Energy of a Simple Harmonic oscillator / SHM

let us take a particle having mass m is in SHM with displacement x then;

$$\text{Potential Energy } (U) = \int -F \cdot dx$$

$$= \int_0^x -(-)kx \cdot dx$$

$$= \frac{1}{2}kx^2$$

$$(U) = \frac{1}{2}k A^2 \sin^2(\omega t + \phi) \quad \text{—— (i)}$$

And.

$$K.E = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$$

$$= \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \phi).$$

$$= \frac{1}{2}kA^2\cos^2(\omega t + \phi) \quad \text{(ii)} \quad \left[\because k = m\omega^2\right]$$

Now; The total energy (TE) = P.E + K.E

$$= \frac{1}{2}kA^2\sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

$$= \frac{1}{2}kA^2$$

Q. Show that average k.E is equal to average P.E is equal t(o) $\frac{1}{2}$ of the total energy of a SHM.

Ans:- We have.

$$k.E = \frac{1}{2} kA^2 \cos^2(\omega t + \phi)$$

Average k.E is,

$$\langle k.E \rangle = \frac{1}{T}\int_0^T (k.E)\,dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2} kA^2 \cos^2(\omega t + \phi)\,dt$$

For simplicity we assume $\phi = 0$

$$\therefore \langle k.E \rangle = \frac{1}{T}\int_0^T \frac{1}{2} kA^2 \cos^2(\omega t)\,dt$$

$$= \frac{1}{2T} kA^2 \int_0^T \frac{\cos 2(\omega t) + 1}{2}\,dt$$

$$= \frac{1}{4T} kA^2 \int_0^T 1 + \cos 2(\omega t)\,dt$$

$$= \frac{1}{4T} kA^2 \left[t + \frac{\sin 2(\omega t)}{2\omega} \right]_0^T$$

$$= \frac{1}{4T} kA^2 \left[T + \frac{\sin 2\left(\frac{2\pi}{\cancel{T}} \times \cancel{T}\right)}{2 \cdot \frac{2\pi}{T}} \right]$$

$$= \frac{1}{4T} kA^2 \left[T + \frac{\sin 4\pi}{4\pi/T} \right]$$

$$= \frac{kA^2}{2T} \left[\frac{T}{2} + \frac{1}{2} \frac{T \sin 4\pi}{4\pi} \right]$$

$$= \frac{kA^2}{2\cancel{T}} \times \frac{\cancel{T}}{2} \qquad \left[\because \langle \cos^2\theta \rangle = \langle \sin^2\theta \rangle = 1/2 \right]$$

$$= kA^2/4 \quad \text{—} (1)$$

Similarly Average P.E is

$$\langle P.E\rangle = \frac{1}{T}\int_0^T (P.E)\,dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}kA^2\sin^2(\omega t+\phi)\,dt$$

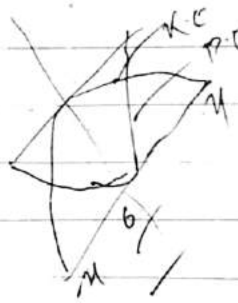
For simplicity we assume $\phi = 0$.

$$\langle P.E\rangle = \frac{1}{2T}kA^2\int_0^T \frac{1-\cos 2\omega t}{2}\,dt$$

$$= \frac{kA^2}{2T}\left\{\int_0^T \frac{1}{2}dt - \cancelto{0}{\int_0^T \frac{\cos 2\omega t}{2}dt}\right\}$$

$$= \frac{kA^2 \times T}{4T}$$

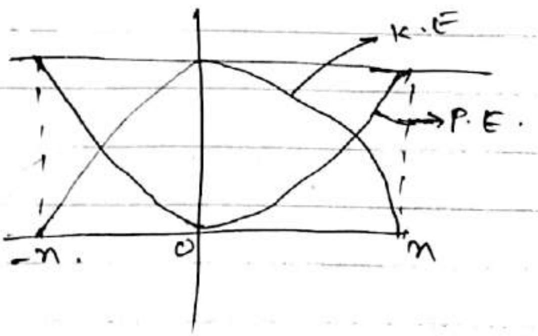
$$= \frac{kA^2}{4} \quad \text{— (ii)}$$



$$\langle K.E\rangle = \langle P.E\rangle = \frac{1}{2}\langle T.E\rangle$$

or, $\frac{kA^2}{4} = \frac{kA^2}{4} = \frac{1}{2}\times\frac{1}{2}kA^2$

$\therefore \frac{kA^2}{4} = \frac{kA^2}{4} = \frac{kA^2}{4}$ proved.



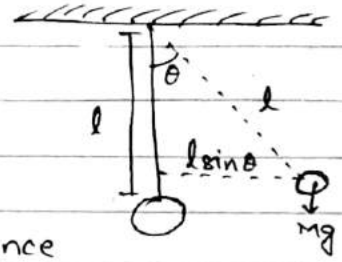
Examples of SHM

① Simple Pendulum.

1 # Simple Pendulum.

Let us take a bob having point mass 'm' is oscillating about a horizontal axis with angular displacement 'θ' and moment of inertia about the point of suspension is $I = ml^2$ where;

l is the length of suspension.



Then the torque acting on the bob is:

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} \quad \text{—— (i)}$$

and restoring torque/force = force × ⊥er distance

$$= -mgl\sin\theta \text{ —— (ii)}$$

equating eqn (i) & (ii).

$$\Rightarrow I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

For small angle $\sin\theta \simeq \theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{mgl}{I}\theta = 0$$

Comparing above eqn with general differential eqn of SHM.

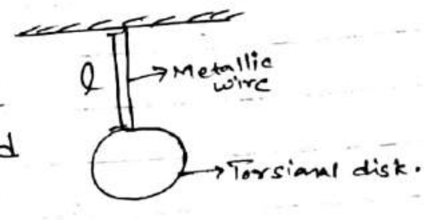
$$\frac{mgl}{I} = \omega^2$$

$$\Rightarrow T = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{ml^2}{mgl}}$$

$$\therefore T = 2\pi\sqrt{l/g}.$$

2. Torsional Pendulum.

Let us take a torsional disc having moment of inertial 'I' is attached with metallic wire having length 'l' and modulus of rigidity of wire is 'η'. Then after the disc is rotated such that the wire is twisted and we calculate the time period.



Let 'θ' be the twisting angle then torque acting is

$$\tau \propto \theta$$

or, $\tau = -c\theta$ —— (i)

where $c = \frac{\pi \eta r^4}{2l}$ is the torsional constant. Also, r is the radius of metallic wire.

We have;

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} \text{ —— (ii)}$$

Equating eqn (i) & (ii) we get;

$$I\frac{d^2\theta}{dt^2} + c\theta = 0 \Rightarrow \frac{d^2\theta}{dt^2} + \frac{c\theta}{I} = 0$$

Comparing above equation with general diffn eqn of SHM $\frac{d^2x}{dt^2} + \omega^2 x = 0$ we get;

$$\omega^2 = \frac{c}{I}$$

We know; $\omega = \frac{2\pi}{T}$

or, $\sqrt{\frac{c}{I}} = \frac{2\pi}{T}$

$\therefore T = 2\pi\sqrt{\frac{I}{c}}$ is the required time period for the torsional pendulum

It shows that time period of torsional pendulum doesn't depend on the accln due to gravity.

Let us take a ring having moment of inertial 'I₁' is kept above the disc and the metallic wire is twisted then the time period is given by:

$$T_1 = 2\pi\sqrt{\frac{I+I_1}{C}}$$

→ Applications

(i) Calculation of modulus of rigidity of wire.

Now, the time period of torsional pendulum is

$$T = 2\pi\sqrt{\frac{I}{C}} \quad \text{— (i)}$$

when ring is kept above the torsional pendulum then the time period is given by

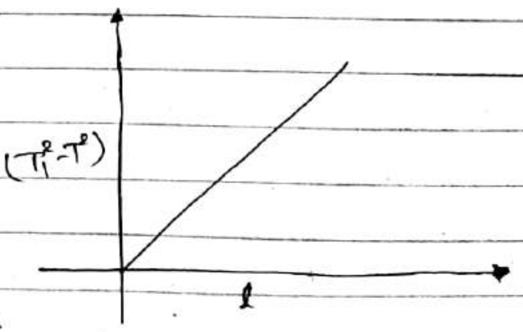
$$T_1 = 2\pi\sqrt{\frac{I+I_1}{C}} \quad \text{— (ii)}$$

~~Comparing~~ Squaring eqn (i) & (ii) and subtracting (i) from (ii) we get;

$$T_1^2 - T^2 = 4\pi^2\left[\frac{I+I_1}{C} - \frac{I}{C}\right]$$

$$\Rightarrow T_1^2 - T^2 = 4\pi^2\frac{I_1}{C}$$

$$\text{or, } T_1^2 - T^2 = \frac{4\pi^2 I_1}{\left(\frac{\pi\eta r^4}{2l}\right)}$$



$$\therefore \eta = \frac{8\pi I_1 l}{(T_1^2 - T^2) r^4}$$

which is the modulus of rigidity of metallic wire at constant Temperature.

Unit of η is dyne/cm²

(ii) To find the moment of inertia of non-uniform irregular rigid body.

We have the time period for torsional pendulum is

$$T = 2\pi\sqrt{\frac{I}{C}} \quad \text{—— (i)}$$

When ring is kept above the torsional pendulum time period is given by

$$T_1 = 2\pi\sqrt{\frac{I+I_1}{C}} \quad \text{—— (ii)}$$

Let us take a non uniform rigid body having moment of Inertia 'I_2' is kept on the disc and twisted such that the time period is

$$T_2 = 2\pi\sqrt{\frac{I+I_2}{C}} \quad \text{—— (iii)}$$

Squaring (i), (ii) & (iii) and divided by (ii) & (iii) by (i)

$$\Rightarrow \quad \frac{T_1^2}{T^2} = \frac{I+I_1}{I} \quad \text{and}$$

$$\frac{T_2^2}{T^2} = \frac{I+I_2}{I}$$

$$\frac{T_1^2 - T^2}{T^2} = \frac{I_1}{I} \quad \text{—— (iv)}$$

and $$\frac{T_2^2 - T^2}{T^2} = \frac{I_2}{I} \quad \text{—— (v)}$$

Dividing eqn (iv) by (v)

$$\frac{T_1^2 - T^2}{T_2^2 - T^2} = \frac{I_1}{I_2}$$

$$\Rightarrow I_2 = I_1\left(\frac{T_2^2 - T^2}{T_1^2 - T^2}\right) \quad \text{—— (vi)}$$

Eqn (vi) gives the moment of inertia of non-uniform rigid body.

(3) Physical Pendulum:-

A rigid body which is capable to oscillates about a horizontal axis is called physical pendulum or compound pendulum.

Let us take a rigid body which is capable to oscillates about a horizontal axis with moment of inertia 'I' as shown in figure. 'S' is the point of suspension, 'O' is the point of CG. The distance betn point of suspension and centre of gravity is called length of suspension (l). When it is oscillate at an angle 'θ' Then the restoring torque act on the pendulum is

$$\tau = -mgl\sin\theta \quad \text{——} (i)$$

We have;

$$\tau = I\alpha = \frac{Id^2\theta}{dt^2} \quad \text{——} (ii)$$

~~Compari~~ Equating eqn (i) & (ii)

$$\frac{Id^2\theta}{dt^2} + mgl\sin\theta = 0$$

$$\therefore d^2\theta/dt^2 + \frac{mgl}{I}\theta = 0 \quad (\text{For small angle } \sin\theta \simeq \theta)$$

Comparing it with SHM differential equation we get;

$$\omega^2 = \frac{mgl}{I}$$

We know, $\omega = \frac{2\pi}{T}$ $\quad \therefore T = \frac{2\pi}{\sqrt{\frac{mgl}{I}}} = 2\pi\sqrt{\frac{I}{mgl}}$

Now, the moment of inertia of the compound pendulum is

$I = Mk^2 + Ml^2$ where; k is the radius of gyration in which the time period of compound pendulum is minimum.

Then the time period is;

$$T = 2\pi\sqrt{\frac{mk^2+ml^2}{mgl}}$$

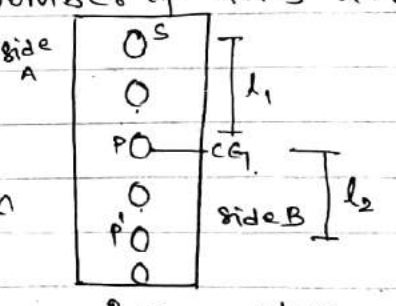
$$\therefore T = 2\pi\sqrt{\frac{k^2+l^2}{gl}}$$

$$\therefore T = 2\pi\sqrt{\frac{k^2/l + l}{g}}$$

(4) Bar Pendulum:

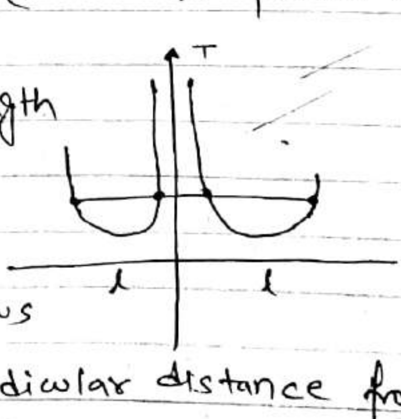
It is the type of physical pendulum.

It is the metallic bar in which large number of holes are created which is at equidistance from each other.



Bar. pendulum.

Let S be the point of suspension, (P) be the point of centre of gravity. The distance betn point of suspension and centre of gravity is called length of suspension. Let us take another point (P') which is at distance k^2/l_1 from the centre of gravity is called length of oscillation. Then the time period of oscillation is:



$$T = 2\pi\sqrt{\frac{k^2/l_1 + l_1}{g}}$$ where k is a radius of gyration i.e the perpendicular distance from the centre of mass to axis of rotation.

Case I: Show that there are four collinear points in Bar pendulum.

Let us take a side A of bar pendulum in which the length of suspension is l_1, then the time period is given by

$$T = 2\pi\sqrt{\frac{\frac{k^2}{l_1}+l_1}{g}}$$

$$\Rightarrow T^2 = 4\pi^2\left(\frac{k^2+l_1^2}{gl_1}\right)$$

$$\Rightarrow l_1^2 - \frac{gl_1T^2}{4\pi^2} + k^2 = 0.$$

$$\therefore l_1 = \frac{\frac{gT^2}{4\pi^2} \pm \sqrt{\left(\frac{gT^2}{4\pi^2}\right)^2 - 4k^2}}{2}$$

It shows that there are two length in same time period for side A. Similarly for side B we can obtain that there are two length in same time period. From this relation we can say that there are four collinear points in Bar pendulum.

Case II:- To find the radius of Gyration.

Let l_1 be the length of suspension and $l_2 = k^2/l_1$ is the length of oscillation. Then the time period of oscillation is

$$T = 2\pi\sqrt{\frac{\frac{k^2}{l_1}+l_1}{g}}$$

Now, $\frac{k^2}{l_1} + l_1 = l_2 + l_1$

$$\Rightarrow k^2 = l_1 l_2$$

$$\Rightarrow k = \sqrt{l_1 l_2}$$

Case III: To find the condition in which time period is maximum & minimum.

We have:

$$T = 2\pi\sqrt{\frac{\frac{k^2}{l_1}+l_1}{g}}$$

The term $\frac{k^2}{l_1}+l_1$ represent the length of the pendulum. The time period is maximum when $\frac{k^2}{l_1}+l_1$ is maximum. It is possible only when $l_1=\infty$ or 0 but practically infinite length of suspension is not posible such that the time period is maximum when $l_1=0$ i.e when length of suspension is 0, It means length of suspension and CG coincide with each other.

The time period is minimum when k^2/l_1+l_1 is minimum.

$$\frac{dT}{dl_1} = \frac{d(k^2/l_1+l_1)}{dl_1} = 0$$

$$\text{or, } -\frac{k^2}{l_1^2}+1 = 0$$

$$\Rightarrow k^2 = l_1^2$$

$$\Rightarrow k = l_1 \quad \text{——— (1)}$$

Again we have $k=\sqrt{l_1 l_2}$

$$\Rightarrow l_1 l_2 = l_1^2$$

$$\Rightarrow l_2 = l_1 \quad \text{——— (2)}$$

from eqn (1) it shows that the time period is minimum when length of suspension is equal to radius of gyration or from eqn (ii) it shows that the time period is minimum when length of suspension is equal to length of oscillation.

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Case IV: Show that point of suspension and point of oscillation are interchange.

let l_1 be the length of suspension and l_2 be the length of oscillation for side A, then time period.

$$T_1 = 2\pi\sqrt{\frac{\frac{k^2}{l_1}+l_1}{g}} \quad \text{--- (i)}$$

If we interchange the point of suspension and point of oscillation then the time period are

$$T_2 = 2\pi\sqrt{\frac{\frac{k^2}{l_2}+l_2}{g}} \quad \text{--- (ii)}.$$

We have;

$$k^2 = l_1 l_2$$

Putting this value in eq^n (i) & (ii) we get;

$$T_1 = 2\pi\sqrt{\frac{\frac{\cancel{l_1} l_2}{\cancel{l_1}}+l_1}{g}} = 2\pi\sqrt{\frac{l_2+l_1}{g}}$$

$$T_2 = 2\pi\sqrt{\frac{l_1+l_2}{g}}$$

$$\therefore T_1 = T_2$$

Hence the point of suspension and point of osillation can be interchange.

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Types of Oscillation.

1) Free oscillation:

The oscillation in which the total energy of amplitude remain constant is called free oscillation. It exists only in vaccum. In free oscillation resistive force doesnot exist in that medium.

Now, the K.E of oscillation having displacement x is

$$K.E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$$

Potential energy $(P.E) = \frac{1}{2}kx^2$

$\therefore$ Total energy $= \frac{1}{2}kx^2 + \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 =$ constant.

Differentiating w.r.t time on both sides we get;

$$\frac{1}{\cancel{2}}k\,\cancel{2}x\frac{dx}{dt} + \frac{1}{\cancel{2}}m\,\cancel{2}\frac{dx}{dt}\left(\frac{d^2x}{dt^2}\right) = 0$$

$$\Rightarrow kx + m\left(\frac{d^2x}{dt^2}\right) = 0$$

$$\Rightarrow m\frac{d^2x}{dt^2} + kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{where } \omega^2 = k/m$$

We have $\omega = 2\pi f$

$\therefore f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ is the reqd freqn of free oscillation

Put cos= "

2) Damped oscillation/vibration.

The oscillation in which amplitude or energy decreases with increase in time due to internal and external resistance factor is called damped oscillation. The external force doesnot applied on the damped oscillation.

The resisting force is $-b\frac{dx}{dt}$ where b is a damping force constant and x is the displacement at any instant of time t. The SI unit of damping force constant (b) is kg/s.

Let us take a particle having mass (m) is in damping oscillation/vibration, x be the displacement at any instant of time, then

$$F = -kx - b\frac{dx}{dt}$$

$$\Rightarrow m\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt} = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \quad \text{--- (1)}$$

Let $x = A_0e^{\lambda t}$ is the general solution of eqn (1) that it should satisfy the eqn (1)

$$\Rightarrow \frac{d^2(A_0e^{\lambda t})}{dt^2} + \frac{b}{m}\frac{d}{dt}(A_0e^{\lambda t}) + \frac{k}{m}A_0e^{\lambda t} = 0$$

$$\Rightarrow A_0\lambda^2e^{\lambda t} + \frac{b}{m}A_0\lambda e^{\lambda t} + \frac{k}{m}A_0e^{\lambda t} = 0$$

$$\Rightarrow \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$$

$$\Rightarrow m\lambda^2 + b\lambda + k = 0$$ which is quadratic in λ, then

$$\therefore \lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

$$= \frac{-b}{2m} \pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Then $x = A_0 e^{\lambda t}$

$$= A_0 e^{-b/2m\,t} \cdot e^{\pm\left(i\sqrt{k/m - b^2/4m^2}\right)t}$$

$$= A_0 e^{-b/2m\,t} \cos\omega' t \;(\text{or } \sin\omega' t) \quad \text{——— (ii)}$$

where $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ is the angular freq^n of damped oscillation..

⇒ frequency of damped oscillation

$$f' = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad \text{—— (iii)}$$

[where k is force constant
b is resistive force constant]

Eqn (ii) shows that the amplitude of damped oscillation decreases exponentially with time, where A_0 is the initial amplitude.

