Inviscid Hypersonic Flow Past Smooth Symmetric Bodies

S H MASLEN*
Martin Company, Baltimore, Md

A simple inverse method is presented for computing the entire inviscid flow field in the shock layer enveloping a sufficiently smooth axisymmetric or planar body moving at hypersonic speed The method is inverse and involves the solution in a Von Mises plane using the stream function and distance along the shock as independent variables A simple, accurate approximation to the lateral momentum equation permits the solution to proceed numerically by quadrature A straightforward application to nonequilibrium flow is discussed Numerical examples are compared with more exact results in the subsonic and transonic regions and with both the method of characteristics and with the hypersonic small disturbance results farther back Agreement is excellent A final example solves accurately, and without undue effort, a direct problem

Nomenclature

h = enthalpy, divided by $\frac{1}{2}w_\infty^2$
L = characteristic length
M_∞ = freestream Mach number
p = pressure, divided by $\rho_\infty w_\infty^2$
r = lateral dimension, divided by L (Fig 1)
$R(x)$ = shock radius of curvature divided by L
S = entropy
u,v = velocities in x,y direction divided by w_∞
w_∞ = freestream velocity
x,y = distance along and normal (inward) to shock, divided by L (Fig 1)
$z(x)$ = streamwise coordinate of projection of shock point onto axis of symmetry, divided by L (Fig 1)
γ = ratio of specific heats
$\theta(x)$ = local shock angle
ρ = density, divided by ρ_∞
σ = 0 or 1 for plane or axisymmetric flow
ψ = stream function, divided by $\rho_\infty w_\infty L^{\sigma+1}$

Subscripts

ψ,x = partial derivatives with respect to ψ or x
s = shock
∞ = freestream value
$s0$ = normal shock

I Introduction

DETAILED analysis of the axisymmetric or planar hypersonic flow past vehicles can be carried out by the method of characteristics combined with some one of the many schemes valid for the subsonic region (for example, Van Dyke[1] or Swigart[2]) However, on many occasions, such calculations are unduly tedious Hence, a number of approximate alternatives are often used

The simplest such approximation is the Newtonian It can be applied to any case regardless of symmetry However, fairly large flow angles are needed for useful results A correction for centrifugal pressure effects helps The Newtonian scheme has the great advantage over the others to be discussed in that it is direct and one which solves for a given body, not a given shock

The constant density method (see, e g , Chap IV of Hayes and Probstein[3]) is in a sense an extension of the Newtonian It assumes the density in the shock layer to be uniform and is, therefore, of interest in the stagnation region of a blunt body or behind a straight shock This is really an inverse method because the shock shape is assumed

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* Chief, Aerophysics Research, Research Department Associate Fellow Member AIAA

Another class of inverse solutions uses the hypersonic small disturbance theory [3–4] This method derives some results from blast wave analysis Useful answers have been obtained for slender bodies at very high Mach number Notable recent contributions with particular reference to the entropy layer have been made by Sychev[6] and Yakura[7] The latter's work has been extended by Lee[8] Lee's results are for power (law) shocks although this is not a necessary condition for use of the slender body theory The results have been developed primarily for the $M = \infty$ case although the second term in an expansion in inverse powers of M^2 has been found for some flows[4 5] Using an integral approach, Chernyi (Chap V)[5] has given results for blunted slender wedges and cones

The results thus far discussed have all been obtained for a perfect gas or by approximating a more nearly real one by juggling the ratio of specific heats This assumption is necessary if analytic results are sought

Cheng[9] has given an analysis of the same problem, but including viscous effects His scheme, as well as the present one, depends on having a thin shock layer As will be seen, the present analysis is simpler and more accurate However, Cheng's relative complication arises because his formulation allows for the viscous effects, which are his primary concern

All of the approximate analyses of hypersonic flow rest on several somewhat unrelated idealizations Some of these are essential to get results, others are associated with a desire for analytic solutions, and others are merely conveniences In the present work, some effort has been made to minimize these idealizations and to make clear what is necessary and what is not The aim is to provide a fairly simple means of solving the flow over a body in a hypersonic stream without being restricted to a small part of the flow field and without any forced restriction on the chemistry This aim has been, for the most part, realized As will be seen, the method involves using what amounts to a streamline coordinate system and a single approximation regarding the variation of pressure across the shock layer It is an inverse method

II Analysis

We consider the inviscid flow of a gas about a body moving at very high speed In such a case, the layer of gas between the shock and body is, happily, thin Furthermore, qualitatively speaking, the flow is approximately parallel to the shock or, for that matter, the body, except near a stagnation point For the present no consideration is given to nonequilibrium flow Extension to such flows will be discussed later

