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EXACT PENALTY FUNCTIONS IN CONSTRAINED OPTIMIZATION*

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Abstract. In this paper formal definitions of exactness for penalty functions are introduced and sufficient conditions for a penalty function to be exact according to these definitions are stated, thus providing a unified framework for the study of both nondifferentiable and continuously differentiable penalty functions. In this framework the best-known classes of exact penalty functions are analyzed, and new results are established concerning the correspondence between the solutions of the constrained problem and the unconstrained minimizers of the penalty functions.

Key words. exact penalty functions, nonlinear programming, constrained optimization

AMS(MOS) subject classifications. 49D30, 49D37, 90C30

1. Introduction. A considerable amount of investigation, both from the theoretical and the computational point of view, has been devoted to methods that attempt to solve nonlinear programming problems by means of a single minimization of an unconstrained function. Methods of this kind are usually termed *exact penalty methods*, as opposed to the *sequential penalty methods*, which include the quadratic penalty method and the method of multipliers (see, e.g., [4], [23], and [26]).

We can subdivide exact penalty methods into two classes: methods based on *exact penalty functions* and methods based on *exact augmented Lagrangian functions*. In our terminology, the term "exact penalty function" is used when the variables of the unconstrained problem are in the same space as the variables of the original constrained problem, whereas the term "exact augmented Lagrangian function" is used when the unconstrained problem has to be minimized on the product space of the problem variables and of the multipliers.

Exact penalty functions can be subdivided, in turn, into two main classes: *nondifferentiable* exact penalty functions and *continuously differentiable* exact penalty functions.

Nondifferentiable exact penalty functions were introduced for the first time in [39] and have been widely investigated in recent years (see, e.g., [1], [2], [5]-[10], [22], [25], [29], and [35]). Continuously differentiable exact penalty functions were introduced in [24] for equality constrained problems and in [28] for problems with inequality constraints; further contributions have been given in [14], [15], and [34].

Exact augmented Lagrangian functions were introduced in [11] and [12] and have been further investigated in [3], [4], [19]-[21], [31], and [38].

In this paper we restrict our attention to exact penalty functions, with the aim of providing a unified framework which applies both to the nondifferentiable and to the continuously differentiable case.

We start from the introduction of formal definitions of various kinds of exactness that attempt to capture the most relevant aspects of the notion of exactness in the context of constrained optimization. This is motivated by the fact that in the current

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literature the term *exact penalty function* seems to be used without a definite agreement on its meaning. In particular, as noted in [29], most of the literature on this subject is mainly concerned with conditions that ensure that the penalty function has a local (global) minimum at a local (global) minimum point of the constrained problem. On the other hand, since the penalty approach is an attempt to solve a constrained problem by the minimization of an unconstrained function, this characterization is fully satisfactory only when both the constrained problem and the penalty function are convex. In the nonconvex case, the study of converse properties appears to be of greater interest, as they ensure that local (global) minimizers of the penalty function are local (global) solutions of the constrained problem.

Moreover, again in the nonconvex case, a distinction has to be made between properties of exactness pertaining to global solutions and properties pertaining to local solutions. It will be shown that, for the same penalty function, different kinds of exactness can be established under different regularity requirements on the problem constraints.

Finally, the correspondence between the constrained and the unconstrained minimization problem can only be established with reference to a compact set containing the problem solutions, and this must be carefully taken into account in the analysis of the properties of exactness.

The formal definitions mentioned so far constitute the basis for the development of sufficient conditions for a penalty function to be exact according to some specified notion of exactness. In particular, we establish sufficient conditions which apply both to the nondifferentiable and to the continuously differentiable case, thus providing a unified framework for the analysis and the construction of exact penalty functions. In this framework, we consider the best-known classes of exact penalty functions, and we provide a complete analysis of their properties, recovering known results and establishing new ones.

The paper is organized as follows. Section 2 contains the problem statement, basic notation, and preliminary results. In § 3 we formalize the definitions of various kinds of exactness of penalty functions, which are classified as *weak exactness*, *exactness*, *strong exactness*, and *global* (*weak*, *strong*) *exactness*. Section 4 deals with nondifferentiable penalty functions: we analyze the properties of l_q exact penalty functions as well as those of the globally exact nondifferentiable penalty function considered in [16]. In § 5 we study continuously differentiable exact penalty functions, and we introduce a globally exact differentiable penalty function for mixed equality and inequality constrained problems by extending the results given in [15].

Computational aspects are beyond the scope of this paper. We refer, e.g., to [3], [4], [9], [18], [21], [27], [28], [33], [34], [36], and [37] for some algorithmic applications of exact penalty functions.

2. Problem statement, basic notation and preliminary results. The problem considered here is the general nonlinear programming problem:

$$\text{(P)} \qquad \begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } g(x) \leqq 0, \qquad h(x) = 0, \end{aligned}$$

where $f:\mathbb{R}^n \to \mathbb{R}$, $g:\mathbb{R}^n \to \mathbb{R}^m$, $h:\mathbb{R}^n \to \mathbb{R}^p$, $p \leqq n$ are continuously differentiable functions and the feasible set

$$\mathcal{F} := \{x \in \mathbb{R}^n : g(x) \leqq 0,\ h(x) = 0\}$$

is assumed to be nonempty.

We denote by $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{L}_{\mathcal{P}}$, respectively, the set of global solutions and the set of local solutions of problem (P) and we assume that $\mathcal{G}_{\mathcal{P}}$ is nonempty.

The Lagrangian function associated with problem (P) is the function $L:\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ defined by

$$L(x, \lambda, \mu) := f(x) + \lambda' g(x) + \mu' h(x).$$

A *Kuhn–Tucker triple* for problem (P) is a triple $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^p$ such that

$$\begin{aligned} &\nabla_x L(\bar{x}, \bar{\lambda}, \bar{\mu}) = 0, \\ &\bar{\lambda}' g(\bar{x}) = 0, \\ &\bar{\lambda} \geqq 0, \\ &g(\bar{x}) \leqq 0, \\ &h(\bar{x}) = 0. \end{aligned}$$

We denote by $\mathcal{T}$ the set

$$\begin{aligned} \mathcal{T} := \{\bar{x} \in \mathbb{R}^n\text{: there exist } (\bar{\lambda}, \bar{\mu}) \text{ such that} \\ (\bar{x}, \bar{\lambda}, \bar{\mu}) \text{ is a K-T triple for problem (P)}\}. \end{aligned}$$

For any $x \in \mathbb{R}^n$ we define the index sets:

$$\begin{aligned} I_0(x) &:= \{i\colon g_i(x) = 0\} \\ I_+(x) &:= (i\colon g_i(x) \geqq 0\}. \end{aligned}$$

We adopt the following terminology.

The *linear independence constraint qualification* (LICQ) holds at $x \in \mathbb{R}^n$ if the gradients $\nabla g_i(x)$, $i \in I_0(x)$, $\nabla h_j(x)$, $j = 1, \cdots, p$ are linearly independent.

The *Mangasarian–Fromovitz constraint qualification* (MFCQ) holds at $x \in \mathbb{R}^n$ if $\nabla h_j(x)$, $j = 1, \cdots, p$ are linearly independent and there exists a $z \in \mathbb{R}^n$ such that

$$\begin{aligned} \nabla g_i(x)' z < 0, \qquad & i \in I_0(x) \\ \nabla h_j(x)' z = 0, \qquad & j = 1, \cdots, p. \end{aligned}$$

It can be shown, by using the theorems of the alternative [32] that the MFCQ can be restated as follows.

The MFCQ holds at $x \in \mathbb{R}^n$ if there exist no u_i, $i \in I_0(x)$, and v_j, $j = 1, \cdots, p$ such that

$$\begin{aligned} \sum_{i \in I_0(x)} u_i \nabla g_i(x) + \sum_{j=1}^{p} v_j \nabla h_j(x) = 0, \\ u_i \geqq 0, \qquad i \in I_0(x), \\ (u_i, i \in I_0(x), v_j, \qquad j = 1, \cdots, p) \neq 0. \end{aligned}$$

In some cases we shall make use of a stronger constraint qualification, which is stated in the following equivalent formulations.

The *extended Mangasarian–Fromovitz constraint qualification* (EMFCQ) holds at $x \in \mathbb{R}^n$ if $\nabla h_j(x)$, $j = 1, \cdots, p$ are linearly independent and there exists a $z \in \mathbb{R}^n$ such that

$$\begin{aligned} \nabla g_i(x)' z < 0, \qquad & i \in I_+(x) \\ \nabla h_j(x)' z = 0, \qquad & j = 1, \cdots, p. \end{aligned}$$

The EMFCQ holds at $x \in \mathbb{R}^n$ if there exist no u_i, $i \in I_+(x)$, and $v_j, j = 1, \cdots, p$ such that

$$\sum_{i \in I_+(x)} u_i \nabla g_i(x) + \sum_{j=1}^{p} v_j \nabla h_j(x) = 0,$$
$$u_i \geqq 0, \qquad i \in I_+(x),$$
$$(u_i, i \in I_+(x), v_j, \qquad j = 1, \cdots, p) \neq 0.$$

It can be noted that the LICQ implies the MFCQ and that the EMFCQ implies the MFCQ.

It is known that if $\bar{x}$ is a local solution of problem (P) and if the MFCQ holds at $\bar{x}$, then $\bar{x} \in \mathcal{T}$, that is, there exist K-T multipliers $(\bar{\lambda}, \bar{\mu})$ associated with $\bar{x}$.

We recall that a nonempty set $\mathscr{C}^* \subseteq \mathscr{C}$ is called an *isolated set* of $\mathscr{C}$ if there exists a closed set $\mathscr{H}$ such that $\mathscr{C}^*$ is contained in the interior $\mathring{\mathscr{H}}$ of $\mathscr{H}$ and such that if $x \in \mathscr{H} - \mathscr{C}^*$, then $x \notin \mathscr{C}$. Isolated sets of local minimum points possess the property stated in the following lemma, which is proved in [23].

LEMMA 1. *Let $\mathscr{C}^*$ be an isolated compact set of local minimum points of problem* (P), *corresponding to the local minimum value f^*; then there exists a compact set $\mathscr{H} \subset \mathbb{R}^n$, such that $\mathscr{C}^* \subset \mathring{\mathscr{H}}$, and for any point $x \in \mathscr{H} \cap \mathscr{F}$, if $x \notin \mathscr{C}^*$, then $f(x) > f^*$.*

We also state the following lemma, which for $q \geqq 2$ is an obvious consequence of the equivalence of the norms $\|\cdot\|_q$ and $\|\cdot\|_{q-1}$ on $\mathbb{R}^n$.

LEMMA 2. *Let $q \in \mathbb{R}$, $1 < q < \infty$. Then, there exists a number $\mu > 0$ such that for all $z \in \mathbb{R}^n$, we have*:

$$\sum_{i=1}^{n} |z_i|^{q-1} \geqq \mu \|z\|_q^{q-1}.$$

Proof. The assertion follows from a more general result on positive homogeneous continuous functions ([30, Thm. 5.4.4]). □

In the sequel we shall be concerned with compact perturbations of the feasible set. In particular, we shall consider the case in which $\mathscr{F}$ is compact and there exists a vector $\beta = (\alpha_0, \alpha')'$ with $\alpha_0 \in \mathbb{R}$, $\alpha \in \mathbb{R}^m$, $\beta > 0$, such that the set

$$\mathscr{S}_\beta := \{x \in \mathbb{R}^n \colon g(x) \leqq \alpha, \|h(x)\|_2^2 \leqq \alpha_0\}$$

obtained by relaxing the problem constraints is compact. It can be shown, by extending a similar result given in [15] for the inequality constrained case, that under the following assumptions:

(i) there exists a $\bar{\beta} \in \mathbb{R}^{m+1}$, $\bar{\beta} > 0$, such that $\mathscr{S}_{\bar{\beta}}$ is compact,

(ii) the MFCQ holds on $\mathscr{F}$,

there exists a compact set S_β, with $\beta > 0$, where the EMFCQ is satisfied.

We make use of the following notation. Given the set $\mathscr{A}$, we denote by $\mathring{\mathscr{A}}$, $\partial \mathscr{A}$, and $\bar{\mathscr{A}}$, respectively, the interior, the boundary, and the closure of $\mathscr{A}$. Given a vector u with components u_i, $i = 1, \cdots, m$ we denote by u^+ the vector with components:

$$u_i^+ := \max[0, u_i], \qquad i = 1, \cdots, m$$

and by U the diagonal matrix defined by:

$$U := \operatorname{diag}(u_i), \qquad i = 1, \cdots, m.$$

Given a function $F: \mathbb{R}^n \to \mathbb{R}$, we denote by $DF(x, d)$ the directional derivative of F at x along the direction d. We say that $\bar{x}$ is a *critical point* of F if $DF(x, d) \geqq 0$ for all $d \in \mathbb{R}^n$. If $\bar{x}$ is a critical point of F and F is differentiable at $\bar{x}$, we have $\nabla F(\bar{x}) = 0$; in this case we say that $\bar{x}$ is a *stationary point* of F.

Finally, we denote by $\mathscr{B}(\bar{x}; \rho)$ the open ball around $\bar{x}$ with radius $\rho > 0$.

3. Definitions of exactness for penalty functions. Roughly speaking, an *exact penalty function* for problem (P) is a function $F(x; \varepsilon)$, where $\varepsilon > 0$ is a *penalty parameter*, with the property that there is an appropriate parameter choice such that a single unconstrained minimization of $F(x; \varepsilon)$ yields a solution to problem (P). In particular, we require that there is an *easy* way for finding correct parameter values by imposing that exactness is retained for all ε ranging on some set of nonzero measure. More specifically, we take $\varepsilon \in (0, \varepsilon^*]$ where $\varepsilon^* > 0$ is a suitable *threshold value*.

In practice, the existence of a threshold value for the parameter ε, and hence the possibility of constructing the exact penalty function $F(x; \varepsilon)$, can only be established with reference to some compact set $\mathscr{D}$. Therefore, instead of problem (P) we shall consider the following problem.

$$(\tilde{\mathrm{P}}) \qquad \text{minimize } f(x), \qquad x \in \mathscr{F} \cap \mathscr{D},$$

where $\mathscr{D}$ is a compact subset of $\mathbb{R}^n$ such that $\mathscr{F} \cap \mathring{\mathscr{D}} \neq \emptyset$. It can be observed that if $\mathscr{F} \subset \mathscr{D}$, then problem ($\tilde{\mathrm{P}}$) and problem (P) are equivalent.

