Metric & Topological Spaces - Definitions

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Definition 1 (Metric Space)**.** A *metric space* (X, d) is a set X and a function (*metric*) $d : X^2 \to \mathbb{R}$ such that

(i) $d(x, y) \geq 0$ for all $x, y \in X$

(ii) $d(x, y) = 0$ if and only if $x = y$

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iv) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

Definition 2 (Normed Vector Space)**.** A *normed* vector space $(V, \|\|)$ is a vector space V (over $\mathbb{F}$) and a map (*norm*) $N : V \to \mathbb{R}$ (with $N(\mathbf{u}) = \|\mathbf{u}\|$) such that

(i) $\|\mathbf{u}\| \geq 0$ for all $\mathbf{u} \in V$

(ii) If $\|\mathbf{u}\| = 0$ then $\mathbf{u} = \mathbf{0}$

(iii) If $\lambda \in \mathbb{F}$ and $\mathbf{u} \in V$ then $\|\lambda\mathbf{u}\| = |\lambda| \, \|\mathbf{u}\|$

(iv) If $\mathbf{u}, \mathbf{v} \in V$ then $\|\mathbf{u}\| + \|\mathbf{v}\| \geq \|\mathbf{u} + \mathbf{v}\|$

Definition 3 (Inner Product Space)**.** An *inner product space* $(V, \langle , \rangle)$ is a vector space V over $\mathbb{R}$ and a map (*inner product*) $M : V \times V \to \mathbb{R}$ (with $M(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$) such that

(i) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$

(ii) If $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ then $\mathbf{u} = \mathbf{0}$

(iii) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

(iv) $\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$

(v) $\langle \lambda\mathbf{u}, \mathbf{v} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle$

Definition 4 (Discrete Metric)**.** The *discrete metric* on a set X is the function $d : X^2 \to \mathbb{R}$ given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Definition 5 (Continuity)**.** Let (X, d) and (Y, ρ) be metric spaces. Then $f : X \to Y$ is *continuous* if given $\epsilon > 0$ and $x \in X$ we can find a $\delta(t, \epsilon)$ such that $d(t, s) < \delta(t, \epsilon) \implies \rho(f(t), f(s)) < \epsilon$.

Definition 6 (Open Set)**.** Let (X, d) be a metric space. Then a subset E is *open* in X if given any $x \in E$ we can find a δ (depending on x) such that $d(x, y) < \delta \implies y \in E$.

Definition 7 (Open Ball)**.** Let (X, d) be a metric space. For $\delta > 0$ and $x \in X$, we have the open ball with centre x and radius δ

$$B(x, \delta) = \{y : d(x, y) < \delta\}$$

Definition 8 (Limit)**.** Let x_n be a sequence in a metric space (X, d). Then $x_n \to x$ if $d(x_n, x) \to 0$.

Definition 9 (Closed Set)**.** Let (X, d) be a metric space. Then a subset F is *closed* in X if for every sequence $x_n \in F$, $x_n \to x \implies x \in F$.

Definition 10 (Topological Space)**.** A *topological space* (X, τ) is a set X and a collection of subsets of X (*topology*) τ such that

(i) $\emptyset, X \in \tau$

(ii) $U_\alpha \in \tau \ \ \forall \alpha \in A \implies \cup_{\alpha \in A} U_\alpha \in \tau$

(iii) $U_j \in \tau$ for all $1 \leq j \leq n \implies \cap_{j=1}^{n} U_j \in \tau$

Definition 11 (Continuity)**.** Let (X, τ) and (Y, σ) be topological spaces. Then $f : X \to Y$ is continuous if $U \in \sigma \implies f^{-1}(U) \in \tau$.

Definition 12 (Closed Set)**.** Let (X, τ) be a topological space. Then a set F in X is *closed* if its complement is open.

Definition 13 (Interior)**.** Let (X, τ) be a topological space and A a subset of X. Then the *interior* of A is $\text{Int}A = \bigcup \{U \in \tau : U \subseteq A\}$.

Definition 14 (Closure)**.** Let (X, τ) be a topological space and A a subset of X. Then the *closure* of A is $\text{Cl}A = \bigcap \{F \text{ closed} : F \supseteq A\}$.

