Introduction

I have folded origami in my spare time since 5th grade and one of my favorite structures has been the sonobe unit. Each sonobe unit consists of two flaps and two pockets (Fig. 1). By combining three units a triangular pyramid can be made (Fig. 2)[2]. These pyramids can then be arranged in different manners to create more complex geometric solids.

While playing with the units recently I built a structure consisting of a combination of two sets of six triangular pyramids meeting at a central point. Together, the equilateral bases of the pyramids formed a hexagon (Fig. 3).

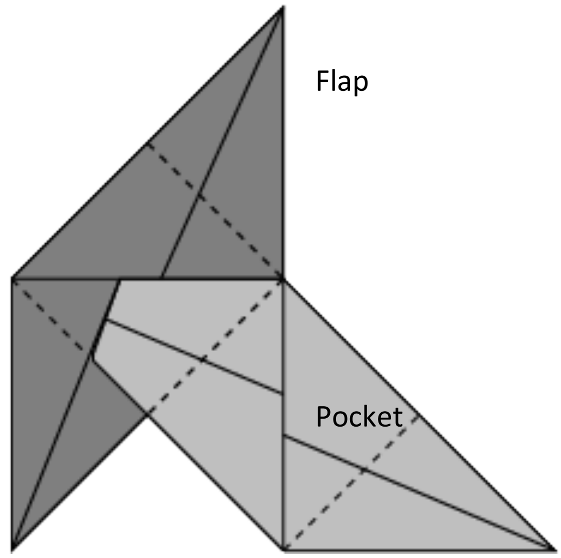


Figure 1[1]
A composition of two sonobe units showing the locations of flap and pocket

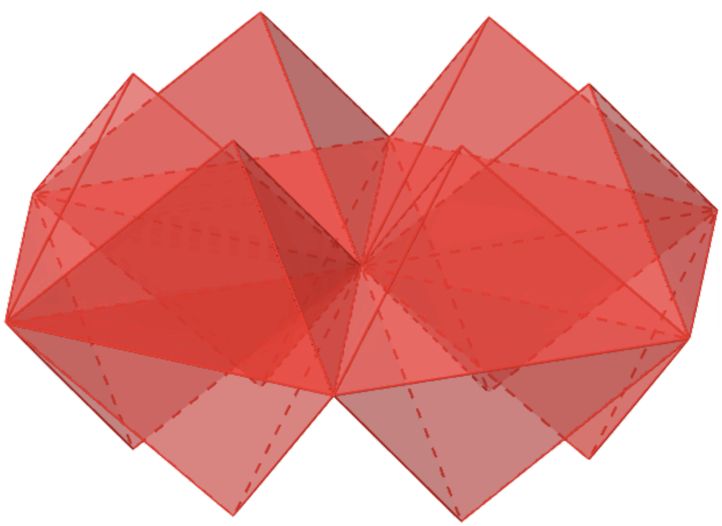


Figure 3
A structure formed from a combination of two sets of six triangular pyramids based upon sonobe units

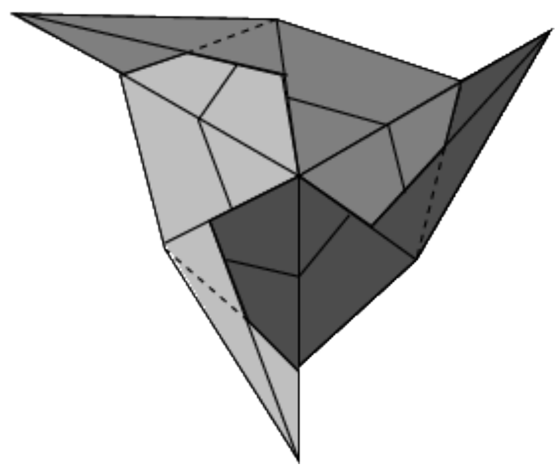


Figure 2[2]
A composition of three sonobe units to form a triangular pyramid

[1] "Triangular flaps of one module fit into pockets of another module." Digital image. Sonobe Origami Polyhedra – Activity Directions. Accessed January 18, 2017.
[2] A half completed sonobe hexahedron. Digital image. Sonobe Origami Polyhedra – Activity Directions. Accessed January 18, 2017. https://riverbendmath.org/modules/Origami/Sonobe_Polyhedra/Activity_Directions/Sonobe9F.png.

I was intrigued by the geometry of the figure and realized that similar solids could be generated with other regular polygons as bases. For a triangular base, the figure would consist of six triangular pyramids. The bases of the pyramids would be 30-30-120 isosceles triangles (Fig. 4). For a square base, the figure would consist of eight triangular pyramids. The bases of the pyramids would be 45-45-90 isosceles triangles (Fig. 5). Any polygon with n sides, where $n \in \mathbb{N}$ and $n \geq 3$, could give rise to a solid consisting of two sets of n triangular pyramids with isosceles bases.

