Complex Analysis: lecture notes

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1 Introduction

1.1 C1123 Introduction

Welcome to C1223! I hope you'll enjoy this very interesting and unique course at Mines ParisTech.

It is a great opportunity to learn about an important field in mathematics (including Applied Mathematics) and one of the last opportunities for you to stretch your "fundamental" math skills before you become engineers.

Whether you've appreciated math courses in the past or not, there should be something in this course for you. The theory is both intuitive (we'll be drawing a lot) and beautiful (things are simple).

1.1.1 Pre-requisites

There are very few prerequisites for this course apart from basic linear algebra (vector spaces, ...), basic topology (open and closed sets, ...) and analysis (Taylor series, ...).

The textbook is mostly self-contained and I'll happily provide any background you may be lacking (or have forgotten) during the lectures.

1.1.2 Methodological reminders

The "lectures" for C1223 last an hour and are directly followed by a problem sesssion (with another instructor). There are two such sessions per day for a week so the course is in fact quite intense.

Some advice:

Stay focused! The course is fast-paced and you won't have much time to absorb the material between sessions! However, rest assured that the material is tailor-made to fit to this format. Just follow the flow and the ideas will progressively mature over the week and following weeks before the exam. I'll try to provide context as we go.

Ask questions! The goal is for everyone to be able to make progress and asking questions definitely contributes to this. As soon as the first person asks his or hers, I'm confident the rest will follow!

Draw! In Complex Analysis, a picture by proof is often more than half way to a solution. In fact, without drawings some parts of the material would be virtually impossible to present.

Finally, give feedback! This is a learning experience for me at least as much as it is for you. If something is unclear or poorly explained, ask questions! And if the pace is too slow or too fast, speak up! I'm happy to accomodate your requests.

1.1.3 About this document

Before we start, a few words about this document:

First, what is it not?

A replacement textbook S.B.'s text is extremely well written and we'll be using it as the main tool for the course. It contains the material, exercices and their solutions.

A cheat sheet You should be getting one of those at the end of the course to help in your review before the exams.

What is it then?

A summary I try to present the most important elements of the text, without any proofs but with illustrations, examples, counter-examples and heuristics.

A study-guide If you can master everything that is here, you should be fine.

My notes I wrote this while preparing for TA'ing this course, so actually, it is mostly for me. I hope it will be useful for you too though.

A final note (and disclaimer): these pages are not a subset of the textbook material nor are they a superset... The syllabus will be explicited by the Professors responsible for the course. Nothing here is official nor endorsed by them. The material I've added are mostly reminders from prerequisites and extra counter-examples... basically, what I had scribbled in the margin of my textbook when I took the course.

2 First sesssion: Complex-differentiability

Contents

1. Reminders
   (a) topology: open sets (definition and examples)
   (b) $\mathbb{C}$, $\mathbb{C}$-vector spaces, $\mathbb{C}$-linearity
   (c) (Analysis in $\mathbb{R}$: characterisation of differentiability)
2. Complex differentiability:
   (a) How do we define complex-differentiabiity?
   (b) the complex-derivative?
   (c) the complex-differential?
   (d) What is the difference between real-differentiability and complex-differentiability?
3. Complex differentiability in practice
   (a) Usual theorems
   (b) Cauchy-Reimann equations: two formulations

2.1 Quick reminders

2.1.1 Open sets

Definition (Open set)**.** *A set* $\Omega \subset A$ *is open in* A *when:*$\forall a \in \Omega, \exists r > 0, \forall y \in \mathcal{B}_r(a), y \in \Omega$

Alternatively:

- $\forall a \in \Omega, \exists r > 0, \forall y \in A, ||y - a|| < r \implies y \in \Omega$
- $\mathring{\Omega} = \Omega$

In practice:

1. It can be "clear": the goal of this course is not to show that sets are open.
2. Write Ω as the inverse image of an open set by a continuous map. For example, the open unit ball is the inverse image of $]0, 1[$ by $x \in E \mapsto ||x||$.

