Data Structures and Algorithms
EE2008/IM1001

Coverage

	Topics
Weeks 1-5 A/P Low Chor Ping icplow@ntu.edu.sg S2-B2c-86	Introduction
	Principles of Algorithm Analysis
	Data Structures
Weeks 6-13 A/P Huang Guangbin egbhuang@ntu.edu.sg Ast Prof Tay Wee Peng wptay@ntu.edu.sg S2.2-B2-51	Sorting
	Searching
	Algorithm Design Techniques

Assessments

- Grading will be based on
  - 2 Homework Assignments,
  - 2 Lab Assignments
  - 1 continuous assessment,
  - Final exam
- 1st Homework Assignment
  - Given out in week 4 by your tutors during tutorial sessions
  - Hand in to your tutors for grading in week 5 during tutorial sessions
- 2nd Homework Assignment
  - Given out in week 11 by your tutors during tutorial sessions
  - Hand in to your tutors for grading in week 12 during tutorial sessions

Late Policy

- Late homework assignment submissions will be penalized 10% each day it is late.
- The penalty begins at the beginning of class the day the assignment is due.
- The 10% penalties will continue for 3 full days at which point no more late submissions will be graded.

Plagiarism Policy

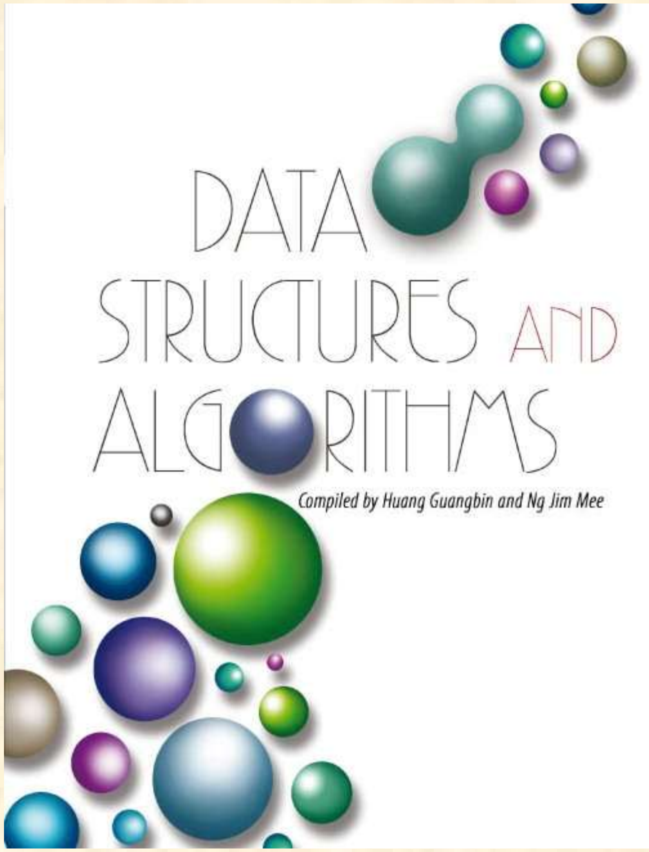
- The actual write-up must be done entirely by yourself.
- You cannot directly copy or slightly change other students’ solutions
- If you cheat on an assignment, both you and the person who helped you will receive a lower grade or the fail grade F.

Continuous Assessment

- Continuous Assessment will be held on week 7.
- The venue and time will be notified in due course.
- Tutors will take attendance during CA
  - Bring along student ID
  - Students without any ID (with photo) are to sign against their names in attendance list. Need to show ID to tutor later before marks can be accepted
- Absentees should contact tutors within one week of the CA
- Absentees with valid reasons or MCs can request to take a separate CA within 2 weeks of CA. Tutor to decide if request can be acceded
- If absentees do not contact the tutor, zero marks will be awarded

Text Book:

- GB Huang and JM Ng (eds), Data Structures and Algorithms, Pearson Education, 2007


DATA STRUCTURES AND ALGORITHMS
Compiled by Huang Guangbin and Ng Jim Mee

Reference Book:

- Richard Johnsonbaugh and Marcus Schaefer, Algorithms, Prentice Hall, 2004. ISBN 0131228536
- Anany Levitin, Introduction to The Design and Analysis of Algorithms, 2nd ed., Addison Welsey, 2007.

  ISBN 0-321-36413-9

Other resources:

- Edventure:

  http://edventure.ntu.edu.sg

What are Algorithms & Data Structures??

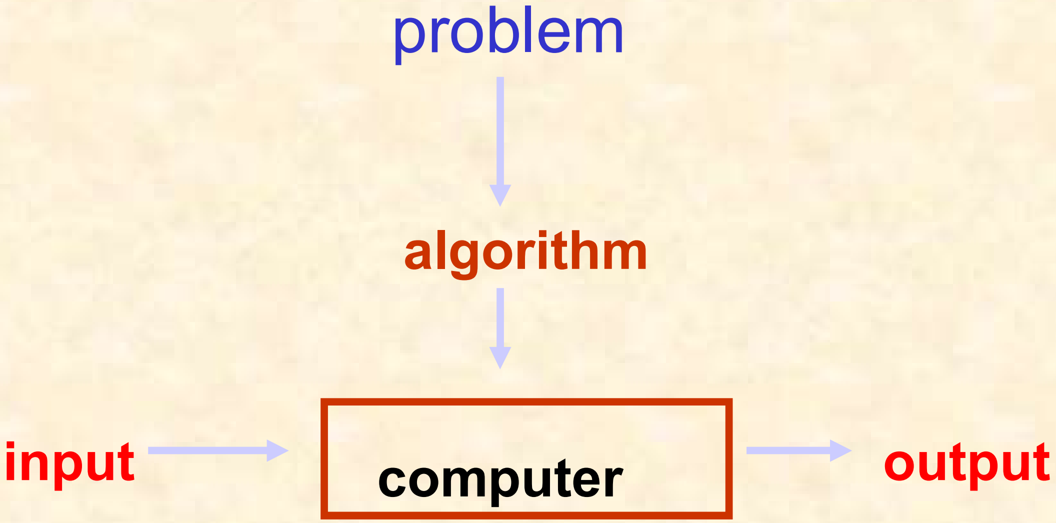
What is the use of Algorithms ?

What is the use of data structures ?

What is the inter-relationship between algorithms & data structures ?

What is the use of Algorithms ???

For solving a problem



problem

algorithm

input

computer

output

Algorithm

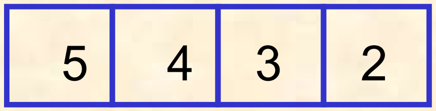
- **a sequence of instructions for solving a problem**

What is the use of Data Structure ?? Basic Data Structures

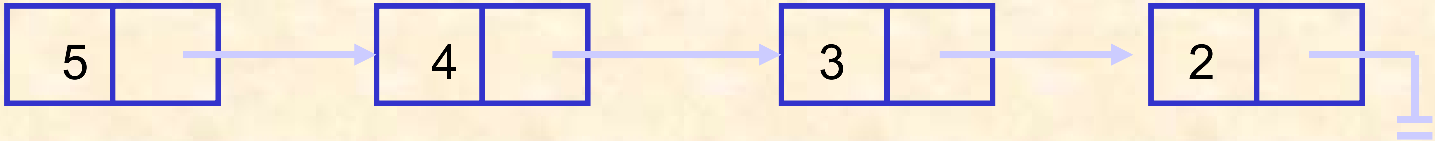
Data Structure

ways of organizing data to promote efficient processing

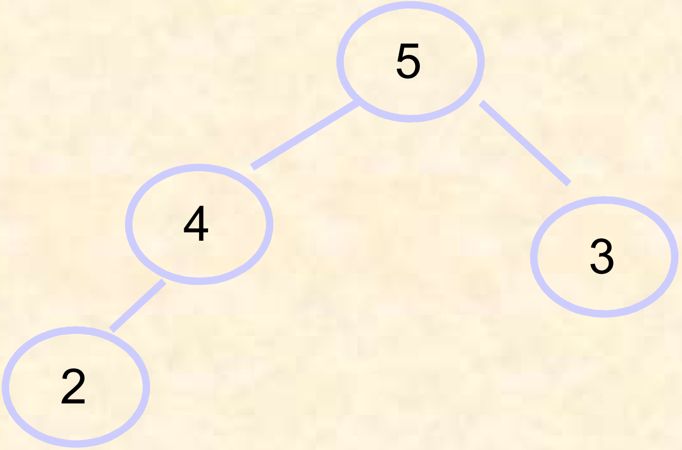
Array



Linked List



Max Heap

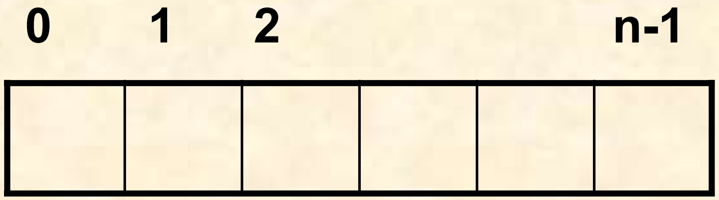


What is the Relationship b/w Algorithms & Data Structure

Choice of Data Structures will affect efficiency of algorithm

- Example
  - **Input : a set of n numbers**
  - **Output: the kth element of the input list**

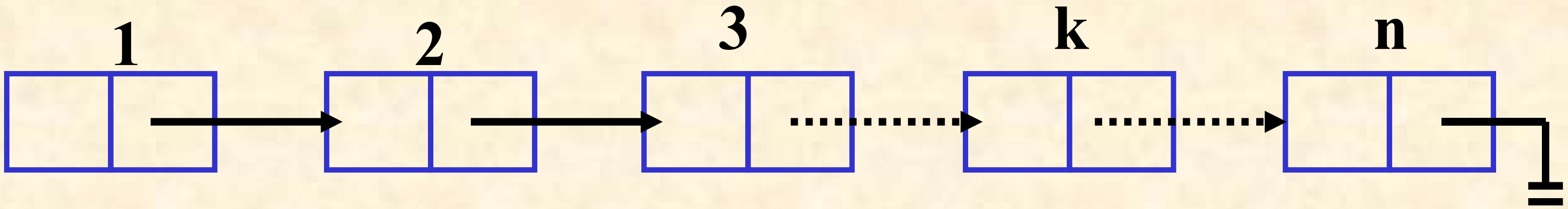
Array X



kth element = X(k-1)

Need 1 operation

Linked List

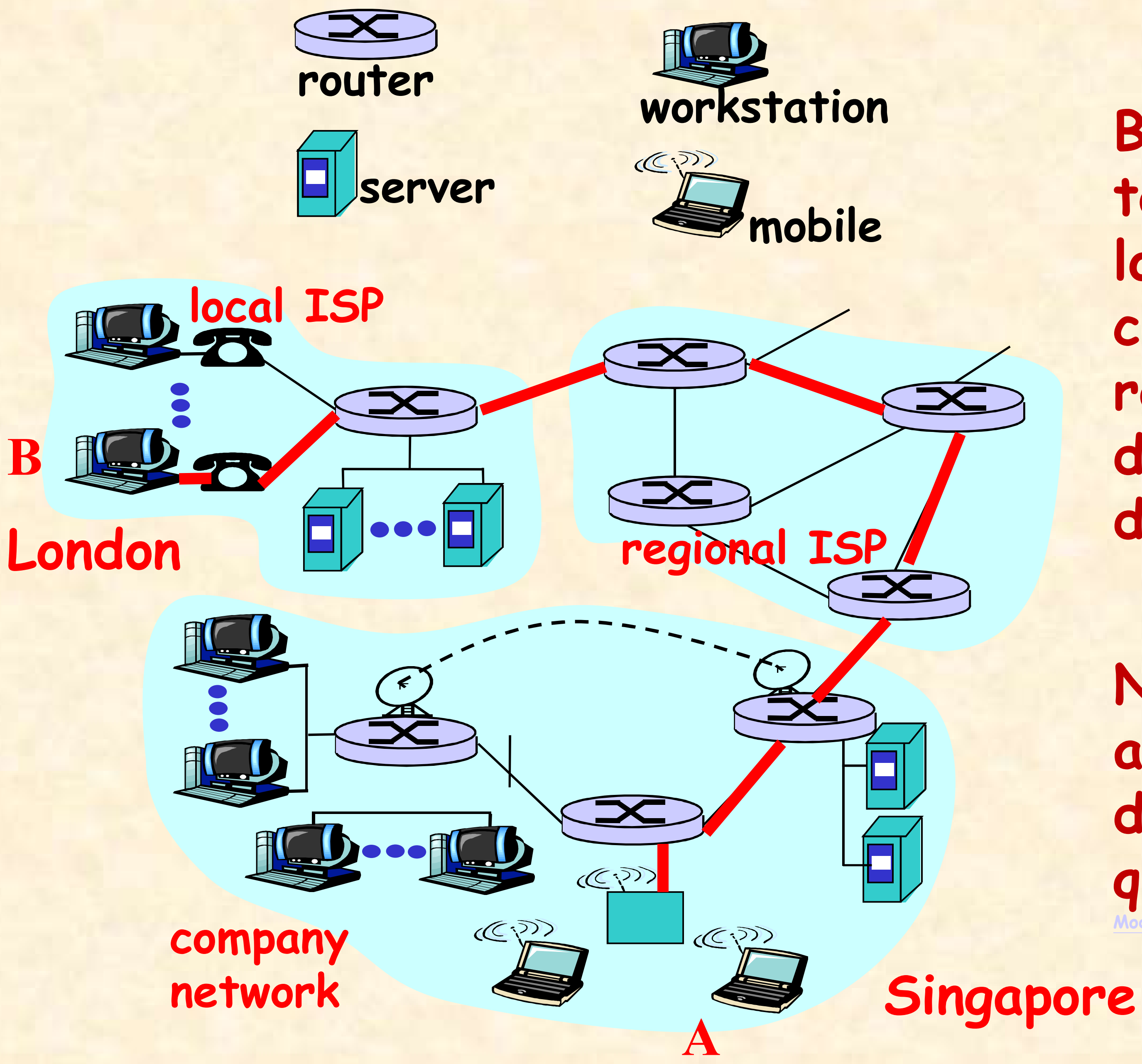


Need k-1 operations

Given a problem: choose appropriate data structure & algorithm to solve problem efficiently

Role of Algorithms/Data Structures in Modern World

- Algorithms are needed to solve real world problems Outline Principles of Algorithm Analysis Slide 61 Asy...
  - Internet Communication Slide 14
    - Because the Internet topology and network load is constantly changing, network routes must be discovered dynamically.
  - Network Security Slide 15
    - Prevent eavesdropping, modification, insertion or deletion of messages transmitted across the network
  - VLSI Design Automation . . . VLSI DESIGN Role of Algorithms/Data Structures in Modern...

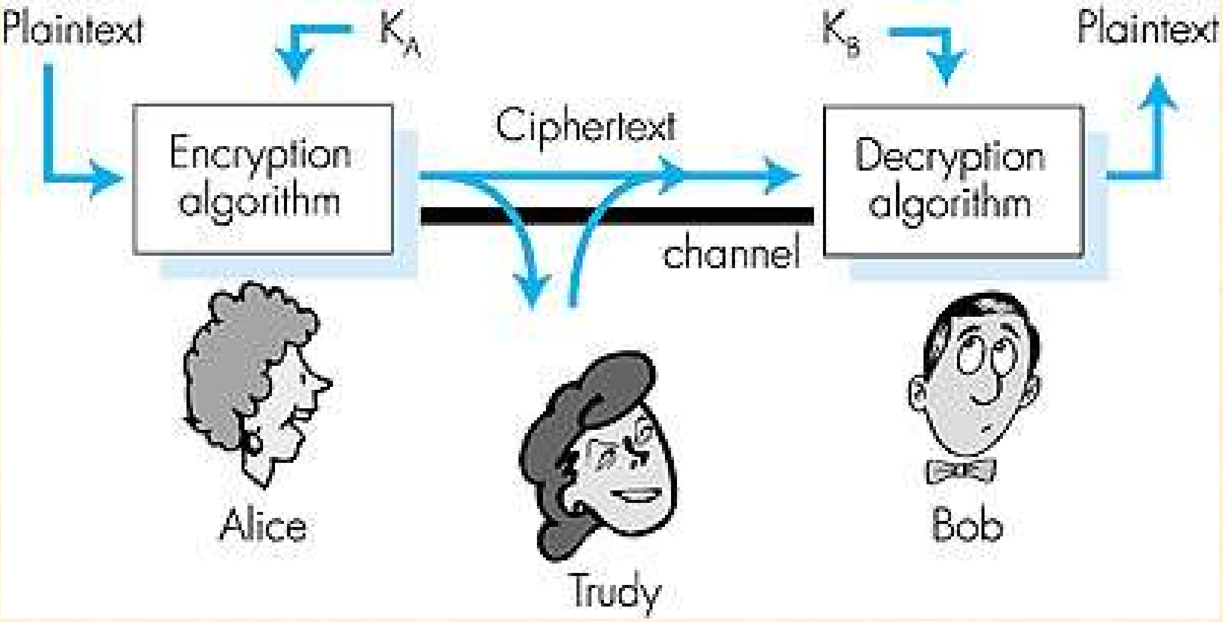


Because the Internet topology and network load is constantly changing, network routes must be discovered dynamically

Need efficient algorithms to discover new routes quickly

Role of Algorithms/Data Structures in Modern World

Network Security



Encryption is the conversion of data into a form, called a **Ciphertext**, that cannot be easily understood by unauthorized people.

Decryption is the process of converting encrypted data back into its original form, so it can be understood.

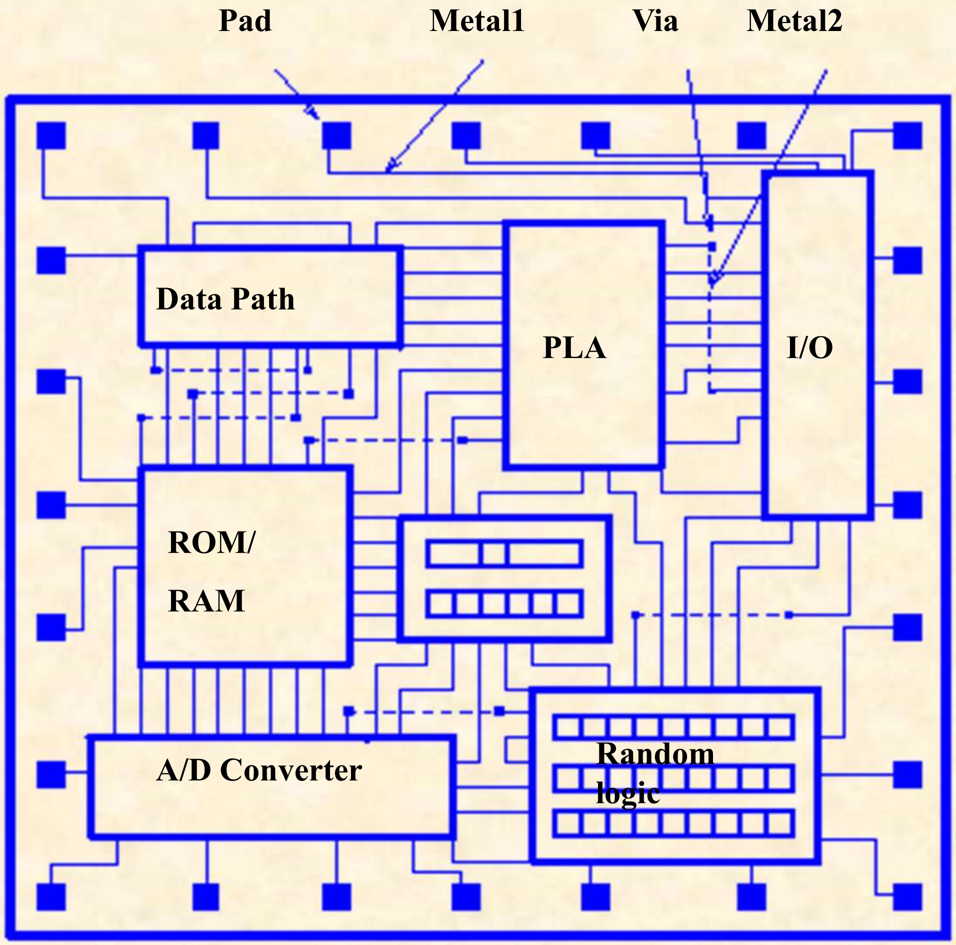
Need algorithms to encrypt and decrypt transmitted data

Role of Algorithms/Data Structures in Modern World

VLSI DESIGN

Role of Algorithms/Data Structures in Modern World

- Devices in VLSI Circuits
  - Transistors
  - Logic gates and cells
  - Function blocks
- How to interconnect all the devices
  - Efficiently
  - Low power
  - Low cost
- **Need intelligent algorithms**



Outline Principles of Algorithm Analysis Slide 61 Asymptotics.pptx

- Overview of algorithms and data structures
- Basic mathematics for analysis of algorithms
- Asymptotic notation and Analysis of algorithms
    - This is an extremely useful mathematical technique because it simplifies greatly the analysis of algorithms
- Basic Recurrence Recursion: An Example Outline Principles of Algorithm ...
    - Recursion is useful for problems that can be represented by a simpler version of the same problem. Notes_Part2_Jan09.pptx
- Basic Data Structures

Introduction

Overview

- What is an Algorithm?
  - A clearly specified set of instructions to be followed to solve a problem
    - takes a set of values as input and
    - produces a value or a set of values, as output

- Algorithms operate on data that are stored in a well defined structure called ***data structure***, e.g. array, linked list, etc.
  - Data structure: a way of storing data so that the data can be used efficiently
    - a carefully chosen data structure will allow a more efficient algorithm to be used

- ***Program*** = ***Algorithm*** + ***Data Structure***

Typical properties of Algorithms

- Input: The algorithm receives *input*.
- Output: produce *output*.
- Precision: steps are precisely stated.
- Determinism: results of each step are unique and are determined only by the inputs and results of the preceding steps.
- Finiteness: it **terminates**.
- Correctness: The output produced is correct.
- Generality: apply to a set of inputs.

