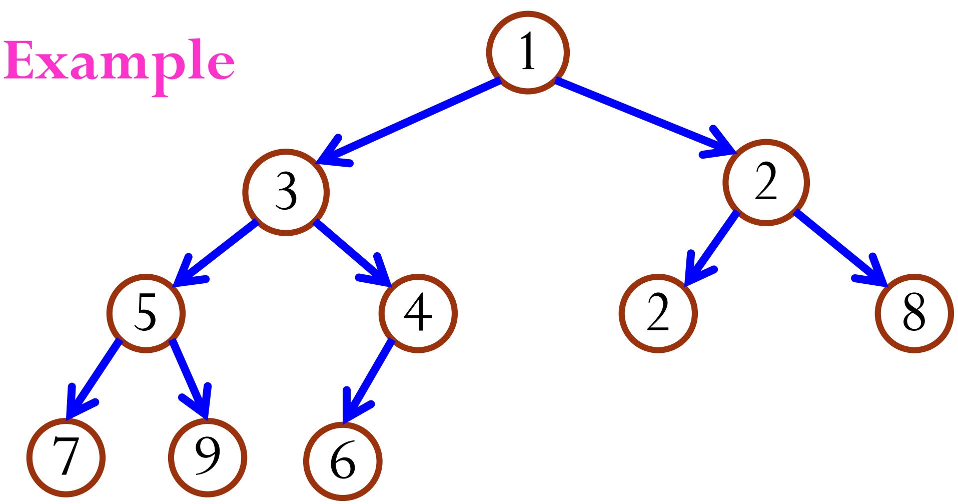
- A **binary heap** is a **complete binary tree**.



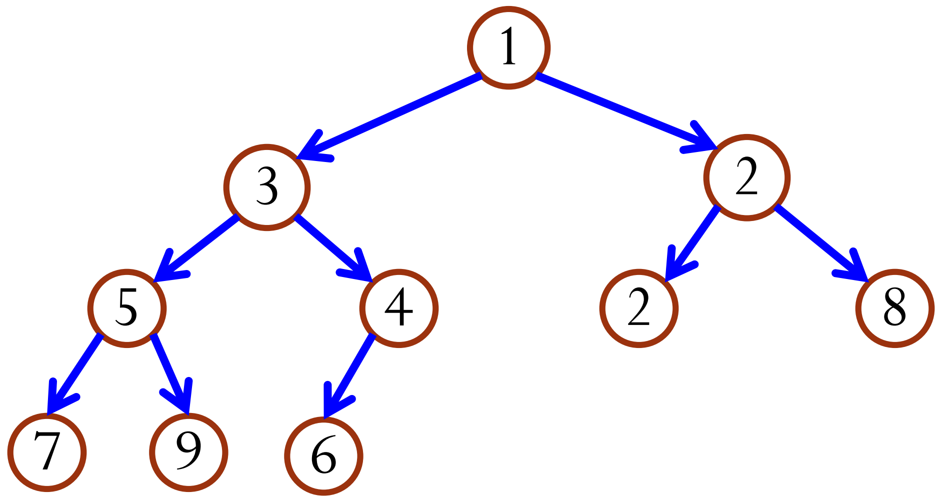
Min Heap

- A min heap is
  - a **binary heap**, and
  - a tree where for **any** node v, the key of v is smaller than or equal to ($\leq$) the keys of any **descendants** of v.
- Property: The key of the root of **any** subtree is always the smallest among all the keys in that subtree.

Example



Min Heap



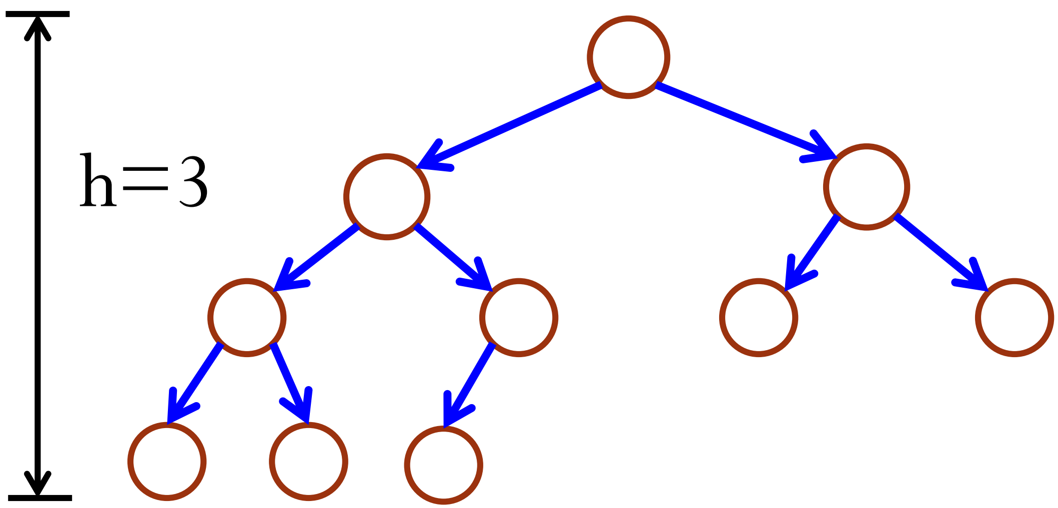
- However, the keys of nodes **across** subtrees have no required relationship.
  - Different from binary search trees, which we will show later.

What's the Height of a Heap of n Nodes?

