Analysis II Proofs - Uniform Convergence

Prop: Let $f_n, f : E \to \mathbb{R}$. Then $f_n \to f$ uniformly on E iff $\sup_{x \in E} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

Proof: use definition of uniform convergence

Proof: $f_n \to f$ uniformly on $E \iff \forall \varepsilon > 0 \ \exists N$ s.t. $\forall n \geq N, \forall x \in E$, $|f_n(x) - f(x)| \leq \varepsilon \iff \forall \varepsilon > 0 \ \exists N$ s.t. $\forall n \geq N \ \sup_{x \in E} |f_n(x) - f(x)| \leq \varepsilon$
$\iff \sup_{x \in E} |f_n(x) - f(x)| \to 0$. $\square$

Thm: (Cauchy criterion for uniform convergence). Let $f_n : E \to \mathbb{R}$. Then (f_n) converges uniformly on E iff
$$\forall \varepsilon > 0 \ \exists N \text{ s.t. } n, m \geq N, x \in E \Rightarrow |f_n(x) - f_m(x)| < \varepsilon. \quad (*)$$

Proof: $\Rightarrow$: use triangle inequality. $\Leftarrow$: get pointwise limit from Cauchy criterion for $\mathbb{R}$, then let $m \to \infty$, use proposition above

Proof: $\Rightarrow$: Suppose $f_n \to f$ uniformly. Then $\forall \varepsilon > 0 \ \exists N$ s.t. $\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon/2$. Then for $m, n \geq N$, $x \in E$,
$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$
$\Leftarrow$: Suppose $(*)$ holds. Then $(f_i(x))_{i=1}^{\infty}$ is a Cauchy sequence of reals for each x, so $\lim_{i \to \infty} f_i(x)$ exists. (see Analysis I). Calling this limit $f(x)$, have $f_n(x) \to f(x)$ pointwise. Letting $m \to \infty$ in $(*)$ gives that $\forall \varepsilon > 0 \ \exists N$ s.t. $n \geq N$, $x \in E \Rightarrow |f_n(x) - f(x)| \leq \varepsilon$. Hence $\sup_{x \in E} |f_n(x) - f(x)| \to 0$. $\square$

Thm: Let $f_n, f : E \to \mathbb{R}$, suppose $f_n \to f$ uniformly on E. If $x \in E$ is a point of continuity for each f_n then x is a point of continuity for f.

Proof: fix N s.t. $\sup_{x \in E} |f(x) - f_N(x)|$ is small, apply continuity of f_N at x.

Proof: Let $\varepsilon > 0$. Choose N s.t. $\sup_{x \in E} |f_N(x) - f(x)| < \varepsilon/3$. Since f_N is continuous at x, can choose $\delta > 0$ s.t. $|f_N(y) - f_N(x)| < \varepsilon/3$ whenever $|y - x| < \delta$.

Then
$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &\leq 2 \sup_{x \in E} |f(x) - f_N(x)| + |f_N(y) - f_N(x)| \\ &< 2\tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon \end{aligned}$$
provided $|y - x| < \delta$. $\square$

Thm: Let $f_n, f : [a,b] \to \mathbb{R}$, suppose f_n, f integrable on $[a,b]$ and $f_n \to f$ uniformly on $[a,b]$. Then $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.

Proof: use properties of Riemann integrals

Proof: $\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \leq \int_a^b |f_n(x) - f(x)|\,dx$

$\leq (b-a) \sup_{x \in [a,b]} |f_n(x) - f(x)| \to 0.$ $\square$

Thm: Let $f_n : [a,b] \to \mathbb{R}$ be differentiable. Suppose that $(f_n(c))$ converges for some $c \in [a,b]$ and (f_n') converges uniformly on $[a,b]$. Then $f_n \to f$ uniformly on $[a,b]$ for some f differentiable with $f'(x) = \lim_{n\to\infty} f_n'(x)$ $\forall x \in [a,b]$.

Proof: show that f_n converges uniformly using convergence of $f_n(c)$ and MVT applied to $(f_n - f_m)$. Define $\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}$ for $t \neq x$, $f_n'(x)$ for $t = x$, show ϕ_n converges uniformly by Cauchy criterion to some ϕ, use continuity of ϕ.

Proof: Let $\varepsilon > 0$. Choose N s.t. for $m, n \geq N$, $|f_n(c) - f_m(c)| < \varepsilon/2$ and $\sup_{t \in [a,b]} |f_n'(t) - f_m'(t)| < \min\left\{\frac{\varepsilon}{2(b-a)}, \varepsilon\right\}$. Then for $m, n \geq N$,

$|(f_n(t) - f_m(t)) - (f_n(c) - f_m(c))| = |(f_n - f_m)'(d)(t-c)|$ some d by MVT

$\leq \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon/2.$

So for $m, n \geq N$, $\sup_{t \in [a,b]} |f_n(t) - f_m(t)| \leq |f_n(c) - f_m(c)| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\therefore$ (f_n) is uniformly Cauchy so $f_n \to f$ uniformly to some f.

Fix $x \in [a,b]$, define $\phi_n(t) = \begin{cases} \frac{f_n(t) - f_n(x)}{t - x} & t \neq x \\ f_n'(x) & t = x \end{cases}$ which is

continuous on $[a,b]$ for each n. For $t \neq x$ and $m, n \geq N$ (as before),

$|\phi_n(t) - \phi_m(t)| = \left|\frac{1}{t-x}\left((f_n(t) - f_m(t)) - (f_n(x) - f_m(x))\right)\right| \leq \varepsilon$ by applying MVT as above. At $t = x$, $|\phi_n(x) - \phi_m(x)| = |f_n'(x) - f_m'(x)| < \varepsilon$.

So (ϕ_n) is uniformly Cauchy sequence of continuous function so converges to some continuous function $\phi : [a,b] \to \mathbb{R}$. Since ϕ is (in particular) continuous at x, $\phi(x) = \lim_{t \to x} \phi(t)$. But for $t \neq x$, $\phi(t) = \lim_{n\to\infty} \phi_n(t) = \frac{f(t) - f(x)}{t - x}$. $\therefore$ $f'(x) = \lim_{n\to\infty} f_n'(x)$. $\square$

Analysis II Proofs

Thm: (Weierstrass M-test). Let $g_n : E \to \mathbb{R}$ for $n = 0, 1, 2, \ldots$. Suppose $\exists M_n \geq 0$ s.t. $|g_n(x)| \leq M_n \; \forall x \in E$ and $\sum_{n=0}^{\infty} M_n$ converges. Then $\sum_{n=0}^{\infty} g_n$ converges absolutely uniformly on E.

Proof: show that partial sums $\sum_{j=0}^{n} |g_j|$ converge uniformly by Cauchy criterion

Proof: Let $\varepsilon > 0$. Choose N s.t. $\sum_{n=N}^{\infty} M_n < \varepsilon$. Define $f_n = \sum_{j=0}^{n} |g_j|$.
Then for $n > m \geq N$, $\sup_{x \in E} |f_n(x) - f_m(x)| = \sup_{x \in E} \sum_{j=m+1}^{n} |g_j(x)|$

$$\leq \sum_{j=m+1}^{n} M_j < \varepsilon$$

So (f_n) converges uniformly by the Cauchy criterion. □

Thm: (Radius of convergence). Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series. Then $\exists! R \in [0, \infty]$ s.t.

(i) If $|x-a| < R$ then $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges absolutely.

(ii) If $|x-a| > R$ then $\sum_{n=0}^{\infty} c_n (x-a)^n$ diverges.

(iii) If $R > 0$ and $0 < r < R$ then $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges absolutely uniformly on the set $\{x : |x-a| \leq r\}$.

Proof: apply M-test with $M_n = |c_n| r^n$

Proof: (i) and (ii) are proven in Analysis I. For (iii), note that taking $x = a + r$ in (i) gives $\sum_{n=0}^{\infty} |c_n| r^n$ converges. Also, if $|x-a| \leq r$ then $|c_n||x-a|^n \leq |c_n| r^n$. Hence by Weierstrass M-test, $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges absolutely uniformly on $\{x : |x-a| \leq r\}$. □

Thm: (Term-wise differentiation of power series). Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series, with r.o.c. $R > 0$. Then

(i) The derived series $\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ has r.o.c. R.

(ii) If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ on $\{x : |x-a| < R\}$ then f is differentiable on $\{x : |x-a| < R\}$ with $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \; \forall x \in \{x : |x-a| < R\}$

Proof: WLOG assume $a = 0$. To show r.o.c. of $\sum n c_n x^{n-1}$ is R, take $|x| < R$ and choose y s.t. $|x| < |y| < R$, then use $(c_n y^n)$ bounded and apply ratio + comparison test. For (ii), apply Thm about uniform limit of differentiable functions to partial sums on $\{|x| \leq r\}$ for $0 < r < R$.