Some Well-known Computational Problems

- Sorting
- Searching
- Graph Algorithms
  - Shortest paths in a graph
  - Minimum spanning tree

Sorting

- Given a sequence of numbers
- Rearrange the numbers in increasing order or decreasing order

Example: 11, 7, 14, 1, 5, 9, 10

Sort in increasing order

11, 7, 14, 1, 5, 9, 10 → 1, 5, 7, 9, 10, 11, 14

An application: Sorting a deck of cards

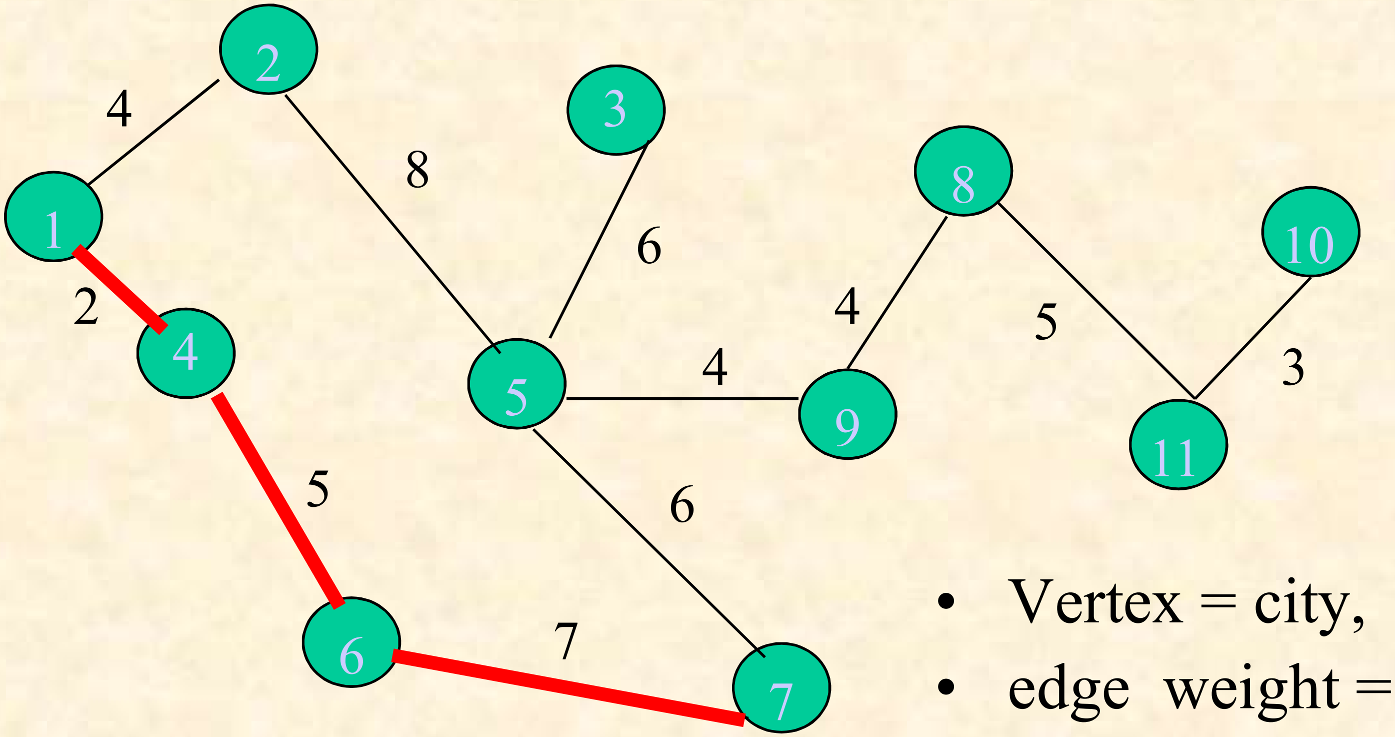
Searching

- What is Searching?
  - retrieving information from a large amount of previously stored information

- What are the applications of searching?
  - **Banking**
    - keep track of all customers' account balances and to search through them to check for various types of transactions

  - **Search engine**: such as Google™
    - need to look for relevant pages on the Web containing a given keyword
    - how does Google find the documents matching your query so fast?

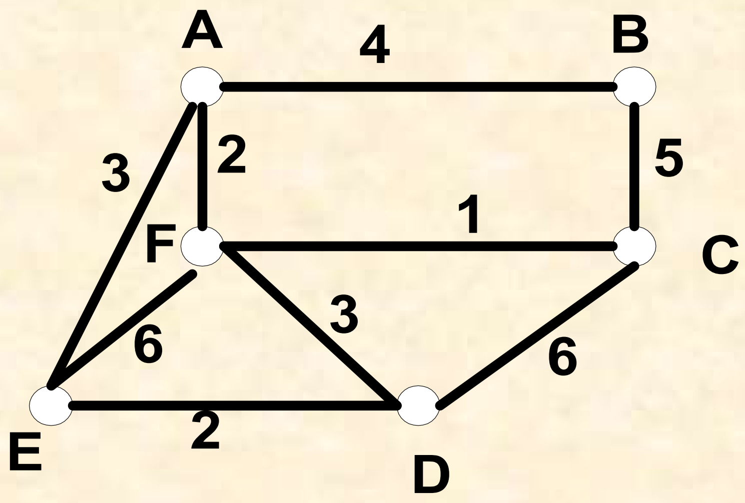
Shortest Paths : An Example

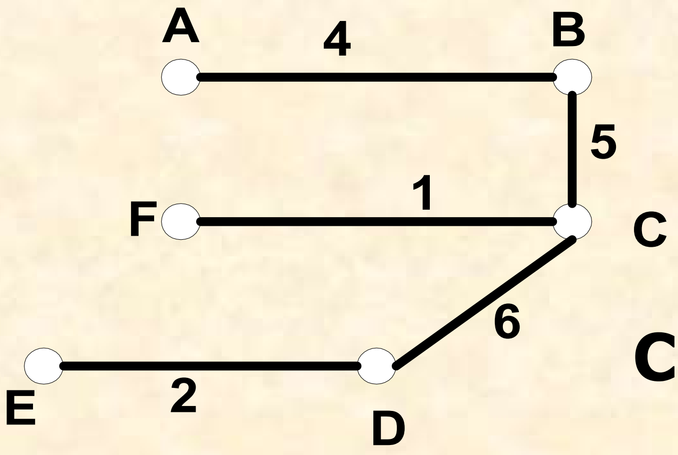


- Vertex = city,
- edge weight = driving distance/time.

What is the shortest path from 1 to 7?

Minimum Spanning Tree



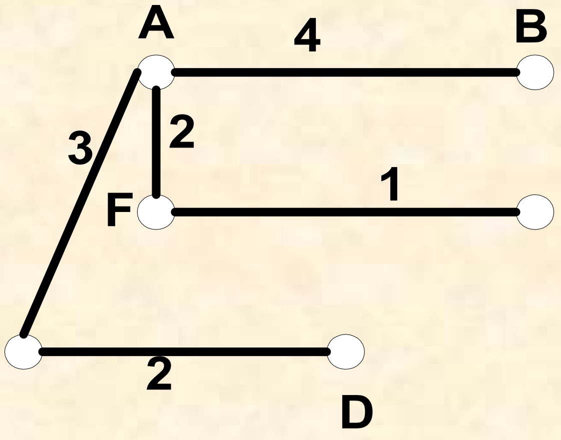


Cost = 18

Each node represents a city

Weight of each edge: cost of building a road connecting two cities

Problem: to build enough roads so that each pair of cities will be connected and to use the lowest cost possible



Cost = 12

Sorting Example Slide 176 Example - Insertion sort Slide 26

- An Example: Sort a set of elements into increasing order
  11, 7, 14, 1, 5, 9, 10 → 1, 5, 7, 9, 10, 11, 14

Insertion sort Insertion Sort: Best Case

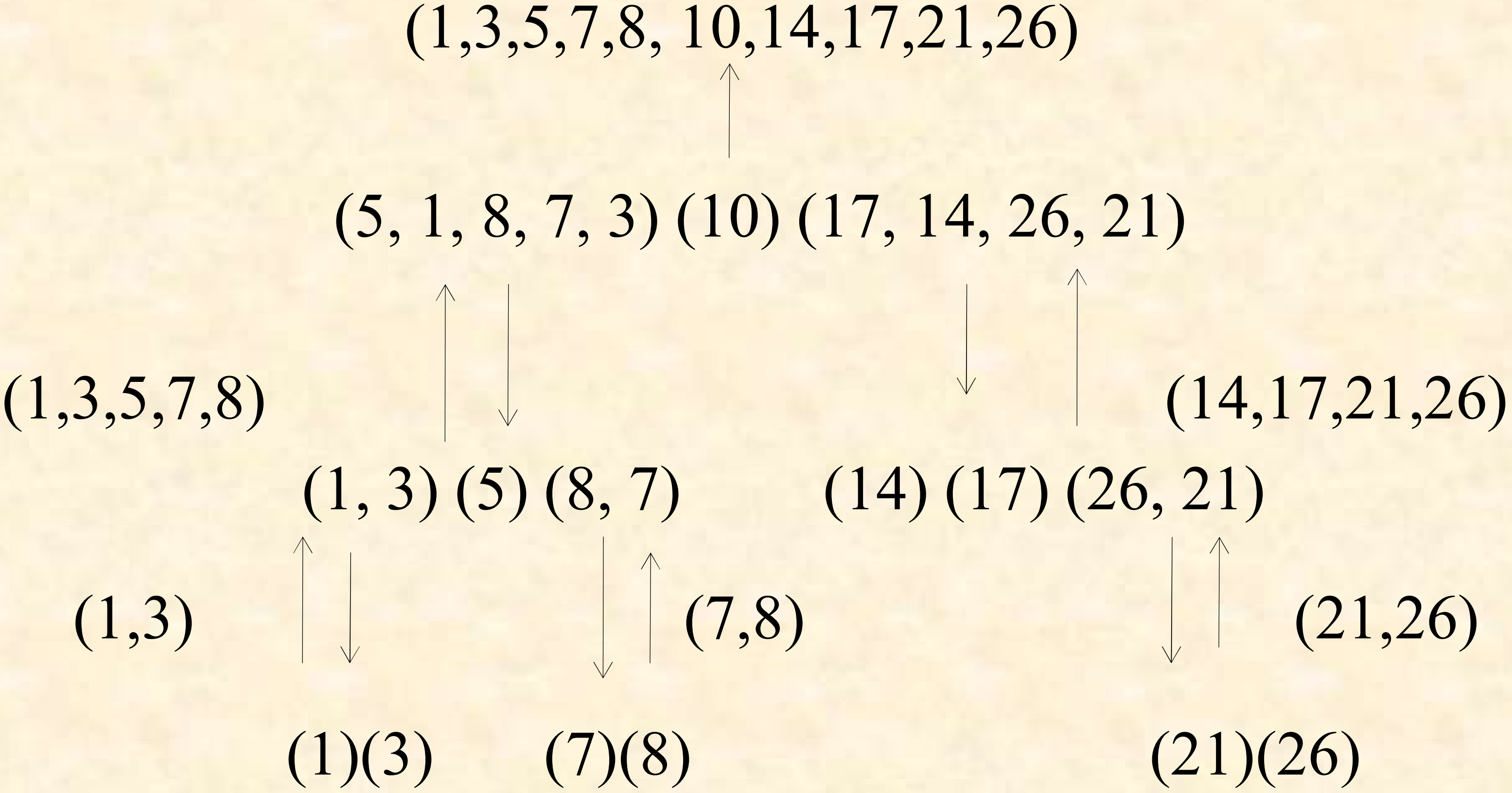
- Examine the number from left to right one by one, and insert the number into an appropriate place in an already sorted sequence
- Insertion of a number is done by comparing it with the numbers in the sorted sequence
- The comparison stops when the correct position is determined

Sorted sequence	**Unsorted sequence**
11	7, 14, 1, 5, 9, 10
7, 11	14, 1, 5, 9, 10
7, 11, 14	1, 5, 9, 10
1, 7, 11, 14	5, 9, 10
1, 5, 7, 11, 14	9, 10
1, 5, 7, 9, 11, 14	10
1, 5, 7, 9, 10, 11, 14	

Quick Sort: 10, 5, 1, 17, 14, 8, 7, 26, 21, 3

- Use the first element to divide the numbers into 3 subsets: (smaller than 10) (10) (greater than 10)
  - (5, 1, 8, 7, 3) (10) (17, 14, 26, 21)
- Next, sort the left and right subsets in the similar way
  - (1, 3) (5) (8, 7) and (14) (17) (26, 21)
- After sorting (8, 7) and (26, 21), and combining the sorted subsets, we have
  - (1, 3, 5, 7, 8) and (14, 17, 21, 26)
- Finally, combining the sorted subsets, we have
  - 1, 3, 5, 7, 8, 10, 14, 17, 21, 26

Quick Sort: 10, 5, 1, 17, 14, 8, 7, 26, 21, 3



(1,3,5,7,8, 10,14,17,21,26)

(5, 1, 8, 7, 3) (10) (17, 14, 26, 21)

(1,3,5,7,8)

(14,17,21,26)

(1, 3) (5) (8, 7)

(14) (17) (26, 21)

(1,3)

(7,8)

(21,26)

(1)(3)

(7)(8)

(21)(26)

Comparing the Performance of Insertion Sort and Quick Sort (time in sec) Slide 78

n	Insertion sort	Quick sort
10	.000001	.000002
100	.000106	.000025
1,000	.011240	.000365
10,000	1.047	.004612
100,000	110.492	.058481
1,000,000	NA	.6842

This example shows that a same problem can be solved by more than one algorithms with different level of efficiencies

When *n* is large, insertion sort will take a longer time to execute than Quick Sort even when a much faster computer is used.

Basic Mathematics for Algorithms

Sets

- A set is a collection of objects, called its members or elements
- If an object x is a member of S, **we write $x \in S$**
- If x is not a member of S, **we write $x \notin S$**
- If S contains elements a, b, c, **we write $S = \{ a , b, c \}$**
  - is $a \in S$?
  - is b $\in S$?
  - is $c \in S$?
  - is $d \in S$?

- $\varnothing$: denotes an empty set (a set with no members)

Sets (contd)

- If all the elements of a set A are contained in a set B, **then we write A $\subseteq$ B (A is a subset of B)**
- If A $\subseteq$ B but A $\neq$ B

  then we write A$\subset$ B (A is a proper subset of B)

Set Operations

- The intersection of sets *A* and *B*
  - $A \cap B = \{x : x \in A \text{ and } x \in B\}$
  - Eg; $A = \{1,2,3\}$, $B = \{2,3,5,6\}$
    - $A \cap B = \{ 2, 3\}$
- The union of sets *A* and *B*
  - $A \cup B = \{ x : x \in A \text{ or } x \in B\}$
  - Eg: $A = \{1,2,3\}$, $B = \{2,3,5,6\}$
    - $A \cup B = \{1,2,3,5,6\}$
- The difference between 2 sets *A* and *B*

  $A - B = \{ x : x \in A \text{ and } x \notin B \}$

  Eg: $A = \{1,2,3\}$, $B = \{2,3,5,6\}$ **, then $\boldsymbol{A - B = \{ 1\}}$**

Definitions and Notations

- Basic notation:
  - Z : set of integers {…., -2, -1, 0, 1, 2, …}
  - *N*: set of natural numbers {1,2,…}
  - *R*: set of real numbers Numbers Definitions and Notations
  - floor: $\lfloor x \rfloor$ = largest integer less than or equal to *x*. $\lfloor 4.9 \rfloor$=4
  - ceiling: $\lceil x \rceil$ = smallest integer greater than or equal to *x*. $\lceil 7.1 \rceil$=8
- Polynomials: Big-Oh Rules
  - A polynomial of degree *n* is a function of the form

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$$

    with $c_n \neq 0$. The numbers c_i are called coefficients.

  - Example, a polynomial of degree 5.

$$p(x) = 3x^5 - 12x^3 + 9x + 4$$

Intervals

- If a and b are numbers, $[a, b]$ is called a ***closed interval*** and is defined to be the set $\{x \mid a \le x \le b\}$
  - if we are restricted to integers, [a, b] is the set:

    $\{x \mid x$ is an integer and $a \le x \le b\}$

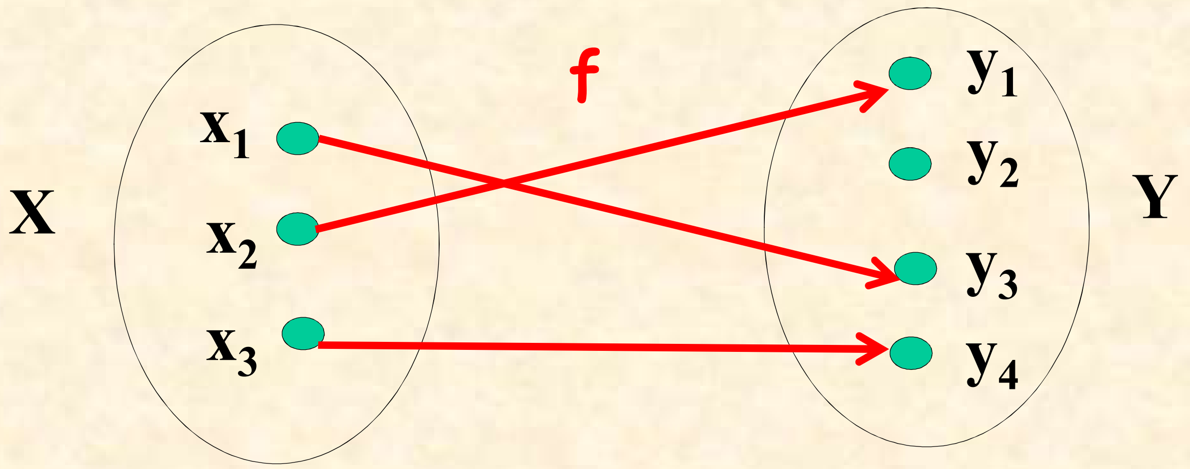
    - E.g. $[2,6] = \{2,3,4,5,6\}$
  - if we are restricted to real numbers, [a, b] is the set:

    $\{x \mid x$ is a real number and $a \le x \le b\}$

    - e.g. $2 \in [2,6]$, $\pi \in [2,6]$
- If a and b are number, (a, b) is called an ***open interval*** and is defined to be the set $\{x \mid a < x < b\}$
- $[a, b)$ and $(a, b]$ are called ***half-open interval***.
  - $[a, b) : \{x \mid a \le x < b\}$; $(a, b] : \{x \mid a < x \le b\}$

Functions

- A function from a set X to a set Y is a relationship between the elements of X and the elements of Y
- Property of the relationship : each element of X is related to a unique element of Y
- The notation $f : X \longrightarrow Y$ means that f is a function from X to Y



$f(x_1) = y_3$

$f(x_2) = y_1$

$f(x_3) = y_4$

- Sequences:
  - A ***finite sequence*** *a* is a function from the set {0,1, .., n} to a set *X*.
    - The sequence is typically denoted as $a_0, a_1, \ldots$, the subscript *i* in a_i is the index of the sequence. Sequences Slide 50
  - An ***infinite sequence*** *a* is a function from the set {0, 1, …} to a set *X*. The sequence is denoted $a_0, a_1, \ldots$, or $\{a_i\}_{i=0}^{\infty}$
    - E.g. $a_i = 3i + 1$, $i \geq 0$, is a infinite sequence,
    - 4, 7, 10, …
  - *a* is an **increasing sequence** if $\forall i, a_i < a_{i+1}$
  - *a* is a **decreasing sequence** if $\forall i, a_i > a_{i+1}$
  - *a* is a **non-increasing sequence** if $\forall i, a_i \geq a_{i+1}$, e.g. 100,90,74,74,56
  - *a* is a **non-decreasing sequence** if $\forall i, a_i \leq a_{i+1}$
- Strings:
  - A string or word is a finite sequence $t_0, t_1, \ldots, t_n$, where $t_i \in X$, and is written as $t_0t_1\ldots t_n$.
    - E.g. X={a,b,c}, t_0=b, t_1=a, t_2=a, t_3=c, the string is written as baac.