Case I:

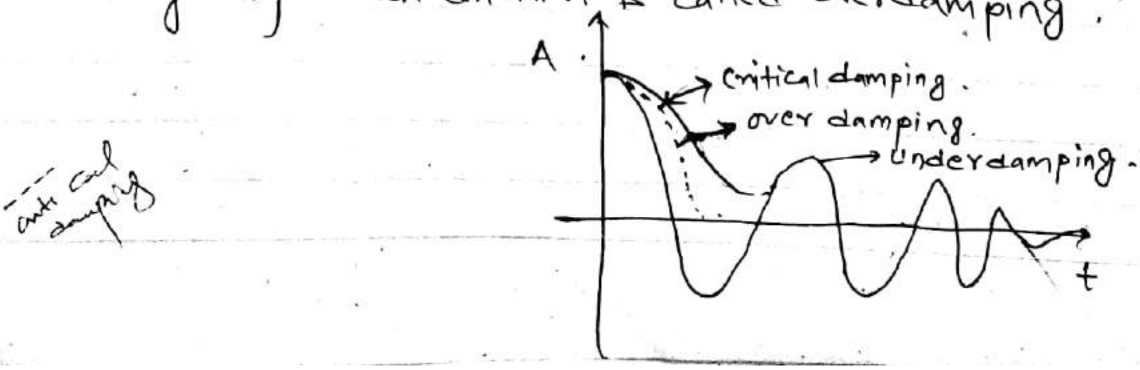
When $k/m = b^2/4m^2$ this condition is called critical damping in which the frequency of oscillation is suddenly die out.

Case II:

When $k/m > b^2/4m^2$ this condition is called underdamping in which the oscillation is die out after a long time.

Case III:

When $k/m < b^2/4m^2$ then ~~the condition~~ the frequency of damped oscillation is imaginary which condition is called overdamping.



The energy of damped oscillation is

$$E = \frac{1}{2} k A_0^2 e^{-b/2m t \cdot 2}$$

$$= \frac{1}{2} k A_0^2 e^{-b/m t}.$$

Quality factor of damped oscillation.

Quality factor is the measurement of sharpness of resonance. In other forms;

$$\text{Quality factor } (Q) = 2\pi \frac{\text{Energy stored}}{\text{Energy lossed per cycle}}$$

$$= 2\pi \frac{\text{Energy stored}}{\text{power loss} \times \text{Time period}}$$

Now;

$$\text{Power loss} = -\frac{dE}{dt}$$

We have, $E = \frac{1}{2} k A_0^2 e^{-b/m t}$

$$\frac{dE}{dt} = \frac{1}{2} k A_0^2 \left(\frac{-b}{m}\right) e^{-b/m t}$$

$$\therefore \frac{dE}{dt} = -\frac{b}{m} E$$

$$\therefore \text{Power loss } (P) = \frac{-dE}{dt} = \frac{b}{m} E$$

$$\therefore Q = \frac{2\pi E}{\frac{b}{m} E \cdot T} = \frac{2\pi}{T}\left(\frac{m}{b}\right) = \frac{\omega m}{b}$$ which is the quality factor of damped oscillation shows that quality factor is inversely proportional to resistive force constant (b)

Forced / Driven oscillation,

When external periodic force is applied to remove or reduce the damping nature of oscillation is called force or driven oscillation.

The general differential equation of forced / driven oscillation is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \sin \omega' t$$

where b is resistive force constant. F_0 is applied force & $\omega' = 2\pi f'$ where f' is the applied frequency. x is the displacement of particle having mass m at any instant of time t.

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin\omega' t \quad \text{——} \ (1)$$

It is inhomogeneous equation such that let $x = A\sin(\omega' t - \phi)$ is the general solution of eqn (1) where ~~is the natural freqn of the body~~ then;

$$\frac{dx}{dt} = A\omega' \cos(\omega' t - \phi)$$

$$\frac{d^2x}{dt^2} = -A\omega'^2 \sin(\omega' t - \phi)$$

$$\Rightarrow -A\omega'^2\sin(\omega' t - \phi) + \frac{b}{m}A\omega'\cos(\omega' t - \phi) + \frac{k}{m}A\sin(\omega' t - \phi) = \frac{F_0}{m}\sin(\omega' t - \phi + \phi).$$

$$\Rightarrow \sin(\omega' t - \phi)\left[A\frac{k}{m} - A\omega'^2\right] + \frac{b}{m}A\omega'\cos(\omega' t - \phi) = \frac{F_0}{m}\left[\sin(\omega' t - \phi)\cos\phi + \cos(\omega' t - \phi)\sin\phi\right]$$

To hold this equation the coefficient of $\cos(\omega' t - \phi)$ & $\sin(\omega' t - \phi)$ should be equal

$$\Rightarrow A\frac{k}{m} - A\omega'^2 = \frac{F_0}{m}\cos\phi \quad \text{——} \ (ii)$$

$$\frac{b}{m}A\omega' = \frac{F_0}{m}\sin\phi \quad \text{——} \ (iii).$$

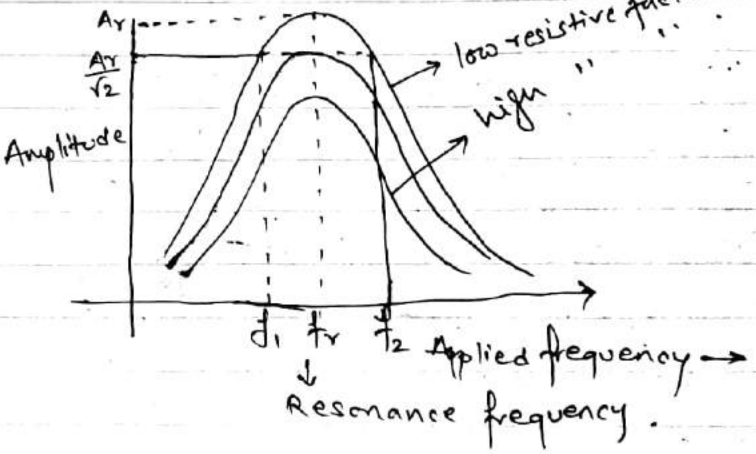
Squaring & adding (ii) & (iii) we get:

$$\frac{F_0^2}{m^2}(\cos^2\phi + \sin^2\phi) = \left[\frac{b^2}{m^2}\omega'^2 + \left(\frac{k}{m} - \omega'^2\right)^2\right] A^2$$

$$\Rightarrow A^2 = \frac{F_0^2}{m^2\left[\frac{b^2}{m^2}\omega'^2 + \left(\frac{k}{m} - \omega'^2\right)^2\right]}$$

$$\Rightarrow A = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega'^2)^2 + \frac{b^2}{m^2}\omega'^2}} \quad \text{where } \omega_0^2 = k/m$$

which is the amplitude of forced oscillation. When $\omega_0 = \omega'$ then the amplitude is maximum which is the condition of resonance.



f_r is resonance frequency.
f_1 is lower cut off frequency.
f_2 is higher cut off frequency.

Now,
The Quality factor of forced oscillation is:

$$Q = \frac{2\pi f_r}{\underbrace{f_2 - f_1}_{\text{band width}}}$$

Electromagnetic Oscillation.

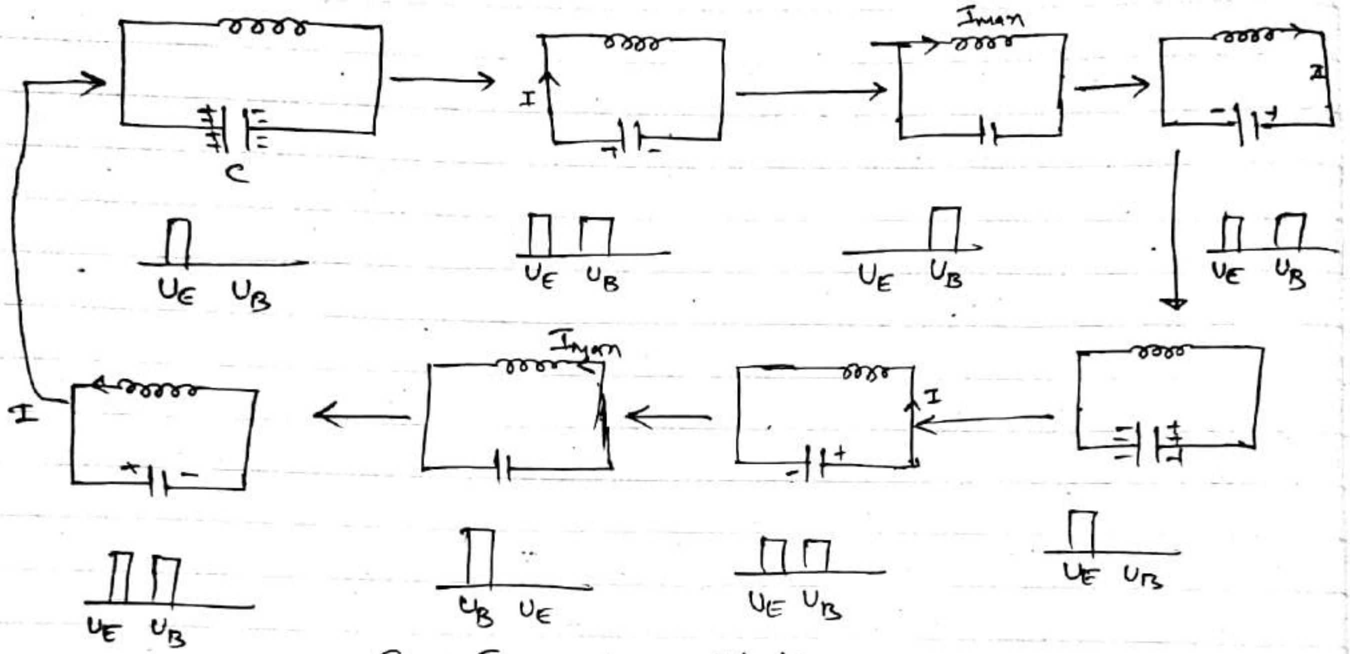


Fig: Free LC oscillation.

→ Capacitor:- It is a device which is used to stored the energy in the form of electrical energy. The energy stored in the capacitor is $U_E = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$ $[\because Q = CV]$

where Q is the charge of capacitor at any instant of time t.

The capacity to stored the energy of the capacitor is called capacitance of the capacitor. Unit of capacitance is farad (F)

→ Inductor:-

It is a device which is used to stored the energy in the form of magnetic energy and it is also used to control the current. The energy stored in the inductor having inductance L is

$$U_B = \frac{1}{2} LI^2$$

. Unit of inductor henry (H).

→ Free LC oscillation.

Let us take a capacitor having capacitance 'c' is fully charged initially then it is connected in series with inductor having inductance (L) then the capacitor starts to discharge and the oscillation occurs as shown in figure.

It is assume that there is no any resisting or energy loss factor. Now, the energy stored in the capacitor at any instant of time t is

$$U_E = \frac{Q^2}{2C}$$

and Energy stored in the inductor is $U_B = \frac{1}{2} LI^2$.

Since there is no any energy loss factor, total energy remain constant.

$\therefore$ $U_E + U_B = \text{constant}$.

$$\frac{Q^2}{2C} + \frac{1}{2} LI^2 = \text{constant} \quad \text{—— (1)}.$$

Diff. (1) w.r.t time,

$$\frac{1}{2C} \cdot 2Q \cdot \frac{dQ}{dt} + \frac{1}{2} L \cdot 2I \cdot \frac{dI}{dt} = 0$$

$$\Rightarrow \frac{Q}{C} \frac{dQ}{dt} + LI \frac{dI}{dt} = 0.$$

$$\Rightarrow \frac{Q}{C}\frac{dQ}{dt} + L\cdot\frac{dQ}{dt}\cdot\frac{d^2Q}{dt^2} = 0 \quad \left[\because I = \frac{dQ}{dt}\right]$$

$$\Rightarrow \frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0 \text{ ——— (ii).}$$

Let the general soln of eqn (ii) be

$Q = Q_0\cos(\omega t + \phi)$

$$\frac{dQ}{dt} = -\omega Q_0 \sin(\omega t + \phi)$$

$$\therefore \frac{d^2Q}{dt^2} = -\omega^2 Q_0 \cos(\omega t + \phi).$$

Then eqn (ii) becomes,

$$-\omega^2 Q_0\cos(\omega t + \phi) + \frac{Q_0\cos(\omega t + \phi)}{LC} = 0$$

or, $-\omega^2 + \frac{1}{LC} = 0$

or, $\omega^2 = \frac{1}{LC}$

or, $f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$

$\therefore f = \frac{1}{2\pi}\cdot\frac{1}{\sqrt{LC}}$

Now; $I_{max} = \left(\frac{dQ}{dt}\right)_{max} = \omega Q_0 = \frac{Q_0}{\sqrt{LC}} = \frac{CV}{\sqrt{LC}}$ ——— (*)

Then energy stored in the capacitor.

$$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{Q_0^2\cos^2(\omega t + \phi)}{2C}$$

$$\Rightarrow U_{E,max} = \frac{Q_0^2}{2C} \text{ ——— (iii)}$$

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$$\langle U_E \rangle = \frac{Q_0^2}{4C} \quad \left[\because \langle \cos^2(\omega t+\phi) \rangle = \tfrac{1}{2}\right]$$

Energy stored in the inductor.

$$U_B = \frac{1}{2} L I^2$$

$$= \frac{1}{2} L \left(\frac{dQ}{dt}\right)^2$$

$$= \frac{1}{2} L \cdot \left[-\omega Q_0 \sin(\omega t+\phi)\right]^2$$

$$= \frac{1}{2} L \omega^2 Q_0^2 \sin^2(\omega t+\phi)$$

$$= \frac{1}{2} L \cdot \frac{1}{LC} Q_0^2 \sin^2(\omega t+\phi) \quad \left(\because \omega^2 = \tfrac{1}{LC}\right).$$

$$= \frac{Q_0^2}{2C} \sin^2(\omega t+\phi)$$

$$\therefore U_{B,max} = \frac{Q_0^2}{2C} \quad \text{——— (iv)}$$

$$\Rightarrow \langle U_B \rangle = \frac{Q_0^2}{4C} \quad \left[\because \langle \sin^2(\omega t+\phi) \rangle = \tfrac{1}{2}\right]$$

Now, Total energy,

$$U_T = U_E + U_B$$

$$= \frac{Q_0^2}{2C} \cos^2(\omega t+\phi) + \frac{Q_0^2}{2C} \sin^2(\omega t+\phi)$$

$$\therefore U_T = \frac{Q_0^2}{2C} \quad \text{——— (v)}$$

It shows that from above equation the maximum energy stored in the capacitor is i.e maximum electric energy stored is equal to maximum energy stored in the inductor i.e Magnetic energy stored which is equal to the ~~total~~ energy

The average electric energy stored in capacitor is equal to average energy stored in the Inductor which is equal to $\frac{1}{2}$ of the total energy

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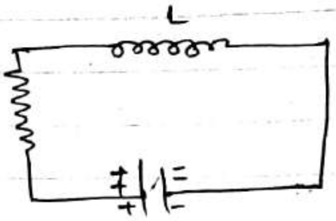
Damped LCR oscillation:-



Let us take a capacitor having capacitance C is fully charged initially connected in series with resistor having resistance R and inductor having inductance L as shown in figure. Then the capacitor starts to discharge and oscillation occurs but due to the presence of resistance the energy is lossed with increase in time such that it is called damped oscillation.

Now, the voltage dropped across R is IR

" " " " L is $L\frac{dI}{dt}$

and At capacitor $V = Q/C$.

$\therefore L\frac{dI}{dt} + IR + \frac{Q}{C} = 0$ at any instant of time t;

$$\Rightarrow L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$\Rightarrow \frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0$$

Let $Q = Q_0 e^{\lambda t}$ be the general soln of eqn ①, where λ is arbitray constant which is to be determine

$$\Rightarrow Q_0\lambda^2 e^{\lambda t} + \frac{R}{L}Q_0\lambda e^{\lambda t} + \frac{Q_0 e^{\lambda t}}{LC} = 0$$

$$\Rightarrow \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

or, $LC\lambda^2 + RC\lambda + 1 = 0$

$$\therefore \lambda = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

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2072 batch

Then $Q = Q_0 e^{\lambda t}$

$$= \cancel{Q_0 e^{-R/2L} \cdot e}$$

$$= Q_0 e^{-R/2L\, t} \cdot e^{\pm \left(i\sqrt{1/LC - R^2/4L^2}\right)t}$$

$$= Q_0 e^{-\frac{R}{2L}t} \cos\omega' t \quad \text{——} \ (1)$$

where $\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

$$\Rightarrow f' = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$\Rightarrow T' = \frac{1}{f'}$$

Eq^n (1) is the general solution of damped LCR oscillation.

Case I:

when $1/LC > R^2/4L^2$, the sol^n is oscillatory.

Case II: $\frac{1}{LC} = \frac{R^2}{4L^2}$ the sol^n is oscillatory but suddenly die-out.

Case III:

When $1/LC < R^2/4L^2$, the sol^n is non-oscillatory.

Q1. A ckt has $L = 1.2\,mH$, $C = 1.6\,\mu F$, $R = 1.5\,\Omega$

a) After what time t will the amplitude of the charge oscillation dropped through one half of its initial value and also how many periods of oscillation does this corresponds.

Q2. A $2\,\mu F$ capacitor is charged upto 100 Volt then the battery is disconnected and the capacitor is connected to the 60 mH with across LC oscillation. Calculate the maximum current, total energy and average magnetic energy stored in the inductor.

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$x = A\sin(\omega' t - \phi)$

(d) solution

Given $L = 1.2\,mH = 1.2\times10^{-3}\,H$

$C = 1.6\,\mu F = 1.6\times10^{-6}\,F$

$R = 1.5\,\Omega$

Now; We have:

$$Q = Q_0 e^{-\frac{Rt}{2L}}$$

$$Q = Q_0 e^{-\frac{Rt}{2L}}$$

By question

$$\frac{Q_0}{2} = Q_0 e^{-Rt/2L}$$

or, $\frac{1}{2} = e^{-\frac{Rt}{2L}}$

or, $\ln(1/2) = -\frac{Rt}{2L}\ln e$

or, $-0.693 = -\frac{1.5\times t}{2\times1.2\times10^{-3}} \times 1$

$\therefore\ t = 1.1088\times10^{-3}$ sec.

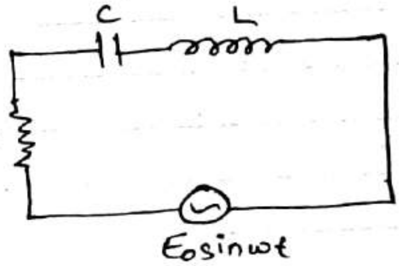
Now;

We have; $f' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \cdot \frac{1}{2\pi}$

$$= \sqrt{\frac{1}{1.2\times10^{-3}\times1.6\times10^{-6}} - \frac{(1.5)^2}{4\times(1.2\times10^{-3})^2}} \times \frac{1}{2\pi}$$

Forced / driven LCR oscillation.

Let us take a resistance 'R' inductance 'L' and capacitor having Capacitance 'c' is connected in series with AC source having frequency 'ω'.



Then;

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = E_0 \sin \omega t \quad (\text{where } \omega = 2\pi f,\ f \text{ is applied frequency})$$

$$\Rightarrow \frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = \frac{E_0}{L}\sin \omega t \quad \text{—— (1)}$$

Also; $\frac{dQ}{dt} = I_0 \cos(\omega t - \phi)$ where $I_0 = Q_0 \omega$.