Select **all** the correct answers:

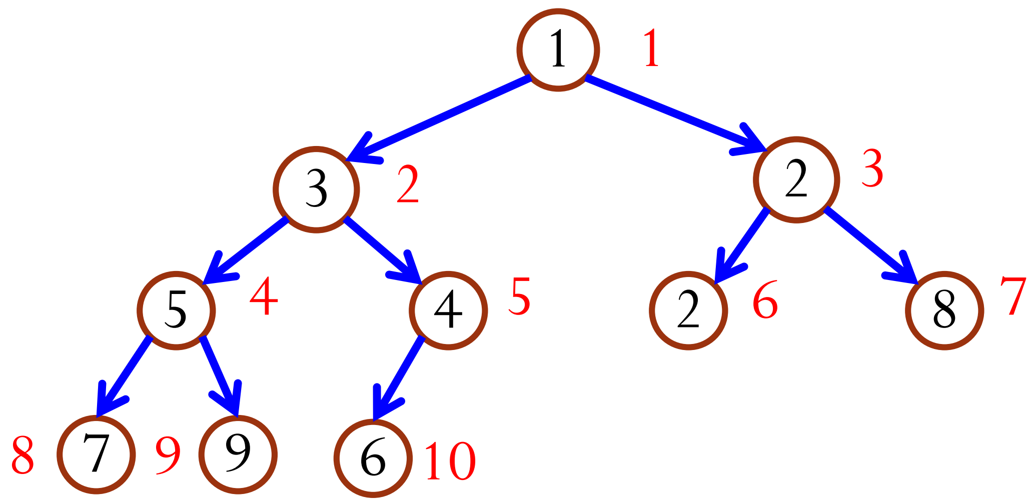
A. $\lfloor \log_2(n+1) \rfloor - 1$ **B.** $\lceil \log_2 n \rceil - 1$

C. $\lceil \log_2(n+1) \rceil - 1$ **D.** $\lfloor \log_2 n \rfloor$



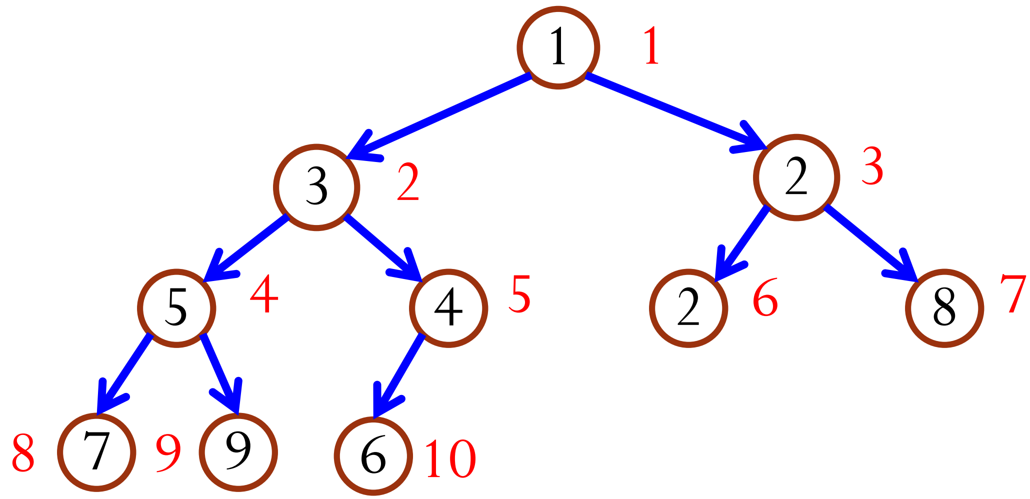
Binary Heap Implementation as an Array

- Store the elements in an array in the order produced by a level-order traversal.
- The first element is stored at index 1.



1	3	2	5	4	2	8	7	9	6
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

Index Relation



Index relation allows us to move up and down a heap easily.

- A node at index i ($i \neq 1$) has its parent at index $\lfloor i/2 \rfloor$.
- Assume the number of nodes is n. A node at index i ($2i \leq n$) has its left child at $2i$.
  - If $2i > n$, it has no left child.
- A node at index i ($2i + 1 \leq n$) has its right child at $2i + 1$.
  - If $2i + 1 > n$, it has no right child.

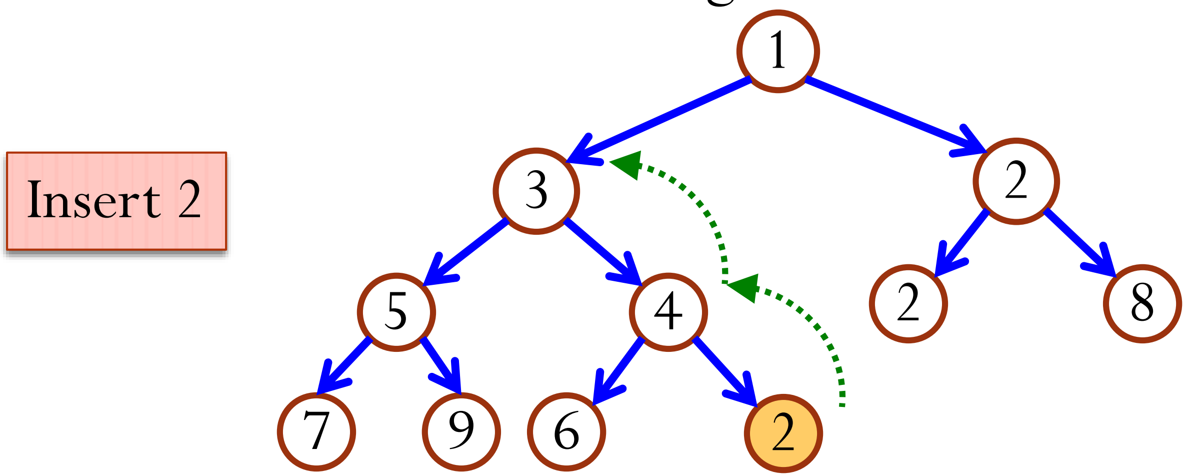
Min Heap Implementation

- We also have a **`size`** variable to keep the number of nodes in the heap.
  - The heap elements are stored in heap[1], heap[2], …, heap[size].