Sequences Slide 50

- A finite sequence
  - f : {0,1,2,…,n} --> X
  - $f(i) = a_i \in X$
  - $a_0, a_1, a_2, \ldots, a_n$
- An infinite sequence
  - f : {0,1,2,…,} -> X
  - $a_0, a_1, a_2, \ldots$
- 1,3,5,7, … (an increasing sequence)
- 100, 98, 76, 55, … (a decreasing sequence)
- 100, 98, 76, 76, 55, 54, … (a non-increasing sequence)
- 1, 3, 5, 5, 7, 8, … (a non-decreasing sequence)
- If X is a set of alphabets, f:{0,1,2, …,n} ->X forms a string of alphabets or words
  - Eg X = {r,s,t}, f(0) = r, f(1) = s, f(2)=r, f(3)=t, the string of alphabets or word formed is rsrt

Boolean (Logic) Variables

- Boolean variables have only two possible values: T, F
- Boolean operators: ***or*** (∨), ***and*** (∧), ***not*** (‾)
- Precedence
  - ***not*** (‾) (highest precedence)
  - ***and*** (∧)
  - ***or*** (∨) (lowest precedence)
- Example: $p \vee q \wedge \bar{r}$

  is interpreted as *p or (q and (not r))*

p	q	$p \vee q$	$p \wedge q$
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	F

Suppose p = T, q = T and r = T

Then $\bar{r} = \mathbf{F}$

(q and (not r)) = **F**

p or (q and (not r)) = **T**

- Factorials:

  n! = *n* × (*n*-1) × … × 3 × 2 × 1

  Example: *5!* = 5 × 4 × 3 × 2 × 1 = 120

- Binomial Coefficient $\binom{n}{k}$
  - Provides the number of ***combinations*** of selecting *k* elements from a set with *n* elements
  - For *n* ≥ *k* ≥ 0, the number of *k*-element subsets of an *n*-element set is given by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

  - Combinations Slide 52
  - Example:

$$\binom{5}{2} = \frac{5!}{3!2!} = \frac{5\times4\times3\times2\times1}{(3\times2\times1)\times(2\times1)} = 10$$

Combinations Slide 52

- $\binom{n}{k}$: the number of ways of selecting k elements from a set of n elements

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

- Example: Given a set of numbers {1,2,3,4}, how many subsets of two numbers can be formed?
  - {1,2}, {1,3}, {1,4}
  - {2,3}, {2,4}
  - {3,4}

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \times 3 \times 2 \times 1}{(2 \times 1)(2 \times 1)} = 6$$

- Logarithms:
  - Recall: if $b^x = n,\ \log_b n = x$ **Definition of logarithm**
  - Laws of logarithms: Suppose that b>0, and $b \neq 1$, then

$$b^{\log_b x} = x$$

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b(\frac{x}{y}) = \log_b x - \log_b y$$

$$\log_b x^y = y \log_b x$$

$$if\ \ a > 0\ and\ a \neq 1, \log_a x = \frac{\log_b x}{\log_b a}$$ **Change of base**

$$if\ \ b > 1\ and\ x > y > 0, \log_b x > \log_b y$$

  - Note: $\log_2$ is usually written as lg

Examples

1. Let b=2 and x=8, then $\log_2 x$=3

   Now $b^{\log_b x} = 2^3 = 8 = x$

2. $\log_{10}32 = 1.505$, $\log_{10}2 = 0.301$

$$\log_2 32 = \frac{\log_{10} 32}{\log_{10} 2} = \frac{1.505}{0.301} = 5$$

Note: 2^5=32, $\log_2 32$=5

Basic Probability

- S = sample space
  - (a set of all possible outcomes of an experiment)
  - Eg: consider an experiment of flipping a coin
    - Two possible outcomes : head or tail
    - The chance of getting a head is 0.5
      - P(getting a head) = 0.5
    - The chance of getting a tail is 0.5
      - P(getting a tail) = 0.5
    - The P(getting a head or a tail) = 0.5 + 0.5 = 1
- each subset A of S is called an event
  - $0 \leq P(A) \leq 1$
  - P(S) = 1

Random Variables & Expectation

- Random variables : variables whose values depend on the outcome of some experiment
- Example : X is a random variable with possible values {1,2,3}. Suppose each of the possible values if equally likely to occur
  - Then the expected (average) value of X = $\frac{1+2+3}{3}$
  - E(X) = $(1)\frac{1}{3} + (2)\frac{1}{3} + (3)\frac{1}{3}$
- In general, given a random variable X, the expected (average) value of X is

$$E(X) = \sum_r rP(X=r)$$

Summation

Slide 120 Slide 122

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

$$\sum_{i=1}^{n} a = \underbrace{a + a + a + \cdots a}_{n\ times} = na$$

$$\sum_{i=p}^{q} a = \underbrace{a + a + a + \cdots a}_{q-p+1\ \ times} = (q - p + 1)a$$

Sum Manipulation Rules

$$\sum_{i=p}^{u} ca_i = c\sum_{i=p}^{u} a_i$$

$$\sum_{i=p}^{u} (a_i \pm b_i) = \sum_{i=p}^{u} a_i \pm \sum_{i=p}^{u} b_i$$

$$\sum_{i=p}^{u} a_i = \sum_{i=p}^{r} a_i + \sum_{i=r+1}^{u} a_i \quad \textbf{where } p \le r \le u$$

$$\sum_{i=1}^{10} 3i = 3\sum_{i=1}^{10} i$$

$$\sum_{k=1}^{100} 3^k + k^2 = \sum_{k=1}^{100} 3^k + \sum_{k=1}^{100} k^2$$

$$\sum_{i=1}^{1000} i = \sum_{i=1}^{100} i + \sum_{i=101}^{1000} i$$

Mathematical Induction

- Mathematical induction can be used to prove a statement is true for all natural numbers $n \geq n_o$ if
  - (Basis of induction) The statement is true for $n = n_o$; and
  - (Induction step) Assuming that the statement is true for $n = k, k \geq n_o$, prove that the statement is true for $n = k + 1$.
  - Let p(n) : Statement to be proven true for $n \geq n_o$

- Example: Prove that Induction: Example Slide 120 Slide 122

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \qquad \text{for all } n \geq 1$$

  - Basis step: Show that the equation is true for $n = 1$.

$$LHS = \sum_{i=1}^{1} i = 1 \qquad RHS = \frac{1(1+1)}{2} = 1$$

  - Inductive step: assume that equation is true for n, and prove that it is true for $n + 1$.

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$\Rightarrow \sum_{i=1}^{n+1} i = \frac{n(n+1) + 2(n+1)}{2}$$

$$\Rightarrow \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

Induction: Example

- P(n) : $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all n ≥ 1

- Inductive step: Assume P(k) is true

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

- Need to prove P(k+1) is true

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

- We have $$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$
- Need to show $$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$
- $$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

We have

Need to show

- Example: Prove that

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \text{for all } n \ge 1$$

  - Basis step: Show that the equation is true for $n = 1$.

$$LHS = \sum_{i=1}^{1} i^2 = 1^2 = 1 \qquad RHS = \frac{1(1+1)(2+1)}{6} = 1$$

  - Inductive step: assume that equation is true for n, and prove that it is true for $n + 1$. Induction Example 2

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$\Rightarrow \sum_{i=1}^{n+1} i^2 = \frac{(n+1)[n(2n+1) + 6(n+1)]}{6}$$

$$\Rightarrow \sum_{i=1}^{n+1} i^2 = \frac{(n+1)(2n^2 + 7n + 6)}{6}$$

$$\Rightarrow \sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$\Rightarrow \sum_{i=1}^{n+1} i^2 = \frac{(n+1)[(n+1)+1][(2(n+1)+1]}{6}$$

Induction Example

- P(n) : $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- Assume P(k) is true

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

- Need to prove P(k+1) is true

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

- We have $\displaystyle\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

- Need to show : $\displaystyle\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

We have

Need to show :

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2+7k+6]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Arithmetic Series

- **arithmetic sequence (series)** is a sequence of numbers such that the difference between the consecutive terms is constant.
  - Eg 1, 3, 5, 7, 9, 11,… (common difference is 2)
- General form
  - 1st term = a
  - 2^{nd} term = a + d
  - 3^{rd} term = a + 2d
  - nth term = a +(n-1)d

Geometric Series

- a **geometric series** is a series with a constant ratio between successive terms
  - Eg: 2, 4, 8, 16, 32,… (common ratio = 2)
- General form
  - 1st term = a
  - 2nd term = ar
  - 3rd term = $ar^2 + 2d$
  - nth term = $ar^{(n-1)}$

- Arithmetic series Arithmetic Series Slide 59 Slide 100 Series Slide 42 Slide 112 Slide 114

$$\sum_{k=1}^{n}\big(a+(k-1)d\big) = a+(a+d)+(a+2d)+\cdots+(a+(n-1)d) = \frac{n}{2}\big(2a+(n-1)d\big)$$

Special case : $$\sum_{i=1}^{n} i = 1+2+\ldots+n = \frac{n(n+1)}{2}$$

- Geometric series Geometric Series : Example Slide 42 Induction Example 4

$$\sum_{k=0}^{n} ar^k = a+ar+\ldots+ar^n = \frac{a(r^{n+1}-1)}{r-1}$$ **r > 1**

- Geometric Series Slide 59

Note $$\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$$ for $0 < r < 1$

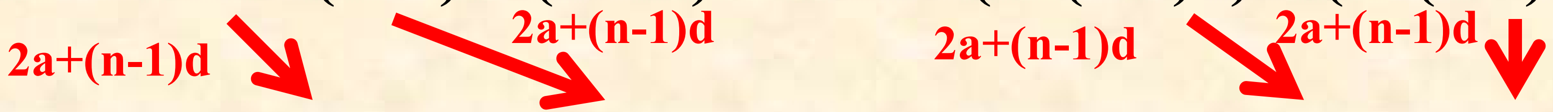
- Telescoping series $$\sum_{k=1}^{n}(a_k - a_{k-1}) = a_n - a_0$$ Telescoping Series Slide 59

Arithmetic Series

Slide 59

$$\sum_{k=1}^{n}\left(a+(k-1)d\right)=a+(a+d)+(a+2d)+\cdots+(a+(n-1)d)=\frac{n}{2}\left(2a+(n-1)d\right)$$

Let T = a + (a+d) + (a+2d) + … + + (a+(n-1)d) + (a+(n-1)d)



2a+(n-1)d

2a+(n-1)d

2a+(n-1)d

2a+(n-1)d

Note that T = (a+(n-1)d) + (a+(n-2)d) + … + (a+2d) +(a+d) + a

2T = n[2a+(n-1)d]

$$T = \frac{n}{2}[2a+(n-1)d]$$

Arithemetic Series: A Special Case Slide 45

$$\sum_{k=1}^{n}\left(a+(k-1)d\right) = a+(a+d)+(a+2d)+\cdots+(a+(n-1)d) = \frac{n}{2}\left(2a+(n-1)d\right)$$

When a = 1 and d = 1, we have : 1 + 2 + 3 + … + n

$$= \frac{n}{2}[2+(n-1)]$$

$$= \frac{n(n+1)}{2}$$

We have earlier proven that

$$\sum_{i=1}^{n} i = 1+2+\ldots+n = \frac{n(n+1)}{2}$$

A special case of an Arithmetic Series

Geometric Series : Example

Slide 42

$$\sum_{k=0}^{10} 2(3)^k = 2(3)^1 + 2(3)^2 + 2(3)^3 + \cdots + 2(3)^{10}$$

$$= \frac{2\left(3^{10+1} - 1\right)}{3 - 1}$$

$$= \frac{2\left(3^{11} - 1\right)}{2}$$

$$= 3^{11} - 1$$

$$\sum_{k=0}^{n} ar^k = \frac{a\left(r^{n+1} - 1\right)}{r - 1}$$

Induction Example

- P(n) : $$\sum_{i=0}^{n} ar^{i} = \frac{a(r^{n+1}-1)}{r-1}$$

- Basic Step: n=0

  - LHS = $\sum_{i=0}^{0} ar^{i} = a$

  - RHS = $\frac{a(r^{0+1}-1)}{r-1} = a$

Induction Example

- Assume P(k) is true $\sum_{i=0}^{k} ar^i = \frac{a(r^{k+1}-1)}{r-1}$
- Need to prove P(k+1) is true

$$\sum_{i=0}^{k+1} ar^i = \frac{a(r^{k+2}-1)}{r-1}$$

- We have $\sum_{i=0}^{k} ar^i = \frac{a(r^{k+1}-1)}{r-1}$

Slide 59

- Need to show : $\sum_{i=0}^{k+1} ar^i = \frac{a(r^{k+2}-1)}{r-1}$

$$\sum_{i=1}^{k+1} ar^i = \sum_{i=1}^{k} ar^i + ar^{k+1} \quad = \frac{a(r^{k+1}-1)}{r-1} + ar^{k+1}$$

$$= \frac{a(r^{k+1}-1) + ar^{k+1}(r-1)}{r-1}$$

$$= \frac{a(r^{k+1}-1) + ar^{k+2} - ar^{k+1}}{r-1}$$

$$= \frac{a(r^{k+2}-1)}{r-1}$$

Geometric Series Slide 59

$$Let\ \ P = \sum_{k=1}^{n} r^{k} = 1 + r + r^{2} + r^{3} + \ldots + r^{n}$$

$$rP = \qquad r + r^{2} + r^{3} + \ldots + r^{n} + r^{n+1}$$

$$P - rP = 1 - r^{n+1}$$

$$P(1 - r) = 1 - r^{n+1}$$

$$P = \frac{(1 - r^{n+1})}{1 - r}$$

Telescoping Series

Slide 59

$$\sum_{k=1}^{n}\left(a_k - a_{k-1}\right) = (\cancel{a_1} - a_0) + (\cancel{a_2} - \cancel{a_1}) + (\cancel{a_3} - \cancel{a_2}) + \ldots + (\cancel{a_{n-1}} - \cancel{a_{n-2}}) + (a_n - \cancel{a_{n-1}})$$

$$= a_n - a_0$$

Describing Algorithms : Using Pseudocode

Algorithm

- a sequence of instructions for solving a problem

Pseudocode

- High-level description of an algo.
- More structured than English prose
- Less detailed than a program
- 3 basic control constructs: sequence, selection and iteration
- Control constructs must be reflected clearly by some standard conventions (many options are available)

Example: find the largest element in an array

```
Algorithm arrayMax($A$, $n$)
  Input array $A$ of $n$ integers
  Output largest element in $A$

  $Max = A[0]$
  for $i = 1$ to $n - 1$ do
    if $A[i] > Max$ then
      $Max = A[i]$
  return $Max$
```

	0	1	2	3	4
A					

Pseudocode Details

- Control flow

  if … **then** … **[else** …]

  while … **do** …

  repeat … **until** …

  for … **do** …

- Algorithm declaration

  Algorithm ***name*** (***arg***, ***arg***,…)

  Input …

  Output …

- Calling another algo.

  Algo_called (***arg***, ***arg***,…)

- Return value

  return ***expression***

- Expressions

  = Assignment **x = b**

  == Equality testing **if x == b**

  n^2 Superscripts and other mathematical formatting allowed

Pseudocode: Selection

```
if (condition)
      action1;
```

```
if (condition)
      action1;
else
      action2;
```

```
if (b > x) // if b is larger than x, update x
      x = b
if (c > x) // if c is larger than x, update x
      x = c;
return x
```

```
if ( x ≥ 0) {
      x = x + 1;
      a = b + c;
}
```

Pseudocode: while loop

```
while (condition)
    action;
```

	1	2	3	4	5
s	-3	20	450		

- ✓ If *condition* is true, action is executed.
- ✓ The process is repeated until condition becomes false

```
// Find the max value in an array using a while loop
array_max1 (s) {
     large = s[1];
     i = 2;
     while (i ≤ s.last ) {
        if (s[i] > large) // larger value found
           large = s[i];
        i = i + 1;
     }
     return large;
}
```

Pseudocode: for loop

```
for (var = init to limit)
     action;
```

1	2	3	4	5
-3	20	450		

s

- ✓ The variable *var* is first set to the value *init*. If *var* ≤ *limit*, action is executed and 1 is added to *var*.
- ✓ The process is repeated until *var* > *limit*

```
// Find the max value in an array using a for loop
array_max2 (s) {
     large = s[1];
     for (i = 2 to s.last ) {
          if (s[i] > large) // larger value found
               large = s[i];
     }
     return large;
}
```

Pseudocode: for loop

```
for (var = init downto limit)
    action;
```

- ✓ Action is executed as long as *var* ≥ *limit*, and updating is performed by subtracting 1 from *var*

Write an algorithm that returns the *index* of the last occurrence of the value *key* in the array s[1], . . , s[n]. If *key* is not in the array, the algorithm returns the value 0. Pseudocode: for loop

```
find_last_key (s, key) {
     for (i = s.last downto 1 ) {
        if (s[i] == key ) // key value found
           return i;
     }
     return 0;
}
```

- ✓ == is used to test equal
- ✓ = is assignment

Pseudocode: for loop

	0	1	2	3	4
S	2	3	2	23	

Key = 2

Find index of last occurrence of key in array S

```
find_last_key (s, key) {
     for (i = s.last downto 1 ) {
        if (s[i] == key ) // key value found
           return i;
     }
     return 0;
}
```

Example 1

- Write an algorithm than compute the sum of n consecutive integers which starts from 1 (i.e. sum = 1 + 2 +3 + …. + n)

```
total(n)
    sum = 0;
    for (i = 1 to n ) {
        sum = sum + i;
    }
    return sum;
```

i	***sum***
1	**1**
2	**1+2**
3	**1+2+3**
n	**1+2+3 + …+ n**

Example 2

Write an algorithm to change a numeric grade to a pass/no pass grade
(pass grade $\geq$ 50%)

```
Grading(marks)
    if (marks ≥ 50){
       grade = “pass”;
    else
       grade = “fail”
    }
   return grade;
```

Example 3

- Write an algorithm that returns the smallest value in the array s[1], s[2], …, s[n]. Use a for loop.

```
Array_min(s)
     min = s[1];
     for (i = 2 to n ) {
        if (s[i] < min)
          min = s[i];
     }
     return min;
```

	1	2	3	4	5
s					

Example 4

- Write an algorithm than returns the smallest value in the array s[1], s[2], …, s[n]. Use a while loop.

```
Array_min(s)
     min = s[1];
     i = 2;
     while (i ≤ n ) {
        if (s[i] < min)
          min = s[i];
        i = i + 1;
     }
     return min;
```

Example 5

- Write an algorithm that delete the ith element of an array with *n* elements (1≤ *i* ≤ *n*)

```
Delete(s,i)
        s[i] = s[n];
        n = n-1;
        return;
```

1	2	3	4	5	6
q	a	n	s	w	u

$i = 3$

$n = 5$

Example 6

- Write an algorithm that returns the index of the first occurrence of the value *key* in the array *s*[1], …, *s*[*n*]. If *key* is not in the array, the algorithm returns the value 0.

```
find_first_key{
    for $i = 1$ to $n$
        if ($s[i] == key$)
            return $i$
    return 0
```