Let $Q = Q_0 \sin(\omega t - \phi)$ be the general solution of eqn (1)

Then;

$$\Rightarrow -Q_0\omega^2 \sin(\omega t - \phi) + \frac{R}{L} Q_0 \omega \cos(\omega t - \phi) + \frac{Q_0 \sin(\omega t - \phi)}{LC} = \frac{E_0}{L}\sin \omega t$$

$$\Rightarrow -Q_0\omega^2 \sin(\omega t - \phi) + \frac{R}{L} Q_0 \omega \cos(\omega t - \phi) + \frac{Q_0}{LC}\sin(\omega t - \phi) = \frac{E_0}{L}\sin(\omega t - \phi + \phi)$$

$$\Rightarrow -Q_0\omega^2 \sin(\omega t - \phi) + \frac{R}{L} Q_0 \omega \cos(\omega t - \phi) + \frac{Q_0}{LC}\sin(\omega t - \phi) = \frac{E_0}{L}\left[\sin(\omega t - \phi)\cos\phi + \cos(\omega t - \phi)\sin\phi\right]$$

To hold this equation coefficient of $\cos(\omega t - \phi)$ and $\sin(\omega t - \phi)$ must be equal

$$-Q_0\omega^2 + \frac{Q_0}{LC} = \frac{E_0 \cos\phi}{L} \quad \text{—— (i)}$$

$$\frac{Q_0 R \omega}{L} = \frac{E_0 \sin\phi}{L} \quad \text{—— (ii)}$$

Squaring eqn (i) & (ii) & adding (i) & (ii) we get:

$$\left(\frac{Q_0}{LC} - Q_0\omega^2\right)^2 + \left(\frac{Q_0 R\omega}{L}\right)^2 = \frac{E_0^2}{L^2}$$

$$Q_0^2\left(\left(\frac{1}{LC}-\omega^2\right)^2+\frac{\omega^2R^2}{L^2}\right)=\frac{E_0^2}{L^2}$$

$$\left(\frac{1}{\omega C}-\frac{L}{\omega}\right)^2$$

$$\Rightarrow Q_0=\frac{E_0/L}{\sqrt{\frac{R^2\omega^2}{L^2}+\left(\frac{1}{LC}-\omega^2\right)^2}}$$

Now current amplitude;

$$I_0=Q_0\omega$$

$$I_0=\frac{\frac{E_0}{L}\cdot\omega}{\sqrt{\frac{R^2\omega^2}{L^2}+\left(\frac{1}{LC}-\omega^2\right)^2}}=\frac{E_0\cdot\omega/L}{\sqrt{\left\{R^2+\left(\frac{1}{LC}\cdot\frac{L}{\omega}-\omega^2\cdot\frac{L}{\omega}\right)^2\right\}\frac{\omega^2}{L^2}}}$$

$$\Rightarrow I_0=\frac{E_0}{\sqrt{R^2+\left(1/\omega C-\omega L\right)^2}}$$

$$\therefore I_0=\frac{E_0}{\sqrt{R^2+(X_L-X_C)^2}}$$

where $X_L=\omega L=2\pi fL$ is inductive reactance
$X_C=1/\omega C=1/2\pi fC$ is capacitive reactance

$\therefore Z=\sqrt{R^2+(X_L-X_C)^2}$ is impedence of LCR circuit.

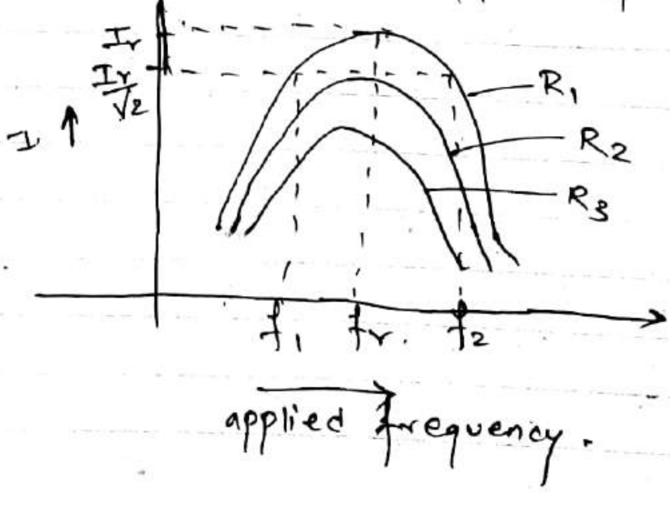
When $X_L=X_C$ then the amplitude ie ~~is~~ current is maximum which is called the resonance current. In resonance the natural frequency of body coincide with the applied frequency

At resonance

$$X_L=X_C$$

$$\Rightarrow \omega L=\frac{1}{\omega C}$$

$$\therefore f_{res}=\frac{1}{2\pi\sqrt{LC}}=f_r.$$



Quality factor.

$$Q=\frac{2\pi f_r}{f_2-f_1}$$

frequency of source = 50 Hz.

1. If $E_0 = 10V$, $L = 30 mH$, $C = 1000 \mu F$, $R = 100 \Omega$. Calculate the voltage drop across inductance, resonance current and quality factor

solution;

Given $E_0 = 10 V$

$L = 30 mH = 30 \times 10^{-3} H$

$C = 1000 \mu F = 1000 \times 10^{-6} F$

$R = 100 \Omega$

$I_0 = \frac{E_0}{\sqrt{R^2 + (X_L - X_C)^2}}$

$I_0 X_L$

$I_r = \frac{E_0}{R}$

To find Voltage across inductance $(V_L) = ?$

Resonance current $(I_0) = ?$

Quality factor $(Q) = ?$

Now;

$$I_0 = \frac{E_0}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{10}{\sqrt{100^2 + \left(2\pi \times 50 \times 30 \times 10^{-3} - \frac{1}{2\pi \times 50 \times 10^{-3}}\right)^2}}$$

$\therefore I_0 = 0.0998 A$.

Now;

Voltage across inductance $(V_L) = I_0 X_L$

$= 0.0998 \times 2\pi \times 50 \times 30 \times 10^{-3}$

$= 0.94 V$.

Resonance current $(I_r) = \frac{E_0}{R} = \frac{10}{100} = 0.1 A$.

Quality factor $(Q) = \frac{2\pi f_r}{f_2 - f_1}$

Resonance frequency $(f_r) = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{30 \times 10^{-3} \times 10^{-3}}}$

$\therefore f_r = 29.05 Hz$.

$(f_1 - f_2)$ can't be determine directly.

② The amplitude of damped oscillation decreases by 3% in one cycle. What factor of the energy of oscillation is lost in each oscillation?

solution;

we have; $A = A_0 e^{-b/2m\,t}$.

Since A is decreased by 3%.

Also; $E = \frac{1}{2}kA^2$

$\therefore E \propto A^2$

$(e^x)^2$

e^{2x}

6%.

$A = 6\%\cdot A$

$E = \frac{1}{2}kA^2$

$A = A - 3\%\text{ of }A$

$= \frac{97A}{100}$

loss of ene = $\boxed{0.06\%}$

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③ A simple pendulum of length 50 cm and mass 70gm is suspended in car that is travelling with constant speed 60 m/s around a circle of radius 120m. If the pendulum undergoes small oscillation in a radial direction about its equilibrium position. Calculate the time period & frequency of oscillation.

Solution:

$$g_{eff} = \sqrt{\left(\frac{v^2}{r}\right)^2 + g^2}$$

$$= \sqrt{\left(\frac{60^2}{120}\right)^2 + 9.81^2}$$

$$= 31.56.$$

$$T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 0.79 sec.$$

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Acoustic:

Reverberation:-

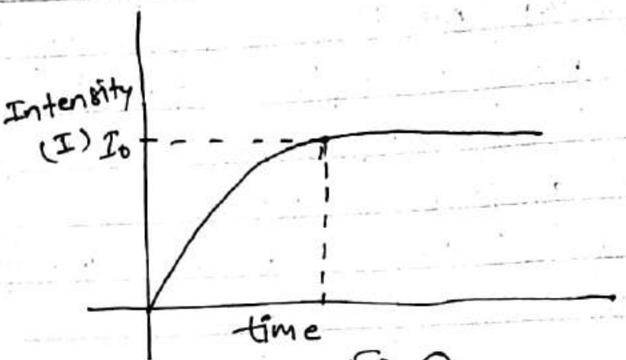

Fig ①

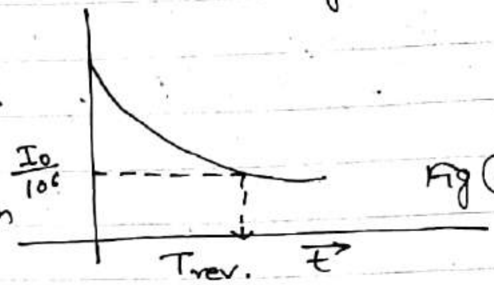

Fig (ii).

Due to the presence of reflecting wall, ceiling, floor & closed window and door in a hall the sound waves are reflected, such that there may be interference of sound waves which produced unclear in sound and even if the source of sound is stopped, the listener's also listen the sound which is due to reflection from the different part is called reverberation.

The time in which the listener's listen the sound after the source of sound is stopped is called reverberation time, in which Intensity falls to $I_0/10^6$ where I_0 is the intensity when the source is just stopped as shown in fig (ii).

Absorption Co-efficient:-

Let $S_1, S_2 \ldots S_n$ be the surface area of absorber having absorption co-efficient $\alpha_1, \alpha_2 \ldots \alpha_n$ then the average absorption coefficient is

$$\alpha_{av} = \frac{\sum_{i=1}^{n} \alpha_i S_i}{\sum_{i=1} S_i}$$

Sabine's Relation to find the Reverberation time of a hall.

It is an imperical relation to find the reverberation time of a hall by using its parameter in terms of volume, surface area and absorption force.

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Let us take a hall having volume(V), Total surface area(S) and mean absorption coefficient of a hall is (α). Let I be the intensity of sound in the hall, δI be the decrease in intensity after the source is stopped at time δt then

$$\delta I = -\alpha n I \delta t$$

where n be the no. of reflections per second in that hall.

Now Mean free path / The distance travelled by the wave between two succesive reflection is:

$$(\lambda_{mean}) = \frac{4V}{S}$$

and relaxation time / the time between two successive reflection of wave is

$$T = \frac{4V}{S\vartheta}$$ where ϑ be the velocity of sound wave in that hall

$$\therefore n = \frac{1}{\tau} = \frac{S\vartheta}{4V}$$

Putting this value in ①

$$\delta I = -\frac{S\vartheta}{4V} \alpha I \delta t$$

$$\Rightarrow \frac{\delta I}{I} = \frac{-S\vartheta}{4V} \alpha \delta t$$

$$\int_{I_0}^{I_t} \frac{\delta I}{I} = \frac{-S\vartheta}{4V} \alpha \int_0^{T_{rev}} \delta t$$

or, $$\ln\left(\frac{I_t}{I_0}\right) = \frac{-S v \alpha}{4V} T_{rev}$$

Also; $I_t = I_0/10^6$ $\vartheta = 350\,m/s$.

$$\therefore T_{rev} = \frac{-4V}{S\vartheta\alpha} \ln\left(\frac{I_0}{10^6} \times \frac{1}{I_0}\right)$$

$$T_{rev} = \frac{-4V}{S \times 350 \cdot \alpha} \ln\left(\frac{1}{10^6}\right)$$ $\therefore T_{rev} = \frac{1.57 \times 10^{-1} V}{S\alpha} = \frac{0.158 V}{\alpha S}$

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In FPS system

$$T_{rev} = \frac{0.05V}{\alpha S}.$$

This is a reverbaration time of a hall with volume (V), surface area (S) and absorption coefficient α.

Now,

Intensity level.

$$I_L = 10\log_{10}\left(\frac{I}{I_0}\right)$$ where I_0 is reference intensity (standard intensity).

Q. Calculate the reverberation time in a hall measuring 40x10x20 ft. with the following parameter, 7500 sq.ft of plaster, $\alpha_1 = 0.03$, 400 sq. ft of glass $\alpha_2 = 0.025$, 6000 sq.ft of wood and floor. $\alpha_3 = 0.06$, 600 seat $\alpha_4 = 0.03$, audience of 500 person $\alpha_5 = 4.0$

Solution;

$$\alpha_{av} = \frac{\alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3}{S_1 + S_2 + S_3} = \frac{0.03\times7500 + 400\times0.025 + 6000\times0.06}{7500 + 400 + 6000}$$

$$= 0.0928$$

$$\alpha = 0.0928 + \underset{\times 600}{0.03} + \underset{\times 500}{4} = 2018.04$$

$$T = \frac{0.05V}{\alpha S} = \frac{0.05\times40\times10\times20}{2018.04\times13900}$$

$$= 1.4\times10^{-5}\text{ sec}.$$

(2)

	Surface	Area	absorption coefficient
a	Plastered wall	98	0.03
b	Plastered ceiling	144	0.04
c	Wooden door.	15	0.06
d	Cushioned chairs	88	1.0
e	Audience	150	4.70

Volume of a hall = 1500m3

Calculate Trev.

solution:

$$\alpha_{av} = \frac{\alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3}{S_1 + S_2 \quad S_3}$$

$= 98 \times 0.03 + 144 \times 0.04 +$

③ If a hall has dimension 30x20x50m. Calculate mean free path and relaxation time.

Solution;

Given; Volume of a hall (v) = 30x20x50
= 30000m3

Total Area of a hall (S) = ~~30x20 = 600m3~~
= 2(30x20 + 20x50 + 30x50)
= 6200m²

Now;

Mean free path (λ_{mean}) = $\frac{4V}{S} = \frac{4\times30000}{6200}$ = 19.35m

Relaxation time (τ) = $\frac{4V}{S\vartheta} = \frac{4\times30000}{6200\times350}$ = 0.066sec.

④ If an empty hall has reverberation time 1.50s. If 500 audiance are present in the hall the reverberation time falls to 1.46sec. Calculat the no. of person present in the hall if the reverberation time of that hall falls 1.40sec.

Solution;

Reverberation time for empty hall (T'_{rev}) = 1.50s

Reverberation time if 500 person are present (T''_{rev}) = 1.46sec.

No. of person (n = ?) if Trev falls to 1.40sec.

$$1.50 = \frac{0.158V}{\alpha S}$$

$$1.46 = \frac{0.158V}{(\alpha + 500\alpha_1)S}$$

$$1.40 = \frac{0.158V}{(\alpha + n\alpha_1)S}.$$

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$$\frac{1.46}{1.50} = \frac{\alpha}{\alpha + 500\alpha_1}$$

$$1.027\alpha = \alpha + 500\alpha_1$$

or, $2.7 \times 10^{-2}\alpha = 500\alpha_1$

or, $5.479 \times 10^{-5}\alpha = \alpha_1$ ——— (i)

$$\frac{1.40}{1.50} = \frac{\alpha}{\alpha + n\alpha_1}$$

$$1.071\alpha = \alpha + n\alpha_1$$

or, $1.071\alpha - \alpha = n\alpha_1$

or, $7.14 \times 10^{-2}\alpha = n\alpha_1$

or, $\dfrac{7.14 \times 10^{-2}\alpha}{n} = \alpha_1$ ——— (ii)

Then from (i) & (ii)

$$\frac{5.479 \times 10^{-5} \cancel{\alpha} \times n}{7.14 \times 10^{-2} \cancel{\alpha}} = 1$$

or, $n = 1303$.

Optical Fiber

It is a cylindrical device consists of core having refractive index (n_1) and cladding having refractive index (n_2) where $n_1 > n_2$. It works in the principal of total internal reflection. The glass material is used to construct the optical fiber. The total internal reflection occurs when angle of incidence is equal to the critical angle and angle of refraction is equal to $90°$.

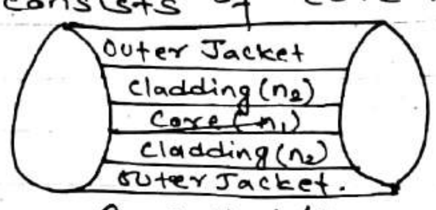


fig: Optical fiber.

From Snell's law

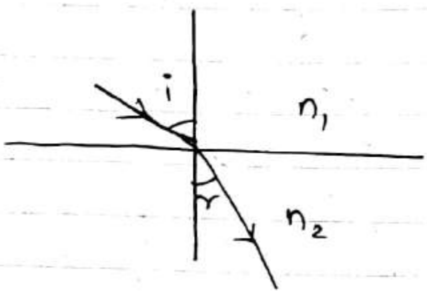
$$n_1 \sin i = n_2 \sin r$$

For total internal reflection

$$i = c \ \& \ r = 90°$$

$$\therefore n_1 \sin c = n_2 \sin 90$$

$$\therefore \sin c = \frac{n_2}{n_1}$$



The internal reflection occurs when the wave tranvels from denser medium to rarer medium.

Let us take an optical fiber in which the refractive index of core is n_1 and cladding is n_2 where $n_1 > n_2$. When input signal is incident on the core with angle of incidence i and angle of refraction on the core is θ as shown in figure. The signal is totally reflected from the cladding surface as shown in figure.

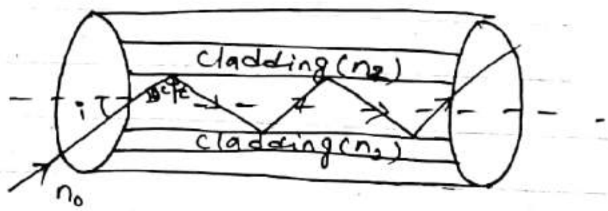


fig: Light propogation in optical fibre.

Then from snell's law.

$n_0 \sin i = n_1 \sin\theta$ [where i is called acceptance angle]

or, $\frac{n_0}{n_1} \sin i = \sin(90-c)$ [$\because \theta = 90-c$]

$\Rightarrow \frac{n_0}{n_1} \sin i = \cos c$ ——— (i)

And $n_1 \sin c = n_2 \sin 90$

$\Rightarrow \sin c = \frac{n_2}{n_1}$ ——— (ii) [$\because$ where c is the critical angle between core & cladding surface]

Squaring eqn (i) & (ii) & add.

$\sin^2 c + \cos^2 c = \frac{n_2^2}{n_1^2} + \frac{n_0^2 \sin^2 i}{n_1^2}$

$\Rightarrow \frac{n_0^2 \sin^2 i}{n_1^2} = 1 - \frac{n_2^2}{n_1^2}$

$\Rightarrow \sin i = \sqrt{\frac{n_1^2 - n_2^2}{n_0^2}}$

For air $n_0 = 1$

$\therefore \sin i = \sqrt{n_1^2 - n_2^2}$

Now, Numerical aperture is the power of gathering of wave.

$\therefore NA = \sqrt{n_1^2 - n_2^2}$ ——— (iii).

Now the fractional refractive inden change:-

$\Delta = \frac{n_1^2 - n_2^2}{2n_1^2}$ ——— (iv). $= \frac{n_1 - n_2}{n_1}$ [if $n_1 \simeq n_2$].

$\Rightarrow NA = n\sqrt{2\Delta}$ ——— (v).

Normalized frequency or v-number

$V = \frac{2\pi}{\lambda} r(NA)$ where r radius of optical fiber, λ is wavelength of signal.

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① The refractive index of core and cladding is 1.48 & 1.46 respectively. Calculate the acceptance angle, critical angle, Numerical aperature, fraction refractive index change. If the wave length of signal is 600nm and diameter of optical fibre is 98µm. Calculate the V-number.

Solution:

Given, Refractive index of core $(n_1) = 1.48$

" " " cladding $(n_2) = 1.46$.

a) $\sin i = \sqrt{n_1^2 - n_2^2}$

$\sin i = \sqrt{1.48^2 - 1.46^2}$

$\therefore i = 14.03°$.

b) Critical angle = ?

$\sin c = \dfrac{n_2}{n_1}$

$\therefore c = \sin^{-1}\left(\dfrac{1.46}{1.48}\right) = 80.56$

c) Numerical Aperature $(NA) = \sqrt{n_1^2 - n_2^2} = 0.242$

d) fraction refractive index $(\Delta) = \dfrac{n_1^2 - n_2^2}{2n_1^2} = \dfrac{(0.242)^2}{2 \times 1.48^2} = 0.0134$

e) V-number = ?

$\lambda = 600nm = 600 \times 10^{-9} m$

$d = 98 \mu m \Rightarrow r = 49 \mu m = 49 \times 10^{-6} m$

$\therefore V = \dfrac{2\pi}{\lambda} \cdot r (NA)$.

② An optical fiber has a numerical aperature of 0.22. The core has refractive inden 1.60. Calculate the acceptance angle in water having refractive inden 1.33. Also calculate Critical angle, fractional refractive inden.

Solution;

Given

Numerical aperture (NA) = 0.22.

Refractive inden of core (n_1) = 1.60

Refractive inden of water (n) = 1.33.

Now;

$$NA = \sqrt{n_1^2 - n_2^2}$$

or, $(0.22)^2 = (1.60)^2 - n_2^2$

$\therefore n_2 = 1.589$

Now;

$$\sin c = \frac{n_2}{n_1}$$

$$\therefore c = \sin^{-1}\left(\frac{1.58}{1.60}\right) = 80.93.$$

Now; $\frac{n}{n_1} \sin i = \cos c$

or, $\frac{1.33}{1.60} \sin(i) = \cos 80.93$

$\therefore i = 10.93$.

$$\Delta = \frac{n_1^2 - n_2^2}{2n_1^2} = \frac{1.60^2 - (1.584)^2}{2 \times 1.60^2}$$

$= 9.95 \times 10^{-3}$.

