Complex Analysis

Lectured by Prof Anthony Scholl

Lent 2015

1 Complex differentiation

Recall,

Defintion.
- $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ is the open disc with radius $r > 0$ about $a \in \mathbb{C}$.
- $U \subset \mathbb{C}$ is *open* if for all $a \in U$, there exists $r > 0$ such that $D(a, r) \subset U$.
- A *curve* in $\mathbb{C}$ is a continuous map $\gamma : [a, b] \to \mathbb{C}$ (with $a < b \in \mathbb{R}$). It is *continuously differentiable* (or C') if γ' exists and is continuous on $[a, b]$ (with 1-sided derivatives at endpoints).
- A non-empty open set $U \subset \mathbb{C}$ is *path-connected* if for all $z, w \in U$, there exists a curve $\gamma : [a, b] \to U$ with $\gamma(a) = z$ and $\gamma(b) = w$. (Note that if such a curve exists, then there exists another curve in U with the same endpoints which is a finite sequence of line segments.)

Defintion (Domain)**.** A *domain* is a non-empty path-connected open subset of $\mathbb{C}$.

In this course we consider complex-valued functions $f : U \to \mathbb{C}$ where U is some domain. Note that $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, so we can write this as

$$f(x + iy) = u(x, y) + iv(x, y)$$

where u, v are real-valued (the *real* and *imaginary parts* of f, respectively).

Defintion (Derivative)**.** The function f is *differentiable* at $a \in U$ if the limit

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists, and this limit is known as the *derivative* of f at $z = a$.

Defintion (Holomorphic function)**.** The function f is *homomorphic at* $a \in U$ if there exists $r > 0$ such that f is differentiable at all points of $D(a, r)$.

We say f is *homomorphic on* U if it is differentiable (equivalently, homomorphic) at every $a \in U$.

The complex derivative satisfies the same formal rules (derivatives of sum, product, quotient, chain rule, inverse functions, etc.) as in real calculus of one variable. (Replace absolute value with modulus in the proofs!)

Defintion (Entire function)**.** An *entire function* is a holomorphic function $f : \mathbb{C} \to \mathbb{C}$.

Example.
- **Polynomials**. $f(z) = z^n$ $(n \geq 0)$ is holomorphic on $\mathbb{C}$, with $f'(z) = nz^{n-1}$.
- **Rational functions**. If $p(z), q(z)$ are polynomials with $q \not\equiv 0$ then $f(z) = \frac{p(z)}{q(z)}$ is a holomorphic function on $U = \mathbb{C} \setminus \{a \in \mathbb{C} : q(a) = 0\}$.

We can compare complex differentiation with differentiation in $\mathbb{R}^2$. Recall that if $U \subset \mathbb{R}^2$ is open and $u : U \to \mathbb{R}$ is a function, then u is differentiable at $(c, d) \in U$ if $\exists \lambda, \mu \in \mathbb{R}$ such that

$$\frac{u(x,y) - [u(c,d) + \lambda(x-c) + \mu(y-d)]}{\sqrt{(x-c)^2 + (y-d)^2}} \to 0 \quad \text{as } (x,y) \to (c,d)$$

in which case $D_u(c,d) = (\lambda, \mu) \in \mathbb{R}^2$ is the derivative of u at (c,d). In particular, $\lambda = u_x(c,d)$ and $\mu = u_y(c,d)$ are the partial derivatives.

The following theorem connects these derivatives.

Theorem 1. Let U be open and $f : U \to \mathbb{C}$ a function with $f(x+iy) = u(x,y) + iv(x,y)$. Then f is differentiable at $a = c+id \in U$ if and only if the functions u, v are differentiable at (c,d) *and*

$$u_x(c,d) = v_y(c,d), \quad u_y(c,d) = -v_x(c,d) \ . \tag{1}$$

If this holds, then

$$f'(a) = u_x(c,d) + iv_x(c,d) \ .$$

The equations (1) are known as the *Cauchy-Riemann (C-R) equations.*

Proof. f is differentiable at $a = c + id$ with derivative $f'(a) = p + qi$ if and only if

$$\lim_{z \to a} \frac{f(z) - f(a) - (p+qi)(z-a)}{|z-a|} = 0$$

or equivalently both (real part)

$$\lim_{(x,y)\to(c,d)} \frac{u(x,y) - u(c,d) - [p(x-c) - q(y-d)]}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$

and (imaginary part)

$$\lim_{(x,y)\to(c,d)} \frac{v(x,y) - v(c,d) - [q(x-c) + p(y-d)]}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$

(since $(p+qi)(z-a) = p(x-c) + q(y-d) + i[q(x-c) + p(y-d)]$).

Hence f is differentiable at $z = a$ with $f'(a) = p+qi$ if and only if u, v are differentiable at (c,d) with

$$D_u(c,d) = (p, -q)$$
$$D_v(c,d) = (q, p)$$

from which equations (1) follow. $\square$

Example. Let $f(z) = \bar{z} = x - iy$. Then $u = x$ and $v = -y$, so

$$\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and so the Cauchy-Riemann equations (1) don't hold, hence f is nowhere differentiable.

Remark. Note that

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} (c, d) = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$$

is the matrix of multiplication by $p + qi \in \mathbb{C}$ on $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}$ with respect to the basis $\{1, i\}$.

Remark. Later, we will show that if f is holomorphic, then so is its derivative, so all partial derivatives of u, v exist and are continuous. Then differentiating (1) gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$$

i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and similarly for v, so the real and imaginary parts of a holomorphic function are *harmonic functions* - they satisfy Laplace's equation.

Corollary 2. Let $f = u + iv : U \to \mathbb{C}$ be a function. Suppose that u, v have continuous partial derivatives on U and satisfy the $C - R$ equations (1) on U. Then f is holomorphic on U.

Proof. Recall from Analysis II that the hypotheses imply that u, v are differentiable on U, so we can apply Theorem 1. □

Corollary 3. Let $f : U \to \mathbb{C}$ be holomorphic on a *domain* U. Then if $f'(z) = 0$ for all $z \in U$, the function f is *constant*.

Proof. Writing $f = u + iv$, the hypotheses imply that $u, v : U \to \mathbb{R}^2$ have zero derivatives. Hence as U is path connected, u, v are constant (see Analysis II), so f is constant. □

2 Power series

Recall,

Theorem 4. Let $(c_n)_{n \geq 0}$ be a sequence of complex numbers. Then

- there exists a (unique) $R \in [0, \infty]$ such that the series $\sum_{n=0}^{\infty} c_n (z - a)^n$ converges absolutely if $|z - a| < R$ and diverges if $|z - a| > R$;
- if $0 \leq r < R$, the series converges uniformly on $\{|z - a| \leq r\}$;
- the radius of convergence R is given by $\sup \{r \geq 0 : \sum |c_n| r^n \text{ is convergent}\}$.

Proof. See Analysis I. □

Convergent power series may be differentiated term-by-term within their radius of convergence.

Theorem 5. Let $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ be a complex power series with radius of convergence $R > 0$. Then

(i) f is *holomorphic* on $D(a, R)$;

(ii) f' is given by the series

$$\sum_{n=1}^{\infty} nc_n(z-a)^{n-1} , \tag{2}$$

which also has radius of convergence R;

(iii) f has derivatives of all orders on $D(a, R)$ and $f^{(n)}(a) = n!\, c_n$;

(iv) if f vanishes identically on some $D(a, r)$ with $0 < r < R$, then it vanishes on $D(a, R)$ and $c_n = 0$ for all n.

Proof. WLOG assume $a = 0$ (can change variables).

We first prove that the derived series (2) has radius of convergence R. Since $|nc_n| \geq |c_n|$ for $n \geq 1$, its radius of convergence is at most R. Let $0 < r < R$. Then since $\sum c_n r^n$ converges, by the ratio test we have $\left|\frac{c_{n+1}}{c_n} r\right| \to l < 1$. Then

$$\left|\frac{(n+1)c_{n+1}r^n}{nc_n r^{n-1}}\right| = \left|\frac{n+1}{n}\right| \left|\frac{c_{n+1}}{c_n} r\right| \to l < 1$$

so $\sum nc_n r^{n-1}$ converges by the ratio test. Hence the radius of convergence of the derived series is R.

We now want to show that this is the derivative. Consider (for $|z|, |w| < R$),

$$g(z, w) = \begin{cases} \frac{f(z)-f(w)}{z-w} = \sum_{n=1}^{\infty} c_n \left(\frac{z^n - w^n}{z-w}\right) & \text{if } z \neq w \\ \sum_{n=1}^{\infty} nc_n w^{n-1} & \text{if } z = w \end{cases}$$

We need to show that g is continuous at $z = w$ (so then $\lim_{z \to w} \frac{f(z)-f(w)}{z-w} = \sum_n c_n w^{n-1}$).

Write

$$g(z, w) = \sum_{n=1}^{\infty} c_n h_n(z, w) \tag{3}$$

where $h_n(z, w) = \sum_{j=0}^{n-1} z^j w^{(n-1)-j}$ (i.e. a GP with sum $\frac{(z/w)^n - 1}{z/w - 1} = \frac{z^n - w^n}{z-w}$ when $z \neq w$). We want to show that the series (3) is uniformly convergent on $\{(z, w) : |z|, |w| \leq r\}$ for any $r < R$. Note that the nth term is bounded by

$$|c_n| \sum_{j=0}^{n-1} |z|^j |w|^{n-1-j} \leq |c_n| n r^{n-1} = M_n$$

and we have already shown that $\sum M_n$ is convergent. Hence by the Weierstass M-test (3) is uniformly convergent on this set. Thus the limit $g(z, w)$ is continuous on $\{|z|, |w| \leq r\}$, so is continuous on $\{|z|, |w| < R\}$. Hence if $|w| < R$ then $f'(w) = \sum_{n=1}^{\infty} nc_n w^{n-1}$ exists.

By induction, derivatives of all orders exist, and it is clear that $f^{(n)}(a) = n! c_n$.

Finally, if $f \equiv 0$ on some $D(a, r)$ then all $f^{(n)} \equiv 0$ on $D(a, r)$, so $c_n = \frac{1}{n!} f^{(n)}(a) = 0$ for all n. □

Defintion (Exponential function)**.** We define $\exp z = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Proposition 6. (i) $\exp z$ is *entire*, and $\exp' z = \exp z$.

(ii) For all $z, w \in \mathbb{C}$, $\exp z + w = \exp z \exp w$, and $\exp z \neq 0$.

(iii) If $z = x + iy$ then $\exp z = e^x (\cos y + y \sin y)$.

Proof. (i) The radius of convergence of $\exp(z)$ is ∞.

(ii) Let $w \in \mathbb{C}$ and set $F(z) = \exp(z+w)\exp(-z)$. Then $F'(z) = 0$ (as $\exp' = \exp$), so F is constant, i.e. $F(z) = F(0) = exp(w)$. Letting $w = -z$ (and noting that $\exp(0) = 1$), we get $\exp(z)\exp(-z) = 1$, so $\exp(z+w) = \exp(z)\exp(w)$, and $\exp(z) \neq 0$.

(iii) By the above, $\exp(z) = \exp(x)\exp(iy)$ and comparing the standard series for $e^x, \cos(y), \sin(y)$ gives the result.

□

Proposition 7. (i) $\exp z = 1 \iff x \in 2\pi i\mathbb{Z}$.

(ii) If $w \in \mathbb{C} \setminus \{0\}$ then there exists $z \in \mathbb{C}$ with $e^z = w$.

Proof. (i) Proof from IA Groups, construct a homomorphism $(\mathbb{C}, +) \to (\mathbb{C} \setminus \{0\}, \times)$ via exp, which has kernel $2\pi i\mathbb{Z}$.