- Operations
  - **`isEmpty: return size==0;`**
  - **`size: return size;`**
  - **`getMin: return heap[1];`**

Procedure of enqueue

- Insert **`newItem`** as the rightmost leaf of the tree.

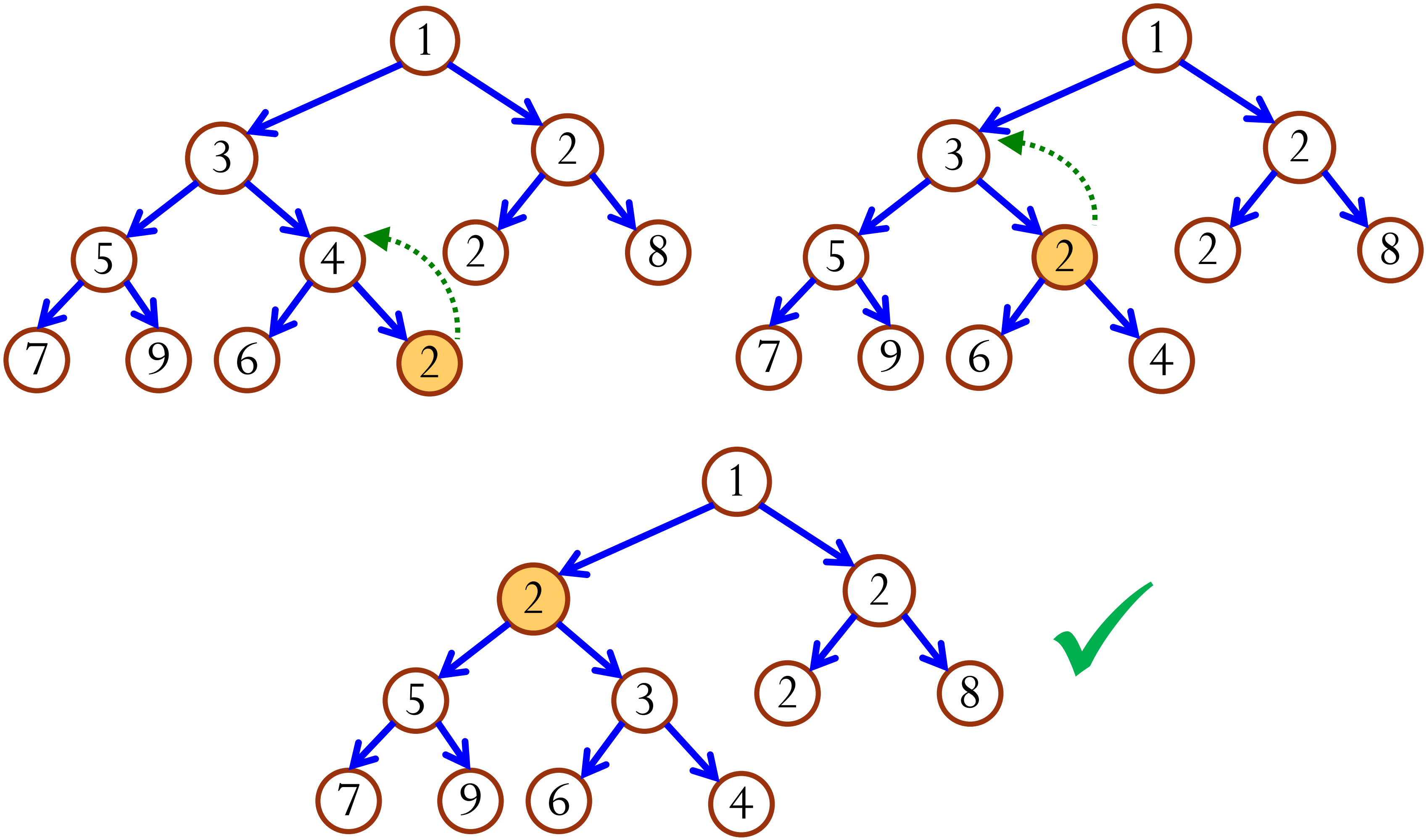



```
heap[++size] = newItem;
```

- The tree may no longer be a heap at this point!
- **Percolate up** **`newItem`** to an appropriate spot in the heap to restore the heap property.

Percolate Up

Illustration



Percolate Up

Code

```
void minHeap::percolateUp(int id) {
  while(id > 1 && heap[id/2] > heap[id]) {
    swap(heap[id], heap[id/2]);
    id = id/2;
  }
}
```

- Pass index (`id`) of array element that needs to be percolated up.
- Swap the given node with its parent and move up to parent until:
  - we reach the root at position 1, or
  - the parent has a smaller or equal key.

