Pergamon

Mathl. Comput. Modelling Vol. 24, No. 2, pp. 103–114, 1996

0895-7177/96 $15.00 + 0.00

PII: S0895-7177(96)00094-5

Derivation of Multi-Dimensional Ellipsoidal Convex Model for Experimental Data

L. P. ZHU AND I. ELISHAKOFF
Department of Mechanical Engineering, Florida Atlantic University
Boca Raton, FL 33431, U.S.A.

J. H. STARNES, JR.
NASA Langley Research Center, Hampton, VA 23664-5225, U.S.A.

(Received June 1994; accepted March 1995)

Abstract—This paper deals with determination of the best ellipsoidal model fitting the available limited experimental data. The problem is defined as that of finding the minimum volume ellipsoid containing all experimental data. A general transformation matrix for the rotation of N-dimensional coordinate system is first obtained by the Gramm-Schmidt orthogonalization procedure. The use of this matrix makes it possible to search in all possible directions to find an ellipsoid with a minimum volume. The general procedure is illustrated by examples in which the real data is utilized. An invariance property of the response with uncertain parameters of different physical nature is also discussed.

Keywords—Convex modeling, Experimental data, Uncertainty analysis.

1. INTRODUCTION

The availability of uncertain, limited, information for the parameters either in a structure or in a excitation to which the structure is subjected, or in both, is often encountered in various branches of engineering; this is partially due to high cost of the measurements. For this case, Ben-Haim and Elishakoff [1,2] developed a novel approach, dubbed as *convex modelling*, to analyze vibration and buckling of beams, plates, and shells due to uncertain excitation or uncertain geometrical parameters. When the excitation is of the stochastic nature with some imbedded uncertain but nonstochastic parameters, the method of random vibration must be combined with convex modelling. These considerations led Elishakoff and Colombi [3] to propose a new, hybrid probabilistic and convex-theoretic approach to analyze dynamic response of structures. In the special case when the set of uncertain parameters is an ellipsoid, closed-form solutions were derived for the upper and lower bounds of the mean-square displacements of the structure. The direct comparison of probabilistic and convex analyses was performed by Elishakoff, Cai and Starnes [4].

Uncertainty modeling by methods alternative to probabilistic modeling was dealt in various contexts by Leitmann [5], Chernousko [6] and Schweppe [7]. Applications to structures include papers by Drenick [8], Shinozuka [9], Deodatis and Shinozuka [10], Lindberg [11], Köylüoğlu, Cakmak and Nielsen [12]. The analysis of structures based on convex model for uncertain parameters consists of two parts: one is the formulation of a deterministic objective function, namely, the stress, strain or displacement of the structure; the other is modeling of uncertain parameters,

The study reported herein was supported by the NASA Langley Center. This support is gratefully appreciated.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX which are represented as belonging to a convex set. In this paper, we focus on the second part, namely, determination of convex model. It is obvious that the response or the buckling load of the structure depend on the associated convex model. Hence, the problem of determining the best convex model for a limited available information about the uncertainty parameters becomes of paramount importance. In the previous studies [1,2], for numerical convenience, the axes of the ellipsoids appearing in the convex models for the uncertain parameters were taken as directed along the coordinate axes. Thus, the results obtained based on such ellipsoidal set may be somewhat conservative for engineering design, since the volume of this ellipsoid may not possess the minimal property amongst all possible ellipsoids which can be constructed. In this study, we abandon this restrictive assumption. A general transformation matrix for rotation of N-dimensional coordinate system is first obtained by the Gramm-Schmidt orthogonalization procedure. The use of this matrix makes it possible to search in all directions to find an ellipsoid with a minimum volume. The general procedure is illustrated by an example in which real data is utilized.

2. N-DIMENSIONAL ELLIPSOIDAL CONVEX MODEL

Assume that there are N uncertain parameters a_i ($i = 1, 2, \ldots, N$) describing either in the structural properties or in the excitation. These parameters constitute an N-dimensional parameter space, namely, $\mathbf{a}^{\mathrm{T}} = \{a_1, a_2, \ldots, a_N\}$. Assume that we have limited information on these parameters, represented by M experimental points, $\mathbf{a}^{(r)}$ ($r = 1, 2, \ldots, M$) in this N-dimensional space. The convex model assumes that all these experimental points belong to an ellipsoid

$$(\mathbf{a} - \mathbf{a}^0)^{\mathrm{T}} \mathbf{G} (\mathbf{a} - \mathbf{a}^0) \le 1, \tag{1}$$

where $\mathbf{a}^0$ is the state vector of the central point of the ellipsoid, and $\mathbf{G}$ is its characteristic matrix

$$\mathbf{G} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1N} \\ g_{21} & g_{22} & \cdots & g_{2N} \\ \vdots & \vdots & & \vdots \\ g_{N1} & g_{N2} & \cdots & g_{NN} \end{bmatrix}, \tag{2}$$

which determines the size and the orientation of the ellipsoid. The matrix $\mathbf{G}$ is diagonal only when the axes of the ellipsoid are directed along the axes of the coordinates. Once $\mathbf{G}$ and $\mathbf{a}_0$ are found, the ellipsoid is determined. The best ellipsoidal convex model in the class of the ellipsoids is identified with the one which contains all given experimental points but has a minimum volume. The possible steps to search for such an ellipsoid include the rotation of the coordinate system and construction of an ellipsoid whose axes are along the principle axes in the rotated coordinate system. If we can rotate the coordinate system in all possible directions, there must exist a direction along which the ellipsoid has a minimum volume. The proposed algorithm to achieve this goal is described as follows.