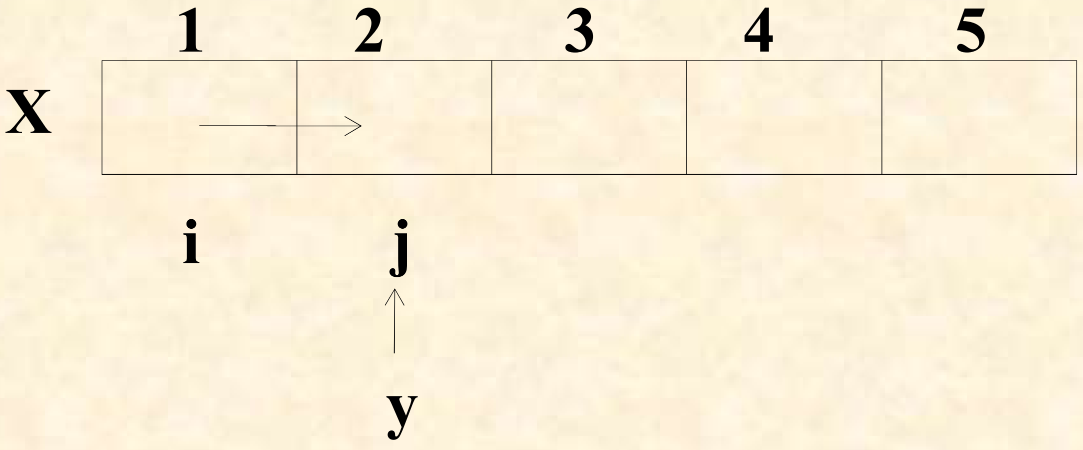
Example 7

Algorithm Insertion_Sort (example) Insertion Sort

Input: $x_1,x_2,\ldots,x_n$

Output: The sorted sequence of $x_1,x_2,\ldots,x_n$

```
for  (j = 2 to n) {      //start from the 2nd item of the unsorted list
     y = x[j]            //compare with the numbers on sorted
     i = j-1             //list from right (big) to left (small)
     while (y<x[i] and i > 0) {
          x[i+1] = x[i]
          i = i-1
     }
     x[i+1] = y
}
```



Example 8

- Write an algorithm to compute a^n

```
exp($a$,$n$){
    $i = 1$
    $pow = 1$
    while ($i \leq n$) {
        $pow = pow * a$
        $i = i + 1$
    }
    return $pow$
}
```

i	***pow***
1	**1**
2	**a**
3	$\mathbf{a^2}$
4	$\mathbf{a^3}$
n	$\mathbf{a^{n-1}}$
n+1	$\mathbf{a^n}$

Example 9

- Write an algorithm the returns the index of the last occurrence of the largest element in the array s[1],…s[n].

```
Find_last_largest(s)
     index = 1;
     for (i = 2 to n ) {
        if (s[i] ≥ s[index])
           index = i;
     }
     return index;
```

index	s[index]
3 1	11 11

	1	2	3	4
s	11	5	11	9

Example 10

- Write an algorithm that returns the index of the first item that is less than its predecessor in the array s[1],…,s[n]

```
Find_out_of_order(s)
     for (i = 2 to n ) {
        if (s[i] < s[i-1])
          return i;
     }
     return 0;
```

	1	2	3	4
s	11	5	11	9

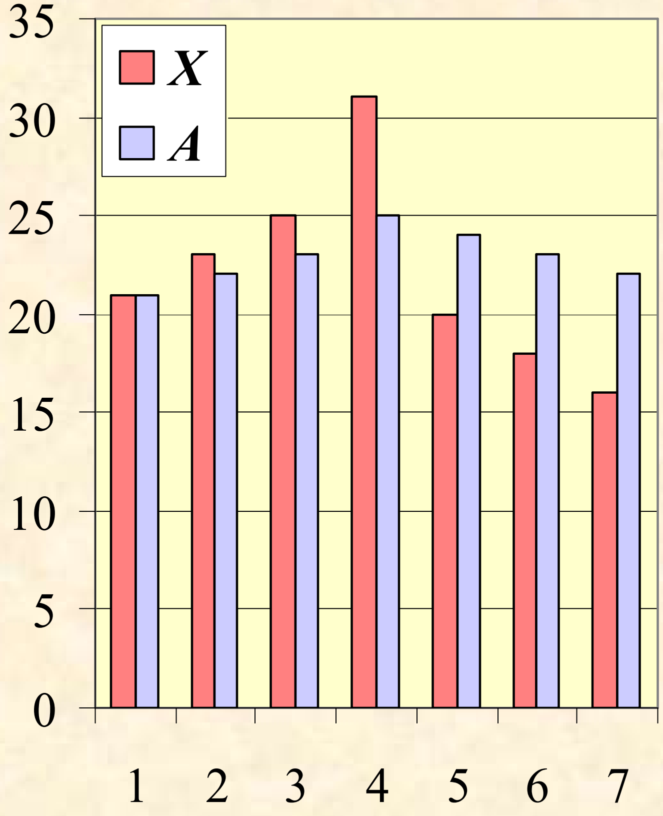
	1	2	3	4
s	11	12	15	19

Example 11 :Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages

- The ***i***-th prefix average of an array ***X*** is average of the first (***i*** + 1) elements of ***X*:**

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$



Prefix Averages (Quadratic)

- **The following algorithm computes prefix averages in quadratic time by applying the definition**

```
Algorithm prefixAverages1($X$, $n$)
    Input array $X$ of $n$ integers
    Output array $A$ of prefix averages of $X$
    $A \leftarrow$ new array of $n$ integers
    for $i = 0$ to $n - 1$ do
        $s = X[0]$
        for $j = 1$ to $i$ do
            $s = s + X[j]$
        $A[i] = s / (i + 1)$
    return $A$
```

Prefix Averages (Linear)

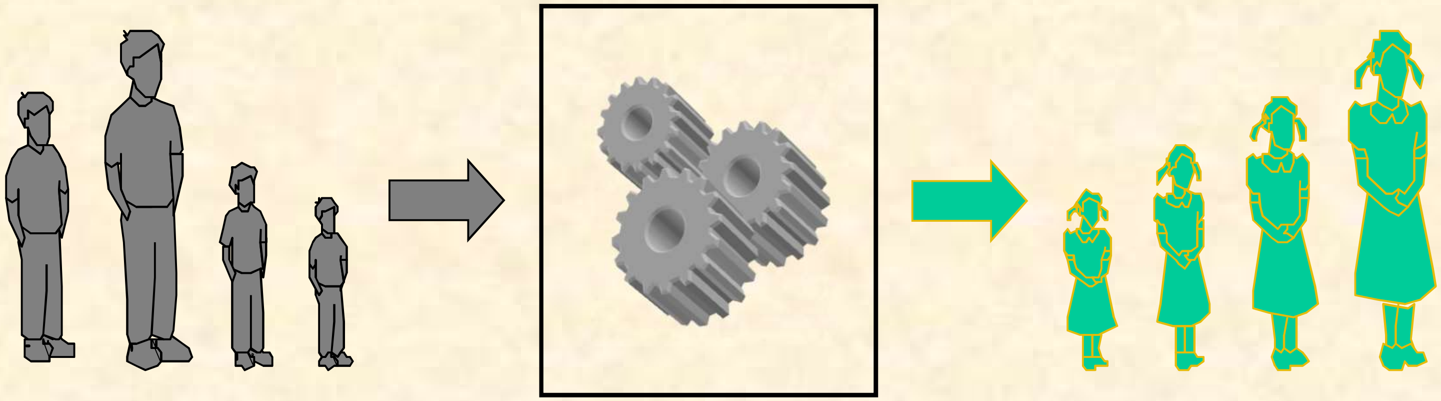
- The following algorithm computes prefix averages in linear time by keeping a running sum

```
Algorithm prefixAverages2(X, n)
    Input array $X$ of $n$ integers
    Output array $A$ of prefix averages of $X$
    $A \leftarrow$ new array of $n$ integers
    $s = 0$
    for $i =$ 0 to $n - 1$ do
        $s = s + X[i]$
        $A[i] = s / (i + 1)$
    return $A$
```

	0	1	2
s			

i	s	A[i]
0	$X[0]$	$S/1$
1	$X[0] + X[1]$	$S/2$
2	$X[0] + X[1] + X[2]$	$S/3$

Analysis of Algorithms



Input Algorithm Output

Foundations of Algorithm Analysis and Data Structures

- Algorithm Analysis:
  - How to predict an algorithm's performance
  - How well an algorithm scales up
  - How to compare different algorithms for a problem

- Data Structures
  - How to efficiently store, access, manage data
  - Data structures effect algorithm's performance

How to Measure Algorithm Performance?

- How to Measure Algorithm Performance?
  - Time efficiency: How long does it take to execute the algorithm or how many instructions does the algorithm execute?
  - Space efficiency: How much memory does the algorithm need during execution?

- **Time Efficiency is the dominant standard**.
  - Quantifiable and easy to compare
  - Often the critical bottleneck

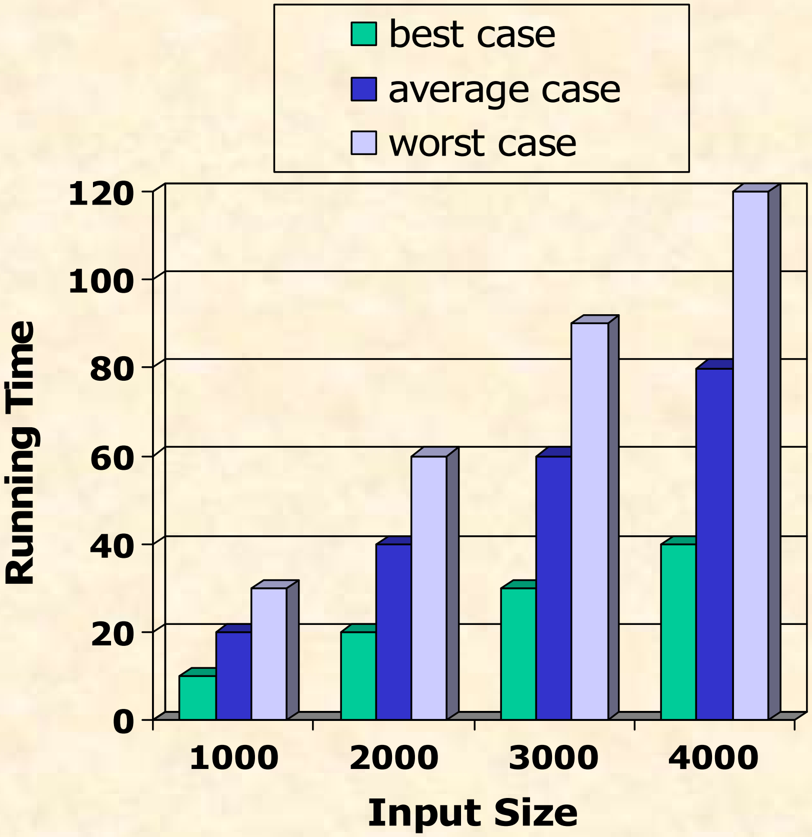
Time Efficiency

- Measures
  - how long does it take to execute the algorithm ?
  - how many instructions does the algorithm execute?

- Approaches
  - Empirical Studies
    - write programs to implement algorithms
    - measure the running time
  - Theoretical Analysis
    - count how many instructions are executed

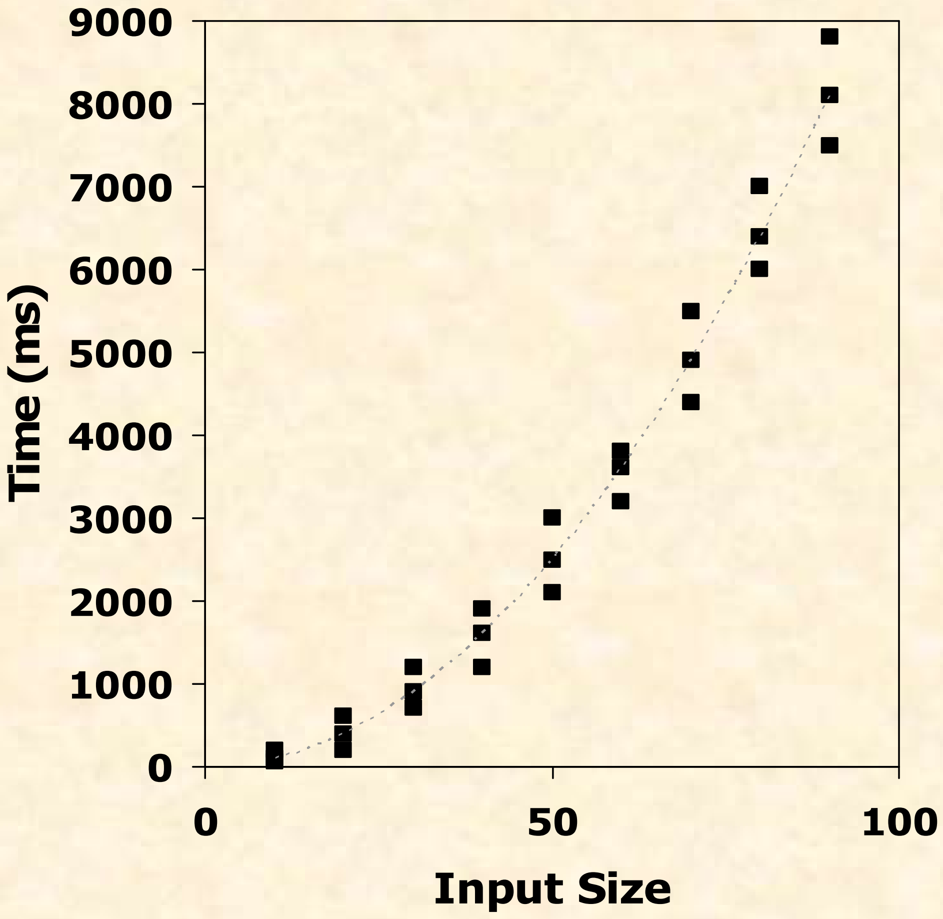
Empirical Studies: Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



Empirical Studies: Implementation

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a method like System.currentTimeMillis() to get an accurate measure of the actual running time
- Plot the results



Limitations of Experiments

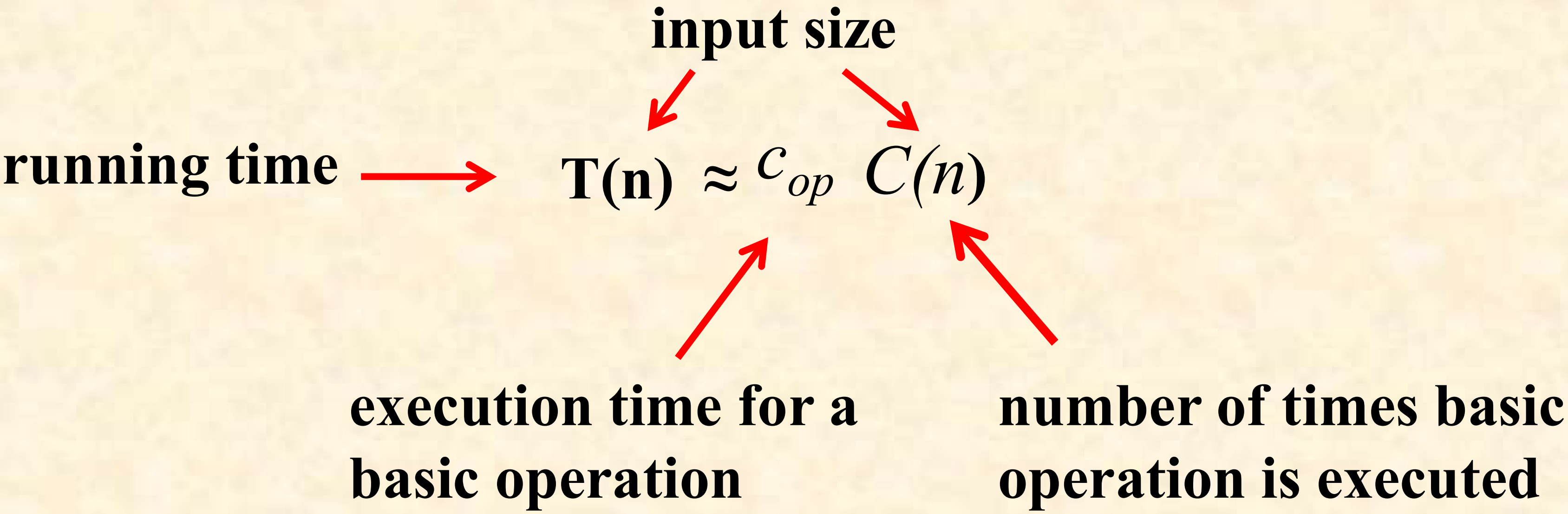
- It is necessary to implement the algorithm, which may be difficult and time consuming
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used

➢ One of the main reasons for studying algorithm analysis

Theoretical analysis of time efficiency

Time efficiency is analyzed by determining the number of repetitions of the *basic operation* as a function of *input size*

- *Basic operation:* the operation that contributes most towards the running time of the algorithm.



input size

running time

$$T(n) \approx c_{op}\, C(n)$$

execution time for a basic operation

number of times basic operation is executed

Theoretical Analysis

Running time : based on number of basic operations performed by the algorithm

- Uses a ***high-level description*** of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, *n*.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Time Efficiency

Time efficiency can be evaluated based on three scenarios: Slide 22

- Worst case: W(*n*) – maximum number of basic operations over inputs of size *n*

- Best case: B(*n*) – minimum number of basic operations over inputs of size *n*

- Average case: A(*n*) – "average" over inputs of size *n*

Basic Operations [Algorithm analysis Basic Operations]

- *Basic operation:* the operation that contributes most towards the running time of the algorithm
- Identifiable in pseudocode
- Largely independent from the programming language

Basic operation examples

Algorithm analysis

Problem	Basic Operation
Searching for key in a list of n items	Key comparison
Multiplying two matrices Matrix Multiplication	Multiplication of two numbers
Sorting a set of numbers	Key comparison
Check if a given integer is a prime number	Division

	0	1	2	3	4
S	2	3	2	23	

Prime Numbers

A **prime number** (or a **prime**) is a natural number greater than 1 that has no positive divisors other than 1 and itself.

Examples: 2, 5, 7, 11, 13 …

Testing for Prime Numbers : Trial Division

Given an input number *n*, check whether any integer *m* from 2 to *n* − 1 evenly divides *n* (the division leaves no remainder). If *n* is not divisible by any *m* then it is a prime number)

For example, to test whether 5 is prime, test whether 5 is divisible by 2, or 3, or 4. Since a prime is only divisible by 1 and itself, if we reach 4 without finding a divisor, then we have proven that 5 is prime.

Counting Basic Operations

- By inspecting the pseudocode, we can determine the maximum number of basic operations executed by an algorithm, as a function of the input size

```
Algorithm arrayMax(A, n)                    # operations
    Max = A[0]                                   1
    for i = 1 to n – 1 do                      n – 1
        if A[i] > Max then                     n – 1
            Max = A[i]                         n – 1
    { increment counter i }                    n – 1
    return Max                                   1
```

Estimating Running Time

- Algorithm ***arrayMax*** executes n-1 basic operations in the worst case. Define:

  $\boldsymbol{a}$ = Time taken by the fastest primitive operation

  $\boldsymbol{b}$ = Time taken by the slowest primitive operation

- Let $\boldsymbol{T}(\boldsymbol{n})$ be worst-case time of ***arrayMax.*** Then

$$\boldsymbol{a}\,(\boldsymbol{n} - 1) \leq \boldsymbol{T}(\boldsymbol{n}) \leq \boldsymbol{b}(\boldsymbol{n} - 1)$$

- Hence, the running time $\boldsymbol{T}(\boldsymbol{n})$ is bounded by two linear functions

Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects ***T***(***n***) by a constant factor, but
  - Does not alter the growth rate of ***T***(***n***)
- The linear growth rate of the running time ***T***(***n***) is an intrinsic property of algorithm ***arrayMax***
- Algorithms can be compared based on the running times, *T(n)*

Independent of hardware/software environment

Caters for all possible inputs

Principles of Algorithm Analysis

- Two algorithms, *A* and *B*, for solving a given problem.
  - Let the running times of the algorithms to be $T_a(n)$ and $T_b(n)$
  - Suppose the problem size is n_0 and $T_a(n_0) < T_b(n_0)$.
    - Then algorithm *A* is better than algorithm *B* for **problem size *n_0***. (not good enough)

❖ If $T_a(n) < T_b(n)$ for all n, $n \geq n_0$, then algorithm *A* is better than algorithm *B* regardless of the problem size.