enqueue

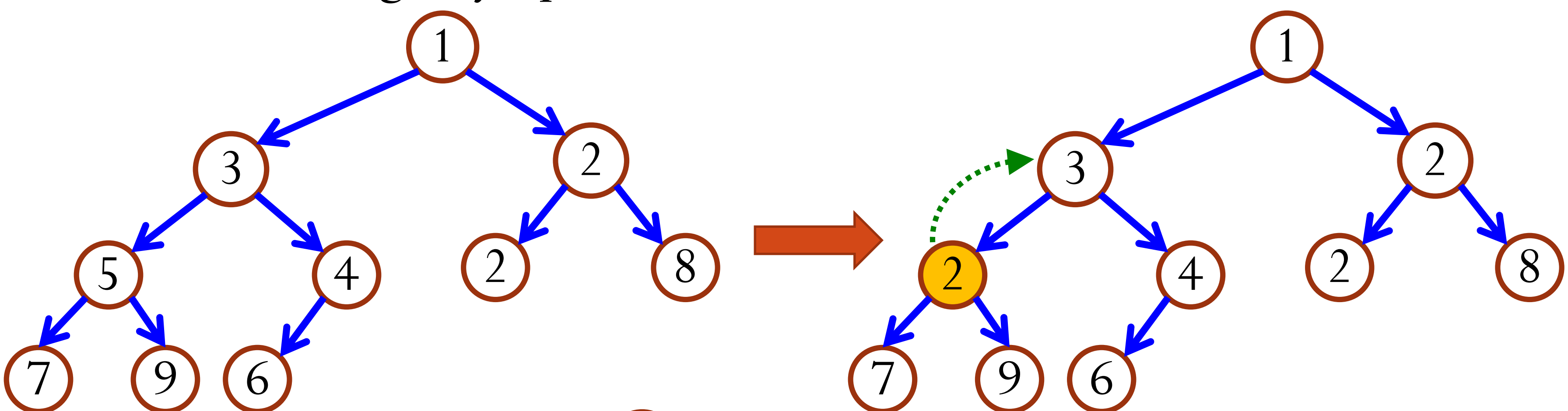
Code

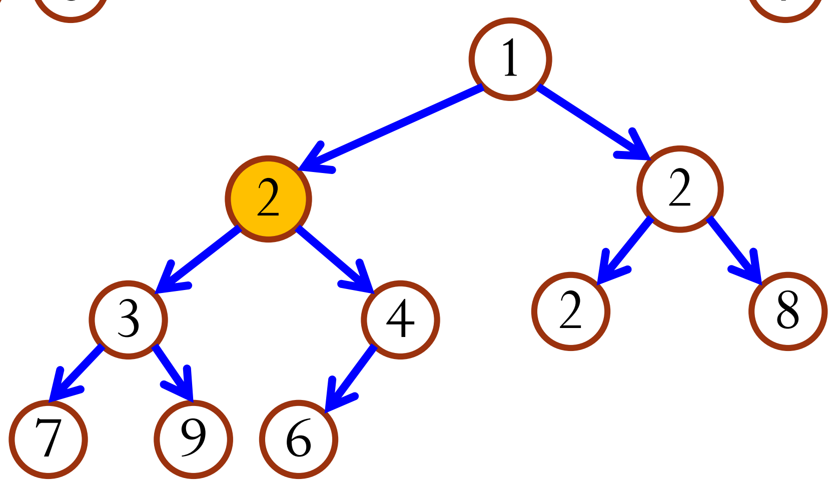
```
void minHeap::enqueue(Item newItem) {
  heap[++size] = newItem;
  percolateUp(size);
}
```

- What is the time complexity?
  - $O(\log n)$

Aside: Decrease Key

- Percolating-up can also be exploited to implement the decreasing-key operation



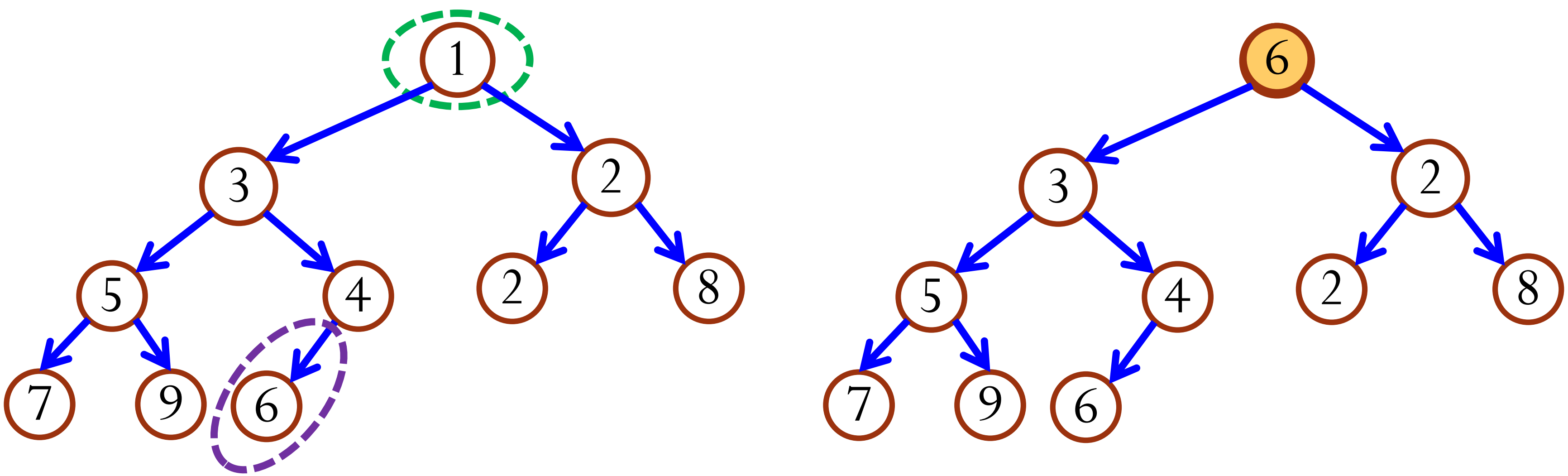


Time complexity: $O(\log n)$

Procedure of dequeueMin

- The min item is at the root. Save that item to be returned.
- Move the item in the rightmost leaf of the tree to the root.