(ii) Proof follows easily from $e^{x+iy} = e^x(\cos(y) + i\sin(y))$.

□

Defintion (Trigonometric & hyperbolic functions)**.** For any $z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$
$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \quad \sinh z = \frac{e^{z} - e^{-z}}{2}.$$

These are entire functions, with $\sin' = cos$ etc. as expected. The usual addition formulae hold for complex values. We also have $\sin z = 0 \iff z \in \pi\mathbb{Z}$. However, *beware*: $|\cos z| \not\leq 1$ in general ($\cos iy = \cosh y \to \infty$ as $y \to \infty$).

Defintion (Logarithm)**.** If $z \in \mathbb{C}$, we say that $w \in \mathbb{C}$ is a *logarithm* of z if $\exp w = z$.

By the proposition above, z has a logarithm if and only if $z \neq 0$. If $z \neq 0$, it has an infinite number of logarithms, differing by integer multiples of $2\pi i$. There's no preferred choice of an individual logarithm.

Defintion (Branch of the logarithm)**.** Let $U \subset \mathbb{C}$ be an open set and $0 \notin U$. A continuous function $\lambda : U \to \mathbb{C}$ is a *branch of the logarithm* if $\exp \lambda(z) = z$ for all $z \in U$ (i.e. if $\lambda(z)$ is a logarithm of z for all $z \in U$.)

Defintion (Principal branch of the logarithm)**.** Let $U = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\} = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. The *principal branch* of the logarithm is the function $\mathrm{Log} : U \to \mathbb{C}$ given by $\mathrm{Log}\, z = \ln|z| + i \arg z$ where $\arg z$ is chosen in $(-\pi, \pi]$.

Proposition 8. (i) Log is holomorphic on $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ with derivative $\frac{1}{z}$.

(ii) If $|z| < 1$ then $\operatorname{Log}(1+z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$.

Proof. (i) Apply the inverse function rule. If $w = \operatorname{Log} z$, then since $z = \exp w$ and $\frac{\mathrm{d}z}{\mathrm{d}w} = \exp w = z \neq 0$, we have Log is differentiable with derivative $\operatorname{Log}'(z) = \left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^{-1} = \frac{1}{z}$.

(ii) The power series has radius of convergence $R = 1$. Differentiating both sides gives (for $|z| < 1$)

$$\frac{\mathrm{d}}{\mathrm{d}z}(LHS) = \frac{1}{1+z}$$
$$\frac{\mathrm{d}}{\mathrm{d}z}(RHS) = \sum_{n=0}^{\infty}(-1)^n z^n = \frac{1}{1+z}$$

So the derivative of $LHS - RHS$ is 0 on the domain $D(0,1)$, hence constant. Since $\operatorname{Log}(1) = 0$, we have $LHS = RHS$.

$\square$

Remark. Log does *not* extend to a continuous function on $\mathbb{C} \setminus \{0\}$. To see this, take $z = e^{i\theta}$ for $\theta \in (-\pi, \pi)$. Then

$$\operatorname{Log}(z) = i\theta \to \begin{cases} -\pi i & \text{as } \theta \to -\pi \\ +\pi i & \text{as } \theta \to +\pi \end{cases}$$

so $\lim_{z \to -1} \operatorname{Log}(z)$ does not exist.

We can use the above to define complex powers z^α for any complex number α. For $z \in U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, define

$$z^\alpha = e^{\alpha \operatorname{Log} z} .$$

3 Conformal mapping

Defintion (Conformal)**.** We say that a (continuous) mapping $f : U \to \mathbb{C}$ (with U open) is *conformal* at a point if it preserves angles (and their sign).

What does it mean to preserve angles? Suppose that f is continuous and complex differentiable at $w \in U$ with $f'(w) \neq 0$. Let $\gamma : [-1,1] \to U$ be a C' curve through w, with $\gamma(0) = w$, and suppose that $\gamma'(0) \neq 0$.

The angle that γ makes with the horizontal (more precisely, the line through w parallel to the real axis) is the argument of the tangent to the curve at w, given by $\arg \gamma'(0)$.

Let δ be the image of γ under the map f, that is $\delta(t) = f(\gamma(t))$. Then by the chain rule we have $\delta'(0) = f'(\gamma(0))\gamma'(0) = f'(w)\gamma'(0)$. Thus the angle that δ makes with the horizontal is $\arg \delta'(0) = \arg \gamma'(0) + \arg f'(w) + 2\pi n$ for some $n \in \mathbb{Z}$. That is, under the mapping f, the angle has been increased by an amount $\arg f'(w)$ which is *independent of the curve* γ. All curves through w are rotated by the same amount $\arg f'(w)$.

This implies that angles between curves are preserved: if γ_1 and γ_2 pass through w at angles θ_1 and θ_2 respectively, then $f(\gamma_1)$ and $f(\gamma_2)$ pass through w at angles $\theta_1 + \phi$ and $\theta_2 + \phi$ respectively (where $\phi = \arg f'(w)$), so the difference in angles $\theta_1 - \theta_2 = (\theta_1 + \phi) - (\theta_2 + \phi)$ is preserved.

Remark. If $f : U \to \mathbb{C}$ is holomorphic on U, then it is conformal on $U \setminus \{w \in U : f'(w) = 0\}$.

Conversely, if $f = u + iv : U \to \mathbb{C}$ with u, v differentiable and not both of Du, Dv zero at $w = c + di$, then if f is conformal at w, it is complex differentiable at w.

Defintion (Conformal equivalence)**.** Let D be a domain and $f : D \to \mathbb{C}$ holomorphic, with $f'(z) \neq 0$ for all $z \in D$ and f injective on D, so $f : D \to D' = f(D)$. We say that f is a *conformal equivalence* between D and D'. (The inverse function $f^{-1} : D' \to D$ is then a conformal equivalence in the other direction.)

Example. (i) M obius transformations. Define $f(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Then f maps $\mathbb{CP} = \mathbb{C} \cup \{\infty\}$ (the *Riemann sphere*) to itself.

(ii) $f(z) = z^n$ $(n \geq 2)$ has $f'(z) = nz^{n-1} \neq 0$ (for $z \neq 0$), so f is conformal everywhere except at 0.

(iii) exp and Log.

Theorem 9 (*Riemann Mapping Theorem*)**.** Let $D \subset \mathbb{C}$ be a domain bounded by a simple (no loops) closed curve [more generally, we can take any simply-connected domain different from $\mathbb{C}$]. Then there is a conformal equivalence $f : D \simeq D(0, 1)$.

4 Complex integration I

4.1 Integral along curves

Let $f : [a, b] \to \mathbb{C}$ be continuous. Then we can define the integral of f in the obvious way,

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b \operatorname{Re} f(t)\,\mathrm{d}t + i\int_a^b \operatorname{Im} f(t)\,\mathrm{d}t\ .$$

Theorem 10.

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \leq (b-a) \sup_{a \leq t \leq b} |f(t)|$$

with equality if and only if f is constant (assuming f is continuous).

Proof. There is nothing to prove if $\int f = 0$.

Otherwise, let $\theta = \arg \int_a^b f(t)\,\mathrm{d}t$ and $M = \sup_{a \leq t \leq b} |f(t)|$. Then

$$\begin{aligned}
\left|\int_a^b f(t)\,\mathrm{d}t\right| &= \int_a^b e^{-i\theta} f(t)\,\mathrm{d}t \\
&= \int_a^b \operatorname{Re}(e^{-i\theta} f(t))\,\mathrm{d}t \\
&\leq \int_a^b |f(t)|\,\mathrm{d}t \\
&\leq M(b-a)
\end{aligned}$$

If equality holds, then both these '$\leq$' must be '$=$'. The second is iff $|f(t)| = M$, i.e. iff f is constant. The first is iff $\arg f(t) = \theta$, ie. $\arg f$ is constant for all t. Thus we have total equality if and only if f is constant on $[a, b]$. ☐

Defintion (Length of curve)**.** The length of a C' curve $\gamma : [a, b] \to \mathbb{C}$ is given by

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)| \, \mathrm{d}t$$

Remark. We say that γ is simple if $\gamma(t_1) \neq \gamma(t_2)$ unless $t_1 = t_2$ or $\{t_1, t_2\} = \{a, b\}$. If γ is simple then length(γ) is just the arclength of the parametrised curve γ.

Defintion (Integral along curve)**.** Let $f : U \to \mathbb{C}$ be continuous and $\gamma : [a, b] \to U$ a C' curve. Then the *integral* of f along γ is

$$\int_\gamma f(z) \, \mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}t \ .$$

4.2 Basic properties

- Linearity. For $c_1, c_2 \in \mathbb{C}$,

$$\int_\gamma c_1 f_1 + c_2 f_2 \, \mathrm{d}z = c_1 \int_\gamma f_1 \, \mathrm{d}z + c_2 \int_\gamma f_2 \, \mathrm{d}z$$

- Additivity. If $a < a' < b$ and $\gamma_1 : [a, a'] \to U$, $\gamma_2 : [a', b] \to U$ are given by $\gamma_i(t) = \gamma(t)$ then

$$\int_\gamma f \, \mathrm{d}z = \int_{\gamma_1} f \, \mathrm{d}z + \int_{\gamma_2} f \, \mathrm{d}z$$

- Inverse path. If $(-\gamma) : [-b, -a] \to U$ is given by $(-\gamma)(t) = \gamma(-t)$ then

$$\int_{(-\gamma)} f \, \mathrm{d}z = - \int_\gamma f \, \mathrm{d}z$$

- Re-parametrisation . If $\phi : [a', b'] \to [a, b]$ is C' with $\phi(a') = a$, $\phi(b') = b$ and $\delta = \gamma \circ \phi : [a', b'] \to U$ then

$$\int_\gamma f \, \mathrm{d}z = \int_\delta f \, \mathrm{d}z$$

Remark. The last two properties follow from the chain rule for $\delta'(t)$. In particular, the last point means that we may assume that $[a, b] = [0, 1]$.

4.3 Piecewise C' curves

Suppose that $a = a_0 < a_1 < \ldots < a_n = b$ and $\gamma : [a, b] \to \mathbb{C}$ is continuous and its restriction to each $[a_i, a_{i+1}]$ (for $i = 0, 1, \ldots, n-1$) is C'. Then we say that γ is *piecewise continuously differentiable*, or *piecewise C'*, and can define $\int_\gamma f \, \mathrm{d}z$ in the natural way.

If $\gamma_i : [a_i, a_{i+1}] \to \mathbb{C}$ are the restrictions of γ to $[a_i, a_{i+1}]$ then

$$\int_\gamma f \, \mathrm{d}z = \sum_{i=0}^{n-1} \int_{\gamma_i} f \, \mathrm{d}z \ .$$

These piecewise C' curves naturally arise as concatenated curves. Let $\gamma : [a,b] \to \mathbb{C}$, $\delta : [c,d] \to \mathbb{C}$ with $\gamma(b) = \delta(c)$. Then their *sum* is the curve $(\gamma + \delta) : [a, b+d-c] \to \mathbb{C}$ given by

$$t \mapsto \begin{cases} \gamma(t) & \text{if } t \in [a,b] \\ \delta(t+c-b) & \text{if } t \in [b, b+d-c] \end{cases}$$

If γ, δ are C', then their sum is piecewise C'. Unless stated otherwise, we now use the term *curve* to denote *piecewise C' curve*.

Proposition 11. For $f : U \to \mathbb{C}$ continuous and a curve $\gamma : [a,b] \to U$,

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \leq \operatorname{length}(\gamma) \sup_\gamma |f|$$

where $\sup_\gamma |f| = \sup_{t \in [a,b]} |f(\gamma(t))|$ exists as $f, \gamma, |\cdot|$ are continuous.

