Analysis I - Definitions

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1 Introduction

Definition 1 (Field)**.** A *field* is a set X with two binary operations, usually denoted $+$ and $\times$, with all the familiar properties of addition and multiplication:

- $+$ and $\times$ are commutative and associative and have identities, denoted 0 and 1 respectively.
- Every $x \in X$ has an inverse under $+$, denoted $-x$ and every $x \neq 0$ has an inverse under $\times$, denoted $\frac{1}{x}$.
- $\times$ is distributive over $+$.

Definition 2 (Total ordering)**.** A *total ordering* $<$ on a set X is a relation with the following two properties:

i) $<$ is transitive (i.e. $\forall x, y, z \in X \quad x < y$ and $y < z \implies x < z$).

ii) For every $x, y \in X$ exactly one of the following statements holds: $x < y$ or $x = y$ or $y < x$.

A *totally ordered set* is a set together with a total ordering.

Definition 3 (Ordered field)**.** An *ordered field* is a field F together with a total ordering $<$ such that:

i) $\forall x, y, z \in F \quad x < y \implies x + z < y + z$

ii) $\forall x, y, z, \in F \quad x < y$ and $z > 0 \implies xz < yz$

Definition 4 (Least upper bound)**.** Let X be an ordered set and let $A \subset X$. An element $u \in X$ is an *upper bound* for A if $\forall a \in Aa \leq u$.

An element $s \in X$ is the *least* upper bound for A if:

i) s is an upper bound for A.

ii) $\forall t < s :\ t$ is not an upper bound for A (i.e. $\forall t < s\ \exists a \in A\ a > t$).

An ordered field F has the *least upper bound property* if for every $A \subset F$: if $A \neq \emptyset$ and A has an upper bound, then A has a least upper bound.

2 Sequences and convergence

Definition 5 (Bounded)**.** A sequence (a_n) of real numbers is

$$\left\{\begin{array}{c}\text{bounded}\\ \text{bounded above}\\ \text{bounded below}\end{array}\right\} \text{ if } \exists M \text{ such that } \forall n \left\{\begin{array}{c}|a_N| \leq M\\ a_n \leq M\\ a_n \geq M\end{array}\right\}$$

A sequence is *eventually bounded* if $\exists M, N$ such that $\forall n \geq N\ |a_n| \leq M$.

Definition 6 (Convergence)**.** Let (a_n) be a real sequence and let $a \in \mathbb{R}$. We say that (a_n) *converges to* a or *tends to* a if

$$\forall \epsilon > 0\ \exists N\ \forall n \geq N\ |a_n - a| < \epsilon$$

Definition 7 (Monotone sequence)**.** A sequence (a_n) is

$$\left\{\begin{array}{c}\text{increasing}\\ \text{strictly increasing}\\ \text{decreasing}\\ \text{strictly decreasing}\end{array}\right\} \text{ if } \left\{\begin{array}{c}a_n \leq a_{n+1}\\ a_n < a_{n+1}\\ a_n \geq a_{n+1}\\ a_n > a_{n+1}\end{array}\right\}$$

It is (strictly) monotone if it is (strictly) increasing or (strictly) decreasing.

An ordered field F has the *monotone sequences property* if every increasing sequence in F that is bounded above converges.

Definition 8 (Subsequence)**.** Let (a_n) be a sequence. A *subsequence* of (a_n) is a sequence of the form $(a_{n_k})_{k=1}^{\infty}$ where $n_1 < n_2 < n_3 < \dots$.

Definition 9 (Cauchy sequence)**.** A sequence (a_n) is *Cauchy* if

$$\forall \epsilon > 0\ \exists N\ \forall p, q \geq N\ |a_p - a_q| < \epsilon$$

3 Infinite series

Definition 10 (Partial sum)**.** Let (a_n) be a sequence. The *Nth partial sum* is

$$S_N = \sum_{n=1}^{N} a_n$$

If the sequence (S_N) converges to a limit S, we say that $\sum_{n=1}^{\infty} = S$.

Definition 11 (Absolute convergence)**.** A series $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges.

Definition 12 (Unconditional convergence)**.** A series $\sum a_n$ converges *unconditionally* if $\sum a_{\pi(n)}$ converges for every permutation (i.e. bijection) $\pi : \mathbb{N} \to \mathbb{N}$.

4 Continuous functions

Definition 13 (Continuity)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and let $x \in \mathbb{R}$. We say that f is *continuous at x* if

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall y \;\; |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

We say that f is *continuous* if it is continuous at x for every $x \in \mathbb{R}$.

Definition 14 (Increasing & decreasing functions)**.** Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then f is

$$\left\{\begin{matrix} \text{increasing} \\ \text{strictly increasing} \\ \text{decreasing} \\ \text{strictly decreasing} \end{matrix}\right\} \text{ if for every } x, y \in A \text{ with } x < y \text{ we have } \left\{\begin{matrix} f(x) \leq f(y) \\ f(x) < f(y) \\ f(x) \geq f(y) \\ f(x) > f(y) \end{matrix}\right\}$$

Definition 15 (Bounded function)**.** Let $f : A \to \mathbb{R}$. Then f is *bounded* if $\exists M$ such that $|f(x)| \leq M$ for every $x \in A$. (Similarly for bounded above and below.)