2.1. Transformation Matrix for Rotation of Coordinate System

We first construct the transformation matrix for rotation of an N-dimensional coordinate system. The new coordinates $\mathbf{b}$ are related with original ones $\mathbf{a}$ as follows

$$\mathbf{b} = \mathbf{T}_N \mathbf{a}, \tag{3}$$

where $\mathbf{T}_N$ is a transformation matrix

$$\mathbf{T}_N = \mathbf{T}_N (\theta_1, \theta_2, \ldots, \theta_{N-1}), \tag{4}$$

which represents a $N \times N$ square matrix and is dependent on $N-1$ parameters, $\theta_1, \theta_2, \dots, \theta_{N-1}$, for the general N-dimensional case; for example, for the two-dimensional case there is one parameter θ_1 in $\mathbf{T}_2$, whereas for the three-dimensional case there are two parameters θ_1 and θ_2 in $\mathbf{T}_3$.

The transformation matrix $\mathbf{T}_N$ can be constructed by any set of orthogonal vectors. One of the approaches to generate these orthogonal vectors is the Gramm-Schmidt orthogonalization procedure, which is briefly presented as follows: assume $\mathbf{V}_k$ $(k = 1, \dots, N)$ to be a set of linear independent vectors. We first normalize one of the vectors, say, $\mathbf{V}_1$

$$\mathbf{U}_1 = \frac{\mathbf{V}_1}{\sqrt{\mathbf{V}_1^\top \mathbf{V}_1}}. \tag{5}$$

Let a new vector $\mathbf{W}_2$ be defined as

$$\mathbf{W}_2 = \mathbf{V}_2 - c_1 \mathbf{U}_1. \tag{6}$$

We require orthogonality of $\mathbf{W}_2$ to $\mathbf{U}_1$

$$\mathbf{U}_1^\top \mathbf{W}_2 = 0, \tag{7}$$

from which the constant c_1 is obtained

$$c_1 = \mathbf{U}_1^\top \mathbf{V}_2. \tag{8}$$

Again, we normalize $\mathbf{W}_2$ to yield

$$\mathbf{U}_2 = \frac{\mathbf{W}_2}{\sqrt{\mathbf{W}_2^\top \mathbf{W}_2}}. \tag{9}$$

Analogously, the general form of the k^{th} orthogonal vector and its normalized form can be obtained as follows:

$$\begin{aligned} \mathbf{W}_k &= \mathbf{V}_k - \sum_{i=1}^{k-1} \left(\mathbf{U}_i^\top \mathbf{V}_k\right) \mathbf{U}_i \neq 0, \qquad k \leq N, \\ \mathbf{U}_k &= \frac{\mathbf{W}_k}{\sqrt{\mathbf{W}_k^\top \mathbf{W}_k}}. \end{aligned} \tag{10}$$

The initial vectors $\mathbf{V}_k$ can be chosen from any set of linear independent vectors. Here a set of $\mathbf{V}_k$ are chosen as follows:

$$\begin{gathered} \mathbf{V}_1 = \left\{ \begin{array}{c} \cos\theta_1 \\ \sin\theta_1 \cos\theta_2 \\ \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ \vdots \\ \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-2} \cos\theta_{N-1} \\ \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-2} \sin\theta_{N-1} \end{array} \right\}; \qquad \mathbf{V}_k = \left\{ \begin{array}{c} 0 \\ \vdots \\ \delta_{i,k-1} \\ \vdots \\ \vdots \\ 0 \end{array} \right\}, \\ 0 \leq \theta_{k-1} \leq \frac{\pi}{2}, \qquad (i = 1, 2, \dots, N;\ k = 2, 3, \dots, N), \end{gathered} \tag{11}$$

where $\delta_{i,m}$ is the Kronecker delta, and the components in $\mathbf{V}_1$ are chosen to be the spherical coordinates in N-dimensional space [14]. Obviously, the vector $\mathbf{V}_k$ satisfies the following relation

$$\mathbf{V}_k^\top \mathbf{V}_k = 1; \qquad (k = 1, 2, \dots, N). \tag{12}$$

By using equation (10), a set of orthogonal vectors $\mathbf{U}_k$ ($k = 1, 2, \ldots, N$) is derived as follows:

$$\mathbf{U}_1 = \mathbf{V}_1; \qquad \mathbf{U}_k = \begin{Bmatrix} \mathbf{O}_{k-2} \\ \bar{\mathbf{U}}_k \end{Bmatrix}; \qquad k = 2, 3, \ldots, N, \tag{13}$$

where $\mathbf{O}_{k-2}$ is a vector with $k-2$ zero components and $\bar{\mathbf{U}}_k$ is a vector with $N-k+2$ components

$$\bar{\mathbf{U}}_k = \begin{Bmatrix} -\sin\theta_{k-1} \\ \cos\theta_{k-1}\cos\theta_k \\ \vdots \\ \cos\theta_{k-1}\sin\theta_k\cdots\sin\theta_{N-2}\cos\theta_{N-1} \\ \cos\theta_{k-1}\sin\theta_k\cdots\sin\theta_{N-2}\sin\theta_{N-1} \end{Bmatrix}. \tag{14}$$

