Analysis II - Definitions

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1 Uniform convergence

Definition 1 (Pointwise convergence). Let E be a set and $f_n, f : E \to \mathbb{R}$. We say that f_n converges to f pointwise if for each x the sequence of numbers $(f_n(x))$ converges to $f(x)$.

Definition 2 (Uniform convergence). Let E be a set and $f_n, f : E \to \mathbb{R}$. We say that f_n converges to f uniformly if

$$\forall \epsilon > 0 \quad \exists N = N(\epsilon) \quad \text{such that} \quad \forall n \geq N,\ \forall x \in E \quad |f_n(x) - f(x)| < \epsilon$$

Definition 3 (Convergence of series). Let $g_n : E \to \mathbb{R}$ for $n = 0, 1, 2, \ldots$ be functions. Let $x \in E$. The series $\sum_{n=0}^{\infty} g_n$ converges

- at x if the sequence $f_n = \sum_{j=0}^{n} g_j : E \to \mathbb{R}$ converges at x.
- *absolutely* at x if $\sum_{n=0}^{\infty} |g_n|$ converges at x.
- *uniformly* on E if the partial sums f_n converge uniformly on E.
- *absolutely uniformly* on E if $\sum_{n=0}^{\infty} |g_n|$ converges uniformly on E.

2 Uniform continuity

Definition 4 (Uniform continuity). Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Then f is uniformly continuous on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in E$ then $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Definition 5 (Upper and lower sums). Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $a = a_0 < a_1 < \ldots < a_n = b$ be a partition P of $[a, b]$. Then the upper sum is

$$U(P, f) = \sum_{j=0}^{n-1} (a_{j+1} - a_j) \sup_{[a_j, a_{j+1}]} f$$

and the lower sum is

$$L(P, f) = \sum_{j=0}^{n-1} (a_{j+1} - a_j) \inf_{[a_j, a_{j+1}]} f$$

Definition 6 (Riemann integral)**.** Let $f : [a, b] \to \mathbb{R}$ be bounded. Then the upper integral is

$$I^*(f) = \inf_p U(P, f)$$

and the lower integral is

$$I_*(f) = \sup_p L(P, f)$$

If $I_I(f) = I^*(f)$ then f is *Riemann integrable* and we define the *Riemann integral* to be the common value, denoted $\int_a^b f(x)dx$.

Definition 7 (Riemann integral in $\mathbb{R}^n$ and $\mathbb{C}$)**.** Let $f : [a, b] \to \mathbb{R}^n$ and write $f(x) = (f_1(x), \ldots, f_n(x))$. Then f is Riemann integrable if f_j is Riemann integrable for each $j = 1, \ldots n$, and $\int_a^b f(x)dx = (\int_a^b f_1(x)dx, \ldots, \int_a^b f_n(x)dx)$.

Let $f : [a, b] \to \mathbb{C}$. Then f is Riemann integrable if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are Riemann integrable, and $\int_a^b f(x)dx = \int_a^b \mathrm{Re}(f(x))dx + i \int_a^b \mathrm{Im}(f(x))dx$.

Definition 8 (Measure zero)**.** Let $E \subset \mathbb{R}$. Then E has *measure zero* if $\forall \epsilon > 0$ there exists a countable collection of open intervals $I_j = (a_j, b_j)$ such that $E \subset \bigcup_{j=1}^{\infty} I_j$ and $\sum_{j=1}^{\infty} |I_j| < \epsilon$ (where $|I_j| = b_j - a_j$).

3 Normed spaces

Definition 9 (Norm)**.** Let V be a vector space. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}$ satisfying the following properties:

(i) $\|v\| \geq 0$ for all $v \in V$

(ii) $\|v\| = 0$ if and only if $v = 0$

(iii) $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{R}$, $v \in V$

(iv) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$

A *normed space* is a pair $(V, \|\cdot\|)$ where V is a vector space and $\|\cdot\|$ is a norm.