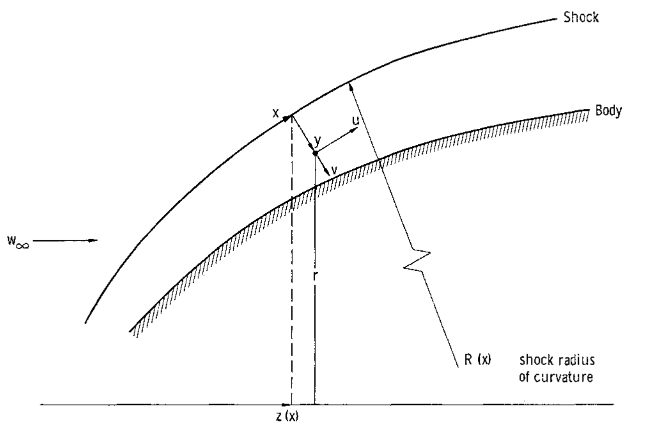

Fig 1 Geometry of the flow

To take advantage of these general properties of the motion, one can proceed as follows Consider the equations of inviscid motion written in a curvilinear system (Fig 1) where one coordinate x is the distance along the shock and the other y is the inward normal Now make a von Mises transformation so that the independent variables become x and the stream function ψ, and introduce nondimensional coordinates (distances are divided by some characteristic length L, velocities and density by their freestream values w_∞, ρ_∞, pressure by $\rho_\infty w_\infty^2$, the enthalpy by $\frac{1}{2}w_\infty^2$, and the stream function by $\rho_\infty w_\infty L^{\sigma+1}$)

The lateral momentum equation becomes

$$r^\sigma p_\psi = \frac{u}{(R-y)} + \frac{v_x}{1-(y/R)} \tag{1}$$

The energy equation is simply

$$S = S(\psi) \tag{2}$$

From the two momentum equations, the energy equation and the condition that the upstream flow is isoenergetic,

$$u^2 = v_{s0}^2 + h_{s0} - v^2 - h \tag{3}$$

where the subscript $s0$ refers to conditions immediately behind a normal shock Conservation of mass becomes

$$r^\sigma y_\psi = -(1/\rho u) \tag{4}$$

The definition of the transformation yields

$$v = \frac{u}{1-(y/R)}\,y_x \tag{5}$$

In addition, there is a state equation

$$h = h(S,p) \tag{6}$$

or

$$\rho = \rho(S,p)$$

This form of state equation is critical in the method Because the entropy is constant along streamlines and is known at the shock, it is, in the von Mises plane (x,ψ), known everywhere behind a given shock Then, if the pressure is found, and this will be done simply, the remaining thermodynamic variables follow from Eq (6) For a perfect gas, Eqs (6) are, noting Eq (2),

$$h = [2\gamma/(\gamma-1)](p/\rho) \tag{7}$$

$$\rho = p^{1/\gamma} f(\psi) \tag{8}$$

Boundary conditions must be added to these equations As is the usual case, these will be taken as the Rankine-Hugoniot conditions at a shock, which must be specified well enough to define its position, slope, and curvature This means, unfortunately, an inverse method

Now we come to the crux of the whole method Because, as discussed earlier, there is a thin shock layer wherein the flow is nearly parallel to the shock, Eq (1) is approximately

$$P_\psi = u_s/Rr_s^{\sigma} \tag{9}$$

where the subscript s refers to conditions immediately behind the shock If one performs an expansion in inverse powers of the shock layer density, the first term in the expansion is u/Rr^σ For a point inside the shock layer, r, u, and $(R-y)$ will actually be smaller than r_s, u_s, and R Also, v_x is always small and negative Some numerical checks show that these assorted errors tend to cancel each other so that Eq (9) is quite accurate and is in fact better than the consistent approximation $P_\psi = u/Rr^\sigma$ However, the test of Eq (9) lies in comparison with results obtained in other ways Equation (9) is apparently very inaccurate near the axis of symmetry behind a detached shock A later discussion will show that this causes no difficulty

Now Eq (9) can be integrated

$$p(x,\psi) = p_s(x) + \frac{u_s(x)}{R(x)r_s^{\sigma}(x)}[\psi - \psi_s(x)] \tag{10a}$$

which is, noting the geometry and the shock condition $u_s(x) = \cos\theta_s(x)$,

$$p(x,\psi) = p_s(x) + (\psi - \psi_s)(r_{s\,xx}/r_s^{\sigma}) \tag{10b}$$

When applied to the hypersonic slender body, this is the same as a result given in Cole's Eq (4 3) [10] In that limit, it agrees on the surface with the Newtonian formulas with centrifugal effect In the present formulation, the expression is to be applied throughout the shock layer and without restriction to slender bodies We note, in passing, that the approximation used is in no way connected with the chemistry of the flow

Next, since the entropy is constant on each streamline and is therefore a known function of ψ, the density can be found everywhere in x,ψ space [from Eqs (2) and (6)] Then $h(x,\psi)$ follows from Eq (6) The accuracy with which ρ and h are determined is limited by the approximation for the pressure However, the variation of ρ (and also h) across the shock layer is governed more by the changes in entropy [Eq (2)] than by those in pressure, so that the density and enthalpy are, in a sense, known more accurately than pressure

