Evolution of my internship work on causality in ML

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Some formulae

The formula for the response for each node was of the form :

$$\left\langle \hat{R}^2_{j\to i}(t) \right\rangle_\xi = Tr\left\{ \boldsymbol{e_i}^\top A(t)\left(\mathbb{1} - \hat{G}^{(D)}\right)\boldsymbol{e_j}\boldsymbol{e_j}^\top \left(\mathbb{1} - \hat{G}^{(D)}\right) A(t)^\top \boldsymbol{e_i} \right\} +$$

$$+ \alpha \frac{\sigma^2}{N} Tr\left[\left(\mathbb{1} - \hat{G}^{(D)}\right)\boldsymbol{e_j}\boldsymbol{e_j}^\top \hat{G}^{(D)}\right] \times$$

$$\boldsymbol{e_i}^\top \left(\sum_\tau A(t-\tau-1)A^\top(t-\tau-1) \right) \boldsymbol{e_i}$$

where $\hat{G}^{(D)} = \left(\mathbb{1} + \alpha \hat{C}_0\right)^{-1}$ is a resolvent-like quantity showing out in the calculation by introducing the L2 regularizer while $A(t)$ is the adjacency matrix of the graph sampled.

Some formulae

By doing the sum over each node, supposing the two matrices in the variance terms are independent (= freeness ?) :

$$\left\langle \hat{R}^2(t) \right\rangle = \sum_{i,j} \left\langle \hat{R}^2_{j\to i}(t) \right\rangle$$

$$= Tr\left[A(t)\left(\mathbb{1} - G^{(D)}\right)\left(\mathbb{1} - G^{(D)}\right)A^\top(t)\right] +$$

$$+ \alpha\frac{\sigma^2}{N} Tr\left[\left(\mathbb{1} - G^{(D)}\right)G^{(D)}\right] \times$$

$$Tr\left[\sum_{\tau=0}^{t-1} A(t-\tau-1)A^\top(t-\tau-1)\right]$$

Some formulae

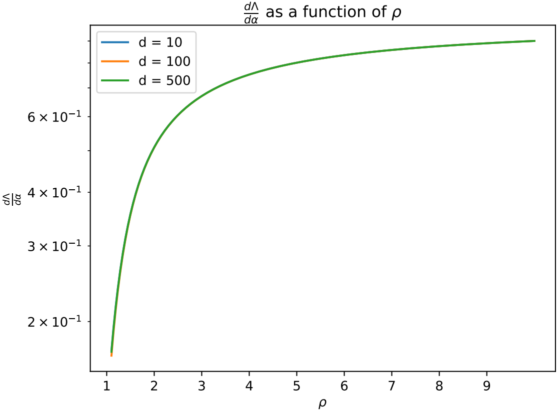
We are facing some terms squared in G. By using the identity $\frac{\partial}{\partial\alpha} G^{(D)} = -\frac{1}{\alpha}\left(\mathbb{1} - G^{(D)}\right) G^{(D)}$, applying the chain rule and going back to the initial equation, we end up with :

$$
\begin{aligned}
\left\langle \hat{R}^2(t) \right\rangle \to\ & Tr\left[A(t)A^\top(t)\right] - Tr\left[A(t)G[\Lambda]A^\top(t)\right] + \\
& - \frac{\alpha}{\Lambda}\frac{\partial\Lambda}{\partial\alpha} Tr\left[A(t)G[\Lambda]\left(\mathbb{1} - G[\Lambda]\right)A^\top(t)\right] + \\
& + \frac{\alpha^2}{\Lambda}\frac{\sigma^2}{N}\frac{\partial\Lambda}{\partial\alpha} Tr\left[G[\Lambda]\left(\mathbb{1} - G[\Lambda]\right)\right] \times \\
& Tr\left[\sum_{\tau=0}^{t-1} A(t-\tau-1)A^\top(t-\tau-1)\right]
\end{aligned}
$$

where $G[\Lambda]$ is the deterministic equivalent of our resolvent-like quantity.

Some formulae

The term $\frac{\partial \Lambda}{\partial \alpha}$ is always smaller than 1. When ρ is big, this is close to 1.



In this plot, $\alpha = 10$.

Correction of the response formula for each node

I did not introduced the prefactor $\frac{\alpha}{\Lambda}\partial\Lambda/\partial\alpha$ in the calculation of the theoretical response formula in the old computations. I chose to add it now in order to add some corrections. I have not been able to demonstrate the formula but it should be similar to the formula for the response for all nodes.

Correction of the response formula for each node

By analogy with the formula for the total response, one may write :

$$\left\langle \hat{R}^2_{j\to i}(t)\right\rangle \to Tr\left[\boldsymbol{e_i}^\top A(t)\left(\mathbb{1}-G[\Lambda]\right)\boldsymbol{e_j}\boldsymbol{e_j}^\top A^\top(t)\boldsymbol{e_i}\right]+$$
$$-\frac{\alpha}{\Lambda}\frac{\partial\Lambda}{\partial\alpha}Tr\left[\boldsymbol{e_i}^\top A(t)\left(\mathbb{1}-G[\Lambda]\right)\boldsymbol{e_j}\boldsymbol{e_j}^\top G[\Lambda]A^\top(t)\boldsymbol{e_i}\right]+$$
$$+\frac{\alpha^2}{\Lambda}\frac{\sigma^2}{N}\frac{\partial\Lambda}{\partial\alpha}Tr\left[\left(\mathbb{1}-G[\Lambda]\boldsymbol{e_j}\boldsymbol{e_j}^\top\right)G[\Lambda]\right]\times$$
$$Tr\left[\sum_{\tau=0}^{t-1}A(t-\tau-1)A^\top(t-\tau-1)\right]$$

Correction of the response formula for each node

Reason on why I have not been able to compute theoretically this formula :

Before, we used the identity $-\frac{1}{\alpha}\left(\mathbb{1} - G\right) G = \frac{\partial}{\partial \alpha} G$.

Now, we have $-\frac{1}{\alpha}\left(\mathbb{1} - G\right) \boldsymbol{e_j} \boldsymbol{e_j}^\top G$. However, $\boldsymbol{e_j} \boldsymbol{e_j}^\top$ do not commute with G !