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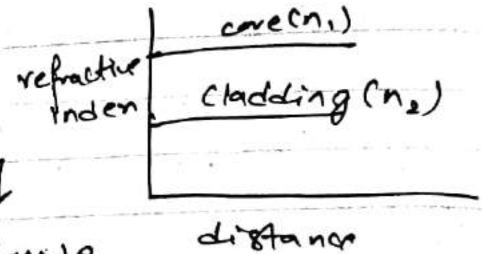
Types of optical fiber.

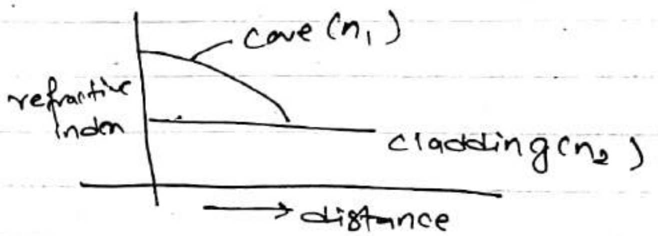
(1) Step index optical fiber.



Monomode step index	Multimode step index.
- Laser diode is used.	- Light emitting diode is used.

The refractive index of core and cladding is fixed and $n_1 > n_2$.

(2) Graded index optical fiber.



Power loss in optical fiber

$$P_{loss} = \frac{1}{L} 10 \log_{10} \frac{P_{in}}{P_{out}}$$

where L is in km.

Wave Motion :-

The disturbance in medium produce wave. Generally, there are two types of wave.

a) Mechanical wave.

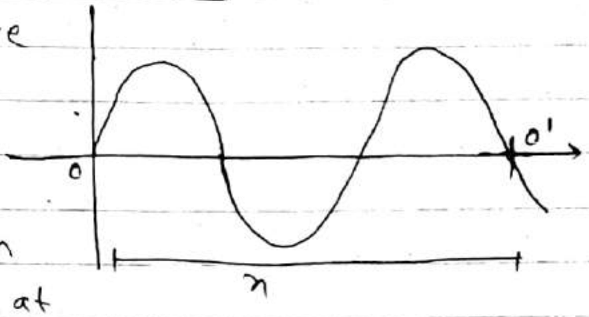
b) Non-Mechanical wave or electromagnetic wave.

The wave which carry the energy and amplitude remains constant is called progressive wave.

Let us consider a progressive wave, let a particle is in motion at O, then the displacement of particle is

$y = A\sin\omega t$.

and the another particle is in motion at O' then the displacement of particle at O' is



$y = A\sin(\omega t - \phi)$

where $\phi = \frac{2\pi x}{\lambda}$

Then; $y = A\sin\left(\omega t - \frac{2\pi x}{\lambda}\right)$

$= A\sin\left(2\pi f t - \frac{2\pi x}{\lambda}\right)$

$= A\sin\frac{2\pi}{\lambda}(f\lambda t - x)$

$= A\sin\frac{2\pi}{\lambda}(vt - x)$ —— (1)

where v is the velocity of wave.

Eqn (1) is the progressive wave equation moving along positive x-direction with wave velocity v.

Similarly the progressive wave equation moving along $-ve$ x axis with velocity v is

$y = A\sin\frac{2\pi}{\lambda}(vt + x)$ —— (ii)

General differential equation of wave:-

We have,

$$y = A\sin\frac{2\pi}{\lambda}(vt - x)$$

$$\frac{dy}{dt} = \frac{2\pi v}{\lambda} A\cos\frac{2\pi}{\lambda}(vt - x)$$

$$\frac{d^2y}{dt^2} = \left(\frac{2\pi v}{\lambda}\right)^2 (-) A\sin\frac{2\pi}{\lambda}(vt - x) \quad \text{—— (i)}$$

and

$$\frac{dy}{dx} = (-)\frac{2\pi}{\lambda} A\cos\frac{2\pi}{\lambda}(vt - x)$$

$$\frac{d^2y}{dx^2} = -\left(\frac{2\pi}{\lambda}\right)^2 A\sin\frac{2\pi}{\lambda}(vt - x) \quad \text{—— (ii)}$$

From eqn (i) & (ii).

$$\boxed{\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2} \quad \text{—— (iii)}}$$

Eqn (iii) is the general differential eqn of wave motion in one direction.

$$\nabla = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$\nabla \rightarrow$ Nebula

$\nabla^2 \Rightarrow$ Laplacian.

In three dimension:

$$v^2 \nabla^2 y = \frac{d^2y}{dt^2}$$

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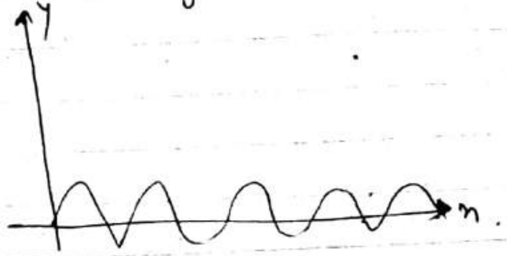
* Show that the velocity of particle is equal to the velocity of wave times the -ve slope of the curve in the string

Ans:-

We have;

$$y = A\sin(\omega t - \phi)$$

Also; $y = A \sin \frac{2\pi}{\lambda}(vt - x)$



where v is the wave velocity.

Now; $v_p = \frac{dy}{dt} = \frac{2\pi}{\lambda} \cdot v\, A \cos\frac{2\pi}{\lambda}(vt-x)$ —— (i)

Also; $\frac{dy}{dx} = (-)\frac{2\pi}{\lambda} A \cos \frac{2\pi}{\lambda}(vt-x)$

or, $-\frac{dy}{dx} = \frac{2\pi}{\lambda} A \cos\frac{2\pi}{\lambda}(vt-x)$. —— (ii)

$$\boxed{v_p = -v\frac{dy}{dx}}$$

$\therefore$ $v_p = A\frac{2\pi}{\lambda} v \cos\frac{2\pi}{\lambda}(vt-x)$

$\frac{d^2y}{dt^2} = a_p = (-)\left(\frac{2\pi}{\lambda}v\right)^2 A \sin\frac{2\pi}{\lambda}(vt-x)$

$\therefore a_p = \frac{d^2y}{dt^2} = -\omega^2 y$.

Hence the particle of a wave vibrate in a SHM.

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$\sqrt{\sqrt{}}X$

Energy, Power and Intensity of progressive wave.

Now, the displacement of particle at any instant of time t.

$$y = A\sin\frac{2\pi}{\lambda}(vt-x)$$

Now, the potential energy

$$U = -\int_0^y F\,dy$$

$$F = m\frac{d^2y}{dt^2} = m\left(\frac{2\pi}{\lambda}v\right)^2(-)A\sin\frac{2\pi}{\lambda}(vt-x)$$

$$\Rightarrow F = -m\left(\frac{2\pi}{\lambda}v\right)^2 y$$

∴ Potential energy per unit volume.

$$U_{volume} = \frac{U}{V} = -\int_0^y \frac{F}{V}\,dy$$

$$= \left(\frac{2\pi v}{\lambda}\right)^2 \rho \int_0^y y\cdot dy$$

$$= \left(\frac{2\pi}{\lambda}v\right)^2 \rho\,\frac{y^2}{2}$$

$$U_{volume} = \left(\frac{2\pi}{\lambda}v\right)^2 \frac{\rho}{2}\cdot A^2\sin^2\frac{2\pi}{\lambda}(vt-x) \quad \text{—— (i)}$$

Now. kinetic energy per unit volume.

$$KE_{volume} = \frac{K.E}{V} = \frac{1}{2}\frac{mv^2}{V}$$

$$= \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2$$

$$= \frac{1}{2}\rho\left(\frac{2\pi}{\lambda}v\,A\cos\frac{2\pi}{\lambda}(vt-x)\right)^2$$

$$= \frac{1}{2}\rho\left(\frac{2\pi}{\lambda}v\right)^2 A^2\cos^2\frac{2\pi}{\lambda}(vt-x) \quad \text{—— (ii)}$$

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Total energy per unit volume is

$$(T.E)_{volume} = (K.E)_{volume} + (P.E)_{volume}$$

$$= \frac{\rho}{2}\left(\frac{2\pi}{\lambda}v\right)^2 A^2 \cos^2 \frac{2\pi}{\lambda}(vt-x) + \frac{\rho}{2}\left(\frac{2\pi}{\lambda}v\right)^2 A^2 \sin^2 2\pi/\lambda\,(vt-x)$$

$$= \frac{\rho}{2}\left(\frac{2\pi}{\lambda}v\right)^2 A^2.$$

$$= 2\pi^2 \rho A^2 f^2.$$

Now, the total energy

$$E = 2\pi^2 \rho A^2 f^2 \cdot V \quad \text{where } V \text{ is volume.}$$

$$= 2\pi^2 \rho A^2 f^2 \cdot a \times l \quad \text{where } a \text{ is area.}$$

$$= 2\pi^2 \rho A^2 f^2 \cdot a \times v \times t$$

$$\text{Power} = \frac{E}{t} = P = 2\pi^2 \rho A^2 f^2 a v$$

And intensity $I = P/a = 2\pi^2 \rho A^2 f^2 v$.

Intensity level $(\alpha) = 10 \log_{10}(I/I_0)$

Q. A source of sound has 500Hz frequency and amplitude 0.45cm. Calculate the energy flow across a square of unit area per sec. The velocity of sound in air is 332m/s. and density of air is $1.29\,kg/m^3$.

Solution.

Given; $f = 500Hz$

$A = 0.45cm = 0.45\times10^{-2}m.$

$V = 332m/s$

$\rho = 1.29kg/m^3.$

$$E = 2\pi^2 \rho A^2 f^2 \cdot a \times v \times t$$

$$\frac{E}{at} = 2\pi^2 \times 1.29 \times (0.45)^2 \times 10^{-4} \times (500)^2 \times 332$$

$$\therefore I = 4.28\times10^{4}\,J/m^2s.$$

Q. How much acoustic power enters the window of area $1.58\,m^2$ where the standard intensity level is $10^{-16}\,W cm^{-2}$. The window opens on a street where the street noise result in an intensity level at the window of 60 dB

Solution.

Area $(A) = 1.58\,m^2$

Intensity level $(I_0) = 10^{-16}\,W/cm^2 = \dfrac{10^{-16}}{10^{-4}}\,W/m^2 = 10^{-12}\,W/m^2$.

We have $\alpha = \log_{10}\left(\dfrac{I}{I_0}\right)$

or, $60 = 10\log_{10}\left(\dfrac{I}{10^{-12}}\right)$

or, $I = 10^{-06}\,W/m^2$.

Now;

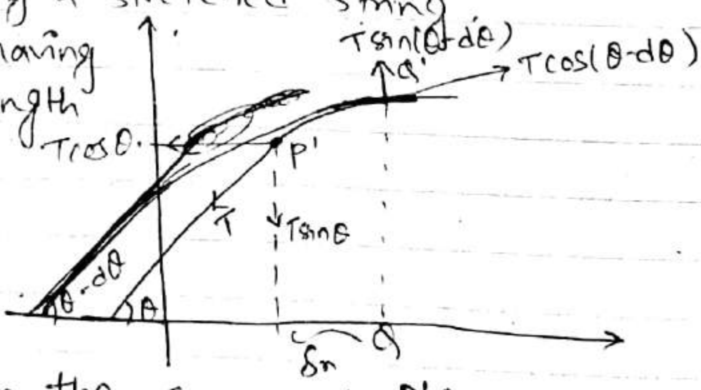
$P = I \times a$

$= 10^{-16} \times 1.58$

$= 1.58 \times 10^{-06}\,W$.

Velocity of transverse wave along a stretched string

Let us take a string having linear mass density 'σ' with length δx. When tension T is applied on the string it starts to vibrate and generate the transverse wave as shown in figure.



Horizontal force acting on the component P'Q' is

$= T\cos(\theta-d\theta) - T\cos\theta$

Since $d\theta$ is very small $\cos(\theta-d\theta) \simeq \cos\theta$.

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∴ Net horizontal force = 0.

And

Net vertical force acting on the component P'Q' is

$T\sin\theta - T\sin(\theta - d\theta)$

for small angle θ

$\sin\theta \simeq \tan\theta$ and $\sin(\theta \cdot d\theta) \simeq \tan(\theta - d\theta)$

∴ Net vertical force acting on the element $= T[\tan\theta - \tan(\theta \cdot d\theta)]$

$\tan\theta = \frac{dy}{dx}$ = slope at P'

Now, slope at Q' $= \frac{dy}{dx} - \frac{d}{dx}\left(\frac{dy}{dx}\right)\delta x$

∴ Net vertical force $= \left\{\frac{dy}{dx} - \left[\frac{dy}{dx} - \frac{d^2y}{dx^2} \cdot \delta x\right]\right\} T.$

$= T\frac{d^2y}{dx^2}\delta x$ —— ①.

Now from Newton's 2nd law.

$F = ma = \sigma\delta x \cdot \frac{d^2y}{dt^2}$ —— ⅱ

Equating ① & ⅱ

$T\frac{d^2y}{dx^2}\delta x = \sigma\delta x\frac{d^2y}{dt^2}$

$\Rightarrow \frac{d^2y}{dt^2} = \frac{T}{\sigma}\frac{d^2y}{dx^2}$

Comparing it with $\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}$ we get

$v = \sqrt{\frac{T}{\sigma}}$

It shows that the velocity of transverse wave in stretched string is depends on the applied tension and linear density of the material

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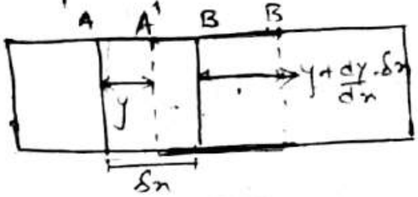
velocity of longitudinal wave in Isotropic rod:-

The medium whose physical properties doesnot changes with direction i.e. same in all direction is called isotropic medium.



Anisotropic medium is the medium whose physical properties changes with direction.

Let us take a uniform rod having cross sectional area 'α' and density is 'ρ'. Let us take two arbitary plane section A and B before passing the wave. When wave is passed in the rod the plane section shifts A to A' and B to B' as shown in fig.

The increase in length after passing the wave is

$$A'B' - AB$$

$$= \cancel{y} + \frac{dy}{dx}.\delta x + \delta x - \cancel{y} - \delta x$$

$$= \delta x . \frac{dy}{dx}.$$

Now the strain $= \dfrac{\frac{dy}{dx}.\delta x}{\delta x} = \dfrac{dy}{dx}.$

Young's Modulum $(Y) = \dfrac{F/\alpha}{dy/dx}$ =

$$\therefore F = \alpha Y \frac{dy}{dx}$$

Similarly, in section B' to B

$$F + \frac{dF}{dx}.\delta x$$

$$\Rightarrow \alpha Y \frac{dy}{dx} + \frac{dF}{dx}.\delta x$$

$$\Rightarrow \alpha Y \frac{dy}{dx} + \alpha Y \frac{d^2y}{dx^2} \cdot \delta x$$

The resultant force is $\alpha Y \frac{d^2y}{dx^2} \delta x$ —— (i).

We have $F = ma$

$$= \alpha \delta x . \rho \frac{d^2y}{dt^2} \quad \text{—— (ii)}.$$

Equating (i) & (ii) we get;

$$\frac{d^2y}{dt^2} = \frac{Y}{\rho} \frac{d^2y}{dx^2}$$

$\Rightarrow v = \sqrt{\frac{Y}{\rho}}$ where Y is young's modulus of elasticity.

For fluid or gases medium.

$v = \sqrt{\frac{K}{\rho}}$ where k is bulk modulus.

Q. The speed of transverse wave on a string in 150m/s when the tension T is 105N. What value must the tension be changed to raised the wave speed of 190m/s.

Solution:

We have;

$$v = \sqrt{T/\sigma}.$$

$$150 = \frac{\sqrt{105}}{\sqrt{\sigma}} \quad \text{—— (1)}$$

For condition 2.

$$190 = \frac{\sqrt{T}}{\sqrt{\sigma}} \quad \text{—— (ii)}.$$

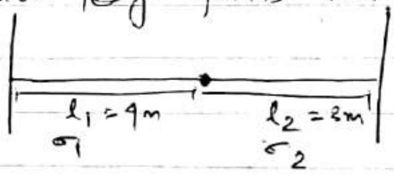
Dividing eqn (i) by (ii).

$$\frac{150}{190} = \sqrt{\frac{105}{T_2}}$$

$$\therefore T_2 = 168.46\,N.$$

$$\Delta T = 168.46 - 105$$
$$= 63.4667\,N.$$

Q. In figure two strings have been tight together with the knot and the ~~stress~~ stretched bet^n two rigid support. The spring has linear density $\sigma_1 = 1.5\times10^{-4}$ kg/m and $\sigma_2 = 2.9\times10^{-4}$ kg/m. $l_1 = 4$m and $l_2 = 8$m and 400 N tension is applied which ~~plug~~ pulse ~~[illegible]~~ reaches the first ?



(a)

Solution;

we have, $v = \sqrt{\frac{T}{\sigma_1}}$

or, $\frac{l_1}{t_1} = \sqrt{\frac{400}{1.5\times10^{-4}}}$

$\therefore t_1 = 4\times\sqrt{\frac{1.5\times10^{-4}}{400}} = 0.00244$sec.

Now, $v_2 = \sqrt{\frac{T}{\sigma_2}}$

or, $t_2 = 8\times\sqrt{\frac{2.9\times10^{-4}}{400}} = 0.00255$sec.

Since the time taken by the spring having length 4m reaches first.

Phase wave/Velocity.

Let us take a particle moving with velocity v then the displacement of particle can be written as

$$y = A\sin\underbrace{(\omega t - kx)}_{\text{phase}}$$

The phase is constant $\omega t - kx = \text{constant}$ —— ①

Diff. ① w.r.t time.

$$\omega - k\frac{dx}{dt} = 0$$

$$\Rightarrow \frac{dx}{dt} = \frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = f\lambda$$

∴ Phase velocity $V_{phase} = \frac{dx}{dt} = f\lambda$

Now in case of matter wave

We have;

$$mc^2 = hf.$$

$$\Rightarrow f = \frac{mc^2}{h}$$

and $\lambda = \frac{h}{mv}$.

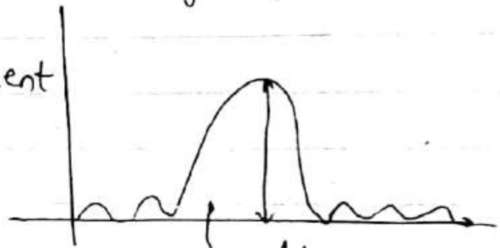
$$\therefore V_{phase} = \frac{mc^2}{h} \cdot \frac{h}{mv} = \frac{c^2}{v}$$ —— (ii)

v – velocity of particle

Eqn (ii) shows that the waves left behind the particle which is impossible such that matter wave is not a phase wave.

Group wave/wave packet:

When matter moves in certain velocity it generates two or more than two waves having slightly different wavelength such that there is interference between waves due to the constructive interferance the amplitude is maximum which is taken as matter wave and it is called group wave or wave packet.



matter wave (wave packet group wave).

Let us take the displacement of particle at any instant time with frequency f and wave no. k then

$$y_1 = a\sin(\omega t - kx)$$

Again take another wave having frequency $(f+\delta f)$ and wave no $(k+\delta k)$ then

$$y_2 = a\sin[(\omega+\delta\omega)t - (k+\delta k)x]$$

$\therefore$ Resultant wave becomes,

$$y = y_1 + y_2$$

$$= a[\sin(\omega t - kx) + \sin[(\omega+\delta\omega)t - (k+\delta k)x]]$$

$$= a\,2\sin\left(\frac{(\omega t-kx) + [(\omega+\delta\omega)t - (k+\delta k)x]}{2}\right)\cos\left(\frac{(\omega t-kx) - [(\omega+\delta\omega)t - (k+\delta k)x]}{2}\right)$$

or, $$y = 2a\sin\left(\frac{\omega t - \cancel{kx} + \omega t + t\delta\omega - \cancel{kx} - \delta kx}{2}\right)\cos\left(\frac{\omega t - kx - \omega t - \delta\omega t + kx + \delta kx}{2}\right)$$

or, $$y = 2a\sin\left(\frac{2\omega t + t\delta\omega - \delta kx}{2}\right)\cos\left(\frac{\delta kx - \delta\omega t}{2}\right)$$

4 2 1 55

or, $y = 2a \sin(\omega t - kx) \cos\left(\frac{\delta k.x - \delta\omega.t}{2}\right)$ $\left[\because \begin{matrix} 2\omega + \delta\omega \simeq 2\omega \\ 2k + \delta k \simeq 2k \end{matrix}\right]$

$\therefore y = A \sin(\omega t - kx) \cos\left(\frac{\delta k.x - \delta\omega.t}{2}\right)$ $[\because A = 2a]$

The resultant amplitude to be maximum

if $\delta k.x - \delta\omega.t = 0$

$\Rightarrow \frac{x}{t} = \frac{\delta\omega}{\delta k}$

$\Rightarrow$ group velocity $v_g = \frac{d\omega}{dk}$ Taking lim $\delta k \to 0$.