Proof: WLOG assume $a=0$. (i) Let $\sum_{n=1}^{\infty} n c_n x^{n-1}$ have r.o.c. R'; so clearly $R' \leq R$. To show $R'=R$, let $|x|<R$ and choose y with $|x|<|y|<R$. Then $\sum_{n=0}^{\infty} c_n y^n$ converges so $(c_n y^n)$ is bounded, say $|c_n y^n| < M$ $\forall n$. Then $|c_n x^n| = |c_n y^n| \left|\frac{x}{y}\right|^n \leq M\left|\frac{x}{y}\right|^n$ so
$\sum_{n=1}^{\infty} |n c_n x^{n-1}| = \frac{1}{|x|}\sum_{n=1}^{\infty} n|c_n x^n| \leq \frac{M}{|x|}\sum_{n=1}^{\infty} n\left|\frac{x}{y}\right|^n$ which converges by the ratio test since $(n+1)\left|\frac{x}{y}\right|^{n+1}/n\left|\frac{x}{y}\right|^n \to \left|\frac{x}{y}\right| < 1$. Hence the derived series converges absolutely for $|x|<R$, i.e. $R'=R$.

(ii) Let $0<r<R$ and define $f_n(x) = \sum_{j=0}^{n} c_j x^j$, so that $f_n \to f$ uniformly on $\{|x|<r\}$ where $f(x) = \sum_{n=0}^{\infty} c_n x^n$. We just showed that $f_n'(x) = \sum_{j=1}^{n} j c_j x^{j-1}$ converges uniformly on $\{|x|<r\}$ too, so by the earlier Thm on uniform limits of differentiable functions,

$$f'(x) = \lim_{n\to\infty} f_n'(x) = \sum_{j=1}^{\infty} j c_j x^{j-1}.$$

If $|x|<R$, we can pick r with $|x|<r<R$ and apply the above, so this derivative holds for all $\{x : |x|<R\}$.

Analysis II Proofs – Uniform Continuity

Thm: If $f:[a,b]\to\mathbb{R}$ is continuous then f is uniformly continuous.
Proof: by contradiction and Bolzano-Weierstrass
Proof: Suppose otherwise. Then $\exists \varepsilon>0$ s.t. for $n=1,2,3,\dots$ we can construct $x_n, y_n \in [a,b]$ with $|x_n-y_n|<\frac{1}{n}$ but $|f(x_n)-f(y_n)|\geq\varepsilon$. Since (x_n) is bounded, by Bolzano-Weierstrass there is a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ with x_{n_k} tending to some $x\in[a,b]$. Then

$$|y_{n_k}-x|\leq|y_{n_k}-x_{n_k}|+|x_{n_k}-x|\to 0$$

so $y_{n_k}\to x$ also. Since f is continuous, $f(x_{n_k})\to f(x)$ and $f(y_{n_k})\to f(x)$, contradicting $|f(x_{n_k})-f(y_{n_k})|\geq\varepsilon$. □

Thm: Let $f:[a,b]\to[A,B]$ be integrable and $g:[A,B]\to\mathbb{R}$ continuous. Then $g\circ f:[a,b]\to\mathbb{R}$ is integrable.
Proof: split $U(P,g\circ f)-L(P,g\circ f)$ sum into sum over $j\in J=\{j:\sup_{[a_j,a_{j+1}]}f-\inf_{[a_j,a_{j+1}]}f<\delta\}$ and $j\notin J$, using uniform continuity and boundedness of g to bound.
Proof: Let $\varepsilon>0$. Since g is continuous, it is uniformly continuous. So choose $\delta>0$ (with $\delta<\varepsilon$) s.t. $x,y\in[A,B]$, $|x-y|<\delta \Rightarrow |g(x)-g(y)|<\varepsilon$.
Since f is integrable, choose $P=\{a=a_0<a_1<\dots<a_n=b\}$ s.t.

$$U(P,f)-L(P,f)<\delta^2.$$

Let $J=\{j:\sup f-\inf f<\delta\}$ (we write just sup, inf assuming over $[a_j,a_{j+1}]$)

- For $j\in J$, $\sup g\circ f-\inf g\circ f<\varepsilon$ by choice of δ.
- Let $K=\sup_{[A,B]}|g|$. Then for $j\notin J$, $\sup g\circ f-\inf g\circ f\leq 2K$, and
$$\delta\sum_{j\notin J}(a_{j+1}-a_j)\leq\sum_{j\notin J}(a_{j+1}-a_j)(\sup f-\inf f)<\delta^2 \text{ by choice of } P,$$
so $\sum_{j\notin J}(a_{j+1}-a_j)<\delta$.

Now, $U(P,g\circ f)-L(P,g\circ f)$

$$\begin{aligned}&=\sum_{j\in J}(a_{j+1}-a_j)(\sup g\circ f-\inf g\circ f)+\sum_{j\notin J}(a_{j+1}-a_j)(\sup g\circ f-\inf g\circ f)\\&\leq\varepsilon(b-a)\qquad\qquad+\qquad 2K\delta\\&<\varepsilon(b-a+2K)\end{aligned}$$

$\therefore g\circ f$ is integrable by Riemann's criterion. □

Cor: If $g:[a,b]\to\mathbb{R}$ is continuous then g is integrable.

Proof: Take $f(x)=x$ in previous Theorem.

Thm: Let $f_n:[a,b]\to\mathbb{R}$ be a sequence of integrable functions converging uniformly to a function $f:[a,b]\to\mathbb{R}$. Then f is integrable and $\int_a^b f_n \to \int_a^b f$.

Proof: define $c_n=\sup_{[a,b]}|f(x)-f_n(x)|$, bound $U(P,f)$ by $U(P,f_n)+c_n(b-a)$ and similarly for $L(P,f_n)$. Then choose n to make c_n sufficiently small, and for this n a partition making $U(P,f_n)-L(P,f_n)$ small.

Proof: Define $c_n=\sup_{[a,b]}|f(x)-f_n(x)|$. Note that $c_n\to 0$ as $n\to\infty$ and $\forall x\in[a,b]$, $f_n(x)-c_n\le f(x)\le f_n(x)+c_n$.

Now if $P=\{a_0<\ldots<a_n\}$ is a partition of $[a,b]$,

$$U(P,f)=\sum_{j=0}^{n-1}(a_{j+1}-a_j)\sup_{[a_j,a_{j+1}]}f \le \sum_{j=0}^{n-1}(a_{j+1}-a_j)\left(\sup_{[a_j,a_{j+1}]}f_n+c_n\right)$$
$$\le U(P,f_n)+c_n(b-a)$$

Similarly, $L(P,f)\ge L(P,f_n)-c_n(b-a)$.

Given $\varepsilon>0$, choose n s.t. $2(b-a)c_n<\varepsilon/2$ and then choose P s.t. $U(P,f_n)-L(P,f_n)<\varepsilon/2$. Then

$$U(P,f)-L(P,f)\le U(P,f_n)-L(P,f_n)+2(b-a)c_n$$
$$<\varepsilon/2+\varepsilon/2=\varepsilon \quad \text{so } f \text{ is integrable.}$$

We showed $\int f_n\to\int f$ before. $\square$

Prop: If $f:[a,b]\to\mathbb{R}^n$ is integrable, then $\|f(x)\|$ is integrable and $\left\|\int_a^b f(x)\,dx\right\|\le\int_a^b\|f(x)\|\,dx$.

Proof: use result that composition of integrable function with continuous function is integrable; for inequality apply Cauchy-Schwarz to $\|v\|^2=\|\int_a^b f(x)\,dx\|^2$.

Proof: Let $f=(f_1,\ldots,f_n)$ so $\|f(x)\|=\left(\sum_{j=1}^n(f_j(x))^2\right)^{1/2}$. Since $(\ )^2$ and $(\ )^{1/2}$ are continuous and (finite) sums of integrable functions are integrable, from an earlier result on composition of functions, $\|f(x)\|$ is integrable.

Define $v_j=\int_a^b f_j(x)\,dx$ so that $v=(v_1,\ldots,v_n)=\int_a^b f(x)\,dx$. For $v\neq 0$ (which is trivial), have $\|v\|=\sum_{j=1}^n v_jv_j=\sum_{j=1}^n v_j\int_a^b f_j(x)\,dx=\int_a^b\sum_{j=1}^n v_jf_j(x)\,dx$

$$\le\int_a^b\|v\|\|f(x)\|\,dx \quad \text{by Cauchy-Schwarz.}$$

Result follows on division by $\|v\|$. $\square$

Analysis II Proofs – $\mathbb{R}^n$ as a normed space

Prop: Let $x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)}) \in \mathbb{R}^n$ for $k = 1, 2, \dots$. Then (wrt $\|\cdot\|_2$ norm), $x^{(k)} \to x = (x_1, \dots, x_n) \in \mathbb{R}^n$ iff $x_j^{(k)} \to x_j$ as $k \to \infty$ for each $1 \leq j \leq n$.