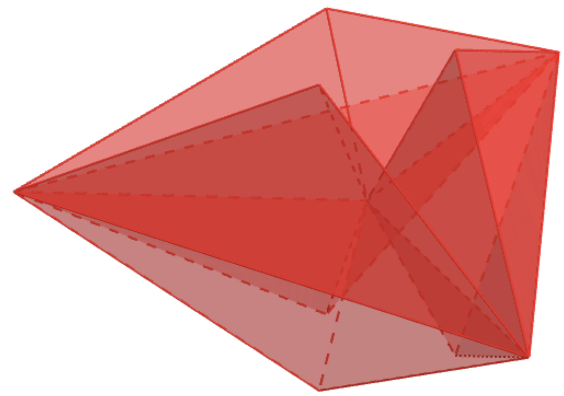


Figure 4
A structure formed from the composition of two sets of three triangular pyramids based on an equilateral triangle

Considering the shapes, I found it of interest to investigate their change in volume as n increases and eventually approaches ∞. In order to compare their volumes, each figure's polygonal base can be inscribed in a circle of radius r where $r \in \mathbb{R}^+$. For a constant r, increasing n would pack more triangles (and thus pyramids) into the polygon and circle. In order for this to occur, the interior angle θ where $0 < \theta \leq 120^o$ would have to decrease (Fig. 6). For this investigation graphs and diagrams were developed using Autograph and Geogebra.

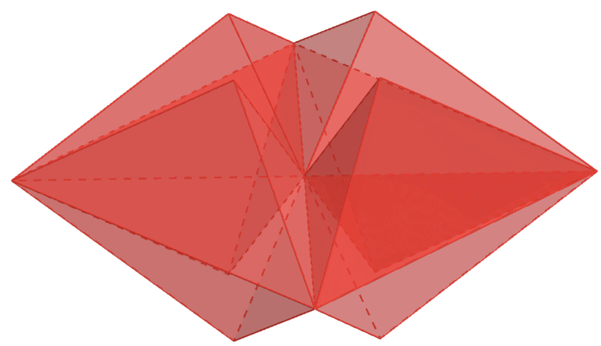
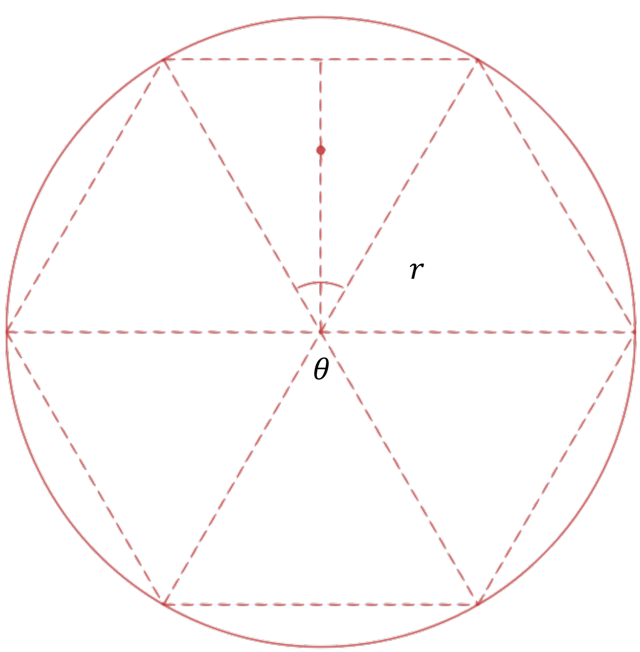


Figure 5
A structure formed from the composition of two sets of four triangular pyramids based on a square

Figure 6
Base formed by an inscribed regular hexagon



Volume of a Composition

I will first develop a formula for the total volume of any such n sided regular polygon based solid. For this investigation, diagrams will be based upon the hexagonal structure, as it was the stimulus. Each polygon with n sides is inscribed in a circle of radius r and then divided into n congruent, isosceles triangles (Fig. 6). The angle formed by the two sides of the triangle meeting at the origin (the central point of the figure for convenience sake) is given by θ. The total volume of the figure can be found by multiplying the volume of one of the pyramids by $2n$.

For a polygon with n sides, θ is given by the equation

$$\theta = \frac{2\pi}{n} \tag{1}$$

Since the triangles are isosceles the area of one triangle can be found using the following equation, considering that $\sin(\theta)$ is always positive over the domain of θ

$$A = \frac{1}{2}r^2 \sin(\theta) \tag{2}$$

The volume of a pyramid with base area A and height h is then

$$V = \frac{1}{3}Ah \tag{3}$$

Therefore the total volume V_T of any structure is

$$V_T = 2nV \tag{4}$$

When written in terms of n by substituting (1) in to (2), (2) in to (3), and (3) in to (4) the total volume V_T is

$$V_T = \frac{n}{3}r^2 \sin\left(\frac{2\pi}{n}\right)h \tag{5}$$

In order to develop values for the volumes of the various polygon-based figures, the height h and radius r will initially be set to one and two respectively, which was the measured ratio in the figure for the stimulus.

	Volume of figures based on n sided polygons						
Number of sides	3	4	5	6	9	12	15
Volume (3 sig. figs)	3.46	5.33	6.34	6.93	7.71	8.00	8.13

Table 1-Figure volume for a series of test values generated using (5)

As seen by the data from table 1, the volumes of the figures increase by smaller and smaller increments as the number of sides grows. This is doubly shown by the graph (Fig. 7) evidencing asymptotic behavior. Therefore it becomes of interest to investigate the value of the volume as n approaches infinity.