Definition 10 (Example norms)**.**

1. $\mathbb{R}^n$ with the *Euclidean norm*

$$\|(x_1, \ldots, x_n)\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$$

2. $\mathbb{R}^n$ with the *taxicab norm*

$$\|(x_1, \ldots, x_n)\|_1 = \left(\sum_{i=1}^n |x_i|\right)$$

3. $\mathbb{R}^n$ with $(1 \leq p < \infty)$

$$\|(x_1, \ldots, x_n)\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$$

4. $\mathbb{R}^n$ with
$$\|(x_1,\ldots,x_n)\|_\infty = \sup\{|x_1|,\ldots,|x_n|\}$$
5. $C([a,b])$ with
$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)\|$$
6. $C([a,b])$ with
$$\|f\|_1 = \int_a^b |f(x)|dx$$
7. $C([a,b])$ with
$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$$

Definition 11 (Lipschitz qquivalence of norms)**.** Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space V are (Lipschitz) *equivalent* if there are numbers $0 < a < b$ such that $a\|v\| \leq \|v\|' \leq b\|v\|$ for all $v \in V$.

Definition 12 (Bounded)**.** Let $(V, \|\cdot\|)$ be a normed space. A subset $E \subseteq V$ is *bounded* if there exists $R > 0$ such that $\|v\| < R$ for all $v \in E$.

Definition 13 (Convergence)**.** Let $(V, \|\cdot\|)$ be a normed space. If $x_k, x \in V$ for $k = 1, 2, 3, \ldots$, then we say $x_k \to x$ if $\|x_k - x\| \to 0$ as $k \to \infty$.

Definition 14 (Equicontinuity)**.** A family of functions $\mathcal{F} \subseteq C([0,1])$ is *equicontinuous* if $\forall\epsilon > 0$, $\exists\delta(\epsilon) > 0$ such that $f \in \mathcal{F}$, $x, y \in [0,1]$ with $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

Definition 15 (Cauchy sequence)**.** Let $(V, \|\cdot\|)$ be a normed space. A sequence (x_k) in V is Cauchy if $\forall\epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n, m \geq N \implies \|x_n - x_m\| < \epsilon$.

Definition 16 (Completeness)**.** A normed space $(V, \|\cdot\|)$ is *complete* if every Cauchy sequence in V converges to an element in V.

Definition 17 (Open ball)**.** Let $(V, \|\cdot\|)$ be a normed space, $y \in V$ and $r > 0$. The *open ball* with radius r and centre y is $B_r(y) = \{x \in V : \|x - y\| < r\}$.

Definition 18 (Open)**.** A subset $E \subseteq V$ is *open* if $\forall y \in E$, $\exists r > 0$ such that $B_r(y) \subseteq E$.

Definition 19 (Limit point)**.** Let $(V, \|\cdot\|)$ be a normed space, and $E \subseteq V$. An element $x \in V$ is a *limit point* of E if there exists a sequence $(x_k) \in E$ such that $x_k \neq x$ $\forall k$ and $x_k \to x$.

Definition 20 (Closed)**.** A subset $E \subseteq V$ is *closed* if it contains all of its limit points.

Definition 21 (Continuity)**.** Let $(V, \|\cdot\|)$, $(V', \|\cdot\|')$ be normed spaces, $E \subseteq V$ and $f : V \to V'$ a mapping. Then f is *continuous* at $x \in E$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $y \in E$, $\|y - x\| < \delta \implies \|f(x) - f(y)\|' < \epsilon$.

Equivalently, $f(B_\delta^V(x) \cap E) \subseteq B_\epsilon^{V'}(f(x)) \iff B_\delta^V(x) \cap E \subseteq f^{-1}(B_\epsilon^{V'}(f(x)))$.

We say f is *continuous* if it is continuous at each $x \in E$.

Definition 22 (Uniform continuity)**.** Let $E \subseteq V$ and $f : E \to V'$. Then f is *uniformly continuous* if $\forall\epsilon > 0, \exists\delta = \delta(\epsilon) > 0$ such that
$$x, y \in E, \|x - y\| < \delta \implies \|f(x) - f(y)\|' < \epsilon.$$

Definition 23 (Lipschitz function)**.** A function $f : E \to V'$ is said to be *Lipschitz* if there exists a fixed constant $c > 0$ such that $\|f(x) - f(y)\|' \le c\|x - y\|$ for all $x, y \in E$.