Proof: $\|x^{(k)} - x\|_2 = \sum_{j=1}^n (x_j^{(k)} - x_j)^2 \to 0$ iff $x_j^{(k)} - x_j \to 0$ $\forall 1 \leq j \leq n$. □

Prop: Suppose $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms on a vector space V. Then (i) A set $E \subseteq V$ is bounded wrt $\|\cdot\|$ iff it is bounded wrt $\|\cdot\|'$.

(ii) $x_k \to x$ wrt $\|\cdot\|$ iff $x_k \to x$ wrt $\|\cdot\|'$

Proof: direct from definitions

Proof: Let $a < a < b$ be such that $a\|v\| \leq \|v\|' \leq b\|v\|$ for all $v \in V$.

(i) If $\|v\| < R$ $\forall v \in E$ then $\|v\|' < bR$ $\forall v \in E$. Conversely, if $\|v\|' < R'$ $\forall v \in E$ then $\|v\| < R/a$ $\forall v \in E$.

(ii) We have $a\|x_k - x\| \leq \|x_k - x\|' \leq b\|x_k - x\|$ so it is clear that $\|x_k - x\| \to 0$ iff $\|x_k - x\|' \to 0$. □

Thm: (Bolzano-Weierstrass for $\mathbb{R}^n$) Any bounded sequence in $\mathbb{R}^n$ (with the Euclidean norm) has a convergent subsequence.

Proof: by induction: construct convergent subsequence of first $(n-1)$ components, and sub-subsequence of corresponding n^{th} component.

Proof: By induction on n. The case $n=1$ is usual Bolzano-Weierstrass in $\mathbb{R}$. Let $n \geq 2$, suppose the theorem holds in $\mathbb{R}^{n-1}$. Let $x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)}) \in \mathbb{R}^n$ be a bounded sequence, say $\|x^{(k)}\| < R$ for each $k = 1, 2, \dots$.
Then clearly if we define $y^{(k)} = (x_1^{(k)}, \dots, x_{n-1}^{(k)})$, we have $\|y^{(k)}\| < R$ for all k, so by the induction hypothesis there exists a subsequence $(y^{(k_j)})$ of $(y^{(k)})$ and $y \in \mathbb{R}^{n-1}$ s.t. $y^{(k_j)} \to y = (x_1, \dots, x_{n-1})$ in $\mathbb{R}^{n-1}$.
Also, $\{x_n^{k_j}\}_{j=1}^\infty$ is a bounded sequence of reals, so there exists a subsequence $\{x_n^{k_{j_l}}\}_{l=1}^\infty$ of $(x_n^{k_j})_{j=1}^\infty$ and $x_n \in \mathbb{R}$ s.t. $x_n^{k_{j_l}} \to x_n$.
Thus $x_i^{k_{j_l}} \to x_i$ as $l \to \infty$ for each $1 \leq i \leq n$, so since coordinate-wise convergence is equivalent to convergence in $\|\cdot\|$, we have that

$x^{(k_{j_l})} \to x = (x_1, \dots, x_n)$ as $l \to \infty$. □

Prop: Let $(V, \|\cdot\|)$ be a normed space. If $x_n \to x$ then (x_n) is Cauchy.

Proof: Let $\varepsilon > 0$. Choose N s.t. $\forall n \geq N$, $\|x_n - x\| < \varepsilon/2$. Now if $m, n \geq N$,
$\|x_n - x_m\| \leq \|x_n - x\| + \|x - x_m\| < \varepsilon$. $\square$

Prop: Any Cauchy sequence is bounded.

Proof: take $\varepsilon = 1$ in definition of Cauchy sequence.

Proof: Let $x_n \in V$ be Cauchy. Choose N s.t. $n \geq N \Rightarrow \|x_n - x_N\| < 1$.
Then $\|x_n\| \leq \|x_n - x_N\| + \|x_N\| \leq 1 + \|x_N\|$ $\forall n \geq N$. Thus (x_n) is eventually bounded, so bounded. $\square$

Thm: $\mathbb{R}^n$ (with the Euclidean norm) is complete.

Proof: use completeness of $\mathbb{R}$

Proof: Let $x^{(k)} \in \mathbb{R}^n$ be Cauchy, where $x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)})$.
Then $\forall \varepsilon > 0$, $\exists N$ s.t. $j, k \geq N \Rightarrow \|x^{(j)} - x^{(k)}\| < \varepsilon \Rightarrow |x_i^{(j)} - x_i^{(k)}| < \varepsilon$ for each $1 \leq i \leq n$, so $(x_i^{(k)})_{k=1}^{\infty}$ is a Cauchy sequence of reals. Thus $x_i^{(k)} \to x_i$ for some $x_i \in \mathbb{R}$, and so $x^{(k)} \to x = (x_1, \dots, x_n)$ since coordinate-wise convergence is equivalent to convergence in $\|\cdot\|$. $\square$

Thm: $C([a,b])$ with the norm $\|\cdot\|_\infty$ is complete.

Proof: use Cauchy criterion for uniform convergence; uniform limit of continuous functions is continuous.

Proof: If $(f_k) \in C([a,b])$ is Cauchy then (f_k) converges uniformly to some function $f: [a,b] \to \mathbb{R}$ by the Cauchy criterion for uniform convergence. Since the uniform limit of continuous functions is continuous, $f \in C([a,b])$. $\square$

Prop: Open balls are open.

Proof: Let $x \in B_r(y)$, set $\rho = r - \|x - y\|$. Now if $z \in B_\rho(x)$,
$\|z - y\| \leq \|z - x\| + \|x - y\| < \rho + \|x - y\| = r$.
Hence $B_\rho(x) \subset B_r(y)$, so $B_r(y)$ is open. $\square$

Analysis II Proofs – $\mathbb{R}^n$ as a normed space

Prop: A subset $E \subseteq V$ is closed iff its complement $V \setminus E$ is open.

Proof: from definitions, using fact that x is not a limit point of E if $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap E = \emptyset$.

Proof: $\Rightarrow$: Suppose E is closed, i.e. it contains all its limit points. Pick $x \in V \setminus E$. Since x is not a limit point of E, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap E = \emptyset$. Hence $B_\varepsilon(x) \subseteq V \setminus E$, so $V \setminus E$ is open.

$\Leftarrow$: Suppose $V \setminus E$ is open. Let $x \in V$ be a limit point of E. If $x \notin E$ then $x \in V \setminus E$ so since $V \setminus E$ is open $\exists r > 0$ s.t. $B_r(x) \subseteq V \setminus E$. But then x can't be a limit point of E, so this is a contradiction. $\square$

Thm: (i) The union of any collection of open subsets is open.
(ii) The intersection of a finite number of open subsets is open.
(iii) $\emptyset$ and V are open.

Proof: direct from definitions

Proof: (i) Let E_α, $\alpha \in A$ be open. Let $E = \bigcup_{\alpha \in A} E_\alpha$. If $x \in E$, then $x \in E_\beta$, some β. Since E_β is open, $\exists r > 0$ s.t. $B_r(x) \subseteq E_\beta \subseteq E$.

(ii) Let E_j, $j = 1, \dots, n$ be open. Let $E = \bigcap_{j=1}^n E_j$. If $x \in E$, then $x \in E_j$ $\forall 1 \leq j \leq n$, so $\exists r_j > 0$ s.t. $B_{r_j}(x) \subseteq E_j$. Choose $r = \min\{r_1, \dots, r_n\} > 0$, so that $B_r(x) \subseteq E$.

(iii) Vacuously holds from definition of open.

Cor: (i) The intersection of any collection of closed subsets is closed.
(ii) The union of a finite number of closed subsets is closed.
(iii) $\emptyset$ and V are closed.

Proof: Take complements and apply previous theorem.

Thm: Let $E \subseteq V$, $f: E \to V'$. Then f is continuous at $x \in E$ iff $f(x_n) \to f(x)$ for every sequence (x_n) in E with $x_n \to x$.

Proof: $\Rightarrow$: direct from definition, $\Leftarrow$: by contradiction.

Proof: $\Rightarrow$: Suppose f is continuous at x, let $x_n \in E$ with $x_n \to x$. Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $y \in E$, $\|y - x\| < \delta \Rightarrow \|f(y) - f(x)\|' < \varepsilon$. Also, $\exists N$ s.t. $n \geq N \Rightarrow \|x_n - x\| < \delta$, so for $n \geq N$, $\|f(x_n) - f(x)\|' < \varepsilon$. Thus $f(x_n) \to f(x)$.