Proof. Let $M = \sup_\gamma |f|$. Then

$$\begin{aligned} \left| \int_\gamma f(z)\,\mathrm{d}z \right| &= \left| \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \right| \\ &\leq \int_a^b |f(\gamma(t))\gamma'(t)|\,\mathrm{d}t \quad \text{by Theorem 10} \\ &\leq \int_a^b M|\gamma'(t)|\,\mathrm{d}t \\ &= M \operatorname{length}(\gamma) \end{aligned}$$

Note that if γ is not C', we need to break $[a,b]$ into intervals on which it is C' to apply Theorem 10. □

Proposition 12. Let $f_n : U \to \mathbb{C}$ (for $n \in \mathbb{N}$) and $f : U \to \mathbb{C}$ be continuous. Suppose $\gamma : [a,b] \to U$ is a curve such that $f_n \to f$ uniformly on the image of γ. Then

$$\int_\gamma f_n(z)\,\mathrm{d}z \to \int_\gamma f(z)\,\mathrm{d}z \ .$$

Proof. The condition is that $f_n(\gamma(t)) \to f(\gamma(t))$ uniformly on $[a,b]$.

Let $M_n = \sup_\gamma |f - f_n|$. By the definition of uniform convergence, we have $M_n \to 0$ as $n \to \infty$. Then by the previous result,

$$\left| \int_\gamma f\,\mathrm{d}z - \int_\gamma f_n\,\mathrm{d}z \right| \leq M_n \operatorname{length}(\gamma) \to 0 \text{ as } n \to \infty$$

(where length(γ) is finite since $\int_a^b |\gamma'(t)|\,\mathrm{d}t \leq (b-a)\sup|\gamma'| < \infty$ as γ is C'). □

Theorem 13 (Fundamental Theorem of Calculus)**.** Let $U \subset \mathbb{C}$ be open, and suppose that $F : U \to \mathbb{C}$ is holomorphic (and F' is continuous). Then for any curve $\gamma : [a,b] \to U$,

$$\int_\gamma F'(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)) \ .$$

In particular, if γ is closed, i.e. $\gamma(a) = \gamma(b)$ then $\int_\gamma F'(z)\,\mathrm{d}z = 0$.

Proof.

$$\begin{aligned}\int_\gamma F'(z)\,\mathrm{d}z &= \int_a^b F'(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t))\,\mathrm{d}t \\ &= F(\gamma(b)) - F(\gamma(a))\end{aligned}$$

by the standard Fundamental Theorem of Calculus for $\mathbb{R}$. □

Corollary 14. If $f : U \to \mathbb{C}$ is continuous and is the derivative of a holomorphic function on U, then for $\gamma : [a,b] \to U$ the integral $\int_\gamma f\,\mathrm{d}z$ depends only on $\gamma(a), \gamma(b)$ and vanishes if γ is closed.

Proof. Say $f = F'$ where F is holomorphic. Then

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_\gamma F'(z)\,\mathrm{d}z \\ &= \int_a^b F'(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t))\,\mathrm{d}t \\ &= F(\gamma(b)) - F(\gamma(a))\end{aligned}$$

by the standard Fundamental Theorem of Calculus for $\mathbb{R}$.

(In general, for γ piecewise C' write $\gamma = \sum \gamma_i$ with each γ_i C' and apply the above.) □

The *converse* of the Fundamental Theorem of Calculus is true.

Theorem 15. Let $f : D \to \mathbb{C}$ be continuous on a domain D. If $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed curve γ in D, then there exists a holomorphic function $F : D \to \mathbb{C}$ with $F'(z) = f(z)$.

Proof. Pick $a_0 \in D$, and for each $w \in D$, pick any curve $\gamma_w : [0,1] \to D$ with $\gamma_w(0) = a_0$ and $\gamma_w(1) = w$. Define $F(w) = \int_{\gamma_w} f(z)\,\mathrm{d}z$. We show directly that $F'(w) = f(w)$ by considering a small displacement h from w.

Fix $w \in D$ and choose $r > 0$ such that $D(w,r) \subset D$. For $|h| < r$ let $\delta_h : [0,1] \to D(w,r)$ be the line segment $\delta_h(t) = w + th$. If we set $\gamma = \gamma_w + \delta_h + (-\gamma_{w+h})$ then by the hypothesis $\int_\gamma f(z)\,\mathrm{d}z = 0$. This says that $F(w+h) \equiv \int_{\gamma_{w+h}} f(z)\,\mathrm{d}z = F(w) + \int_{\delta_h} f(z)\,\mathrm{d}z$. So

$$\begin{aligned}\left|\frac{F(w+h) - F(w)}{h} - f(w)\right| &= \left|\frac{1}{h}\int_{\delta_h} f(z)\,\mathrm{d}z - f(w)\right| \\ &= \frac{1}{|h|}\left|\int_{\delta_h} f(z) - f(w)\,\mathrm{d}z\right| \\ &\leq \frac{1}{|h|}\,\mathrm{length}(\delta_h)\sup_{\delta_h}|f(z) - f(w)| \\ &\leq \sup_{|z-w|\leq|h|}|f(z) - f(w)| \\ &\to 0 \text{ as } |h| \to 0\end{aligned}$$

since f is continuous. Hence $F'(w)$ exists and equals $f(w)$. □

Defintion (Convex and starlike)**.** A domain D is

- *convex* if $\forall a, b \in D$ and $\forall t \in [0,1]$, we have $at + (1-t)b \in D$;
- *starlike* if $\exists a_0 \in D$ such that $\forall b \in D$ and $\forall t \in [0,1]$, we have $a_0 t + (1-t)b \in D$.

Lemma 1. Let D be a disc (or more generally and convex or 'starlike' domain), and $f : D \to \mathbb{C}$ continuous. Suppose that for every *triangle* γ in D, $\int_\gamma f \, \mathrm{d}z = 0$. Then there exists a holomorphic function F on D with $F' = f$.

Proof. Follow the same proof as for Theorem 15 but taking γ_w to be the *straight line* path from a_0 to w, which exists by assumption. $\square$

4.4 Cauchy's Theorem for a disc

Cauchy's Theorem says that *under suitable hypotheses* on D and/or γ, if $f : D \to \mathbb{C}$ is holomorphic and $\gamma : [a,b] \to D$ closed, then $\int_\gamma f \, \mathrm{d}z = 0$. The most basic version is for a disc D. We first prove some intermediary results.

Remark. We need some hypotheses since if γ is the unit circle then $\int_\gamma \frac{1}{z} \, \mathrm{d}z = 2\pi i \neq 0$.

Remark. We define a triangle $\triangle = \triangle(a,b,c)$ in the obvious way, as $\{\lambda a + \mu b + \nu c : \lambda, \mu, \nu \geq 0, \lambda + \mu + \nu = 1\}$. Then $\partial\triangle$ is the sum of the line segments $[a,b]$, $[b,c]$ and $[c,a]$, a simple closed curve.

Theorem 16 (Cauchy's Theorem for a triangle, a.k.a. Goursat's Lemma)**.** Let U be open, $f : U \to \mathbb{C}$ holomorphic, and $\triangle \subset U$ a triangle. Then $\int_{\partial\triangle} f \, \mathrm{d}z = 0$.

Proof. Let $L = \text{length}(\delta\triangle)$, $I = |\int_{\partial\triangle} f(z) \, \mathrm{d}z|$. By bisecting its sides, subdivide $\triangle$ into 4 subtriangles $\triangle^{(1)}, \ldots, \triangle^{(4)}$. Then $\int_{\partial\triangle} f(z) \, \mathrm{d}z = \sum_{j=1}^4 \int_{\partial\triangle^{(j)}} f(z) \, \mathrm{d}z$ since the integrals on shared edges of $\triangle^{(j)}$ cancel each other in pairs (since they go in opposite directions).

Hence for some j, we have $|\int_{\partial\triangle^{(j)}} f(z) \, \mathrm{d}z| \geq \frac{1}{4}I$, set $\triangle_1 = \triangle^{(j)}$. Subdividing this triangle and repeating, we obtain a sequence of nested triangles $\triangle = \triangle_0 \supset \triangle_1 \supset \triangle_2 \supset \ldots$ such that $\text{length}(\partial\triangle_n) = \frac{L}{2^n}$ and $|\int_{\partial\triangle_n} f(z) \, \mathrm{d}z| \geq \frac{I}{4^n}$.

We now show that $\bigcap_{n\geq 0} \triangle_n = \{w\}$ is a single point. Take a sequence $w_n \in \triangle_n$. Then for $m \geq n$, $w_m \in \triangle_n$ (since triangles are nested), so $|w_n - w_m| \leq \text{length}(\partial\triangle_n) \to 0$ as $n \to \infty$. Hence (w_n) is Cauchy so converges; if $w = \lim_{n\to\infty}(w_n)$ then $w \in \triangle_n$ for each n as each $\triangle_n$ is closed. Since $\text{length}(\partial\triangle_n) \to 0$, no other point can belong to $\bigcap_{n\geq 0} \triangle_n$.

Since $f(z)$ is differentiable at $z = w$, the function

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} - f'(w) & z \neq w \\ 0 & z = w \end{cases}$$

is continuous on D. Then

$$\begin{aligned}
\frac{I}{4^n} &\leq \left| \int_{\partial\triangle_n} f(z) \, \mathrm{d}z \right| \\
&= \left| \int_{\partial\triangle_n} f(w) + f'(w)(z-w) \, \mathrm{d}z + \int_{\partial\triangle_n} g(z)(z-w) \, \mathrm{d}z \right| \\
&= \left| \int_{\partial\triangle_n} g(z)(z-w) \, \mathrm{d}z \right|
\end{aligned}$$

since $f(w) + f'(w)(z-w)$ has an anti-derivative on $\mathbb{C}$ (it's a linear function in z), so the integral vanishes by (the corollary to) the Fundamental Theorem of Calculus. Now by Proposition 11 and noting that since $w \in \triangle_n$, $\sup_{\triangle_n}(z-w) \leq \text{length}(\partial\triangle_n)$,

$$\begin{aligned}\frac{I}{4^n} &\leq \text{length}(\partial\triangle_n) \sup_{\triangle_n} |g(z)| \, \text{length}(\partial\triangle_n) \\ &= \frac{L^2}{4^n} \sup_{\triangle_n} |g(z)|\end{aligned}$$

Hence $I \leq L^2 \sup_{\triangle_n} |g(z)| \to 0$ as $n \to \infty$ since $g(w) = 0$ (and g is continuous). □

Theorem 17. Let $S \subset U$ be a finite subset and assume $f : U \to \mathbb{C}$ is holomorphic on $U \setminus S$ and continuous on U. Then for every $\triangle \subset U$, $\int_{\partial\triangle} f \, \mathrm{d}z$.

Proof. Let $M = \sup_{\triangle} |f| < \infty$ since $\triangle$ is compact. As in the proof of Theorem 16, subdivide $\triangle$ into subtriangles $\triangle'$ with $\text{length}(\triangle') = \frac{L}{2^n}$, where $L = \text{length}(\triangle)$. For $\triangle'$ with $\triangle' \cap S = \emptyset$, $\int_{\partial\triangle'} f(z) \, \mathrm{d}z = 0$ by Theorem 16. Note that each element of S can belong to at most 6 subtriangles (if it lies at the point where 6 meet), so

$$\begin{aligned}\left| \int_{\partial\triangle} f(z) \, \mathrm{d}z \right| &= \left| \sum_{\substack{\triangle' \text{ with} \\ \triangle' \cap S \neq \emptyset}} \int_{\partial\triangle'} f(z) \, \mathrm{d}z \right| \\ &\leq \sum_{\substack{\triangle' \text{ with} \\ \triangle' \cap S \neq \emptyset}} \text{length}(\partial\triangle') \sup_{\partial\triangle'} |f| \\ &\leq 6 \times |S| \times 2^{-n} LM \to 0 \text{ as } n \to \infty\end{aligned}$$

□

Theorem 18 (Cauchy's Theorem for a disc, or convex or starlike domain)**.** Let D be a disc (or convex or starlike domain), and $f : D \to \mathbb{C}$ holomorphic. Then for every closed curve γ in D, $\int_\gamma f \, \mathrm{d}z = 0$.