- Note: Our primarily concern is to estimate the time of an algorithm instead of computing its exact time
  - a useful measurement of time can be obtained by counting the ***fundamental, dominating steps*** of the algorithm Slide 20
  - E.g. counting the number of comparisons in a sorting algo.

- Let
  - c_{op} = execution time of an algorithm's basic operation on a computer
  - $C(n)$ = number of times the basic operation is executed for this algorithm **Operation Count**
  - $T(n)$ = running time of this algo. on the computer

$$T(n) \approx c_{op}\, C(n)$$

- Assume that for an algorithm, the operation count $C(n) = ½\, n(n\text{-}1)$. How much longer will the algo. run if its input size is doubled?

$$C(n) = ½\, n(n\text{-}1) \approx ½\, n^2$$

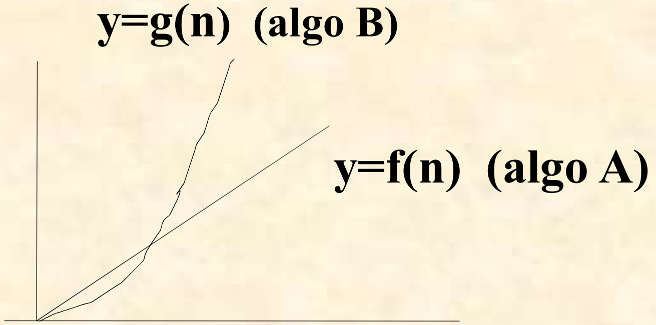
$$\frac{T(2n)}{T(n)} = \frac{c_{op}C(2n)}{c_{op}C(n)} \approx \frac{\frac{1}{2}(2n)^2}{\frac{1}{2}(n)^2} = 4$$

When the input size is doubled, the run time is 4 times longer.

Order of Growth:

How fast does the operation count increase as the input size increases?

- For algorithm analysis, emphasize on the operation count's **order of growth** for large input sizes (*asymptotic behaviour*) Slide 26
  - Note: the difference in running times on small inputs cannot really distinguish efficient algorithms from inefficient ones
    - Interested in large values of input, n
- To compare and rank the order of growth (for comparing the efficiency of different algorithms), three notations have been defined: Asymptotics.pptx Example Properties of Asymptotic
  - O (big oh), Ω (big omega) and Θ (big theta)



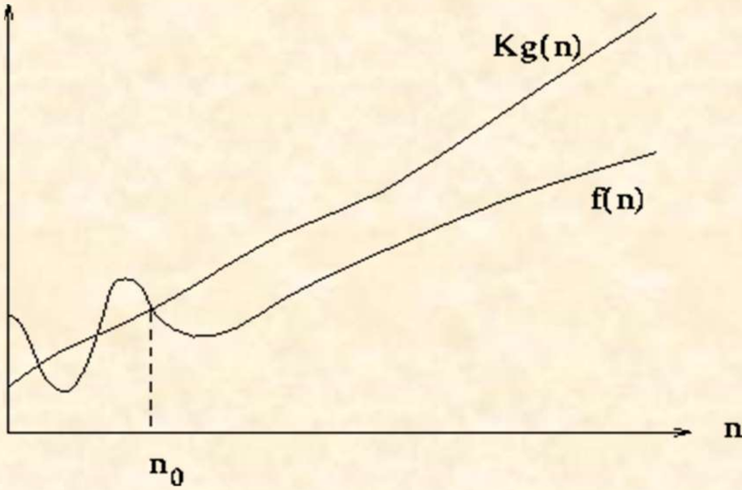
Worst-case analysis

Compare order growth of operations count under worst-case scenario

Asymptotic Analysis

Asymptotic Notation

- **In comparing algorithms, in general, we consider the *asymptotic* behavior of the two algorithms for large problem sizes, under worst-case.**
- ***Big-Oh*** notation: a notation used for characterizing the asymptotic behavior of functions.
  - gives an asymptotic upper bound for a function to within a constant factor Big-Oh Asymptotic Notation Big-Oh Asymptotic Notation
- f(n) = O(g(n))
  - The growth rate of f(n) is less than or equal to the growth rate of g(n)
  - g(n) is an upper bound on f(n)



Big-Oh Notation

- ***Big-Oh*** notation
  - gives an asymptotic upper bound for a function to within a constant factor Big-Oh Asymptotic Notation Big-Oh Asymptotic Notation

Compare the order of growth of operation count for large inputs

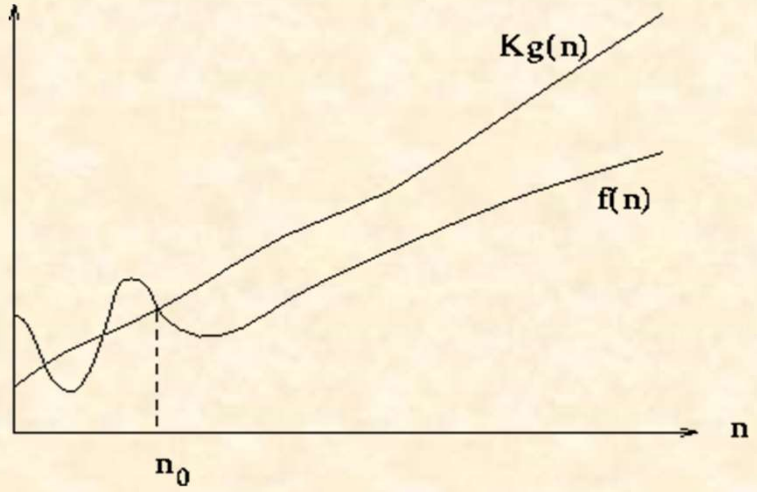
Definition: Given non-negative functions

- **$f(n)$ and $g(n)$, we say that**

$f(n)=O(g(n))$

if there exists an integer n_0 and a constant $k>0$ such that

$f(n)\leq kg(n)$ for all integers $n\geq n_0$

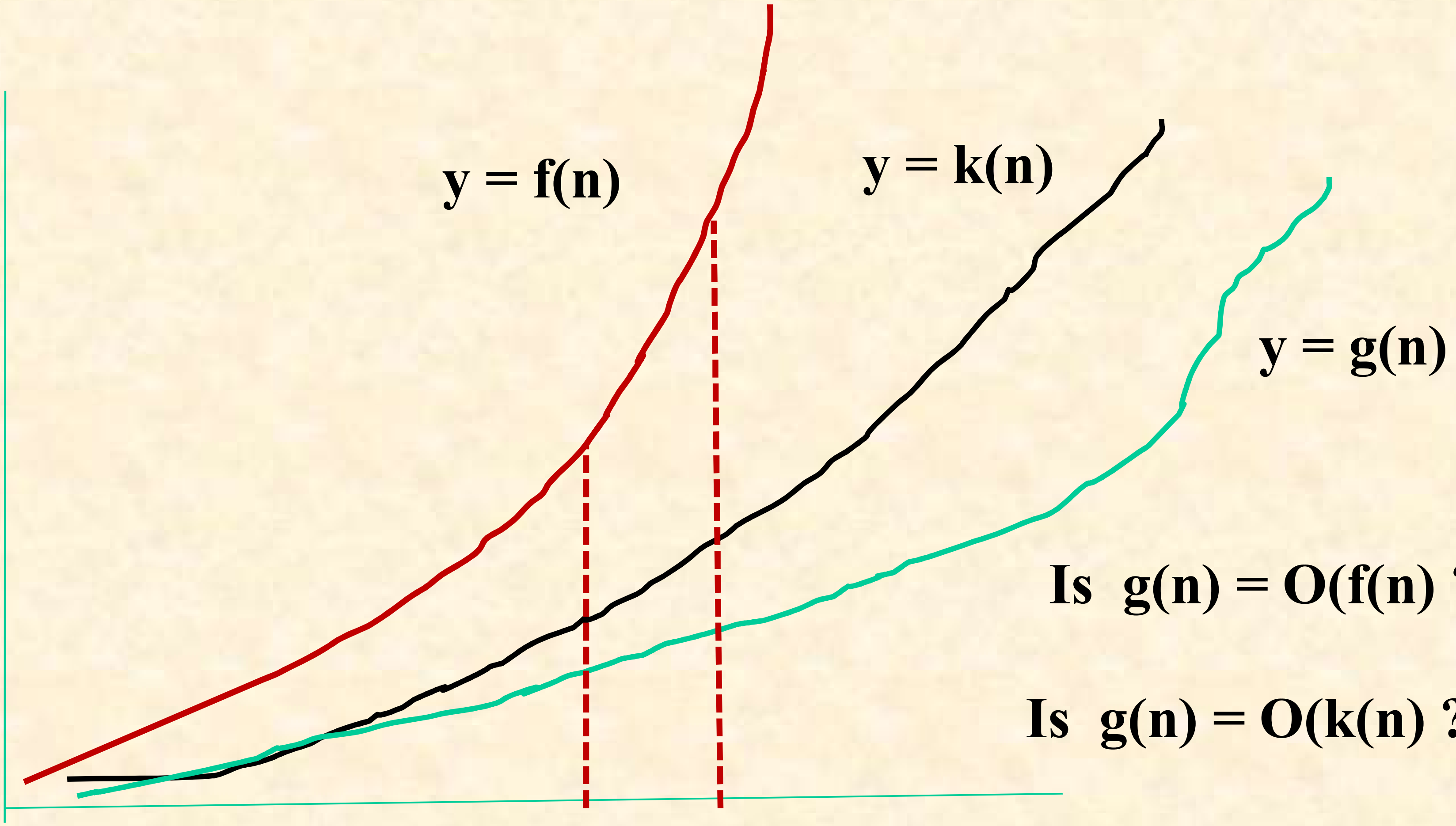


- **$f(n)=O(g(n))$: $f(n)$ is of order at most $g(n)$ or $f(n)$ is big oh of $g(n)$**

f(n) grows no faster than g(n) for large n

Big-Oh

Asymptotic Notation



Is g(n) = O(f(n)) ?

Is g(n) = O(k(n)) ?

Big-Oh

Asymptotic Notation

- Example

$$\frac{T(2n)}{T(n)} = \frac{c(2n)^2}{c(n)^2} = 4$$

  - f(n) = O(n^2)
    - Growth rate of f(n) is less than or equal to n^2
    - Time or number of operations does not exceed c.n^2 on any input of size n (n suitably large).
    - So, the time or number of operations is expected to quadruple each time n is doubled
    - O(n^2) : reads "order n-squared" or "Big-oh n-squared"

Example

- Example: $2n + 10$ is $O(n)$ Big-Oh: Example 1 Example

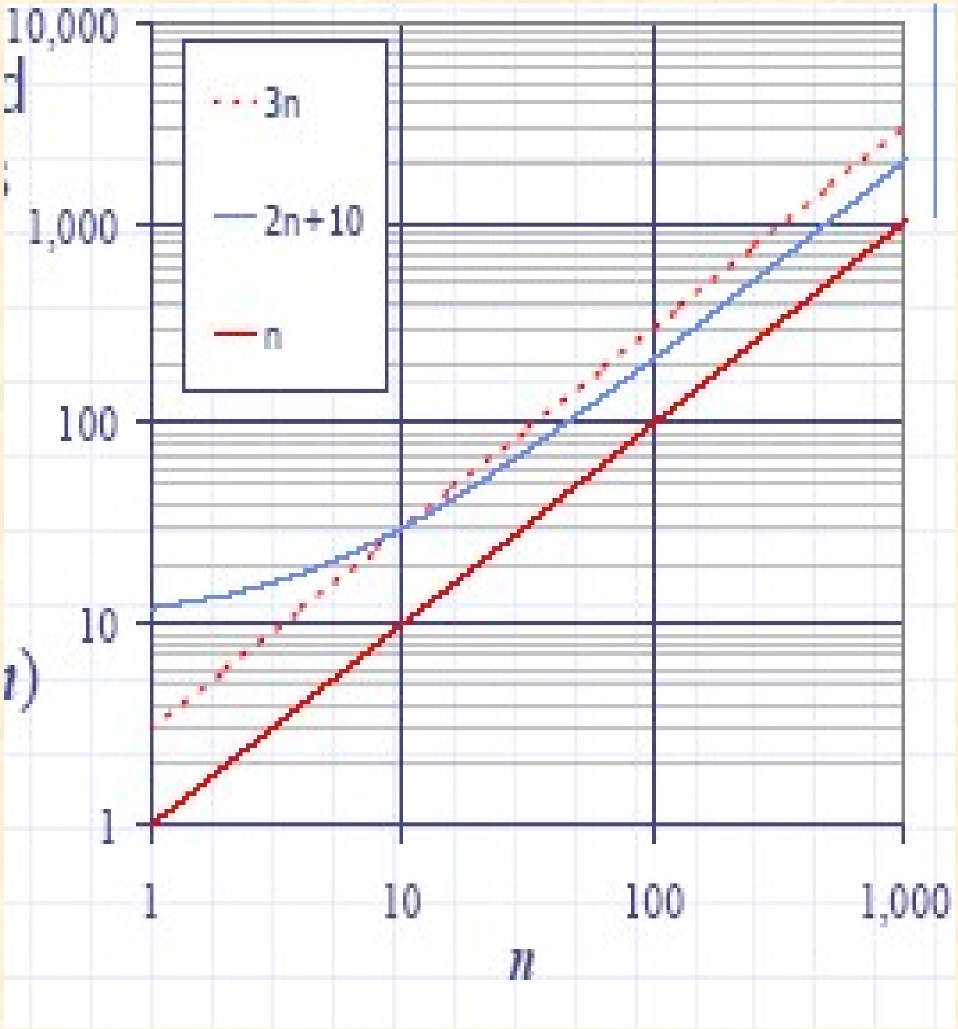
  $2n + 10 \le kn$ for $n \ge n_0$

  $10 \le (k - 2)\, n$

  $n \ge 10/(k - 2)$

Pick $k = 3$ and $n_0 = 10$, so

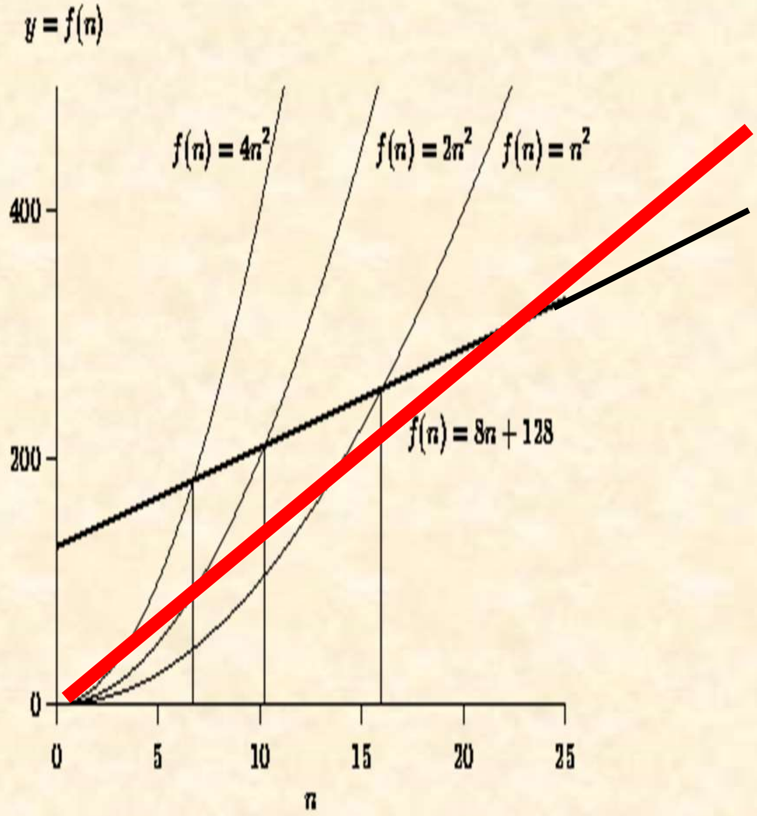
$2n + 10 \le 3n$ for $n \ge 10$



- Example: Given $f(n)=8n+128$ and $g(n) = n^2$. Is $f(n) = O(g(n))$? Big Oh : Example 2 Slide 68
- We need to find an integer n_0 and a constant $k>0$ such that for all integers $n \geq n_0$, $f(n) \leq kn^2$.
- We must get $kn^2 - 8n - 128 \geq 0$
  - If we set $k=1$, we have
    $(n-16)(n+8) \geq 0$;
    hence, we need $(n-16) \geq 0$,
    i.e. $n_0 = 16$, k=1.
- Of course, there are many other values of k and n_0 that will do. For example, $k=2$ and $n_0 = 11$.
- If $g(n) = n$, prove that $f(n) = O(g(n))$. Big Oh : Example 2 Slide 68

$f(n) = O(n^2)$

$f(n) = O(n)$ ← **Best answer, asymptotically tight**



Example

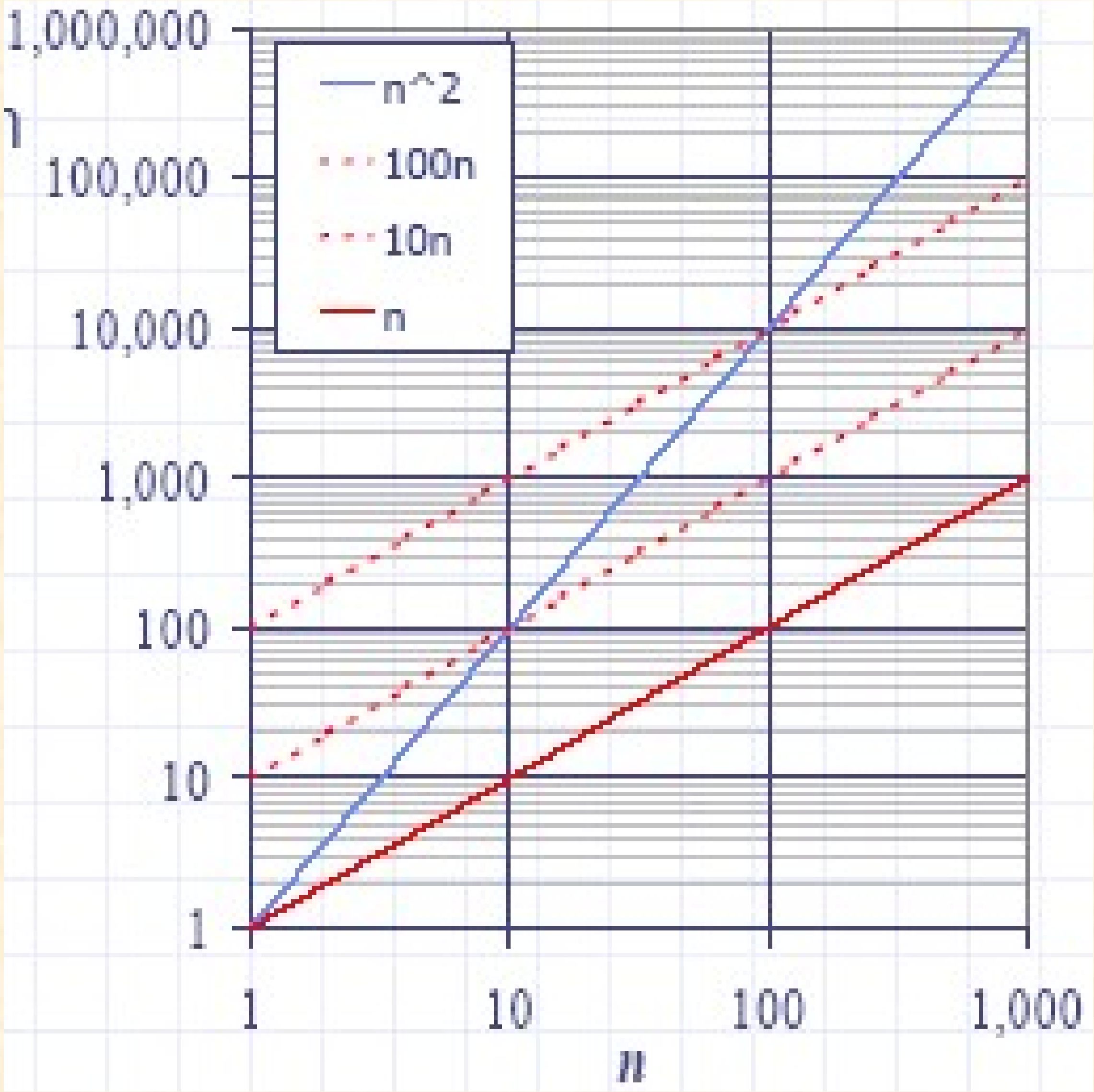
Big Oh : Example 3 Example Big Oh4.pptx

- Example: the function $\boldsymbol{n}^2$ is not $\boldsymbol{O}(\boldsymbol{n})$