$\Leftarrow$: Suppose f is not continuous at x. Then $\exists \varepsilon > 0$ s.t. for every $n \in \mathbb{N}$ we can construct $x_n \in E$ with $\|x_n - x\| < \frac{1}{n}$ but $\|f(x_n) - f(x)\|' \geq \varepsilon$. This gives a sequence $(x_n) \in E$ with $x_n \to x$ but $f(x_n) \not\to f(x)$, a contradiction. $\square$

Prop: Let $(V, \|\cdot\|)$ be a normed space. Then $\|\cdot\| : V \to \mathbb{R}$ is Lipschitz continuous.

Proof: apply the triangle inequality twice, swapping x and y.

Proof: Let $x, y \in V$. By the triangle inequality,

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\| \Rightarrow \|x\| - \|y\| \leq \|x - y\|.$$

Reversing x and y above gives $\|y\| - \|x\| \leq \|x - y\|$. Hence $\big|\|x\| - \|y\|\big| \leq \|x - y\|$. (This is the reverse triangle inequality.) $\square$

Lemma: (i) If $(V, \|\cdot\|)$ is a normed space and $E \subseteq V$ is sequentially compact then E is closed and bounded.

(ii) A closed, bounded subset $E \subseteq \mathbb{R}^n$ is sequentially compact.

Proof: (i) Bounded: by contradiction. Closed: direct from definitions.

(ii) apply Bolzano-Weierstrass.

Proof: (i) Bounded: suppose otherwise, then $\exists x_n \in E$ with $\|x_n\| \to \infty$. Since E is sequentially compact, (x_n) has a convergent subsequence, a contradiction.

Closed: let $x \in V$ be a limit point of E, so $\exists (x_n) \in E$ with $x_n \to x$. Since E is sequentially compact, (x_n) has a subsequence (x_{n_j}) converging to a point in E. But (x_{n_j}) must converge to x, so $x \in E$.

(ii) Let $(x_n) \in E$. Since E is bounded, by the Bolzano-Weierstrass theorem, there is a convergent subsequence (x_{n_j}) with say $x_{n_j} \to x \in \mathbb{R}^n$. But since E is closed, $x \in E$ (either $x = x_{n_j}$, some j or x is a limit point.) $\square$

Analysis II Proofs – $\mathbb{R}^n$ as a normed space

Thm: Let $(V, \|\cdot\|)$ be a normed space and $K \subseteq \mathbb{R}^n$ closed and bounded. Suppose $f: K \to V$ is continuous (where $\mathbb{R}^n$ has Euclidean norm).
Then (i) f is uniformly continuous
(ii) $f(K)$ is sequentially compact
(iii) $f(K)$ is closed and bounded.

Proof: (i) same as for $f:[a,b] \to \mathbb{R}$ (by contradiction: suppose otherwise, construct $x_n, y_n \in K$ with $\|x_n - y_n\|_2 < 1/n$ but $\|f(x_n) - f(y_n)\| \geq \varepsilon$, apply Bolzano-Weierstrass
(ii) trace back an arbitrary sequence $y_n \in f(K)$, apply lemma to K giving that it is sequentially compact, so continuity gives the limit of (y_n).
(iii) follows from lemma (sequentially compact $\Rightarrow$ closed and bounded.
(ii) Let (y_n) be a sequence in $f(K)$. Then $\exists x_n \in K$ s.t. $f(x_n) = y_n$. By the lemma, K is sequentially compact, so there exists a subsequence (x_{n_j}) and $x \in K$ s.t. $x_{n_j} \to x$ as $j \to \infty$. Since f is continuous, $f(x_{n_j}) \to f(x) \in f(K)$. Thus y_n converges to a point in $f(K)$, i.e. $f(K)$ is sequentially compact. □

Cor: If $K \subseteq \mathbb{R}^n$ is closed and bounded and $f: K \to \mathbb{R}$ continuous then f attains its sup and inf.
Proof: Since $f(K)$ is bounded, either $\sup f(K) \in f(K)$ or $\sup f(K)$ is a limit point of $f(K)$. Since $f(K)$ is closed, we must have $f(K)$ contains $\sup f(K)$. Similarly for inf (or consider $-f$). □

Thm: Any two norms on $\mathbb{R}^n$ are Lipschitz equivalent.
Proof: show any norm is equivalent to Euclidean norm, apply transitivity. Using triangle inequality and Cauchy-Schwarz, show $\|x\| \leq b\|x\|_2$ so that the map $\|\cdot\|: (\mathbb{R}^n, \|\cdot\|_2) \to \mathbb{R}$ is Lipschitz (so continuous) as $|\|x\| - \|y\|| \leq \|x-y\| \leq b\|x-y\|_2$.
The unit sphere under $\|\cdot\|_2$ is closed and bounded, so its image under the map $\|\cdot\|$ has an infimum, a. If $x \in \mathbb{R}^n$, $x/\|x\|_2$ is on unit sphere, so $a \leq \| \frac{x}{\|x\|_2} \|$

Proof: Let $\{e_1, \dots, e_n\}$ be a basis for $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, write $x = \sum_{j=1}^n x_j e_j$.

Then $\|x\| = \|\sum_{j=1}^n x_j e_j\| \leq \sum_{j=1}^n |x_j| \|e_j\|$ by triangle inequality

$\leq (\sum_{j=1}^n |x_j|^2)^{1/2} (\sum_{j=1}^n \|e_j\|^2)^{1/2}$ by Cauchy-Schwarz.

Thus $\|x\| \leq B\|x\|_2$ for constant B. Also, $|\|x\| - \|y\|| \leq \|x-y\| \leq B\|x-y\|_2$, and so the map $\|\cdot\| : (\mathbb{R}^n, \|\cdot\|_2) \to \mathbb{R}$ is Lipschitz and so continuous.

Let $S = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ be the unit sphere, which is clearly bounded, and is closed since the map $\|\cdot\|_2$ is continuous so preserves the unit modulus.

Thus by the corollary, the image of S under the map $\|\cdot\|$ is bounded with say $A = \inf_{x \in S} \|x\|$. Note that $A > 0$ since the infimum is attached and $0 \notin S$. If $x \in \mathbb{R}^n$, then $x/\|x\|_2 \in S$, so $A \leq \| x/\|x\|_2 \|$, thus we have $A\|x\|_2 \leq \|x\| \leq B\|x\|_2$, so $\|\cdot\|$ and $\|\cdot\|_2$ are Lipschitz equivalent.

By transitivity of Lipschitz equivalence, all norms on $\mathbb{R}^n$ are equivalent. $\square$

Cor: Any two norms on a finite dimensional vector space are Lipschitz equivalent.

Proof: biject the vector space with $\mathbb{R}^n$ via the coordinates at some basis, define equivalent norms on $\mathbb{R}^n$ and apply theorem.

Proof: Let V be a vector space with basis $\{v_1, \dots, v_n\}$. Let $\|\cdot\|_\alpha, \|\cdot\|_\beta$ be norms on V. Consider the bijection $\phi : \mathbb{R}^n \to V$ given by $(x_1, \dots, x_n) \mapsto \sum_{j=1}^n x_j v_j$.

Define corresponding norms $\|(x_1, \dots, x_n)\|'_\alpha \equiv \|\sum_{j=1}^n x_j v_j\|_\alpha$ and similarly $\|(x_1, \dots, x_n)\|'_\beta \equiv \|\sum_{j=1}^n x_j v_j\|_\beta$ (one should check that these are norms!)

Applying the theorem gives $a\|x\|'_\alpha \leq \|x\|'_\beta \leq b\|x\|'_\alpha$ which corresponds to $a\|v\|_\alpha \leq \|v\|_\beta \leq b\|v\|_\alpha$ in V. $\square$

Analysis II Proofs - Metric spaces

Thm: (i) The union of any collection of open sets is open.
(ii) The intersection of a finite collection of open sets is open.
(iii) The empty set $\emptyset$ and the whole set X are open.
Proof: same as for normed spaces

Thm: A subset $E \subseteq X$ is closed iff its complement $X \setminus E$ is open.
Proof: same as for normed spaces

Thm: (i) The intersection of any collection of closed sets is closed
(ii) The union of a finite collection of closed sets is closed.
(iii) The empty set $\emptyset$ and the whole set X are closed.
Proof: same as for normed spaces.

Prop: A singleton $\{x\} \subseteq X$ is a closed subset.
Proof: there exists an open ball around any other point not containing x.
Proof: Let $y \in X \setminus \{x\}$, set $r = d(x,y) > 0$. Then $B_r(y) \subseteq X \setminus \{x\}$. Thus $X \setminus \{x\}$ is open, so $\{x\}$ is closed. $\square$

Cor: Any finite set is closed.
Proof: take the union of finitely many singletons.