We denote by $\mathscr{G}_{\tilde{\mathscr{P}}}$ and $L_{\tilde{\mathscr{P}}}$, respectively, the set of global solutions and the set of local solutions of problem ($\tilde{\mathrm{P}}$), that is:

$$\mathscr{G}_{\tilde{\mathscr{P}}} := \{x \in \mathscr{F} \cap \mathscr{D} : f(x) \leqq f(y), \text{ for all } y \in \mathscr{F} \cap \mathscr{D}\}$$

$$\mathscr{L}_{\tilde{\mathscr{P}}} := \{x \in \mathscr{F} \cap \mathscr{D} : \text{ for some } \rho > 0 \; f(x) \leqq f(y), \text{ for all } y \in \mathscr{F} \cap \mathscr{D} \cap \mathscr{B}(x; \rho)\}.$$

We have, obviously, that $\mathscr{L}_{\tilde{\mathscr{P}}} \cap \mathring{\mathscr{D}} = \mathscr{L}_{\mathscr{P}} \cap \mathring{\mathscr{D}}$; moreover, if $\mathscr{G}_{\mathscr{P}} \cap \mathring{\mathscr{D}} \neq \emptyset$, we have also $\mathscr{G}_{\mathscr{P}} \cap \mathring{\mathscr{D}} = \mathscr{G}_{\tilde{\mathscr{P}}} \cap \mathring{\mathscr{D}}$.

For any given $\varepsilon > 0$, let $F(x; \varepsilon)$ be a continuous real function defined on a set $\mathscr{E}$, such that $\mathring{\mathscr{D}} \subseteq \mathscr{E} \subseteq \mathscr{D}$ and consider the following problem.

$$(\mathrm{Q}) \qquad \text{minimize } F(x; \varepsilon), \qquad x \in \mathring{\mathscr{D}}.$$

Since $\mathring{\mathscr{D}}$ is an open set, any local solution of problem (Q), provided it exists, is unconstrained; thus problem (Q) can be considered as an essentially unconstrained problem. The sets of global and local solutions of problem (Q) are denoted, respectively, by $\mathscr{G}_{\mathscr{Q}}(\varepsilon)$ and $\mathscr{L}_{\mathscr{Q}}(\varepsilon)$:

$$\mathscr{G}_{\mathscr{Q}}(\varepsilon) := \{x \in \mathring{\mathscr{D}} : F(x; \varepsilon) \leqq F(y; \varepsilon), \text{ for all } y \in \mathring{\mathscr{D}}\}$$

$$\mathscr{L}_{\mathscr{Q}}(\varepsilon) := \{x \in \mathring{\mathscr{D}} : \text{ for some } \rho > 0 \; F(x; \varepsilon) \leqq F(y; \varepsilon), \text{ for all } y \in \mathring{\mathscr{D}} \cap \mathscr{B}(x; \rho)\}.$$

There are different kinds of relationships between problem ($\tilde{\mathrm{P}}$) and problem (Q), which can be associated with different notions of exactness.

A first possibility is that of considering a correspondence between global minimizers of problem ($\tilde{\mathrm{P}}$) and global minimizers of problem (Q). This correspondence is established formally in the following definition.

DEFINITION 1. We say that the function $F(x; \varepsilon)$ is a *weakly exact penalty function* for problem (P) with respect to the set $\mathscr{D}$ if there exists an $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, any global solution of problem ($\tilde{\mathrm{P}}$) is a global minimum point of problem (Q) and conversely; that is if for some $\varepsilon^* > 0$:

$$\mathscr{G}_{\tilde{\mathscr{P}}} = \mathscr{G}_{\mathscr{Q}}(\varepsilon), \quad \text{for all } \varepsilon \in (0, \varepsilon^*].$$

The property stated above guarantees that the constrained problem can actually be solved over $\mathscr{D}$ by means of the *global unconstrained minimization* of $F(x; \varepsilon)$ for sufficiently small values of the parameter ε.

We remark that if all global solutions of problem (P) are contained in $\mathring{\mathscr{D}}$, then problem (P) and problem ($\tilde{\mathrm{P}}$) possess the same global solutions. In this case, weak exactness implies that global solutions of problem (P) and global minimizers of problem (Q) are the same.

The notion of exactness expressed by Definition 1 appears to be of limited value for general nonlinear programming problems, since it does not give a meaning to local minimizers of the penalty function, while unconstrained minimization algorithms determine only local minimizers. Therefore, we introduce a further requirement concerning local minimizers which gives rise to a stronger notion of exactness.

DEFINITION 2. We say that the function $F(x; \varepsilon)$ is an *exact penalty function* for problem (P) with respect to the set $\mathscr{D}$ if there exists an $\varepsilon^*>0$ such that, for all $\varepsilon\in(0, \varepsilon^*]$, $\mathscr{G}_{\tilde{\mathscr{P}}}=\mathscr{G}_{\mathscr{Q}}(\varepsilon)$ and, moreover, any local unconstrained minimizer of problem (Q) is a local solution of problem (P), that is:

$$\mathscr{L}_{\mathscr{Q}}(\varepsilon)\subseteq\mathscr{L}_{\mathscr{P}},\quad \text{for all } \varepsilon\in(0, \varepsilon^*].$$

It must be remarked that the notion of exactness given in Definition 2 does not require that all local solutions of problem (P) in $\mathring{\mathscr{D}}$ correspond to local minimizers of the exact penalty functions. A one-to-one correspondence of local minimizers does not seem to be required, in practice, to give a meaning to the notion of exactness, since the condition $\mathscr{G}_{\tilde{\mathscr{P}}}=\mathscr{G}_{\mathscr{Q}}(\varepsilon)$ ensures that global solutions of problem $(\tilde{\mathrm{P}})$ are preserved. However, for the classes of exact penalty functions considered in the sequel, it will be shown that this correspondence can be established, also, at least with reference to isolated compact sets of local minimizers of problem (P) contained in $\mathring{\mathscr{D}}$. Thus, we can also consider the following definition.

DEFINITION 3. We say that the function $F(x; \varepsilon)$ is a *strongly exact penalty function* for problem (P) with respect to the set $\mathscr{D}$ if there exists an $\varepsilon^*>0$ such that, for all $\varepsilon\in(0, \varepsilon^*]$, $\mathscr{G}_{\tilde{\mathscr{P}}}=\mathscr{G}_{\mathscr{Q}}(\varepsilon)$, $\mathscr{L}_{\mathscr{Q}}(\varepsilon)\subseteq\mathscr{L}_{\mathscr{P}}$, and, moreover, any local solution of problem (P) belonging to $\mathring{\mathscr{D}}$ is a local unconstrained minimizer of $F(x; \varepsilon)$, that is:

$$\mathscr{L}_{\mathscr{P}}\cap\mathring{\mathscr{D}}\subseteq\mathscr{L}_{\mathscr{Q}}(\varepsilon)\quad \text{for all } \varepsilon\in(0, \varepsilon^*].$$

The properties considered in the preceding definitions do not characterize the behavior of $F(x; \varepsilon)$ on the boundary of $\mathring{\mathscr{D}}$. Although this may be irrelevant from the conceptual point of view in connection with the notion of exactness, it may assume a considerable interest from the computational point of view, when unconstrained descent methods are employed for the minimization of $F(x; \varepsilon)$. In fact, it may happen that there exist points of $\mathring{\mathscr{D}}$ such that a descent path for $F(x; \varepsilon)$ that originates at some of these points crosses the boundary of $\mathring{\mathscr{D}}$. This implies that the sequence of points produced by an unconstrained algorithm may be attracted toward a stationary point of $F(x; \varepsilon)$ out of $\mathring{\mathscr{D}}$ or may not admit a limit point. Therefore, it could be difficult to construct minimizing sequences for $F(x; \varepsilon)$ which are globally convergent on $\mathring{\mathscr{D}}$ toward the solutions of the constrained problem. In order to avoid this difficulty, it is necessary to impose further conditions on $F(x; \varepsilon)$, and we are led to introduce the notion of *global exactness* of a penalty function.

DEFINITION 4. The function $F(x; \varepsilon)$ is said to be a *globally* (*weakly*, *strongly*) *exact penalty function* for problem (P) with respect to the set $\mathscr{D}$ if it is (weakly, strongly) exact and, moreover, for any $\varepsilon>0$ and for any $\hat{x}\in\partial\mathscr{D}$ there exists a neighborhood $\mathscr{B}(\hat{x}, \rho)$ such that if $\{x_k\}\subset\mathring{\mathscr{D}}$ and $\lim_{k\to\infty} x_k=\hat{x}$, we have:

$$\liminf_{k\to\infty} F(x_k; \varepsilon)>F(x; \varepsilon),$$

for all $x\in\mathscr{B}(\hat{x}; \rho)\cap\mathring{\mathscr{D}}$.

The condition given above excludes the existence of minimizing sequences for $F(x; \varepsilon)$ originating in $\mathring{\mathscr{D}}$ that have limit points on the boundary. In fact, we can state the following proposition.

PROPOSITION 1. *Let $F(x;\varepsilon)$ be a (weakly, strongly) globally exact penalty function with respect to the set $\mathscr{D}$ and let $\{x_k\}\subset \mathring{\mathscr{D}}$ be a sequence such that $F(x_{k+1};\varepsilon)\leqq F(x_k;\varepsilon)$. Then, any limit point of $\{x_k\}$ belongs to $\mathring{\mathscr{D}}$.*

Proof. By the compactness of $\mathscr{D}$ there exists a subsequence, which we relabel $\{x_k\}$, such that $x_k\to\hat{x}\in\mathscr{D}$. Reasoning by contradiction, assume that $\hat{x}\in\partial\mathscr{D}$. Then, recalling Definition 4, we have, for sufficiently large values of j, $\liminf_{k\to\infty} F(x_k;\varepsilon)>F(x_j;\varepsilon)$, which contradicts the assumption $F(x_k;\varepsilon)\leqq F(x_j;\varepsilon)$ for all $k\geqq j$. □

4. Sufficient conditions for exactness. In this section we state sufficient conditions which imply that a penalty function $F(x;\varepsilon)$ possesses some of the properties of exactness considered in the preceding section. Everywhere below we suppose that the following assumption holds.

Assumption (A1). Any global solution of problem $(\tilde{\mathrm{P}})$ belongs to the set $\mathring{\mathscr{D}}$, that is: $\mathscr{G}_{\tilde{\mathscr{P}}}\subset\mathring{\mathscr{D}}$.

We note that Assumption (A1) concerns the selection of the set $\mathscr{D}$ and implies that $\mathscr{G}_{\tilde{\mathscr{P}}}\subset\mathscr{L}_{\mathscr{P}}$; it can be satisfied, in particular, by a proper choice of $\mathscr{D}$, whenever the global solutions of problem (P) belong to a bounded subset of $\mathscr{F}$.

Let $\mathscr{K}$ be the subset of $\mathscr{F}\cap\mathring{\mathscr{D}}$ where the function $F(x;\varepsilon)$ takes the same values of $f(x)$, that is:

$$\mathscr{K}:=\{x\in\mathscr{F}\cap\mathring{\mathscr{D}}:\ F(x;\varepsilon)=f(x)\text{ for all }\varepsilon>0\}. \tag{1}$$

The next theorem establishes a sufficient condition for $F(x;\varepsilon)$ to be a weakly exact penalty function in the sense of Definition 1.

THEOREM 1. *Let $F(x;\varepsilon)$ be such that the following conditions are satisfied.*

(a_1) *For any $\varepsilon>0$, the function $F(x;\varepsilon)$ admits a global minimum point on a set $\mathscr{E}$, such that $\mathring{\mathscr{D}}\subseteq\mathscr{E}\subseteq\mathscr{D}$.*

(a_2) *If $\{\varepsilon_k\}$ and $\{x_k\}\subseteq\mathscr{E}$ are sequences such that $\lim_{k\to\infty}\varepsilon_k=0$, $\lim_{k\to\infty}x_k=\hat{x}\in\mathscr{D}$ and $\limsup_{k\to\infty}F(x_k,\varepsilon_k)<\infty$, we have $\hat{x}\in\mathscr{F}\cap\mathscr{D}$ and $f(\hat{x})\leqq\limsup_{k\to\infty}F(x_k,\varepsilon_k)$.*

(a_3) $\mathscr{G}_{\tilde{\mathscr{P}}}\subseteq\mathscr{K}$.