Definition 15 (Dense subset)**.** Let (X, τ) be a topological space and F a closed subset of X. Then $A \subseteq X$ is a *dense* subset of F if $\text{Cl}A = F$.

Definition 16 (Homeomorphism)**.** Two topological spaces (X, τ) and (Y, σ) are *homeomorphic* if there exists a bijection $\theta : X \to Y$ such that θ and θ^{-1} are continuous. Then θ is a *homeomorphism*.

Definition 17 (Cauchy)**.** Let (X, d) be a metric space, and x_n a sequence in X. Then x_n is *Cauchy* if given $\epsilon > 0$ there exists N such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$.

Definition 18 (Complete)**.** Let (X, d) be a metric space. Then (X, d) is *complete* if every Cauchy sequence in X converges.

Definition 19 (Subspace topology)**.** Let (X, τ) be a topological space and $Y \subseteq X$. Then the *subspace topology* τ_Y on Y is the smallest topology on Y for which the inclusion map ($j : Y \to X$ given by $j(y) = y$ for all $y \in Y$) is continuous.

Definition 20 (Product topology)**.** Let (X, τ) and (Y, σ) be topological spaces. Then the *product topology* μ on $X \times Y$ is the smallest topology on $X \times Y$ for which the projection maps $\pi_X : X \times Y \to X : \pi_X(x, y) = x$ and $\pi_Y : X \times Y \to Y : \pi_Y(x, y) = y$ are continuous.

Definition 21 (Quotient topology). Let (X, τ) be a topological space and $\sim$ an equivalence relation on X. Let $q : X \to X/\sim$ be given by $q(x) = [x]$. Then the quotient topology is the largest topology σ on $X/\sim$ for which q is continuous, i.e. $\sigma = \{U \subseteq X/\sim : q^{-1}(U) \in \tau\}$.

Definition 22 (Hausdorff space). Let (X, τ) be a topological space. Then (X, τ) is *Hausdorff* if, whenever $x, y \in X$ and $x \neq y$, we can find $U, V \in \tau$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Definition 23 (Open neighbourhood). Let (X, τ) be a topological space and $x \in U \in \tau$. Then U is an *open neighbourhood* of x.

Definition 24 (Compact space). Let (X, τ) be a topological space. Then (X, τ) is *compact* if, whenever we have a collection U_α of open sets $[\alpha \in A]$ with $\bigcup_{\alpha \in A} U_\alpha = X$, we can find a finite subcollection $U_{\alpha(1)}, U_{\alpha(2)}, \ldots, U_{\alpha(n)}$ with $\alpha(j) \in A$ $[1 \leq j \leq n]$ such that $\bigcup_{j=1}^{n} U_{\alpha(j)} = X$.

Definition 25 (Compact subset). Let (X, τ) be a topological space and $Y \subseteq X$. Then Y is *compact* if the subspace topology on Y is compact.

Definition 26 (Sequentially compact). Let (X, d) be a metric space. Then (X, d) is *sequentially compact* if every sequence in X has a convergent subsequence.

Definition 27 (Connected & disconnected spaces). Let (X, τ) be a topological space. Then (X, τ) is *disconnected* if there exist non-empty open sets U and V such that $U \cup V = Y$ and $U \cap V = \emptyset$. A space which is not disconnected is called *connected*.

Definition 28 (Connected & disconnected subsets). Let (X, τ) be a topological space and $E \subseteq X$. Then E is *connected* if the subspace topology on E is connected. Similarly for disconnectedness.

Definition 29 (Path connected). Let (X, τ) be a topological space. Then $x, y \in X$ are *path connected* if (giving $[0, 1]$ the standard topology) there exists a continuous function $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. The function γ is called a *path* from x to y.

Definition 30 (Neighbourhood). Let (X, τ) be a topological space. For $x \in X$, we say that N is a *neighbourhood* of x if there exists $U \in \tau$ with $x \in U \subseteq N$.

Definition 31 (Basis). Let X be a set. A collection $\mathcal{B}$ of subsets is called a *basis* if

(i) $\bigcup_{B \in \mathcal{B}} B = X$

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.