Next, because of the geometry of the flow, v (and v_{s0}) can be neglected in Eq (3) and that equation solved for u This approximation is exact on the axis of symmetry and will be very accurate elsewhere It will certainly be valid when Eq (9) is To get an estimate of the error, we can evaluate $(v_{s0}^2 - v^2)/u^2$ at the shock For a perfect gas, this is exactly

$$\frac{v_{s0}^2 - v_s^2}{u_s^2} = \frac{2(\gamma-1)}{(\gamma+1)^2 M_\infty^2}\left[2 + \frac{\gamma-1}{2}M_\infty^2(1+\sin^2\theta)\right]$$

which is always very small Hence, we replace Eq (3) with

$$u = (h_{s0} - h)^{1/2} \tag{11}$$

Now Eq (4) can be integrated Observe first, from Fig 1, that

$$r = r_s(x) - y\cos\theta_s(x) \tag{12}$$

Then Eq (4) yields

$$y - \frac{\sigma y^2 \cos\theta_s}{2r_s} = \frac{1}{r_s^{\sigma}}\int_\psi^{\psi_s}\frac{d\psi}{\rho u} \tag{13}$$

or

$$r^{\sigma+1} = r_s^{\sigma+1} - (\sigma+1)u_s\int_\psi^{\psi_s}\frac{d\psi}{\rho u}$$

As a first approximation, Eq (13) is

$$y = \frac{1}{r^{\sigma}} \int_{\psi}^{\psi} \frac{d\psi}{\rho u} \tag{14}$$

This essentially gives a complete determination of the flow field To get a new pressure distribution, one can, in principle, iterate the process by solving Eq (5) for v and putting the result, together with Eq (13), into Eq (1) However, unless a machine computation is contemplated, this is not very practical except in the special case of conical flow That case will be discussed later on

A Stagnation Region

In this section, the aim is not to show a new way to solve the problem in the subsonic region, but rather simply to demonstrate that the present scheme does not go to pieces as the stagnation point behind a detached shock is approached There are several difficulties in this region with the formulation that has been given First, the transformation is ill-behaved on the axis of symmetry (both ψ and x are zero along it) Second, the streamline curvature is opposite to that of the shock so that the pressure should rise rather than fall, as indicated by Eq (9), on proceeding from shock to body Third, as the axis of symmetry is approached, $v^2/u^2 \to \infty$ However, let us ignore these difficulties and proceed directly with the method as given in the previous section Consider for convenience, only a perfect gas and a spherical shock The pressure and density at the shock are

$$\left.\begin{aligned} p_s &= \frac{2}{\gamma+1}\sin^2\theta - \frac{\gamma-1}{(\gamma+1)\gamma M_\infty^2} \approx \\ &\qquad \frac{2}{\gamma+1}(1-\phi^2) - \frac{\gamma-1}{(\gamma+1)\gamma M_\infty^2} \\ \rho &= \frac{[(\gamma+1)/2]M_\infty^2\sin^2\theta_s}{1+[(\gamma-1)/2]M_\infty^2\sin^2\theta} \approx \\ &\frac{[(\gamma+1)/2]M_\infty^2}{1+[(\gamma-1)/2]M_\infty^2}\left\{1 - \frac{\phi^2}{1+[(\gamma-1)/2]M_\infty^2}\right\} \end{aligned}\right\} \tag{15}$$

where $[(\pi/2) - \theta\,] = \phi$ and is small

From Eq (10a), since $\phi(x) = x/R$, and $r \approx x$,

$$p = p\,(x) + \frac{\cos\theta_s(x)}{r\,(x)R(x)}[\psi - \psi\,(x)] \approx p\,(x) + \frac{1}{R^2}[\psi - \psi\,(x)] \tag{16}$$

Then the density is, from Eq (8),

$$\rho(x,\psi) = \rho\,(\psi)\left[\frac{p(x,\psi)}{p\,(\psi)}\right]^{1/\gamma}$$

or, using Eqs (15) and (16),

$$\rho(x,\psi) = \frac{[(\gamma+1)/2]M_\infty^2}{1+[(\gamma-1)/2]M_\infty^2} \times \left\{1 - \frac{\phi^2(\psi)}{1+[(\gamma-1)/2]M_\infty^2} + \frac{[(\gamma+1)/2R^2][\psi - \psi_s(x)] + \phi^2(\psi) - \phi^2(x)}{\gamma\{1 - [(\gamma-1)/2\gamma M_\infty^2]\}}\right\}$$