* Relationship between particle velocity and group velocity.

$\because$ We have,

group velocity $v_g = \frac{d\omega}{dk} = \frac{d(\hbar\omega)}{d(\hbar k)} = \frac{dE}{dP}$

we have $E = \frac{P^2}{2m}$

$\Rightarrow \frac{dE}{dP} = \frac{P}{m} = \frac{mv}{m} = v$

$\Rightarrow v_g = v$ where v is particle velocity

It shows that in matter waves the group velocity is equal to particle velocity.

* Relationship between group velocity and phase velocity.

We have, $v_g = \frac{d\omega}{dk} = \frac{d\omega}{d\lambda} \cdot \frac{d\lambda}{dk}$

Again we have

$\omega = 2\pi f = \frac{2\pi V_{phase}}{\lambda}$

$$\Rightarrow \frac{d\omega}{d\lambda} = V_{phase}\left(-\frac{2\pi}{\lambda^2}\right) + \frac{2\pi}{\lambda}\frac{dV_{phase}}{d\lambda}.$$

And

$$k = \frac{2\pi}{\lambda}$$

$$\frac{dk}{d\lambda} = \frac{-2\pi}{\lambda^2}$$

Now;

$$v_g = \frac{d\omega}{d\lambda}\cdot\frac{d\lambda}{dk} = \frac{V_{phase}\left(-2\pi/\lambda^2\right) + \frac{2\pi}{\lambda}\frac{dV_{phase}}{d\lambda}}{\frac{-2\pi}{\lambda^2}}$$

$$= \frac{\frac{\cancel{2\pi}}{\cancel{\lambda}}\left(\frac{-1}{\lambda}V_{phase} + \frac{dV_{phase}}{d\lambda}\right)}{-\frac{\cancel{2\pi}}{\cancel{\lambda^2}}}$$

$$V_g = \left(V_{phase} - \lambda\frac{dV_{phase}}{d\lambda}\right)$$

It shows that the group velocity is less than the phase velocity in dispersive medium. In non dispersive medium;

$$\frac{dV_{phase}}{d\lambda} = 0$$

$$\therefore \quad V_g = V_{phase}.$$

$h = 6.62 \times 10^{-34}$ J.s

$\hbar = \frac{h}{2\pi} = 1.05 \times 10^{-34}$ J.s.

Photon & Matter Wave

The quanta of electromagnetic wave or light wave is called photon. The minimum packet of energy is called quanta. The quanta of sound wave is called phonon.

Matter wave (De-Broglie wave).

When any object or particle is in motion they generates the wave which is called matter wave. Let us take a photon; the energy of the photon is given by plank's quantum theory is

$$E = hf = \hbar\omega \quad \text{—— (i)}$$

Now from photoelectric effect, the photon acts as a particle, so the energy of the photon from Einstein's energy mass relation is

$$E = mc^2 \quad \text{—— (ii)}$$

Equating eq^n (i) and (ii) we get,

$$hf = mc^2$$

$$h \cdot \frac{c}{\lambda} = mc^2$$

$$\therefore \lambda = \frac{h}{mc} \quad \text{—— (iii)}$$

If a particle having mass m moving with velocity v, then

$$\lambda = \frac{h}{mv}$$

① If you are moving with velocity 20km per hour. Calculate the debroglie wavelength.

Solution;

$$Mass(m) = 55kg.$$

$$v = 20 km/hr = \frac{20 \times 1000}{3600 \times 60} = 5.556$$

Now;

$$\lambda = \frac{h}{mv} = \frac{6.62 \times 10^{-34}}{55 \times 5.556} = 2.16 \times 10^{-36} m$$

② Calculate the de-broglie's wave length of an electron if 10kev ~~~~ is supplied on electron.

Solution;

We know, $\frac{1}{2} mv^2 = eV$

{ if $v > 10^7$ or 10^8 m is change }

or, $v = \sqrt{\frac{2eV}{m}}$

$$\therefore \boxed{\lambda = \frac{h}{\sqrt{2meV}}} = \cancel{\frac{6.62 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31}}}}$$

$$\therefore v = \sqrt{\frac{2 \times 10 \times 10^3 \times 1.6 \times 10^{-19}}{9.1 \times 10^{-31}}} = 5.93 \times 10^7 m/s.$$

Now;

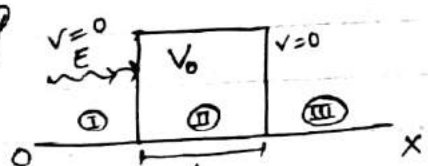
$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{9.1 \times 10^{-31}}{\sqrt{1 - \frac{(5.93 \times 10^7)^2}{(3 \times 10^8)^2}}} = 9.28 \times 10^{-31} kg$$

$$\therefore \lambda = \frac{h}{mv} = \frac{6.64 \times 10^{-34}}{9.28 \times 10^{-31} \times 5.93 \times 10^7}$$

$$= 1.20 \times 10^{-11} m.$$

Potential Barrier / Tunneling effect.

Let us take a particle having energy E is incident on a potential barrier having height V_0 where $E < V_0$. Classically the particle totally reflected back from $x=0$. But in quantum mechanics there is some probability of transition of a particle from the barrier having width l. This effect is called tunnelling effect. For example: α-decay, β-decay and tunneling diode.



In (I) region the SWE can be written as:

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \psi_I(x)}{\partial x^2} + 0 = E\,\psi_I(x)$$

$$\Rightarrow \frac{\partial^2 \psi_I(x)}{\partial x^2} + \frac{2mE}{\hbar^2}\psi_I(x) = 0$$

Let $\alpha^2 = \frac{2mE}{\hbar^2}$

Then; $$\frac{\partial^2 \psi_I(x)}{\partial x^2} + \alpha^2 \psi_I(x) = 0 \quad \text{--- (i)}$$

The general soln of eqn (i) is

$$\psi_I(x) = Ae^{i\alpha x} + Be^{-i\alpha x} \quad \text{--- (ii)}$$

The first term $Ae^{i\alpha x}$ is the amplitude of incident wave and the second term $Be^{-i\alpha x}$ is the amplitude of wave reflected from $x=0$.

In (II) region the SWE can be written as

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \psi_{II}(x)}{\partial x^2} + V_0(x)\,\psi_{II}(x) = E\,\psi_{(II)}x$$

$$\Rightarrow \frac{\partial^2 \psi_{II}(x)}{\partial x^2} - \frac{2m(V_0 - E)}{\hbar^2}\psi_{II}(x) = 0 \quad \text{--- (iii)}$$

~~The general soln of eqn (iii) is $\psi_{II}(x)$~~ Let $\beta^2 = \frac{2m(V_0-E)}{\hbar^2}$

then $\frac{\partial^2 \psi_{II}(x)}{\partial x^2} - \beta^2 \psi_{II}(x) = 0$

The general solution is

$$\psi_{II}(x) = Ce^{\beta x} + De^{-\beta x} \quad \text{——— (iv)}$$

The first term $Ce^{\beta x}$ gives the amplitude of incident wave on $x=L$ and the second term $De^{-\beta x}$ gives the amplitude of reflected wave from $x=L$.

Now, the SWE for region (III) is

$$\frac{\partial^2 \psi_{III}(x)}{\partial x^2} + \frac{2mE}{\hbar^2}\psi_{III}(x) = 0$$

Let $\alpha^2 = \frac{2mE}{\hbar^2}$

$$\psi_{III}(x) = Fe^{i\alpha x} + Ge^{-i\alpha x}$$

Since there is no any barrier in third region then there is no possibility of reflection of wave such that $Ge^{-i\alpha x} = 0$

Hence. $\psi_{III}(x) = Fe^{i\alpha x}$ ——— (v).

Using boundary condition we get Transmission-coefficient.

$$T = \left|\frac{F}{A}\right|^2 = \frac{16E(V_0-E)}{V_0^2} e^{-2\beta l}$$

where; $\beta = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

l = width of potential Barrier.

E = Energy of particle.

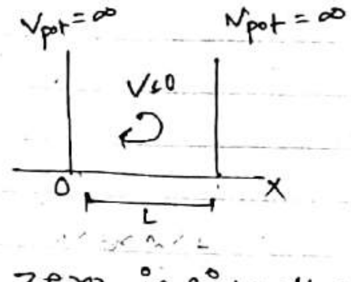
V_0 = barrier potential/height.

m = mass of incident particle.

Imp.

Particle in a box / Infinitely deep potential well

Let us take a particle having mass m is confined in an infinitely deep potential well such that the particle moves in to and fro motion entire the well as shown in figure. The potential energy is zero inside the well



$V = 0; \quad 0 < x < L$

$V = \infty \quad x \leq 0 \ \& \ x \geq L$

Now, we have schrodinger time independent wave equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + V_{pot}\cdot\psi(x) = E\psi(x)$$

Since $V_{pot}=0$, inside the well, then it becomes:

$$\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{2mE}{\hbar^2}\psi(x) = 0$$

Let $k^2 = \dfrac{2mE}{\hbar^2}$

Then, $\dfrac{\partial^2 \psi(x)}{\partial x^2} + k^2 \psi(x) = 0$ ——— (1)

The general soln of eqn (1) is

$$\psi(x) = A\sin kx + B\cos kx$$

Now, $\psi(x) = 0 \Big|_{x=0}$

$0 = A\cdot 0 + B$

$\therefore B = 0$

Again $\psi(x) = 0 \Big|_{x=L}$

$0 = A\sin Lk + B\cos Lk$

$\Rightarrow A\sin kL = 0 \quad (\because A \neq 0)$

$\Rightarrow \sin kL = 0$

$\Rightarrow \sin kL = \sin n\pi$

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$\Rightarrow k = \frac{n\pi}{L}$

We have $k^2 = \frac{2mE}{\hbar^2}$

$\therefore \frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{L^2}$

$\Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$ (energy eigen value) where $n = 1, 2, 3$

which gives the energy Eigen value of a particle confined in a infinitely deep potential wave along x-axis having width L. It shows that energy of particle is quantized .... contd.

①. An electron is confined to an infinite height box of size 0.1nm. Calculate the ground state energy of the electron. How this energy can be put to the third energy state.

Solution;

$L = 0.1 nm = 0.1 \times 10^{-9} m.$

For ground state, $n = 1$

$E_1 = \frac{1^2 \times \pi^2 \times \hbar^2}{2mL^2} =$

For third state, $n = 3$

$E_3 = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{3^2 \times \pi^2 \times 6.69 \times 10^{-39}}{2\pi \times 9.1 \times 10^{-31} \times (0.1 \times 10^{-9})^2} = 1.03 \times 10^{18} J.$

(2) If a football having mass 500gm is confined in a Dashrath Rangasala having width 1km. Calculate the velocity of the football.

Solution

Mass(m) = 500×10^{-3} kg.

$n = 1$ (for ground state)

$L = 1 km = 1000 m$

$$\therefore E_n = \frac{n^2 \pi^2 \hbar^2}{2 m L^2} = \frac{1^2 \times \pi^2 \times (1.05 \times 10^{-34})^2}{2 \times 500 \times 10^{-3} \times (1000)^2} = 1.08 \times 10^{-73} J$$

Now;

$$E_n = \frac{1}{2} m v^2$$

or, $1.08 \times 10^{-73} = \frac{1}{2} \times 500 \times 10^{-3} \times v^2$

$$\therefore v = 6.57 \times 10^{-37} m/s.$$

contd.....

The wavefunction of the particle inside the infinitely deep potential well is

$$\psi(x) = A \sin kx$$

To find the normalization constant A;

$$\int_0^L \psi^*(x)\, \psi(x) dx = 1 \qquad \int_0^L$$

or, $\int_0^L A^2 \sin^2 kx \, dx = 1$

or, $A^2 \left[\int_0^L \frac{1 - \cos 2kx}{2} dx\right] = 1$

or, $A^2 \left[\frac{1}{2} \times L - \frac{1}{2} \int_0^L \cos 2kx dx\right] = 1$

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or, $A^2\left[\frac{L}{2}-\frac{1}{2}\left[\frac{\sin 2kx}{2k}\right]_0^L\right]=1$

~~or, $A^2\left[\frac{L}{2}-\frac{1}{4k}\left(\sin 2k\right)\right]=1$~~

~~or, $A^2\left[\frac{L}{2}-\frac{1}{4\times\frac{n\pi}{L}}\sin 2\frac{n\pi}{L}\right]=$~~

~~or $A^2\left[\frac{L}{2}-\frac{1}{4\pi n}\sin 2\frac{n\pi}{L}\right]=$~~

$A^2\left[\frac{L}{2}-\frac{1}{2}\left(\frac{\sin 2kx}{2k}\right)_0^L\right]=1$

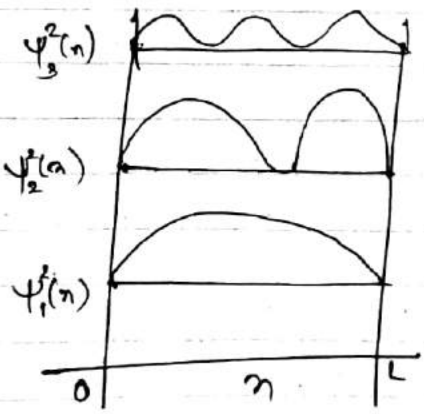
or, $A^2\left[\frac{L}{2}-\frac{1}{2}\times\frac{1}{\frac{2n\pi}{L}}\left(\sin 2\frac{n\pi}{L}\times L-\sin 0\right)\right]=1$

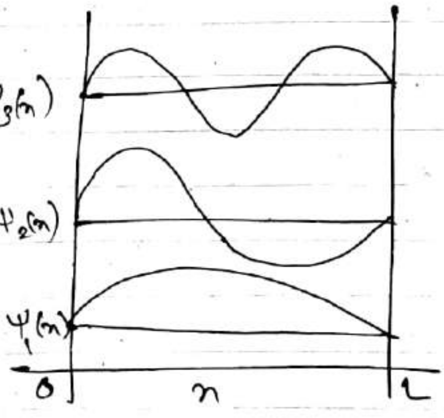
or, $A^2\left[\frac{L}{2}-\frac{L}{2}\times\frac{1}{2n\pi}\left(\sin 2n\pi\right)\right]$

or $A^2=\frac{1}{L/2}$ since $n=1,2,3\cdots$

$\therefore A=\sqrt{2/L}$

$\therefore \psi_n(x)=\sqrt{2/L}\sin\frac{n\pi}{L}x$





Let R be the reflection co-efficient, then

$$R+T=1$$
$$\Rightarrow R=1-T$$

(1) If an electron beam having energy 5ev is incident on a potential barrier having height 6ev where the width of the potential barrier is 15A°. Calculate the reflection coeff and transmission coefficient.

solution;

Given; Energy of electron $(E)=5eV=5\times1.6\times10^{-19}J$

~~Energy~~ Height or barrier potential $(V_0)=6ev.=6\times1.6\times10^{-19}J$

Width of potential barrier $(l)=15A°=15\times10^{-10}m$.

Now,

$$\beta=\sqrt{\frac{2m(V_0-E)}{\hbar^2}}=\sqrt{\frac{2\times M_e\times(6\times1.6\times10^{-19}-5\times1.6\times10^{-19})}{(1.05\times10^{-34})^2}}$$

$$=5.26\times10^{-8}$$

Now; Transmission coeff $(T)=\frac{16E(V_0-E)}{V_0^2}e^{-2\beta l}$

$$=\frac{16\times5\times(1.6\times10^{-19})(6\times1.6\times10^{-19}-5\times1.6\times10^{-19})}{(6\times1.6\times10^{-19})^2}\times e^{-(2\times5.26\times10^{-8}\times15\times10^{-10})}$$

$$=3.32\times10^{-9}.$$

$$=$$

$\therefore$ Reflection coefficient $(R)=1-T$

$$=1-3.32\times10^{-9}$$

$$=9.99\times10^{-1}.$$

Heisenberg Uncertainity principle,

Any two canonically conjugate variable cannot be measure accurately and simultaneously which is called heisenberg Uncertainity principle. We have group velocity;

$$V_g = \frac{d\omega}{dk} = \hbar\frac{d\omega}{dp} = V_{particle} = \frac{dx}{dt}$$

for small increments

$$\frac{\Delta x}{\Delta t} = \hbar\frac{\Delta\omega}{\Delta p}$$

Now,

$$\Delta\omega . \Delta t \geq 1$$

$$\Rightarrow \Delta x . \Delta P_x \geq \hbar$$

Direct Current.

The rate of flow of charge is called current.

DC.	AC.
1) The polarity of DC is constant.	1) The polarity of AC changes with time.
2) frequency remains constant.	2) frequency changes.

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Ohm's law

"At a constant physical condition like tempr, pressure etc the current through the conductor is directly proportionalE to the potential difference applied across the end of the conductor".

Mathematically:

$$\Rightarrow V \propto I$$

$V = IR$ where R is a proportionality constant called resistance at constant tempr.

If a p.d V is applied on the conductor then the current flow in the conductor is given by

$$V = IR$$

Let us take a conductor having length l and cross-sectional area A. then the current flowing in the conductor is.

$$I = \frac{q}{t} = \frac{Ne}{t} = \frac{neAl}{t} = V_d enA$$ where n is charge carrier density.

N is no. of charge carriers.

Current Density $(J) = \frac{I}{A}$

$= \quad . \quad \Rightarrow I = \int J \cdot dA$

$\therefore \quad n = \frac{\rho N_A}{M}$ where

M = Molar mass

ρ = Density of material

N_A = Avogadro's number

Resistance and Resistivity.

Resistance In conductor is due to the collision of electrons. Let us take a conductor having length 'l'. If the p.d V is applied at the two ends of the conductor then:

$$E = \frac{V}{l}$$

The modified ohm's law is

$J = \sigma E$ where σ is conductivity.

$$\therefore \frac{J}{\sigma} = \frac{V}{l}$$

$$\Rightarrow V = \frac{lJ}{\sigma} = \frac{lI}{A\sigma} = IR$$

$$\Rightarrow R = \frac{l}{A\sigma} = \frac{\rho l}{A}$$ where ρ is resistivity of the material

$$\Rightarrow R = \frac{\rho l^2}{A \times l}$$

$$\Rightarrow \boxed{R \propto l^2}$$

① Calculate the current flowing between the R and R/2 where R is radius of wire. If current density is $10^{-4} A/m^2$ & diameter of conductor is 75mm.

Solution.

$J = 10^{-4} A/m^2$

$d = 75mm = 37.5mm = 37.5 \times 10^{-3} m$

$\therefore$ ~~$R = 37.5 \times 10^{-3} m$~~

~~$R/2 = 18.75 \times 10^{-3} m$~~

$\cancel{A = J \times A}$

$\cancel{= 10^{-4} \times \pi \times (37.5 \times 10^{-3})^2}$

$A = \pi r^2$

$dA = 2\pi r \, dr$

$$I = \int_{r/2}^{r} J \cdot 2\pi r \, dr$$

$$= 2\pi \times J \int_{r/2}^{r} r \, dr$$

$$= \frac{2\pi \times 10^{-4}}{2} \left[r^2 \right]_{r/2}^{r}$$

$$= 3.14 \times 10^{-4} \left(r^2 - (r/2)^2 \right)$$

$$= 3.14 \times 10^{-4} \left((37.5 \times 10^{-3})^2 - \left(\frac{37.5 \times 10^{-3}}{2} \right)^2 \right)$$

$$= 3.31 \times 10^{-7} A.$$

② If a wire having length l is enloged to $5l$. calculate the %change in resistance.

Soln

we know; $R \propto l^2$

$$\frac{R_1}{R_2} = \left(\frac{l_1}{5 l_1} \right)^2 \qquad \frac{R_1}{R_2} = \frac{1}{25}$$

$$\therefore R_2 = 25 R_1$$

Now,

$$\Delta R = \frac{25 R_1 - R_1}{R_1} \times 100\%$$

$$= 2400\%.$$

Show that $J = \sigma E$

Soln.

$$J = \frac{I}{A} = \frac{V}{RA} = \frac{V}{\frac{\rho l}{\cancel{A}}.\cancel{A}} = \frac{1}{\rho}E = \sigma E \quad (\because E = V/L).$$

Microscopic view of ohm's law.