  $\boldsymbol{n}^2 \leq \boldsymbol{kn}$

  $\boldsymbol{n} \leq \boldsymbol{k}$

  – The above inequality cannot be satisfied since $\boldsymbol{k}$ must be a constant



More Big-Oh Examples

- $7n$-2
  - $7n$-2 is $O(n)$
  - need $k > 0$ and $n_0 \geq 1$ such that $7n\text{-}2 \leq kn$ for n $\geq n_0$
  - this is true for $k = 7$ and $n_0 = 1$
- $3n^3 + 20n^2 + 5$ Big Oh : Example 5
  - $3n^3 + 20n^2 + 5$ is $O(n^3)$
  - need $k > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq kn^3$ for $n \geq n_0$
  - this is true for $k = 4$ and $n_0 = 25$
- $3\ lg\ n + 5$ Big Oh : Example 6 More Big-Oh Examples
  - $3\ lg\ n + 5$ is $O(lg\ n)$
  - need $k > 0$ and $n_0 \geq 1$ such that $3\ lg\ n + 5 \leq k\ lg\ n$ for n $\geq n_0$
  - this is true for $k = 8$ and $n_0 = 2$

Big Oh : Example

- Given $f(n) = 3n^3 + 20n^2 + 5$ show that $f(n) = O(n^3)$
- Need to find constants k and n_0 such that

$$3n^3 + 20n^2 + 5 \le kn^3 \text{ for } n \ge n_0$$

$$(k-3)n^3 - 20n^2 - 5 \ge 0 \text{ for } n \ge n_0$$

Let k = 4:

$$\begin{aligned} & n^3 - 20n^2 - 5 \\ \ge\ & n^3 - 20n^2 - 5n^2 \\ =\ & n^2(n-25) \\ \ge\ & 0 \qquad \text{for } n \ge 25 \end{aligned}$$

Hence by choosing k=4 and $n_0 = 25$, we have

$$3n^3 + 20n^2 + 5 \le kn^3 \text{ for } n \ge n_0$$

Another Proof

More Big-Oh Examples

- $3n^3 + 20n^2 + 5$

$\leq 3n^3 + 20n^3 + 5n^3$ for $n \geq 1$

$= 28n^3$

Hence by choosing k=28 and $n_0 = 1$, we have

$3n^3 + 20n^2 + 5 \leq kn^3$ for all $n \geq n_0$

So, $3n^3 + 20n^2 + 5 = O(n^3)$

Big Oh : Example More Big-Oh Examples

- Given f(n) = 3 $lg\ n$ + 5 show that f(n) = O($lg\ n$)
- Need to find constants k and n_0 such that

$$3\ lg\ n + 5 \le k\ lg\ n \quad \text{for } n \ge n_0$$

$$(k\text{-}3)\ lg\ n - 5 \ge 0 \text{ for } n \ge n_0$$

Let k = 8:

$$5\ lg\ n - 5$$

$$\ge 0 \qquad for\ all\ n \ge 2$$

lgn $\ge$ 1 for n $\ge$ 2

Hence by choosing k=8 and n_0 = 2, we have

$$3\ lg\ n + 5 \le k\ lg\ n \text{ for } n \ge n_0$$

Big-Oh Rules

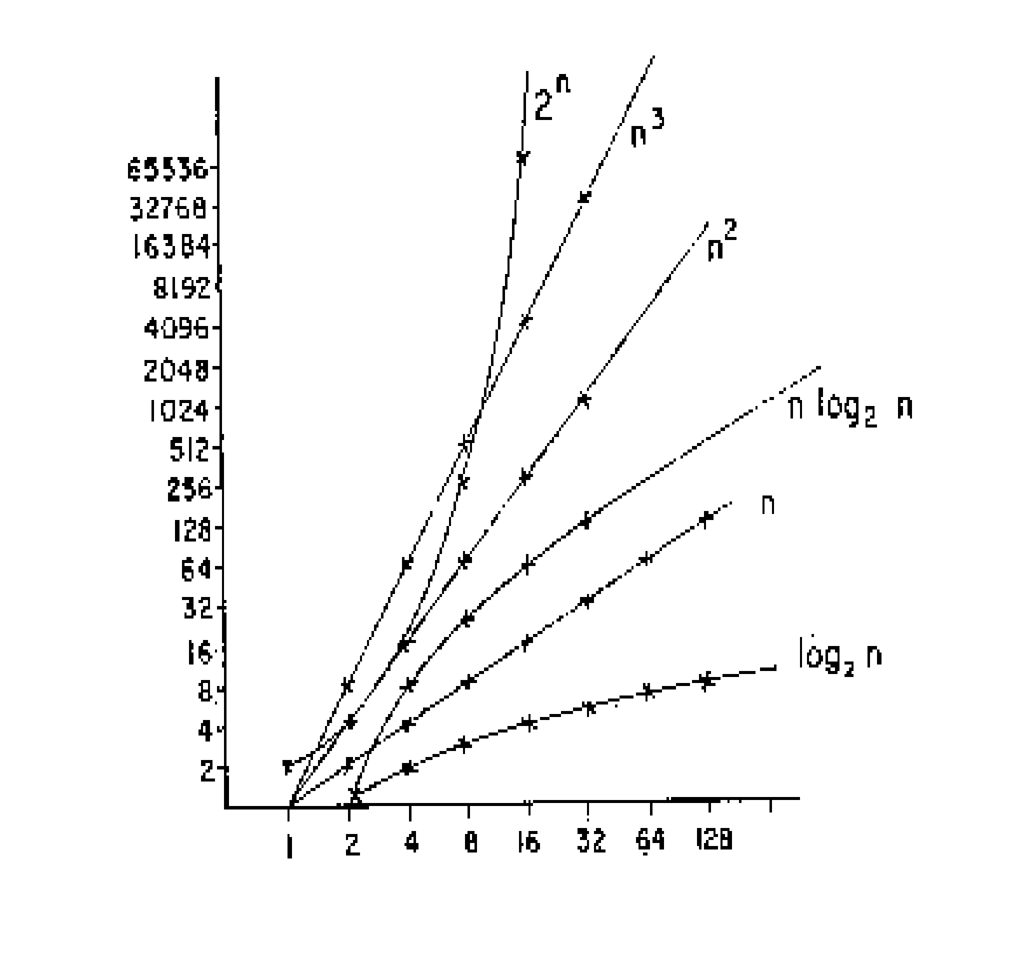
- If is ***f***(***n***) a polynomial of degree ***d***, then ***f***(***n***) is ***O***(***n***^{d}), i.e., $f(n) = c_d n^d + c_{d-1} n^{d-1} + \ldots + c_1 d + c_0$
  1. Drop lower-order terms
  2. Drop constant factors Polynomial.pptx
- Use the smallest possible class of functions
  - Say "100***n*** is ***O***(***n***)" instead of "100***n*** is ***O***(***n***^{2})"
- Use the simplest expression of the class
  - Say "3***n*** + 5 is ***O***(***n***)" instead of "3***n*** + 5 is ***O***(3***n***)"

Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$\boldsymbol{f}(\boldsymbol{n})$ is $\boldsymbol{O}(\boldsymbol{g}(\boldsymbol{n}))$" means that the growth rate of $\boldsymbol{f}(\boldsymbol{n})$ is no more than the growth rate of $\boldsymbol{g}(\boldsymbol{n})$
- We can use the big-Oh notation to rank functions according to their growth rate

Seven Important Functions

- The amount of time required to execute an algorithm usually depends on the input size, *n*
- Seven functions that often appear in algorithm analysis:
  - Constant ≈ ***1*** constant.pptx
  - Logarithmic ≈ log ***n***
  - Linear ≈ ***n***
  - N-Log-N ≈ ***n*** log ***n***
  - Quadratic ≈ $\boldsymbol{n}^2$
  - Cubic ≈ $\boldsymbol{n}^3$
  - Exponential ≈ $\mathbf{2}^n$



- The following table shows the running times computed for a very conservative scenario. Check your algorithm
  - We assume that the constant, c, is one cycle of a 100 MHz clock. This table shows the running times we can expect even if only one instruction is done for each element of the input

	n=1	n=8	n= 1k	n= 1024k
$O(1)$	10ns	10ns	10ns	10ns
$O(\lg n)$	10ns	30ns	100ns	200ns
$O(n)$	10ns	80ns	1.02 µs	10.05ms
$O(n\lg n)$	10ns	240ns	10.2 µs	210ms
$O(n^2)$	10ns	640ns	102 µs	3.05hours
$O(n^3)$	10ns	5.12µs	10.7s	365years
$O(2^n)$	10ns	2.56µs	10^{293}years	$10^{10^{15}}$years

Constant time C ==> $C \le C(1)$ for $n \ge 1$ ==> $C = O(1)$

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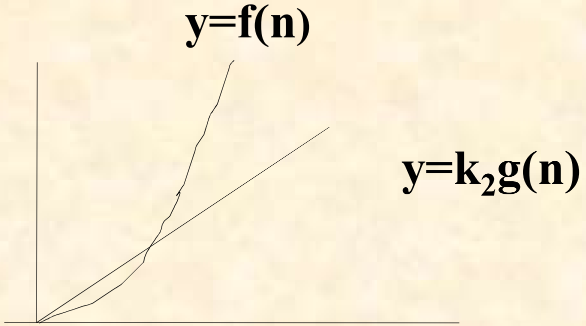
Asymptotic notation



- $f(n) = \Omega(g(n))$
  - $f(n)$ is of order **at least** $g(n)$ or $f(n)$ is omega of $g(n)$ if there exist constants k_2 and n_2 such that $f(n) \geq k_2\, g(n)$ for all $n \geq n_2$
- $f(n) = \Theta(g(n))$
  - $f(n)$ is of order $g(n)$ or $f(n)$ is theta of $g(n)$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
- Hence,
  - if $f(n) = O(g(n))$, g is an asymptotic upper bound for f.
  - if $f(n) = \Omega(g(n))$, g is an asymptotic lower bound for f.
  - if $f(n) = \Theta(g(n))$, g is an asymptotic tight bound for f.

Exercise: Show that for $f(n)=8n+128$, $g(n)=n$, $f(n) = \Omega(g(n))$. Big-Omega

Asymptotic notation

Big-oh

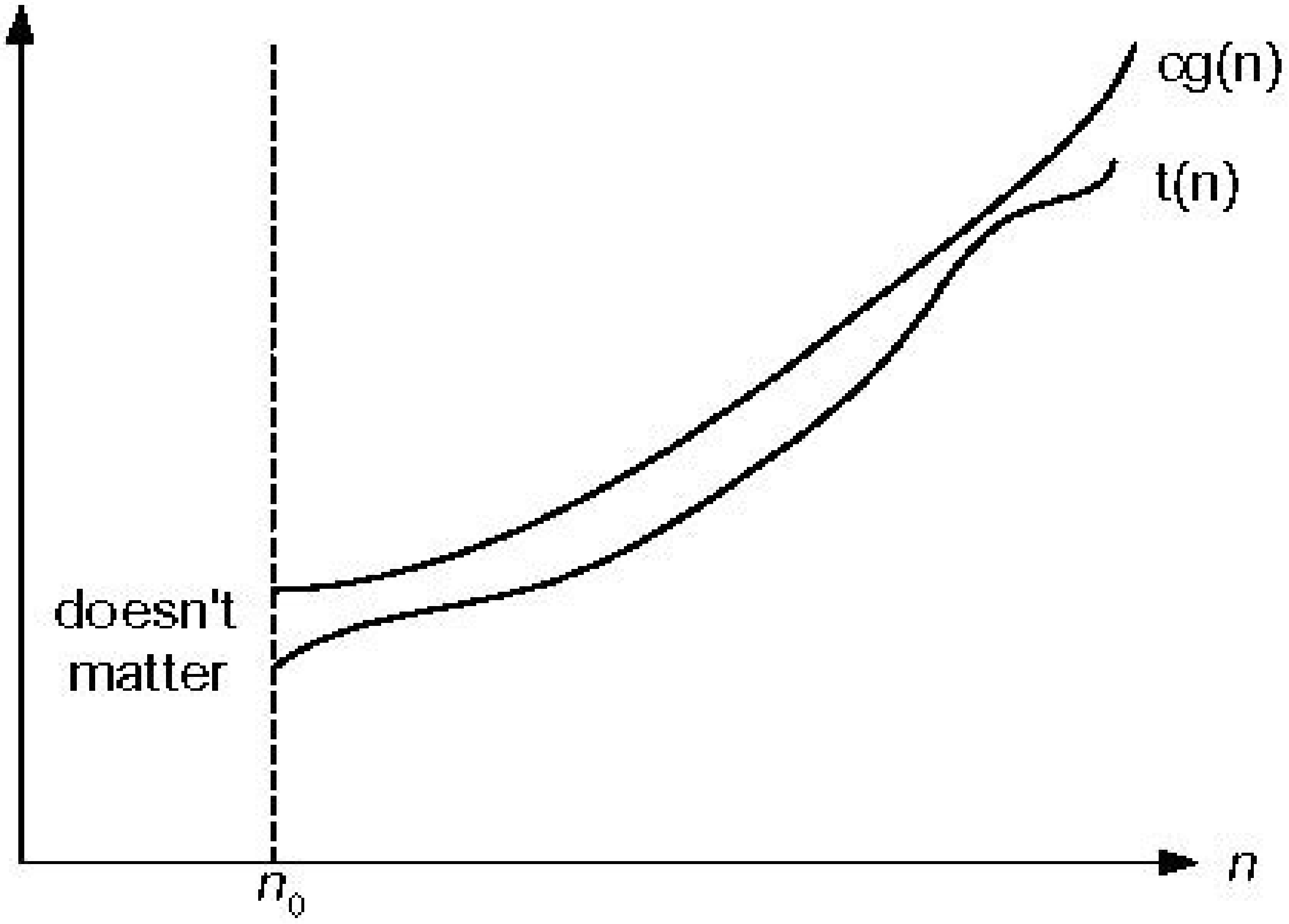


Figure 2.1 Big-oh notation: $t(n) \in O(g(n))$

Big-omega

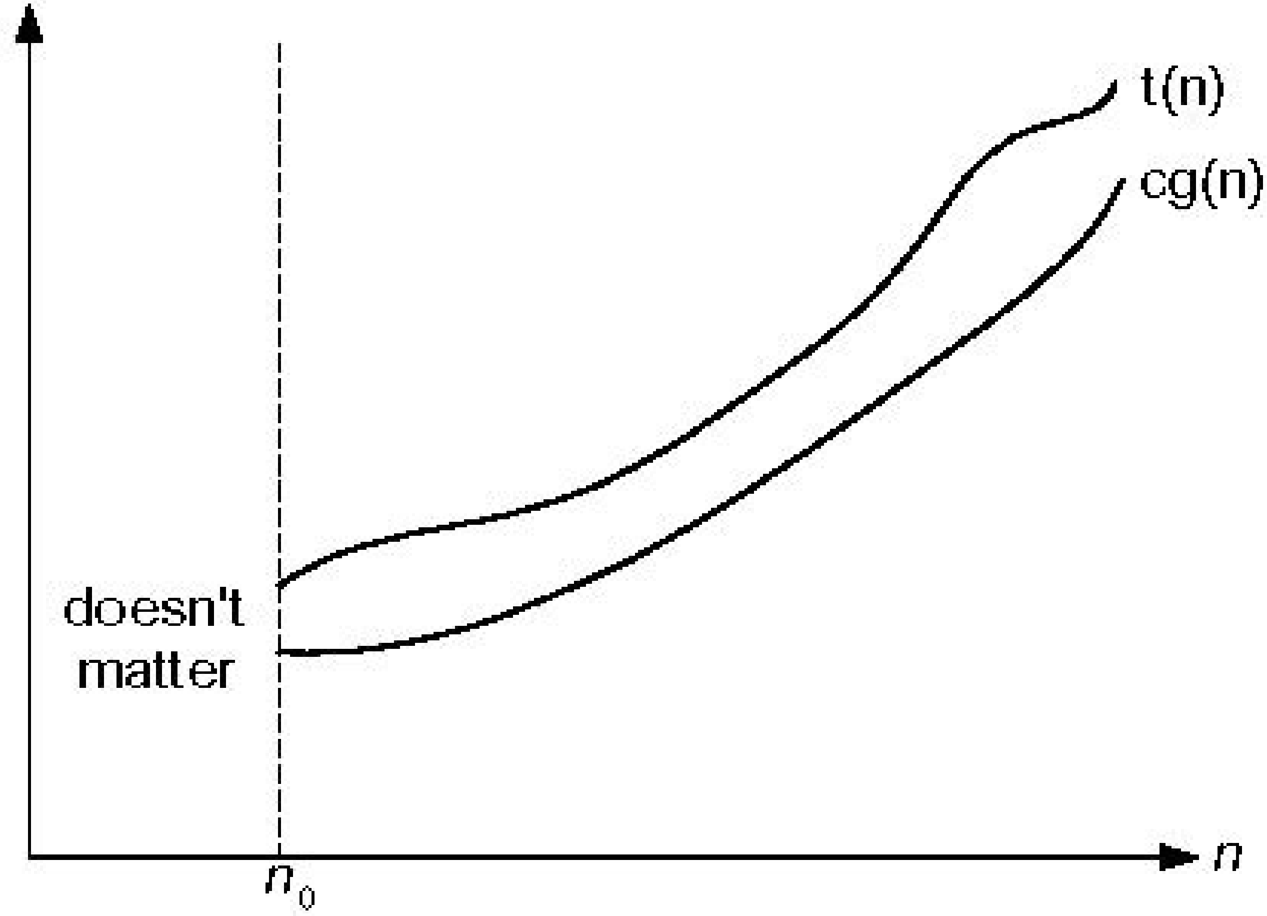


Fig. 2.2 Big-omega notation: $t(n) \in \Omega(g(n))$

Big-theta

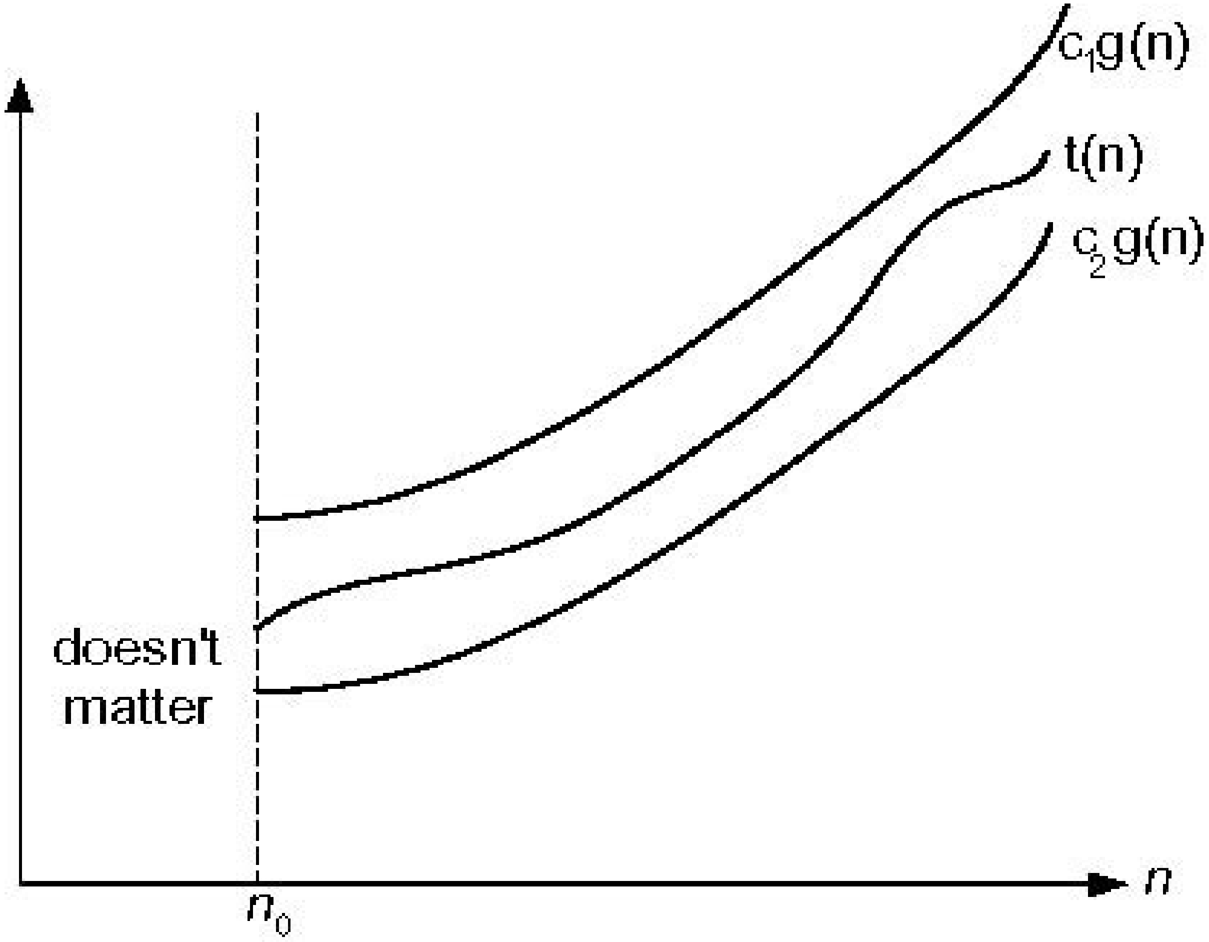


Figure 2.3 Big-theta notation: $t(n) \in \Theta(g(n))$

Example

Given $f(n)=60n^2+5n+1$, $g(n)=n^2$, prove that $60n^2+5n+1= \Theta(n^2)$

Proof:

$60n^2+5n+1 \leq 60n^2+5n^2+n^2 = 66n^2$ for all $n \geq 1$

Hence,

we can take $k_1 =66$, and $n_1=1$, and conclude that $f(n) = O(n^2)$

Since $60n^2+5n+1\geq 60n^2$, for all $\mathrm{n} \geq 1$,

we can take $k_2 = 60$, and $n_2=1$, and conclude that $f(n) = \Omega(n^2)$

Based on the above, we have $60n^2+5n+1= \Theta(n^2)$

Example

Given $f(n)=2n + 3 \lg n$, $g(n)=n$, prove that $2n + 3 \lg n = \Theta(n)$

Proof:

$2n + 3 \lg n \le 2n+3n = 5n$ for all $n \ge 1$

Hence,

we can take $k_1 = 5$, and $n_1=1$, and conclude that $f(n) = O(n)$

Since $2n + 3 \lg n \ge 2n$, for all $n \ge 1$,

we can take $k_2 = 2$, and $n_2=1$, and conclude that $f(n) = \Omega(n)$

Therefore, we have $2n + 3 \lg n = \Theta(n)$

Example

Given $f(n)=1+2+\ldots+n$, show that $f(n) = \Theta(n^2)$

Proof:

$1+2+\ldots+n \le n+n+\ldots+n = n \cdot n = n^2$ for all $n \ge 1$

Hence,

we can take $k_1 = 1$, and $n_1=1$, and conclude that $f(n) = O(n^2)$

Since $1+2+\ldots+n = n(n+1)/2$, for all $n \ge 1$,

$n(n+1)/2 = (n^2+n)/2 \ge n^2/2$

we can take $k_2 = ½$, and $n_2=1$, and conclude that $f(n) = \Omega(n^2)$

Therefore, we have $f(n) = \Theta(n^2)$

Properties of Asymptotic

Suppose we know that $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

- What can we say about the asymptotic behavior of the *sum* and the product of $f_1(n)$ and $f_2(n)$?

- Theorem 3.1: $f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$ Slide 145
- Theorem 3.2: $f_1(n) \times f_2(n) = O(g_1(n) \times g_2(n))$ Thm3_2.pptx

- Consider the functions $f_1(n) = n^3 + n^2 + n + 1 = O(n^3)$ and $f_2(n) = n^2 + n + 1 = O(n^2)$
  - By Theorem 3.2 , the asymptotic behavior of the product $f_1(n) \times f_2(n)$ is $O(n^3 \times n^2) = O(n^5)$.

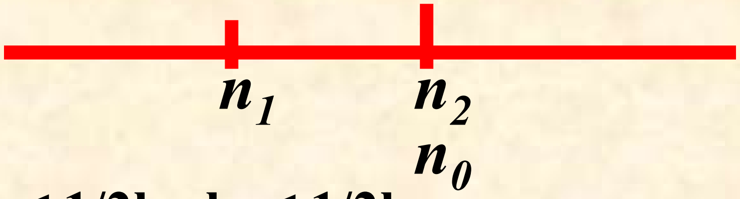
Proof of theorem 3.1:

Properties of Asymptotic

- Given that f1(n) = O(g1(n)) & f2(n) = O(g2(n))
  - So by definition, there exist two integers, n_1 and n_2, and two constants k_1 and k_2 such that

$f_1(n) \leq k_1\, g_1(n)$ for $n \geq n_1$ and $f_2(n) \leq k_2\, g_2(n)$ for $n \geq n_2$

- Let $n_0 = \max(n_1, n_2)$



$n \geq n_0 \;\Rightarrow\; n \geq n_1$

$n \geq n_0 \;\Rightarrow\; n \geq n_2$

- $k_0 = 2\max(k_1, k_2)$ ← $\mathbf{k_1 \leq 1/2k_0}$ $\mathbf{k_2 \leq 1/2k_0}$
- Note that

$\mathbf{k_0 \geq 2k_1}$?

$\mathbf{k_0 \geq 2k_2}$?

$f_1(n) \leq k_1\, g_1(n)$ for $n \geq n_0$

and $f_2(n) \leq k_2\, g_2(n)$ for $n \geq n_0$

So, $f_1(n) + f_2(n) \leq k_1\, g_1(n) + k_2\, g_2(n)$ for $n \geq n_0$

$\leq k_0\,(g_1(n) + g_2(n))/2$

$\leq k_0 \max(g_1(n), g_2(n))$

$\mathbf{g_1(n) \leq \max(g_1(n), g_2(n))}$

$\mathbf{g_2(n) \leq \max(g_1(n), g_2(n))}$

- Thus $f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$

$\mathbf{g_1(n) + g_2(n) \leq 2\max(g_1(n), g_2(n))}$

Proof of theorem 3.1:

- By definition, there exist two integers, n_1 and n_2, and two constants k_1 and k_2 such that

  $f_1(n) \le k_1\, g_1(n)$ for $n \ge n_1$ and $f_2(n) \le k_2\, g_2(n)$ for $n \ge n_2$

- Let $n_0 = \max(n_1, n_2)$, $k_0 = 2\max(k_1, k_2)$, consider the sum for $f_1(n) + f_2(n)$ for $n \ge n_0$,

$$\begin{aligned} f_1(n) + f_2(n) &\le k_1\, g_1(n) + k_2\, g_2(n) \qquad \text{for } n \ge n_0 \\ &\le k_0\,(g_1(n) + g_2(n))/2 \\ &\le k_0 \max(g_1(n), g_2(n)) \end{aligned}$$

- Thus $f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$ Properties of Asymptotic

Proof of theorem 3.2: Properties of Asymptotic

- By definition, there exist two integers, n_1 and n_2, and two constants k_1 and k_2 such that

  $f_1(n) \le k_1\, g_1(n)$ for $n \ge n_1$ and $f_2(n) \le k_2\, g_2(n)$ for $n \ge n_2$

- Let $n_0 = \max(n_1, n_2)$, $k_0 = k_{1*}\, k_{2,}$
- Note that

$$f_1(n) \le k_1\, g_1(n) \qquad \text{for } n \ge n_0$$

and

$$f_2(n) \le k_2\, g_2(n) \qquad \text{for } n \ge n_0$$

$$\begin{aligned} f_1(n) * f_2(n) &\le k_1\, g_1(n) * k_2\, g_2(n) \qquad \text{for } n \ge n_0 \\ &= k_1\, k_2\, g_1(n) g_2(n) \\ &= k_0\, g_1(n) g_2(n) \end{aligned}$$

- Thus $f_1(n) * f_2(n) = O(g_1(n) * g_2(n))$

Algorithm analysis Basic Operations

- Concern primarily with the running time and memory space needed to execute a program
- Factors that affect the running time of a program.
  - The algorithm used, the input data, and the computer system
  - The performance of a computer is determined by
    - processor used (type and speed),
    - memory available (cache and RAM) and disk access speed,
    - the programming language used, and
    - the computer operating system software.
- A detailed analysis which takes all of these factors into account is a very difficult and time-consuming
  - only consider a simple model of algorithm independent of the others since we are concerned with the estimating the time of an algorithm

Algorithm analysis - example

- A simple algorithm:

```
for $i$ =1 to $n$
    for $j$ = 1 to $n$
        $x = x+1$
```

i	**j**	**No of time "x=x+1" executed**
1	**1**	**1**
	2	**1**
	3	**1**
	n	**1**

- What is the number of times the statement $x = x+1$ being executed?
  - The running time of the inner loop j = 1 to n is $O(n)$
  - The running time of the outer loop i = 1 to n is $O(n)$
  - As they are nested loops, the total time is $O(n^2)$

- Example: $S_j = \sum_{i=0}^{j} a_i$ $S_0 = \sum_{i=0}^{0} a_i = a_0$ $S_1 = \sum_{i=0}^{1} a_i = a_0 + a_1$
- An algorithm to compute the series of summations is given in following:

```
1   for $j$ = 0 to $n$-1
2       sum = 0
3       for $i$ = 0 to $j$
4           sum = sum + a[$i$]
5       s[$j$] = sum
```

j	number of executions of inner loop
0	1
1	2
2	3
n-1	n

- The inner-most loop (steps 3 and 4) is executed for 1 to n times, i.e. $1 + 2 + \ldots + n = n(n+1)/2$ times **$n(n+1)/2 = 1/2[n^2 + n]$**
- Therefore, the total running time of the program is $O(n^2)$.

General Plan for Algo. Analysis

1. Decide on parameter *n* indicating *input size*
2. Identify algorithm's *basic operation*
3. Determine *worst* case for input of size *n*
   - May also need to determine the *average* and *best* cases
4. Set up a sum expressing the number of times the algo's basic operation is executed
5. Simplify the sum using standard formulas and rules to determine big-Oh of the algo's running time

General Rules to determine Running Time

1. For loops Loops General Rules to determine Running Time
   - The running time of the statements inside the loop times the number of iterations
2. Nested loops
   - The running time of the statements inside the nested loop times the product of the sizes of all the loops
   - An example which is of $O(n^2)$

```
for i = 1 to n
    for j = 1 to n
        k = k +1
```

3. Consecutive statements
   - Simply add the running time
   - Example: the fragment has $O(n)$ work followed by $O(n^2)$, which has running time being $O(n^2)$

```
for i = 1 to n                ←    O(n)
    a[i] = 0
for i = 1 to n
    for j = 1 to n            ←    O(n^2)
        a[i] = a[j]+i+j
```

$$\mathbf{k_1\, n + k_2\, n^2 = O(n^2)}$$

   - Note: actually only the max. is counted

4. If (condition) S1
   else S2
   - The running time is the larger of the running times of S1 and S2

Example - Fibonacci numbers

- 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, … Fibonacci Numbers Example - Fibonacci numbers

  $f_n = f_{n-1} + f_{n-2},\ n \geq 2$ $f_0=0,\ f_1=1$

- Algorithm 1: fibo_1(n)

```
previous = 0
result = 1
for $i$ = 2 to n
    fib = result + previous
    previous = result
    result = fib
return (fib)
```

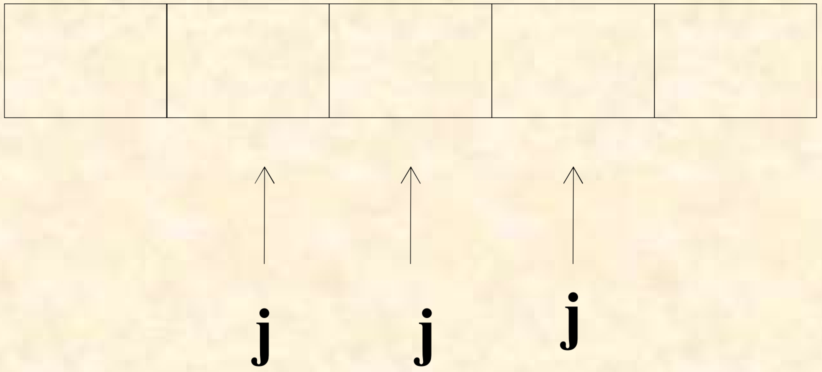
	f_{i-2}	f_{i-1}	f_i
i	**previous**	**result**	**fib**
2	**0**	**1**	**1**
3	**1**	**1**	**2**
4	**1**	**2**	**3**
5	**2**	**3**	**5**

- Running time: $O(n)$

Example - Insertion sort

Slide 26

```
for  (j = 2 to n) {          //start from the 2nd item of the unsorted list
        y = x[j]             //compare with the numbers on sorted
        i = j-1              //list from right (big) to left (small)
        while (y<x[i] and i > 0) {
                x[i+1] = x[i]
                i = i-1
        }
        x[i+1] = x
}
```



- while loop: executed for 1 to *n*-1 time

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} \approx \frac{n^2}{2} = O(n^2)$$

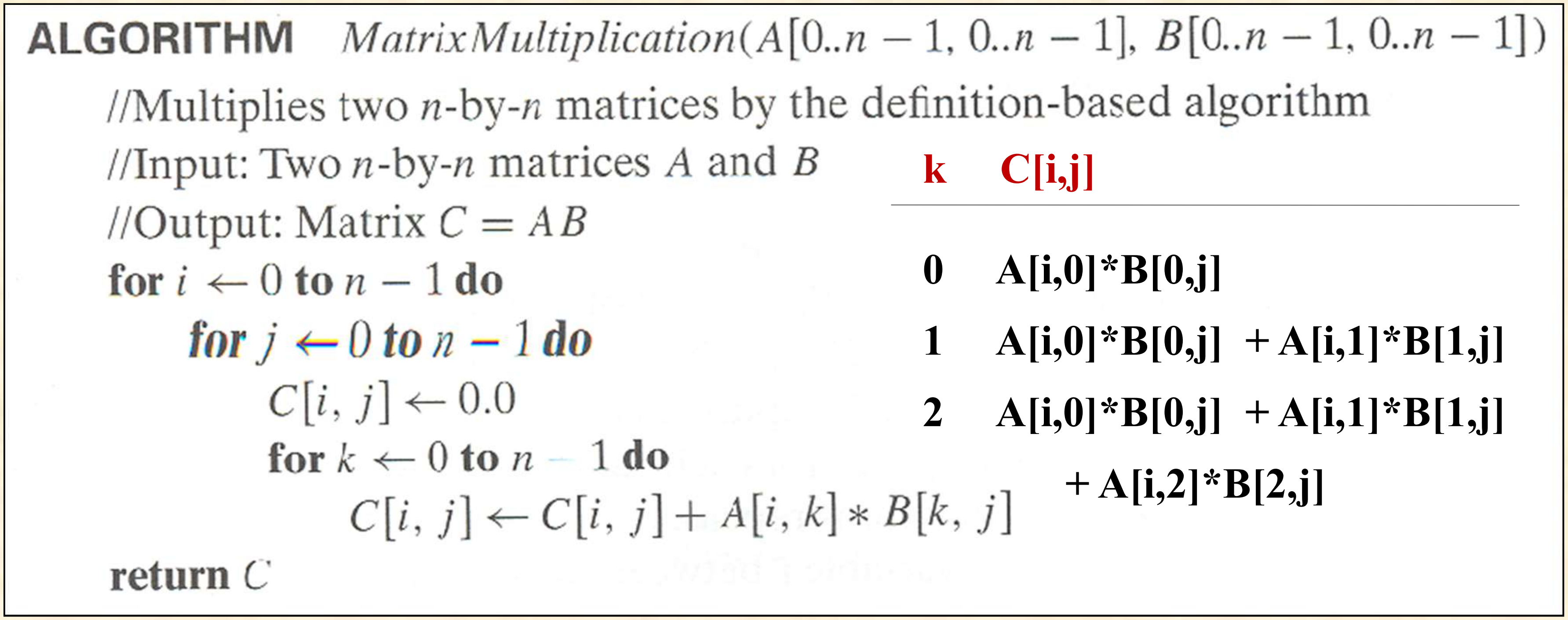
- Running time: *O(n²)*

j	Max number of comparisons
2	1
3	2
4	3
n	n-1

Example: Given 2 *n*-by-*n* matrices, *A* and *B*, find the time efficiency of the algo. for computing their product *C*=*AB*.

- Note: $C[i,j] = A[i,0]B[0,j] + \ldots + A[i,k]B[k,j] + A[i,n-1]B[n-1,j]$

for $0 \le i,j \le n-1$ Slide 150



```
ALGORITHM MatrixMultiplication($A[0..n-1, 0..n-1], B[0..n-1, 0..n-1]$)
    //Multiplies two n-by-n matrices by the definition-based algorithm
    //Input: Two n-by-n matrices A and B
    //Output: Matrix $C = AB$
    for $i \leftarrow 0$ to $n-1$ do
        for $j \leftarrow 0$ to $n-1$ do
            $C[i, j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ do
                $C[i, j] \leftarrow C[i, j] + A[i, k] * B[k, j]$
    return C
```

k	C[i,j]
0	A[i,0]*B[0,j]
1	A[i,0]*B[0,j] + A[i,1]*B[1,j]
2	A[i,0]*B[0,j] + A[i,1]*B[1,j] + A[i,2]*B[2,j]

Matrix Multiplication

Basic operation examples

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x & p \\ y & q \\ z & r \end{bmatrix}$$

$$= \begin{bmatrix} ax + by + cz & ap + bq + cr \\ dx + ey + fz & dp + eq + fr \\ gx + hy + iz & gp + bq + ir \end{bmatrix}$$

x

$$\text{row}_i\,(\mathbf{A}) \rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{array}{ccccc} & & \text{col}_j\,(\mathbf{B}) & & \\ & & \downarrow & & \\ \left[b_{11} \right. & \cdots & b_{1j} & \cdots & \left. b_{1p} \right] \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{array}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

$$\text{row}_i\,(\mathbf{A}) \bullet \text{col}_j\,(\mathbf{B}) = \sum_{\mathbf{k=1}}^{\mathbf{n}} a_{ik}\, b_{kj}$$

- Note that the basic operation is the multiplication at the most inner loop (with index *k*)
  - The number of operations to get a specific value for C[i,j] is

$$\sum_{k=0}^{n-1} 1$$

- The total number of operations is

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} 1 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} n^2 = n^3$$

- The running time of the algo. is $O(n^3)$

- Example: Maximum Subsequence Sum Problem
- Given n integers, A_1, A_2, …, A_n, find the max value of $\sum_{k=i}^{j} A_k$
  - E.g. For -2, 11, -4, 13, -5, -2, the answer is 20 (A_2 to A_4)
  - Add the elements in an array by making multiple passes

```
maxSum = 0
for $i$ = 1 to $n$
    for $j$ = $i$ to $n$
        thisSum = 0
        for $k$ = $i$ to $j$
            thisSum = thisSum + $A[k]$
        if thisSum > maxSum
            maxSum = thisSum
```

i	j	thisSum
1	1	-2
	2	-2+11
	3	-2+11+(-4)
	4	-2+11+(-4)+13
	5	-2+11+(-4)+13+(-5)
	6	-2+11+(-4)+13+(-5)+(-2)
2	2	-11
	3	-11+(-4)

- **The total number of operations is** $\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=i}^{j} 1$

Given n integers, A_1, A_2, …, A_n, find the max value of $\sum_{k=i}^{j} A_k$ E.g. -2, 11, -4, 13, -5, -2

- -2
- -2 + 11 = 9
- -2 + 11 + (-4) = 5
- -2 + 11 +(-4) + 13 = 18
- -2 + 11 + (-4) + 13 + (-5) = 13
- -2 + 11 + (-4) + 13 +(-5) + (-2) = 11
- -11
- 11 + (-4) = 7
- $\boxed{11 + (-4) + 13 = 20}$
- 11 + (-4) + 13 + (-5) = 15
- 11 + (-4) + 13 + (-5) + (-2) = 13