Then, if these are put into Eq (11), using Eq (7), there follows after some reduction

$$u^2(x,\psi) = \frac{4(\gamma-1)}{(\gamma+1)^2}\left[1 + \frac{2}{(\gamma-1)M_\infty^2}\right] \times \left[\phi^2(x) + \frac{\phi^2(\psi)(M_\infty^2-1)^2}{2M_\infty^2\{1+[(\gamma-1)/2]M_\infty^2\}} - \frac{\gamma+1}{2R^2}[\psi - \psi\,(x)]\right]$$

or, since $\psi = r^2/2$, $r \approx x$, and $\phi = x/R$, so that $\phi^2(\psi) = 2\psi/R^2$,

$$\frac{R^2}{x^2}u^2(x,\psi) = \frac{4(\gamma-1)}{(\gamma+1)^2}\left[1 + \frac{2}{(\gamma-1)M_\infty^2}\right] \times \left[1 + \frac{2\psi}{x^2}\frac{(M_\infty^2-1)^2}{2M_\infty^2\{1+[(\gamma-1)/2]M_\infty^2\}} - \frac{\gamma+1}{4}\left(\frac{2\psi}{x^2} - 1\right)\right] = \frac{1}{\rho_0 C}\left[1 + B\frac{2\psi}{x^2} - C\left(\frac{2\psi}{x^2} - 1\right)\right] \tag{17}$$

where B and C are defined by comparing the last two equations Now the transformation back to physical space can be performed From Eq (13),

$$\frac{y(x,\psi)}{R} = \frac{1}{rR}\int_{\psi}^{\psi_s}\frac{d\psi}{\rho u} \approx \frac{1}{\rho_0}\int_{2\psi/x^2}^{1}\frac{d(2\psi/x^2)}{2(Ru/x)} = \left(\frac{C}{\rho_0}\right)^{1/2}\left[\frac{1-(2\psi/x^2)}{(1+B)^{1/2} + [1 + C + (B-C)(2\psi/x^2)]^{1/2}}\right] \tag{18}$$

or, at the wall, in the limit of infinite Mach number,

$$\frac{y(x,0)}{R} = \frac{\gamma-1}{(4\gamma)^{1/2}}\left\{\frac{1}{1+[(\gamma-1)(\gamma+5)/4\gamma]^{1/2}}\right\} \tag{19}$$

This is almost exactly the constant density result [Hayes and Probstein,[3] Eq (4 4 16)] Why? From the present point of view, it is good luck Near the axis of symmetry the density is well known so that, to find the standoff distance, the problem is to find u The pressure [Eq (16)] is in error partly because the wrong value of u was used [Eq (16) uses $u \approx x/R$ in disagreement with Eq (17)], but mainly because the v_x term was omitted This term will primarily contribute a rise in pressure from shock to body along the stagnation streamline However, this effect must be mostly cancelled by the term $v_0^2 - v^2$, which was omitted in the equation for u [Eq (11)] As a happy result, the expression for u is essentially correct

Hence, in the stagnation region the present method essentially reduces to that of constant density However, unless it is carried further, the pressure at the stagnation point is too low This next approximation to that pressure is easily carried out From Eq (18), $y(x,\psi) = f(\eta)$, where $\eta = 2\psi/x^2$ so that, from Eq (5),

$$v \approx u(\partial f/\partial x) = -(2u\eta/x)(df/d\eta)$$

but, from Eq (19),

$$df/d\eta = -x/2\rho_0 u$$

so that

$$v = \eta/\rho_0$$

Then, from Eq (1),

$$p_\psi \to v_x/r = -2\eta/x^2\rho_{s0}$$

Hence, the pressure at the stagnation point is

$$p(0,0) = p\,(0) + \int_0^{x^2/2}\frac{2\eta d\psi}{x^2\rho_0} = p\,(0) + \frac{1}{2\rho_0} \tag{20}$$

This agrees with the constant density result

B Nonequilibrium Effects

Application of the present method to a nonequilibrium case presents no great difficulty In the first place, the problem is solved in streamline coordinates so that particles are readily tracked Second, because the flow essentially parallels the shock (except near the axis), the distance along a streamline is given by the change in x To the present approximation,

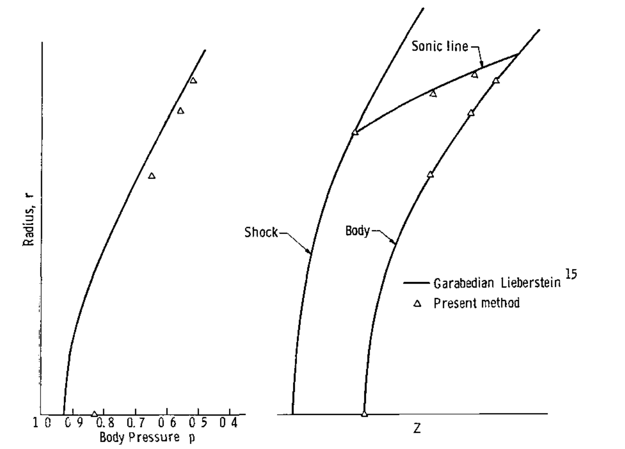

Fig 2 Results for an almost spherical body (M_∞ = 5 8, γ = 1 4)

the pressure distribution [Eqs (10)] for a given shock is a known function of x,ψ and depends only on the chemistry of the shock, not on that of the field behind the shock