Proof. By Theorem 16, $\int_{\partial\triangle} f(z) \, \mathrm{d}z = 0$ for every triangle $\triangle \subset D$. Then by Lemma 1, there exists a holomorphic function $F : D \to \mathbb{C}$ with $F'(z) = f(z)$, so by the Fundamental Theorem of Calculus, $\int_\gamma f(z) \, \mathrm{d}z = 0$ for every closed curve γ in D. □

4.5 Cauchy's integral formula

Theorem 19. Let $D = D(a, r)$ be a disc and $f : D \to \mathbb{C}$ holomorphic. If $w \in D$ and $|a - w| < \rho < r$ then

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} \, \mathrm{d}z$$

where γ is the circle $|z - a| = \rho$, i.e. $\gamma : [0,1] \to \mathbb{C}$ given by $\gamma(t) = a + \rho e^{2\pi i t}$.

Proof. Note that Theorem 18 holds if f is merely continuous at a finite set of points (and holomorphic elsewhere), but using Theorem 17 instead of Theorem 16 in the proof.

Let

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z = w \end{cases}$$

which is holomorphic on $D \setminus \{w\}$ and continuous on D. Then $\int_\gamma g(z)\,\mathrm{d}z = 0$, so

$$\begin{aligned} \int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z &= \int_\gamma \frac{f(w)}{z-w}\,\mathrm{d}z \\ &= \int_\gamma f(w) \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}\,\mathrm{d}z \end{aligned}$$

since $\sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} = \frac{1}{z-a} / \left(1 - \frac{w-a}{z-a}\right) = \frac{1}{z-w}$, converging uniformly on $|z-a| = \rho$ (by the Weierstrass M-test since $\left|\frac{w-a}{z-a}\right| = \frac{|w-a|}{\rho} < 1$). (See Proposition 12.)

By uniform convergence, we can swap limits and integrate term-by-term, so

$$\int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z = \sum_{n=0}^{\infty} f(w)(w-a)^n \int_\gamma \frac{1}{(z-a)^{n+1}}\,\mathrm{d}z$$

and we have that

$$\int_\gamma \frac{1}{(z-a)^{n+1}}\,\mathrm{d}z = \begin{cases} 0 & \text{if } n \neq 0 \\ 2\pi i & \text{if } n = 0 \end{cases}$$

so only the $n = 0$ term remains, and the result follows. □

Corollary 20 (Mean Value Property)**.** If $f : D(w, r) \to \mathbb{C}$ is holomorphic, then $\forall \rho \in (0, r)$,

$$f(w) = \int_0^1 f\left(w + \rho e^{2\pi i t}\right) \mathrm{d}t$$

Proof. Take $a = w$ in Theorem 19, so

$$\begin{aligned} 2\pi i f(w) &= \int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z \\ &= \int_0^1 \frac{f(\gamma(t))\gamma'(t)}{\gamma(t) - w}\,\mathrm{d}t \\ &= \int_0^1 \frac{f(w + \rho e^{2\pi i t})\rho 2\pi i e^{2\pi i t}}{\rho e^{2\pi i t}}\,\mathrm{d}t \\ &= 2\pi i \int_0^1 f(w + \rho e^{2\pi i t})\,\mathrm{d}t \end{aligned}$$

□

4.6 Applications of Cauchy integral formula

Theorem 21 (Liouville's Theorem)**.** Every bounded entire function is constant.

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and bounded, so there exists M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then if $w \in \mathbb{C}$ and $R > |w|$, by Cauchy's Integral Formula,

$$\begin{aligned}
|f(w) - f(0)| &= \frac{1}{2\pi} \left| \int_{|z|=R} f(z) \left(\frac{1}{z-w} - \frac{1}{z} \right) \mathrm{d}z \right| \\
&= \frac{1}{2\pi} \left| \int_{|z|=R} \frac{w f(z)}{z(z-w)} \mathrm{d}z \right| \\
&\leq \frac{1}{2\pi} 2\pi R \frac{|w|M}{R(R-|w|)} \to 0 \text{ as } R \to \infty
\end{aligned}$$

But $f(w)$ is independent of R, so $f(w) = f(0)$ for all $w \in \mathbb{C}$, i.e. f is constant. □

Theorem 22 (Fundamental Theorem of Algebra)**.** Every non-constant polynomial (with real or complex coefficients) has a complex root.

Proof. Let $P(z)$ be a non-constant polynomial. Then $|P(z)| \to \infty$ as $|z| \to \infty$, so there exists R such that $|P(z)| \geq 1$ for all z with $|z| \geq R$. Let $f(z) = \frac{1}{P(z)}$, which is holomorphic except at the zeroes of $P(z)$. Suppose that $P(z)$ has no zeroes, so that f is entire. Then since f is continuous, $|f(z)|$ is bounded for $|z| \leq R$ and we have $|f(z)| \leq 1$ for $|z| \geq R$, so $|f(z)|$ is bounded on $\mathbb{C}$. Then by Liouville's Theorem, f is constant, which is a contradiction. □

Theorem 23 (Local maximum modulus principle)**.** Let $f : D(a,r) \to \mathbb{C}$ be holomorphic. If $\forall z \in D(a,r)$ we have $|f(z)| \leq |f(a)|$ (i.e. a maximises the modulus of f locally) then f is constant.

Proof. By the Mean-Value Property, for $0 < \rho < r$,

$$\begin{aligned}
|f(a)| &= \left| \int_0^1 f(a + \rho e^{2\pi i t}) \, \mathrm{d}t \right| \\
&\leq \sup_{|z-a|=\rho} |f(z)|
\end{aligned}$$

so by the hypothesis, $|f(z)| = |f(a)|$ for all z with $|z - a| = \rho$ (equality follows from Theorem 10). But ρ is arbitrary, so $|f|$ is constant on $D(a,r)$, so f is constant (by the Cauchy-Riemann equations, for example). □

5 Taylor expansion

Example. Recall that if $f : (-1,1) \to \mathbb{R}$ is infinitely differentiable, its Taylor series at $x = 0$ need not converge to f (it need not even converge! For example, take

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

then $f^{(n)}(0) = 0$ for all n.

We don't have this problem over $\mathbb{C}$.

Theorem 24. Let $f : D(a,r) \to \mathbb{C}$ be holomorphic. Then f has a representation as a power series, converging on $D(a,r)$, as

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z$$

for any $0 < \rho < r$.

Proof. By Cauchy's Integral Formula, if $|w-a| < \rho < r$ then

$$f(w) = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-w} \, \mathrm{d}z$$

and we have that

$$\frac{1}{z-w} = \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}$$

converging uniformly for $|z-a| = \rho$ (see proof of Theorem 19). Then

$$f(w) = \frac{1}{2\pi i} \int_{|z-a|=\rho} \sum_{n=0}^{\infty} f(z) \frac{(w-a)^n}{(z-a)^{n+1}} \, \mathrm{d}z$$

and as $|f|$ is bounded on $|z-a| = \rho$, by uniform convergence we can exchange the integral and summation, so

$$f(w) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z \right) (w-a)^n$$

for all $w \in D(a,\rho)$ where $\rho < r$. So f has a convergence power series representation on $D(a,\rho)$ for each $\rho < r$. Since power series are unique, these series are the same for all ρ, so represent f on $D(a,r)$. $\square$

Corollary 25. If $f : U \to \mathbb{C}$ is holomorphic on an open $U \subset \mathbb{C}$ then f has derivatives of all orders.

Proof. Since convergent power series have derivatives of all orders, by Taylor's Theorem, f has derivatives of all orders on any $D(a,r) \subset U$. $\square$

Defintion (Analytic)**.** We say that $f : U \to \mathbb{C}$ is *analytic* if $\forall a \in U, \exists r > 0$ such that $D(a,r) \subset U$ and f has a representation by a convergent power series on $D(a,r)$.

Remark. With the above definition, the Theorem says that holomorphic $\iff$ analytic.

We now get a converse to Cauchy's Theorem.

Corollary 26 (Movera's Theorem)**.** Let $D = D(a,r)$ and $f : D \to \mathbb{C}$ be continuous, with $\int_\gamma f \, \mathrm{d}z = 0$ for any closed curve γ in D (in fact, triangles suffice). Then f is holomorphic.

Proof. By the converse to the Fundamental Theorem of Calculus (Theorem 15), $f = F'$ for some holomorphic F. Then by Corollary 25, f is holomorphic (since F has derivatives of all orders). $\square$

Lemma 2 (*Easy case of Fubini's Theorem*)**.** Let $f[a,b] \times [c,d] \to \mathbb{R}$ be continuous. Then the functions $f_1(x) = \int_c^d f(x,y)\,\mathrm{d}y$ and $f_2(y) = \int_a^b f(x,y)\,\mathrm{d}x$ are continuous and $\int_a^b f_1(x)\,\mathrm{d}x = \int_c^d f_2(y)\,\mathrm{d}y$.

Corollary 27. Let $U \subset \mathbb{C}$ be open and $a < b \in \mathbb{R}$. Let $\phi : U \times [a,b] \to \mathbb{C}$ be continuous such that for all $t \in [a,b]$ the function $z \mapsto \phi(z,t)$ is holomorphic. Then $f(z) = \int_a^b \phi(z,t)\,\mathrm{d}t$ is holomorphic on U.

Proof. Note that f is holomorphic at $w \in U$ if and only if f is holomorphic at $w \in D(w,r) \subset U$, so WLOG U is a disc.

Let $\gamma = \sum \gamma_i : [0,1] \to U$ be a piecewise C' closed curve. By Lemma 2,

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \sum_i \int_{\gamma_i} f(z)\,\mathrm{d}z \\
&= \sum_i \int_0^1 \left(\int_a^b \phi(\gamma_i(s),t)\,\mathrm{d}t\right) \gamma_i'(s)\,\mathrm{d}s \\
&= \int_a^b \left(\sum_i \int_0^1 \phi(\gamma_i(s),t)\gamma_i'(s)\,\mathrm{d}s\right) \mathrm{d}t \\
&= \int_a^b \left(\int_\gamma \phi(z,t)\,\mathrm{d}z\right) \mathrm{d}t \\
&= 0
\end{aligned}$$

by Cauchy's Theorem for a disc. $\square$

Example. The gamma function

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,\mathrm{d}t$$

is holomorphic for $\mathrm{Re}(z) \geq 1$.

Defintion (Order of zeroes)**.** If $f : D(w,r) \to \mathbb{C}$ is holomorphic and not identically zero, then $f(z) = \sum_{n=0}^\infty c_n(z-w)^n$ with not all c_n vanishing. Taking $m = \min\{n \geq 0 : c_n \neq 0\}$, the least n for which $f^{(n)}(w) \neq 0$, we have $f(z) = (z-w)^m g(z)$ where $g(z)$ is holormophic on $D(a,r)$ and $g(w) = c_m \neq 0$. If $m > 0$, we say that f has a zero of order m at $z = w$.