Definition 24 (Sequentially compact)**.** Let $(V, \|\cdot\|)$ be a normed space. A subset $K \subseteq V$ is said to be *sequentially compact* if every sequence (x_n) in K has a subsequence $(x_{n_j})_{j=1}^{\infty}$ converging to an element in K.

4 Metric spaces

Definition 25 (Metric)**.** A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ such that for any $x, y, z \in X$,

(i) $d(x, y) \ge 0$ and $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \le d(x, z) + d(z, y)$.

Definition 26 (Example metrics)**.** 1. $\mathbb{R}^n$ with the *Euclidean metric*

$$d(x, y) = \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}}$$

2. If $(V, \|\cdot\|)$ is a normed space, $d(x, y) = \|x - y\|$ is the metric derived from the norm.

3. The discrete metric on any set X

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \ne y \end{cases}$$

Definition 27 (Lipschitz equivalence of metrics)**.** Two metrics d and d' on a set X are *Lipschitz equivalent* if there exist numbers A, B with $0 < A < B$ such that $Ad(x, y) \le d'(x, y) \le Bd(x, y)$ for all $x, y \in X$.

Definition 28 (Subspace metric)**.** If (X, d) is a metric space and $Y \subseteq X$, then the restriction $d|_{Y \times Y} : Y \times Y \to \mathbb{R}$ is a metric on Y called the *subspace metric* (or the *induced* metric).

Definition 29 (Continuity)**.** Let (X, d) and (X', d') be metric spaces. Then $f : X \to X'$ is *continuous* at $x \in X$ if $\forall \epsilon > 0$, $\exists \delta = \delta(x, t) > 0$ such that $y \in X, d(y, x) < \delta \implies d'(f(y), f(x)) < \epsilon$. We say that f is continuous if it is continuous at every $x \in X$.

Definition 30 (Uniform continuity)**.** Let (X, d) and (X', d') be metric spaces. Then $f : X \to X'$ is *uniformly continuous* if $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ such that $x, y \in X, d(x, y) < \delta \implies d'(f(x), f(y)) < \epsilon$.

Definition 31 (Lipschitz)**.** Let (X, d) and (X', d') be metric spaces. Then $f : X \to X'$ is *Lipschitz* if there exists $c > 0$ such that $d'(f(x), f(y)) \le cd(x, y)$ for all $x, y \in X$.

Definition 32 (Open ball)**.** For a metric space (X, d), $a \in X$ and $r > 0$, define the open ball with centre a and radius r to be $B_r(a) = \{x \in X : d(x, a) < r\}$.

Definition 33 (Open subset)**.** A subset $U \subseteq X$ is *open* if $\forall x \in U, \exists r > 0$ such that $B_r(x) \subseteq U$.

Definition 34 (Neighbourhood)**.** For $x \in X$, a *neighbourhood* of x is an open set containing x.

Definition 35 (Limit point)**.** Let (X, d) be a metric space and $E \subseteq X$. A point $x \in X$ is a *limit point* of E if $\forall \epsilon > 0, \exists y \in E$ such that $0 < d(x, y) < \epsilon$.

Equivalently, $\forall \epsilon > 0$, $(B_\epsilon(x) \setminus \{x\}) \cap E \neq \emptyset$.

Definition 36 (Closed)**.** A subset $E \subseteq X$ is closed if E contains all of its limit points.

Definition 37 (Convergence)**.** Let (X, d) be a metric space. Let $x_n, x \in X$ for $n \in \mathbb{N}$. We say that $x_n \to x$ if $\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}$ such that $n \geq N \implies d(x_n, x) < \epsilon$.

Definition 38 (Homeomorphism)**.** Let (X, d) and (X', d') be metric spaces. A map $h : X \to X'$ is a *homeomorphism* if h is a bijection with h and h^{-1} both continuous.