When the flow is neither frozen nor in equilibrium, Eqs (2) and (6) do not hold In the present notation, the energy equation is

$$\rho h_x = 2p_x \tag{21}$$

The state equation is

$$h = h(p,\rho,c_i) \qquad i = 1 \quad N \tag{22}$$

where c_i is the mass fraction of the ith species To these must be added N chemical rate equations:

$$\rho u(c_i)_x = \sigma_i \qquad i = 1 \quad N \tag{23}$$

where σ is a chemical souıce function [11]

Equations (21–23), together with the isoenergetic condition [Eq (11)], and the known pressure distribution, [Eq (10)], form a determinate system of ordinary differential equations which can be integrated along each streamline This yields, in particular, the density and velocity distribution in x,ψ space Then, by integrating Eq (4), the transformation back to physical space is performed Observe that the chemistry calculation has been separated from the remainder of the problem A calculation somewhat similar to this has been carried out for a stagnation region by Freeman[12] using a simple dissociating gas

III Examples

A Comparison with Garabedian and Lieberstein

As a test of the procedure in the neighborhood of the sonic line, the case given by Garabedian and Lieberstein[13] for the flow of a perfect gas (γ = 1 4) at M_∞ = 5 8 past an approximately spherical body has been computed Figure 2 shows the results, which compare sonic lines, surface coordinates, and surface pressure for the given shock The points on the axis of symmetry follow from Eqs (15–18) Agreement is, for practical purposes, excellent although it is clear that the surface pressure is progressively more underestimated as the stagnation point is approached This is to be expected since, as discussed previously, the approximate differential equation for the pressure, Eq (9), gives a surface pressure less than that at the shock Note that the next approximation to the surface pressure, Eq (20), is in excellent agreement with the more exact result

B Paıaboloidal Shock

Consider a perfect gas with γ = 1 4 and $M_\infty = \infty$ The equation of the shock is $r = (2z)^{1/2}$ From this, at the shock, using the Rankine-Hugoniot conditions,

$$p = \frac{2}{\gamma+1}\left(\frac{1}{2z+1}\right) \qquad \rho = \frac{\gamma+1}{\gamma-1}$$

$$u = \left(\frac{2z}{1+2z}\right)^{1/2} \qquad \psi = z$$

Then, from Eqs (10, 2, 11, and 13), successively,

$$p = p_s\{1 + K[(\psi/\psi_s) - 1]\} \tag{24}$$

wheıe

$$\begin{aligned} K &= \frac{\gamma+1}{4}\left(\frac{2z}{2z+1}\right) \\ \rho &= \frac{\gamma+1}{\gamma-1}\left[\frac{\gamma+1}{2}\,p(1+2\psi)\right]^{1/\gamma} \\ u &= \left[\frac{4\gamma}{(\gamma+1)^2} - \frac{2\gamma}{\gamma-1}\,\frac{P}{\rho}\right]^{1/2} \\ r &= \left[2z - 2u\int_{\psi_s}^{\psi}\frac{d\psi}{\rho u}\right]^{1/2} \end{aligned} \tag{25}$$

The results of these integrations are shown in Figs 3 and 4 Comparison is made near the front with the results of Van Dyke[1] and Swigart[2] and farther back with those of Yakura[7] and Sychev[6] Agreement is very good Cheng[9] has also solved this problem (see his Fig 5 2), his calculations ex tending to $z \approx 1$ 0 His predicted body shape agrees as well with the "exact" results as do the present ones However, his surface pressure is high and the discrepancy tends to increase with z Also, his predicted sonic line differs substantially from the "exact" result with which the present method gives excellent agreement (not shown)

Finally, as shown in Fig 5, there is good agreement in all respects between the profiles across the shock layer as found in this investigation and by Yakura Actually the pressure gradient at the shock can be found exactly in terms of p, ρ, u, and v and their derivatives along the shock (Note that such derivatives are for constant y, not ψ) For the case in Fig 5, the present method, as expected, gives this moıe accurately than Yakura's first-order result These profiles

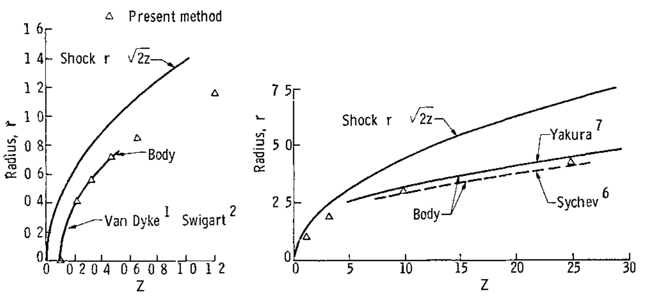

Fig 3 Body associated with paraboloidal shock ($M_\infty = \infty$, γ = 1 4)