Drawing. *Visually, this means you can get as close as you want to the complement of the open set* Ω *and your neighbors are still in* Ω*.*

Example. *A few examples of open sets are:* $\mathbb{R}$, $\mathbb{C}$, $]0, 1[$. $[0, 1]$ *and* $[0, 1[$ *are not open.*

2.1.2 The complex plane & complex functions

Definitions and properties:

1. Definition: $\{x + iy, x \in \mathbb{R}, y \in \mathbb{R}\}$, i such that $i^2 = 1$.
2. Exponential, trigonometric forms:
$$\exists R > 0, \theta, x + iy = R(\cos\theta + i\sin\theta) = R\exp(i\theta)$$
3. Conjugate: $z = x + iy \mapsto \bar{z} = x - iy$, $z = Re^{i\theta} \mapsto \bar{z} = Re^{-i\theta}$
4. ...

Exercise 1. *On the structure of $\mathbb{C}$ as a vector space.*

1. *Show that the complex plane $\mathbb{C}$ is a complex vector space.*
2. *Give a basis.*
3. *What changes if we consider it to be a real vector space?*

Exercise 2. *Give an example of a $\mathbb{C}$-linear function. Give a counter-example (an $\mathbb{R}$-linear function from $\mathbb{C}$ to $\mathbb{C}$ that is not $\mathbb{C}$-linear.)*

2.2 Complex-differentiable, derivative and differential

In this section, Ω is an open subset of $\mathbb{C}$.

Definition (Complex-differentiability & Derivative)**.** *Let z an interior point in Ω. Then, f is* complex-differentiable *at z iff the* complex derivative *of f exists in $\mathbb{C}$, as defined by*

$$f'(z) = \lim_{h \to 0, h \in \mathbb{C}} \frac{f(z+h) - f(z)}{h}$$

Note. *Let's insist on the $h \in \mathbb{C}$ in the definition. This means that the limit exists no matter the direction from which we are "arriving" in $\mathbb{C}$. Also, no matter the direction, the value is the same.*

Note. *The previous definition is a* local *property. Let's generalize it to Ω below.*

Definition (Holomorphic)**.** *$f : \mathbb{C} \to \mathbb{C}$ is* holomorphic *if $\forall z \in \Omega$, it is* complex-differentiable*, i.e.* complex-differentiable everywhere.

Example. *Some examples of holomorphic functions. Also, a counter-example.*

- *Some "obvious" examples: constant functions, identity, affine, ...*
- *($\star$) Show the inverse function is holomorphic where it is defined.*
- *($\star$) A counter example: the complex conjuguate function is nowhere complex-differentiable.*

Definition. *Let $f : A \subset \mathbb{C} \to \mathbb{C}$.*

Let $z \in A$ as interior point.

We define the complex-differential *(or $\mathbb{C}$-differential) of f in z a complex-linear, continous operator $df_x : \mathbb{C} \to \mathbb{C}$ that verifies:*

$$\lim_{h \to 0, h \in \mathbb{C}} \frac{||f(z+h) - f(z) - df_z(h)||}{||h||} = 0$$

Equivalently:

$$f(z+h) = f(z) + df_z(h) + o(h)$$

Note. *Recall that*

$$g(h) = o(h) \iff \lim_{h \to 0} \frac{g(h)}{h} = 0$$

Note. *This result extends to $f : E \to F$ where E and F are complex, normed vector spaces. For example, $\mathbb{C}^n$.*

Now, an important theorem that reflects the structure of $\mathbb{C}$:

Theorem. *Let $f : A \subset \mathbb{C} \to \mathbb{C}$. Let $z \in \mathring{A}$.*

The complex-differential of f df_z exists iff the derivative $f'(z)$ exists.

In this case, we have:

$$\forall h \in \mathbb{C}, df_z(h) = f'(z)h$$

Note. *This is linked to our view of $\mathbb{C}$ as a complex vector space, of dimension 1.*

2.3 Complex-differentiability in practice

2.3.1 Calculus

- $\mathbb{C}$-linear combination
- Product
- Chain rule
- Quotient
- Polynomials
- Rational fractions

In practice: *calculate like in* $\mathbb{R}$.

These "usual" properties allow us to verify that functions are $\mathbb{C}$-differentiable or holomorphic easily, when they are products and compositions (for example).

2.3.2 Cauchy-Reimann Equations

We can rephrase the definitions as:

Theorem. *f is complex differentiable on Ω if and only if both:*

1. *f is real-differentiable on Ω (i.e. its differential exists, but maybe isn't $\mathbb{C}-linear$)*
2. *its real-differential is $\mathbb{C}$-linear.*

But the second condition is not easy to prove without expliciting the differential. The Cauchy-Riemann equations offer an alternative formulation of this condition.