Prop: The sequence $x_n \to x$ iff for any neighbourhood V of x in X, $x_n \in V$ for all but finitely many n.
Proof: direct from definitions (for $\Leftarrow$ note that $B_\varepsilon(x)$ is a neighbourhood of x).
Proof: $\Rightarrow$: Suppose $x_n \to x$, let V be a neighbourhood of x. Then $\exists r > 0$ s.t. $B_r(x) \subseteq V$. Also, $\exists N$ s.t. $n \geq N \Rightarrow d(x_n, x) < r \Rightarrow x_n \in B_r(x) \subseteq V$.
$\Leftarrow$: Note that $B_\varepsilon(x)$ is a neighbourhood of x for any $\varepsilon > 0$. Hence for each $\varepsilon > 0$, $x_n \in B_\varepsilon(x)$ for all but finitely many n, i.e. for all $n \geq N$ for some N. $\square$

Thm: Let (X,d) and (X',d') be metric spaces and $f: X \to X'$ a mapping. The following are equivalent:

(i) f is continuous

(ii) If $x_n, x \in X$ with $x_n \to x$, then $f(x_n) \to f(x)$.

(iii) For any open set V in X', $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is open in X.

(iv) For any closed set F in X', $f^{-1}(F)$ is closed in X.

Proof: (i)$\Leftrightarrow$(ii) is the same as for normed spaces.

(iii)$\Leftrightarrow$(iv) since $f^{-1}(X' \setminus E) = X \setminus f^{-1}(E)$ so can take complements.

(i)$\Leftrightarrow$(iii) can be proven by noting that f is continuous at a point $x \in X$ iff $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

Proof: (i)$\Rightarrow$(iii): Suppose f is continuous, V open in X' and $x \in f^{-1}(V)$. Since $f(x) \in V$, which is open, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(f(x)) \subseteq V$. Then by the continuity of f at x, $\exists \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq V$. Hence $B_\delta(x) \subseteq f^{-1}(V)$, i.e. there is an open ball around every point in $f^{-1}(V)$.

(iii)$\Rightarrow$(i): Let $x \in X$ and $\varepsilon > 0$. Set $V = B_\varepsilon(f(x))$. Then V is open in X' and hence $f^{-1}(V)$ is open in X. But $x \in f^{-1}(V)$ by construction, so $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(V) \Rightarrow f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. $\square$

Cor: Let (X,d), (X',d') and (X'',d'') be metric spaces. Suppose $f: X \to X'$ and $g: X' \to X''$ are continuous. Then $g \circ f: X \to X''$ is continuous.

Proof: note that $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ and apply theorem.

Proof: Let V be open in X''. Then $g^{-1}(V)$ is open in X', so $f^{-1}(g^{-1}(V))$ is open in X. $\square$

Thm: Let (X,d) and (X',d') be metric spaces. If $f: X \to X'$ is uniformly continuous and (x_n) is Cauchy in X then $(f(x_n))$ is Cauchy in X'.

Proof: direct from definitions.

Proof: Since f is continuous, given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $x, y \in X$, $d(x,y) < \delta$ $\Rightarrow d'(f(x), f(y)) < \varepsilon$. Since (x_n) is Cauchy, $\exists N$ s.t. $n, m \geq N \Rightarrow d(x_n, x_m) < \delta$ $\Rightarrow d'(f(x_n), f(x_m)) < \varepsilon$, i.e. $(f(x_n))$ is Cauchy. $\square$

Analysis II Proofs – Metric spaces

Prop: If (x_n) is convergent in X then (x_n) is Cauchy.

Proof: as for $\mathbb{R}$, straight from definitions

Proof: Suppose $x_n \to x$, let $\varepsilon > 0$. Then $\exists N$ s.t. $n \geq N \Rightarrow d(x_n, x) < \varepsilon/2$.
Now for $n, m \geq N$, $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \varepsilon$. □

Prop: If (x_n) is Cauchy and has a subsequence (x_{n_j}) converging to $x \in X$ then (x_n) converges to x.

Proof: direct from definitions

Proof: Let $\varepsilon > 0$. Choose N s.t. $n, m \geq N \Rightarrow d(x_n, x_m) < \varepsilon/2$. Choose j_0 with $n_{j_0} \geq N$ s.t. $d(x_{n_{j_0}}, x) < \varepsilon/2$. Now, for $n \geq N$,
$d(x_n, x) \leq d(x_{n_{j_0}}, x) + d(x_{n_{j_0}}, x_n) < \varepsilon$. □

Prop: Let (X, d) be a metric space and $Y \subseteq X$.
(i) If $(Y, d|_{Y \times Y})$ is complete then Y is closed in X.
(ii) If (X, d) is complete, then $(Y, d|_{Y\times Y})$ is complete iff Y is closed in X.

Proof: direct from definitions

Proof: (i) Let $x \in X$ be a limit point of Y, so $\exists x_n \in Y$ with $x_n \to x$. Then (x_n) is Cauchy in X and hence in Y (with the subspace metric). Since $(Y, d|_{Y\times Y})$ is complete the limit x is in Y, so Y is closed.
(ii) $\Rightarrow$: see above

$\Leftarrow$: Let (x_n) be Cauchy in $(Y, d|_{Y\times Y})$, so (x_n) is Cauchy in X. Since X is complete, $\exists x \in X$ s.t. $x_n \to x$. But x is a limit point of Y and Y is closed, so $x \in Y$. □

Prop: If (X, d) is sequentially compact then X is complete and bounded.

Proof: complete: by definitions, earlier Prop.; bounded: by contradiction.

Proof: Let (x_n) be Cauchy in X. By sequential compactness, (x_n) has a convergent subsequence. By earlier proposition, (x_n) converges to the same value.
Suppose X is not bounded. Then $\exists x_0 \in X$ and $x_n \in X$ s.t. $d(x_n, x_0) \geq n$ $(n = 1, 2, \dots)$.
Then for $n > m$, $d(x_n, x_m) \geq d(x_n, x_0) - d(x_m, x_0) \geq n - m \geq 1$, so (x_n) cannot have a convergent subsequence. □

Analysis II Proofs - The Contraction Mapping Theorem

Thm: (Contraction mapping theorem) If f is a contraction mapping on a complete metric space X then f has a unique fixed point in X i.e. $\exists!\, x \in X$ with $f(x) = x$.

Proof: Uniqueness: contradiction; Existence: pick $x_0 \in X$, construct $x_{n+1} = f(x_n)$ - show this is Cauchy and apply conditions in theorem.

Proof: Uniqueness: Suppose $f(x) = x$ and $f(y) = y$. Then

$d(x,y) = d(f(x), f(y)) \leq \lambda d(x,y)$. Since $\lambda < 0$, must have $d(x,y) = 0$.

Existence: Pick $x_0 \in X$ and recursively define $x_{n+1} = f(x_n)$ $(n = 0, 1, 2, \ldots)$.

Now $d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \lambda d(x_n, x_{n-1}) \leq \ldots \leq \lambda^n d(x_1, x_0)$.

Then if $m > n$ by the triangle inequality we have

$$d(x_m, x_n) \leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j) \leq d(x_1, x_0) \sum_{j=n}^{m-1} \lambda^j \leq d(x_1, x_0) \sum_{j=n}^{\infty} \lambda^j$$
$$= \frac{\lambda^n}{1-\lambda} d(x_1, x_0) \to 0 \text{ as } n \to \infty.$$

Hence (x_n) is Cauchy. Since X is complete, $\exists x \in X$ s.t. $x_n \to x$. Since f is continuous, $f(x_n) \to f(x)$. But $f(x_n) = x_{n+1} \to x$. Since limits are unique in metric spaces, $f(x) = x$. □

Cor: Let $f: X \to X$ be a mapping on a complete metric space X. Suppose $f^{(m)} \equiv f \circ f \circ \ldots \circ f : X \to X$ is a contraction. Then f has a unique fixed point.

Proof: if x is a fixed point of $f^{(m)}$, so is $f(x)$ - by uniqueness $f(x) = x$.

Proof: By the contraction mapping theorem $f^{(m)}$ has a unique fixed point $x \in X$. So $f^{(m)}(x) = x \Rightarrow f^{(m+1)}(x) = f(x) \Rightarrow f^{(m)}(f(x)) = f(x)$, i.e. $f(x)$ is a fixed point of $f^{(m)}$ and so by uniqueness, $f(x) = x$. This fixed point of f is unique since any fixed point of f is also a fixed point of $f^{(m)}$. □

Analysis II Proofs – Differentiation from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Prop: (derivative well-defined): Let $U \subseteq \mathbb{R}^n$ be open, $a \in U$, $f: U \to \mathbb{R}^m$. Suppose there exist linear maps A_1, A_2 s.t. $\lim_{h \to 0} \frac{f(a+h) - f(a) - A_i(h)}{\|h\|} = 0$ for $i = 1, 2$. Then $A_1 = A_2$.