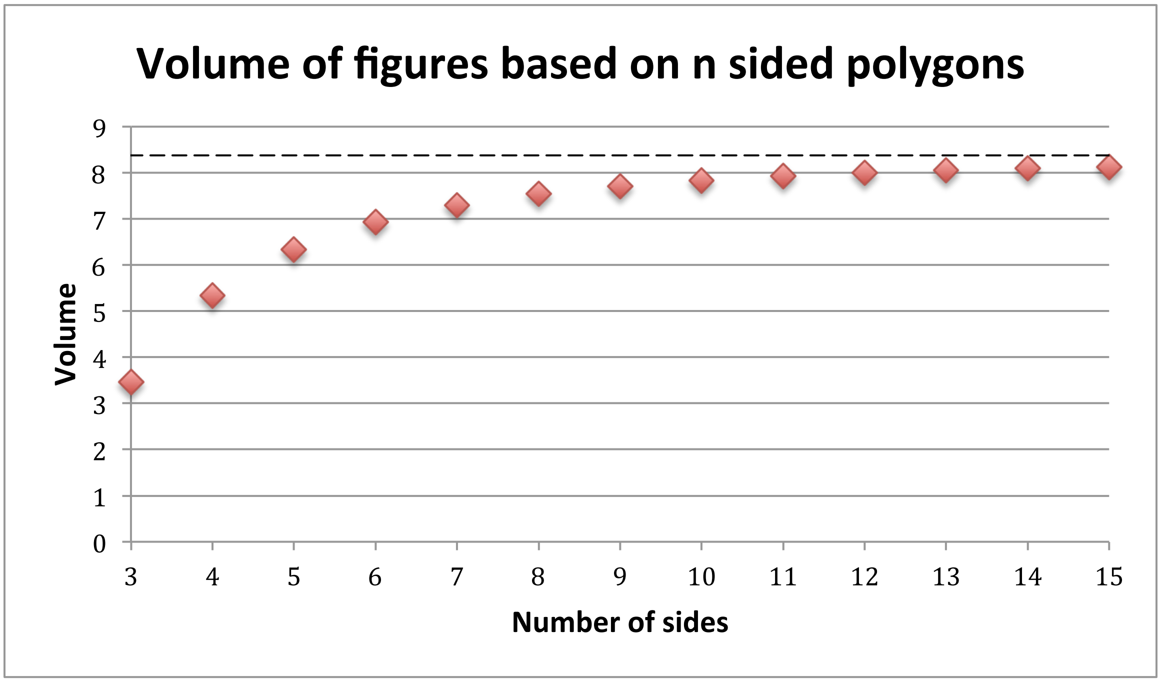


Figure 7
A graph of volume for figures with polygonal bases of varying numbers of sides. The asymptotic behavior is denoted by the dashed line

The volume V_{∞_1}, the limit of V_T as n approaches infinity, is equal to

$$V_{\infty_1} = \lim_{n\to\infty}\left(\frac{n}{3}r^2\sin\left(\frac{2\pi}{n}\right)h\right)$$

Since $\lim_{\frac{2\pi}{n}\to 0}\left(\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}\right) = 1$, V_{∞_1} can be rewritten and found to be equal to

$$V_{\infty_1} = \lim_{\frac{2\pi}{n} \to 0} \left(\frac{1}{3 \cdot \frac{1}{n}} \cdot \frac{2\pi}{2\pi} r^2 \sin\left(\frac{2\pi}{n}\right) h \right)$$

$$= \lim_{\frac{2\pi}{n} \to 0} \left(\frac{2\pi}{3} r^2 h \cdot \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \right)$$

$$= \frac{2\pi}{3} r^2 h$$

For a radius r of two and a height h of one as before, V_{∞_1} is equal to

$$V_{\infty_1} = \frac{8\pi}{3} \approx 8.378 \tag{6}$$

Validation of Limit Using Disk and Washer Rotation

As n approaches infinity, the pyramids become increasingly thin eventually resembling triangles. Therefore, it appears to be possible to verify the limit by finding the volume of revolution of a triangle in the x-z plane with base length r as before and height h where $h \in \mathbb{R}^+$ around the z-axis (Fig. 8).

In order to calculate the volume of revolution, the sides of the triangle must be represented by linear functions p and q. Where p passes through the origin and the apex and q passes through the apex and the intersection of r and the side opposite the origin (Fig. 5).

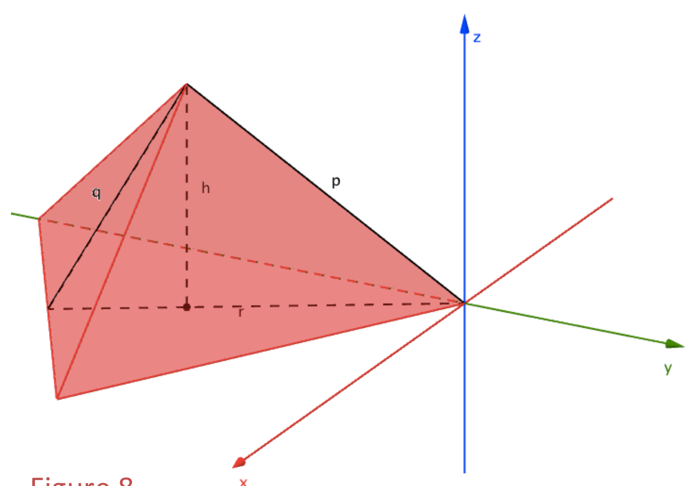


Figure 8
One of the pyramids from the original hexagon based structure and the triangle with base r and height h to be rotated around the z-axis

In the original figure I designed, the pyramid's apex was placed above the triangular base's center. For an equilateral triangle, all the triangular centers (circumcenter, incenter, centroid, and orthocenter) coincide but this is not the case for other triangles. The location along the base of the apex, however, appears to be inconsequential since the volume of a pyramid only depends on the value of the height not its position in accordance with the formula $V_{pyr} = \frac{1}{3}Bh$.