The intuition behind the theorem is to look at what happens to the real and imaginary parts of f when we move the real and imaginary parts of z

If we write:

1. if $f : \mathbb{C} \to \mathbb{C}$, we have $u : \mathbb{C} \to \mathbb{R}$ and $v : \mathbb{C} \to \mathbb{R}$ such that:

   $$\forall z \in \mathbb{C}, f(z) = u(z) + iv(z)$$

   Of course, u and v are the real and imaginary parts of $f(z)$.

2. and, notice a similar property about z:

   $$\forall z \in \mathbb{C}, z = x + iy$$

   where x and y are the imaginary parts of z.

we can then study the following quantities:

- $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$
- $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$

Theorem (Cauchy-Riemann Equations)**.** *f is complex differentiable on Ω if and only if both:*

1. *f is real-differentiable (i.e. its differential exists, but maybe isn't $\mathbb{C}-linear$)*
2. *its real-differential is $\mathbb{C}$-linear.*

The second condition (2) can be replaced by one of the following properties:

(a) $\forall z, df_z(i) = idf_z(1)$

(b) *f verifies the **complex Cauchy-Riemann equation**:*

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$$

(b) *f verifies the* ***scalar Cauchy-Riemann equations:***

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Theorem (Cauchy-Riemann Equations (alternate))**.** *f is complex differentiable on Ω if and only if one of the following conditions hold:*

(a) *$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist, are* ***continuous*** *and verify the* ***complex Cauchy-Riemann equation***

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$$

(b) *$\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ exist, are* ***continous*** *and verify the* ***scalar Cauchy-Riemann equations***

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Exercise 3 ($\star$)**.** *Show that the complex exponential and logarithm maps are holomorphic where they are defined.*

3 Second sesssion: Line Integrals & Primitives

The goal of this chapter is to derive an equivalent to the fundamental theorem of calculus for functions of a complex variable.

Recall that is $\mathbb{R}$ one generally uses a rectangular approximation along the segment one integrates over. In $\mathbb{C}$, the principle is the same: we'll present how we can build paths on $\mathbb{C}$ and then how we can integrate over them by approximating the paths.

Contents

3.1 Paths

3.1.1 Definition, Vocabulary & Examples

Definition (Path)**.** *A path is a continuous function from* $[0,1]$ *to* $\mathbb{C}$*.*

Note. *The intuition behind a path is that it is parametrized by time, or another variable independent of the complex plane.*

Exercise 4 (Unit circle)**.** *Give a path whose image is the unit circle.*

Exercise 5 (Straight line)**.** *Give a path such that* $\gamma(0) = a$ *and* $\gamma(1) = b$ *where* $a, b \in \mathbb{C}$*.*

Note. *Paths are oriented!*

Exercise 6 (Reverse path)**.** *Given a path* γ *how would you describe its reverse path* $\gamma^{\leftarrow}$*.*

3.1.2 Image of a path

Definition (Image of a path)**.** *The image (or trajectory, or trace) of a path* γ *is* $\gamma([0,1])$*, all the points reached by the path.*

Definition (Path on a subset)**.** *A path is* on $A \subset \mathbb{C}$ *if its image is a subset of* A*, i.e.* $\gamma([0,1]) \subset A$*.*

Exercise 7. *The image of a path is compact.*

Solution. γ is continuous and $[0,1]$ is compact (closed and bounded, in a finite dimensional space), so $\gamma([0,1])$ is compact. $\square$

3.1.3 Path concatenation

Here we present the vocabulary and properties necessary to build more complex paths.

Definition (Consecutive paths)**.** *Two paths γ_1 and γ_2 are consecutive if $\gamma_1(1) = \gamma_2(0)$, i.e. if γ_1 terminates (terminal point) where γ_2 begins (initial point).*

Definition (Path concatenation)**.** *Let $t_0 = 0 < t_1 < \cdots < t_{n-1} < 1 = t_n$ be a partition of $[0,1]$ (these are our endpoints).*

Let γ_1, ..., γ_n n consecutive paths (these are the paths.)