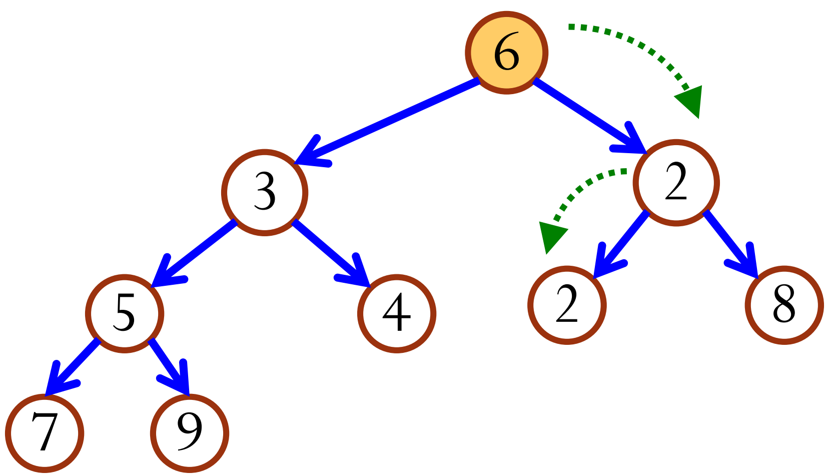
```
swap(heap[1], heap[size--]);
```



- The tree may no longer be a heap at this point!

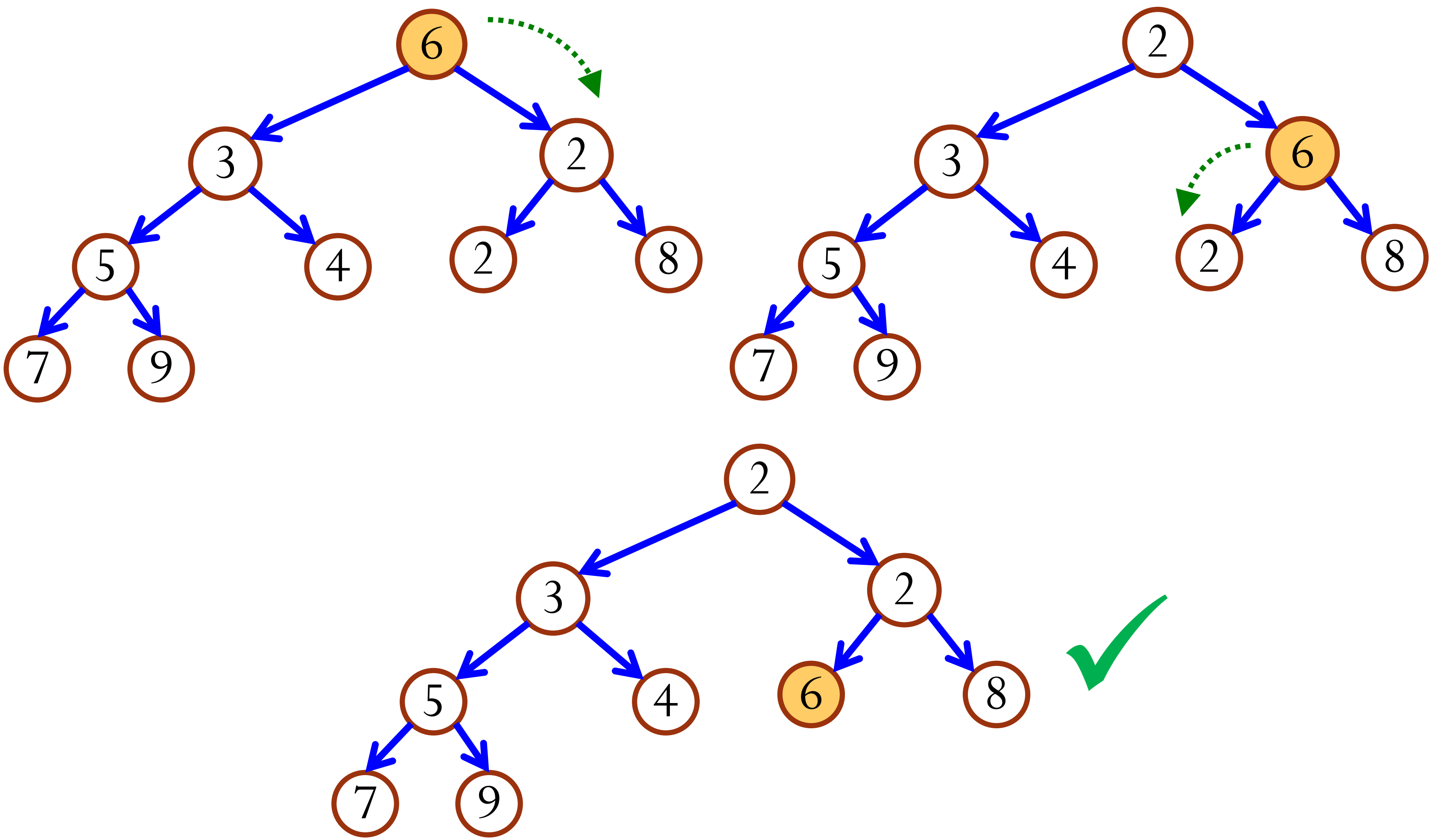
Procedure of dequeueMin

- **Percolate down** the recently moved item at the root to its proper place to restore heap property.
  - For each subtree, if the root has a **larger** search key than **either of its children**, swap the item in the root with that of the the **smaller** child.



Percolate Down

Illustration



Percolate Down

Code

```
void minHeap::percolateDown(int id) {
  for(j = 2*id; j <= size; j = 2*id) {
    if(j < size && heap[j] > heap[j+1]) j++;
    if(heap[id] <= heap[j]) break;
    swap(heap[id], heap[j]);
    id = j;
  }
}
```

find the smaller child

- Pass index (**`id`**) of array element that needs to be percolated down.
- Swap the key in the given node with the smallest key among the node's children, moving down to that child, until:
  - we reach a leaf node, or
  - both children have larger (or equal) key

dequeueMin

Code

```
Item minHeap::dequeueMin() {
  swap(heap[1], heap[size--]);
  percolateDown(1);
  return heap[size+1];
}
```

- What is the time complexity?
  - $O(\log n)$

Outline

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- Min Heap Initialization and Application

Initializing a Min Heap

- How do we initialize a min heap from a set of items?