In order to have consistency with the stimulus, the apex will be placed above one of the triangular centers. As n approaches infinity, leading to thinner triangles, the positions of the centers move to certain limits. The centroid has a limit of $\frac{2}{3}r$, the circumcenter has a limit of $\frac{1}{2}r$, and both the orthocenter and incenter have limits of r. The reason behind this behavior is beyond the scope of the investigation. However, for the purpose of verifying the limit from (6), the apex will be placed above the incenter as it will always remain inside the triangle[3], which is consistent with the initial stimulus and the desired shape of the structure.

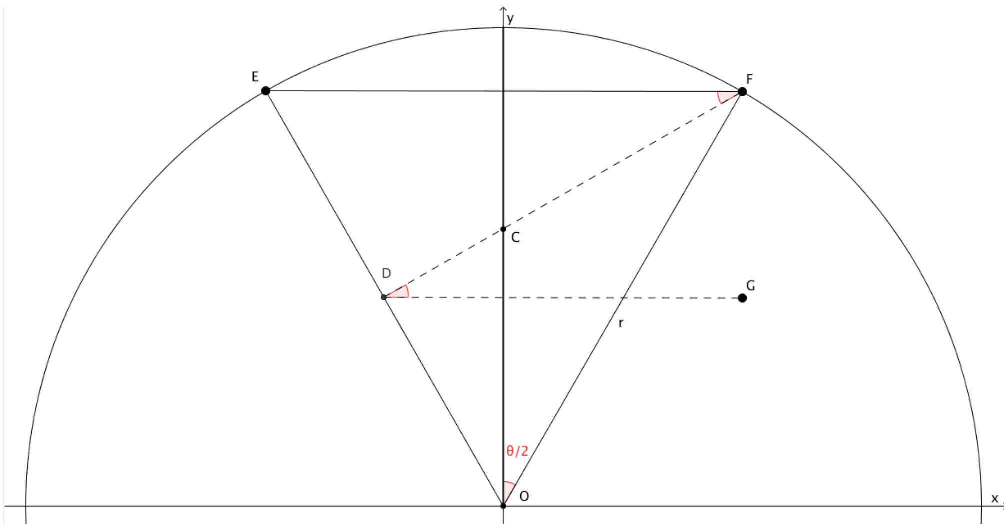


Figure 9
The incenter C of a triangu component of the inscrib polygon. $\overline{DG}$ has been construct to be parallel to $\overline{EF}$.

The coordinates C of the incenter can be found through the intersection of the angle bisector of $\angle EFO$ $\overline{FD}$ with the y-axis (which itself is the angle bisector of $\angle FOE$) (Fig. 9). The x and y values of point F are given respectively by the following equations with θ and r having been defined previously

$$F_x = r\sin\left(\frac{\theta}{2}\right) \tag{7}$$

$$F_y = r\cos\left(\frac{\theta}{2}\right) \tag{8}$$

Since $\angle EFO = \theta$ and ΔOEF is isosceles, half of $\angle EFO$ is equal to

$$\frac{1}{2}\angle EFO = \frac{1}{2}\cdot\frac{\pi-\theta}{2} = \frac{\pi-\theta}{4} \tag{9}$$

Since $\overline{DG}$ is constructed to be parallel to $\overline{EF}$, $\frac{1}{2}\angle EFO = \angle FDG$ by alternate interior angle theorem. The linear equation for the angle bisector of $\angle EFO$ in point-slope form is therefore

[3] "Triangle incenter, description and properties." Math Open Reference. 2011. Accessed January 19, 2017. http://www.mathopenref.com/triangleincenter.html.

$$y = \tan\left(\frac{\pi - \theta}{4}\right)(x - F_x) + F_y \tag{10}$$

Finally, the position of the incenter along the y-axis C_y can be found by substituting zero for x in (10)

$$C_y = -\tan\left(\frac{\pi - \theta}{4}\right)F_x + F_y \tag{11}$$

When written in terms of n by substituting (1), (7), and (8) into (11) the position C_y is then

$$C_y = -r\tan\left(\frac{\pi - \frac{2\pi}{n}}{4}\right)\sin\left(\frac{\pi}{n}\right) + r\cos\left(\frac{\pi}{n}\right)$$

The position C_∞ of the incenter is equal to the limit of C_y as n approaches infinity

$$C_\infty = \lim_{n\to\infty}\left(-r\tan\left(\frac{\pi - \frac{2\pi}{n}}{4}\right)\sin\left(\frac{\pi}{n}\right) + r\cos\left(\frac{\pi}{n}\right)\right)$$