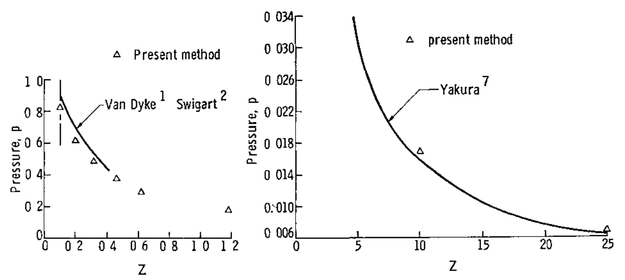

Fig 4 Surface pressure on body supporting a paraboloidal shock ($M_\infty = \infty$, γ = 1 4)

also demonstrate why the approximate pressure equation, Eq (9), is so satisfactory The stream function distribution (not shown) has almost the same shape as the density profile and reaches half its shock value at 83% of the distance to the shock from the body, so that most of the integration occurs near the shock

It might be observed that, for a general power law shock ($r = 2^{1/2}z^m$), the pressure distribution has the form of Eq (24), but with

$$K = \frac{\gamma + 1}{2(\sigma + 1)}\left(\frac{1 - m}{m}\right)\left[\frac{1}{1 + 2m^2z^{2(m-1)}}\right] \tag{26}$$

or, if $m < 1$ and $z \gg 1$,

$$K \to \frac{\gamma + 1}{2(\sigma + 1)}\left(\frac{1 - m}{m}\right) \tag{26a}$$

This last result is the same as that of Chernyi[14] and is compared by Mirels[4] [see Eqs (3 6, 3 9, 5 16) and Table 1 of Ref 4] with the numerical solution of the slender body equations Agreement with regard to surface pressure is within 4% in all cases We observe also that, if K exceeds unity, the surface pressure is negative Hence we require, from Eq (26),

$$m > [(\gamma + 1)/(\gamma + 3 + 2\sigma)]$$

Lees and Kubota[15] give, as a minimum value for m for the existence of a similarity solution, the same expression but with $\gamma = 1$

Equation (24) and, more generally, Eq (26) demonstrate that for a given shock the local surface pressure distribution far back is uninfluenced by the entropy layer associated with small leading edge bluntness This is, of course, not a new result [1]

C Hemisphere-Cylinder

This example was undertaken for two reasons First, it is a direct problem; the body is given Second, it is a severe test of the approximations because, in the neighborhood of the shoulder, the shock and body angles differ markedly (on the order of 30°), making the approximation suspect The example computed is that of a hemisphere-cylinder in a perfect gas ($\gamma = 1\ 4$) at $M_\infty = 7\ 7$, a case which has been measured by Kubota[15, 16] and verified theoretically by Inouye and Lomax[17] using the method of characteristics and Fuller's variation of the Van Dyke method for the stagnation region

In the present analysis, the solution was obtained as follows Using experience obtained in previous examples, as well as the solutions of other authors, an initial estimate of the shock was made for the subsonic-transonic region Small variations yielded the desired body Three tries sufficed The solution was continued by assuming, at subsequent downstream stations, new values of the shock radius of curvature From this, by successive integrations, the shock slope and position are obtained by using $1/R = d \sin\theta /dz$, $\tan\theta_s = dr_s/dz$ The curvature, rather than the radius of the shock, was used as the basic quantity because the related properties could be found by integration rather than differentiation, which reduces accuracy Convergence on the desired body point was rapid, two or sometimes three guesses being enough Initial estimates of the radius of curvature were made by requiring that it have a smooth variation along the shock The results were obtained using a net of 54 points (to $Z = 6$), although this is certainly an unnecessarily large number Of these, there were 9 body and 9 shock points The x interval to be used is governed entirely by what is required to derive the shock position and slope from its curvature The shock shape and surface pressure distribution for this example is shown in Fig 6 Comparison is made with the results of Inouye and Lomax[17] and Kubota [16] Agreement is certainly satisfactory

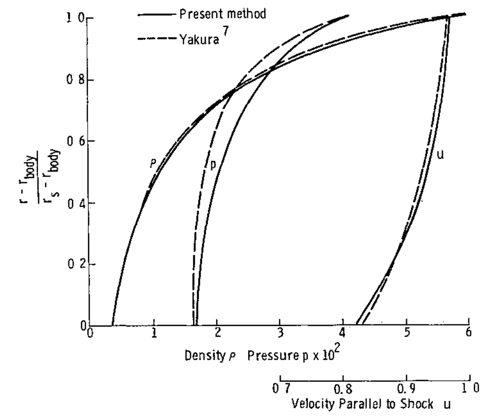

Fig 5 Pressure, density, and velocity profiles at $z = 10$ for paraboloidal shock $r_s = (2z)^{1/2}$ at $M_\infty = \infty$, $\gamma = 1\ 4$