Thus, the transformation matrix $\mathbf{T}_N$ for the rotation of the N-dimensional coordinate system reads

$$\mathbf{T}_N(\theta_1, \theta_2, \ldots, \theta_{N-1}) = [\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_N], \tag{15}$$

which is an orthogonal matrix. From the general N-dimensional transformation matrix, the specific cases can be obtained. For $N = 2$

$$\mathbf{T}_2(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}. \tag{16}$$

For $N = 3$

$$\mathbf{T}_3(\theta_1, \theta_2) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_2 \\ \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 & \cos\theta_2 \end{bmatrix}. \tag{17}$$

The coordinate system for $\mathbf{T}_2(\theta_1)$ is shown in Figure 1. Note that the $\mathbf{T}_3(\theta_1, \theta_2)$ can be represented as the product of two transformation matrices

$$\mathbf{T}_3(\theta_1, \theta_2) = \mathbf{T}_3^{(1)}(\theta_2)\, \mathbf{T}_3^{(2)}(\theta_1), \tag{18}$$

where

$$\mathbf{T}_3^{(1)}(\theta_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & \sin\theta_2 & \cos\theta_2 \end{bmatrix}; \tag{19}$$

$$\mathbf{T}_3^{(2)}(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{20}$$

which represent two consecutive rotations around axis 1 and axis 3′, respectively (see Figure 3). The rotational coordinate systems corresponding to $\mathbf{T}_3^{(1)}(\theta_2)$ and $\mathbf{T}_3^{(2)}(\theta_1)$ are given in Figures 2a and 2b, respectively.

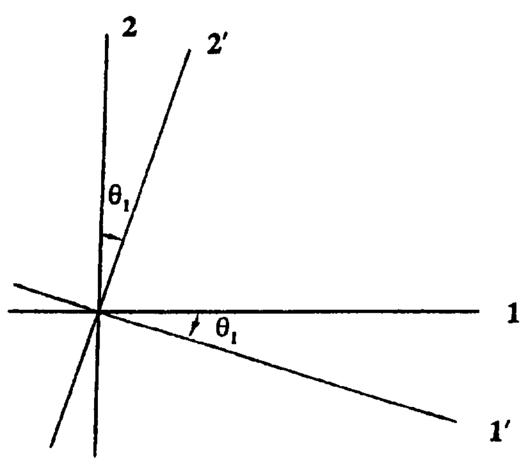
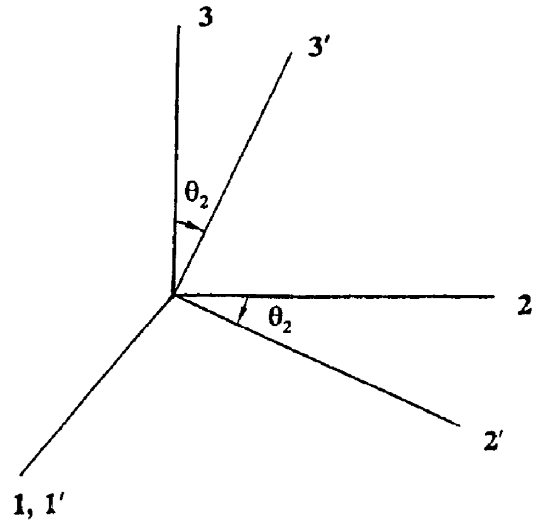
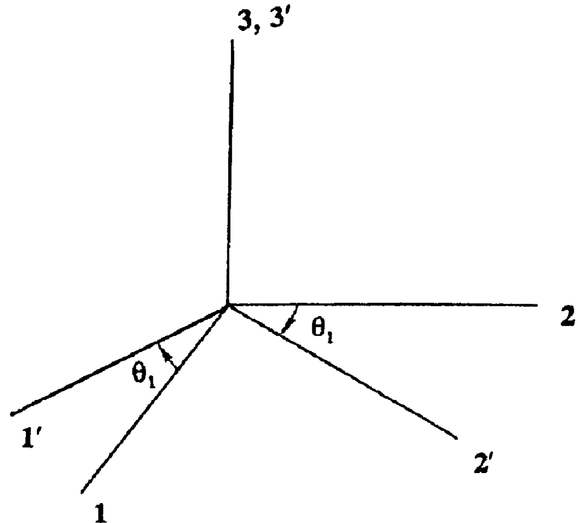


Figure 1. Rotation of coordinate system associated with $\mathbf{T}_2(\theta_1)$.



(a) Rotation of coordinate system associated with $\mathrm{T}_3^{(1)}(\theta_2)$.



(b) Rotation of coordinate system associated with $\mathrm{T}_3^{(2)}(\theta_1)$.

Figure 2.

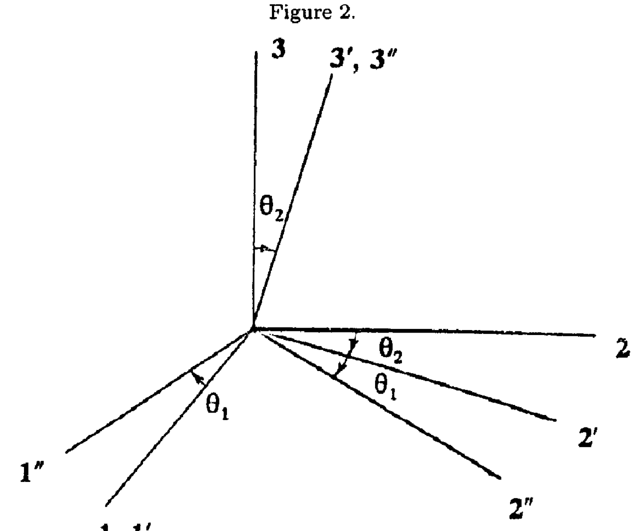


Figure 3. Rotation of coordinate system associated with $\mathbf{T}_3(\theta_1, \theta_2)$.