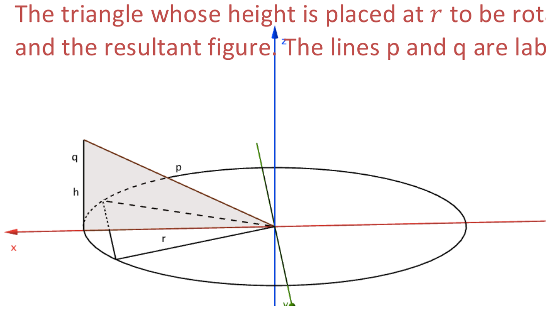
Knowing that $\lim_{n\to\infty}\frac{\pi}{n} = 0$, C_∞ can be rewritten and found to be equal to

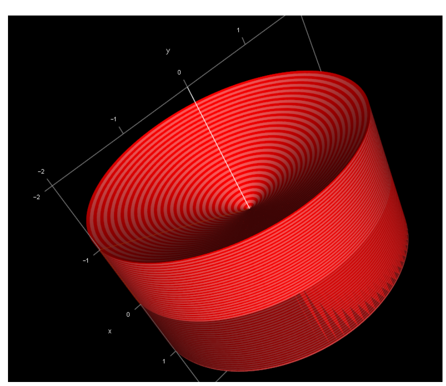
$$C_\infty = \lim_{\frac{\pi}{n}\to 0}\left(-r\tan\left(\frac{\pi - \frac{2\pi}{n}}{4}\right)\sin\left(\frac{\pi}{n}\right)\right) + \lim_{\frac{\pi}{n}\to 0}\left(r\cos\left(\frac{\pi}{n}\right)\right)$$
$$= 0 + r(1) = r$$

Since $C_\infty = r$, the limiting shape of the pyramid will be a right triangle whose height is placed at the circumference of the circle. Subsequently, the volume of revolution can be found by rotating this triangle with base length r and height h where the height originates from the incenter $(r, 0, 0)$ as n approaches infinity around the z-axis. This process would be the same as revolving a function in the x-y plane about the y-axis. This leads to a figure akin to removing two cones from a cylinder (Fig. 10). The equations for the line p through the origin and apex $(r, 0, h)$ and line q through the point $(r, 0, 0)$ and apex $(r, 0, h)$ are respectively

$$p : x = -\frac{h}{r}z$$
$$q : x = r$$

Figure 10
The triangle whose height is placed at r to be rotated and the resultant figure. The lines p and q are labeled.





Therefore the figure's volume V_{fig10} using the disk and washer method is

$$V_{fig10} = 2\pi \int_0^h \left((r)^2 - \left(\frac{-r}{h}z\right)^2\right) dz$$

When the height and radius are set to one and two respectively as previously the volume is equal to

$$\begin{aligned} V_{fig10} &= 2\pi \int_0^1 \left((2)^2 - \left(\frac{-2}{1}z\right)^2\right) dz \\ &= 2\pi \int_0^1 (4 - 4z^2)\, dz \\ &= 2\pi \left(4z - \frac{4}{3}z^3\right)\Big|_0^1 = \frac{16\pi}{3} \end{aligned}$$

Conflicting Answers

Somehow two conflicting answers are reached. I decided to look at another rotational volume to see if there was an error in my earlier method for calculating the volume of the figure using pyramids. This time I placed the height above the origin for each pyramid (Fig. 11). As n approaches infinity, the triangles would have their heights above the origin and when rotated about the z-axis form a composition of two cones (Fig. 12). This results in a new function for p through the origin and apex $(0, 0, h)$ and q through the point $(r, 0, 0)$ and apex $(0, 0, h)$ and a new integral for the volume of revolution V_{fig12}

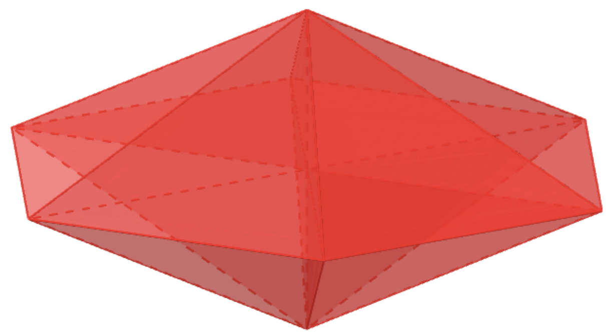


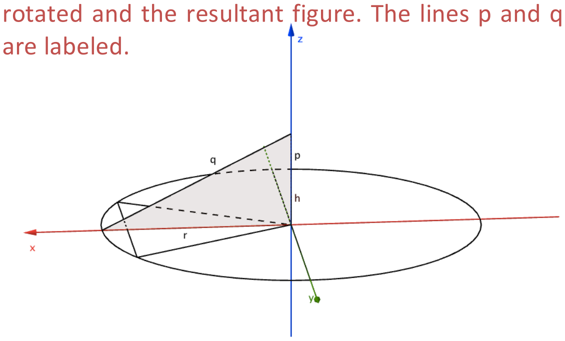
Figure 11
The hexagon based structure where the heights of the pyramids are at the origin.