(a_4) *For any $\hat{x}\in\mathscr{G}_{\tilde{\mathscr{P}}}$ there exist numbers $\varepsilon(\hat{x})>0$ and $\sigma(\hat{x})>0$ such that, for all $\varepsilon\in(0,\varepsilon(\hat{x})]$, if $\mathscr{G}_{\mathscr{Q}}(\varepsilon)\neq\varnothing$ and $x_\varepsilon\in\mathscr{G}_{\mathscr{Q}}(\varepsilon)$ is a global minimum point of problem* (Q) *satisfying $\|x_\varepsilon-\hat{x}\|\leqq\sigma(\hat{x})$, we have $x_\varepsilon\in\mathscr{K}$.*

Then, $F(x;\varepsilon)$ is a weakly exact penalty function for problem (P) *with respect to the set $\mathscr{D}$.*

Proof. We show first that there exists an $\varepsilon^*>0$ such that, for all $\varepsilon\in(0,\varepsilon^*]$ we have $\mathscr{G}_{\mathscr{Q}}(\varepsilon)\neq\varnothing$ and $\mathscr{G}_{\mathscr{Q}}(\varepsilon)\subseteq\mathscr{G}_{\tilde{\mathscr{P}}}$. Recalling condition ($a_1$), we have that, for any given $\varepsilon>0$, there exists a point $x_\varepsilon^*\in\mathscr{E}$ such that:

$$F(x_\varepsilon^*;\varepsilon)=\min_{x\in\mathscr{E}}F(x;\varepsilon).$$

We prove, by contradiction, that there exists an $\varepsilon^*>0$ such that, for all $\varepsilon\in(0,\varepsilon^*]$ the point x_ε^* is a global solution to problem $(\tilde{\mathrm{P}})$. Suppose that this assertion is false. Then, for any integer k there must exist an $\varepsilon_k\leqq 1/k$ and a global minimizer x_k of $F(x;\varepsilon_k)$ on $\mathscr{E}$ such that x_k is not a global solution of problem $(\tilde{\mathrm{P}})$. Let $\tilde{x}$ be a global minimizer of problem $(\tilde{\mathrm{P}})$; then, by (a_3) we have $\tilde{x}\in\mathscr{K}$, so that, by definition of the set $\mathscr{K}$, we have:

$$F(\tilde{x};\varepsilon_k)=f(\tilde{x}).$$

Then, as $\mathscr{K}\subseteq\mathscr{F}\cap\mathring{\mathscr{D}}\subseteq\mathscr{E}$, we can write:

$$F(x_k,\varepsilon_k)=\min_{x\in\mathscr{E}}F(x;\varepsilon_k)\leqq F(\tilde{x};\varepsilon_k)=f(\tilde{x}). \tag{2}$$

Since $\mathcal{D}$ is compact and $\mathcal{E} \subseteq \mathcal{D}$, there exists a convergent subsequence, which we relabel $\{x_k\}$, such that $\lim_{k\to\infty} x_k = \hat{x} \in \mathcal{D}$. By (2) we have:

$$\limsup_{k\to\infty} F(x_k, \varepsilon_k) \leqq f(\tilde{x}),$$

so that (a_2) implies:

$$\hat{x} \in \mathcal{F} \cap \mathcal{D} \quad \text{and} \quad f(\hat{x}) \leqq f(\tilde{x}),$$

whence it follows that $\hat{x}$ is a global minimum point of problem ($\tilde{\text{P}}$). Recalling Assumption (A1), we have $\hat{x} \in \mathring{\mathcal{D}}$ and therefore, since $\lim_{k\to\infty} x_k = \hat{x}$, it follows that for sufficiently large values of k, say $k \geqq k_0$, the point x_k belongs to $\mathring{\mathcal{D}}$. As $\mathring{\mathcal{D}} \subseteq \mathcal{E}$, this implies that

$$F(x_k, \varepsilon_k) = \min_{x\in\mathcal{E}} F(x; \varepsilon_k) \leqq \min_{x\in\mathring{\mathcal{D}}} F(x; \varepsilon_k),$$

that is: $\mathcal{G}_{\mathcal{Q}}(\varepsilon_k) \neq \emptyset$ and $x_k \in \mathcal{G}_{\mathcal{Q}}(\varepsilon_k)$ for $k \geqq k_0$. Moreover, since $\hat{x} \in \mathcal{G}_{\tilde{\mathcal{P}}}$, $\lim_{k\to\infty} \varepsilon_k = 0$ and $\lim_{k\to\infty} x_k = \hat{x}$, we have that there exists an integer $k_1 \geqq k_0$ such that, for all $k \geqq k_1$, $x_k \in \mathcal{G}_{\mathcal{Q}}(\varepsilon_k)$, $\varepsilon_k \leqq \varepsilon(\hat{x})$ and $\|x_k - \hat{x}\| \leqq \sigma(\hat{x})$, where $\varepsilon(\hat{x})$ and $\sigma(\hat{x})$ are the numbers considered in condition (a_4). Therefore (a_4) implies that $x_k \in \mathcal{K}$ for all $k \geqq k_1$, so that, by (2), we obtain:

$$f(x_k) = F(x_k; \varepsilon_k) \leqq f(\tilde{x}), \qquad k \geqq k_1.$$

Hence, $x_k \in \mathcal{F} \cap \mathring{\mathcal{D}}$ is both a global minimum point of $F(x; \varepsilon_k)$ on $\mathcal{E}$ and a global minimum point of problem ($\tilde{\text{P}}$) and this contradicts our original assumption. It can be concluded that there exists an $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ any global minimizer x_ε^* of $F(x; \varepsilon)$ on $\mathcal{E}$ is a global solution to problem ($\tilde{\text{P}}$). On the other hand, by Assumption (A1), the global solutions of problem ($\tilde{\text{P}}$) are in $\mathring{\mathcal{D}}$ and hence, for all $\varepsilon \in (0, \varepsilon^*]$, we have that $x_\varepsilon^* \in \mathring{\mathcal{D}}$ is a global minimizer for problem (Q).

Thus we have proved that for $\varepsilon \in (0, \varepsilon^*]$, $\mathcal{G}_{\mathcal{Q}}(\varepsilon) \neq \emptyset$ and $\mathcal{G}_{\mathcal{Q}}(\varepsilon) \subseteq \mathcal{G}_{\tilde{\mathcal{P}}}$. Now let $\varepsilon \in (0, \varepsilon^*]$ and let x_ε be any point in $\mathcal{G}_{\mathcal{Q}}(\varepsilon) \subseteq \mathcal{G}_{\tilde{\mathcal{P}}}$. By condition ($a_3$) we have $\mathcal{G}_{\mathcal{Q}}(\varepsilon) \subseteq \mathcal{K}$ so that:

$$f(x_\varepsilon) = F(x_\varepsilon; \varepsilon). \tag{3}$$

If $\bar{x}$ is another global minimizer of Problem ($\tilde{\text{P}}$), again by (a_3), we have

$$f(\bar{x}) = F(\bar{x}; \varepsilon). \tag{4}$$

Therefore, as $f(x_\varepsilon) = f(\bar{x})$, (3) and (4) imply that $F(\bar{x}; \varepsilon) = F(x_\varepsilon; \varepsilon)$ and this proves that $\bar{x}$ is a global solution to problem (Q). Thus, $\mathcal{G}_{\tilde{\mathcal{P}}} \subseteq \mathcal{G}_{\mathcal{Q}}(\varepsilon)$ for all $\varepsilon \in (0, \varepsilon^*]$ and this completes the proof. □

A short discussion of conditions (a_1)-(a_4) is in order.

Condition (a_1) requires the existence of a global minimizer of $F(x; \varepsilon)$ on the set $\mathcal{E}$. Two cases are of interest: the case $\mathcal{E} = \mathcal{D}$ and the case $\mathcal{E} = \mathring{\mathcal{D}}$. When $\mathcal{E} = \mathcal{D}$, recalling that $\mathcal{D}$ is compact, the existence of a global minimizer is ensured by the continuity of $F(x; \varepsilon)$; if $\mathcal{E} = \mathring{\mathcal{D}}$, we must specify some further condition on the behavior of $F(x; \varepsilon)$ on $\partial\mathcal{D}$. We shall address this problem later in connection with sufficient conditions for global exactness.

Condition (a_2) indicates the role played by the penalty parameter ε; it requires, in particular, that, as ε goes to zero, if the penalty function remains bounded from above, the constraints are satisfied in the limit.

With regard to (a_3), we may note that this condition is satisfied whenever, for all $\varepsilon > 0$:

$$F(x; \varepsilon) = f(x), \quad \text{for } x \in \mathcal{F} \cap \mathring{\mathcal{D}}.$$

In fact, in this case we have, by (1), that $\mathcal{K} = \mathcal{F} \cap \mathring{\mathcal{D}}$. The different classes of exact penalty functions considered in the sequel are associated with different characterizations of $\mathcal{K}$. In particular, in the case of nondifferentiable penalty functions, we have $\mathcal{K} = \mathcal{F} \cap \mathring{\mathcal{D}}$. However, this requirement would be too strong to allow the construction of continuously differentiable exact penalty functions, as will be apparent from the content of § 5. Thus, in the case of continuously differentiable exact penalty functions, the set $\mathcal{K}$ turns out to be a subset of $\mathcal{F} \cap \mathring{\mathcal{D}}$ containing a region where suitable necessary optimality conditions for problem ($\tilde{\mathrm{P}}$) are satisfied.

Finally, we may note from the proof of Theorem 1 that condition (a_4) is of major relevance in order to establish the properties of exactness considered, since the first three conditions are usually satisfied in the case of sequential penalty functions also.

We give now a sufficient condition for $F(x; \varepsilon)$ to be an exact penalty function in the sense of Definition 2, which is obtained by replacing (a_4) of Theorem 1 with a stronger condition and by imposing that $F(x; \varepsilon)$ is bounded above by $f(x)$ on the set $\mathcal{F} \cap \mathring{\mathcal{D}}$.

More specifically, we state the following theorem.

THEOREM 2. *Let* $F(x; \varepsilon)$ *be such that conditions* (a_1)–(a_3) *of Theorem* 1 *are satisfied and assume further that the following conditions hold.*

(a_5) *There exists an* $\bar{\varepsilon} > 0$ *such that, for all* $\varepsilon \in (0, \bar{\varepsilon}]$, *if* $\mathcal{L}_{\mathcal{D}}(\varepsilon) \neq \emptyset$ *and* $x_\varepsilon \in \mathcal{L}_{\mathcal{D}}(\varepsilon)$, *we have* $x_\varepsilon \in \mathcal{K}$;

(a_6) $F(x; \varepsilon) \leqq f(x)$ *for all* $\varepsilon > 0$ *and* $x \in \mathcal{F} \cap \mathring{\mathcal{D}}$.

Then, $F(x; \varepsilon)$ *is an exact penalty function for problem* (P) *with respect to the set* $\mathcal{D}$, *in the sense of Definition* 2.

Proof. We observe first that condition (a_5) is stronger than condition (a_4) of Theorem 1 so that the function $F(x; \varepsilon)$ is a weakly exact penalty function in the sense of Definition 1. Hence, there exists an $\tilde{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \tilde{\varepsilon}]$ we have $\mathcal{G}_{\mathcal{D}}(\varepsilon) = \mathcal{G}_{\mathcal{F}}$ and this implies $\mathcal{L}_{\mathcal{D}}(\varepsilon) \neq \emptyset$. Now let $\varepsilon^* = \min[\tilde{\varepsilon}, \bar{\varepsilon}]$, where $\bar{\varepsilon}$ is the number considered in condition (a_5), let $\varepsilon \in (0, \varepsilon^*]$, and assume that $x_\varepsilon \in \mathcal{L}_{\mathcal{D}}(\varepsilon)$. Then, by ($a_5$), we have $x_\varepsilon \in \mathcal{K}$ so that we get $F(x_\varepsilon; \varepsilon) = f(x_\varepsilon)$. This implies that for $\varepsilon \in (0, \varepsilon^*]$ and for some $\rho > 0$ it can be written:

$$f(x_\varepsilon) \leqq F(x; \varepsilon) \quad \text{for all } x \in \mathcal{F} \cap \mathcal{B}(x_\varepsilon; \rho). \tag{5}$$

Hence, by (5) and (a_6) we have:

$$f(x_\varepsilon) \leqq f(x) \quad \text{for all } x \in \mathcal{F} \cap \mathcal{B}(x_\varepsilon; \rho),$$

so that x_ε is a local minimizer of problem (P). □

As already observed in the proof of the preceding theorem, condition (a_5) is considerably stronger than (a_4) of Theorem 1; it requires, in particular, that for sufficiently small values of ε *any local minimum point of problem* (Q) *is a feasible point for problem* (P). It will be shown in the sequel that satisfaction of condition (a_5) requires the introduction of a suitable constraint qualification in problem (P).

In order to give sufficient conditions for strong exactness, we now establish a condition which ensures that isolated compact sets of local minimizers of problem (P) correspond to local unconstrained minimizers of $F(x; \varepsilon)$ for sufficiently small values of ε.

PROPOSITION 2. *Let* $\mathcal{C}^*$ *be a nonempty isolated compact set of local minimum points of problem* (P) *corresponding to the local minimum value* f^*, *such that* $\mathcal{C}^* \subset \mathcal{F} \cap \mathring{\mathcal{D}}$. *Let* $F(x; \varepsilon)$ *be such that the following conditions are satisfied.*

(b_1) *If* $\{\varepsilon_k\}$ *and* $\{x_k\} \subset \mathring{\mathcal{D}}$ *are sequences such that* $\lim_{k\to\infty} \varepsilon_k = 0$, $\lim_{k\to\infty} x_k = \hat{x} \in \mathcal{D}$, *and* $\limsup_{k\to\infty} F(x_k, \varepsilon_k) < \infty$, *we have* $\hat{x} \in \mathcal{F} \cap \mathcal{D}$ *and* $f(\hat{x}) \leqq \limsup_{k\to\infty} F(x_k, \varepsilon_k)$.

(b_2) $\mathscr{C}^* \subseteq \mathscr{K}$.

(b_3) *For any* $\tilde{x} \in \mathscr{C}^*$ *there exist numbers* $\varepsilon(\tilde{x}) > 0$ *and* $\sigma(\tilde{x}) > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon(\tilde{x})]$, *if* $\mathscr{L}_{\mathscr{Q}}(\varepsilon) \neq \emptyset$ *and* $x_\varepsilon \in \mathscr{L}_{\mathscr{Q}}(\varepsilon)$ *is a local minimum point of problem* (Q) *satisfying* $\|x_\varepsilon - \tilde{x}\| \leqq \sigma(\tilde{x})$, *we have* $x_\varepsilon \in \mathscr{K}$.

Then, there exists an $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*]$, $\bar{x} \in \mathscr{C}^*$ *implies that* $\bar{x}$ *is a local minimum point of problem* (Q).

Proof. Let $\mathscr{H}$ be the compact set considered in Lemma 1. Since $\mathscr{C}^* \subset \mathring{\mathscr{D}} \cap \mathring{\mathscr{H}}$, we can find a compact set $\mathscr{R} \subset \mathbb{R}^n$ satisfying

$$\mathscr{C}^* \subset \mathring{\mathscr{R}} \quad \text{and} \quad \mathscr{R} \subset \mathring{\mathscr{D}} \cap \mathring{\mathscr{H}},$$

such that

$$f(x) > f^* \quad \text{for all } x \in \mathscr{F} \cap \mathscr{R}, \qquad x \notin \mathscr{C}^*. \tag{6}$$

Now consider the following problem.

$$\text{minimize } f(x), \qquad x \in \mathscr{F} \cap \mathscr{R}. \tag{7}$$

Then, by (6), we have that $\mathscr{C}^* \subset \mathring{\mathscr{R}}$ is the set of global solutions of problem (7). Recalling Theorem 1, it can be easily verified that the function $F(x; \varepsilon)$ is weakly exact with respect to the set $\mathscr{R}$, and hence there exists an $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$, $\bar{x} \in \mathscr{C}^*$ implies that $\bar{x}$ is a global minimum point of the following problem.

$$\text{minimize } F(x; \varepsilon), \qquad x \in \mathring{\mathscr{R}}. \tag{8}$$

On the other hand, since $\mathring{\mathscr{R}} \subset \mathring{\mathscr{D}}$, any global solution of problem (8) is a local minimizer of problem (Q) and this completes the proof. $\square$

Using the preceding result, we can establish a sufficient condition for strong exactness under the following assumption on problem (P).