Theorem 28 (Zeroes of analytic functions are isolated)**.** Let $f : D(w,r) \to \mathbb{C}$ be holomorphic and not identically zero. Then there exists $\rho \in (0,r)$ such that for all $z \neq w$ with $|z-w| < \rho$ we have $f(z) \neq 0$. In particular, we can't have a sequence of distinct w_i with $(w_i) \to w$ and $f(w_i) = 0$, unless f is identically zero.

Proof. Write $f(z) = (z-w)^m g(z)$ with $m \geq 0$ and $g(z)$ holomorphic with $g(w) \neq 0$. Then g is continuous, so $\exists \rho > 0$ such that if $|z-w| < \rho$ then $|g(z) - g(w)| < g(w)$. So $g(z) \neq 0$ for all z with $|z-w| < \rho$, hence $f(z) \neq 0$ for all z with $0 < |z-w| < \rho$. $\square$

5.1 Analytic continuation

There are many ways in general to extend a function on $[0,1] \to \mathbb{R}$. For analytic functions, this doesn't happen.

Theorem 29 (Uniqueness of analytic continuation)**.** Let $D' \subset D \subset \mathbb{C}$ be domains and $f : D' \to \mathbb{C}$ analytic. Then there exists *at most* one analytic function $g : D \to \mathbb{C}$ whose restriction to D' is f (i.e. such that $\forall z \in D', f(z) = g(z)$). If g exists, it is said to be an *analytic continuation* of f.

Proof. Suppose that $g_1, g_2 : D \to \mathbb{C}$ both restrict to f on D'. Then $h = g_1 - g_2 : D \to \mathbb{C}$ is holomorphic with $h(z) = 0 \; \forall z \in D'$.

Define

$$D_0 = \{w \in D : h = 0 \text{ on some } D(w,r) \subset D\}$$
$$D_1 = \{w \in D : h^{(n)}(w) \neq 0 \text{ for some } n \geq 0\}$$

Then for any $w \in D$, h has a power series expansion on some $D(w,r) \subset D$, and this vanishes if and only if all $h^{(n)}(w) = 0$. Hence $D = D_0 \cup D_1$ and $D_0 \cap D_1 = \emptyset$.

Note that D_0 is open (since if $w \in D_0$, then h restricted to $D(w,r)$ is identically 0, so $D(w,r) \subset D_0$ as well). Also, D_1 is open (since each $h^{(n)}$ is continuous). Since D is path connected, it is connected, so D is not the union of two disjoint non-empty open subsets. Since D_0 contains D' it is non-empty, so $D_0 = D$, i.e. $h = 0$ on D. □

Example. The function $f(z) = \sum_{n=0}^{\infty} z^n$ is analytic on $D(0,1)$ and on no larger $D(0,r)$. But it has an analytic continuation to $D = \mathbb{C} \setminus \{1\}$, namely $g(z) = \frac{1}{1-z}$.

Example. The function $f(z) = \sum_{n=0}^{\infty} z^{n^2}$ is analytic on $D(0,1)$. One can show that there is *no* domain $D \supsetneq D(0,1)$ to which f can be analytically continued. We say that $\{|z| = 1\}$ is a *natural boundary* for the power series f.

6 Complex integration II

6.1 Winding numbers

Let $\gamma : [a,b] \to \mathbb{C}$ be a continuous curve (not necessarily piecewise C') and $w \in \mathbb{C} \setminus \{\text{image of } \gamma\}$.

Defintion. Suppose that $\gamma(t) = w + r(t)e^{i\theta(t)}$ where $r, \theta : [a,b] \to \mathbb{R}$ are continuous with $r(t) > 0$. The *winding number* or *index* of γ about w is

$$I(\gamma; w) = \frac{1}{2\pi}\left[\theta(b) - \theta(a)\right]$$

Remark. Note that if γ is closed then $I(\gamma; w) \in \mathbb{Z}$.

For this definition to be well-defined, we should prove that such a θ exists.

Theorem 30. There exists such a continuous function $\theta(t)$, which is well-defined up to addition of an integer multiple of 2π.

Proof. We consider first the simple case where γ lies in a half-plane not containing w, then reduce to this case.

If $\mathrm{Re}(\gamma(t)-w) > 0$ (so γ lies in the half-plane to the right of w), then we can define $\theta(t) = \arg(\gamma(t)-w)$ using the principal branch of arg which is continuous on $\{\mathrm{Re}(z) > 0\}$. Similarly, if $\mathrm{Re}(\frac{\gamma(t)-w}{e^{i\alpha}}) > 0$ (i.e. the above situation rotated by α), we can define $\theta(t) = \alpha + \arg(\frac{\gamma(t)-w}{e^{i\alpha}})$.

In general, let $r(t) = |\gamma - w|$ and $\rho = \inf_{t\in[a,b]} r(t)$. Since $r(t)$ is a positive continuous function on a closed, bounded interval, we have $\rho > 0$. As γ is continuous (and hence uniformly continuous) on $[a,b]$, there exists $a = a_0 < a_1 < \ldots < a_n = b$ such that $\forall t \in [a_{i-1}, a_i]$, we have $|\gamma(t) - \gamma(a_i)| < \rho$.

Now because $|\gamma(a_i) - w| \geq \rho$ (by construction of ρ), the image of $[a_{i-1}, a_i]$ under γ lies in the disc $\{z \in \mathbb{C} : |z - \gamma(a_i)| < \rho\}$ which doesn't contain w - thus it is contained in a half-plane not containing w. So as above, we can construct a continuous $\theta_i : [a_{i-1}, a_i] \to \mathbb{R}$ such that $\forall t \in [a_{i-1}, a_i]$ we have $\gamma(t) = w + r(t)e^{i\theta_i(t)}$. By changing each θ_i by a suitable multiple of 2π, we can arrange for $\theta_i(a_i) = \theta_{i+1}(a_i)$. Then together these θ_i define the desired θ. $\square$

Remark. (i) If γ is closed and image(γ) is contained in a $\frac{1}{2}$-plane not containing w, then $I(\gamma; w) = 0$.

(ii) If γ, γ_1 and $|\gamma_1(t) - \gamma(t)| < \rho$ where $\rho = \inf_{[a,b]} |\gamma(t) - w|$, then $I(\gamma; w) = I(\gamma_1; w)$.

Proposition 31. Let $\gamma : [a,b] \to \mathbb{C} \setminus \{w\}$ be piecewise C' and closed. Then

$$I(\gamma; w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-w}\, \mathrm{d}z \ .$$

Proof. Write $\gamma(t) = w + r(t)e^{i\theta(t)}$. Then r and θ are piecewise C' and

$$\begin{aligned} \frac{1}{2\pi i}\int_\gamma \frac{1}{z-w}\,\mathrm{d}z &= \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)-w}\,\mathrm{d}t \\ &= \frac{1}{2\pi i}\int_a^b \frac{r'(t)}{r(t)} + i\theta'(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi i}\left[\log(r(t)) + i\theta(t)\right]_a^b \\ &= \frac{1}{2\pi}[\theta(b) - \theta(a)] \end{aligned}$$

since $r(a) = r(b)$ as γ is closed. $\square$

Defintion. Let $U \subset \mathbb{C}$ be an open set.

(i) A closed curve $\gamma : [a,b] \to U$ is *homologous to* 0 (in U) if $\forall w \notin U$ we have $I(\gamma, w) = 0$.

(ii) U is *simply-connected* if every closed γ in U is homologous to 0.

6.2 Cauchy's Theorem and integral formula

Theorem 32. Let $D \subset \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ holomorphic, and $\gamma : [a, b] \to \mathbb{C}$ a closed piecewise C' curve. Suppose that γ is homologous to 0 in D. Then

(i) $\int_\gamma f \, \mathrm{d}z = 0$

(ii) If $w \in D \setminus \text{image}(\gamma)$ then $\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} \, \mathrm{d}z = I(\gamma; w) f(w)$.

Proof. (i) Apply the second part of the theorem to the function $g(z) = (z - w)f(z)$, which has $g(w) = 0$.

(ii) Define $g : D \times D \to \mathbb{C}$ by

$$g(z, w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z = w \end{cases}$$

Since f is analytic, g is continuous and for fixed w is an analytic function of z (see proof of Theorem 5).

Now consider

$$h(w) = \begin{cases} \int_\gamma g(z, w) \, \mathrm{d}z & w \in D \\ \int_\gamma \frac{f(z)}{z-w} \, \mathrm{d}z & w \in E = \{w \in \mathbb{C} \setminus \text{image}(\gamma) : I(\gamma, w) = 0\} \end{cases}$$

Note that these expressions for h agree on $D \cap E$ since if $w \in D \cap E = \{w \in D \setminus \text{image}(\gamma) : I(\gamma, w) = 0\}$ then $\int_\gamma g(z, w) \, \mathrm{d}z = \int_\gamma \frac{f(z)}{z-w} \, \mathrm{d}z + f(w) \int_\gamma \frac{1}{z-w} \, \mathrm{d}z = \int_\gamma \frac{f(z)}{z-w} \, \mathrm{d}z + f(w)I(\gamma, w) = \int_\gamma \frac{f(z)}{z-w} \, \mathrm{d}z$. Also, since γ is homologous to 0 on D, we have $\mathbb{C} = D \cup E$.

By Corollary 27, we see that h is holomorphic on $\mathbb{C}$. Let $R = \sup |\gamma(t)|$. Then if $|w| > R$, γ lies in a half-plane not containing w, so $I(\gamma, w) = 0$, i.e. $w \in E$. Also,

$$|h(w)| \leq \text{length}(\gamma) \sup_\gamma \frac{|f|}{|w| - R} \to 0 \text{ as } |w| \to \infty$$

so h is bounded on $\mathbb{C}$, and hence by Liouville's Theorem h is constant and this constant is 0 (since the above limit is 0).

Now if $w \in D \setminus \text{image}(\gamma) = D \cap E$,

$$\begin{aligned} 0 &= \int_\gamma g(z, w) \, \mathrm{d}z \\ &= \int_\gamma \frac{f(z)}{z - w} \, \mathrm{d}z - f(w) \int_\gamma \frac{1}{z - w} \, \mathrm{d}z \\ &= \int_\gamma \frac{f(z)}{z - w} \, \mathrm{d}z - 2\pi i f(w) I(\gamma, w) \end{aligned}$$

from which the result follows.

□

We can define a cycle Γ to be a finite sum of closed curves

$$\Gamma = \gamma_1 + \ldots + \gamma_n$$

with each γ_i closed, and define the winding number

$$I(\Gamma, w) = \sum_i I(\gamma_i, w) \quad \text{if } w \notin \cup_i \operatorname{image}(\gamma_i) \ .$$

The same proof then shows that Theorem 32 holds for cycles which are homologous to 0 (i.e. $I(\Gamma, w) = 0 \ \forall w \notin D$).

Corollary 33 (Cauchy's Theorem for a simple-connected domain)**.** Suppose D is simply connected (i.e. $I(\gamma, w) = 0$ for all closed γ in D and for all $w \notin D$). Then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every holomorphic $f : D \to \mathbb{C}$ and closed γ in D.

6.3 Laurent expansion and residue theorem

Theorem 34. Let $f : A = \{z \in \mathbb{C} : r < |z-a| < R\} \to \mathbb{C}$ be holomorphic, where $0 \le r < R \le \infty$. Then

(i) f has a unique convergent series expansion (the *Laurent expansion*) on A as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} c_{-n}(z-a)^{-n} \ .$$

(ii) $\forall \rho \in (r, R)$,

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z \ .$$

(iii) If $r < \rho' < \rho < R$, the series converges uniformly on the closed annulus $\{z \in \mathbb{C} : \rho' \le |z-a| \le \rho\}$.