When an electric field E is supplied on a conductor then the force experienced by eletron is

$$F = eE = \frac{eJ}{\sigma} \quad \text{---} (i).$$

We have from Newton's law

$$F = ma$$

$$= m.\frac{v_d}{\tau} \rightarrow \text{drift velocity} \quad \text{---} (ii).$$

where m is mass of electron

τ = Relaxation time / mean free time.

Equating eqn (i) & (ii)

$$\frac{eJ}{\sigma} = \frac{mv_d}{\tau}$$

or, $$\sigma = \frac{ne^2\tau}{m} \quad [\because J = nev_d]$$

is conductivity of the electron.

We have;

$$\frac{1}{2}mv_d^2 = \frac{3}{2}K_BT$$

The mean free path $\lambda = v_d \times \tau$.

And Mobility(μ) = $\left|\frac{v_d}{E}\right|$.

①. A current of 1.2×10^{-8} A exist in a copper wire having diameter 3mm. Calculate the current density, electrical conductivity, mobility of electrons, charge carrier density, Mean free path, ($\rho = 9$ gm/cc, $M = 63.5$ gm) and each cu atom has 1 free electron. (Mean free time = 10^{-8} sec)

solution,

$I = 1.2 \times 10^{-8}$ A.

$d = 3mm$ & $r = 1.5 \times 10^{-3} = 0.0015m$.

a) $J = \frac{I}{A} = \frac{1.2 \times 10^{-8}}{\pi \times (0.0015)^2} = 1.69 \times 10^{-3} A/m^2$

b) $n = \frac{\rho N_A}{M} = \frac{9 \times 10^{-3} \times 6.02 \times 10^{23}}{63.5 \times 10^{-6} \times 10^{-3}} = 8.5 \times 10^{28}$

c) $\sigma = \frac{ne^2\tau}{m_e} = \frac{8.5 \times 10^{28} \times (1.6 \times 10^{-19})^2 \times 10^{-8}}{9.1 \times 10^{-31}}$

Semi Conductor:-

The material whose conductivity are between conductor and insulator. at low tempr. the semiconductor act as insulator and in at high tempr it acts as conductor. Main there are two type of semiconductor.

1) Intrinsic or pure semiconductor:-

The Sillicon & Germanium are the intrinsic semiconduc

2) Entrinsic or impure semiconductor:-

When pure semiconductor is dopped by pentavale or trivalent atom like Phosphorus & Boron then there conductivity is increases such type of semiconductor are called entrinsic or impure semiconductor.

a) N-type semiconductor;

When pentavalent atom is dopped into Sillicon or German then electron acts as majority charge carriers which are called N-type Semiconductor

b) P-type Semiconductor.

When trivalent atom is dopped into Si or Ge then holes acts as majority charge carriers which is called p-type Semiconductor..

Conductivity of Semi-conductor.

$$\sigma = n_i e(\mu_e + \mu_h)$$

where μ_h = mobility of hole

μ_e = Mobility of electron.

$$n_i = \sqrt{np}$$

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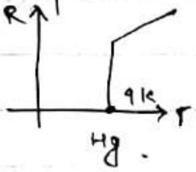
Super Conductor.

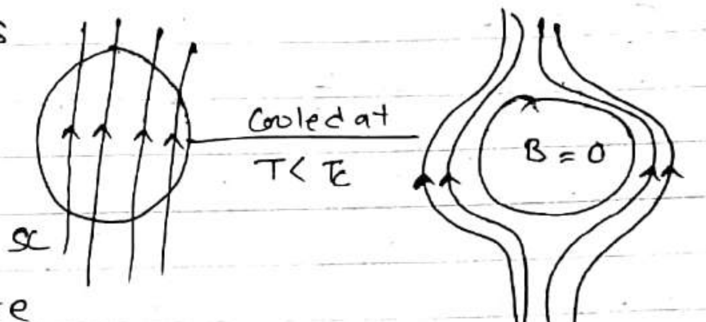
When some metal or alloy cooled at very low temperature i.e below the critical tempr then there resistivity or resistance suddenly drops to zero. This phenomenon is called super conductivity and the material which show this phenomenon is called Super conductor.

for eg. Mercury



Meissner's effect.

When superconductors are placed in magnetic field and cooled at very low temperature i.e below the critical temperature then the magnetic field lines are ejected from the superconductor which shows the diamagnetic property of the superconductor which is called Meisner's effect.



Let H be the applied magnetic field on the superconductor and M be the magnetization then

Total magnetic field;

$$B = \mu_0 (H + M)$$

Since $B = 0$

Then $H + M = 0$

$\therefore$ Magnetic susceptibility

$$\chi = \frac{M}{H} = -1$$ shows diamagnetic property.

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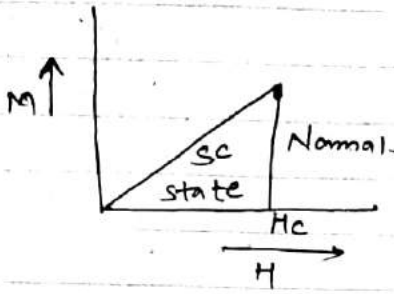
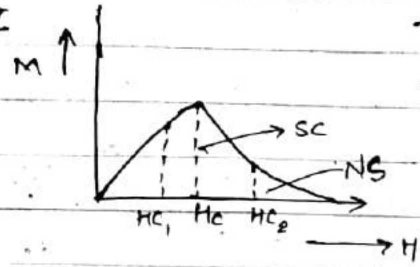
Types of Super Conductor -

(1) Type I.



The super conductor which exactly follow the meisner's effect and there exist only one vortex state is called type I superconductor.

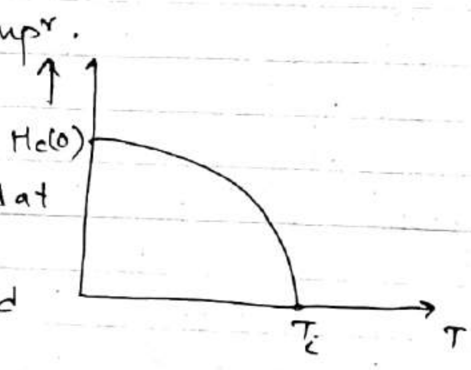
(2) Type II



The superconductor which doesnot follow the meisner's effect and have two or more that two vortex state is called type II superconductor.

Critical magnetic field and critical tempr.

$$H_c(T) = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$



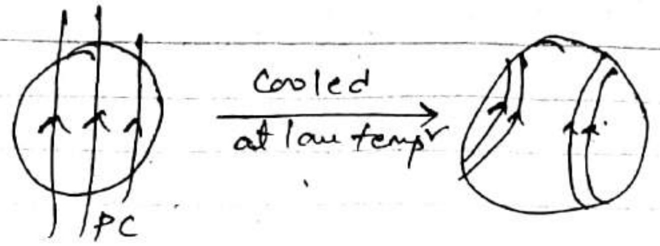
where $H_c(0)$ = Critical magnetic field at 0K

$H_c(T)$ = Critical magnetic field at tempr T

T_c = Critical tempr at zero magnetic field.

Perfect Conductor.

- The conductor which has zero resistance at any tempr. It is an idealized concept. It is used to correct the mathematical relation. When a perfect conductor is placed in magnetic field and cooled at very low tempr then the magnetic field lines are freezes inside it.



Induced magnetic field / Modified Ampere's law / Displacement current

From Faraday law of induction, we have

$$\oint \vec{E}.d\vec{l} = -\frac{\partial \phi_B}{\partial t} \quad \text{where } \phi_B = BA = \oint \vec{B}.d\vec{A} \text{ is magnetic flux.}$$

Similarly;

Maxwell suppose that the change in electric flux produced magnetic field, why not.

Then:

$$\boxed{\oint \vec{B}.d\vec{l} = \mu_0 \epsilon_0 \frac{\partial \phi_E}{\partial t}}$$

where $\epsilon_0 \frac{\partial \phi_E}{\partial t}$ is called displacement current

where $\phi_E = EA = \oint \vec{E}.d\vec{A}$ Electric flux.

Then Modified Ampere's law becomes:

$$\oint \vec{B}\,d\vec{l} = \mu_0(I + I_d) \quad \text{where } I_d = \epsilon_0 \frac{\partial \phi_E}{\partial t}$$

→ What is displacement current & show that the displacement current is equal to the conduction current in a circularly parallel plate capacitor.

Ans: Displacement current is a fictitious current. The displacement current is given by

$$I_d = \epsilon_0 \frac{\partial \phi_E}{\partial t} \text{ where } \phi_E = EA$$
$$= \oint \vec{E}.d\vec{A}$$

Let us take two circullarly parallel plate capacitor having area(A) and seperated by a finite distance 'd'. When they are connected with the varying electric fied then the charge on the capacitor at any instant of time 't' can be written as

$q = CV$ where C is the capacitance of the capacitor.

Then current is given by

$$I = \frac{dq}{dt} = C\frac{dV}{dt} = \frac{\epsilon_0 A}{d}\frac{dV}{dt}$$

$$= \epsilon_0 A \frac{d(V/d)}{dt}$$

$$= \epsilon_0 A \frac{dE}{dt}$$

$$= \epsilon_0 \frac{\partial \phi_E}{\partial t} \quad \text{where } \phi_E = EA$$

$$\therefore \boxed{I = I_d}$$

Q. Calculate the displacement current betn the capacitor plates of area $1.5 \times 10^{-2} m^2$ and rate of electric field change is 1.5×10^{12} V/ms and also calculate induced magnetic fied?

Solution.

Given $A = 1.5 \times 10^{-2} m^2$

$dE/dt = 1.5 \times 10^{12}$ V/ms

$\epsilon_0 = 8.85 \times 10^{-12}$ F/m.

Now, $I_d = \epsilon_0 \frac{\partial \phi_E}{\partial t}$

$$= \epsilon_0 A \, dE/dt$$

$$= 8.85 \times 10^{-12} \times 1.5 \times 10^{-2} \times 1.5 \times 10^{12}$$

$$= 0.199 A.$$

Now, $\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{\partial \phi_E}{\partial t} = \mu_0 I_d$

$$B = \frac{\mu_0 I_d}{2\pi r} = \frac{4\pi \times 10^{-7} \times 0.199}{2\pi \times \sqrt{\frac{1.5 \times 10^{-2}}{\pi}}}$$

$= 5.75 \times 10^{-7}\ T$.

Q. A variable field of 10^{12} V/m/s is applied to a parallel plate capacitor with circular plate of diameter 10cm. Calculate the displacement current density, current and induced magnetic field.

Solution,

$dE/dt = 10^{12}$ V/ms

diameter (d) = 10cm = 5cm = 5×10^{-2} m = 0.05m

$I_d = ?$

$J_d = ?$

$B = ?$

Now, $I_d = \epsilon_0 A \frac{dE}{dt} = 8.85 \times 10^{-12} \times \pi \times (0.05)^2 \times 10^{12}$

$= 6.9 \times 10^{-2}$ A.

$$J_D = \frac{I_D}{A} = \frac{6.9\times10^{-2}}{\pi\times(0.05)^2} = 8.785\,A/m^2.$$

$$B = \frac{\mu_0 I_d}{2\pi r} = 2.76\times10^{-7}\,T.$$

Electromagnetic wave.

1) Maxwell Equations in integral form

The basic ~~form~~ law of electricity and Magnetism in which the Ampere's law is modified by Maxwell are called Maxwell equation.

<i> $\oint_S \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0} = \oint \frac{\rho\, dv}{\epsilon_0}$

which is the guass law of electricity. It states that the total electric flux inside a closed surface is equal to the charge enclosed by the surface.

<ii> $\oint_S \vec{B}\cdot d\vec{A} = 0$

which is the guass law of magnetism. It states that the total magnetic flux inside a closed surface is equal to zero i.e there is no existence of magnetic monopole/existing of dipole.

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<iii> $\oint_l \vec{E}\cdot\vec{dl} = -\frac{\partial \phi_B}{\partial t} = \frac{-\partial}{\partial t}\oint_S \vec{B}\cdot\vec{dA}$

which is the Faraday's law of induction. It states that the change in magnetic flux produces induced emf.

<iv> $\oint_l \vec{B}\cdot\vec{dl} = \mu_0(I+I_d)$

$= \mu_0\left(\int_S \vec{J}\cdot\vec{dA} + \epsilon_0 \frac{\partial}{\partial t}\oint_S \vec{E}\cdot\vec{dA}\right)$

which is the modification of Ampere's law by Maxwell. It states that the change in electric flux also produces the magnetic field

Conversion of Maxwell equation into differential form.

(i) We have

$$\oint_S \vec{E}\cdot\vec{dA} = \frac{q}{\epsilon_0} = \oint_V \frac{\rho\, dv}{\epsilon_0}$$

Using Gauss divergence theorem.

$$\oint_S \vec{E}\cdot\vec{dA} = \oint_V (\nabla\cdot\vec{E})dv.$$

$$\therefore \oint_V (\nabla\cdot\vec{E})dv = \int_V \frac{\rho\, dv}{\epsilon_0}$$

$$\Rightarrow \nabla\cdot\vec{E} = \rho/\epsilon_0 \quad \text{(1)}$$

Since $D = \epsilon_0\vec{E}$

$\nabla\cdot D = \rho$

is the required Maxwell I[st] equation in differential form.

Note:

$\nabla \times \nabla\phi = 0$

$\nabla \cdot (\nabla \times \vec{A}) = 0$

$\oint_S \vec{F} \cdot d\vec{A} = \oint_V (\nabla \cdot \vec{F}) dv$

↳ Gauss divergence theorem

$\oint_l \vec{F} \cdot d\vec{l} = \oint_S (\nabla \times \vec{F}) \cdot d\vec{A}$

↳ Stoke's theorem

(ii) We have

$$\oint_S \vec{B} \cdot d\vec{A} = 0$$

Using Gauss div. Theorem $\oint_S \vec{B} \cdot d\vec{A} = \oint_V (\nabla \cdot \vec{B}) dv$

$$\oint (\nabla \cdot \vec{B}) dv = 0$$

$$\Rightarrow \nabla \cdot \vec{B} = 0 \quad \text{— (ii)}$$

Maxwell 2nd eqn in differential forms.

(iii) We have

$$\oint_l \vec{E} \cdot d\vec{l} = \frac{-\partial \phi_B}{\partial t} = \oint_S \frac{-\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

Now

$$\oint_l \vec{E} \cdot d\vec{l} = \oint_S (\nabla \times \vec{E}) \cdot d\vec{A} \quad \text{Using Stoke's theorem}$$

$$\therefore \oint_S (\nabla \times \vec{E}) \cdot d\vec{A} = \oint_S \frac{-\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

$\Rightarrow \nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$ is Maxwell third equation in diff. form

(iv) We have

$$\oint_l \vec{B} \cdot d\vec{l} = \mu_0 (I + I_d)$$

$$= \mu_0 \oint_S \vec{J} \cdot d\vec{A} + \mu_0 \epsilon_0 \oint_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}$$

Using stoke's theorem

$$\oint_l \vec{B} \cdot d\vec{l} = \oint_S (\nabla \times \vec{B}) \cdot d\vec{A}$$

$$\therefore \oint_S (\nabla \times \vec{B}) d\vec{A} = \mu_0 \oint_S \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \cdot d\vec{A}$$

$\varepsilon_0 E = D$
$B = \mu_0 H$

$\Rightarrow$ $\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$ is Maxwell 4th eqn in diff. form.

since $B = \mu_0 H$
$$\nabla \times \vec{H} = \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$
$\vec{D} = \varepsilon_0 \vec{E}$ is displacement vector.

→ Write the Maxwell equation in differential form and then convert it into integral form.

Q. Show that charge conservation theorem. $\nabla\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$

Sol^n

we have from Maxwell 4th eq^n is

$$\nabla\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) \quad \text{--- (1)}$$

Taking divergence on both side of eq^n (1).

$$\nabla\cdot(\nabla\times\vec{B}) = \mu_0\left[\nabla\cdot\vec{J} + \varepsilon_0\frac{\partial}{\partial t}(\nabla\cdot\vec{E})\right]$$

or, $$0 = \mu_0\left[\nabla\cdot\vec{J} + \varepsilon_0\frac{\partial}{\partial t}(\nabla\cdot\vec{E})\right]$$

or, $$\nabla\cdot\vec{J} + \varepsilon_0\frac{\partial}{\partial t}(\nabla\cdot\vec{E}) = 0$$

or, $$\nabla\cdot\vec{J} + \varepsilon_0\frac{\partial}{\partial t}(\rho/\varepsilon_0) = 0 \quad (\because \nabla\cdot\vec{E} = \rho/\varepsilon_0)$$

or, $$\nabla\cdot\vec{J} + \frac{\partial(\rho)}{\partial t} = 0$$

$$\therefore \nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \quad \text{proved.}$$

2067 V.V.I

Maxwell Equation's in free space and plane wave or em wave in free space

In free space the charge density 'ρ' and the current density (J) is zero. Such that Maxwell equation becomes:-

$$\nabla\cdot\vec{E} = 0 \quad \text{--- (i)}$$

$$\nabla\cdot\vec{B} = 0 \quad \text{--- (ii)}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{--- (iii)}$$

$$\nabla\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad \text{--- (iv)}$$

To find the em/plane wave equations. Taking curl on bothsides of eqn (iii).

$$\nabla\times(\nabla\times\vec{E}) = \frac{-\partial}{\partial t}(\nabla\times\vec{B})$$

$$\Rightarrow \overset{0}{\cancel{(\nabla\cdot\vec{E})}}\nabla - \nabla^2\vec{E} = \frac{-\partial}{\partial t}\mu_0\varepsilon_0\left(\frac{\partial\vec{E}}{\partial t}\right)$$

$$\Rightarrow \nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{—— (v)}$$

Similarly; In terms of magnetic field vector B :-

$$\nabla\times(\nabla\times\vec{B}) = \nabla\times\left\{\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right\}$$

(B a)c
(a b)c

$$\Rightarrow \overset{0}{\cancel{\nabla(\nabla\cdot\vec{B})}} - \vec{B}\nabla^2 = \mu_0\varepsilon_0 \frac{\partial(\nabla\times\vec{E})}{\partial t}$$

$$\Rightarrow -\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\frac{-\partial\vec{B}}{\partial t}\right)$$

$$\Rightarrow \nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}. \quad \text{—— (vi)}.$$

Comparing eqn (v) & (vi). with general diff. wave equation.

$$v^2 = \frac{1}{\mu_0\varepsilon_0}$$

$$\therefore v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{4\pi\times10^{-7}\times 8.85\times10^{-12}}} = 3\times10^8\,m/s$$

It shows that the em wave travels with speed of light (c) in free space.

The general soln of (v) & (vi) is

$$E = E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$B = B_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

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In one dimension.

$$E = E_0 e^{i(kx-\omega t)}$$

$$B = B_0 e^{i(kx-\omega t)}$$

If the wave is sinusoidal

$$E = E_0 \sin(kx-\omega t)$$

$$B = B_0 \sin(kx-\omega t)$$

Q. Show that $E/B = E_0/B_0 = c$.

We have Maxwell 3rd eqn is

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

In one dimension it becomes; $\frac{\partial \vec{E}}{\partial x} = -\frac{\partial \vec{B}}{\partial t}$

Now we have $E = E_0 e^{i(kx-\omega t)}$

$B = B_0 e^{i(kx-\omega t)}$

or, $E_0 e^{i(kx-\omega t)} \cdot \cancel{i}k = (-) B_0 e^{i(kx-\omega t)} \cdot \omega \cancel{i}$

or, $\frac{E_0}{B_0} = \frac{\omega}{k} = \frac{2\pi f}{\frac{2\pi}{\lambda}} = \lambda f = c$

$$\therefore \frac{E_0}{B_0} = c.$$

Q. Show that magnetic energy density is equal to electric energy density.

Solution

We have, Electric energy density $(U_E) = \frac{1}{2}\epsilon_0 E^2$ ——— (i)

Magnetic energy density $(U_B) = \frac{B^2}{2\mu_0}$ ——— (ii).