Assumption (A2). There exists a finite number of isolated compact sets $\mathscr{C}^*(f_i^*)$, $i = 1, \cdots, r$ of local minimum points of problem (P) corresponding to the local minimum values f_i^*, such that $\mathscr{C}^*(f_i^*) \subset \mathring{\mathscr{D}}$ and

$$\mathscr{L}_{\mathscr{P}} \cap \mathring{\mathscr{D}} = \bigcup_{i=1}^{r} \mathscr{C}^*(f_i^*).$$

Assumption (A2) requires that any local minimizer of problem (P) in $\mathring{\mathscr{D}}$ belongs to an isolated compact set of local minimizers and that the number of these sets contained in $\mathring{\mathscr{D}}$ is finite.

Then, we have the following theorem.

THEOREM 3. *Suppose that Assumption* (A2) *holds. Let* $F(x; \varepsilon)$ *be such that the following conditions are satisfied.*

(c_1) *For any* $\varepsilon > 0$, *the function* $F(x; \varepsilon)$ *admits a global minimum point on a set* $\mathscr{E}$, *such that* $\mathring{\mathscr{D}} \subseteq \mathscr{E} \subseteq \mathscr{D}$.

(c_2) *If* $\{\varepsilon_k\}$ *and* $\{x_k\} \subset \mathscr{E}$ *are sequences such that* $\lim_{k\to\infty} \varepsilon_k = 0$, $\lim_{k\to\infty} x_k = \hat{x} \in \mathscr{D}$, *and* $\limsup_{k\to\infty} F(x_k, \varepsilon_k) < \infty$, *we have* $\hat{x} \in \mathscr{F} \cap \mathscr{D}$ *and* $f(\hat{x}) \leqq \limsup_{k\to\infty} F(x_k, \varepsilon_k)$.

(c_3) $\mathscr{L}_{\mathscr{P}} \cap \mathring{\mathscr{D}} \subseteq \mathscr{K}$.

(c_4) *There exists an* $\bar{\varepsilon} > 0$ *such that, for all* $\varepsilon \in (0, \bar{\varepsilon}]$, *if* $\mathscr{L}_{\mathscr{Q}}(\varepsilon) \neq \emptyset$ *and* $x_\varepsilon \in \mathscr{L}_{\mathscr{Q}}(\varepsilon)$, *we have* $x_\varepsilon \in \mathscr{K}$;

(c_5) $F(x; \varepsilon) \leqq f(x)$ *for all* $\varepsilon > 0$ *and* $x \in \mathscr{F} \cap \mathring{\mathscr{D}}$.

Then, $F(x; \varepsilon)$ *is a strongly exact penalty function for problem* (P) *with respect to the set* $\mathscr{D}$.

Proof. By Assumption (A1) we have $\mathscr{G}_{\tilde{\mathscr{P}}} \subseteq \mathscr{L}_{\mathscr{P}} \cap \mathring{\mathscr{D}}$; then, noting that the set of conditions (c_1)–(c_5) implies the conditions stated in Theorem 2, we have that the function $F(x;\varepsilon)$ is exact in the sense of Definition 2 for some threshold value $\varepsilon_0>0$ of the penalty parameter. On the other hand, for $i=1,\cdots,r$ we have $\mathscr{C}^*(f_i^*)\subseteq\mathscr{K}$, so that, since conditions (c_1)-(c_5) also imply conditions (b_1)-(b_3) of Proposition 2, there exist values $\varepsilon_i>0$, $i=1,\cdots,r$ such that for all $\varepsilon\in(0,\varepsilon_i]$ we can assert that $\mathscr{C}^*(f_i^*)\subseteq\mathscr{L}_{\mathscr{D}}(\varepsilon)$. Thus, by letting $\varepsilon^*=\min_{0\leqq i\leqq r}\varepsilon_i$ we have that $F(x;\varepsilon)$ is exact, and, moreover, by Assumption (A2) we have

$$\mathscr{L}_{\mathscr{P}}\cap\mathring{\mathscr{D}}=\bigcup_{i=1}^{r}\mathscr{C}^*(f_i^*)\subseteq\mathscr{L}_{\mathscr{D}}(\varepsilon)$$

for all $\varepsilon\in(0,\varepsilon^*]$, so that the function $F(x;\varepsilon)$ is strongly exact in the sense of Definition 3. $\square$

5. Nondifferentiable exact penalty functions. In this section we shall make use of the sufficient conditions given before in order to establish the exactness of the best-known class of nondifferentiable penalty functions.

We suppose that Assumption (A1) stated in § 4 is satisfied, that is, $\mathscr{G}_{\bar{\mathscr{F}}}\subset\mathring{\mathscr{D}}$.

We consider the class of nondifferentiable penalty functions defined by

$$J_q(x;\varepsilon):=f(x)+\frac{1}{\varepsilon}\,\|[g^+(x)'h(x)']'\|_q,$$

where $1\leqq q\leqq\infty$. In particular, we have

$$J_q(x;\varepsilon)=f(x)+\frac{1}{\varepsilon}\left[\sum_{i=1}^{m}(g_i^+(x))^q+\sum_{j=1}^{p}|h_j(x)|^q\right]^{1/q},$$

for $1\leqq q<\infty$, and

$$J_\infty(x;\varepsilon)=f(x)+\frac{1}{\varepsilon}\max\,[g_1^+(x),\cdots,g_m^+(x),|h_1(x)|,\cdots,|h_p(x)|].$$

For this class of functions the set $\mathscr{K}$ defined in (1) is obviously obtained as

$$\mathscr{K}=\mathscr{F}\cap\mathring{\mathscr{D}}.$$

The expression of the directional derivative $DJ_q(x,d;\varepsilon)$ of $J_q(x;\varepsilon)$ is given in the following proposition, which is proved in full detail in [16].

PROPOSITION 3. *For all $\varepsilon>0$ and $d\in\mathbb{R}^n$, the function $J_q(x;\varepsilon)$ admits a directional derivative $DJ_q(x,d;\varepsilon)$.*

Let

$$\xi_i(x,d):=\begin{cases}\nabla g_i(x)'d, & \text{if } g_i(x)>0;\\ (\nabla g_i(x)'d)^+, & \text{if } g_i(x)=0;\\ 0, & \text{if } g_i(x)<0\end{cases}\tag{9}$$

and

$$\zeta_j(x,d):=\begin{cases}\nabla h_j(x)'d, & \text{if } h_j(x)>0;\\ |\nabla h_j(x)'d|, & \text{if } h_j(x)=0;\\ -\nabla h_j(x)'d, & \text{if } h_j(x)<0.\end{cases}\tag{10}$$

Then, we have:

(a) *for $q=1$:*

$$DJ_1(x,d;\varepsilon)=\nabla f(x)'d+\frac{1}{\varepsilon}\left(\sum_{i=1}^{m}\xi_i(x,d)+\sum_{j=1}^{p}\zeta_j(x,d)\right);$$

(b) *for* $1<q<\infty$, $x\notin\mathcal{F}$:

$$DJ_q(x, d;\varepsilon)=\nabla f(x)'d + \frac{1}{\varepsilon\|[g^+(x)'h(x)']'\|_q^{q-1}}\left[\sum_{i=1}^{m}(g_i^+(x))^{q-1}\xi_i(x,d)+\sum_{j=1}^{p}|h_j(x)|^{q-1}\zeta_j(x,d)\right];$$

(c) *for* $1<q<\infty$, $x\in\mathcal{F}$:

$$DJ_q(x, d;\varepsilon)=\nabla f(x)'d+\frac{1}{\varepsilon}\left[\sum_{i=1}^{m}(\xi_i(x,d))^q+\sum_{j=1}^{p}(\zeta_j(x,d))^q\right]^{1/q};$$

(d) *for* $q=\infty$:

$$DJ_\infty(x, d;\varepsilon)=\nabla f(x)'d+\frac{1}{\varepsilon}\max\left[\{\xi_i(x,d), i\in I_1(x)\},\{\zeta_j(x,d), j\in I_2(x)\}\right],$$

where

$$I_1(x):=\{i: g_i^+(x)=\|[g^+(x)'h(x)']'\|_\infty\}$$
$$I_2(x):=\{j: |h_j(x)|=\|[g^+(x)'h(x)']'\|_\infty\}.$$

The next two propositions, which have been proved in [17], play a significant role in establishing the exactness properties of $J_q(x;\varepsilon)$.

PROPOSITION 4. *Let* $\hat{x}\in\mathcal{F}$ *and assume that the* MFCQ *holds at* $\hat{x}$. *Then, there exist numbers* $\varepsilon(\hat{x})>0$ *and* $\sigma(\hat{x})>0$ *such that, for all* $\varepsilon\in(0,\varepsilon(\hat{x})]$, *if* x_ε *is a critical point of* $J_q(x;\varepsilon)$ *satisfying* $\|x_\varepsilon-\hat{x}\|\leqq\sigma(\hat{x})$, *we have* $x_\varepsilon\in\mathcal{F}$.

PROPOSITION 5. *Assume that the* EMFCQ *holds on* $\mathcal{D}$. *Then, there exists an* $\bar{\varepsilon}>0$ *such that, for all* $\varepsilon\in(0,\bar{\varepsilon}]$, *if* $x_\varepsilon\in\mathcal{D}$ *is a critical point of* $J_q(x;\varepsilon)$, *we have* $x_\varepsilon\in\mathcal{F}$.

We can now prove that, under suitable assumptions on problem (P), the function $J_q(x;\varepsilon)$ satisfies the sufficient conditions of exactness stated in the preceding section.

THEOREM 4. (a) *Assume that the* MFCQ *is satisfied at every global minimum point of problem* ($\tilde{\mathrm{P}}$). *Then, the function* $J_q(x;\varepsilon)$ *is a weakly exact penalty function for problem* (P) *with respect to the set* $\mathcal{D}$.

(b) *Assume that the* EMFCQ *is satisfied on* $\mathcal{D}$. *Then, the function* $J_q(x;\varepsilon)$ *is an exact penalty function for problem* (P) *with respect to the set* $\mathcal{D}$; *moreover, if Assumption* (A2) *holds, the function* $J_q(x;\varepsilon)$ *is a strongly exact penalty function for problem* (P) *with respect to the set* $\mathcal{D}$.

Proof. We show first that conditions (a_1)–(a_4) of Theorem 1 are satisfied.

Let $\mathcal{E}=\mathcal{D}$; then, (a_1) follows from the continuity of $J_q(x;\varepsilon)$ and the compactness of $\mathcal{D}$.

With regard to condition (a_2), let $\{\varepsilon_k\}$ and $\{x_k\}\subset\mathcal{D}$ be sequences such that $\varepsilon_k>0$, $\lim_{k\to\infty}\varepsilon_k=0$, $\lim_{k\to\infty}x_k=\hat{x}\in\mathcal{D}$, and assume that

$$\limsup_{k\to\infty} J_q(x_k;\varepsilon_k)=\eta<\infty.$$

This implies

$$f(\hat{x})+\limsup_{k\to\infty}\frac{1}{\varepsilon_k}\|[g^+(x_k)'h(x_k)']'\|_q=\eta,$$

whence it follows that

$$\|[g^+(\hat{x})'h(\hat{x})']'\|_q=0 \quad\text{and}\quad f(\hat{x})\leqq\eta$$

so that (a_2) is satisfied.

Condition (a_3) follows from the definition of $J_q(x; \varepsilon)$, since

$$J_q(x; \varepsilon) = f(x) \quad \text{for all } x \in \mathcal{F}.$$

Finally, condition (a_4) follows from Proposition 4, since any global minimum point of problem (Q) is a critical point of $J_q(x; \varepsilon)$. This concludes the proof of (a).

Consider now the conditions stated in Theorem 2; we have already proved that (a_1)-(a_3) hold. Condition (a_5) is implied by the result given in Proposition 5 and condition (a_6) follows from the definition of $J_q(x; \varepsilon)$. It can be easily verified also that conditions (c_1), (c_2), (c_4), and (c_5) of Theorem 3 reduce to conditions (a_1), (a_2), (a_5), and (a_6), and that condition (c_3) follows from the definition of $J_q(x; \varepsilon)$. Thus (b) follows from Theorems 2 and 3. □

In the next two propositions we report additional results concerning the correspondence between critical points of $J_q(x; \varepsilon)$ and K-T triples of problem (P).

PROPOSITION 6. *Let $\bar{x} \in \mathcal{F}$; then, if $\bar{x}$ is a critical point of $J_q(x; \varepsilon)$ we have $\bar{x} \in \mathcal{T}$. Moreover, if the* EMFCQ *holds on $\mathcal{D}$, there exists an $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$, if $x_\varepsilon \in \mathring{\mathcal{D}}$ is a critical point of $J_q(x; \varepsilon)$, we have $x_\varepsilon \in \mathcal{T}$.*

Proof. Let us define the following set.

$$\mathcal{Z} := \{z \in \mathbb{R}^n : \nabla g_i(\bar{x})'z \leqq 0,\ i \in I_0(\bar{x}),$$
$$\nabla h_j(\bar{x})'z = 0,\ j = 1, \cdots, p,\ \nabla f(\bar{x})'z < 0\}.$$

It is known that, by Farkas' lemma, $\mathcal{Z} = \emptyset$ implies that there exist $\bar{\lambda} \in \mathbb{R}^m$ and $\bar{\mu} \in \mathbb{R}^p$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a K-T triple for problem (P). (See, e.g., [23, p. 18].)