- Simple solution: insert each entry one by one.
  - The worst case time complexity for inserting the k-th item is $O(\log k)$, so creating a heap in this way is $O(n \log n)$.

- Instead, we can do better by putting the entries into a **complete** binary tree and running **percolate down** intelligently.

Initializing a Min Heap

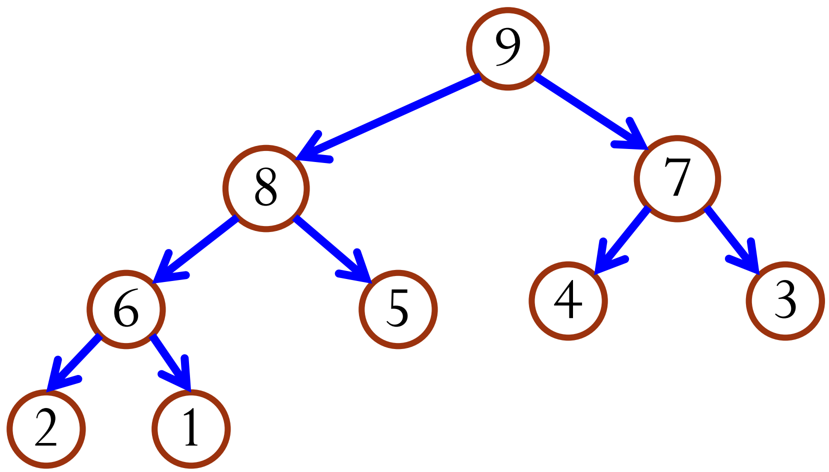
- Put all the items into a complete binary tree.
  - Implemented using an array.

- Starting at the rightmost array position **that has a child**, percolate down all nodes in **reverse** level-order.
  - The rightmost array position **that has a child** is `size/2`.
- Procedure:

```
For i = size/2 down to 1
    percolateDown(i);
```

Initializing a Min Heap
Illustration

- Input items: 9, 8, 7, 6, 5, 4, 3, 2, 1
- First step: put all the items into a complete binary tree.

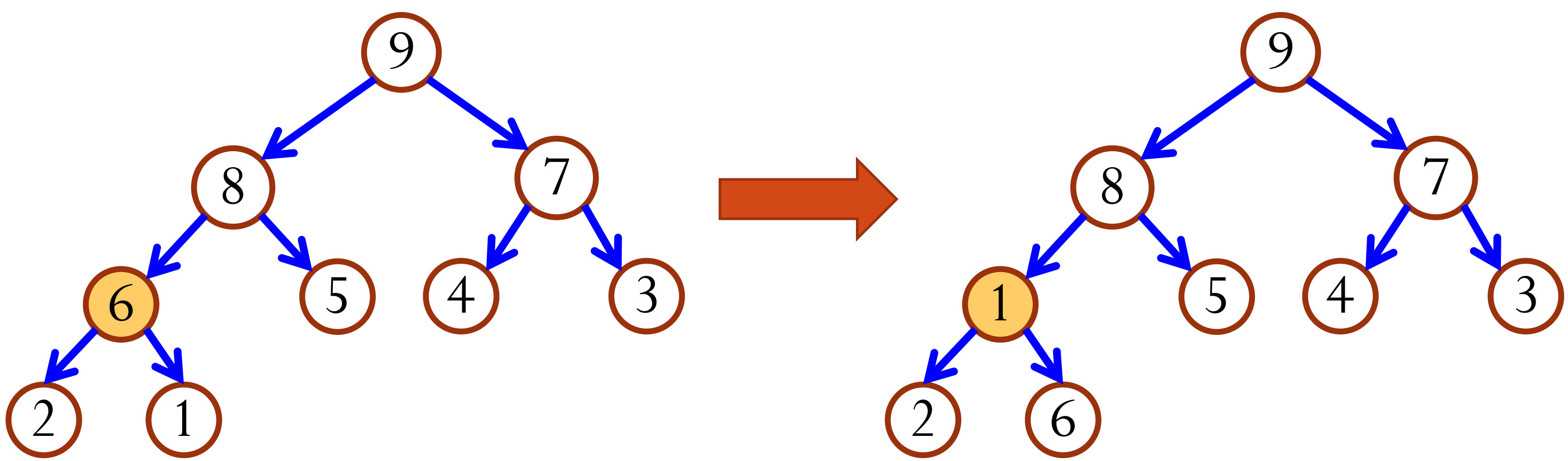


Initializing a Min Heap

Illustration

- Starting at the rightmost array position **that has a child**, percolate down all nodes in **reverse** level-order.