2.2. Ellipsoid in Rotated Coordinate System

By using transformation matrix $\mathbf{T}_N$ given in equation (4), the given M points in the rotated coordinate system will have their new coordinates denoted by $\mathbf{b}^{(r)}$ $(r = 1, 2, \ldots, M)$. To obtain the ellipsoid, let us first examine an N-dimensional box of the form

$$|\mathbf{b} - \mathbf{b}^0| \leq \mathbf{d}, \tag{21}$$

which contains all M points. The vector of semi-axes $\mathbf{d}^\top = \{d_1, d_2, \ldots, d_N\}$ and the vector of central points $\mathbf{b}^{0\,\top} = \{b_1^0, b_2^0, \ldots, b_N^0\}$ of the "box" in the rotated coordinate system are given by

$$\begin{aligned} d_k &= \frac{1}{2}\left[\max_r\left\{b_k^{(r)}\right\} - \min_r\left\{b_k^{(r)}\right\}\right], \\ b_k^0 &= \frac{1}{2}\left[\max_r\left\{b_k^{(r)}\right\} + \min_r\left\{b_k^{(r)}\right\}\right], \end{aligned} \qquad (r = 1, 2, \ldots, M;\ k = 1, 2, \ldots, N). \tag{22}$$

We now enclose this box by an ellipsoid

$$\sum_{k=1}^{N} \frac{\left(b_k - b_k^0\right)^2}{g_k^2} \leq 1, \tag{23}$$

where g_k are the semi-axes of the ellipsoid. There are infinite number of ellipsoids which contain the box given in equation (21). Clearly, the best choice is the one with the minimum volume. The volume of an N-dimensional ellipsoid is given by

$$V_e = C_N \prod_{k=1}^{N} g_k, \tag{24}$$

where C_N is a constant, depending on the dimensionality of the ellipsoid; for example, $C_2 = \pi$, $C_3 = 4\pi/3$, etc; $\prod$ denotes the product. The surface of the ellipsoid should pass through the corner points of the "box" (see equation (21). Therefore,

$$\sum_{k=1}^{N} \frac{d_k^2}{g_k^2} = 1. \tag{25}$$

We are interested in minimizing the volume V_e of the ellipsoid, subject to constraint (25). We use the Lagrange multiplier technique. The Lagrangian reads

$$L = C_N \prod_{k=1}^{N} g_k + \lambda \left(\sum_{k=1}^{N} \frac{d_k^2}{g_k^2} - 1 \right). \tag{26}$$

By requiring

$$\frac{\partial L}{\partial g_k} = 0, \qquad (k = 1, 2, \ldots, N), \tag{27}$$

we obtain a set of equations

$$C_N \prod_{k=1, k \neq i}^{N} g_k - 2\lambda \frac{d_i^2}{g_i^3} = 0, \qquad (i = 1, 2, \ldots, N). \tag{28}$$

Multiplying equation (28) by g_i and summing up the results with respect to i, we obtain

$$NV_e - 2\lambda \sum_{i=1}^{N} \frac{d_i^2}{g_i^2} = 0. \tag{29}$$

Combining equations (25) and (29), we arrive at

$$\lambda = \frac{N}{2} V_e. \tag{30}$$

Substitution of equation (30) into equation (28) results in

$$\frac{V_e}{g_i} - N V_e \frac{d_i^2}{g_i^3} = 0. \tag{31}$$

Since V_e is nonzero, we get

$$g_i = \sqrt{N}\, d_i, \qquad (i = 1, 2, \ldots, N). \tag{32}$$

Thus, once the size of the box equation (21) is known, the semi-axes of the minimum-volume ellipsoid enclosing the box of the experimental data are readily determined by utilizing equation (32). If there are no experimental points at the corner of the box, the size of such an ellipsoid may further be reduced until one of the experimental points reaches the surface of the ellipsoid. The semi-axes of the ellipsoid in this case may be replaced by ηg_k, where the factor η is determined from the condition

$$\eta = \sqrt{\max_r \sum_{k=1}^{N} \frac{\left[b_k^{(r)} - b_k^0\right]^2}{g_k^2}} \le 1; \qquad (r = 1, 2, \ldots, M). \tag{33}$$

If there are some experimental points in the corner of the multidimensional box, the factor η equals unity. The ellipsoid (23) can be rewritten in the form

$$\left\{\mathbf{b} - \mathbf{b}^0\right\}^{\mathrm{T}} \mathbf{D} \left\{\mathbf{b} - \mathbf{b}^0\right\} \le 1, \tag{34}$$

in which $\mathbf{b}^0$ is the vector of central points whose components are given by equation (22), and $\mathbf{D}$ is a diagonal matrix

$$\mathbf{D} = \operatorname{diag}\left\{(\eta g_1)^{-2}, (\eta g_2)^{-2}, \ldots, (\eta g_N)^{-2}\right\}. \tag{35}$$