Proof. (i) Let $w \in A$ and choose ρ_1, ρ_2 with $r < \rho_2 < |w-a| < \rho_1 < R$. Let Γ be the cycle $\Gamma = \gamma_1 - \gamma_2$ where γ_i is the circle $|z-a| = \rho_i$. Then Γ is homologous to 0 in D, so by Cauchy's Integral Formula (as $I(\Gamma, w) = 1$),

$$\begin{aligned} f(w) &= \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{z-w}\,\mathrm{d}z - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(z)}{z-w}\,\mathrm{d}z \\ &= \qquad f_1(w) \qquad + \qquad f_2(w) \end{aligned}$$

As in Taylor's Theorem (Theorem 24), we have that $f_1(w) = \sum_{n=0}^{\infty} c_n (w-a)^n$ where $c_n = \frac{1}{2\pi i}\int_{|z-a|=\rho_1} \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z$.

Note that

$$-\frac{1}{z-w} = \frac{\dfrac{1}{w-a}}{1 - \dfrac{z-a}{w-a}} = \sum_{m=1}^{\infty} \frac{(z-a)^{m-1}}{(w-a)^m}$$

converging uniformly for $\left|\frac{z-a}{w-a}\right| = \frac{\rho_2}{|w-a|} < 1$. Hence $f_2(w) = \sum_{m=1}^{\infty} d_m (w-a)^{-m}$ where $d_m = \frac{1}{2\pi i}\int_{\gamma_2} f(z)(z-a)^{m-1}\,\mathrm{d}z$. Taking $c_{-m} = d_m$ gives the required series expression for f. Uniqueness follows from the expression for c_n in the second part.

(ii) By the third part of the Theorem, we can integrate the series term-by-term along $|z-a| = \rho$ and

$$\int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{m+1}} \, \mathrm{d}z = \sum_{n=-\infty}^{\infty} c_n \int_{|z-a|=\rho} (z-a)^{n-m-1} \, \mathrm{d}z = c_m 2\pi i$$

since all except the $\frac{1}{z-a}$ integrals vanish.

(iii) Let $r < \rho' < \rho < R$. Then $\sum_{n=0}^{\infty} c_n (z-a)^n$ has radius of convergence at least R so it converges uniformly on $\{|z-a| \leq \rho\}$. Also, if $z' = \frac{1}{z-a}$ then $\sum_{m=1}^{\infty} c_{-m}(z')^m$ has radius of convergence at least $\frac{1}{r}$ so it converges uniformly on $\{|z-a| = \frac{1}{|z'|} \geq \rho'\}$. Hence the whole series converges on the given annulus.

□

Remark. If f is holomorphic on the annulus A then $f = f_1 + f_2$ where f_1 is holomorphic for $|z-a| < R$ and f_2 is holomorphic for $|z-a| > r$.

Defintion (Singularities)**.** Consider now the particular case with $r = 0$, so that $f : D(a,R) \setminus \{a\} \to \mathbb{C}$ is holomorphic. We say that f has an *isolated singularity* at $z = a$. Let $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ be its Laurent expansion. There are three possibilities,

(i) $c_n = 0 \ \forall n < 0$. The Laurent expansion is a convergent power series, defining a holomorphic function on $D(a,R)$. We say that f has a *removable singularity* at $z = a$.

(ii) $\exists k > 0$ such that $c_k \neq 0$ but $c_n = 0 \ \forall n < -k$ (i.e. finitely-many c_n are non-zero for $n < 0$). We say that f has a *pole of order k* at $z = a$.

(iii) $c_n \neq 0$ for infinitely many $n < 0$. We say that f has an *essential singularity* at $z = a$.

Example. (i) Removable singularity. $f(z) = \frac{e^z - 1}{z}$ becomes holomorphic at $z = 0$ by defining $f(0) = 1$.

(ii) Pole. $f(z) = \frac{e^z}{z^5} = \frac{1}{z^5} + \frac{1}{z^4} + \frac{1}{2! z^3} + \dots$ has a pole of order 5 at $z = 0$.

(iii) Essential singularity. $f(z) = e^{1/z} = \dots + \frac{1}{2! z^2} + \frac{1}{z} + 1$.

Proposition 35. Let $f : D(a,R) \setminus \{a\} \to \mathbb{C}$ be holomorphic. Then f has a removable singularity at $z = a$ if and only if $(z-a)f(z) \to 0$ as $z \to a$.

Proof. $\implies$ If f has a removable singularity then

$$(z-a)f(z) = c_0(z-a) + c_1(z-a)^2 + \dots \to 0 \quad \text{as } z \to a$$

$\impliedby$ Define

$$g(z) = \begin{cases} (z-a)^2 f(z) & z \neq a \\ 0 & z = a \end{cases}$$

which is differentiable at $z = a$ with $g'(a) = \lim_{z \to a} (z-a) f(z) = 0$. Hence $g(z) = \sum_{n=2}^{\infty} d_n (z-a)^n$, and so $f(z) = \sum_{n=0}^{\infty} d_{n+2} (z-a)^n$ has a removable singularity.

□

Proposition 36. Let $f : D(a,R)\setminus\{a\} \to \mathbb{C}$ be holomorphic. Then f has a pole at $z=a$ if and only if $|f(z)| \to \infty$ as $z \to a$. Moreover, the following are equivalent.

(i) f has a pole of order $k>0$ at $z=a$.

(ii) $f(z) = (z-a)^{-k}g(z)$ with g holomorphic at $z=a$ and $g(a)\neq 0$.

(iii) $f(z) = \frac{1}{h(z)}$ with h holomorphic with a zero of order k at $z=a$.

Proof.

- (i) $\iff$ (ii). If $f(z) = c_{-k}(z-a)^{-k} + \dots$ then $g(z) = c_{-k} + c_{1-k}(z-a) + \dots$ is holomorphic and non-zero at $z=a$. Conversely, if g is holomorphic and non-zero at $z=a$ and $f=(z-a)^{-k}g(z)$, then the Laurent expansion of $f = \frac{1}{(z-a)^k}(\text{Taylor series of } g) = g(a)(z-a)^{-k} + \dots$.

- (ii) $\iff$ (iii). If $f(z) = (z-k)^{-k}g(z)$ with g holomorphic at $z=a$ and $g(a)\neq 0$, then $g(z)$ is non-zero on some $D(a,\epsilon)$ by continuity, so $h(z) = \frac{1}{f(z)} = (z-a)^k\frac{1}{g(z)}$ is holomorphic with a zero of order k at $z=a$. Conversely, if $f(z) = \frac{1}{h(z)}$ with h holomorphic with a zero of order k at $z=a$ then $h(z) = (z-a)^k l(z)$ with $l(z)$ holomorphic and non-zero at $z=a$. Then l is non-zero on some $D(a,\epsilon)$ by continuity, so $g(z) = \frac{1}{l(z)}$ is holomorphic and non-zero at $z=a$, and $f(z) = (z-a)^{-k}g(z)$.

- If f has a pole of order k at $z=a$ then $f=\frac{1}{h}$ where h has a zero of order k by (iii). Then $\frac{1}{f}\to 0$ as $z\to a$, so $|f|\to\infty$ as $z \to a$.

- If $|f|\to\infty$ as $z\to a$, then $f\neq 0$ for $|z-a|<\epsilon$ for some $\epsilon>0$. Hence $g=\frac{1}{f}$ is holomorphic on $D(a,\epsilon)\setminus\{a\}$ and $g(z)\to 0$ as $z\to a$. By Proposition 35, g has a removable singularity at $z=a$, so there exists a holomorphic function $h : D(a,\epsilon)\to\mathbb{C}$ extending g. Then $h(a) = \lim_{z\to a} g(z) = 0$ so h has a zero of some order. Then $f=\frac{1}{h}$ has a pole at $z=a$ by (iii).

$\square$

Corollary 37. Let $f : D(a,R)\setminus\{a\}\to\mathbb{C}$ be holomorphic. Then f has an *essential singularity* at $z=a$ if and only if $|f(z)|$ has no limit (in $\mathbb{R}\cup\{\infty\}$) as $z\to a$.

Proof. If f doesn't have an essential singularity, then either $f(z)$ tends to some limit in $\mathbb{C}$ (if f has a removable singularity at $z=a$), or $|f(z)|\to\infty$ (if f has a pole at $z=a$).

Conversely, suppose $|f|\to l$ as $z\to a$. If $l=\infty$, then f has a pole at $z=a$. If $l<\infty$ then $|f|$ is bounded in a neighbourhood of $z=a$ so $(z-a)f(z)\to 0$ as $z\to a$ so f has a removable singularity at $z=a$ by Proposition 35. $\square$

Defintion (Meromorphic)**.** The function f is *meromorphic* on D if there exists a set $S\subset D$ of isolated points such that $f : D\setminus S\to\mathbb{C}$ is holomorphic, with at worst pole (i.e. poles or removable singularities) at S.

Remark. In some sense, poles are 'not really' singularities: if $f : D\setminus\{a\}\to\mathbb{C}$ is holomorphic with a pole at $z=a$, we can extend it to a map from D to $\mathbb{C}\cup\{\infty\} = \mathbb{CP}'$ (the *Riemann sphere*) by setting $f(a)=\infty$. This is continuous and (in a well-defined sense) is a holomorphic map from D to $\mathbb{CP}'$.

Defintion (Residue & principal part)**.** Let f have an isolated singularity at $z = a$ and Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ for $0 < |z-a| < r$. The *residue* of f at $z = a$ is $\operatorname{Res}(f, z = a) = c_{-1}$, the coefficient of $\frac{1}{z-a}$.

The *principal part* of f at $z = a$ is the series $\sum_{n=-\infty}^{-1} c_n(z-a)^n$.

Proposition 38. Let f be holomorphic on $D = D(a, r) \setminus \{a\}$ and γ a closed curve in D. Then

$$\int_\gamma f \,\mathrm{d}z = 2\pi i \operatorname{Res}(f, z = a) I(\gamma, a) \ .$$

In particular, if $0 < \rho < r$ then $\int_{|z-a|=\rho} f \,\mathrm{d}z = 2\pi i \operatorname{Res}(f, z = a)$.

Proof. There exist ρ, ρ' with $0 < \rho < \rho' < r$ such that $|\gamma(t) - a)| \in [\rho, \rho']$ for all t (the image of the continuous function $\gamma(t) - a$ is bounded). Then the Laurent series for f is uniformly convergent for $\rho \le |z-a| \le \rho'$, so we can integrate term-by-term, noting that

$$\int_\gamma (z-a)^n \,\mathrm{d}z = \begin{cases} 0 & n \neq -1 \\ 2\pi i I(\gamma, a) & n = -1 \end{cases}$$

from which the result follows. □

Theorem 39 (Residue Theorem)**.** Let f be meromorphic on D. Let γ be a closed curve in D which is homologous to 0. Assume that f has no poles on γ and has a finite number of poles $a_1, \ldots, a_m$ in D for which $I(\gamma, a_i) \neq 0$. Then

$$\int_\gamma f \,\mathrm{d}z = 2\pi i \sum_{i=1}^{m} \operatorname{Res}(f, z = a_i) I(\gamma, a_i) \ .$$

Proof. Write $f = g + \sum_{i=1}^{m} g_i$ where g is holomorphic at $\{a_i\}$ and g_i is the principle part of f at a_i. Let $D' = D \setminus \{\text{singularities of } g\}$. By hypothesis, $I(\gamma, b) = 0$ if b is a singularity of g, so γ is homologous to 0 in D'. Then

$$\begin{aligned} \int_\gamma f \,\mathrm{d}z &= \int_\gamma g \,\mathrm{d}z + \sum_i \int_\gamma g_i \,\mathrm{d}z \\ &= 0 \qquad + \sum_i 2\pi i \operatorname{Res}(f, z = a_i) I(\gamma, a_i) \end{aligned}$$

by Cauchy's Theorem and Proposition 38. □

Defintion (Bounding curve)**.** A curve (or cycle) γ *bounds* a domain D if $\forall w \in D$, $I(\gamma, w) = 1$ and $\forall w \notin D \cup \operatorname{image}(\gamma)$, $I(\gamma, w) = 0$.