We prove first that if $\bar{x} \in \mathcal{F}$ is a critical point of $J_q(x; \varepsilon)$, we have $\mathcal{Z} = \emptyset$. In fact, since $\bar{x} \in \mathcal{F}$, we have, by Proposition 3

$$DJ_q(\bar{x}, z; \varepsilon) = \nabla f(\bar{x})'z + \frac{1}{\varepsilon}\left[\sum_{i \in I_0(\bar{x})} [(\nabla g_i(\bar{x})'z)^+]^q + \sum_{j=1}^{p} |\nabla h_j(\bar{x})'z|^q\right]^{1/q},$$

for $1 \leqq q < \infty$, and

$$DJ_\infty(\bar{x}, z; \varepsilon) = \nabla f(\bar{x})'z + \frac{1}{\varepsilon} \max\,[(\nabla g_i(\bar{x})'z)^+,\ i \in I_0(\bar{x}),\ |\nabla h_j(\bar{x})'z|,\ j = 1, \cdots, p].$$

It follows that, whenever $\nabla g_i(\bar{x})'z \leqq 0$, for $i \in I_0(\bar{x})$ and $\nabla h_j(\bar{x})'z = 0$, for $j = 1, \cdots, p$ we have:

$$DJ_q(\bar{x}, z; \varepsilon) = \nabla f(\bar{x})'z.$$

Therefore, as $\bar{x}$ is a critical point of $J_q(x; \varepsilon)$, we have $\nabla f(\bar{x})'z \geqq 0$ for all z satisfying

$$\nabla g_i(\bar{x})'z \leqq 0, \qquad i \in I_0(\bar{x})$$
$$\nabla h_j(\bar{x})'z = 0, \qquad j = 1, \cdots, p$$

and this implies $\mathcal{Z} = \emptyset$, so that $\bar{x} \in \mathcal{T}$. Now, recalling Proposition 5, we have that if the EMFCQ holds on $\mathcal{D}$ there exists an $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ if $x_\varepsilon \in \mathring{\mathcal{D}}$ is a critical point of $J_q(x; \varepsilon)$ we have $x_\varepsilon \in \mathcal{F}$ and hence $x_\varepsilon \in \mathcal{T}$. □

PROPOSITION 7. *Let $(\bar{x}, \bar{\lambda}, \bar{\mu})$ be a* K-T *triple for Problem* (P). *Then*: (a) *$\bar{x}$ is a critical point of $J_q(x; \varepsilon)$, $1 \leqq q < \infty$ for all $\varepsilon > 0$ such that*

$$\bar{\lambda}_i \varepsilon \leqq (m+p)^{(1-q)/q}, \qquad i \in I_0(\bar{x})$$
$$|\bar{\mu}_j| \varepsilon \leqq (m+p)^{(1-q)/q}, \qquad j = 1, \cdots, p.$$

(b) $\bar{x}$ *is a critical point of* $J_\infty(x;\varepsilon)$, *for all* $\varepsilon>0$ *such that*

$$\varepsilon\left[\sum_{i\in I_0(\bar{x})}\bar{\lambda}_i+\sum_{j=1}^{p}|\bar{\mu}_j|\right]\leqq 1.$$

Proof. Since $(\bar{x},\bar{\lambda},\bar{\mu})$ is a K-T triple for problem (P), we can write

$$\nabla f(x)=-\left(\sum_{i\in I_0(\bar{x})}\bar{\lambda}_i\nabla g_i(\bar{x})+\sum_{j=1}^{p}\bar{\mu}_j\nabla h_j(\bar{x})\right). \tag{11}$$

Consider first the case $1\leqq q<\infty$. As $\bar{x}\in\mathcal{F}$, by Proposition 3 we have, for any given $d\in\mathbb{R}^n$:

$$DJ_q(\bar{x},d;\varepsilon)=\nabla f(\bar{x})'d+\frac{1}{\varepsilon}\left[\sum_{i\in I_0(\bar{x})}[(\nabla g_i(\bar{x})'d)^+]^q+\sum_{j=1}^{p}|\nabla h_j(\bar{x})'d|^q\right]^{1/q},$$

so that, by (11):

$$DJ_q(\bar{x},d;\varepsilon)\geqq\frac{1}{\varepsilon}\left[\sum_{i\in I_0(\bar{x})}[(\nabla g_i(\bar{x})'d)^+]^q+\sum_{j=1}^{p}|\nabla h_j(\bar{x})'d|^q\right]^{1/q}$$
$$-\left[\sum_{i\in I_0(\bar{x})}\bar{\lambda}_i(\nabla g_i(\bar{x})'d)^++\sum_{j=1}^{p}|\bar{\mu}_j||\nabla h_j(\bar{x})'d|\right].$$

Using Hölder's inequality, we can write

$$DJ_q(\bar{x},d;\varepsilon)\geqq\sum_{i\in I_0(\bar{x})}\left[\frac{1}{\varepsilon(m+p)^{(q-1)/q}}-\bar{\lambda}_i\right](\nabla g_i(\bar{x})'d)^+$$
$$+\sum_{j=1}^{p}\left[\frac{1}{\varepsilon(m+p)^{(q-1)/q}}-|\bar{\mu}_j|\right]|\nabla h_j(\bar{x})'d|$$

and this implies (a).

Consider now the case $q=\infty$. Since $\bar{x}\in\mathcal{F}$, we have, by Proposition 3:

$$DJ_\infty(\bar{x},d;\varepsilon)=\nabla f(\bar{x})'d+\frac{1}{\varepsilon}\max\left[(\nabla g_i(\bar{x})'d)^+,\ i\in I_0(\bar{x}),\ |\nabla h_j(\bar{x})'d|,\ j=1,\cdots,p\right].$$

By (11), we can write:

$$DJ_\infty(\bar{x},d;\varepsilon)\geqq\frac{1}{\varepsilon}\max\left[(\nabla g_i(\bar{x})'d)^+,\ i\in I_0(\bar{x}),\ |\nabla h_j(\bar{x})'d|,\ j=1,\cdots,p\right]$$
$$-\left[\sum_{i\in I_0(\bar{x})}\bar{\lambda}_i(\nabla g_i(\bar{x})'d)^++\sum_{j=1}^{p}|\bar{\mu}_j|\,|\nabla h_j(\bar{x})'d|\right]$$

whence, noting that

$$\sum_{i\in I_0(\bar{x})}\bar{\lambda}_i(\nabla g_i(\bar{x})'d)^++\sum_{j=1}^{p}|\bar{\mu}_j|\,|\nabla h_j(\bar{x})'d|$$
$$\leqq\left[\sum_{i\in I_0(\bar{x})}\bar{\lambda}_i+\sum_{j=1}^{p}|\bar{\mu}_j|\right]\max\left[(\nabla g_i(\bar{x})'d)^+,\ i\in I_0(\bar{x}),\ |\nabla h_j(\bar{x})'d|,\ j=1,\cdots,p\right]$$

we obtain (b). □

We consider now, under suitable compactness assumptions on the feasible set, the nondifferentiable penalty function with global exactness properties proposed in [16]. This function incorporates a barrier term which goes to infinity on the boundary of a compact perturbation of the feasible set and can be viewed as a generalization of the "M_2" penalty function introduced in [22].

Let

$$\mathcal{S}_\beta := \{x \in \mathbb{R}^n, g(x) \leqq \alpha, \|h(x)\|_2^2 \leqq \alpha_0\}$$

be the set introduced in § 2 and suppose that the following assumption is satisfied.

Assumption (A3). The set $\mathcal{S}_\beta$ is compact.

Obviously, this assumption implies that the feasible set is compact. We can take $\mathcal{D} = \mathcal{S}_\beta$, so that $\mathcal{F} \subset \mathring{\mathcal{D}}$, problem ($\tilde{\text{P}}$) reduces to the original problem (P), and Assumption (A1) of § 4 is satisfied.

Let us introduce the functions:

$$a_0(x) := \alpha_0 - \|h(x)\|_2^2$$
$$a_i(x) := \alpha_i - g_i(x), \qquad i = 1, \cdots, m$$

and denote by $A(x)$ the diagonal matrix:

$$A(x) := \operatorname{diag}(a_i(x)), \qquad i = 1, \cdots, m.$$

We have, obviously, that $a_i(x) > 0$, $i = 0, 1, \cdots, m$, for all $x \in \mathring{\mathcal{D}}$.

Then, we consider the following function

$$Z_q(x;\varepsilon) := f(x) + \frac{1}{\varepsilon} \left\| \left[(A^{-1}(x) g^+(z))' \frac{h(x)'}{a_0(x)} \right]' \right\|_q,$$

where $\varepsilon > 0$ and $1 \leqq q \leqq \infty$. In particular, we have:

$$Z_q(x;\varepsilon) = f(x) + \frac{1}{\varepsilon} \left[\sum_{i=1}^m \left(\frac{g_i^+(x)}{a_i(x)} \right)^q + \frac{1}{a_0(x)^q} \sum_{j=1}^p |h_j(x)|^q \right]^{1/q},$$

for $1 \leqq q < \infty$, and

$$Z_\infty(x;\varepsilon) = f(x) + \frac{1}{\varepsilon} \max \left[\frac{g_1^+(x)}{a_1(x)}, \cdots, \frac{g_m^+(x)}{a_m(x)}, \frac{|h_1(x)|}{a_0(x)}, \cdots, \frac{|h_p(x)|}{a_0(x)} \right].$$

An equivalent expression of $Z_q(x;\varepsilon)$ can be derived by defining the functions

$$\hat{g}_i(x) = \frac{g_i(x)}{a_i(x)}, \qquad i = 1, \cdots, m \tag{12}$$

$$\hat{h}_j(x) = \frac{h_j(x)}{a_0(x)}, \qquad j = 1, \cdots, p. \tag{13}$$

Using (12) and (13) we can write

$$Z_q(x;\varepsilon) = f(x) + \frac{1}{\varepsilon} \|[\hat{g}^+(x)' \hat{h}(x)']'\|_q.$$

Taking this into account, the expression of the directional derivative $DZ_q(x, d; \varepsilon)$ can be obtained from (a)–(d) of Proposition 4 by replacing $g(x)$ with $\hat{g}(x)$ and $h(x)$ with $h(x)$. By this substitution we have for the gradients

$$\nabla \hat{g}_i(x) = \frac{a_i}{a_i^2(x)} \nabla g_i(x), \qquad i = 1, \cdots, m \tag{14}$$

$$\nabla \hat{h}_j(x) = \frac{1}{a_0(x)} \nabla h_j(x) + 2 \frac{h_j(x)}{a_0^2(x)} \frac{\partial h(x)'}{\partial x} h(x), \qquad j = 1, \cdots, p. \tag{15}$$

We can now perform the analysis of the exactness properties of the function $Z_q(x;\varepsilon)$ making use of the sufficient conditions given in the preceding section. The analysis is based on the fact that, by construction, the function $Z_q(x;\,\varepsilon)$ is defined for all $x \in \mathring{\mathcal{D}}$ and goes to infinity for x converging to a point of $\partial\mathcal{D}$; then, by Definition 4 we have that $Z_q(x;\,\varepsilon)$ is a globally (weakly, strongly) exact penalty function for problem (P) with respect to the set $\mathcal{D}$ if it is (weakly, strongly) exact in the sense of Definitions 1, 2, and 3.

The following propositions, which have been established in [16], can be viewed as the analogues of Propositions 5 and 6.

PROPOSITION 8. *Let $\hat{x} \in \mathcal{F}$ and assume that the* MFCQ *holds at $\hat{x}$. Then, there exist numbers $\varepsilon(\hat{x}) > 0$ and $\sigma(\hat{x}) > 0$ such that, for all $\varepsilon \in (0, \varepsilon(\hat{x})]$, if $x_\varepsilon \in \mathring{\mathcal{D}}$ is a critical point of $Z_q(x;\,\varepsilon)$ satisfying $\|x_\varepsilon - \hat{x}\| \leqq \sigma(\hat{x})$, we have $x_\varepsilon \in \mathcal{F}$.*

PROPOSITION 9. *Assume that the* EMFCQ *holds on $\mathcal{D}$. Then, there exists an $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}]$, if $x_\varepsilon \in \mathring{\mathcal{D}}$ is a critical point of $Z_q(x;\,\varepsilon)$, we have $x_\varepsilon \in \mathcal{F}$.*

Using the preceding results, it is possible to establish the properties of exactness of $Z_q(x;\,\varepsilon)$, which are collected in the following theorem.

THEOREM 5. (a) *Assume that the* MFCQ *is satisfied at every global minimum point of problem* (P). *Then, the function $Z_q(x;\,\varepsilon)$ is a globally weakly exact penalty function for problem* (P) *with respect to the set $\mathcal{D}$.*

(b) *Assume that the* EMFCQ *is satisfied on $\mathcal{D}$. Then, the function $Z_q(x;\,\varepsilon)$ is a globally exact penalty function for problem* (P) *with respect to the set $\mathcal{D}$; moreover, if Assumption* (A2) *holds, the function $Z_q(x;\,\varepsilon)$ is a globally strongly exact penalty function for problem* (P) *with respect to the set $\mathcal{D}$.*

Proof. By construction, we have $\lim_{k\to\infty} Z_q(x_k;\,\varepsilon) = \infty$ for any sequence $\{x_k\} \subset \mathring{\mathcal{D}}$ such that $x_k \to y \in \partial\mathcal{D}$. Hence, by Definition 4 we have that $Z_q(x;\,\varepsilon)$ is globally (weakly, strongly) exact if it is (weakly, strongly) exact.

With regard to 5(a), letting $\mathcal{E} = \mathring{\mathcal{D}}$, we can proceed as in the proof of Theorem 4 making use of Proposition 8 in place of Proposition 4; the proof of 5(b) is similar to that of Theorem 4(b) provided that we employ Proposition 9 in place of Proposition 5. □

Finally, we state without proof the relationships between critical points of $Z_q(x;\,\varepsilon)$ and K-T triples of problem (P).

Noting that, for $x \in \mathcal{F}$ we have $\nabla\hat{g}_i(x)'z \leqq 0$ and $\nabla\hat{h}_j(x)'z = 0$ if and only if $\nabla g_i(x)'z \leqq 0$ and $\nabla h_j(x)'z = 0$, and recalling Proposition 9, the proof of Proposition 6 can be easily modified to yield the following result.

PROPOSITION 10. *Let $\bar{x} \in \mathcal{F}$; then, if $\bar{x}$ is a critical point of $Z_q(x;\,\varepsilon)$, we have $\bar{x} \in \mathcal{T}$. Moreover, if the* EMFCQ *holds on $\mathcal{D}$, there exists an $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$, if $x_\varepsilon \in \mathring{\mathcal{D}}$ is a critical point of $Z_q(x;\,\varepsilon)$, we have $x_\varepsilon \in \mathcal{T}$.*

The next proposition is an analogue of Proposition 7 which can be established by taking into account formulas (14) and (15) when considering the expression $DZ_q(x, d;\,\varepsilon)$.