2.3. The Ellipsoid with Minimum Volume

The volume of the ellipsoid now reads

$$V_e = C_N \eta^N \prod_{k=1}^{N} g_k, \tag{36}$$

which is a function of a set of parameters θ_k $(k = 1, 2, \ldots, N-1)$. Therefore, the best ellipsoid among these ellipsoids is the one which contains all given points and possesses the minimum volume, i.e.,

$$V_e = \min_{\theta_1, \theta_2, \ldots, \theta_{N-1}} \left\{V_e\left(\theta_1, \theta_2, \ldots, \theta_{N-1}\right)\right\}. \tag{37}$$

A possible approach to determine this ellipsoid is to search among all possible cases by increasing θ_k $(k = 1, 2, \ldots, N-1)$ from 0 to $\pi/2$ in sufficiently small increments $\Delta\theta_k$, and to compare the volumes of so obtained ellipsoids. Once one finds the ellipsoid with minimum volume in one direction, say θ_k^0 $(k = 1, 2, \ldots, N-1)$, the ellipsoid can be transformed back into the original coordinate system by applying the transformation matrix $\mathbf{T}_N$. Hence, the vector $\mathbf{a}^0$ of central point and the characteristic matrix $\mathbf{G}$ in equation (1) become

$$\mathbf{a}^0 = \mathbf{T}_N^{\mathrm{T}} \mathbf{b}^0, \qquad \mathbf{G} = \mathbf{T}_N^{\mathrm{T}} \mathbf{D} \mathbf{T}_N, \tag{38}$$

where $\mathbf{T}_N = \mathbf{T}_N\left(\theta_1^0, \theta_2^0, \ldots, \theta_{N-1}^0\right)$ is given by equation (15); vector $\mathbf{b}^0$ and matrix $\mathbf{D}$ are given by equations (22) and (35), respectively. It can be shown that $\mathbf{G}$ is a symmetric and positive-definite matrix which is nondiagonal in the general case.

3. NUMERICAL EXAMPLES AND INVARIANCE PROPERTY

As an example, a set of uncertain parameters obtained from real tests on shell bucking [14] are chosen. These parameters represent Fourier coefficients of the half-wave cosine and half-wave sine representations, respectively, of the initial imperfections of shells. Chosen four parameters form the four-dimensional space. The values of Fourier coefficients of initial geometric imperfection derived in eight tests are represented by eight points in this space. They are listed in Tables 1 and 3, respectively.

Table 1. The values of uncertain parameters A_k for half-wave cosine representation.

k	1	2	3	4
$A_k^{(1)}$	1.800×10^{-2}	-5.000×10^{-3}	6.700×10^{-2}	7.000×10^{-3}
$A_k^{(2)}$	3.400×10^{-2}	-3.000×10^{-3}	0.653	2.800×10^{-2}
$A_k^{(3)}$	2.300×10^{-2}	-6.000×10^{-3}	8.300×10^{-2}	2.000×10^{-2}
$A_k^{(4)}$	1.100×10^{-2}	2.000×10^{-3}	-2.300×10^{-2}	0.000
$A_k^{(5)}$	2.000×10^{-3}	1.000×10^{-3}	1.600×10^{-2}	0.000
$A_k^{(6)}$	2.000×10^{-3}	0.000	2.400×10^{-2}	0.000
$A_k^{(7)}$	3.000×10^{-3}	0.000	6.600×10^{-2}	1.000×10^{-3}

Table 2. Minimum volume of ellipsoid obtained by different increments, $\Delta\theta$.

Increment $\Delta\theta$	Orientation θ_i			Volume A_e
	θ_1	θ_2	θ_3	
5°	90°	90°	50°	1.7623×10^{-6}
3°	90°	3°	87°	5.6886×10^{-7}
2°	90°	2°	88°	4.4827×10^{-7}
1°	90°	2°	87°	4.2200×10^{-7}

Table 3. The values of uncertain parameters B_k for half-wave sine.

k	1	2	3	4
$B_k^{(1)}$	3.700×10^{-2}	-1.600×10^{-2}	6.600×10^{-2}	9.000×10^{-3}
$B_k^{(2)}$	2.600×10^{-2}	-1.000×10^{-2}	0.611	4.500×10^{-2}
$B_k^{(3)}$	5.600×10^{-2}	-1.000×10^{-2}	7.500×10^{-2}	-6.000×10^{-3}
$B_k^{(4)}$	2.900×10^{-2}	8.000×10^{-3}	-1.900×10^{-2}	1.000×10^{-3}
$B_k^{(5)}$	9.000×10^{-3}	4.000×10^{-3}	6.000×10^{-3}	1.000×10^{-3}
$B_k^{(6)}$	-2.000×10^{-3}	5.000×10^{-3}	2.000×10^{-2}	0.000
$B_k^{(7)}$	3.000×10^{-3}	5.000×10^{-3}	4.900×10^{-2}	8.000×10^{-3}

Table 4. Minimum volume of ellipsoid obtained by different increments of $\Delta\theta$.