The concatenation of $\gamma_1, \ldots, \gamma_n$ associated to the partition $t_0, \ldots, t_{n-1}$ is the path γ uniquely defined such that: $\boxed{\forall k \in \{0, \ldots, n-1\}, \gamma_{|[t_{k-1},t_k]} = \gamma_k \left(\frac{t-t_{k-1}}{t_k - t_{k-1}}\right)}$

We denote it:

$$\gamma_1|_{t_1} \cdots |_{t_{n-1}} \gamma_n$$

We can simplify notations if the partition is uniform (i.e. $t_k = k/n$):

$$\gamma_1| \ldots |\gamma_n$$

Example (Oriented Polyline)**.** *An oriented polyline is the concatenation of consecutive oriented line segments. We denote them:*

$$[a_0 \to \cdots \to a_n]$$

where $(a_0, \ldots, a_n) \in \mathbb{C}^n$ are "the endpoints".

3.1.4 Paths & Regularity

Until now, the paths we've described were "only" continuous. Integrating over paths is like using changes of variable. Recall that is real analysis, changes of variable generally need to be $\mathcal{C}^1$. Here we define analogous tools in $\mathbb{C}$.

Definition. *A path is rectifiable if it is piece-wise continuously differentiable, i.e. it is differentiable and its differential is continuous.*

Note (Continuous differential)**.** *Let's clarify the meaning of continuous in this case. For this, recall that f's differential at a point z is a linear operator df_z. This operator is continuous with respect to its argument, i.e. $h \mapsto df_z(h)$ is continuous.*

A function f is continously differentiable if the map $z \mapsto df_z$ is continuous.

In the case of paths, this comes down to the path being differentiable and its derivative being continuous.

Note. *There are no conditions on the differentials at the boudaries "between pieces".*

Example. *An oriented polyline is rectifiable.*

Theorem (Continuously differentiable decomposition)**.** *A path γ is rectifiable if and only if there are $\gamma_1, \ldots, \gamma_n$ consecutive continuously differentiable paths and a partition $(t_1, \ldots, t_{n-1})$ of $[0,1]$ such that:*

$$\gamma = \gamma_1|_{t_1} \cdots |_{t_{n-1}} \gamma_n$$

3.1.5 Connected sets

Definition. *An open subset Ω of $\mathbb{C}$ is (path-)connected if for any points $x, y \in \Omega^2$, there is a path on Ω that joins x and y.*

Example. *Some connected open sets:*

- $\mathbb{C}$
- $\mathcal{B}_0(r)$
- $\ldots$

Some open, disconnected sets:

- $\mathbb{C} - \mathbb{R}i$
- $\mathcal{B}_0(r) - \{0\}$
- $\ldots$

Theorem. *An open subset Ω of the complex plane is connected if and only if every pair of points can be joined by a rectifiable path of Ω.*

Sketch of the proof. $\gamma([0,1])$ is compact and $\mathbb{C} - \Omega$ is closed. Thus the difference between the two is strictly positive. By uniform continuity of γ we can build an oriented polyline that approximates it while staying within Ω (same technique as is $\mathbb{R}$). The polyline is a rectifiable path on Ω. $\square$

3.2 Line Integrals

In this section, we define the line integral of $f : \mathbb{C} \to \mathbb{C}$ over a rectifiable path γ as the integral over $[0,1]$ of $f \circ \gamma \times \gamma'$.

Definition. *The line integral along a rectifiable path γ of $f : \mathbb{C} \to \mathbb{C}$, continuous over $\gamma([0,1])$ is:*

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt$$

Note. *We deliberately overlook the fact that γ is only rectifiable and thus γ' is undefined at a finite number of points (almost everywhere).*

This gives us a way of explicitly calculating line integrals! We won't be doing (too) much of that here. However, let's look at a few examples.

Definition (Length of a rectifiable path)**.** *The length of a rectifiable path γ is*

$$\ell(\gamma) = \int_0^1 |\gamma'(t)|dt$$

Note. *Because of the derivative, the "position" of γ in the complex plane has no impact on the length of the path... this is what we'd expect!*

Exercise 8 ($\star$)**.** *Calculate the length of a line segment.*

Exercise 9 ($\star$)**.** *Show that $\ell(\gamma) = \ell(\gamma^{\leftarrow})$. If $\gamma = \gamma_1|_{t_1} \dots |_{t_{n-1}}\gamma_n$ a concatenaiton of consecutive, rectifiable paths, show that:*

$$\ell(\gamma) = \sum_k \ell(\gamma_k)$$

Exercise 10 ($\star$)**.** *If $\gamma = \gamma_1|_{t_1} \dots |_{t_{n-1}}\gamma_n$ a concatenaiton of consecutive, rectifiable paths, show that:*

$$\ell(\gamma) = \sum_k \ell(\gamma_k)$$

Exercise 11 ($\star$)**.** *Calculate the length of an oriented circle of radius r and with n traversals.*