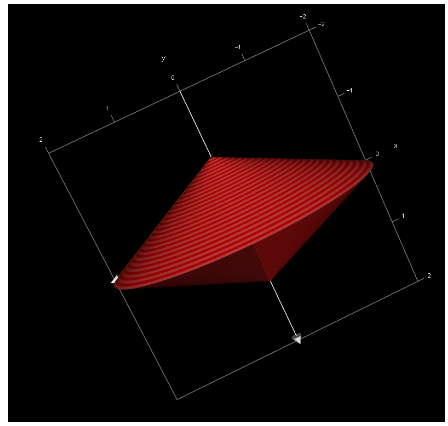
$$p : x = 0$$
$$q : x = \frac{r}{h}z$$
$$V_{fig12} = 2\pi \int_0^h \left(\frac{r}{h}z\right)^2 dz$$

When the height and radius are set to one and two respectively as previously the volume is equal to

$$\begin{aligned} V_{fig12} &= 2\pi \int_0^1 \left(\frac{2}{1}z\right)^2 dz = 2\pi \int_0^1 4z^2 dz \\ &= 2\pi \left(\frac{4}{3}z^3\right)\Big|_0^1 = \frac{8\pi}{3} \end{aligned}$$

Figure 12
The triangle whose height is placed at 0 to be rotated and the resultant figure. The lines p and q are labeled.





The new integral verified the volume from the methodology using the pyramids in (6) raising the question: if the height has no effect on the volume of a pyramid, why would its location when rotating the triangle (which is the limiting shape of the pyramid as n approaches infinity) about the z-axis change the final structure's volume?

Regardless of where I place the height for the pyramids, their volume will approach the same total as n approaches infinity. It should not matter what shape the figure is as long as the height h and radius r remain constant since the figures will always be divided into n equal pyramids. Therefore, a certain portion of the volume must have been ignored or miscalculated. In order to investigate the effect of placing the height in varying locations, I drew a figure depicting two scenarios side by side (Fig. 13).

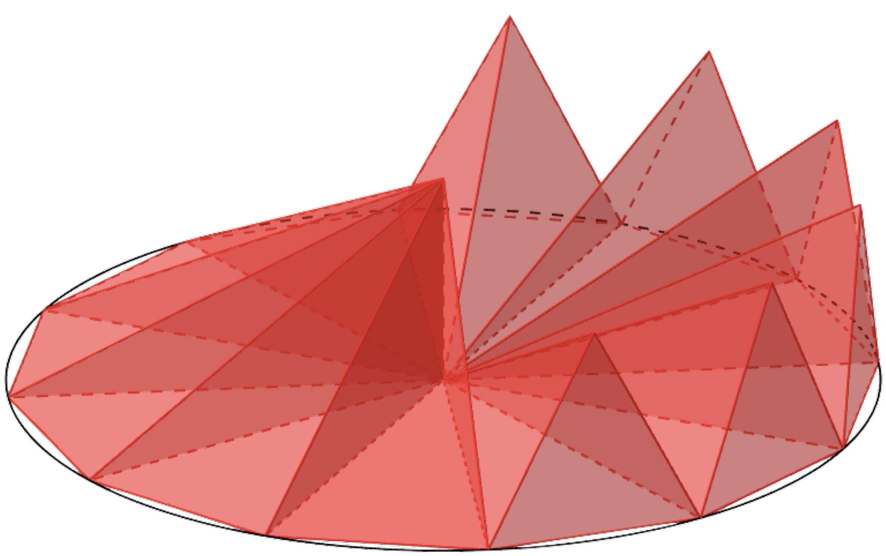


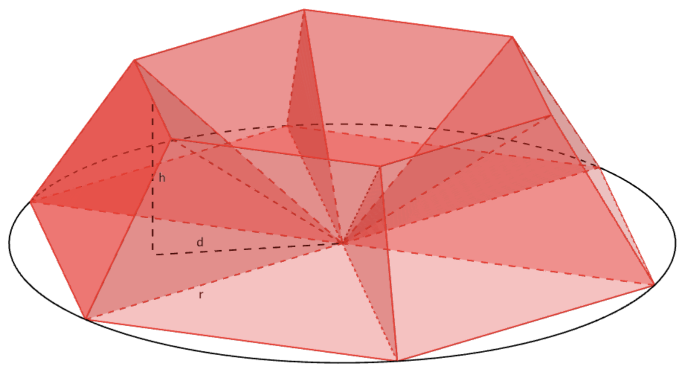
Figure 13
Two variations of pyramids based on the placement of the apex drawn on the same inscribed polygon base

Looking at the figure I realized that when the height was placed at the origin the pyramids connected forming a cone like shape; while if it was placed at the circumference, it generated a series of disconnected pyramids. As n approaches infinity, the gaps between the pyramids still exist for the latter case. Even when infinitesimally small, the gaps prevent the pyramids from ever properly approximating the entirety of the volume leading it to be different from that of the rotated triangle, which sweeps through the entire region leaving no gaps.