Increment $\Delta\theta$	Orientation θ_i			Volume A_e
	θ_1	θ_2	θ_3	
5°	25°	0°	85°	5.2986×10^{-6}
3°	15°	0°	87°	7.2538×10^{-6}
2°	26°	0°	86°	5.0534×10^{-6}
1°	25°	0°	86°	5.0479×10^{-6}

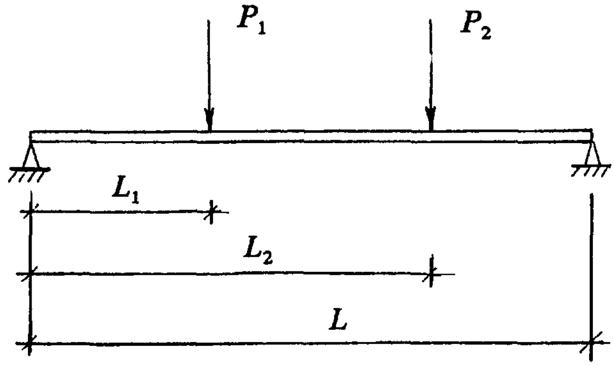


Figure 4. A simply supported beam with concentrated loading.

Tables 2 and 4, respectively, list the minimum volumes and orientations of the ellipsoid obtained by different increments $\Delta\theta$. The size of the increment is reduced so as to achieve the practical numerical convergence. The process of reducing $\Delta\theta$ is completed if the volumes of so constructed ellipsoids change by less than one percent. The minimum volume of an ellipsoid depends on the value of the increment $\Delta\theta$. The axes of the ellipsoid with the minimum volume are usually along the general, nonprincipal directions.

An additional simple example of beam with simply supported ends and under two concentrated loads is shown in Figure 4. We are interested in the invariance property of convex modelling. To this end we assume that two hypothetical independent investigations use two different unit systems, for example, SI system and English system to analyze the same experimental data. The most and least favorable response of a structure based on the convex analysis read [2]

$$\left.\begin{matrix} R_{\max} \\ R_{\min} \end{matrix}\right\} = R(\mathbf{a}^0) \pm \sqrt{\mathbf{r}^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{r}}, \qquad \mathbf{r}^{\mathrm{T}} = \left[\frac{\partial R(\mathbf{a})}{\partial a_1}, \frac{\partial R(\mathbf{a})}{\partial a_2}, \dots, \frac{\partial R(\mathbf{a})}{\partial a_N}\right]_{\mathbf{a}=\mathbf{a}^0}. \tag{39}$$

The invariance property of the convex model will assure that the responses obtained by two different unit systems are identical. Let us investigate a simple example. The bending moment at the midpoint of the simply supported beam (Figure 4) subjected to two concentrated loads P_1 and P_2 reads

$$M = \frac{1}{2}\left[P_1 L_1 + P_2 (L - L_2)\right]. \tag{40}$$

First, we assume that two span lengths $a_1 = L_1$ and $a_2 = L_2$ are uncertain parameters with a limited measurements represented by four points in two-dimensional space as shown in Figures 5 and 6 in different units, and $P_1 = 1\,\mathrm{kN}$ ($0.2248\,\mathrm{klb}_f$), $P_2 = 2\,\mathrm{kN}$ ($0.4496\,\mathrm{klb}_f$) and $L = 3\,\mathrm{m}$ ($9.843\,\mathrm{ft}$). The ellipsoidal convex model and the maximum values of moment at the midpoint of beam are evaluated in two different units as follows: for SI system (m, kN)

$$\begin{gathered} \theta^0 = 65^\circ, \quad \mathbf{a}^0 = \{1.0721\,\mathrm{m}, 2.0003\,\mathrm{m}\}, \\ \mathbf{G} = \begin{bmatrix} 37.1723 & 9.4686 \\ 9.4686 & 21.2821 \end{bmatrix}, \end{gathered} \tag{41}$$

and

$$R_{\max} = M_{\max}^{(\mathrm{SI})} = 1.53575 + 0.27217 = 1.80792\,\mathrm{kN}\cdot\mathrm{m}, \tag{42}$$

for English system (ft, klb_f)

$$\begin{gathered} \theta^0 = 65^\circ, \quad \mathbf{a}^0 = \{3.5176\,\mathrm{ft}, 6.5631\,\mathrm{ft}\}, \\ \mathbf{G} = \begin{bmatrix} 3.45585 & 0.88065 \\ 0.88065 & 1.97794 \end{bmatrix}, \end{gathered} \tag{43}$$