Dividing eq^n (i) by (ii)

$$\cancel{\frac{U_E}{U_B} = \frac{\frac{1}{2}\epsilon_0 E^2}{\frac{B^2}{2\mu_0}} = \frac{1}{2}\epsilon_0 E^2 \times \frac{2\mu_0}{B^2}} = \cancel{\epsilon_0}$$

$$\frac{U_B}{U_E} = \frac{B^2}{2\mu_0} \times \frac{2}{\epsilon_0 E^2}$$

$$= \frac{(B/E)^2}{(\sqrt{\mu_0 \epsilon_0})^2}$$

$$= (1/c)^2 \times c^2$$

$$= 1.$$

Maxwell equations in Non-conducting/dielectric isotropic medium

In dielectric or non conducting medium charge density 'ρ' and current density 'J' is zero and permeability $\mu = \mu_r \mu_0$ and permitivity $\epsilon = \epsilon_r \epsilon_0$

Then the Maxwell equation becomes

$$\nabla \cdot \vec{E} = 0 \quad \text{—— (i)}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{—— (ii)}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{—— (iii)}$$

$$\nabla \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{—— (iv).}$$

To find the e.m/plane wave equation. Taking curl on both sides of eq^n (iii)

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\Rightarrow (\nabla \cdot \vec{E})\nabla - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{—— (v).}$$

Similarly In terms of magnetic fied vector B.

$$\nabla \times (\nabla \times \vec{B}) = \nabla \times \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\therefore \quad \nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{— (vi)}.$$

~~Comparing eqn (v) & (vi) with general diff wave eqn~~

~~$v=$~~

velocity of em wave is $v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_r \mu_0 \cdot \epsilon_r \epsilon_0}}$

$$= \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

Maxwell eqn in conducting medium.

In conducting medium let the ~~current density~~ conductivity be 'σ' and charge density 'ρ' is zero.

Then the maxwell eqn becomes

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu \left[\sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \right].$$

Then to find em/plane wave equation.

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} (\nabla \times \vec{B})$$

$$(\nabla \cdot \vec{E})\nabla - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \mu \left[\sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \right]$$

$$-\nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\therefore \nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}.$$

$$\nabla \times \vec{B} = \mu_0 \left[\vec{J} + \varepsilon_0 \cancel{\frac{\partial \vec{E}}{\partial t}}\right]$$
$$= \mu_0 \vec{J}.$$

Similarly in terms of magnetic field.

$$\nabla^2 \vec{B} = \mu \sigma \frac{\partial \vec{B}}{\partial t} + \mu \varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

Maxwell eqn in slowly varying field

In slowly varying field electric field $\vec{E}$ and magnetic field $\vec{B}$ slowly changes with time so $\frac{\partial \vec{E}}{\partial t} = 0$ and $\frac{\partial \vec{B}}{\partial t} = 0$.

$$\nabla \cdot \vec{E} = \rho/\varepsilon$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = 0$$
$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Then;

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} (\nabla \times \vec{B})$$

$$\Rightarrow (\nabla \cdot \vec{E}) \nabla - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \mu_0 \vec{J}$$

$$\Rightarrow \frac{\rho}{\varepsilon} \nabla - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \mu_0 \vec{J}$$

$$\Rightarrow \frac{\rho}{\varepsilon} \nabla = \nabla^2 \vec{E} + \mu_0 \frac{\partial \vec{J}}{\partial t}.$$

Poynting vector.

When an object is placed in the path of electromagnetic wave then the em wave transfer the energy on the object in the form of electric and magnetic energy. The rate of transfer of energy per unit area is called poynting vector. It can be written as

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

Let us take an object having area A and thickness dx is placed in a path of em wave then at any instant of time t the energy transfer on the object can be written as

$$dU = dU_B + dU_E$$

$$= (U_B + U_E) A\,dx$$

where U_E = electric energy density.
U_B = magnetic " " .

Dividing both sides by dt,

$$\frac{dU}{dt} = (U_E + U_B) \cdot A \frac{dx}{dt}$$

$$\frac{dU}{A\,dt} = (U_E + U_B) \cdot c$$

$$\text{or, } S = \left(\frac{1}{2}\epsilon_0 E^2 + \frac{B^2}{2\mu_0}\right) c$$

$$\text{or } S = \left(\frac{\mu_0 \epsilon_0 E^2 + B^2}{2\mu_0}\right) c$$

$$\text{or } S = \left(\frac{E^2/c^2 + B^2}{2\mu_0}\right) c$$

$$\text{or, } S = \frac{E^2 + B^2 c^2}{2\mu_0 c}$$

We have $E/B = c$

Then $S = \dfrac{2E^2}{2\mu_0 c} = \dfrac{EB}{\mu_0}$

In vector form; $\vec{S} = \dfrac{\vec{E} \times \vec{B}}{\mu_0}$. Since E and B are ⊥er to each other

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The electromagnetic wave is sinusoidal such that we have.

$E = E_0 \sin(kx - \omega t)$

$B = B_0 \sin(kx - \omega t)$

$\therefore S = \frac{E_0 B_0}{\mu_0} \sin^2(kx - \omega t)$

The average poynting vector or average intensity is

$\langle S \rangle = I_{av} = \frac{E_0 B_0}{2\mu_0} \quad \left[\because \langle \sin^2(kx-\omega t)\rangle = \frac{1}{2}\right]$

$\langle S \rangle = \frac{E_0}{\sqrt{2}} \cdot \frac{B_0}{\sqrt{2}} \cdot \frac{1}{\mu_0}$

$= \frac{E_{rms} \cdot B_{rms}}{\mu_0}$

Also; $\langle S \rangle = \frac{E_0^2}{2\mu_0 c} \quad \left[\because E_0/B_0 = c\right]$

$= \frac{E_{rms}^2}{\mu_0 c}$

$= \frac{B_{rms}^2}{\mu_0} \cdot c$

$I_{av} = \frac{c B_0 B_0}{2\mu_0}$

$= \frac{B_0^2 c}{2\mu_0}$

$\therefore I_{av} = \langle S \rangle = P/A$ where P = Power, A = Area.

. The maximum electric field 10m from an isotropic point source of light is 2V/m. What is the max^m value of magnetic field, average intensity and power of source?

solution.

Given: Electric field $(E_0) = 2V/m$

$I_{av} = \langle S \rangle = \frac{E_0^2}{2\mu_0 c} = \frac{2^2}{2 \times 4\pi \times 10^{-7} \times 3 \times 10^8} = 0.0053 W/m^2.$

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$I_{av} = \langle S \rangle = \frac{P}{A}$ Also, $B_0 = \frac{E_0}{c} = \frac{0.0053}{3\times10^8} = 6.667\times10^{-3}$ T.

or, $0.0053 = \frac{P}{4\pi\times10^2}$

$\therefore P = 6.660$ W.

Q. Calculate the maximum electric and magnetic field if solar constant is 1.4 kW/m²

Solution;

$I_{av} = \frac{1.4 kW}{m^2} = 1.4\times10^3$ W/m².

$\therefore$ $I_{av} = \frac{E_0^2}{2\mu_0 c}$.

$1.4\times10^3 = \frac{E_0^2}{2\times4\pi\times10^{-7}\times3\times10^8}$

$\therefore E_0 = \sqrt{1.06\times10^6} = 1.03\times10^3$ V

$I_{av} = \frac{B_0^2 c}{2\mu_0}$

$B_0^2 = \frac{1.4\times10^3\times2\times4\pi\times10^{-7}}{3\times10^8}$

$\therefore B_0 = 3.42\times10^{-6}$ T

1 mile = 1852m.

Q. What is the average magnitude of poynting vector 5 mile from the radio transmitter broad casting isotropically with average power 250 kW?

Soln.

Av. Power $\langle P \rangle = 250\,kW = 250\times10^3\,W$. $\quad r = 5\,mile = 5\times1852 = 9.26\times10^3\,m$

$$I_{av} = \langle S \rangle = \frac{P}{A} = \frac{250\times10^3}{4\pi\times(9.26\times10^3)^2}$$

$$= 2.32\times10^{-4}\,W/m^2.$$

Radiation pressure.

Q. If a intensity of direct sunlight at a point on the surface of earth is $0.85\,kW/m^2$. Calculate the momentum density and radiation pressure (rate of flow of momentum)?

Soln

Given; $I = 0.85\,kW/m^2 = 850\,W/m^2$.

Radiation pressure $(P_r) = I/c$

$$= \frac{850}{3\times10^8}$$

$$= 2.83\times10^{-6}\,pa.$$

Momentum density $= \frac{I}{c^2}$

$$= \frac{850}{(3\times10^8)^2}$$

$$= 9.44\times10^{-15}\,Pa/m^3.$$

$$\frac{\Delta P}{V} = \frac{\Delta E}{C.A.l}$$

$$= \frac{\Delta E}{\Delta t \cdot c \cdot \frac{Al}{\Delta t}}$$

$$= \frac{P}{CAC}$$

$$= I/c^2$$

$W/m^2 \quad \frac{W/m^2}{(m/s)^2} \quad \frac{W}{m^2}\times\frac{s^2}{m^2} \quad \frac{Ws^2}{m^4}$.

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Q. A laser beam having power 10mW and wave length 680nm. The laser beam is focused until its diameter is equal to the ~~a~~ on a sphere. If the sphere has density 5×10^7 kg/m^3. What is the beam intensity at the sphere, the radiation pressure on the sphere. Corresponding force and magnitude of acceleration

Sol^n

$$I = \frac{P}{A} = \frac{10\times10^6}{\pi\times(680\times10^{-9})^2}$$

$$= 6.88\times10^9 \, W/m^2$$

$$\boxed{I = P/A = \frac{P}{4\pi r^2} \quad P_r = I/c \text{ (radiation press)}}$$

Radiation pressure $(P_r) = I/c$

$$= \frac{6.88\times10^9}{3\times10^8}$$

$$= 2.29\times10$$

$I = I/A$

Electromagnetism.

Magnetic field intensity (B) = $\mu_0(H+M)$ and its unit is Tesla.

Magnetic flux:-

The total number of imagninary lines of Magnetic field in a surface is called magnetic flux. Let 'B' be the magnetic field intensity then the magnetic flux

$$\phi_B = BA = \oint \vec{B} \cdot d\vec{A}$$ in any closed surface.

In a closed surface the total no. of incoming magnetic field lines is equal to the total number of outgoing magnetic field lines such that

$$\oint \vec{B} \cdot d\vec{A} = 0$$ which is called guass law of magnetism It states that, "there is no existense of magnetic monopole".

Magnetic field and force:-

Let us consider a charged particle having charge 'q' is in dynamic motion. When magnetic field 'B' is applied on a charged particle Moving with velocity 'v' then the magnetic force experienced by charged particle is

$F_B \propto B$

$F_B \propto q$

$F_B \propto v$

$F_B \propto \sin\theta$

Then, $F_B = Bqv\sin\theta$.

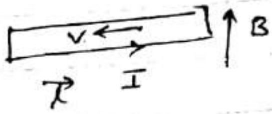
$= q(\vec{v} \times \vec{B})$

$F_e = q\vec{E}$

$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ which is a lorentz force

Magnitic force on a current carrying conductor.

Consider a conductor having length l and area A. Let N be the ~~total~~ total number of electron flowing in a conductor and I be the current flowing in the conductor Then the ~~mag~~ charge is given by



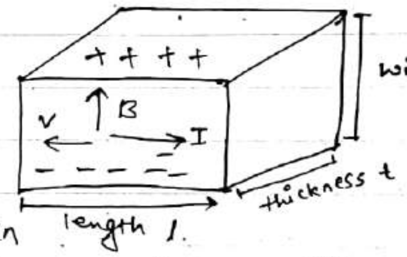
$$q = Ne = nA \times le$$

Then the magnetic force is given by

$$F_B = Bqv\sin\theta$$
$$= BnAlev\sin\theta$$
$$= IBl\sin\theta \quad [\because I = Ven A]$$
$$\therefore \vec{F_B} = I(\vec{l} \times \vec{B})$$
$$\therefore d\vec{F_B} = I(d\vec{l} \times \vec{B})$$ (for elementary length dL)

Hall effect.

Let us ~~take~~ take a conducting stripe having length 'l', width 'd' and thickness 't' as shown in figure. The current I is flowing along lengthwise on x-direction. When magnetic field is applied along z direction that is $\perp^{er}$ to thickness then a voltage is develope across width or along y-direction which is called Hall potential. It is called Hall effect in which hall potential is $\perp^{er}$ to the both 'B' and 'I' at steady state.



The magnetic force enpereince by a change partical is equal to electric force enperience by charge particle.

i.e $$eE_H = Bev$$

where E_H is Hall fold.

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i.e. $\frac{eV_H}{d} = Bev$

$\Rightarrow V_H = Bvd$

or, $V_H = B \cdot \frac{I}{neA} d$

or, $V_H = \frac{BId}{ne\,t \times d}$

$\therefore V_H = \frac{BI}{net}$

* Hall coefficient.

$$R_H = \frac{E_H}{J_x B} = \frac{BV}{J_x B} = \frac{BV}{nevB} = \frac{1}{ne}$$

* Hall Mobility

$$\mu_H = \left|\frac{v}{E}\right| = \frac{J}{neE} = \frac{\sigma}{ne} = R_H \sigma$$

* Hall angle.

$$\theta = \frac{J_y}{J_x} = \frac{\sigma E_y}{J_x} = \frac{\sigma E_H}{J_x}$$

$$= \sigma \frac{E_H}{J_x B} \cdot B$$

$$= \sigma R_H B$$

Hall Resistance

We have, $V_H = \frac{BI}{net}$

$\Rightarrow V_H = IR$ where $R = \frac{B}{net}$

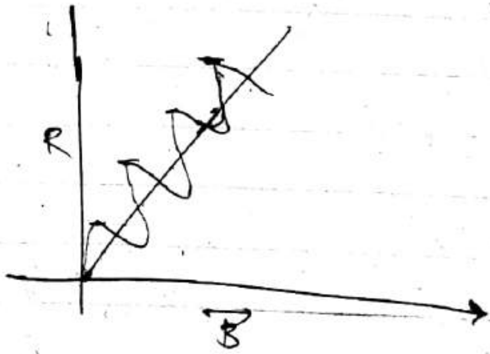

Fig: Hall resistance quantized.

n – charge carrier density $n = \frac{\text{Total charge}}{\text{volume}}$

Q. In a hall experiment a current of 3A sent lengthwise through a conductor of 1cm width, 4cm length and 1μm thick produces a transverse hall voltage of 10μV when a magnetic field of 1.5T is passed perpendicular through the thickness of conductor. Calculate drift velocity, charge carries density, hall field, hall coefficient and hall resistance.

Solution;

Given, $I = 3A$, $l = 4cm = 4\times10^{-2}m$

$d = 1cm = 1\times10^{-2}m$

$t = 1\mu m = 1\times10^{-6}m$

$V_H = 10\mu V = 10\times10^{-6}V$.

$B = 1.5T$

$eE_H = Be\vartheta$

or, $e.\frac{V_H}{d} = Be\vartheta$

$\therefore V_H = B\vartheta d$

Now, $V_H = B v d$

or, $10\times10^{-6} = 1.5\times v\times1\times10^{-2}$

$\therefore v = 6.66\times10^{-4} m/s$.

Now;

$E_H = BV = 1.5\times6.66\times10^{-4} = 10^{-4}V$.

Hall coeff $(R_H) = \frac{1}{ne} = \frac{1}{\left(\frac{BI}{V_Het}\right)\cdot e} = \frac{10\times10^{-6}\times1\times10^{-6}}{1.5\times3}$

$= 2.22\times10^{-12}$.

Hall resistance $(R) = \frac{B}{net} = \frac{1.5}{2.81\times10^{30}\times1.6\times10^{-19}\times1\times10^{-6}} = 3.33\times10^{-6}$

$n = \frac{BI}{V_Het} = \frac{1.5\times3}{10\times10^{-6}\times e\times1\times10^{-6}} = 2.81\times10^{30}/m^3$.

(2) A copper strip having 2.5cm width, 1.5mm thick is placed in a magnetic field of 2T per to the plane of the stripe. If a current of 1.5A is set up into the stripe. Calculate the hall potential difference across a stripe.

Given

$d = 2.5cm = 2.5 \times 10^{-2} m$

$t = 1.5mm = 1.5 \times 10^{-3} m$

$B = 2T$.

$I = 1.5A$

Sol:. $V_H = ?$.

$e E_H = B e \vartheta$

$\cancel{e} \cdot \frac{V_H}{d} = B \cdot \cancel{e} \vartheta$

$\therefore V_H = B \vartheta d$

$V_H = \frac{BI}{net}$.

$A = \pi r^2$

$= \pi \times \left(\frac{1.5 \times 10^{-3}}{2} \right)$.

$I = ven A$

Maximum K.E of particle is

$$(K.E)_{max} = \frac{1}{2} m V_{max}^2$$

$$= \frac{1}{2} \cancel{m} \cdot \frac{B^2 q^2 r_{max}^2}{m^2}$$

$$= \frac{B^2 q^2 r_{max}^2}{2m}$$

$$= \frac{B^2 q^2}{4\pi^2 m^2} \cdot 2\pi^2 m \cdot r_{max}^2$$

$$\therefore K.E_{max} = 2\pi^2 m f^2 r_{max}^2$$

We have:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Due to the constant magnetic field and has limitation on size of Dee's we use synchroton to obtain high energy particles in $f = f_0$ and m be the mass then to obtain the resonance form is constant in sychrotism.

). Suppose a cyclotron is operated at an frequency of 15MHz of ~~a~~ a Dee of radius 55cm. What is the resulting k.E of nuclear? What is the magnetic field needed for the deuterium? $(m_D) = 3.34 \times 10^{-27}$ kg.

Sol^n.

Frequency (f) = 15MHz = 15×10^6 Hz

Radius (r) = 55cm = 55×10^{-2} m.

Now; $K.E_{max} = 2\pi^2 m f^2 r_{max}^2$

$$= 2\pi^2 \times 3.34 \times 10^{-27} \times (15 \times 10^6)^2 \times (55 \times 10^{-2})^2$$

$$= 4.47 \times 10^{-12} J.$$

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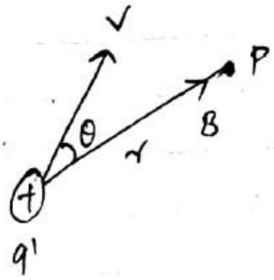
Biot-Savart Law.

$$B = \frac{qv\sin\theta}{r^2}$$

$$B = \frac{qv\sin\theta . r}{r^3}$$



$$\therefore \vec{B} = \frac{q(\vec{v}\times\vec{r})}{|\vec{r^3}|}$$

Let us take a charged particle having charge q moving with velocity v then the magnetic field due to the charged particle at point P which is at distance 'r' from charged particle is given by

$$B \propto q$$
$$B \propto v$$
$$B \propto 1/r^2$$
$$B \propto \sin\theta$$

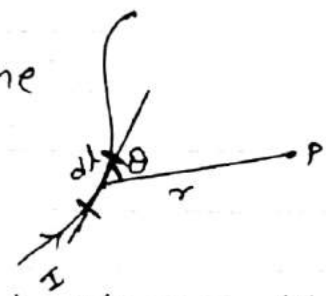
$$\therefore B = \frac{qv\sin\theta}{r^2}\frac{\mu_0}{4\pi}$$

Magnetic field due to current carrying element.

Let us take a conductor carrying current I, let dl be the segment of the conductor then the charge on the segment is

$$= Nq = nAdl\,q$$

where A is the cross sectional area.



Then the magnetic field due to the current element dl is

$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{nAdlqv\sin\theta}{r^2}$$

$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2}$$

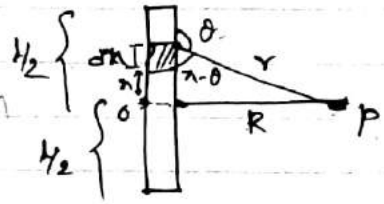
$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$$

$$\vec{B} = \int \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$$

Application of Biot-Savart law.

(1) Magnetic field due to a conductor having length l.