PROPOSITION 11. *Let $(\bar{x}, \bar{\lambda}, \bar{\mu})$ be a K-T triple for problem* (P). *Then*: (a) *$\bar{x}$ is a critical point of $Z_q(x;\,\varepsilon)$, $1 \leqq q < \infty$ for all $\varepsilon > 0$ such that*:

$$\bar{\lambda}_i\varepsilon \leqq \frac{1}{\alpha_i}(m+p)^{(1-q)/q}, \qquad i \in I_0(\bar{x})$$

$$|\bar{\mu}_j|\varepsilon \leqq \frac{1}{\alpha_0}(m+p)^{(1-q)/q}, \qquad j = 1, \cdots, p.$$

(b) *$\bar{x}$ is a critical point of $Z_\infty(x;\,\varepsilon)$, for all $\varepsilon > 0$ such that*:

$$\varepsilon\left[\sum_{i\in I_0(\bar{x})} \bar{\lambda}_i\alpha_i + \sum_{j=1}^{p} |\bar{\mu}_j|\alpha_0\right] \leqq 1.$$

6. Continuously differentiable exact penalty functions. In this section we study a class of continuously differentiable exact penalty functions, making use of the sufficient conditions established in § 4.

The key idea for the construction of continuously differentiable exact penalty functions is that of replacing the multiplier vectors (λ, μ) which appear in the augmented Lagrangian function of Hestenes, Powell, and Rockafellar [4] with continuously differentiable *multiplier functions* $(\lambda(x), \mu(x))$, depending on the problem variables.

Let

$$\mathscr{X} := \{x \in \mathbb{R}^n : \nabla g_i(x),\ i \in I_0(x),\ \nabla h_j(x),\ j = 1, \cdots, p \text{ are linearly independent}\};$$

then, for any $x \in \mathscr{X}$ we can consider the multiplier functions $(\lambda(x), \mu(x))$ introduced in [28], which are obtained by minimizing over $\mathbb{R}^m \times \mathbb{R}^p$ the quadratic function in (λ, μ) defined by:

$$\Psi(\lambda, \mu; x) := \|\nabla_x L(x, \lambda, \mu)\|^2 + \gamma^2 \|G(x)\lambda\|^2,$$

where $\gamma \neq 0$ and

$$G(x) := \operatorname{diag}(g_i(x)).$$

The function $\Psi(\lambda, \mu; x)$ can be viewed as a measure of the violation of the set of K-T necessary conditions:

$$\nabla_x L(x, \lambda, \mu) = 0, \qquad G(x)\lambda = 0.$$

Let $N(x)$ be the $(m+p) \times (m+p)$ matrix defined by:

$$N(x) := \begin{bmatrix} \dfrac{\partial g(x)}{\partial x} \dfrac{\partial g(x)'}{\partial x} + \gamma^2 G^2(x) & \dfrac{\partial g(x)}{\partial x} \dfrac{\partial h(x)'}{\partial x} \\ \dfrac{\partial h(x)}{\partial x} \dfrac{\partial g(x)'}{\partial x} & \dfrac{\partial h(x)}{\partial x} \dfrac{\partial h(x)'}{\partial x} \end{bmatrix}.$$

In the next proposition we recall some known results established in [28].

PROPOSITION 12. *Let* $\bar{x} \in \mathscr{X}$ *and* $\gamma \neq 0$. *Then*: (a) *the matrix* $N(x)$ *is positive definite*;

(b) *there exists a unique minimizer* $(\lambda(x), \mu(x))$ *of the quadratic function in* (λ, μ), $\Psi(\lambda, \mu; x)$, *given by*

$$\begin{bmatrix} \lambda(x) \\ \mu(x) \end{bmatrix} = -N^{-1}(x) \begin{bmatrix} \dfrac{\partial g(x)}{\partial x} \\ \dfrac{\partial h(x)}{\partial x} \end{bmatrix} \nabla f(x);$$

(c) *if* $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathscr{X} \times \mathbb{R}^m \times \mathbb{R}^p$ *is a triple such that* $\nabla_x L(\bar{x}, \bar{\lambda}, \bar{\mu}) = 0$ *and* $G(\bar{x})\bar{\lambda} = 0$, *we have* $\lambda(\bar{x}) = \bar{\lambda}$ *and* $\mu(\bar{x}) = \bar{\mu}$;

(d) *the Jacobian matrices of* $\lambda(x)$ *and* $\mu(x)$ *are given by*

$$\begin{bmatrix} \dfrac{\partial \lambda(x)}{\partial x} \\ \dfrac{\partial \mu(x)}{\partial x} \end{bmatrix} = -N^{-1}(x) \begin{bmatrix} R(x) \\ S(x) \end{bmatrix}, \tag{16}$$

where

$$R(x) := \frac{\partial g(x)}{\partial x} \nabla_x^2 L(x, \lambda(x), \mu(x)) + \sum_{i=1}^{m} e_i^m \nabla_x L(x, \lambda(x), \mu(x))' \nabla^2 g_i(x)$$
$$+ 2\gamma^2 \Lambda(x) G(x) \frac{\partial g(x)}{\partial x}$$

$$S(x) := \frac{\partial h(x)}{\partial x} \nabla_x^2 L(x, \lambda(x), \mu(x)) + \sum_{j=1}^{p} e_j^p \nabla_x L(x, \lambda(x), \mu(x))' \nabla^2 h_j(x)$$

$$\nabla_x L(x, \lambda(x), \mu(x)) := [\nabla_x L(x, \lambda, \mu)]_{\substack{\lambda=\lambda(x)\\ \mu=\mu(x)}}$$

$$\nabla_x^2 L(x, \lambda(x), \mu(x)) := [\nabla_x^2 L(x, \lambda, \mu)]_{\substack{\lambda=\lambda(x)\\ \mu=\mu(x)}}$$

$$\Lambda(x) := \text{diag}\,(\lambda_i(x))$$

and $e_i^m(e_j^p)$ *denote the* ith(jth) *column of the* $m \times m(p \times p)$ *identity matrix.*

Thus we can consider the penalty function introduced in [28], defined by

$$\begin{aligned} W(x; \varepsilon) := f(x) + \lambda(x)'(g(x) + Y(x; \varepsilon)y(x; \varepsilon)) + \frac{1}{\varepsilon} \|g(x) + Y(x; \varepsilon)y(x; \varepsilon)\|^2 \\ + \mu(x)'h(x) + \frac{1}{\varepsilon} \|h(x)\|^2, \end{aligned} \tag{17}$$

where

$$\begin{aligned} y_i(x; \varepsilon) &:= \left\{-\min\left[0, g_i(x) + \frac{\varepsilon}{2}\lambda_i(x)\right]\right\}^{1/2}, \qquad i = 1, \cdots, m \\ Y(x; \varepsilon) &:= \text{diag}\,(y_i(x; \varepsilon)). \end{aligned} \tag{18}$$

It can be verified that the function $W(x; \varepsilon)$ can also be written in the form

$$\begin{aligned} W(x; \varepsilon) = f(x) + \lambda(x)'g(x) + \frac{1}{\varepsilon} \|g(x)\|^2 + \mu(x)'h(x) \\ + \frac{1}{\varepsilon} \|h(x)\|^2 - \frac{1}{4\varepsilon} \sum_{i=1}^{m} \{\min[0, \varepsilon\lambda_i(x) + 2g_i(x)]\}^2. \end{aligned}$$

From the above expression and the differentiability assumptions on the problem functions, it follows that $W(x; \varepsilon)$ is continuously differentiable on $\mathscr{X}$. The expression of $W(x; \varepsilon)$ can be derived by means of the following reasoning.

Consider the transformed problem:

$$\begin{aligned} &\text{minimize}\, f(x) \\ &\text{subject to}\; g(x) + Yy = 0, \qquad h(x) = 0, \end{aligned}$$

where y_i, $i = 1, \cdots, m$ are slack variables and $Y := \text{diag}\,(y_i)$.

Define the augmented Lagrangian function for this problem:

$$L_a(x, y, \lambda, \mu; \varepsilon) := f(x) + \lambda'(g(x) + Yy) + \frac{1}{\varepsilon} \|g(x) + Yy\|^2 + \mu'h(x) + \frac{1}{\varepsilon} \|h(x)\|^2.$$

Then, by substituting $(\lambda(x), \mu(x))$ for (λ, μ) and minimizing with respect to y, we get the function $W(x; \varepsilon)$, that is,

$$W(x; \varepsilon) = L_a(x, y(x; \varepsilon), \lambda(x), \mu(x); \varepsilon) = \min_y L_a(x, y, \lambda(x), \mu(x); \varepsilon).$$

Since, by construction,

$$[\nabla_y L_a(x, y, \lambda, \mu; \varepsilon)]_{\substack{\lambda=\lambda(x)\\ \mu=\mu(x)\\ y=y(x;\, \varepsilon)}} = 0,$$

the gradient expression of $W(x;\varepsilon)$ can be obtained by treating formally $y(x;\varepsilon)$ as a constant vector. Thus, we have:

$$\begin{aligned}\nabla W(x;\varepsilon)=\nabla f(x)&+\frac{\partial g(x)'}{\partial x}\lambda(x)+\frac{\partial h(x)'}{\partial x}\mu(x)\\&+\frac{2}{\varepsilon}\frac{\partial g(x)'}{\partial x}(g(x)+Y(x;\varepsilon)y(x;\varepsilon))+\frac{2}{\varepsilon}\frac{\partial h(x)'}{\partial x}h(x)\\&+\frac{\partial\lambda(x)'}{\partial x}(g(x)+Y(x;\varepsilon)y(x;\varepsilon))+\frac{\partial\mu(x)'}{\partial x}h(x),\end{aligned}$$

where $\partial\lambda(x)/\partial x$ and $\partial\mu(x)/\partial x$ are the Jacobian matrices defined in (16).

Some properties of exactness of the function $W(x;\varepsilon)$ have been established in [15] for inequality constrained problems. Here we perform a more complete analysis for problems with both equality and inequality constraints, making use of the sufficient conditions given in § 4.

We suppose that Assumption (A1) of § 4 is satisfied, that is, $\mathcal{G}_{\bar{\mathcal{F}}}\subset\mathring{\mathcal{D}}$, and that everywhere in this section the following assumption holds.

Assumption (A4). The LICQ is satisfied on $\mathcal{D}$, that is, $\mathcal{D}\subset\mathcal{X}$.

Some immediate consequences of the definition of $W(x;\varepsilon)$ are pointed out in the following proposition.

PROPOSITION 13. *Let* $(\bar{x},\bar{\lambda},\bar{\mu})$ *be a* K-T *triple for problem* (P), *such that* $\bar{x}\in\mathcal{D}$. *Then, for any* $\varepsilon>0$, *we have*: (a) $g(\bar{x})+Y(\bar{x};\varepsilon)y(\bar{x};\varepsilon)=0$;

(b) $W(\bar{x};\varepsilon)=f(\bar{x})$;

(c) $\nabla W(\bar{x};\varepsilon)=0$.

Proof. By Proposition 12 we have $\lambda(\bar{x})=\bar{\lambda}$ and $\mu(\bar{x})=\bar{\mu}$, so that, since $(\bar{x},\bar{\lambda},\bar{\mu})$ is a K-T triple for problem (P), we obtain $\nabla_x L(\bar{x},\lambda(\bar{x}),\mu(\bar{x}))=0$, $\lambda(\bar{x})\geqq 0$ and $\lambda_i(\bar{x})=0$ when $g_i(\bar{x})<0$. Then, (a) is satisfied by definition of $y(\bar{x};\varepsilon)$ and (b) follows directly from (17). Finally, (c) follows from (19), taking (a) into account and noting that, by assumption, $h(\bar{x})=0$ and $\nabla_x L(\bar{x},\lambda(\bar{x}),\mu(\bar{x}))=\nabla_x L(\bar{x},\bar{\lambda},\bar{\mu})$. □

Then, we have the following proposition.

PROPOSITION 14. *Let* $\hat{x}\in\mathcal{F}\cap\mathcal{D}$. *Then, there exist numbers* $\varepsilon(\hat{x})>0$ *and* $\sigma(\hat{x})>0$ *such that, for all* $\varepsilon\in(0,\varepsilon(\hat{x})]$, *if* $x_\varepsilon\in\mathring{\mathcal{D}}$ *is a stationary point of* $W(x;\varepsilon)$ *satisfying* $\|x_\varepsilon-\hat{x}\|\leqq\sigma(\hat{x})$, *we have that* $(x_\varepsilon,\lambda(x_\varepsilon),\mu(x_\varepsilon))$ *is a* K-T *triple for problem* (P).