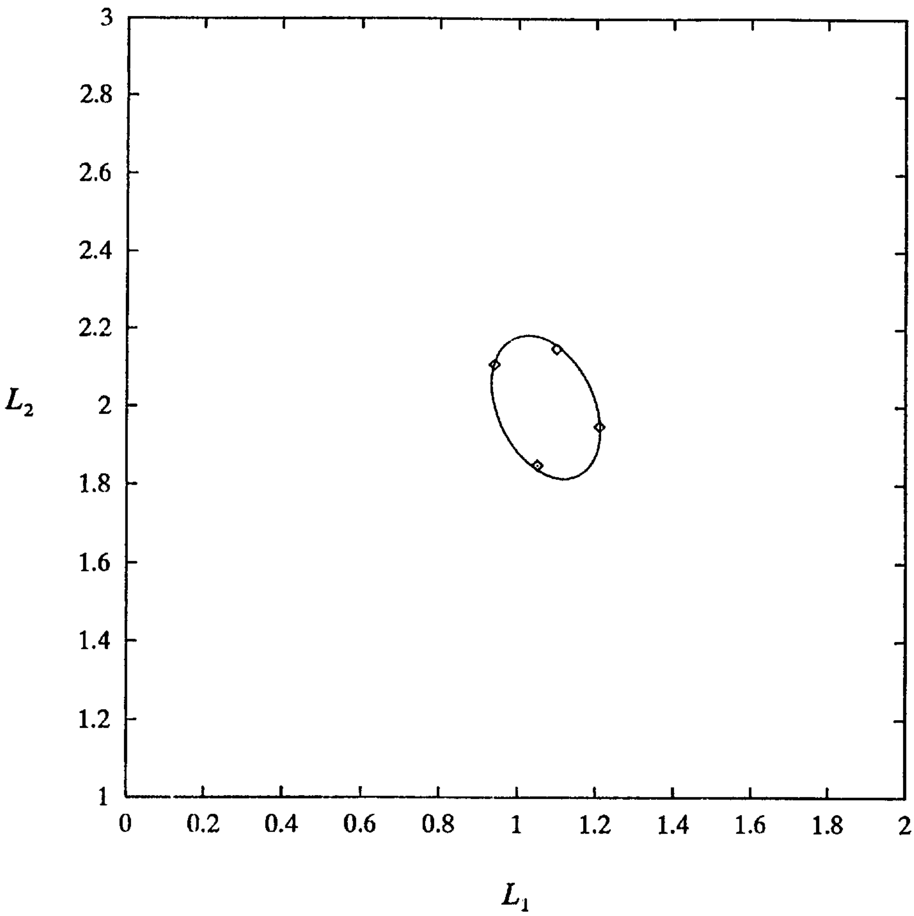


Figure 5. Ellipse of uncertain parameters L_1 and L_2 in SI units (m).

and

$$R_{\max} = M_{\max}^{(\mathrm{EG})} = 1.13272 + 0.20072 = 1.33344\,\mathrm{klb}_f \cdot \mathrm{ft}. \tag{44}$$

Note that the unit factors for force and length λ_P and λ_L are

$$\begin{aligned} \lambda_P &= 1\,\mathrm{kN} = 0.2248\,\mathrm{klb}_f, \\ \lambda_L &= 1\,\mathrm{m} = 3.281\,\mathrm{ft}. \end{aligned} \tag{45}$$

Thus,

$$\lambda_L \lambda_P M_{\max}^{(\mathrm{SI})} = 0.73757 \times 1.80792 = 1.33346\,\mathrm{klb}_f \cdot \mathrm{ft} \equiv M_{\max}^{(\mathrm{EG})}. \tag{46}$$

Is is shown that two results are identical, as expected.

Consider now the case that two uncertain parameters have different dimensions. In order to maintain the invariance property of the responses with different units, nondimensional uncertain parameters are suggested to be used in the convex analysis. For example, assume in our previous example that uncertain parameters are $L_1 = a_1 \lambda_L$ and $P_2 = a_2 \lambda_P$, where a_1 and a_2 are nondimensional uncertain parameters. The other, fixed parameters are $L_2 = 2\lambda_L$, $P_1 = 1\lambda_P$ and $L = 3\lambda_L$. Thus, the bending moment at the midpoint of the beam from (40) becomes, in view of equation (40)

$$\begin{aligned} M(a_1, a_2) &= \frac{1}{2}(a_1 + a_2)\,\lambda_P \lambda_L \equiv m(a_1, a_2)\,\lambda_P \lambda_L, \\ \mathbf{F}^{\mathrm{T}}(a_1, a_2) &= \mathrm{grad}^{\mathrm{T}}\{M(a_1, a_2)\} = \left\{\frac{1}{2}, \frac{1}{2}\right\} \lambda_P \lambda_L \equiv f^{\mathrm{T}}(a_1, a_2)\,\lambda_P \lambda_L, \end{aligned} \tag{47}$$