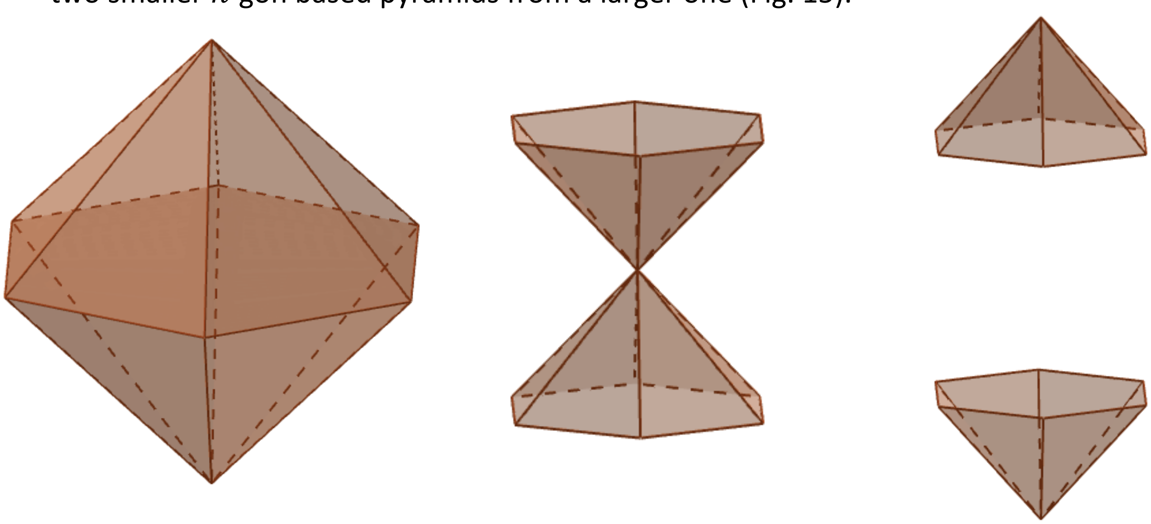
Alternate Approximation

I then wondered if it would be possible to develop an alternate method to use pyramids to reach the volumes of the rotated triangles. Using the previous method as a starting point, I realized that for the volumes to be equal, the pyramids would have to lie alongside each other with no gaps between them. In order to do this while still using n-gons as a base, the pyramids would have to lie on their sides with their apexes at the center of the circle. For the pyramids to connect, they would need to have trapezoidal bases (Fig. 14). Using pyramids of this sort, the placement of h along the base is now of consequence as it would change the angle of the trapezoidal base and thus the height and volume of the trapezoidal pyramid. For this new approximation the additional variable d denoting the position of h along the base must be considered where $0 \leq d \leq r$. For the purpose of clarity, only the top half of the structure is shown for the diagrams in this section.

Figure 14
The structure produced by laying the pyramids on their sides with their apexes at the center of the circle. Only half is shown in order to make the diagram clearer.



Initial attempts to find the volumes of the individual trapezoidal pyramids using the changing height proved complicated leading to the use of a different method. The volume of the new approximation was instead calculated by subtracting the volumes of two smaller n-gon based pyramids from a larger one (Fig. 15).



Composition 1 | Composition 2 | Composition 3

Figure 15
By removing the volumes of the two compositions of pyramids to the right from the left, the volume of the alternate approximation can be calculated

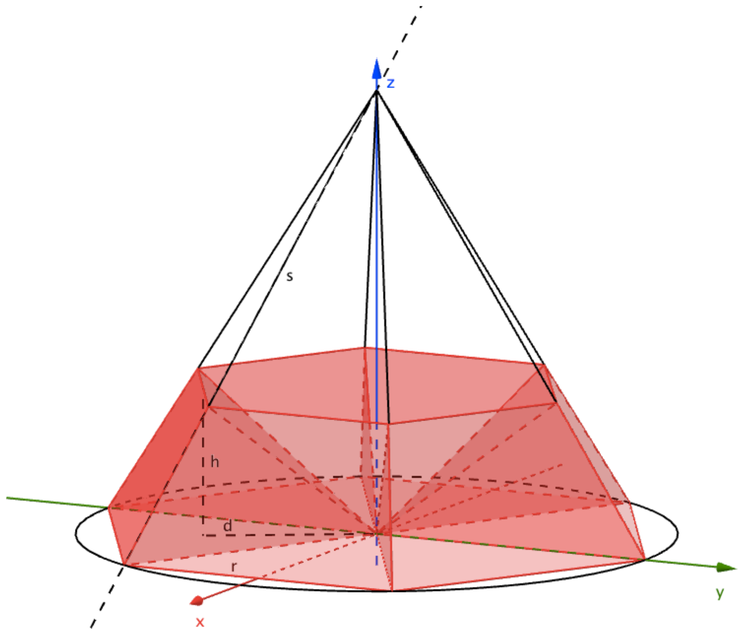


Figure 16
Here the frames of one of the halves of the three compositions are displayed on one diagram. Line s runs along one of the edges of the figure and intersects the z-axis. Composition 1 is the large pyramid. Composition 2 is the small pyramid inset into the figure with height h. Composition 3 is the pyramid cap with a height of the difference between the z-intercept of s and h.