Definition 39 (Cauchy)**.** Let (X, d) be a metric space. A sequence $(x_n) \in X$ is *Cauchy* if $\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}$ such that $n, m \geq N \implies d(x_n, x_m) < \epsilon$.

Definition 40 (Completeness)**.** A metric space (X, d) is *complete* if every Cauchy sequence in X converges to an element in X.

Definition 41 (Sequential compactness)**.** A metric space (X, d) is *sequentially compact* if every sequence in X has a convergent subsequence.

Definition 42. A metric space (X, d) is *bounded* if $X = B_R(x_0)$ for some $x_0 \in X$ and some $R > 0$.

Definition 43 (Totally bounded)**.** A metric space (X, d) is *totally bounded* if for every $\epsilon > 0$ there is a finite subset $\{x_1, x_2, \ldots, x_N\}$ such that $X = \cup_{j=1}^N B_\epsilon(x_j)$.

5 Contraction mapping theorem

Definition 44 (Contraction)**.** Let (X, d) be a metric space. A mapping $f : X \to X$ is a *contraction* if there exists a λ with $0 \leq \lambda < 1$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$.

6 Differentiation in $\mathbb{R}^n$

Definition 45 (Limit of a function)**.** Let (X, d) and (X', d') be metric spaces. Let $E \subseteq X$ and $a \in X$ be a limit point of E. For $b \in X'$ and $f : E \to X'$, we say $f(x)$ tends to b as x tends to a and write $\lim_{x \to a} f(x) = b$ if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } x \in E, 0 < d(x, a) < \delta \implies d'(f(x), b) < \epsilon$$

N.B. f need not be defined at $x = a$.

Definition 46 (Differentiability)**.** Let $U \subseteq \mathbb{R}^n$ be open (where we take $\mathbb{R}^n$ with the Euclidean metric). Let $f : U \to \mathbb{R}^m$ and $a \in U$. Then f is *differentiable* at a if there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Ah}{\|h\|} = 0$$

where 0 is the zero vector in $\mathbb{R}^m$ (and $h \in \mathbb{R}^n$).

When it exists, the map A is called the *derivative* of f at a, and is denoted $Df(a)$.

Definition 47 (Operator norm)**.** For $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, the *operator norm* $\|A\|$ is defined as

$$\|A\| = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} \|Ax\|.$$

Definition 48 (Directional derivative)**.** Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$. For $u \in \mathbb{R}^n$ (the *direction*), $a \in U$, the *directional derivative* of f in the direction u at a is defined by

$$D_u f(a) = \lim_{t \to 0} \frac{f(a+tu) - f(a)}{t}$$

provided the limit exists.

Definition 49 (Partial derivative)**.** Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ and $a \in U$. Then jth *partial derivative* of f at a is the directional derivative $D_{e_j} f(a)$.

Definition 50 (Continuously differentiable)**.** Let $U \subseteq \mathbb{R}^n$ be open. Then $f : U \to \mathbb{R}^m$ is *continuously differentiable* if f is differentiable at every point in U and $Df : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is continuous at every point in U. We say f is C^1 or $f \in C^1(U, \mathbb{R}^m)$ or f is of class C^1.

Definition 51 (Path connectedness)**.** A subset $E \subseteq \mathbb{R}^n$ is *path connected* if for every $x, y \in E$ there exists a continuous (path) $\gamma : [0,1] \to E$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Definition 52 (Higher order partial derivative)**.** Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$. For $i, j \in \{1, \ldots, n\}$, we write

$$D_{ij} f(a) = D_i(D_j f)(a)$$

if the second partial derivative on the right hand side exists.

(This is sometimes also written $D_{ij} f = \frac{\partial^2 f}{\partial x_i \partial x_j}$).

Definition 53 (Second order derivative)**.** Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$. Define the bilinear map $D^2 f(a) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ by

$$D^2 f(a)(u, v) = D(Df)(a)(u)(v)$$

We write $D^2 f(a)[u]^2 = D^2 f(a)(u, u)$.