Proof. Let $x\in\mathcal{X}$; then, by definition of $y(x;\varepsilon)$, we have

$$Y^2(x;\varepsilon)\lambda(x)=-\frac{2}{\varepsilon}Y^2(x;\varepsilon)(g(x)+Y(x;\varepsilon)y(x;\varepsilon));\tag{20}$$

moreover, by definition of $\lambda(x)$ we can write:

$$\begin{aligned}\frac{\partial g(x)}{\partial x}\nabla_x L(x,\lambda(x),\mu(x))&=-\gamma^2G^2(x)\lambda(x)\\&=-\gamma^2G(x)(G(x)+Y^2(x;\varepsilon))\lambda(x)+\gamma^2G(x)Y^2(x;\varepsilon)\lambda(x)\\&=-\gamma^2G(x)\Lambda(x)(g(x)+Y(x;\varepsilon)y(x;\varepsilon))+\gamma^2G(x)Y^2(x;\varepsilon)\lambda(x).\end{aligned}\tag{21}$$

Therefore, by (20) and (21) we get

$$\varepsilon\frac{\partial g(x)}{\partial x}\nabla_x L(x,\lambda(x),\mu(x))=-\gamma^2G(x)(\varepsilon\Lambda(x)+2Y^2(x;\varepsilon))(g(x)+Y(x;\varepsilon)y(x;\varepsilon)),$$

so that, by (19), we can write

$$
\begin{aligned}
\varepsilon \frac{\partial g(x)}{\partial x} \nabla W(x; \varepsilon) = {} & \varepsilon \frac{\partial g(x)}{\partial x} \nabla_x L(x, \lambda(x), \mu(x)) \\
& + \frac{\partial g(x)}{\partial x}\left(2 \frac{\partial g(x)'}{\partial x} + \varepsilon \frac{\partial \lambda(x)'}{\partial x}\right)(g(x) + Y(x; \varepsilon) y(x; \varepsilon)) \\
& + \frac{\partial g(x)}{\partial x}\left(2 \frac{\partial h(x)'}{\partial x} + \varepsilon \frac{\partial \mu(x)'}{\partial x}\right) h(x) \\
= {} & K_{11}(x; \varepsilon)(g(x) + Y(x; \varepsilon) y(x; \varepsilon)) + K_{12}(x; \varepsilon) h(x),
\end{aligned} \tag{22}
$$

where

$$
\begin{aligned}
K_{11}(x; \varepsilon) &:= 2\left(\frac{\partial g(x)}{\partial x} \frac{\partial g(x)'}{\partial x} - \gamma^2 G(x) Y^2(x; \varepsilon)\right) \\
&\quad + \varepsilon\left(\frac{\partial g(x)}{\partial x} \frac{\partial \lambda(x)'}{\partial x} - \gamma^2 G(x) \Lambda(x)\right), \\
K_{12}(x; \varepsilon) &:= 2 \frac{\partial g(x)}{\partial x} \frac{\partial h(x)'}{\partial x} + \varepsilon \frac{\partial g(x)}{\partial x} \frac{\partial \mu(x)'}{\partial x}.
\end{aligned}
$$

Now, by definition of $\mu(x)$, we have

$$\frac{\partial h(x)}{\partial x} \nabla_x L(x, \lambda(x), \mu(x)) = 0$$

and hence, by (19) we can write

$$\varepsilon \frac{\partial h(x)}{\partial x} \nabla W(x; \varepsilon) = K_{21}(x; \varepsilon)(g(x) + Y(x; \varepsilon) y(x; \varepsilon)) + K_{22}(x; \varepsilon) h(x), \tag{23}$$

where

$$
\begin{aligned}
K_{21}(x; \varepsilon) &:= 2 \frac{\partial h(x)}{\partial x} \frac{\partial g(x)'}{\partial x} + \varepsilon \frac{\partial h(x)}{\partial x} \frac{\partial \lambda(x)'}{\partial x} \\
K_{22}(x; \varepsilon) &:= 2 \frac{\partial h(x)}{\partial x} \frac{\partial h(x)'}{\partial x} + \varepsilon \frac{\partial h(x)}{\partial x} \frac{\partial \mu(x)'}{\partial x}.
\end{aligned}
$$

Thus, from (22) and (23) we get, for all $x \in \mathscr{X}$:

$$\varepsilon \begin{bmatrix} \dfrac{\partial g(x)}{\partial x} \\ \dfrac{\partial h(x)}{\partial x} \end{bmatrix} \nabla W(x; \varepsilon) = K(x; \varepsilon) \begin{bmatrix} g(x) + Y(x; \varepsilon) y(x; \varepsilon) \\ h(x) \end{bmatrix}, \tag{24}$$

where $K(x; \varepsilon)$ is the matrix defined by

$$K(x; \varepsilon) := \begin{bmatrix} K_{11}(x; \varepsilon) & K_{12}(x; \varepsilon) \\ K_{21}(x; \varepsilon) & K_{22}(x; \varepsilon) \end{bmatrix}.$$

Let now $\hat{x} \in \mathscr{F} \cap \mathscr{D}$; then, by definition of $y(x; \varepsilon)$ we have

$$Y^2(\hat{x}; 0) = -G(\hat{x}),$$

so that, by definition of $K(x; \varepsilon)$, we get

$$K(\hat{x}; 0) = 2N(\hat{x}).$$

Therefore, since Assumption (A4) implies that $N(x)$ is nonsingular, by continuity there exist numbers $\varepsilon(\hat{x})>0$ and $\sigma(\hat{x})>0$ such that the matrix $K(x;\varepsilon)$ is nonsingular for all $\varepsilon\in[0,\varepsilon(\hat{x})]$ and all x such that $\|x-\hat{x}\|\leqq\sigma(\hat{x})$. Let $\varepsilon\in[0,\varepsilon(\hat{x})]$ and let $x_\varepsilon\in\mathring{\mathcal{D}}$ be a stationary point of $W(x;\varepsilon)$ satisfying $\|x_\varepsilon-\hat{x}\|\leqq\sigma(\hat{x})$. By (24) we have

$$K(x_\varepsilon;\varepsilon)\begin{bmatrix} g(x_\varepsilon)+Y(x_\varepsilon;\varepsilon)y(x_\varepsilon;\varepsilon)\\ h(x_\varepsilon)\end{bmatrix}=0,$$

which implies, as $K(x_\varepsilon;\varepsilon)$ is nonsingular, $h(x_\varepsilon)=0$, and

$$g(x_\varepsilon)+Y(x_\varepsilon;\varepsilon)y(x_\varepsilon;\varepsilon)=0. \tag{25}$$

Therefore, since $\nabla W(x_\varepsilon;\varepsilon)=0$, we have from (19)

$$\nabla_x L(x_\varepsilon,\lambda(x_\varepsilon),\mu(x_\varepsilon))=0; \tag{26}$$

on the other hand, by definition of $\lambda(x)$ and $\mu(x)$, we have

$$\frac{\partial g(x)}{\partial x}\nabla_x L(x_\varepsilon,\lambda(x_\varepsilon),\mu(x_\varepsilon))+\gamma^2G^2(x_\varepsilon)\lambda(x_\varepsilon)=0, \tag{27}$$

and hence, by (26) and (27) we obtain

$$G(x_\varepsilon)\lambda(x_\varepsilon)=0.$$

Finally, if $g_i(x_\varepsilon)=0$ for some i, we have, by (25), $y_i^2(x_\varepsilon;\varepsilon)=0$, which implies, by definition of $y(x;\varepsilon)$, that $\lambda_i(x_\varepsilon)\geqq 0$. Hence, the triple $(x_\varepsilon,\lambda(x_\varepsilon),\mu(x_\varepsilon))$ is a K-T triple for problem (P). □

The next proposition establishes the correspondence between stationary points of $W(x;\varepsilon)$ and K-T triples for problem (P) on the whole set $\mathcal{D}$.

PROPOSITION 15. *Assume that the* EMFCQ *holds on* $\mathcal{D}$. *Then, there exists an* $\varepsilon^*>0$ *such that, for all* $\varepsilon\in(0,\varepsilon^*]$, *if* $x_\varepsilon\in\mathring{\mathcal{D}}$ *is a stationary point of* $W(x;\varepsilon)$, *we have that* $(x_\varepsilon,\lambda(x_\varepsilon),\mu(x_\varepsilon))$ *is a* K-T *triple for problem* (P).

Proof. The proof is by contradiction. Assume that the assertion is false. Then, for any integer k, there exists an $\varepsilon_k\leqq 1/k$ and a point $x_k\in\mathring{\mathcal{D}}$ such that $\nabla W(x_k;\varepsilon_k)=0$, but $(x_k,\lambda(x_k),\mu(x_k))$ is not a K-T triple for problem (P).

Since $\mathcal{D}$ is compact, there exists a convergent subsequence (relabel it again $\{x_k\}$) such that $\lim_{k\to\infty}x_k=\hat{x}\in\mathcal{D}$. Moreover, since $\nabla W(x_k;\varepsilon_k)=0$ for all k and since $\varepsilon_k\to 0$, we have in the limit, by (19):

$$\frac{\partial g(\hat{x})'}{\partial x}(g(\hat{x})+Y(\hat{x};0)y(\hat{x};0))+\frac{\partial h(\hat{x})'}{\partial x}h(\hat{x})=0, \tag{28}$$

where, by definition of $y_i^2(x;\varepsilon)$ we have

$$y_i^2(\hat{x};0)=-\min[0,g_i(\hat{x})],\qquad i=1,\cdots,m.$$

It follows that (28) can be rewritten into the form:

$$\sum_{i\in I_+(\hat{x})}g_i(\hat{x})\nabla g_i(\hat{x})+\sum_{j=1}^{p}h_j(\hat{x})\nabla h_j(\hat{x})=0,$$

where $I_+(\hat{x})=\{i\colon g_i(\hat{x})\geqq 0\}$. Therefore, by the EMFCQ we have $g_i(\hat{x})=0$, $i\in I_+(\hat{x})$, and $h_j(\hat{x})=0, j=1,\cdots,p$ so that $\hat{x}\in\mathcal{F}\cap\mathring{\mathcal{D}}$. On the other hand, by Proposition 14 there exists an integer $\bar{k}$ such that for all $k\geqq\bar{k}$ we have that $(x_k,\lambda(x_k),\mu(x_k))$ is a K-T triple for problem (P) and we get a contradiction. □

The properties of exactness of $W(x;\varepsilon)$ are summarized in the following theorem.

THEOREM 6. (a) *The function* $W(x;\varepsilon)$ *is a weakly exact penalty function for problem* (P) *with respect to the set* $\mathcal{D}$.

(b) *Assume that the* EMFCQ *is satisfied on* $\mathcal{D}$. *Then, the function* $W(x; \varepsilon)$ *is an exact penalty function for problem* (P) *with respect to the set* $\mathcal{D}$; *moreover, if Assumption* (A2) *holds, the function* $W(x; \varepsilon)$ *is a strongly exact penalty function for problem* (P) *with respect to the set* $\mathcal{D}$.

Proof. Let $\mathscr{E} = \mathcal{D}$; we show first that conditions (a_1)-(a_4) of Theorem 1 are satisfied.

It is easily seen that (a_1) follows from the continuity of $W(x; \varepsilon)$ and the compactness of $\mathcal{D}$.

With regard to condition (a_2), let $\{\varepsilon_k\}$ and $\{x_k\} \subset \mathcal{D}$ be sequences such that $\varepsilon_k > 0$, $\lim_{k\to\infty} \varepsilon_k = 0$, $\lim_{k\to\infty} x_k = \hat{x} \in \mathcal{D}$ and assume that

$$\limsup_{k\to\infty} W(x_k; \varepsilon_k) < \infty.$$

By the continuity assumptions we get from (17)

$$g(\hat{x}) + Y(\hat{x}; 0)y(\hat{x}; 0) = 0, \qquad h(\hat{x}) = 0,$$

which imply $\hat{x} \in \mathcal{F}$ and

$$f(\hat{x}) \leqq \limsup_{k\to\infty} W(x_k; \varepsilon_k),$$

so that (a_2) is satisfied.

We observe now that Assumptions (A1) and (A4) imply that $\mathcal{G}_{\mathcal{F}} \subseteq \mathcal{T}$. On the other hand, by (b) of Proposition 13 we obtain $\mathcal{T} \subseteq \mathcal{K}$. Therefore, we have $\mathcal{G}_{\mathcal{F}} \subseteq \mathcal{K}$ and hence condition (a_3) is satisfied.

Finally, condition (a_4) follows from Proposition 14, noting that $\mathcal{T} \subseteq \mathcal{K}$ and that, by the differentiability of $W(x; \varepsilon)$, any point $x_\varepsilon \in \mathcal{G}_{\mathcal{D}}(\varepsilon)$ is a stationary point of $W(x; \varepsilon)$. Thus (a) is proved.

With regard to (b), we have already shown that conditions (a_1)-(a_3) of Theorem 2 are satisfied. Condition (a_5) is implied by Proposition 15. Therefore we must show that (a_6) holds, that is,

$$W(x; \varepsilon) \leqq f(x) \quad \text{for all } \varepsilon > 0 \quad \text{and} \quad x \in \mathcal{F} \cap \mathring{\mathcal{D}}.$$

Let $x \in \mathring{\mathcal{D}}$ be such that $g(x) \leqq 0$ and $h(x) = 0$. Suppose first that $y_i^2(x; \varepsilon) = 0$; this implies, by definition of $y_i^2(x; \varepsilon)$, that

$$2g_i(x) + \varepsilon\lambda_i(x) \geqq 0,$$

so that, since $g_i(x) \leqq 0$, we have

$$\frac{1}{\varepsilon} g_i^2(x) + \lambda_i(x) g_i(x) \leqq \frac{2}{\varepsilon} g_i^2(x) + \lambda_i(x) g_i(x) \leqq 0. \tag{29}$$

Now assume that $y_i^2(x; \varepsilon) > 0$; in this case we obtain

$$g_i(x) + y_i^2(x; \varepsilon) = -\frac{\varepsilon}{2} \lambda_i(x),$$

whence:

$$\frac{1}{\varepsilon} (g_i(x) + y_i^2(x; \varepsilon))^2 + \lambda_i(x)(g_i(x) + y_i^2(x; \varepsilon)) = -\frac{\varepsilon}{4} \lambda_i^2(x) \leqq 0. \tag{30}$$

Therefore, by (29) and (30) we have, for any $i = 1, \cdots, m$:

$$\frac{1}{\varepsilon} (g_i(x) + y_i^2(x; \varepsilon))^2 + \lambda_i(x)(g_i(x) + y_i^2(x; \varepsilon)) \leqq 0,$$

so that, by (17), we obtain $W(x;\varepsilon) \leqq f(x)$ and hence condition (a_6) is satisfied. It can be verified also that conditions (c_1), (c_2), (c_4), and (c_5) of Theorem 3 are satisfied and that condition (c_3) follows from (b) of Proposition 13. Thus (b) follows from Theorems 2 and 3. □

We now consider the construction of a continuously differentiable exact penalty function with global exactness properties, along the same lines followed in the non-differentiable case. As in § 5, we take $\mathscr{D} = \mathscr{S}_\beta$, and we suppose that Assumption (A3) holds. Then, we can define on the set $\mathscr{E} = \mathring{\mathscr{D}}$ the continuously differentiable exact penalty function:

$$\begin{aligned} Z(x;\varepsilon) &:= f(x) + \lambda(x)'(g(x) + \tilde{Y}(x;\varepsilon)\tilde{y}(x;\varepsilon)) \\ &\quad + \frac{1}{\varepsilon}(g(x) + \tilde{Y}(x;\varepsilon)\tilde{y}(x;\varepsilon))'A^{-1}(x)(g(x) + \tilde{Y}(x;\varepsilon)\tilde{y}(x;\varepsilon)) \\ &\quad + \mu(x)'h(x) + \frac{1}{\varepsilon a_0(x)}\|h(x)\|^2, \end{aligned}$$

where $(\lambda(x), \mu(x))$ are the multiplier functions defined in Proposition 12, and

$$\begin{aligned} a_0(x) &:= \alpha_0 - \|h(x)\|_2^2 \\ a_i(x) &:= \alpha_i - g_i(x), \qquad i = 1, \cdots, m \\ A(x) &:= \operatorname{diag}(a_i(x)), \qquad i = 1, \cdots, m \end{aligned}$$

with:

$$\begin{aligned} \tilde{Y}(x;\varepsilon) &:= \operatorname{diag}(\tilde{y}_i(x;\varepsilon)), \qquad i = 1, \cdots, m \\ \tilde{y}_i(x;\varepsilon) &:= \left\{ -\min\left[0, g_i(x) + \frac{\varepsilon}{2} a_i(x)\lambda_i(x)\right]\right\}^{1/2}. \end{aligned}$$

In order to justify the expression of $Z(x;\varepsilon)$ we first consider the equality constrained problem obtained from problem (P) by introducing the vector Yy of squared slack variables into the inequality constraints $g(x) \leqq 0$.