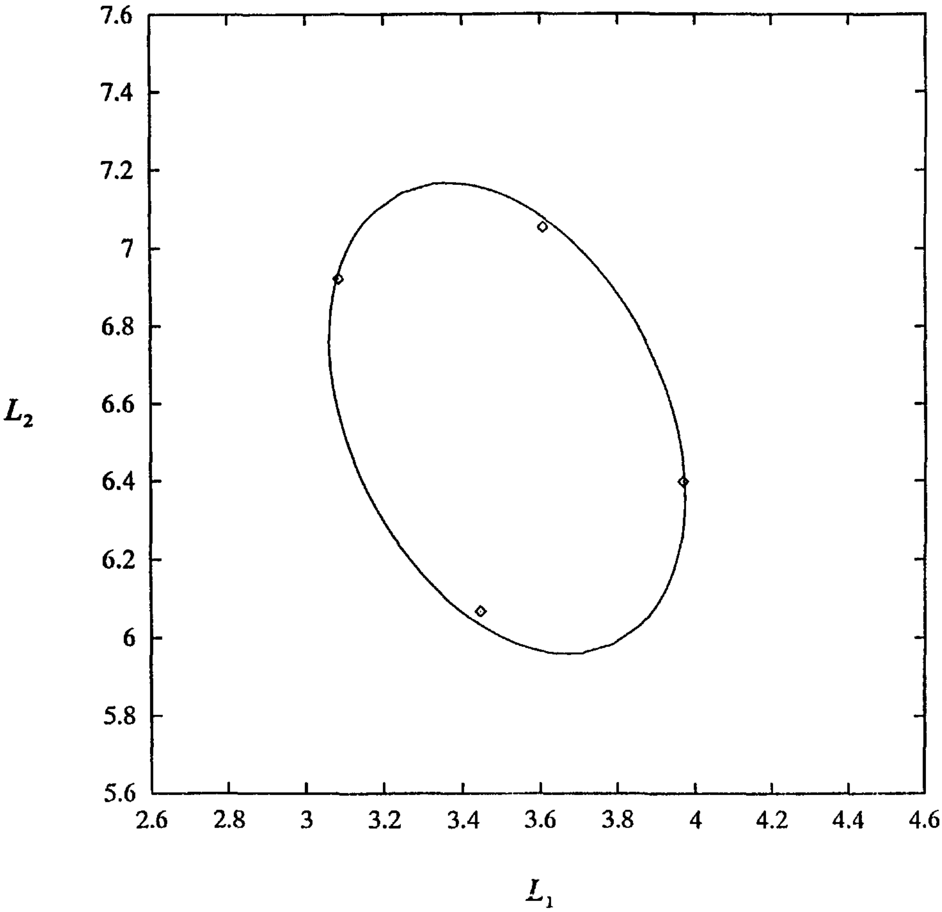


Figure 6. Ellipse of uncertain parameters L_1 and L_2 in customary units (ft).

where $m(a_1, a_2)$ and $f(a_1, a_2)$ have nondimensional values. Since a_1 and a_2 are nondimensional, the $\mathbf{a}^0$ and $\mathbf{G}$ (or $\mathbf{G}^{-1}$) obtained by convex analysis in equation (38) are also nondimensional. Hence, the maximum value of the response becomes, in view of equation (39)

$$\begin{aligned} M_{\max} &= M\left(a_1^0, a_2^0\right) + \sqrt{\mathbf{F}^\top\left(a_1^0, a_2^0\right)\mathbf{G}^{-1}\mathbf{F}\left(a_1^0, a_2^0\right)} \\ &= \left\{m\left(a_1^0, a_2^0\right) + \sqrt{f^\top\left(a_1^0, a_2^0\right)\mathbf{G}^{-1}f\left(a_1^0, a_2^0\right)}\right\}\lambda_L\lambda_P, \end{aligned} \tag{48}$$

where $\mathbf{a}^0 = \{a_1^0, a_2^0\}$ is the nominal vector or central point of ellipse given by equation (38). As is seen the choice of the units can be arbitrary.

4. CONCLUSION

A general transformation matrix for rotation of N-dimensional coordinate system was constructed by using Gramm-Schmidt orthogonalization procedure. It was shown that the use of this matrix makes it possible to search in all directions to find the N-dimensional ellipsoid with a minimum volume. Several numerical examples have been chosen for illustration. It is shown that the axes of the ellipsoid with the minimum volume have the general orientation in the parameter space. The invariance property of the response of structure with uncertain parameters of different units was investigated. It is shown that, with the nondimensional formulation, the invariance property of the response is retained.

REFERENCES

1. Y. Ben-Haim and I. Elishakoff, *Convex Models of Uncertainty in Applied Mechanics*, Elsevier Science, Amsterdam, (1990).

2. I. Elishakoff and Y. Ben-Haim, Dynamics of a thin cylindrical shell under impact with limited deterministic information on its initial imperfections, *Journal of Structural Safety* **8**, 103–112 (1990).
3. I. Elishakoff and P. Colombi, Combination of probabilistic and convex models of uncertainty when scarce knowledge is present on acoustic excitation parameters, *Computer Methods in Applied Mechanics and Engineering* **104**, 187–209 (1993).
4. I. Elishakoff, G.Q. Cai and J.H. Starnes, Jr., Nonlinear buckling of a column with initial imperfection via stochastic and nonstochastic convex modes, *International Journal of Nonlinear Mechanics* **29**, 71–82 (1994).
5. G. Leitmann, One approach to the control of uncertain systems, *Journal of Dynamic Systems, Measurement and Control* **115**, 362–373 (1993).
6. F.L. Chernousko, *State Estimation for Dynamic Systems*, CRC Press, Boca Raton, (1994).
7. F.C. Schweppe, *Uncertain Dynamics Systems*, Prentice Hall, Englewood Cliffs, (1973).
8. R.F. Drenick, Model-free design of aseismic structures, *Journal of Engineering Mechanics* **96**, 483–493 (1970).
9. M. Shinozuka, Maximum structural response to seismic excitations, *Journal of Engineering Mechanics* **96**, 729–738 (1970).
10. G. Deodatis and M. Shinozuka, Bounds on response variability of stochastic systems, *Journal of Engineering Mechanics* **115**, 2543–2563 (1989).
11. H. Lindberg, Convex models for uncertain imperfection control in multimode dynamic buckling, *Journal of Applied Mechanics* **59**, 937–945 (1992).
12. H.U. Köylüoğlu, A.S. Cakmak and S.R. Nielsen, Applications of interval algebra in mechanics for structural uncertain and pattern loading, In *Proceedings of the Second International Conference on Computational Stochastic Mechanics*, (Edited by S.P. Spanos), pp. 125–133, Athens, 1994.
13. G. Masaglia, Choosing a point from the surface of a sphere, *Ann. Math. Stat.* **43**, 645–646 (1972).
14. J. Arbocz and H. Abramovich, The initial imperfection data bank at the Delft University of Technology, Part I, Report LR-290, (December 1979).