- -4
- (-4) + 13 = 9
- (-4) + 13 + (-5) = 4
- (-4) + 13 + (-5) + (-2) = 2
- 13
- 13 + (-5) = 8
- 13 + (-5) + (-2) = 6
- -5
- (-5) + (-2) = -7
- -2

h

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=i}^{j}1$$

$$\sum_{k=i}^{j}1 = j-i+1$$ H

$$\sum_{j=i}^{n}(j-i+1) = 1+2+\ldots+(n-i+1) = \frac{(n-i+1)(n-i+2)}{2}$$

H

$$\sum_{i=1}^{n}\frac{(n-i+1)(n-i+2)}{2} = \frac{n^3+3n^2+2n}{6}$$

- **The running time of the algo. is *O*(*n*^{3})**Maxsubsequence_analysis.pptx

$$\sum_{i=1}^{n}\frac{(n-i+1)(n-i+2)}{2}=\sum_{i=1}^{n}\frac{n^2-ni+2n-ni+i^2-2i+n-i+2}{2}$$

$$=\sum_{i=1}^{n}\frac{n^2-2ni+3n+i^2-3i+2}{2}$$

$$=\frac{n^3-2n\sum_{i=1}^{n} i+3n^2+\sum_{i=1}^{n} i^2-3\sum_{i=1}^{n} i+2n}{2}$$

$$=\frac{n^3-2n\frac{n}{2}(n+1)+3n^2+\frac{n}{6}(n+1)(2n+1)-\frac{3n}{2}(n+1)+2n}{2}$$

$$=\frac{n^3-n^3-n^2+3n^2+\frac{n}{6}(n+1)(2n+1)-\frac{3n}{2}(n+1)+2n}{2}$$

H

$$\sum_{i=1}^{n}\frac{(n-i+1)(n-i+2)}{2}=\frac{n^3-n^3-n^2+3n^2+\frac{n}{6}(n+1)(2n+1)-\frac{3n}{2}(n+1)+2n}{2}$$

$$=\frac{2n^2+\frac{1}{6}(n^2+n)(2n+1)-\frac{9}{6}n(n+1)+2n}{2}$$

$$=\frac{12n^2+(n^2+n)(2n+1)-9n^2-9n+12n}{12}$$

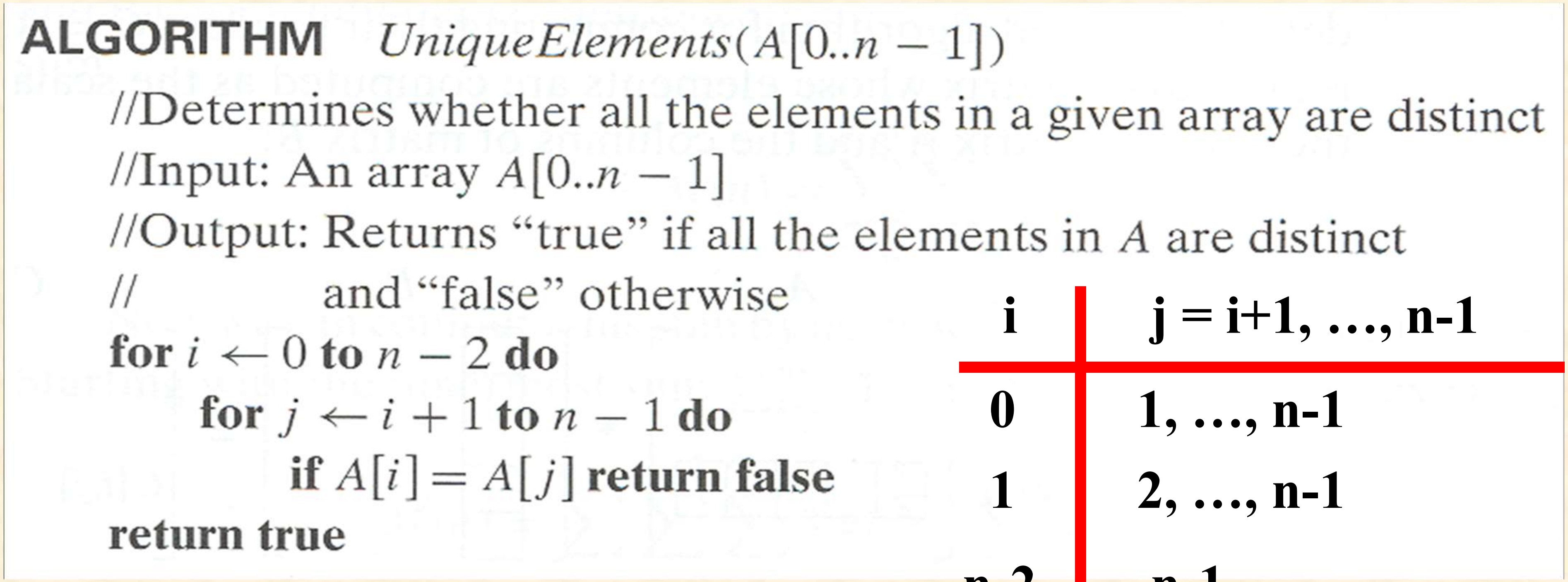
$$=\frac{12n^2+2n^3+3n^2+n-9n^2-9n+12n}{12}$$

$$=\frac{2n^3+6n^2+4n}{12}$$

$$=\frac{n^3+3n^2+2n}{6}$$

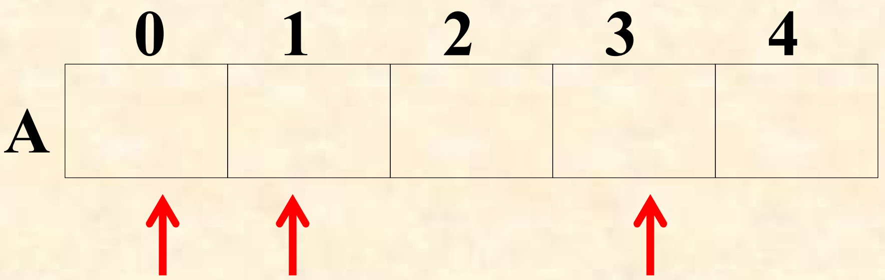
H

Example : Element uniqueness problem



```
ALGORITHM  UniqueElements(A[0..n − 1])
    //Determines whether all the elements in a given array are distinct
    //Input: An array A[0..n − 1]
    //Output: Returns “true” if all the elements in A are distinct
    //          and “false” otherwise
    for i ← 0 to n − 2 do
        for j ← i + 1 to n − 1 do
            if A[i] = A[j] return false
    return true
```

i	j = i+1, …, n-1
0	1, …, n-1
1	2, …, n-1
n-2	n-1



- **The total number of operations is** $$\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}1$$

$$\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} 1$$

$$\sum_{j=i+1}^{n-1} 1 = (n-1) - (i+1) + 1 = n - i - 1$$

H

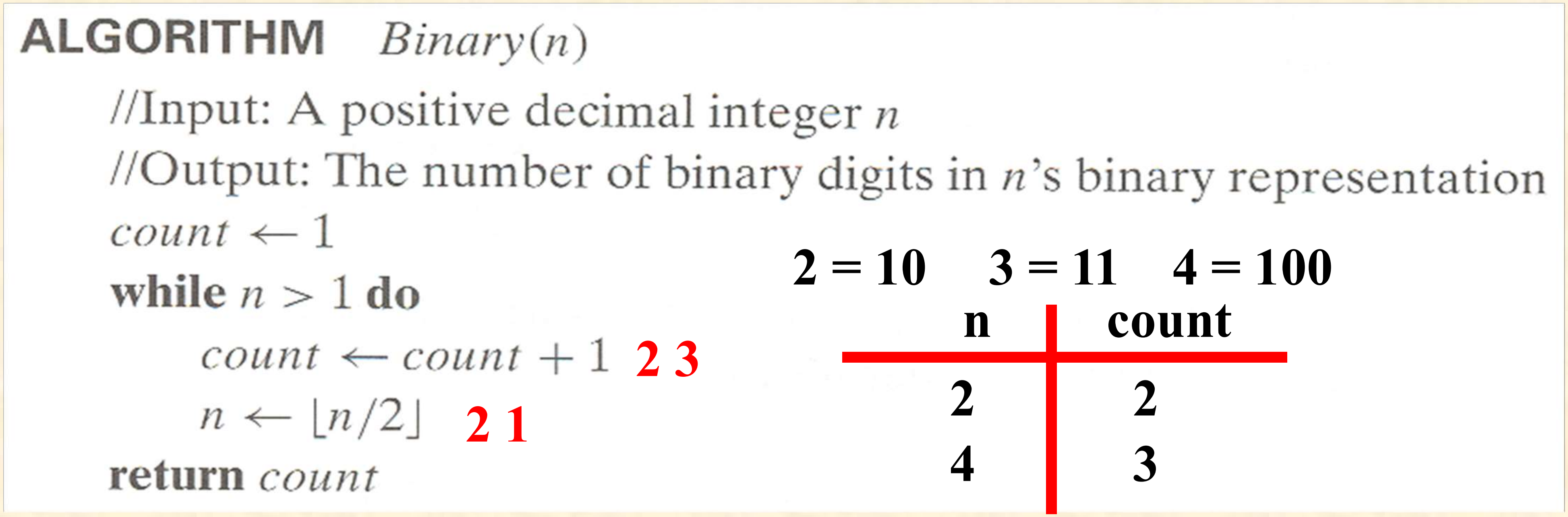
H

$$\sum_{i=0}^{n-2} (n-i-1) = (n-1) + (n-2) + \ldots + (1) = \frac{(n-1)(n-2)}{2}$$

- **The running time of the algo. is *O*(*n*^{2})**

Example : Counting binary digits

Binary Representation Example : Counting binary digits



```
ALGORITHM  Binary(n)
    //Input: A positive decimal integer n
    //Output: The number of binary digits in n's binary representation
    $count \leftarrow 1$
    while $n > 1$ do
        $count \leftarrow count + 1$   2 3
        $n \leftarrow \lfloor n/2 \rfloor$   2 1
    return count
```

2 = 10   3 = 11   4 = 100

n	count
2	2
4	3

Each iteration reduces *n* by a factor of 2

After *i* iterations => *n* will be reduced by 2^i

The halving game: Find integer *i* such that $n / 2^i = 1$.

$$\frac{n}{2^i} = 1 \Rightarrow n = 2^i \qquad \Rightarrow i = \lg n$$

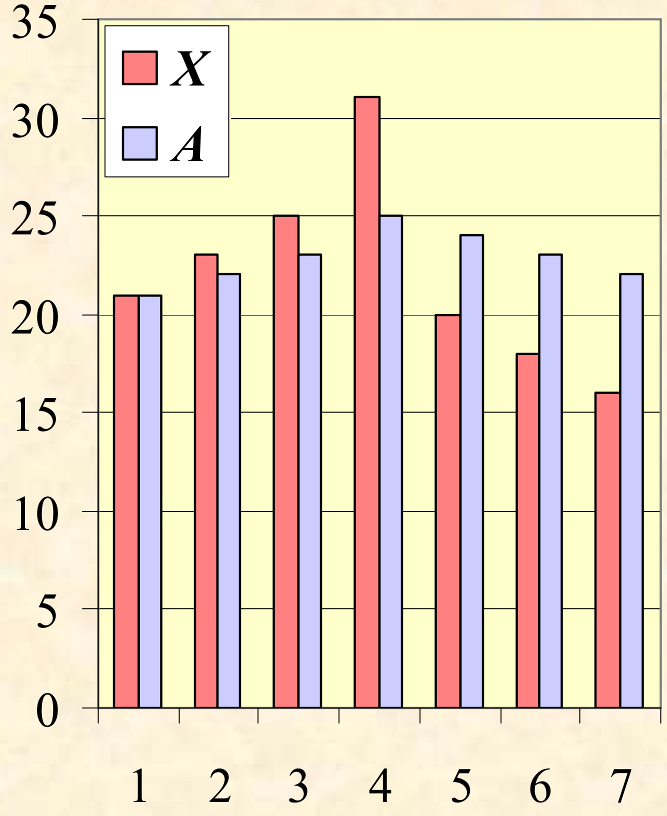
- The running time of the algo. is *O*(*lgn*)

Example :Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages

- The ***i***-th prefix average of an array ***X*** is average of the first (***i*** + 1) elements of ***X*****:**

$$A[\boldsymbol{i}] = (X[0] + X[1] + \dots + X[\boldsymbol{i}])/(i+1)$$



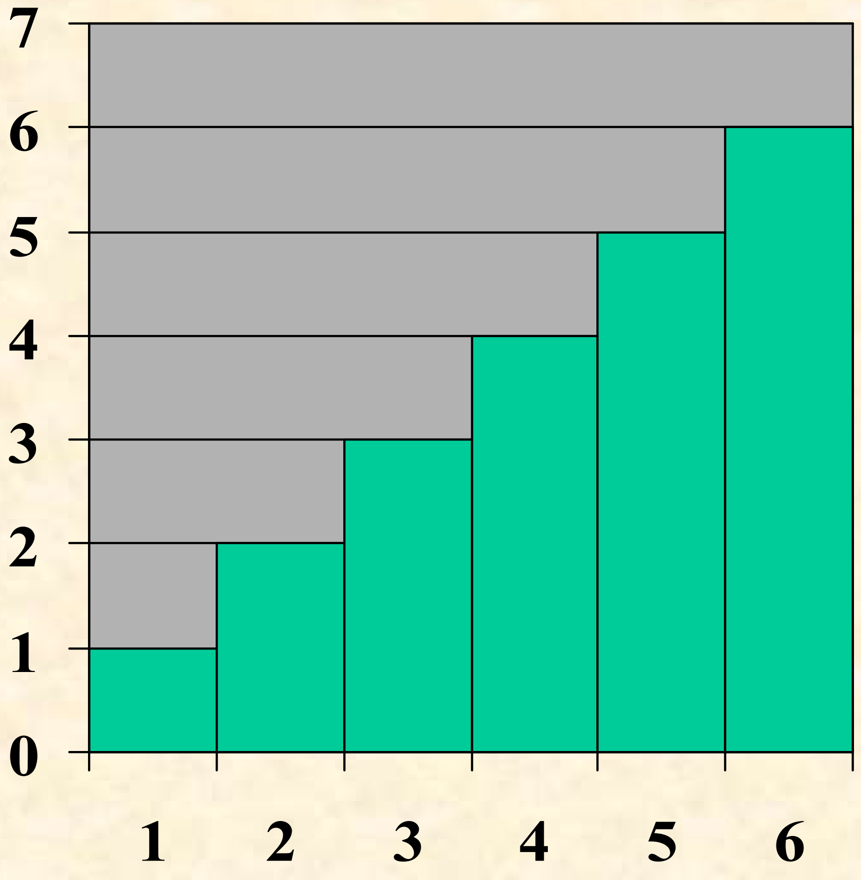
Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

```
Algorithm prefixAverages1($X$, $n$)
    Input array $X$ of $n$ integers
    Output array $A$ of prefix averages of $X$          #operations
    $A \leftarrow$ new array of $n$ integers            $n$
    for $i = 0$ to $n - 1$ do                           $n$
        $s = X[0]$                                      $n$
        for $j = 1$ to $i$ do                           $1 + 2 + \ldots + (n - 1)$
            $s = s + X[j]$                              $1 + 2 + \ldots + (n - 1)$
        $A[i] = s / (i + 1)$                            $n$
    return $A$                                          $1$
```

Arithmetic Progression

- The running time of ***prefixAverages1*** is $\boldsymbol{O}(1 + 2 + \ldots + \boldsymbol{n})$
- The sum of the first $\boldsymbol{n}$ integers is $\boldsymbol{n}(\boldsymbol{n} + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $\boldsymbol{O}(\boldsymbol{n}^2)$ time



Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by keeping a running sum

```
Algorithm prefixAverages2($X$, $n$)
    Input array $X$ of $n$ integers
    Output array $A$ of prefix averages of $X$
    $A \leftarrow$ new array of $n$ integers
    $s = 0$
    for $i = 0$ to $n - 1$ do
        $s = s + X[i]$
        $A[i] = s / (i + 1)$
    return $A$
```

	0	1	2
s			

i	s	A[i]
0	$X[0]$	$S/1$
1	$X[0] + X[1]$	$S/2$
2	$X[0] + X[1] + X[2]$	$S/3$

- Algorithm *prefixAverages2* runs in $O(n)$ time

Prefix Averages (Linear)

- **The following algorithm computes prefix averages in linear time by keeping a running sum**

```
Algorithm prefixAverages2($X$, $n$)
    Input array $X$ of $n$ integers
    Output array $A$ of prefix averages of $X$          #operations
    $A \leftarrow$ new array of $n$ integers                $n$
    $s = 0$                                                 $1$
    for $i = 0$ to $n - 1$ do                               $n$
        $s = s + X[i]$                                      $n$
        $A[i] = s / (i + 1)$                                $n$
    return $A$                                              $1$
```

- **Algorithm *prefixAverages2* runs in $O(n)$ time**

Check your algorithm

- Suppose the running time of a program is $T(n)$, and on the basis of analysis, the worst-case running time of the program is $O(g(n))$.
  - How do you tell from the measurements made that the program behaves as predicted?
  - Following the definition $T(n) \leq kg(n)$ for $n \geq n_0$, compute the ratio $T(n)/g(n)$ for each of value of n in the experiment and observe how the ratio behaves as n increases. Run-Time Analysis Check your algorithm
    - If the ratio diverges, then $g(n)$ is probably too small
    - if the ratio converges to zero, then $g(n)$ is probably too big
    - if the ratio converges to a constant, then the analysis is probably correct.

Run-Time Analysis

Check your algorithm

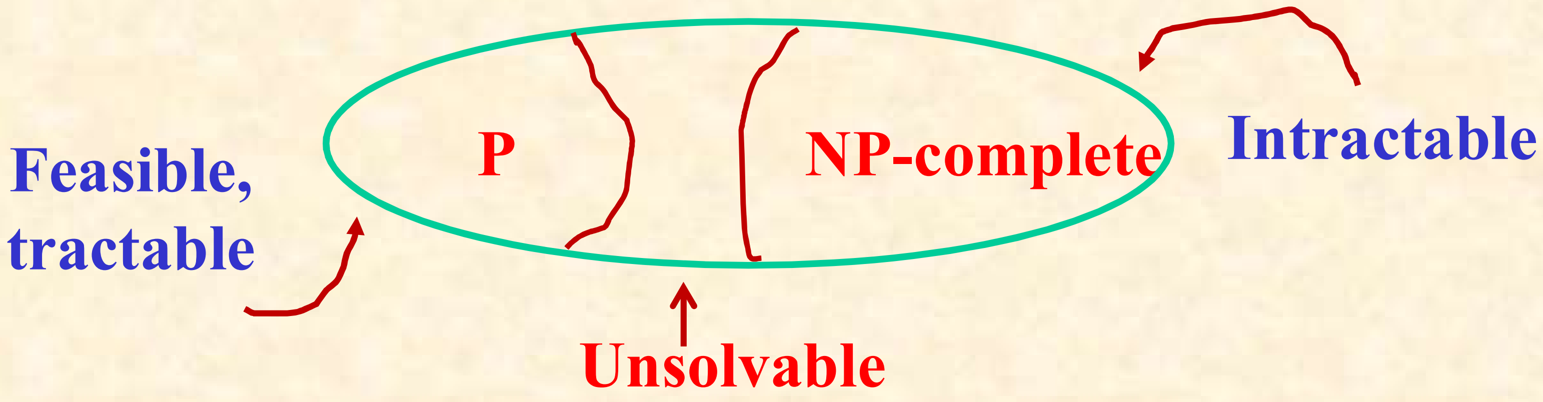
- Running time of algorithm T(n)
- Asymptotic Analysis: O(g(n)) => T(n) ≤ kg(n) for n ≥ n_0
- ratio T(n)/(g(n))diverges
  - as n increases, T(n) increases faster than g(n)
  - Reasons :
    - g(n) is too small; error in analysis of algorithm
- ratio T(n)/(g(n)) converges to zero
  - as n increases, g(n) increase faster than T(n)
    - g(n) is too big, i.e. upper bound on time complexity is not tight
    - Errors in analysis of algorithm
    - Worst-case did not arise in experiment
- ratio T(n)/g(n)) converges to a constant
  - As n increases, both T(n) & g(n) increase at similar rate
    - Analysis of algorithm is probably correct

Algorithm Analysis

- A problem that has *worst-case polynomial-time*, *O(n^k)*, algorithm is considered "***efficient***"
  - The algorithm can be implement and run for a reasonably large input
  - Such problems are called ***feasible*** or ***tractable***
  - Otherwise, the problem is said to be ***intractable*** *Slide 70*
    - Any algorithm, if there is one, will take a long time to execute in the worst case, even for modest sizes of input

Unsolvable and NP-Complete

- Some problems are so hard that they have no algorithms at all
  - ***Unsolvable*** problems
- A large number of solvable problems have an as yet undetermined status
  - They are thought to be intractable but have not been proved to be intractable
  - They are called ***NP-complete*** problems



Feasible, tractable

P

NP-complete

Intractable

Unsolvable