Then, we define the augmented Lagrangian function:

$$\begin{aligned} \tilde{L}_a(x, y, \lambda, \mu; \varepsilon) &:= f(x) + \lambda'(g(x) + Yy) + \frac{1}{\varepsilon}(g(x) + Yy)'A^{-1}(x)(g(x) + Yy) \\ &\quad + \mu'h(x) + \frac{1}{\varepsilon a_0(x)}\|h(x)\|^2, \end{aligned}$$

where the penalty terms are weighted by the barrier functions $A^{-1}(x)$ and $1/a_0(x)$.

Finally, by substituting $(\lambda(x), \mu(x))$ for (λ, μ) and minimizing with respect to y we get the function $Z(x;\varepsilon)$, that is,

$$\begin{aligned} Z(x;\varepsilon) &= \tilde{L}_a(x, \tilde{y}(x;\varepsilon), \lambda(x), \mu(x); \varepsilon) \\ &= \min_y \tilde{L}_a(x, y, \lambda(x), \mu(x); \varepsilon). \end{aligned}$$

By construction, we have

$$[\nabla_y \tilde{L}_a(x, y, \lambda, \mu)]_{\substack{\lambda = \lambda(x) \\ \mu = \mu(x) \\ y = \tilde{y}(x;\varepsilon)}} = 0$$

and hence the gradient expression of $Z(x;\varepsilon)$ can be obtained by taking $\tilde{y}(x;\varepsilon)$ as a constant vector.

Thus, we can write:

$$
\begin{aligned}
\nabla Z(x;\varepsilon) &= \nabla f(x) + \frac{\partial g(x)'}{\partial x}\lambda(x) + \frac{\partial h(x)'}{\partial x}\mu(x) \\
&\quad + \frac{2}{\varepsilon}\frac{\partial g(x)'}{\partial x}A^{-1}(x)(g(x)+\tilde{Y}(x;\varepsilon)\tilde{y}(x;\varepsilon)) \\
&\quad + \frac{1}{\varepsilon}\frac{\partial g(x)'}{\partial x}[G(x)+\tilde{Y}^2(x;\varepsilon)]A^{-2}(x)(g(x)+\tilde{Y}(x;\varepsilon)\tilde{y}(x;\varepsilon)) \\
&\quad + \frac{2}{\varepsilon a_0(x)}\frac{\partial h(x)'}{\partial x}h(x) + \frac{2\|h(x)\|_2^2}{a_0^2(x)}\frac{\partial h(x)'}{\partial x}h(x) \\
&\quad + \frac{\partial \lambda(x)'}{\partial x}(g(x)+\tilde{Y}(x;\varepsilon)\tilde{y}(x;\varepsilon)) + \frac{\partial \mu(x)'}{\partial x}h(x).
\end{aligned} \tag{31}
$$

The study of the properties of exactness of the function $Z(x;\varepsilon)$ can be performed along the same lines followed in the case of the function $W(x;\varepsilon)$, taking into account the expressions of $\tilde{y}(x;\varepsilon)$ and $\nabla Z(x;\varepsilon)$ and noting that $a_i(x)>0$, $i=0,1,\cdots,m$ for $x\in\mathring{\mathscr{D}}$.

In particular, the next two propositions are the analogue of Propositions 13 and 14 and can be proved in a similar way.

PROPOSITION 16. *Let* $(\bar{x},\bar{\lambda},\bar{\mu})$ *be a* K-T *triple for problem* (P), *such that* $\bar{x}\in\mathscr{D}$. *Then, for any* $\varepsilon>0$, *we have*:

(a) $g(\bar{x})+\tilde{Y}(\bar{x};\varepsilon)\tilde{y}(\bar{x};\varepsilon)=0$;

(b) $Z(\bar{x};\varepsilon)=f(\bar{x})$;

(c) $\nabla Z(\bar{x};\varepsilon)=0$.

PROPOSITION 17. *Let* $\hat{x}\in\mathscr{F}\cap\mathscr{D}$. *Then, there exist numbers* $\varepsilon(\hat{x})>0$ *and* $\sigma(\hat{x})>0$ *such that, for all* $\varepsilon\in(0,\varepsilon(\hat{x})]$, *if* $x_\varepsilon\in\mathring{\mathscr{D}}$ *is a stationary point of* $Z(x;\varepsilon)$ *satisfying* $\|x_\varepsilon-\hat{x}\|\leqq\sigma(\hat{x})$, *we have that* $(x_\varepsilon,\lambda(x_\varepsilon),\mu(x_\varepsilon))$ *is a* K-T *triple for problem* (P).

We now need the following lemma which is proved in [15].

LEMMA 3. *Let* $\{\delta_k^{(i)}\}$, $i=1,\cdots,r$ *be r sequences of positive numbers. Then, there exist an index* i^* *and subsequences* $\{\delta_k^{(i)}\}_K$, $i=1,\cdots,r$ *corresponding to the same index set* K, *such that*:

$$\lim_{\substack{k\to\infty\\ k\in K}}\frac{\delta_k^{(i^*)}}{\delta_k^{(i)}}=l_i<+\infty,\quad i=1,\cdots,r.$$

The following proposition establishes the correspondence between stationary points of $Z(x;\varepsilon)$ and K-T triples for problem (P).

PROPOSITION 18. *Assume that the* EMFCQ *holds on* $\mathscr{D}$. *Then, there exists an* $\varepsilon^*>0$ *such that, for all* $\varepsilon\in(0,\varepsilon^*]$, *if* $x_\varepsilon\in\mathring{\mathscr{D}}$ *is a stationary point of* $Z(x;\varepsilon)$, *we have that* $(x_\varepsilon,\lambda(x_\varepsilon),\mu(x_\varepsilon))$ *is a* K-T *triple for problem* (P).

Proof. Reasoning by contradiction, we assume that for any integer k, there exists an $\varepsilon_k\leqq 1/k$ and a point $x_k\in\mathring{\mathscr{D}}$ such that $\nabla Z(x_k;\varepsilon_k)=0$, but $(x_k,\lambda(x_k),\mu(x_k))$ is not a K-T triple for problem (P). Since $\mathscr{D}$ is compact, there exists a convergent subsequence (relabel it again $\{x_k\}$) such that $\lim_{k\to\infty}x_k=\hat{x}\in\mathscr{D}$.

We show first that $\hat{x} \in \mathring{\mathscr{D}}$. In fact, assume that $\hat{x} \in \partial\mathscr{D}$; this implies, by definition of $\mathscr{D}$, that there exists a subset J of $\{0, 1, \cdots, m\}$ such that:

$$\lim_{k\to\infty} a_i(x_k) = 0, \qquad i \in J. \tag{32}$$

By Lemma 3 we can define an index $i^* \in J$ and a subsequence (relabel it again $\{x_k\}$) such that

$$\lim_{k\to\infty} \frac{a_{i^*}(x_k)}{a_i(x_k)} = l_i < +\infty, \qquad i \in J, \tag{33}$$

where, in particular, $l_{i^*} = 1$. Recalling (31), we can write

$$\begin{aligned} 0 &= \varepsilon_k a_{i^*}^2(x_k) \nabla Z(x_k; \varepsilon_k) \\ &= \varepsilon_k a_{i^*}^2 \Bigg[\nabla_x L(x_k, \lambda(x_k), \mu(x_k)) \\ &\qquad + \frac{\partial \lambda(x_k)'}{\partial x} (g(x_k) + \tilde{Y}(x_k; \varepsilon_k) y(x_k; \varepsilon_k)) + \frac{\partial \mu(x_k)'}{\partial x} h(x_k) \Bigg] \\ &\quad + \sum_{i=1}^m \frac{a_{i^*}^2(x_k)}{a_i(x_k)} \left(2 + \frac{g_i(x_k) + \tilde{y}_i^2(x_k; \varepsilon_k)}{a_i(x_k)} \right) (g_i(x_k) + \tilde{y}_i^2(x_k; \varepsilon_k)) \nabla g_i(x_k) \\ &\quad + 2 \sum_{j=1}^p \frac{a_{i^*}^2(x_k)}{a_0(x_k)} \left(1 + \frac{\|h(x_k)\|_2^2}{a_0(x_k)} \right) h_j(x_k) \nabla h_j(x_k). \end{aligned} \tag{34}$$

Taking limits of (34) and recalling (32) we can write

$$\sum_{i=1}^m v_i \nabla g_i(\hat{x}) + \sum_{j=1}^p u_j \nabla h_j(\hat{x}) = 0, \tag{35}$$

where, by (33)

$$v_i = \begin{cases} l_i^2 (g_i(\hat{x}) + \tilde{y}_i^2(\hat{x}; 0))^2, & \text{if } i \in J; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$u_j = \begin{cases} l_0^2 \|h(\hat{x})\|_2^2 h_j(\hat{x}) & \text{if } 0 \in J \\ 0, & \text{otherwise.} \end{cases}$$

Since $i \in J$ and $i \geqq 1$ imply $i \in I_+(\hat{x})$, we can rewrite (35) into the following form:

$$\sum_{i \in I_+(\hat{x})} v_i \nabla g_i(\hat{x}) + \sum_{j=1}^p u_j \nabla h_j(\hat{x}) = 0.$$

This implies, by the EMFCQ, that $v_i = 0$ for $i \in I_+(\hat{x})$ and $u_j = 0$, for $j = 1, \cdots, p$. On the other hand, as $l_{i^*} = 1$, we have either $h(\hat{x}) = 0$ (if $i^* = 0$) or $g_{i^*}(\hat{x}) + \tilde{y}_{i^*}^2(\hat{x}; 0) = 0$ (if $i^* \in \{1, \cdots, m\}$). In both cases we get a contradiction to (32). Then we can conclude that $\hat{x} \in \mathring{\mathscr{D}}$. Therefore, from (31) and (34), taking the limit of $\varepsilon_k \nabla Z(x_k; \varepsilon_k)$ over the subsequence converging to $\hat{x}$, we have

$$\begin{aligned} &\sum_{i=1}^m \left[2 \frac{g_i(\hat{x}) + \tilde{y}_i^2(\hat{x}; 0)}{a_i(\hat{x})} + \frac{(g_i(\hat{x}) + \tilde{y}_i^2(\hat{x}; 0))^2}{a_i^2(\hat{x})} \right] \nabla g_i(\hat{x}) \\ &\quad + \sum_{j=1}^p \left[2 \frac{h_j(\hat{x})}{a_0(\hat{x})} + \frac{\|h(\hat{x})\|_2^2}{a_0^2(\hat{x})} h_j(\hat{x}) \right] \nabla h_j(\hat{x}) = 0. \end{aligned}$$

Noting that, by definition of $\tilde{y}_i(x_k;\varepsilon_k)$, the inequality $g_i(\hat{x})<0$ implies $g_i(\hat{x})+\tilde{y}_i^2(\hat{x};0)=0$, we can write

$$\sum_{i\in I_+(\hat{x})} v_i\nabla g_i(\hat{x})+\sum_{j=1}^{p} u_j\nabla h_j(\hat{x})=0,$$

where now

$$v_i:=2\,\frac{g_i(\hat{x})+\tilde{y}_i^2(\hat{x};0)}{a_i(\hat{x})}+\frac{(g_i(\hat{x})+\tilde{y}_i^2(\hat{x};0))^2}{a_i^2(\hat{x})}\geqq 0$$

and

$$u_j:=2\left[1+\frac{\|h(\hat{x})\|_2^2}{a_0(\hat{x})}\right]\frac{h_j(\hat{x})}{a_0(\hat{x})}.$$

Therefore, again by the EMFCQ, we have $v_i=0,\ i\in I_+(\hat{x})$, and $\mu_j=0, j=1,\cdots,p$ which imply

$$g_i(\hat{x})+\tilde{y}_i^2(\hat{x};0)=0,\qquad i=1,\cdots,m$$

$$h_j(\hat{x})=0,\qquad j=1,\cdots,p,$$

so that $\hat{x}\in\mathcal{F}$. As $\varepsilon_k\to 0$, Proposition 17 implies that for sufficiently large values of k, the triple $(x_k,\lambda(x_k),\mu(x_k))$ is a K-T triple for problem (P) and this yields a contradiction. □

We can now summarize the properties of exactness of $Z(x;\varepsilon)$ in the following theorem.

THEOREM 7. (a) *The function* $Z(x;\varepsilon)$ *is a globally weakly exact penalty function for problem* (P) *with respect to the set* $\mathcal{D}$.

(b) *Assume that the* EMFCQ *is satisfied on* $\mathcal{D}$. *Then, the function* $Z(x;\varepsilon)$ *is a globally exact penalty function for problem* (P) *with respect to the set* $\mathcal{D}$; *moreover, if Assumption* (A2) *holds, the function* $Z(x;\varepsilon)$ *is a globally strongly exact penalty function for problem* (P) *with respect to the set* $\mathcal{D}$.

Proof. By construction, we have $\lim_{k\to\infty} Z(x_k;\varepsilon)=\infty$ for any sequence $\{x_k\}\subset\mathring{\mathcal{D}}$ such that $x_k\to y\in\partial\mathcal{D}$. Hence, by Definition 4 we have that $Z(x;\varepsilon)$ is globally (weakly, strongly) exact if it is (weakly, strongly) exact.

Letting $\mathcal{E}=\mathring{\mathcal{D}}$, assertion (a) can be proved along the same lines followed in the proof of Theorem 6, making use of Propositions 16 and 17 in place of Propositions 13 and 14.

With regard to (b), again letting $\mathcal{E}=\mathring{\mathcal{D}}$, we can proceed, as in the proof of Theorem 6, by employing Proposition 18 in place of Proposition 15 and making use of the inequality:

$$Z(x;\varepsilon)\leqq f(x)\quad\text{for all } x\in\mathcal{F},$$

which can be established in a way similar to that followed in the proof of Theorem 6 for the case of the function $W(x;\varepsilon)$. □

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