The following is a more traditional version of Theorem 39.

Theorem 40. Let γ bound a domain D. Suppose f is meromorphic on an open set containing D and γ, and holomorphic on γ. Then

$$\int_\gamma f \,\mathrm{d}z = 2\pi i \sum_i \operatorname{Res}(f, a_i)$$

where $\{a_i\}$ is the set of poles of f in D.

6.4 Evaluation of definite integrals

The idea is to convert a real integral into a (possibly real part of) a complex integral, then compute it using the residue theorem.

Example.

$$I = \int_0^{2\pi} \frac{1}{5 + 4\cos\theta}\,\mathrm{d}\theta = I$$

Note that

$$\frac{1}{5 + 4\cos\theta} = \frac{1}{5 + 2e^{i\theta} + 2e^{-i\theta}} = \frac{e^{i\theta}}{(2e^{i\theta} + 1)(e^{i\theta} + 2)}$$

Therefore we consider

$$f(z) = \frac{1}{(2z + 1)(z + 2)}$$

so that taking $\gamma : [0, 2\pi] \to \mathbb{C}$ to be the unit circle $\gamma(\theta) = e^{i\theta}$ with $\gamma'(\theta) = ie^{i\theta}$, we have

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^{2\pi} f(\gamma(\theta))\gamma'(\theta)\,\mathrm{d}\theta = i\int_0^{2\pi} \frac{1}{5 + 4\cos\theta}\,\mathrm{d}\theta = iI$$

On the other hand, the function $g(z) = \frac{1}{z+2}$ is holomorphic on $\{|z| < 2\}$, so applying the Cauchy integral formula (at the point $z = -\frac{1}{2}$), we get

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \frac{1/2}{(z + 1/2)(z + 2)}\,\mathrm{d}z = 2\pi i \left[\frac{1/2}{z + 2}\right]_{z=-\frac{1}{2}} = \frac{2\pi i}{3}$$

Hence $I = \frac{2\pi}{3}$.

Note that a similar method will work for integrals of the form $\int_0^{2\pi} \phi(\sin\theta, \cos\theta)$ where ϕ is some rational function. For more complicated examples, it might be necessary to use the residue theorem. In this case, we would have

$$\int_{|z|=1} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}\left(\frac{1}{(2z + 1)(z + 2)}, z = -\frac{1}{2}\right)$$

since f is holomorphic except for simple poles at $z = -\frac{1}{2}$ and $z = -2$.

Suppose that a function f has a pole at $z = a$. We would like to be able to compute $\operatorname{Res}(f, z = a)$. There are several possible methods.

(i) Compute some of the Laurent expansion. For example, one could write

$$f(z) = \frac{g(z)}{h(z)}$$

where g, h have known power series, and use the method of undetermined coefficients.

(ii) If $f(z)$ has a *simple pole*, then

$$f(z) = c_{-1}(z - a)^{-1} + c_0 + \dots$$

so $\operatorname{Res}(f, z = a) = c_{-1} = \lim_{z\to a}(z - a)f(z)$.

(iii) A special case of the above. Suppose $f(z) = \frac{g(z)}{h(z)}$ with g, h holormophic at $z = a$, $g(a) \neq 0$ and $h(a) = 0 \neq h'(a)$ (i.e. h has a simple 0 at $z = a$). Then

$$\operatorname{Res}(f, z = a) = \lim_{z \to a} \frac{(z-a)g(z)}{h(z)} = \frac{g(a)}{h'(a)}$$

(by L'Hôpital's rule).

(iv) If $f(z)$ has a pole of order $k > 1$, then for some holomorphic g with $g(a) \neq 0$,

$$f(z) = (z-a)^{-k} g(z) = (z-a)^{-k} \left[g(a) + \sum_{n \geq 1} \frac{1}{n!} g^{(n)}(a)(z-a)^n \right]$$

so $\operatorname{Res}(f, z = a) = \frac{g^{(k-1)}(a)}{(k-1)!}$.

Example.

$$I = \int_{-\infty}^{\infty} \frac{\cos(mx)}{x^2+1} \, \mathrm{d}x$$

Write $I = \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+1} \, \mathrm{d}x$ (the imaginary part of the RHS is 0 as $\frac{\sin x}{x^2+1}$ is odd). Then

$$I = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{imx}}{x^2+1} \, \mathrm{d}x = \lim_{R \to \infty} \int_{\gamma_1} \frac{e^{imz}}{z^2+1} \, \mathrm{d}z$$

where γ_1 is $[-R, R]$, i.e. $\gamma_1(x) = x$ for $-R \leq x \leq R$.

Let γ_2 be the semi-circle $\gamma_2(t) = Re^{it}$ for $0 \leq t \leq \pi$, and let $\gamma = \gamma_1 + \gamma_2$ be the closed curve. Write $f(z) = \frac{e^{imz}}{z^2+1}$.

We can use the residue theorem to find $\int_\gamma f \, \mathrm{d}z$, and show that $\int_{\gamma_2} f \, \mathrm{d}z \to 0$ as $R \to \infty$, giving I. We show $\int_{\gamma_2} f \, \mathrm{d}z \to 0$ first.

For z on γ_2, we have $|z| = R$ and $\operatorname{Im}(z) \geq 0$, so $|e^{imz}| = e^{-m \operatorname{Im}(z)} \leq 1$ and $|z^2 + 1| \geq R^2 - 1$ (since $|z^2+1| + |-1| \geq |z^2 + 1 - 1| = R^2$ by the triangle inequality). Hence for $R > 1$,

$$\int_{\gamma_2} f(z) \, \mathrm{d}z \leq \underbrace{\pi R}_{\text{length}(\gamma_2)} \times \underbrace{\frac{1}{R^2 - 1}}_{\text{upper bound for } |f|} \to 0 \text{ as } R \to \infty$$

Now we evaluate $\int_\gamma f \, \mathrm{d}z$. Note that f has simple poles at $z = \pm i$ and is holomorphic elsewhere. Then since $I(\gamma, i) = 1$ and $I(\gamma, -i) = 0$, we have

$$I = 2\pi i \operatorname{Res}(f, z = i) = 2\pi i \left[\frac{e^{imz}}{2z} \right]_{z=i} = \frac{\pi}{e^m}$$

where we have used the fact (noted earlier) that if $f = \frac{g}{h}$ then $\operatorname{Res} = \frac{g(a)}{h'(a)}$.

Example.

$$I = \int_0^\infty \frac{\sin x}{x} \, \mathrm{d}x$$

Note that this integral converges (but not absolutely) - you can see this by applying the alternating series test to the integral split into intervals on which sin is positive or negative. Since $\frac{\sin x}{x}$ is even, we write

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x = \frac{1}{2} \operatorname{Im} \left(\lim_{\substack{R \to \infty \\ r \to 0}} \left[\int_r^R \frac{e^{ix}}{x} \, \mathrm{d}x + \int_{-R}^{-r} \frac{e^{ix}}{x} \, \mathrm{d}x \right] \right)$$

Let $f(z) = \frac{e^{iz}}{z}$ which has a pole at $z = 0$ and is holomorphic elsewhere. Let γ_1 and γ_3 be the intervals $[-R, -r]$ and $[r, R]$, and γ_2 and γ_4 the semicircles $|z| = r$ and $|z| = R$ that lie in the upper half plane, oriented such that $\gamma = \sum_i \gamma_i$ forms a closed curve. Specifically,

$$\begin{aligned}
\gamma_1(t) &= t & -R \le t \le -r \\
\gamma_2(t) &= re^{i(\pi - t)} & 0 \le t \le \pi \\
\gamma_3(t) &= t & r \le t \le R \\
\gamma_4(t) &= Re^{it} & 0 \le t \le \pi
\end{aligned}$$

Then Cauchy's integral theorem implies that $\int_\gamma f \, \mathrm{d}z = 0$ for all r, R.

Lemma 3. Let f be holomorphic on $D(a, R) \setminus \{a\}$ with a *simple pole* at $z = a$. Let $\delta_r : [\alpha, \beta] \to \mathbb{C}$ be given by $\delta_r(t) = a + re^{it}$ (where $r > 0$). Then

$$\lim_{r \to 0} \int_{\delta_r} f(z) \, \mathrm{d}z = (\beta - \alpha) i \operatorname{Res}(f, z = a)$$

Proof. Write $f = \frac{c}{z-a} + g(z)$ where $g(z)$ is holomorphic at $z = a$. Then

$$\int_{\delta_r} \frac{c}{z - a} \, \mathrm{d}z = \int_\alpha^\beta \frac{crie^{it}}{re^{it}} \, \mathrm{d}t = (\beta - \alpha) i \underbrace{\operatorname{Res}(f, z = a)}_{=c}$$

and

$$\left| \int_{\delta_r} g(z) \, \mathrm{d}z \right| \le \underbrace{(\beta - \alpha) r}_{\text{length}(\delta_r)} \sup_{\delta_r} |g| \to 0 \text{ as } r \to 0$$

since g is continuous and bounded on $\{|z - a| \le \frac{R}{2}\}$. □

Then by the lemma, $\int_{\gamma_2} f(z) \, \mathrm{d}z = -\int_{(-\gamma_2)} f(z) \, \mathrm{d}z \to -\pi i \operatorname{Res}(f, z = 0) = -\pi i$.

Lemma 4 (Jordan's Lemma)**.** Let f be holomorphic on $\{z : |z| > r\}$ and suppose that $|zf(z)|$ is bounded as $|z| \to \infty$. Then if $\alpha > 0$, and $\gamma_R : [0, \pi] \to \mathbb{C}$ is given by $\gamma_R(t) = Re^{it}$, then

$$\int_{\gamma_R} e^{i\alpha z} f(z) \, \mathrm{d}z \to 0 \text{ as } R \to \infty$$

Proof. Since $|zf(z)|$ is bounded as $|z| \to \infty$, there exists C such that $|f(z)| \le \frac{C}{|z|}$ for $|z|$ sufficiently large. Note that $\sin(t) \ge \frac{2t}{\pi}$ for $0 \le t \le \frac{\pi}{2}$ so if $z = Re^{it}$ with $0 \le t \le \pi$ then

$$|e^{i\alpha z}| = e^{-\alpha R \sin t} \le \begin{cases} e^{-2\alpha R t/\pi} & 0 \le t \le \pi/2 \\ e^{-2\alpha R t'/\pi} & 0 \le t' = \pi - t \le \pi/2 \end{cases}$$

Thus the integral over the part of γ_R with $0 \le t \le \frac{\pi}{2}$ has modulus

$$\begin{aligned}
\left| \int_0^{\frac{\pi}{2}} f(Re^{it}) e^{i\alpha Re^{it}} iRe^{it} \, \mathrm{d}t \right| &\le \int_0^{\frac{\pi}{2}} R \left| f(Re^{it}) \right| e^{-2\alpha Rt/\pi} \, \mathrm{d}t \\
&\le C \int_0^{\frac{\pi}{2}} e^{-2\alpha Rt/\pi} \, \mathrm{d}t \\
&= C \left[-\frac{\pi}{\alpha R} e^{-2\alpha Rt/\pi} \right]_0^{\frac{\pi}{2}} \to 0 \text{ as } R \to \infty
\end{aligned}$$

and similarly for the other part of γ_R. □

Since $f(z) = \frac{1}{z}$ satisfies the hypotheses of the lemma, we get that $\int_{\gamma_4} f(z)\,\mathrm{d}z \to 0$ as $R \to \infty$.