The local underexpansion of pressure predicted near the shoulder and the reduced shock angle and position far back arise from the same source Near the shoulder, the v_x term in the lateral momentum equation [Eq (1)] is not really negligible It should contribute to a greater pressure drop from shock to body and also should permit a smaller local shock radius of curvature This smaller value would then be reflected in larger downstream shock angles and a thicker shock layer In any case, the discrepancies are not serious

Finally, in Fig 7 there is shown the profile shape for the impact pressure at the $Z = 6$ station together with the experimental results and those results from the method of characteristics Except at the shock, the three results agree to within the (small) experimental error

D Conical Flow

This is a pleasant exercise because the results are both simple and very accurate Unfortunately, they are mostly not new but follow from the constant density theory (see, for example, Chap 4 of Hayes and Probstein[3]) The present method, however, does offer a simple means of finding higher approximations to the solution Furthermore, although the first approximation yields constant density, this is not an assumption of the present scheme

The conical property should probably be used at the outset; however, to be consistent, we proceed directly from the

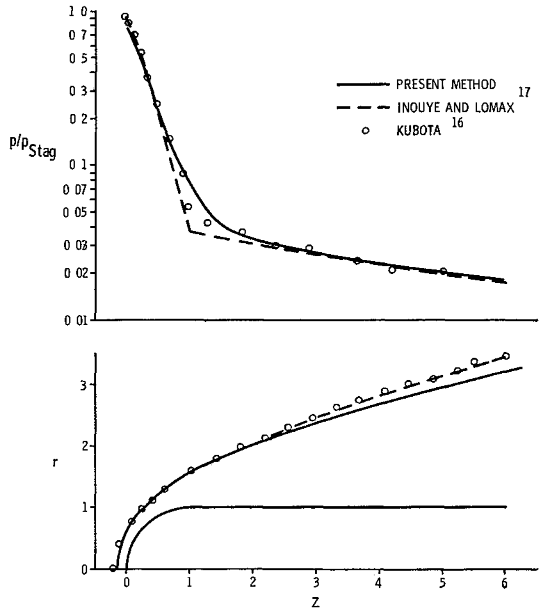

Fig 6 Shock shape and surface pressure for hemisphere cylinder ($M_\infty = 7\ 7$, $\gamma = 1\ 4$)

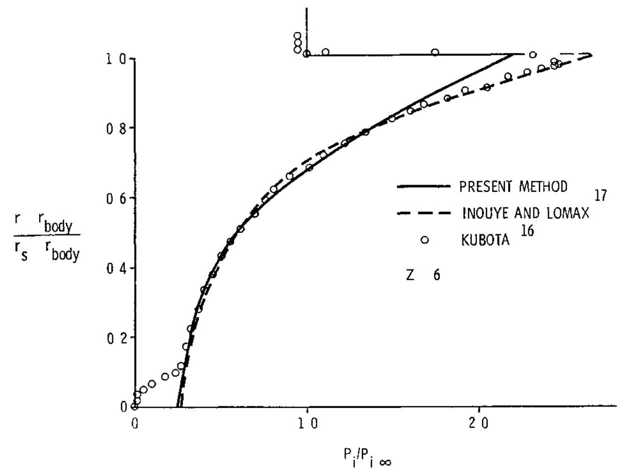

Fig 7 Impact pressure across hemisphere cylinder shock layer ($M_\infty = 7\ 7$, $\gamma = 1\ 4$, $z = 6\ 0$)

forms already displayed Beyond the conical limitation, which requires that the flow be either frozen or in equilibrium, nothing need be assumed about how ıeal the gas is

Because the shock is straight (i e , $R = \infty$), the first approximations to p, ρ, h, and u [Eqs (10, 6, and 11), respectively] are constants, namely, the values immediately behind the shock Then Eq (14) becomes

$$y = (\psi - \psi)/\rho\, u\, r \tag{27}$$

At the body, if θ is the cone angle, this becomes,

$$\tan(\theta - \theta) = \tan\theta /2\rho \tag{28}$$

Observe that, from the shock relations, the difference between the shock angle and flow deflection angle at the shock is exactly $\tan^{-1}[(\tan\theta)/\rho]$, which is just twice $(\theta - \theta)$ as given here

For conical flow, the result can easily be pursued further Because $\psi^{1/2}$ and r vary as x, Eqs (5) and (1) yield, successively,

$$v = (\psi + \psi)/\rho\, r\, x \tag{29}$$

and

$$p - p = \int_{\psi}^{\psi} \frac{v_x d\psi}{r} = -\int_{\psi}^{\psi} \frac{2\psi d\psi}{\rho\, r^{\,2} x^2} = \frac{\psi_s{}^2 - \psi^2}{\rho\, r^{\,2} x^2} \tag{30}$$