Exercise 12. *Give a parametrization for the line integral of f over $[a \to b]$.*

Exercise 13. *Give a parametrization for the line integral of f over the oriented circle of radius r, of center 0 and in the positive direction. And with n traversals?*

3.2.1 Line integral calculus

Complex-linearity

$$\int_\gamma \alpha f + \beta g dz = \alpha \int_\gamma f dz + \beta \int_\gamma g dz$$

Integration along a reverse path

$$\int_\gamma f dz = - \int_{\gamma^{\leftarrow}} f dz$$

Integration over a concatenation If $\gamma = \gamma_1|_{t_1} \dots |_{t_{n-1}}\gamma_n$ a concatenaiton of consecutive, rectifiable paths, γ is rectifiable and:

$$\int_\gamma f dz = \sum_k \int_{\gamma_k} f dz$$

3.2.2 Reparametrization & Changes in variables in line integrals

Here, we present two useful properties of line integrals: first, that we can reparametrize the path along which we are integrating (i.e. scale time) without changing the value of the integral; second, that we can apply classical change in variable formulas under certain conditions.

Theorem (Invariance by reparametrization)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ is a continuous function, and γ a continuously differentiable path.*

Let $\phi : [0,1] \to [0,1]$ an increasing-$\mathcal{C}^1$-diffeomorphism, i.e such that:

- *ϕ is continuously differentiable*
- *ϕ is increasing (i.e. $\phi'(t) > 0$)*
- *$\phi(0) = 0$ and $\phi(1) = 1$.*

Then, if $\mu = \gamma \circ \phi$:

1. *μ is a continuously differentiable path with the same endpoints and image as γ.*
2. *$\ell(\mu) = \ell(\gamma)$*
3. *The line integrals of f over μ and γ are equal:*

$$\int_\mu f dz = \int_\gamma f dz$$

Note. *The intutition behind this is that a path is a trajectory on the complex plane, for example, that of a robot. If two robots follow the same path at different speeds but leave and arrive simulataneously, their distances traveled are the same, and they've covered the same ground.*

Now let's show that the usual change of variable formula holds in complex analysis:

Theorem (Changes of variables in line integrals)**.** *Let Ω be an open subset of $\mathbb{C}$.*
Let $f : \Omega \to \mathbb{C}$ be a holomorphic function.
Let $g : f(\gamma([0, 1])) \to \mathbb{C}$ a continuous function.
Then,

1. *$f \circ$ gamma is a rectifiable path.*
2. *The following change of variables holds:*

$$\int_{f\circ\gamma} g(z)dz = \int_\gamma g \circ f(z)f'(z)dz$$

3.2.3 ML-Inequality & Convergence

An important consequence of the triangular inequality is that the (line) integral can be bounded by the product of the maximum of the integrand and the length of the path (or segment) one integrates over.

In complex analysis, we have the following important result:

Theorem (M-L Inequality)**.** *Let γ be a rectifiable path.*
Let $f : A \subset \mathbb{C} \to \mathbb{C}$ a continuous function.
Then, $\boxed{\left\| \int_\gamma f \right\| \leq \max_{z\in\gamma([0,1])} \|f(z)\| \times \ell(\gamma)}$

Proof. Use the triangular inequality. $\square$

Thanks to this inequality, we can prove that the line integrals of an uniform approximation f_n of f converge towards the the line integral of f, as in $\mathbb{R}$.

Theorem. *For any rectifiable path γ and uniform approximation f_n of f a continuous function (i.e. $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$),*

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz$$

3.3 Primitives

Now that we have defined line integrals, we can identify the main difficulty of defining a primitive of a complex function: the integral depends on the the path use to integrate over! However, derivation is a very local property that does not know what happens far from the derivation point. For these reasons, the fundamental theorem of analysis over $\mathbb{R}$ is ambiguous. Here we adapt it, which yields important results about holomorphic functions.

Definition. *A primitive of a continuous function f defined on a open subset Ω of $\mathbb{C}$ is a holomorphic function defined on Ω such that $g' = f$.*