Proof: Write $B = A_1 - A_2$, show $\lim_{h \to 0} \frac{\|Bh\|}{\|h\|} \to 0$ by triangle inequality. Fix $h \neq 0$, consider $\frac{B(th)}{\|th\|}$ as $t \to 0$, use linearity of B.

Proof: Let $B = A_1 - A_2$, then $\|Bh\| \leq \|A_1 - (f(a+h) - f(a))\| + \|f(a+h) - f(a) - A_2 H\|$ so $\lim_{h \to 0} \frac{\|Bh\|}{\|h\|} = 0$. Fix any $h \neq 0$. Then $\frac{B(th)}{\|th\|} \to 0$ as $t \to 0$, but for $t > 0$ $\frac{B(th)}{\|th\|} = \frac{B(h)}{\|h\|}$ by linearity of B, and this is independent of t, so $Bh = 0$. Since h was arbitrary, must have $B \equiv 0$, i.e. $A_1 = A_2$. $\square$

Prop: If $U \subseteq \mathbb{R}^n$ is open and $f: U \to \mathbb{R}^m$ is differentiable at $a \in U$, then f is continuous at a.

Proof: Let $A = Df(a)$, define $r(h) = f(a+h) - f(a) - Ah$, so that $\frac{\|r(h)\|}{\|h\|} \to 0$ as $h \to 0$.

Proof: Let $A = Df(a)$, define $r(h) = f(a+h) - f(a) - Ah$ so that $\frac{\|r(h)\|}{\|h\|} \to 0$ as $h \to 0$. Then

$$\|f(a+h) - f(a)\| \leq \|Ah + r(h)\|$$
$$\leq \|h\| \left(\frac{\|Ah\|}{\|h\|} + \frac{\|r(h)\|}{\|h\|} \right) \to 0 \quad \text{as } h \to 0$$

$\therefore$ f is continuous at a. $\square$

Prop: If $U \subseteq \mathbb{R}^n$ is open and $f = (f_1, \dots, f_m): U \to \mathbb{R}^m$ then f is differentiable at $a \in U$ iff $f_j: U \to \mathbb{R}$ is differentiable at a for $j = 1, \dots, m$. Moreover, writing $Df(a)$ as an $m \times n$ real matrix, $Df_j(a)$ is the linear map $\mathbb{R}^n \to \mathbb{R}$ corresponding to the j^{th} row of $Df(a)$.

Proof: f is differentiable at a with $A = Df(a)$

$$\Leftrightarrow \lim_{h \to 0} \frac{1}{\|h\|} (f(a+h) - f(a) - Ah) = 0$$

$$\Leftrightarrow \lim_{h \to 0} \left(\frac{1}{\|h\|} (f_1(a+h) - f_1(a) - A_1 h), \dots, \frac{1}{\|h\|} (f_m(a+h) - f_m(a) - A_m h) \right) = 0$$

$$\Leftrightarrow \lim_{h \to 0} \frac{1}{\|h\|} (f_j(a+h) - f_j(a) - A_j h) = 0 \text{ for each } j = 1, \dots, m.$$

$\Leftrightarrow$ f_j is differentiable at a with $A_j = Df_j(a)$ for each $j = 1, \dots, m$.

Prop: If $U \subseteq \mathbb{R}^n$ is open, $f_1, f_2 : U \to \mathbb{R}^m$ are differentiable at $a \in U$ and $c_1, c_2 \in \mathbb{R}$ are constants, then $c_1 f_1 + c_2 f_2 : U \to \mathbb{R}^m$ is differentiable at a with $D(c_1 f_1 + c_2 f_2)(a) = c_1 Df_1(a) + c_2 Df_2(a)$.

Proof: $\frac{1}{\|h\|}\left((c_1f_1+c_2f_2)(a+h) - (c_1f_1+c_2f_2)(a) - (c_1Df_1(a)+c_2Df_2(a))h\right)$
$= \frac{c_1}{\|h\|}(f_1(a+h) - f_1(a) - Df_1(a)h) + \frac{c_2}{\|h\|}(f_2(a+h) - f_2(a) - Df_2(a)h) \to 0$ as $h \to 0$ □

Prop: If $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then A is differentiable at every point $a \in \mathbb{R}^n$ with $DA(a) = A$.

Proof: $\frac{1}{\|h\|}(A(a+h) - A(a) - A(h)) = \frac{1}{\|h\|}(A(a) + A(h) - A(a) - A(h)) = 0$ by linearity. □

Thm: (The Chain Rule). Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open, $f : U \to \mathbb{R}^m$ and $g : V \to \mathbb{R}^p$. Let $a \in U$ with $f(a) \in V$. Suppose that f is differentiable at a and g is differentiable at $f(a)$. Then $g \circ f : U \to \mathbb{R}^p$ is differentiable at a with $D(g\circ f)(a) = Dg(f(a)) \circ Df(a)$.

Proof: Use remainder functions $r(h) = f(a+h) - f(a) - Ah$, $s(k) = g(f(a)+k) - g(f(a)) - Bk$ and appropriate bounds.

Proof: Let $A = Df(a)$ and $B = Dg(f(a))$. Define $r(h) = f(a+h) - f(a) - Ah$ and $s(k) = g(f(a)+k) - g(f(a)) - Bk$, so that $\frac{r(h)}{\|h\|} \to 0$ as $h \to 0$, $\frac{s(k)}{\|k\|} \to 0$ as $k \to 0$.

Now, $(g\circ f)(a+h) - (g\circ f)(a) - BAh$
$$= g(f(a) + Ah + r(h)) - g(f(a)) - BAh \qquad (\text{and taking } k = Ah + r(h))$$
$$= g(f(a)) + B(Ah + r(h)) + s(Ah + r(h)) - g(f(a)) - BAh$$
$$= Br(h) + s(Ah + r(h)) \qquad \text{since } B \text{ is linear}$$

Note that $\frac{\|Br(h)\|}{\|h\|} \leq \|B\| \frac{\|r(h)\|}{\|h\|} \to 0$ as $h \to 0$

and $\|Ah + r(h)\| \leq \left(\|A\| + \frac{\|r(h)\|}{\|h\|}\right)\|h\|$ so $h \to 0 \Rightarrow Ah + r(h) \to 0$

and $\frac{\|s(Ah + r(h))\|}{\|h\|} \leq \left(\|A\| + \frac{\|r(h)\|}{\|h\|}\right) \frac{\|s(Ah + r(h))\|}{\|Ah + r(h)\|} \to 0$ as $h \to 0$.

Hence $\frac{1}{\|h\|}((g\circ f)(a+h) - (g\circ f)(a) - BAh) \to 0$ as $h \to 0$, i.e. $g\circ f$ is differentiable at a with derivative BA. □

Analysis II Proofs – Differentiation from $\mathbb{R}^m$ to $\mathbb{R}^n$

Prop: Let $A, B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and let $\|\cdot\|$ be the operator norm.

(i) $\|A\| < \infty$

(ii) $\|A + B\| \leq \|A\| + \|B\|$

(iii) $\|A\| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Ax\|}{\|x\|}$

(iv) $\|Ax\| \leq \|A\| \|x\|$

(v) $\|BA\| \leq \|B\| \|A\|$

Proof: (i) Since ~~$\|Ax - Ay\| \leq \|A\|$~~ $x \mapsto \|Ax\|$ is continuous and $\{x \in \mathbb{R}^n : \|x\| = 1\}$ is closed and bounded, $\|A\|$ is bounded.

(ii) $\|A+B\| = \sup_{\|x\|=1} \|(A+B)x\| = \sup_{\|x\|=1} \|Ax + Bx\| \leq \sup_{\|x\|=1} (\|Ax\| + \|Bx\|)$
$\leq \sup_{\|x\|=1} \|Ax\| + \sup_{\|x\|=1} \|Bx\| = \|A\| + \|B\|$.

(iii) If $x \in \mathbb{R}^n \setminus \{0\}$ then $x = \lambda y$ with $\lambda > 0$ and $\|y\| = 1$, so $\frac{\|Ax\|}{\|x\|} = \|Ay\|$.

(iv) Write $x = \lambda y$ as above, then $\|Ax\| = \|A\lambda y\| = \lambda \|Ay\| \leq \lambda \|A\| = \|A\| \|x\|$.

(v). $\|BAx\| \leq \|B\| \|Ax\| \leq \|B\| \|A\| \|x\|$, so $\sup_{\|x\|=1} \|BAx\| \leq \|B\| \|A\|$. $\square$

Prop: Let $U \subseteq \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^m$ be differentiable at $a \in U$. Then for any $u \in \mathbb{R}^n$, $D_u f(a)$ exists and is given by $D_u f(a) = Df(a)u$.