Let us consider a conductor having length l carrying current I as shown in figure. Let 'dx' be the small length element. Then the magnetic field due to small length element is



$$dB = \frac{\mu_0}{4\pi} \frac{I\, dx \sin\theta}{r^2}$$

$$\sin\theta = \sin(\pi - \theta) = \frac{R}{\sqrt{R^2 + x^2}}$$

Then,

Total magnetic field $\displaystyle B = 2\int_0^{l/2} \frac{\mu_0}{4\pi} I\, d\vec{x} \cdot \frac{R}{\sqrt{R^2+x^2}} \cdot \frac{1}{r^2}$

$$= 2\int_0^{l/2} \frac{\mu_0}{4\pi} I \frac{d\vec{x}\; R}{(R^2+x^2)^{3/2}}$$

$$= 2\frac{\mu_0}{4\pi} I R \int_0^{l/2} \frac{dx}{(R^2+x^2)^{3/2}}$$

$$= \frac{2\mu_0 I R}{4\pi} \cdot \left[\int_0^{l/2} (R^2+x^2)^{-3/2} \right]$$

Put $x = R \tan\alpha$

$dx = R\sec^2\alpha\, d\alpha$

As $x \to 0$, $\alpha \to 0$

$x \to l/2$, $\alpha \to$

Put $x = R\tan\theta$

$dx = R\sec^2\theta \, d\theta$.

$$\vec{B} = \frac{\mu_0 I}{2\pi} R \int_0^{l/2} (R^2+x^2)^{-3/2} \, dx$$

$$\int (R^2+x^2)^{-3/2} \, dx$$

$$= \int \frac{1}{(R^2+x^2)^{3/2}} \, dx$$

$$= \int \frac{1}{(R^2+R^2\tan^2\theta)^{3/2}} \, dx$$

$$= \int \frac{1}{R^3\sec^3\theta} \, dx$$

$$= \frac{1}{R^3}\int \cos^3\theta \; R\sec^2\theta \, d\theta$$

$$= \frac{1}{R^2}\int \cos\theta \, d\theta$$

$$= \frac{1}{R^2}\sin\theta$$

$$= \frac{\sin\theta}{R^2}$$

$$= \frac{1}{R^2}\left[\frac{R}{\sqrt{R^2+x^2}}\right]_0^{l/2}$$

$$= \frac{1}{R}\left[\frac{1}{\sqrt{R^2+(l/2)^2}} - \frac{1}{\sqrt{R^2}}\right]$$

$$= \frac{1}{R}\left[\frac{2}{\sqrt{4R^2+l^2}} - \frac{1}{R}\right]$$

$$= \frac{2}{R\sqrt{4R^2+l^2}} - \frac{1}{R^2}$$

At the centre of the coil $x=0$ such that magnetic field at centre. $B=\frac{\mu_0 I}{2R}$. If there are N coil having same radius then the M.F due to N coils is

$$B_N = \cancel{\frac{\mu_0 N I}{2R}} \; \frac{\mu_0 N I R^2}{2(x^2+R^2)^{3/2}}$$

Magnetic dipole moment

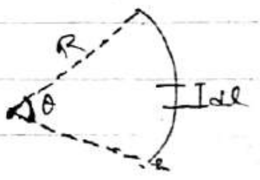
$$M_d = NIA$$

$$\therefore B_x = \frac{\mu_0 N I \pi R^2}{2(x^2+R^2)^{3/2} \cdot \pi}$$

$$\therefore B_x = \frac{\mu_0 M_d}{2\pi (x^2+R^2)^{3/2}}.$$

(3) Magnetic field due to circular arc.

Let us consider a circular arc having radius 'R'. Let 'dl' be the small length element. Then magnetic field due to small length element is



$$dB = \frac{\mu_0}{4\pi} \frac{I\,dl}{R^2}$$

Total magnetic field

$$B = \int dB = \frac{\mu_0 I}{4\pi R^2} \int_0^{\pi/2} dl$$

$$= \frac{\mu_0 I}{4\pi R^2} \int_0^{\pi/2} R\,d\theta \quad (\because dl = R\,d\theta)$$

$$= \frac{\mu_0 I}{4\pi R} \cdot \pi/2$$

$$B = \frac{\mu_0 I}{8R}$$

Ampere's law.

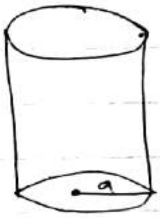
The magnetic field produce by steady current flowing in a closed loop which is symmetrical in nature is given by

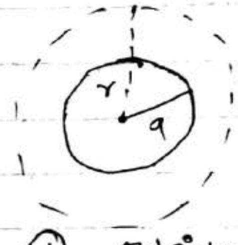
$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I$$

Application.

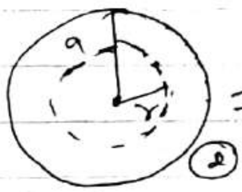
→ Calculate the magnetic field inside and outside a current carrying wire.

Let us take a wire having radius 'a' carrying current I then the magnetic field outside the wire $(r>a)$.. We consider a Amperian loop at a distance of r from the centre. Then





① outside.



Inside ②

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I \quad \text{for } (r>a)$$

$$\text{or } B \cdot 2\pi r = \mu_0 I$$

$$\therefore B = \frac{\mu_0 I}{2\pi r}$$

$\frac{I}{A} = J'/A$

$\frac{I}{\pi a^2} = \frac{I'}{\pi r^2}$

$I' = r^2/a^2 \, I$

$\oint \vec{B} \cdot \vec{dl} = \mu_0 I'$

$\vec{B} \cdot 2\pi r = \mu_0 I'$

$\vec{B} = \mu_0$

Case II : Inside $r<a$

$\oint \vec{B} \cdot \vec{dl} = \mu_0 I'$. [$I'$ is the current enclosed by Amperian's loop in fig ①]

Now, we have the current density J is constant everywhere.

$$J = \frac{I}{A} = \frac{I'}{A'}$$

$$\Rightarrow \frac{I}{\pi a^2} = \frac{I'}{\pi r^2}$$

$$\therefore I' = \frac{r^2 I}{a^2}$$

$$\therefore 2\pi r B = \mu_0 \frac{r^2}{a^2} I \quad \therefore B_{inside} = \frac{\mu_0 I r}{2\pi a^2}.$$

$$\therefore \quad \mathcal{E}_{ind} = -\frac{\partial \phi_B}{\partial t}$$

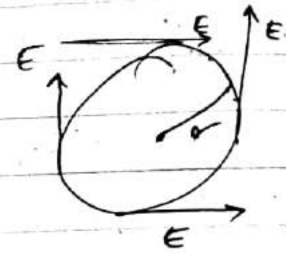
Induced electric field.

Let us take a conducting loop having charge 'q' which is moving along the direction of electric field then the charged particle of the loops experience the electric ~~force~~ force is

$$\vec{F} = q\vec{E}$$

From Faraday's law of induction we have,

$$\mathcal{E}_{ind} = \frac{-\partial \phi_B}{\partial t}$$



The work done by the electric force is

$$W = F \cdot 2\pi r$$

$$W = qE \cdot 2\pi r \quad \text{—— (i)}$$

Workdone by induced emf is

$$W = q\,\mathcal{E}_{ind} \quad \text{—— (ii)}$$

$$\therefore \mathcal{E}_{in} = E\,2\pi r \quad \text{(on equating (i) \& (ii))}$$

$$-\frac{\partial \phi_B}{\partial t} = \oint \vec{E} \cdot \vec{dl}$$

Using Stoke's theorem

$$\oint_l \vec{E} \cdot \vec{dl} = \oint (\nabla \times \vec{E}) \cdot \vec{dA}$$

$$\text{and} \quad \phi_B = \oint \vec{B} \cdot \vec{dA}$$

Now,

$$-\frac{\partial \phi_B}{\partial t} = \oint (\nabla \times \vec{E}) \cdot \vec{dA}$$

$$\text{or} \quad -\frac{\partial (\oint \vec{B} \cdot \vec{dA})}{\partial t} = \oint (\nabla \times \vec{E}) \cdot \vec{dA}$$

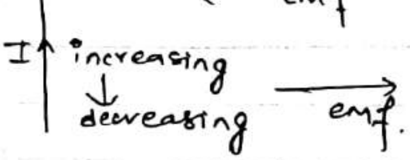
$$\text{or} \quad -\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E}$$

$$\therefore \boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

which shows that the variation in magnetic field on conducting loop gives the induced electric field.

Lenz's law.

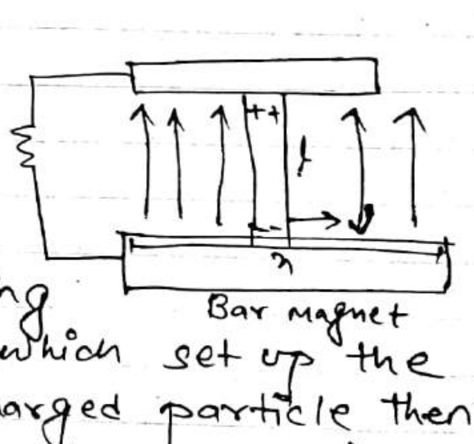
The induced current flows always in such direction as to oppose the change causing it.



Induction and Energy transformation.

let us take a conductor having length 'l' is move on a magnetic field $\vec{B}$ then the magnetic force enperience by the charged particle is

$$\vec{F} = q(\vec{v} \times \vec{B}) \quad \text{--- (i)}$$



The magnetic force is $\perp^{er}$ to both velocity and $\vec{B}$. Such that the electrons of the conductor accumulated at lower surface leaving positive charge on upper surface which set up the potential difference between charged particle then the electric force enperience by charged particle is

$$\vec{F} = q\vec{E} \quad \text{--- (ii)}$$

Taking magnitude and equating (i) & (ii)

$$E = vB$$

Now the potential difference

$$V = El$$

$$\text{or, } V = Bvl \quad \text{--- (iii)}$$

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$-R$

and from ~~figure~~ induced emf
$$\varepsilon_{ind} = -\frac{\partial \phi_B}{\partial t} \quad \text{where} \quad \phi_B = B.A = Bl x$$

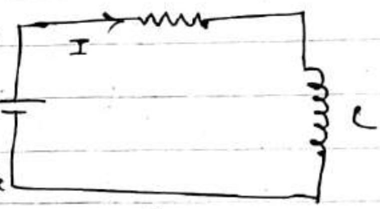
$$\therefore \varepsilon_{ind} = -Bl\frac{dx}{dt} = -Bl v \quad \text{— (iv)}$$

from (iii) & (iv) we get the mechanical power is equal to the electrical power.
$$P = \frac{B^2 l^2 v^2}{R}$$

i.e Mechanical Energy → Electrical energy → Thermal energy

Self Inductance

An induced emf appears in a coil if we change the current in the coil is called self induction.



According to Henry; the total magnetic flux on the coil is directly proportional to the current flowing in the coil. Let 'N' be the number of turns on the coil then
$$N\phi_B \propto I$$
$$\text{(1)} — N\phi_B = LI \quad \text{where } L \text{ is the self inductance}$$

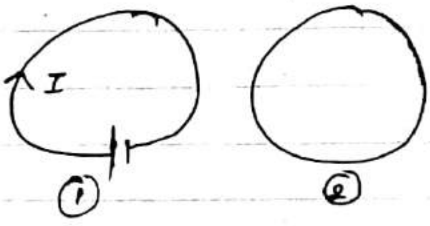
Differentiating (1)
$$N\frac{d\phi_B}{dt} = L\frac{dI}{dt}$$

we have, $\varepsilon_{ind} = -\frac{N\partial\phi_B}{\partial t} = -L\frac{dI}{dt}$,,

Mutual Inductance.

The phenomenan of producing an induced emf in a circuit due to variation of current in a neighbouring circuit is called Mutual Inductance.

Let us take a two current loop ① & ② the current I_1 is flowing on loop ① as shown in figure. Let B_1 be the magnetic field produced at loop ② then the magnetic flux on loop ② due to loop ① is



$$\phi_{21} = \oint \vec{B_1} \cdot \vec{ds_2}$$

$$\phi_{21} = M_{21} I_1$$

Now

$$-\frac{\partial \phi_{21}}{\partial t} = -M_{21} \frac{\partial I_1}{\partial t}$$

$$\therefore \quad E_{21} = -M_{21} \frac{dI_1}{\partial t}.$$

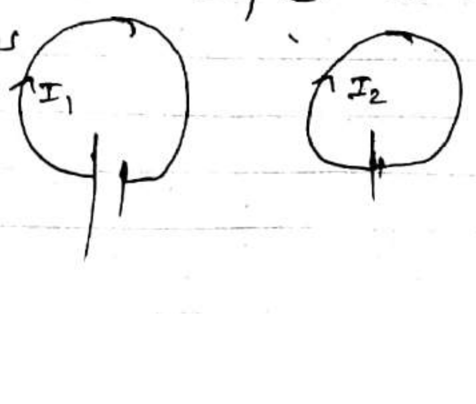
Similarly if current I_2 is flowing on loop② then the induced emf on loop① can be written as

$$E_{12} = -M_{12} \frac{\partial I_2}{\partial t}.$$

Relationship betn Mutual inductance and self inductance

Let us take a two current loops in which current I_1 is flowing in loop① and I_2 is flowing in loop② having self inductance of loop① is L_1 and loop② is L_2

Then Let N_1 be the no. of turns on loop ① and N_2 be the no. of turns on loop ②



$$N_1 \phi_1 = L_1 I_1$$

$$N_2 \phi_2 = L_2 I_2$$

and $N_2 \phi_{21} = M_{21} I_1$
$N_1 \phi_{12} = M_{12} I_2$

$$\frac{N_1 N_2 \phi_{21} \phi_{12}}{I_1 I_2} = M_{12} M_{21}$$

Since $\phi_{21} = \phi_1$ and $\phi_{12} = \phi_2$

With $M_{12} = M_{21} = M$

Then $M^2 = \frac{L_1 I_1 L_2 I_2}{I_1 I_2}$

$\therefore M = \sqrt{L_1 L_2}$

Q. Calculate the self inductance of solenoid and toroid.

Now, the magnetic field on the axis of solenoid having length l, number of turns 'N' and current flowing on solenoid is I the magnetic field B is

$$B = \frac{\mu_0 N I}{l}$$

Let A be the cross sectional area of solenoid then

$$\phi_B = BA = \frac{\mu_0 N I A}{l}$$

$$\text{Total flux} = \phi_{B_{total}} = N\phi_B$$

$$\phi_{B_{total}} = \frac{\mu_0 N^2 I A}{l}$$

$$\mathcal{E}_{ind} = \frac{-\partial \phi_B}{\partial t} = -L\frac{dI}{dt}$$

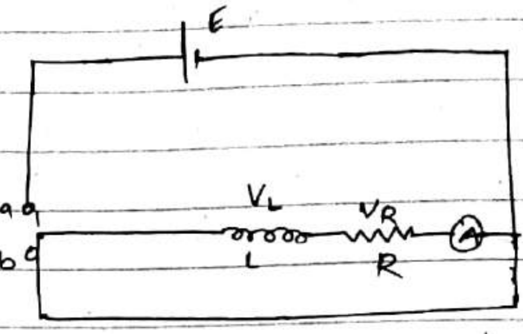
$$\text{or, } \frac{-\partial}{\partial t}\left(\frac{\mu_0 N^2 I A}{l}\right) = -L\frac{dI}{dt}$$

$$\text{or, } \frac{\mu_0 N^2 A}{l} \frac{-\partial I}{\partial t} = -L\frac{dI}{dt}$$

VVI

Series L-R circuit.

Let us take a battery having emf (E), inductor having inductance 'L', Resistor having resistor R are connected in series with ammeter A as shown in figure.



Growing Current:

When terminal a is connected then the current flow through the inductor and resistor, then at instant of time 't' using Kirchoff's law we can write

$$E = V_L + V_R$$

or, $\frac{I_0 R}{} = L\frac{dI}{dt} + IR$ $\qquad L\frac{dI}{dt} = I_0/_R - IR$

$(I_0 - I)R$

$I_0R - IR$

or, $\frac{dI}{dt} = (I_0 - I)\frac{R}{L}$

or, $\frac{dI}{dt} = -(I - I_0)\frac{R}{L}$

or, $\int_0^I \frac{dI}{(I - I_0)} = -\frac{R}{L}\int_0^t dt$

or, $\ln(I - I_0)\Big|_0^I = -\frac{R}{L}\, t\Big|_0^t$

or, $\ln(I - I_0) - \ln(-I_0) = -\frac{R}{L}t$

or, $\ln\left(\frac{I - I_0}{-I_0}\right) = -\frac{R}{L}t$

or, $\frac{I - I_0}{-I_0} = e^{-R/L\, t}$

or, $I = I_0(1 - e^{-R/L\, t})$ which is the growing current in LR circuit

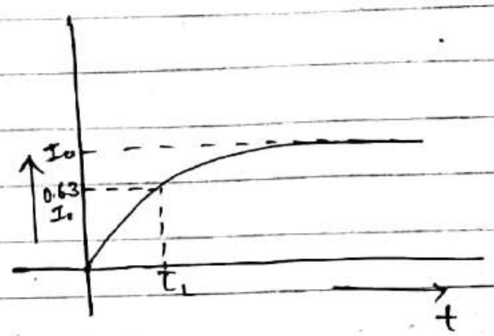
Let $\tau_L = L/R$ is the inductive time constant.

At $t = \tau_L$

$I = 0.63 I_0$

When the current reaches to 63% of final or equillibrium current then the inductive time constant

$\tau_L = t$



Decay current in LR circuit.

When terminal b is connected and a is open then the current starts to decay. Using kirchoff's law:

$$V_L + V_R = 0$$

or, $$L\frac{dI}{dt} + IR = 0$$

or, $$\frac{dI}{I} = -\frac{R}{L}\,dt.$$

or, $$\int_{I_0}^{I} \frac{dI}{I} = -\frac{R}{L}\int_0^t dt$$

or, $$\ln I \Big|_{I_0}^{I} = -\frac{R}{L}t$$

or, $$\ln\left(\frac{I}{I_0}\right) = -\frac{R}{L}t$$

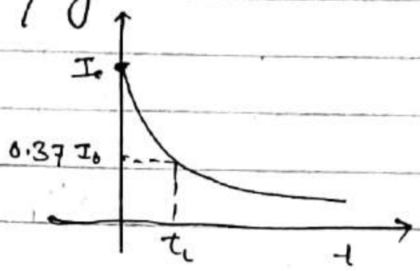
$$\therefore \quad I = I_0 e^{-R/L\,t}$$

$$\therefore \quad I = I_0 e^{-t/\tau_L}$$

Ans $t = \tau_L$

$\therefore I = 0.37 I_0$

The time required to reach the current of 37% of initial current is called inductive time constant in terms of decaying current.



① A solenoid having an inductance of 0.6 μH is connected in series with 1.2 kΩ resistor. If a ~~cell~~ 14V battery is connected across the pair. How long will it take for the current to reach 80% of equillibrium value?, What is the current through the resistor at time $t = \tau_L$.

Solution;

Here; $L = 0.6\,\mu H = 0.6 \times 10^{-6}\,H$

$R = 1.2\,k\Omega = 1.2 \times 10^{3}\,\Omega$

$E = 14\,V.$

We know

$$I = I_0\left(1 - e^{-R/L\, t}\right)$$

or, 80% of $I_0 = I_0\left(1 - e^{-R/L\, t}\right)$

or, $\frac{4}{5} I_0 = I_0\left(1 - e^{-R/L\, t}\right)$

or, $\frac{4}{5} = 1 - e^{-R/L\, t}$

or, $e^{-R/L\, t} = 1/5$

$\therefore -R/L\, t = \ln(1/5)$ $\quad \therefore t = -\frac{0.6 \times 10^{-6} \ln(1/5)}{1.2 \times 10^{3}} = 8.4 \times 10^{-9}$ sec

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$I = I_0 (1 - e^{-R/L t})$

At $t = \tau_L$

Energy stored in a Magnetic field.

Let us take an inductor having inductance L is connected to a generator which terminal voltage can be varied when the current is increasing. Induced emf is

$E_{ind} = -L \frac{dI}{dt}$ appears across the inductor. The

generator must do work to push the charges through the inductor against induced emf. The rate at which work done against the emf is:

$P = IE$

Now Workdone

$W = U_B = \int_0^I P dt = \int_0^I I L \frac{dI}{dt} \cdot dt = \frac{1}{2} L I^2$

which is the energy stored in inductor having inductance

Let us take a solenoid having length 'l' and area 'A' with self inductance 'L' then the magnetic energy density is given by

$$\frac{U_B}{Volume} = \frac{\frac{1}{2}LI^2}{A \times l}$$

We have the self inductance of solenoid.

$$L = \frac{\mu_0 N^2 A}{l}$$

Then $$U_B/V = \frac{1}{2} \times \frac{\mu_0 N^2 A}{l} \times I^2 \times \frac{1}{A \times l}$$

$$= \frac{\mu_0 I^2 N^2 \cancel{A}}{2 \cancel{A} l^2}$$

$$= \frac{\mu_0 N^2 I^2}{2 l^2}$$

Now, the magnetic field of solenoid is $B = \frac{\mu_0 N I}{l}$

$$\therefore \frac{U_B}{Volume} = \frac{B^2}{2\mu_0}$$ ✓