Node at index 9/2 = 4

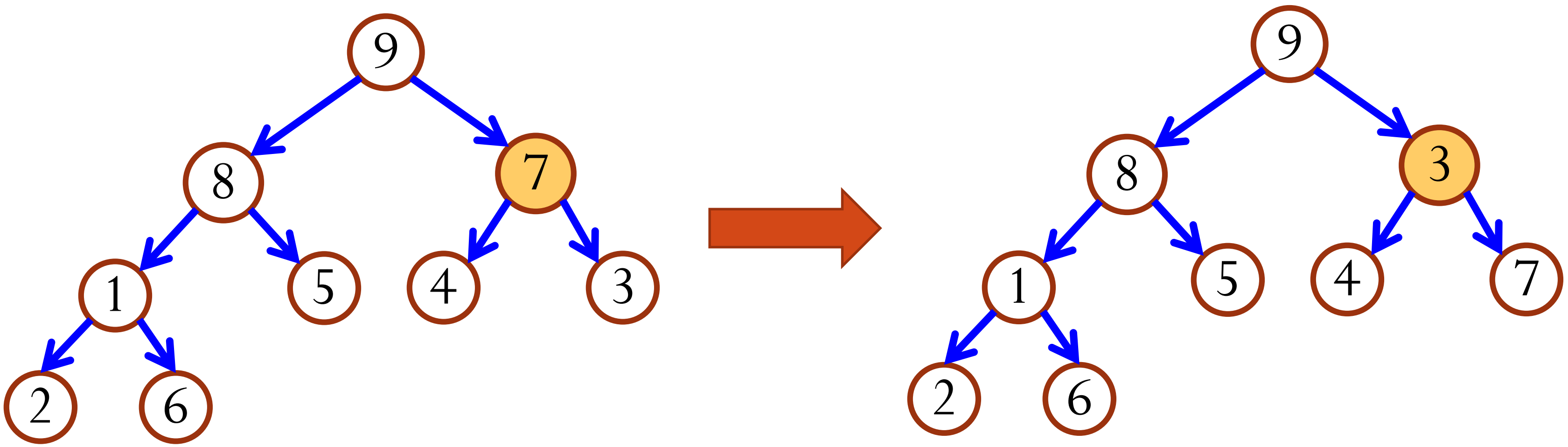


Move to next lower array position.

Initializing a Min Heap

Illustration

Node at index 3

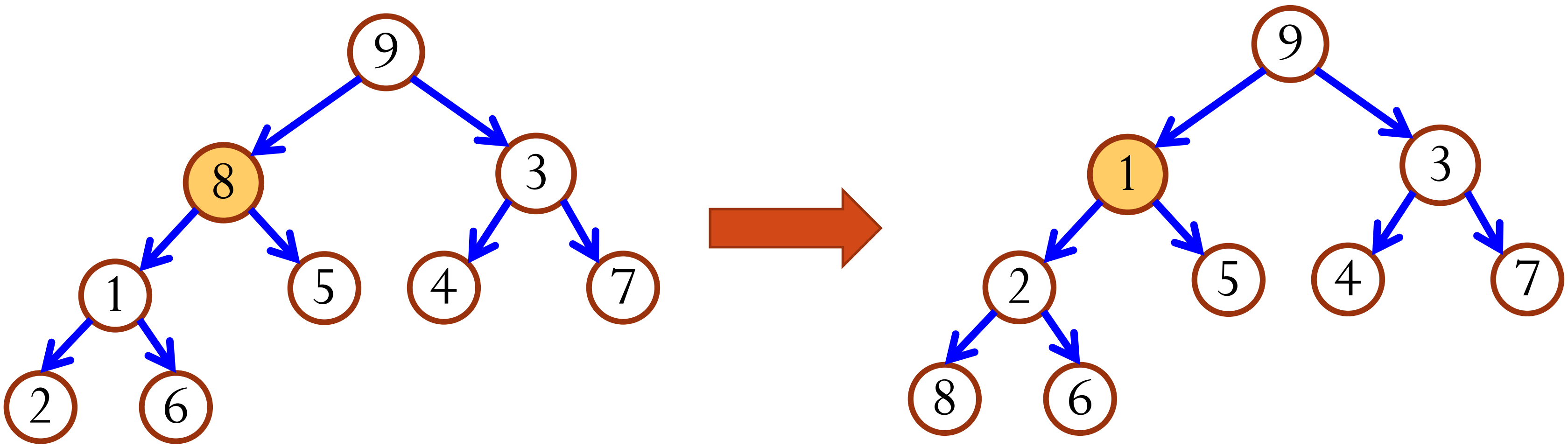


Move to next lower array position.

Initializing a Min Heap

Illustration

Node at index 2

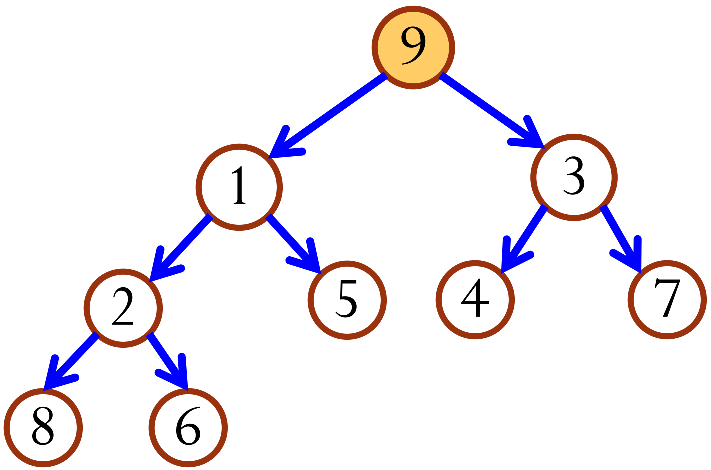


Move to next lower array position.