Proof: define $\phi(t) = a + tu$ and $g(t) = f(\phi(t))$, apply chain rule to g at 0.

Proof: Define $\phi(t) = a + tu$ and $g(t) = f(\phi(t)) = f(a + tu)$ (where $g: (-\delta, \delta) \to \mathbb{R}^m$ for δ ~~sufficiently~~ small.) By the chain rule,

$$g'(0) = Df(\phi(0))\phi'(0) = Df(a)u$$

but also, $g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = D_u f(a)$. $\square$

Prop: Let $U \subseteq \mathbb{R}^n$ be open and $f = (f_1, \dots, f_m): U \to \mathbb{R}^m$. If f is differentiable at $a \in U$ then the partial derivatives $D_j f_i(a)$ exist for $1 \leq i \leq m$, $1 \leq j \leq n$ and the matrix for $Df(a)$ w.r.t. the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ is $\left(\frac{\partial f_i(a)}{\partial x_j}\right)$.

Proof: By an earlier proposition, $f_i: U \to \mathbb{R}$ is differentiable at a for $1 \leq i \leq m$ and $Df(a)$ is the matrix $\begin{pmatrix} Df_1(a) \\ \vdots \\ Df_m(a) \end{pmatrix}$. By the above proposition,

$\frac{\partial f_i}{\partial x_j}(a) = Df_i(a)e_j$ which is the entry in the i^{th} row and j^{th} column of $Df(a)$. $\square$

Thm: Let $U \subseteq \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^m$. Suppose there exists an open ball $B_r(a) \subset U$ s.t. (i) $D_j f_i(x)$ exists $\forall x \in B_r(a)$ for $1 \leq i \leq m$, $1 \leq j \leq n$.
(ii) $D_j f_i$ are continuous at a for $1 \leq i \leq m$, $1 \leq j \leq n$.
Then f is differentiable at a with $Df(a) = (D_j f_i(a))$

Proof: Suffices to prove for $m=1$. For $h=(h_1, \dots, h_n)$ with $\|h\| < r$ define $h^{(j)} = \sum_{i=1}^{j} h_i e_i$, $h^{(0)} = 0$, so $f(a+h) - f(a) = \sum_{j=1}^{n} f(a+h^{(j)}) - f(a+h^{(j-1)})$ and bound each term by MVT

Proof: Suffices to prove for $m=1$ since $f = (f_1, \dots, f_m)$ differentiable iff each f_i differentiable. For $h = (h_1, \dots, h_n)$ with $\|h\| < r$, define $h^{(j)} = \sum_{i=1}^{j} h_i e_i$ $(1 \leq j \leq n)$ and $h^{(0)} = 0$. Then

$f(a+h) - f(a) = \sum_{j=1}^{n} f(a+h^{(j)}) - f(a+h^{(j-1)})$ as this is a telescoping series.

Now, $f(a+h^{(j)}) - f(a+h^{(j-1)}) = f(a+h^{(j-1)} + h_j e_j) - f(a+h^{(j-1)})$

$= h_j D_j f(a+h^{(j-1)} + \theta_j h_j e_j)$ for some $\theta_j \in (0,1)$

by the mean value theorem (thinking of $f(a+h^{(j-1)} + te_j)$ for $t \in (0, h_j)$.)

Then writing $Df(a)h$ for $\sum_{j=1}^{n} D_j f(a) h_j$ and noting that $|h_j| \leq \|h\|$,

$|f(a+h) - f(a) - Df(a)h| \leq \|h\| \sum_{j=1}^{n} |D_j f(a+h^{(j-1)} + \theta_j h_j e_j) - D_j f(a)|$

Given $\varepsilon > 0$, choose $\delta > 0$ s.t. $|D_j f(a+k) - D_j f(a)| < \varepsilon/n$ whenever $\|k\| < \delta$

and note that $\|h\| < \delta \Rightarrow \|h^{(j-1)} + \theta_j h_j e_j\| = \sqrt{(\sum_{i=1}^{j-1} h_i^2) + \theta_j^2 h_j^2} < \delta$, so

for $\|h\| < \delta$, $|f(a+h) - f(a) - Df(a)h| < \|h\| \varepsilon$

$\Rightarrow \frac{1}{\|h\|} |f(a+h) - f(a) - Df(a)| < \varepsilon$, i.e. f is differentiable at a. $\square$

Cor: Let $U \subseteq \mathbb{R}^n$ be open. Then $f: U \to \mathbb{R}^m$ is continuously differentiable on U iff $D_j f_i(x)$ exist for each x and $D_j f_i$ are continuous on U $(1 \leq i \leq m, 1 \leq j \leq n)$.

Proof: $\Rightarrow$ follows from earlier proposition and linear maps being continuous.
$\Leftarrow$ follows from above. $\square$

Thm: Let $f: [a,b] \to \mathbb{R}^m$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose there exists $M \geq 0$ s.t. $\|f'(t)\| \leq M$ for all $t \in (a,b)$. Then $\|f(b) - f(a)\| \leq M(b-a)$.

Proof: Let $v = f(b) - f(a)$, $g(t) = v \cdot f(t)$, consider $|g'(t)|$, apply Cauchy-Schwarz and use 1D MVT

Proof: Let $v = f(b) - f(a)$ and $g: [a,b] \to \mathbb{R}$ given by $g(t) = v \cdot f(t) = \sum_{j=1}^{m} v_j f_j(t)$.

Then $|g'(t)| = |\sum_{j=1}^{m} v_j f_j'(t)| = |v \cdot f'(t)| \leq \|v\| \|f'(t)\|$ by Cauchy-Schwarz

$\leq M \|f(b) - f(a)\|$. Also, $g(b) - g(a) = v \cdot (f(b) - f(a)) = \|f(b) - f(a)\|^2$

and $g(b) - g(a) = g'(t)(b-a)$, some $t \in (a,b)$ by MVT. Hence

$\|f(b) - f(a)\|^2 = |g'(t)|(b-a) \leq (b-a) M \|f(b) - f(a)\|$, result follows. $\square$

Analysis II Proofs – Differentiation from $\mathbb{R}^n \to \mathbb{R}^n$

Thm: Let $f: B_r(a) \to \mathbb{R}^m$ be differentiable at every point in $B_r(a)$. Suppose there exists $M \geq 0$ s.t. $\|Df(x)\| \leq M$ for all $x \in B_r(a)$. If $b \in B_r(a)$ then $\|f(b) - f(a)\| \leq M\|b-a\|$.

Proof: define $\gamma(t) = (1-t)a + tb$ $(0 \leq t \leq 1)$ and $g(t) = f(\gamma(t))$, apply chain rule and then previous theorem.

Proof: Define $\gamma: [0,1] \to \mathbb{R}^n$ by $\gamma(t) = (1-t)a + tb$ (which lies in $B_r(a)$). Let $g(t) = f(\gamma(t))$ which by the chain rule is differentiable with $g'(t) = Df(\gamma(t))\,\gamma'(t)$, so $\|g'(t)\| \leq \|Df(\gamma(t))\|\|\gamma'(t)\| \leq M\|b-a\|$ using properties of operator norms. Now applying previous theorem to g,
$\|f(b) - f(a)\| = \|g(1) - g(0)\| \leq M\|b-a\|(1-0) = M\|b-a\|$. □

Cor: If $f: B_r(a) \to \mathbb{R}^m$ is differentiable with $Df(x) = 0$ for all $x \in B_r(a)$ then f is constant.

Proof: by previous theorem with $M=0$, $\|f(b) - f(a)\| = 0$ for all $b \in B_r(a)$. □

Thm: Let $U \subseteq \mathbb{R}^n$ be open and path connected. If $f: U \to \mathbb{R}^m$ is differentiable with $Df \equiv 0$ on U then f is constant on U.

Proof: Let (wlog $m=1$) γ be a path connecting arbitrary $x, y \in U$, apply Cor. to an open ball around an arbitrary point on γ, so $f(\gamma(t))$ is constant locally hence has zero derivative and can apply 1D MVT.

Proof: Suffices to prove for $m=1$ since $f = (f_1, \ldots, f_m)$ constant iff each f_i constant. Let $x, y \in U$ and $\gamma: [0,1] \to U$ such that $\gamma(0) = x$, $\gamma(1) = y$. Define $g: [0,1] \to \mathbb{R}$ by $g(t) = f(\gamma(t))$, and let $s \in (0,1)$. Since U is open, $\exists \varepsilon > 0$ st. $B_\varepsilon(\gamma(s)) \subseteq U$, and by the above corollary, f is constant on $B_\varepsilon(\gamma(s))$. Since γ is continuous, $\exists \delta > 0$ s.t. $(s-\delta, s+\delta) \subset (0,1)$ and $\gamma(s-\delta, s+\delta) \subset B_\varepsilon(\gamma(s))$. Then g is constant on $(s-\delta, s+\delta)$, so g is differentiable at s with $g'(s) = 0$. Since s is arbitrary, $g'(t) = 0$ on $(0,1)$, hence continuous on $[0,1]$. Then by the (standard) MVT, $g(1) - g(0) = g'(t)(1-0) = 0$, so $f(x) = f(y)$. □

Thm: (Inverse function theorem): Let $U \subseteq \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^n$ continuously differentiable. Let $a \in U$ be s.t. $Df(a)$ is invertible (as a linear map). Then there exist open sets V, W s.t. $f|_V: V \to W$ is a bijection with continuously differentiable inverse.