Thus $\lim_{\substack{R\to\infty \\ r\to 0}} \int_{\gamma_1} f\,\mathrm{d}z + \int_{\gamma_3} f\,\mathrm{d}z = -\lim_{\gamma_2} f\,\mathrm{d}z = \pi i$, so $I = \frac{\pi}{2}$.

Remark. Another way to show that $\int_{\gamma_4} f\,\mathrm{d}z \to 0$ is to integrate by parts,

$$\int_{\gamma_4} \frac{e^{iz}}{z}\,\mathrm{d}z = \underbrace{\left[\frac{1}{i}e^{iz}\frac{1}{z}\right]_{-R}^{R}}_{\substack{=\frac{1}{iR}e^{iR}+\frac{1}{iR}e^{-iR} \\ \to 0 \text{ as } R\to\infty}} - \underbrace{\int_{\gamma_4} \frac{1}{i}e^{iz}\frac{-1}{z^2}\,\mathrm{d}z}_{\to 0 \text{ by naïve estimate}}$$

Example.

$$I = \int_0^\infty \frac{x^\alpha}{1+x^2}$$

where $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$.

Let $f(z) = \frac{z^\alpha}{1+z^2}\,\mathrm{d}z$, where z^α is the branch holomorphic on $\mathbb{C} \setminus i\mathbb{R}_{\leq 0}$ (i.e. if $z = re^{i\theta}$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ then $z^\alpha = r^\alpha e^{i\alpha\theta}$). Let γ_i $(i = 1,2,3,4)$ and $\gamma = \sum \gamma_i$ be as before. Then

- $\int_\gamma f\,\mathrm{d}z = 2\pi i \operatorname{Res}(f, z = i)$.
- $\int_{\gamma_3} f\,\mathrm{d}z \to I$ as $r \to 0, R \to \infty$.
- $\int_{\gamma_1} f\,\mathrm{d}z = -\int_{(-\gamma_1)} \frac{z^\alpha}{1+z^2}\,\mathrm{d}z = -\int_r^R \frac{(-x)^\alpha}{1+x^2}\,\mathrm{d}x = e^{i\pi\alpha}\int_r^R \frac{x^\alpha}{1+x^2}\,\mathrm{d}x \to e^{i\pi\alpha}I$.
- $\left|\int_{\gamma_4} f\,\mathrm{d}z\right| \leq \pi R \times \frac{R^\alpha}{R^2-1} \to 0$ since $0 < \alpha < 1$.
- $\left|\int_{\gamma_2} f\,\mathrm{d}z\right| \leq \pi r \times \frac{r^\alpha}{1-r^2} \to 0$.

Thus

$$2\pi i \operatorname{Res}(f, z = i) = \lim_{\substack{R\to\infty \\ r\to 0}} \int_\gamma f\,\mathrm{d}z = \lim_{\substack{R\to\infty \\ r\to 0}} \left(\int_{\gamma_1} f\,\mathrm{d}z + \int_{\gamma_3} f\,\mathrm{d}z\right) = (1 + e^{\pi i\alpha})I$$

from which we can find I by arithmetic.

In a similar way, one can calculate integrals of functions like $\frac{\log x}{1+x^2}$.

Remark. If the denominator of the function to integrate is no longer even, for example

$$\int_0^\infty \frac{x^\alpha}{x^2+x+3}\,\mathrm{d}x$$

then one can use a similar technique to above but with the 'keyhole contour'.

6.5 The argument principle and Rouché's Theorem

Proposition 41. Suppose that f has a zero (or pole) at $z = a$ of order $k > 0$. Then $\frac{f'(z)}{f(z)}$ has a simple pole at $z = a$ with residue k (or $-k$).

Proof. Suppose f has a zero of order k. Then $f(z) = (z-a)^k g(z)$ with $g(a) \neq 0$, so $\frac{f'(z)}{f(z)} = k\frac{1}{z-a} + \frac{g'(z)}{g(z)}$ which has a simple pole with residue k. Similarly if f has a pole. $\square$

Theorem 42 (Principle of the argument)**.** Let γ be a closed curve (or cycle) bounding a domain D and let f be meromorphic on D and holomorphic on γ. Suppose that f is not identically zero, and has P poles and Z zeroes in D (counted according to multiplicities), but no zeroes on γ. Then

$$Z - P = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z = I(\Gamma, 0)$$

where $\Gamma = f \circ \gamma$ is the image of γ under f.

Proof. Since $f \neq 0$ on γ, Γ is a closed curve in $\mathbb{C} \setminus \{0\}$. Hence $I(\Gamma, 0) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{w} \,\mathrm{d}w = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z$. By the Residue Theorem, this integral is equal to the sum of the residues of $\frac{f'}{f}$ in D, which by Proposition 41 is equal to $Z - P$. □

Remark. The Theorem says that $2\pi(Z - P)$ equals the variation in the argument of $f(z)$ as z traces out γ.

Defintion. Let f be holomorphic at $z = a$ and not identically constant, with $f(a) = b$, say. Then *local degree of f at $z = a$* is $\deg_{z=a} f(z)$, the order of the zero of $f - b$ at $z = a$.

Proposition 43. For any circle $\gamma(t) = a + re^{2\pi i t}$ (for $t \in [0, 1]$), with $r > 0$ sufficiently small,

$$\deg_{z=a} f = I(f \circ \gamma, f(a))$$

Proof. Consider $g(z) = f(z) - f(a)$ which has a zero of order $\deg_{z=a} f$ at $z = a$, and for $\epsilon > 0$ sufficiently small, $g \neq 0$ on $D(a, \epsilon) \setminus \{a\}$ (see Theorem 28 on isolated zeroes). Then we if choose $0 < r < \epsilon$, by the argument principle,

$$I(f \circ \gamma, f(a)) = I(g \circ \gamma, 0) = Z - P = \deg_{z=a} f$$

□

Remark. One could say that 'morally', f behaves like $(z - a)^k$ near $z = a$.

Theorem 44. Let $f : D(a, R) \to \mathbb{C}$ be holomorphic and non-constant with $d = \deg_{z=a} f$. Then for $r > 0$ sufficiently small, there exists $\epsilon > 0$ such that $\forall w \in \mathbb{C}$ with $0 < |w - f(a)| < \epsilon$ the equation $f(z) = w$ has exactly d solutions in $D(a, r) \setminus \{a\}$.

Proof. Let $b = f(a)$ and choose $r > 0$ such that $f(z) - b$ and $f'(z)$ are non-zero for $0 < |z - a| \leq r$ (possible since f is non-constant and zeroes of holomorphic functions are isolated). Let γ be the circle $|z - a| = r$. Then $\Gamma = f \circ \gamma$ is a closed curve whose image doesn't contain b. Thus for some $\epsilon > 0$, Γ does not meet $D(b, \epsilon)$.

Now if $w \in D(b, \epsilon)$ then by the argument principle, the number of zeroes of $f(z) - w$ in $D(a, r)$ is equal to $I(\Gamma, w)$. But $I(\Gamma, w) = d = I(\gamma, b)$ and since $f' \neq 0$ on $D(a, r) \setminus \{a\}$, all the zeroes are simple. □

Corollary 45 (Open mapping theorem)**.** A non-constant holomorphic function $f : U \to \mathbb{C}$ (with $U \subset \mathbb{C}$ open) maps open sets to open sets. We say that f is an *open mapping*.

Proof. It suffices to prove that if $a \in U$ and $r > 0$ is sufficiently small (so that $D(a, r) \subset U$), then there exists $\epsilon > 0$ such that $f(D(a, r)) \supset D(f(a), \epsilon)$. This holds since by Theorem 44, $f(z) = w$ has a solution in $D(a, r)$ for all $w \in D(f(a), \epsilon)$. □

Theorem 46 (Rouché's Theorem)**.** Let γ be a closed curve (or cycle) bounding a domain D. If f, g are holomorphic on $D \cup \text{image}(\gamma)$ and $|f| > |g|$ on γ then f and $f + g$ have the same number of zeroes in D.

Proof. Let $h = \frac{f+g}{f} = 1 + \frac{g}{f}$. Since $|f| > |g|$ on γ, $h \circ \gamma$ is contained in $D(1, 1)$ in the right half-plane. Hence $I(h \circ \gamma, 0) = 0$. Thus by the argument principle, the number of poles of h in D is the numbers of zeroes of h in D. Hence the result. □

6.6 Uniform limits of analytic functions

Remark. Any continuous *real-valued* function on $[0, 1]$ is a uniform limit of polynomials. This is known as the Weierstrass approximation theorem.

Defintion. Let $U \subset \mathbb{C}$ be open and $f_n : U \to \mathbb{C}$ functions. Then (f_n) is *locally uniformly convergent* on U if $\forall a \in U$, $\exists r > 0$ such that $D(a, r) \subset U$ and (f_n) converges uniformly on $D(a, r)$.

Example.
- (i) Consider $f_n = \frac{1}{1-z^n}$ on $U = D(0, 1)$. This is not uniformly convergent on $D(0, 1)$ but it is uniformly convergent on any $D(0, r)$ with $r < 1$. Hence (f_n) is locally uniformly convergent on U.
- (ii) Every convergent power series is locally uniformly convergent on $\{|z - a| = R\}$ where R is its radius of convergence. It may (e.g. $\sum \frac{z^n}{z^2}$) or may not (e.g. $\sum z^n$) be uniformly convergent on this disc.

Remark. A sequence of functions $f_n : U \to \mathbb{C}$ is locally uniformly convergent on U if and only if for every *compact* subset $S \subset U$, (f_n) is uniformly convergent on S.

Theorem 47. Let (f_n) be a locally uniformly convergent sequence of holomorphic function on U. Then the limit function $f = \lim_{n\to\infty} f_n$ is holomorphic and $(f_n') \to f'$ locally uniformly on U.

Proof. Let $a \in U, r > 0$ with $D(a, r) \subset U$ and $(f_n) \to f$ uniformly on $D(a, r)$. It suffices to prove that f is holomorphic on $D(a, r)$ and that $(f_n') \to f'$ locally uniformly on $D(a, r)$.

By Cauchy's Theorem, $\int_\gamma f_n \, \mathrm{d}z = 0$ for any closed curve $\gamma \in D(a, r)$. Then since $f_n \to f$ uniformly on $D(a, r)$, $\int_\gamma f \, \mathrm{d}z = \lim_{n\to\infty} \int_\gamma f_n \, \mathrm{d}z = 0$, so by Movera's Theorem f is holomorphic.

Fix $\epsilon > 0$ and let $w \in D(a, r - \epsilon)$, then by Cauchy's Integral Formula (see Theorem 24),

$$\begin{aligned} |f_n'(w) - f'(w)| &= \frac{1}{2\pi} \left| \int_{|z-w|=\epsilon} \frac{f_n(z) - f(z)}{(z - w)^2} \, \mathrm{d}z \right| \\ &\leq \frac{1}{2\pi} 2\pi\epsilon \sup_{D(a,r)} \frac{|f_n - f|}{\epsilon^2} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

since $\sup_{D(a,r)} |f_n - f| \to 0$ by uniform convergence. So $f_n' \to f'$ uniformly on $D(a, r - \epsilon)$ and hence on $D(a, r)$. □

Remark. Another Weierstrass theorem says that every holomorphic function is a uniform limit of *rational* functions.