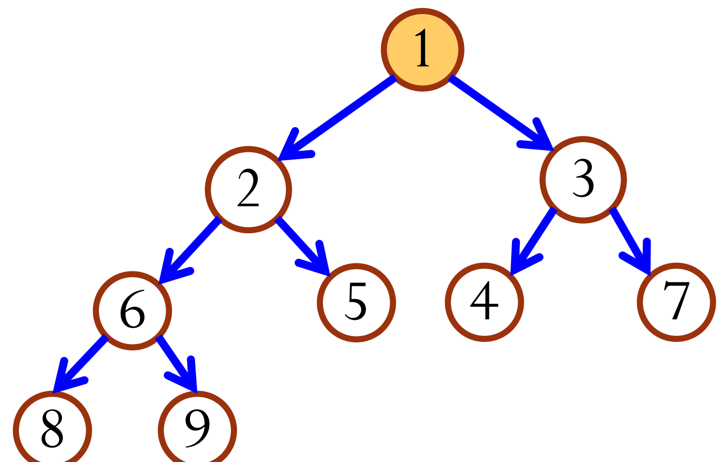
Initializing a Min Heap

Illustration

Node at index 1

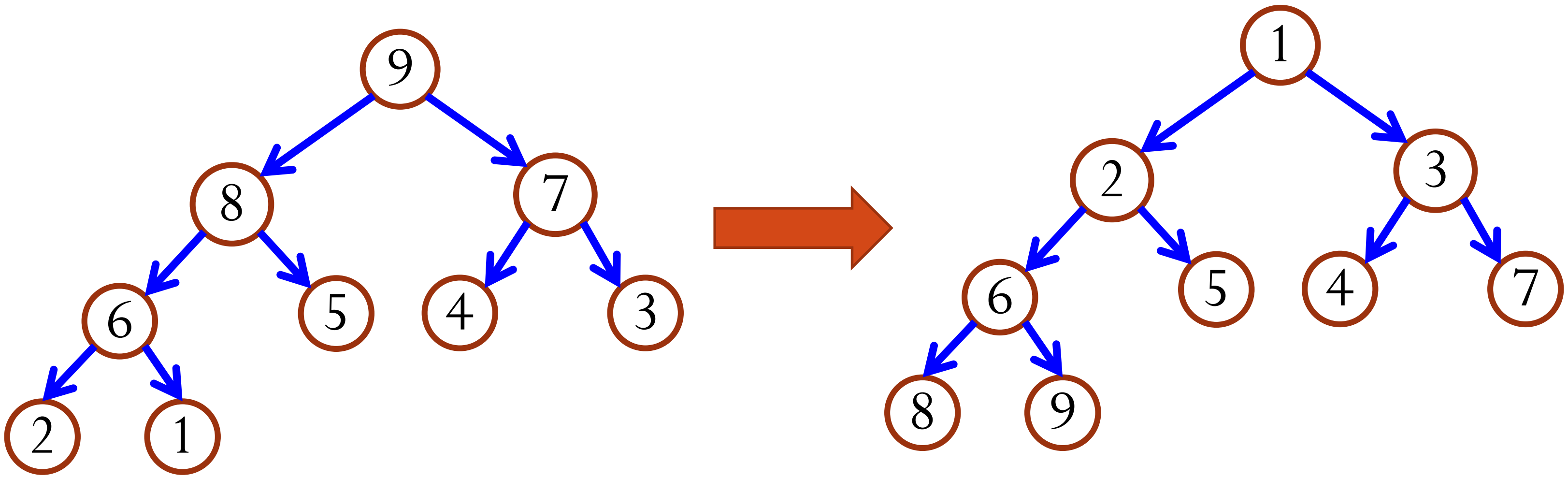


Exercise: What's the result?



Done!

Time Complexity Analysis



- Suppose: the **height** of the heap is h.
- Note: Number of nodes at level k ($0 \le k \le h$) is $\le 2^k$.
- Note: The worst case time complexity of percolating down a node at level k is $O(h-k)$.

Time Complexity Analysis

$$T(h) \leq \sum_{k=0}^{h-1} 2^k O(h-k) = O\left(\sum_{k=0}^{h-1} 2^k (h-k)\right)$$

- What is $S(h) = \sum_{k=0}^{h-1} 2^k (h-k)$?

$$S(h) = 2^0 h + 2^1(h-1) + 2^2(h-2) + \cdots + 2^{h-1} \cdot 1$$

$$2S(h) = 2^1 h + 2^2(h-1) + \cdots + 2^{h-1} \cdot 2 + 2^h \cdot 1$$

$$2S(h) = \quad 2^1 h \quad + 2^2(h-1) + \cdots + 2^{h-1} \cdot 2 + 2^h \cdot 1$$

$$S(h) = 2S(h) - S(h) = 2^1 + 2^2 + \cdots + 2^h - h = 2^{h+1} - 2 - h$$

Time Complexity Analysis

$$T(h) \le O(2^{h+1} - 2 - h)$$

- For a complete binary tree, we have

$$h = \lceil \log_2(n+1) \rceil - 1 \le \log_2(n+1)$$

where n is the number of nodes.

- Therefore, the algorithm for initializing a min heap with n nodes has worst case time complexity $T(n) = O(n)$.
  - Better than the way to enqueue entry one by one.

Application of Heap: Sorting

- Procedure:
  1. Initialize a min heap with all the elements to be sorted

     $\boxed{\text{Complexity: } O(n)}$

  2. Repeatedly call **`dequeueMin`** to extract elements out of the heap.

     $\boxed{\text{Complexity: } O(n \log n)}$

- The resulting elements are sorted by their keys.

- What is the time complexity? $O(n \log n)$
- This is known as **heap sort**.

Application: Median Maintenance

- Input: a sequence of numbers $x_1, x_2, \dots, x_n$, one-by-one
- Output: at each time step i, the median of $x_1, x_2, \dots, x_i$
- Problem: how to do this with $O(\log i)$ time at each step i?
- Hint: using two heaps, one min heap and one max heap
- Key idea: maintain the smallest half ($\left\lceil \frac{n}{2} \right\rceil$) in max heap and the largest half ($\left\lfloor \frac{n}{2} \right\rfloor$) in the min heap
- Question: How do you get the median (i.e., the $\left\lceil \frac{n}{2} \right\rceil$-th smallest item)?
  - Answer: get max from the max heap

How to Insert a New Item?

- <u>Key problem</u>: maintain the **invariant** that the smallest half ($\left\lceil \frac{n}{2} \right\rceil$) in max heap and the largest half ($\left\lfloor \frac{n}{2} \right\rfloor$) in the min heap
  - To maintain balance between the two heaps
- If n (before insertion) is even
  - If new item <= min(minHeap), insert it into maxHeap
  - Else (new item > min(minHeap)), first extract min value from minHeap, then insert that value in maxHeap, and finally insert new item into minHeap
- If n (before insertion) is odd
  - If new item >= max(maxHeap), insert it into minHeap
  - Else (new item < max(maxHeap)), first extract max value from maxHeap, then insert that value in minHeap, and finally insert new item into maxHeap

Time complexity is $O(\log i)$