Equation (30) can be written

$$\frac{p - p_c}{p_s - p} = \left(\frac{\theta - \theta_c}{\theta - \theta_c}\right)^2 \tag{31}$$

the subscript c referring to the cone surface The perturbation in density and speed $(u^2 + v^2)^{1/2}$ also takes the form of Eq (31) Equations (28) and (30) are not new, having been found previously by the constant density theory [Hayes and Probstein,[3] Eqs (4 2 20) and (4 2 23)]

The present method permits further iterations of the solution to be peıformed in a straightforward manner However, they are not very interesting except for the size of the next terms In the next approximation the right side of Eq (27) is multiplied by the factor

$$\left[1 + \frac{1}{4\rho}\left(1 - \frac{2\sin^2\theta_s}{3\rho\, a^{\,2}}\right)\right]$$

where a is the ratio of the speed of sound immediately behind the shock to the freestream velocity Observe that the appearance here of the sound speed is the first point of entrance of the gas model for the region behind the shock For a perfect gas, this factor is

$$\left[1 + \frac{2\gamma - 1}{12\gamma\rho}\right]$$

Clearly, this is a very small correction

Comparison of Eqs (27) and (30) with tabulated data for both real and perfect gases shows that the results are accurate to within about 1%, provided that the component of freestream Mach number normal to the shock exceeds about 3 Also, the variation indicated by Eq (31) is confirmed with remarkable accuracy by Kelly[18] in his Fig 4

IV Comments

The scheme described herein for the calculation of hypersonic flows has the very great virtue that it is simple and straightforward, and, for the examples given, its accuracy is quite adequate for general purposes Furthermore, without modification, the method can be applied throughout the entire shock layer from the axis of symmetry to points far downstream Detailed profiles in the flow are easily found Also, one is not limited to a perfect gas or to extreme Mach number Even nonequilibrium cases can be handled in a straightforward manner, although such a calculation is no longer simple A concrete illustration of the simplicity of the calculation is perhaps in order All the numerical results herein, including all the details of the flow fields (many of which are not presented), represent less than three working days of slide rule calculations

Although it is true that the present method is an inverse one, which means that some sort of iteration scheme must be invoked to solve the flow over a given body, it is clear from the experience with the hemisphere-cylinder example that this presents no great obstacle Further, by working from the radius of curvature of the shock rather than the shock position itself, operations in the x direction are only integrations, not differentiations, so that the step size (in x) can be laıge

Another point of some concern is that integration of the equations of motion is in a direction normal to the shock Actually, for a given shock, that part of the shock up to some point x determines the flow field only up to the characteristic running forward from the point x on the shock However, this apparent discrepancy is resolved by two considerations First, not only the shock position and slope, but also its curvature are required This implies that the shock is really specified for some distance downstream of the nominal point of interest Second, most of the mass flow in the hypersonic part of the field is near the shock For example, in the hemisphere cylinder case, about half the mass flow is in the outer 20% of the shock layer This concentration of the mass varies with the amount of bluntness, but in any case it tends to make the shock relatively unresponsive to variations near the surface The upshot of all this is simply that the method should be satisfactory whenever the shock is smooth, in particular, when its curvatuıe vaıies smoothly This is a rather unrestrictive condition

Speaking loosely, it is apparent that, particularly on examination of the iteration described briefly for the conical flow case, the present analysis forms the leading term in an expansion in inverse powers of ρ However, based on the arguments surrounding the equation for the pressure distribution, as well as the good agreement with more elaborate analyses throughout the flow field, it appears that the result is more accurate than the first term in such an expansion would seem to indicate The method provides a link connecting the constant density analysis applicable near the front with the small disturbance theory valid far back on a slender body It is not, however, restricted by the usual limitations of either method

To repeat, the essential assumption is Eq (9) It requires a shock layer in which the streamlines essentially parallel the shock Also, the change (across the layer) of the cylindrical radius r and main velocity component u are such that the mass weighted pressure gradient P_ψ does not vary across the shock layer The parallel assumption means that the freestieam Mach number must be large In the stagnation region the assumption is false, but not seriously so, and can easily be coirected Further downstream, where the shock angle is still big, the density ratio across the shock is big also Then the shock layer is thin and the assumption is good Far back where the shock is weak, the Mach angle is very small, and the shock will follow the flow regardless of the density ratio These requirements are most important (and usually most valid) relatively near the shock because near the body, in the entropy layer, the density is low and the local mass flow (i e , ψ) is small Finally, the especially simple expression for the pressure distribution has been verified by a direct check of the consistency of the present results with the exact pressure relation, Eq (1) (not shown herein), and is a reflection of the fact that variations across the shock layer are actually quite linear in ψ

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