5 Differentiation

Definition 16 (Limit of a function)**.** Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$. Let $a \in A$. We have:

$$\lim_{x \to a^+} f(x) = l \quad \text{if} \quad \forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in (a, a + \delta) \cap A \;\; |f(x) - l| < \epsilon$$

$$\lim_{x \to a^-} f(x) = l \quad \text{if} \quad \forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in (a - \delta, a) \cap A \;\; |f(x) - l| < \epsilon$$

$$\lim_{x \to a} f(x) = l \quad \text{if} \quad \forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \;\; |x - a| < \delta \text{ and } x \neq a \implies |f(x) - l| < \epsilon$$

Definition 17 (Derivative)**.** Let A be an interval and let $f : A \to \mathbb{R}$. Let $x \in A$. The function f is *differentiable at x* if

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

exists.

This limit is the *derivative* of f at x.

Definition 18 (Little-o notation)**.** We can write $o(h)$ for a function that tends to zero as h tends to zero even if you divide it by h.

Definition 19 (Limit of a function at infinity)**.** Write $\lim_{x \to \infty} = l$ if

$$\forall \epsilon > 0 \;\; \exists M \;\; \forall x \geq M \;\; |f(x) - l| < \epsilon$$

6 Power Series

Definition 20 (Power series)**.** A *power series* is an expression of the form $\sum_{n=0}^{\infty} a_n z^n$, where the a_n and z can be in $\mathbb{R}$ or $\mathbb{C}$.

Definition 21 (Radius of convergence)**.** The *radius of convergence* of a power series $\sum_{n=0}^{\infty} a_n z^n$ is the value R such that $\sum_{n=0}^{\infty} a_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.

Definition 22 (Limsup & liminf)**.** Let (a_n) be a real sequence. Then

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup_{n\geq N} a_n$$

Also

$$\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf_{n\geq N} a_n$$

Definition 23 (Convolution)**.** Let (a_r) and (b_s) be sequences. Then the sequence (c_n) such that $c_n = a_0b_n + a_1b_{n-1} + \ldots + a_nb_0$ is the *convolution* of (a_r) and (b_s).

Definition 24 (Exponential function)**.** The exponential function e^x is given by $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Definition 25 (Trigonometric functions)**.** Define $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2}$ and $\tan z = \frac{\sin z}{\cos z}$.

Definition 26 (Hyperbolic functions)**.** Defined $\cosh z = \frac{e^{z}+e^{-z}}{2}$, $\sinh z = \frac{e^{z}-e^{-z}}{2}$ and $\tanh z = \frac{\sinh z}{\cosh z}$.

7 Riemann Integration

Definition 27 (Dissection)**.** Let a and b be real numbers with $a < b$. A *dissection* of $[a, b]$ is a sequence of the form $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

Definition 28 (Upper and lower sums)**.** Let $f : [a, b] \to \mathbb{R}$ and let $\mathcal{D} = \{x_0 < x_1 < \ldots < x_n\}$ be a dissection of $[a, b]$. The *upper sum* $S_{\mathcal{D}}(f)$ is defined to be

$$\sum_{i=1}^{n}(x_i - x_{i-1}) \sup_{x\in[x_{i-1},x_i]} f(x)$$

. The *lower sum* $s_{\mathcal{D}}(f)$ is defined to be

$$\sum_{i=1}^{n}(x_i - x_{i-1}) \inf_{x\in[x_{i-1},x_i]} f(x)$$

Definition 29 (Refinement)**.** Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be dissections of $[a, b]$. Then $\mathcal{D}_2$ *refines* (or is a *refinement* of) $\mathcal{D}_1$ if every point of $\mathcal{D}_1$ is a point of $\mathcal{D}_2$.

Definition 30 (Riemann integral)**.** Let $f : [a, b] \to \mathbb{R}$. We say that f is *(Riemann) integrable* if

$$\inf_{\mathcal{D}} S_{\mathcal{D}}(f) = \sup_{\mathcal{D}} s_{\mathcal{D}}(f)$$

In this case, we denote the common value by $\int_a^b f(x)\,\mathrm{d}x$.

Definition 31 (Uniform continuity)**.** A function $f : A \to \mathbb{R}$ (where $A \subset R$) is *uniformly continuous* if

$$\forall\epsilon > 0 \quad \exists\delta > 0 \quad \forall x, y \in A \quad |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Definition 32 (Piecewise continuity)**.** A function $f : [a, b] \to \mathbb{R}$ is *piecewise continuous* if there exists $x_1, \ldots, x_k$ in $[a, b]$ with $x_1 < x_2 < \ldots < x_k$ such that f is continuous at x whenever $x \notin \{x_1, \ldots, x_k\}$ and $f(x)$ tends to a limit whenever $x \to x_i^-$ or $x \to x_i^+$.

8 Odds and Ends

Definition 33 (Coverage)**.** Let $\mathcal{A}$ be a collection of subsets of $\mathbb{R}$ and let $X \subset \mathbb{R}$. We say that $\mathcal{A}$ *covers* X if $X \subset \cup_{A \in \mathcal{A}} A$. A *subcover* is a subset $\mathcal{B}$ of $\mathcal{A}$ that covers X.