Proof: Suffices to prove case $Df(a) = I$ (consider $\tilde{f}(x) = (Df(a))^{-1} f(x)$).

- Choose a closed ball $\overline{B_\varepsilon(a)}$ s.t. $\|Df(x) - I\| < \frac{1}{2}$ in this ball and apply CMT to $T_y(x) = x - f(x) + y$ to give inverse, g.
- show g Lipschitz using $T_y(g(y)) = g(y)$ and mean value inequality.

Proof: It suffices to prove the case $Df(a) = I$ (can apply this to $\tilde{f}(x) = (Df(a))^{-1} f(x)$).

Since Df is continuous, $\exists \varepsilon > 0$ s.t. $\overline{B_\varepsilon(a)} \subseteq U$ and $\|Df(x) - I\| < \frac{1}{2}$ for $x \in \overline{B_\varepsilon(a)}$.

Pick $y \in B_{\varepsilon/2}(f(a)) = W$, define $T_y: \overline{B_\varepsilon(a)} \to \mathbb{R}^n$ by $T_y(x) = x - f(x) + y$.

Note that $\|DT_y\| = \|I - Df(x)\| < \frac{1}{2}$, so by the mean value inequality $\|T_y(x_1) - T_y(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|$. Also, for $x \in \overline{B_\varepsilon(a)}$, have

$$\|T_y(x) - a\| \leq \|T_y(x) - T_y(a)\| + \|T_y(a) - a\| \leq \frac{1}{2}\|x - a\| + \|f(a) - y\|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon,$$ so $T_y: \overline{B_\varepsilon(a)} \to B_\varepsilon(a) \subset \overline{B_\varepsilon(a)}$ is a contraction.

Thus by the contraction mapping theorem, $\exists!\, x \in \overline{B_\varepsilon(a)}$ s.t. $T_y(x) = x$. Since T_y maps into $B_\varepsilon(a)$, in fact $x \in B_\varepsilon(a)$. Note $T_y(x) = x \Rightarrow f(x) = y$. Hence have shown that $\forall y \in W$, $\exists$ unique $x \in B_\varepsilon(a)$ with $f(x) = y$. Choosing $V = B_\varepsilon(a) \cap f^{-1}(W)$ which is open, $f|_V: V \to W$ is ~~open~~ a bijection. Let $g: W \to V$ be its inverse.

Note that for $y \in W$, $T_y(g(y)) = g(y) - f(g(y)) + y = g(y) - y + y = g(y)$.

Now for $y_1, y_2 \in W$,

$$\|g(y_1) - g(y_2)\| = \|T_{y_1}(g(y_1)) - T_{y_2}(g(y_2))\|$$
$$\leq \|T_{y_1}(g(y_1)) - T_{y_1}(g(y_2))\| + \|T_{y_1}(g(y_2)) - T_{y_2}(g(y_2))\|$$
$$\leq \underbrace{\tfrac{1}{2}\|g(y_1) - g(y_2)\|}_{\text{by mean value inequality}} + \underbrace{\|y_1 - y_2\|}_{\text{by definition of } T_y}$$

Hence $\|g(y_1) - g(y_2)\| \leq 2\|y_1 - y_2\|$, so g is Lipschitz, so continuous.

Differentiability of g is non-examinable. $\square$

Analysis II Proofs – Differentiation from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Thm: (Implicit function theorem). Let $U \subseteq \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^n$ continuously differentiable. Let $S = \{(x,y) \in U : f(x,y) = 0\}$. Suppose $(a,b) \in S$ with $D_y f$ invertible at (a,b). Then there exist open subsets $V \subseteq U$ containing (a,b) and $W \subseteq \mathbb{R}^m$ containing a, and a continuously differentiable map $g: W \to \mathbb{R}^n$ s.t. $S \cap V = \text{graph}\, g \equiv \{(x, g(x)) : x \in W\}$.

Proof: apply inverse function theorem to $F(x,y) = (x, f(x,y))$ to give C' inverse $G = (G_1, G_2)$. Take $g(x) = G_2(x,0)$ and consider $F \circ G$ and $G \circ F$ to show that $S \cap V$ and graph g are contained in each other.

Proof: Consider $F: U \to \mathbb{R}^m \times \mathbb{R}^n$ given by $F(x,y) = (x, f(x,y))$ which has
$DF = \begin{pmatrix} I_m & 0 \\ D_x f & D_y f \end{pmatrix}$ which is invertible as $D_y f$ is invertible (at (a,b)).

Then by the inverse function theorem, there exists $V \subseteq U$ open containing (a,b) and $Q \subseteq \mathbb{R}^m \times \mathbb{R}^n$, open containing $F(a,b) = (a, f(a,b)) = (a, 0)$ s.t. $F: V \to Q$ is a bijection with C' inverse $G = (G_1, G_2): Q \to V$.

Let $W = \{x \in \mathbb{R}^m : (x,0) \in Q\}$ which is open since Q is open and define $g: W \to \mathbb{R}^n$ by $g(x) = G_2(x,0)$ which is C' since G is C'.
Now if $x \in W \subseteq Q$ then since G is the inverse of F,

$$(x,0) = (F \circ G)(x,0) = (G_1(x,0), f(G_1(x,0), G_2(x,0)))$$

$\Rightarrow G_1(x,0) = x$ and $0 = f(G_1(x,0), G_2(x,0)) = f(x, g(x))$
so $\text{graph}\, g \subseteq S \cap V$.

Similarly, if $(x,y) \in S \cap V$, so $f(x,y) = 0$ then

$$(x,y) = ((G \circ F))(x,y) = (G_1(x, f(x,y)), G_2(x, f(x,y))) = (G_1(x,0), G_2(x,0))$$
$$= (x, g(x))$$

so $S \cap V \subseteq \text{graph}\, g$. $\therefore S \cap V = \text{graph}\, g$. $\square$

Thm: Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$. Suppose there exists $a \in U$ and $r > 0$ s.t. $B_r(a) \subseteq U$ and $D_{ij}f(x)$ and $D_{ji}f(x)$ exist for all $x \in B_r(a)$ and are continuous at a. Then $D_{ij}f(a) = D_{ji}f(a)$.

Proof: consider $h_{ij}(t) = f(a+te_i+te_j) - f(a+te_i) - f(a+te_j) + f(a)$ and use MVT twice to give $h_{ij}(t) = t^2 D_{ij}f(a+\theta_1 te_i + \theta_2 te_j)$ and similar symmetric expression, then let $t \to 0$ using continuity. Suffices to prove for $m=1$ since $D_{ij}f$ symmetric in i and j iff $D_{ij}f_k$ symmetric for each k.

Proof: Define $h_{ij}(t) = f(a+te_i+te_j) - f(a+te_i) - f(a+te_j) + f(a)$.

Consider $g_1 : [0,1] \to \mathbb{R}$ given by $g_1(s) = f(a + ste_i + te_j) - f(a + ste_i)$.

Then by MVT, $h_{ij}(t) = g_1(1) - g_1(0) = g_1'(\theta_1) = t(D_i f(a+\theta_1 te_i + te_j) - D_i f(a+\theta_1 te_i))$ for some $\theta_1 \in (0,1)$.

Now consider $g_2 : [0,1] \to \mathbb{R}$ given by $g_2(s) = tD_i f(a+\theta_1 te_i + ste_j)$.

Then by MVT, $h_{ij}(t) = g_2(1) - g_2(0) = g_2'(\theta_2) = t^2 D_{ji} f(a+\theta_1 te_i + \theta_2 te_j)$ for some $\theta_2 \in (0,1)$.

Similarly, can show $h_{ij}(t) = h_{ji}(t) = t^2 D_{ij} f(a+\tilde{\theta}_1 te_i + \tilde{\theta}_2 te_j)$ so equating get for $t \neq 0$, $D_{ji}f(a+\theta_1 te_i + \theta_2 te_j) = D_{ij}f(a+\tilde{\theta}_1 te_i + \tilde{\theta}_2 te_j)$.

Since both D_{ij} and D_{ji} are continuous at a we can let $t \to 0$ to get $D_{ij}f(a) = D_{ji}f(a)$. $\square$