Having shown earlier in (2) that the base area A of composition 1 is

$$A = \frac{n}{2} r^2 \sin\left(\frac{2\pi}{n}\right)$$

The next step is to find the height of the pyramids in the first composition. This can be done by finding the z-intercept of the line s passing along the edge of one of the faces (Fig. 16). Given that the edge will pass through the point $(d, 0, h)$ and $(r, 0, 0)$ the line s would have the equation

$$s : z = \frac{-h}{r-d}(x - r)$$

The z-intercept of s is therefore

$$z = \frac{rh}{r-d}$$

Together this would make the volume V_{pyr1} of pyramid composition 1 equal to twice the volume of a pyramid with base area A and height z

$$V_{pyr1} = \frac{nr^2}{3} \cdot \frac{rh}{r-d} \sin\left(\frac{2\pi}{n}\right)$$

Using similar logic, the base area A_2 of pyramid composition 2 and 3 is

$$A_2 = \frac{n}{2} d^2 \sin\left(\frac{2\pi}{n}\right)$$

Each pyramid in composition 2 has a height of h therefore the volume V_{pyr2} of the composition is

$$V_{pyr2} = \frac{nhd^2}{3} \sin\left(\frac{2\pi}{n}\right)$$

Each pyramid in composition 3 has a height of $\frac{rh}{r-d} - h$ therefore the volume V_{pyr3} of the composition is

$$V_{pyr3} = \frac{nd^2}{3} \cdot \left(\frac{rh}{r-d} - h\right) \sin\left(\frac{2\pi}{n}\right)$$

Altogether the volume V_{fig14} of figure 14 is then

$$\begin{aligned} V_{fig14} &= V_{pyr1} - V_{pyr2} - V_{pyr3} \\ &= \frac{nr^2}{3} \cdot \frac{rh}{r-d} \sin\left(\frac{2\pi}{n}\right) - \frac{nhd^2}{3} \sin\left(\frac{2\pi}{n}\right) - \frac{nd^2}{3} \cdot \left(\frac{rh}{r-d} - h\right) \sin\left(\frac{2\pi}{n}\right) \\ &= \frac{nrh}{3} \sin\left(\frac{2\pi}{n}\right)\left(\frac{r^2 - d^2}{r-d}\right) \\ &= \frac{nrh}{3} \sin\left(\frac{2\pi}{n}\right)(r+d) \end{aligned} \tag{12}$$

Verifying the New Approximation

In order to verify the new approximation, I calculated the limit V_{∞_2} of V_{fig14} as n approaches infinity.

$$\begin{aligned} V_{\infty_2} &= \lim_{n\to\infty}\left(\frac{nrh}{3} \sin\left(\frac{2\pi}{n}\right)(r+d)\right) \\ &= \lim_{\frac{2\pi}{n}\to 0}\left(\frac{nrh}{3} \sin\left(\frac{2\pi}{n}\right)(r+d)\right) \\ &= \lim_{\frac{2\pi}{n}\to 0}\left(\frac{2\pi rh}{3}(r+d) \cdot \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}\right) \\ &= \frac{2\pi rh}{3}(r+d) \end{aligned}$$

At infinity, pyramid compositions 1, 2, and 3 should resemble assemblies of two cones. If the limit of the approximation is correct, it would be equivalent to the equation developed by finding twice the difference of the formulas for the volume of a cone

V_{comp1} with height $\frac{rh}{r-d}$ and radii r and d and cones V_{comp2} with height h and radius d and V_{comp3} with height $\frac{rh}{r-d} - h$ and radius d.

$$V_{comp1} = \frac{\pi r^2}{3} \cdot \frac{rh}{r-d}$$

$$V_{comp2} = \frac{\pi d^2 h}{3}$$

$$V_{comp3} = \frac{\pi d^2}{3}\left(\frac{rh}{r-d} - h\right)$$

$$\begin{aligned} 2\left(V_{comp1} - V_{comp2} - V_{comp3}\right) &= 2\left(\frac{\pi r^2}{3} \cdot \frac{rh}{r-d} - \frac{\pi d^2 h}{3} - \frac{\pi d^2}{3}\left(\frac{rh}{r-d} - h\right)\right) \\ &= \frac{2\pi h}{3}\left(\frac{r^3}{r-d} - \frac{rd^2}{r-d}\right) \\ &= \frac{2\pi rh}{3}\left(\frac{r^2 - d^2}{r-d}\right) \\ &= \frac{2\pi rh}{3}(r+d) = V_{\infty_2} \end{aligned}$$

Therefore using this method of pyramid approximation, it is possible to take the limit of (12) as n approaches infinity and have the limit equal the volume created by rotating any triangle with base length r, height h, and position of apex d about the z-axis. Since V_{∞_2} is always increasing with respect to d, for this set of figures with r and h being 2 and 1 respectively, the minimum volume is $\frac{8\pi}{3}$ and the maximum volume is $\frac{16\pi}{3}$ for $0 \leq d \leq r$. Interestingly, when these two values are added together it is equal to the volume of a cylinder with a radius and height of 2, which would be the result of visually meshing both cases.

Conclusion

When I began this investigation, I was expecting to find that when the number of sides n approached infinity, the solids would be identical to those developed using rotational volumes as the limiting shape of a pyramid would be a triangle. I thought an identical area under a function would lead to the same volume for the rotated solid regardless of the placement of the height. Through this investigation, I realized otherwise. An infinite number of disconnected pyramids would never be able to mimic these solids of revolution. When rotating triangles about an axis, the position of the height matters, as it will lead to more or less volume being swept out during the rotation. In order to approach these solids of revolution using pyramids, they would have to connect face to face eventually resembling a smooth surface. Beyond the novelty of the initial origami figure, through this investigation, I was able to explore some of the details involved in the infinite subdivision necessary for calculus with solids.