Definition (Vector space)**.** A *vector space* over $\mathbb{F}$ is an abelian group $(V,+)$ equipped with a function $F\times V\to V:(\lambda,v)\mapsto\lambda v$ such that
1. $\lambda(\mu v)=(\lambda\mu)v \quad \forall\lambda,\mu\in\mathbb{F},v\in V$
2. $\lambda(u+v)=\lambda u+\lambda v \quad \forall\lambda\in\mathbb{F},u,v\in V$
3. $(\lambda+\mu)(v)=\lambda v+\mu v \quad \forall\lambda,\mu\in\mathbb{F},v\in V$
4. $1.v=v \quad \forall v\in V$

Definition (Subspace)**.** Let V be a vector space over $\mathbb{F}$. A subset $U\subset V$ is a (linear) *subspace* if
1. $\forall u_1,u_2\in U, \quad u_1+u_2\in U$
2. $\forall u\in U,\forall\lambda\in\mathbb{F}, \quad \lambda u\in U$
3. $0\in U$

Definition (Span)**.** Let V be a vector space over $\mathbb{F}$ and $S\subset V$. Then the *span* of S in V is $\langle S\rangle=\{\sum_{i=1}^n\lambda_i s_i:\lambda_i\in\mathbb{F},s_i\in S,n\geqslant 0\}$.

Definition (Sum of subspaces)**.** Let $U,W\subset V$ be subspaces of a vector space V over $\mathbb{F}$. The *sum* of U and W is $U+W=\{u+w:u\in U,w\in W\}$.

Definition (Spanning set)**.** Let V be a vector space over $\mathbb{F}$. Then $S\subset V$ *spans* V if $\langle S\rangle=V$.

Definition (Linear independence)**.** Let V be a vector space over $\mathbb{F}$. Then $S\subset V$ is *linearly independent* if whenever we have $\sum_{i=1}^n\lambda_i s_i=0$ with $\lambda_i\in\mathbb{F}$ and $s_i\in S$ distinct then $\lambda_i=0$ for all i. If S is not linearly independent, we say that S is *linearly dependent*.

Definition (Basis)**.** Let V be a vector space over $\mathbb{F}$. Then $S\subset V$ is a *basis* if S is linearly independent and spans V.

Definition (Dimension)**.** Let V be a vector space over $\mathbb{F}$. Then if V has a finite basis S we say that V is *finite dimensional* with *dimension* $\dim V=|S|$. Otherwise, we say V is *infinite dimensional*.

Definition (Internal direct sum)**.** Let U,W be subspaces of a vector space V over $\mathbb{F}$. Then V is the (internal) *direct sum* of U and W and we write $V=U\oplus V$ if
- $V=U+W$
- $U\cap W=0$

Equivalently, every element $v\in V$ can be written uniquely as $v=u+w$ for $u\in U$, $w\in W$. We say that U and W are *complementary subspaces* on V.

Definition (External direct sum)**.** Given two vector spaces U,W over $\mathbb{F}$, the (external) *direct sum* $U\oplus W$ of U and W is given by $U\oplus W=\{(u,v):u\in U,w\in W\}$ with coordinate-wise addition and multiplication.

Definition (Linear map)**.** Let U,V be vector spaces over $\mathbb{F}$. A function $\alpha:U\to V$ is a *linear map* if
- $\alpha(u_1+u_2)=\alpha(u_1)+\alpha(u_2) \quad \forall u_1,u_2\in U$
- $\alpha(\lambda u)=\lambda\alpha(u) \quad \forall u\in U,\lambda\in\mathbb{F}$.

The set of all linear maps $U\to V$ is denoted $\mathcal{L}(U,V)$.

Definition (Isomorphism)**.** Let U,V be vector spaces over $\mathbb{F}$. Then U and V are *isomorphic* if there exists $\alpha:U\to V$ linear and $\beta:V\to U$ linear such that $\alpha\beta=id_V$ and $\beta\alpha=id_u$. We say that α and β are isomorphisms.

Definition (Kernel)**.** Let U,V be vector spaces and $\alpha:U\to V$ a linear map. Then the *kernel* of α is

$$\ker\alpha=\{u\in U:\alpha(u)=0\}$$

Definition (Image)**.** Let U,V be vector spaces and $\alpha:U\to V$ a linear map. Then the *image* of α is
$$\text{Im}\alpha=\{\alpha(u):u\in U\}$$

Definition (Nullity)**.** The nullity of a linear map α is $n(\alpha)=\text{dim}\ker\alpha$.

Definition (Rank)**.** The rank of a linear map α is $r(\alpha)=\text{dim}\text{Im}\alpha$.

Definition (Inversion)**.** Let $A\in M_{n,n}(\mathbb{F})$. Then if there exists $B\in M_{n,n}(\mathbb{F})$ such that $BA=AB=I_n$, we say that A is *invertible* and write $B=A^{-1}$.

Definition (Equivalence)**.** Let $A,B\in M_{m,n}(\mathbb{F})$. Then A and B are *equivalent* if there exists $P\in M_{n,n}(\mathbb{F})$ and $Q\in M_{m,m}(\mathbb{F})$ both invertible such that $B=Q^{-1}AP$.

Definition (Rank)**.** Let $A\in M_{m,n}(\mathbb{F})$. Then the *column rank* of A is the dimension of the subspace of $\mathbb{F}^n$ spanned by the columns of A, written $r(A)$. The *row rank* of A is the column rank of A^T, written $r(A^T)$. Note that $r(A)=r(A^T)$ and we refer to their common value as the *rank* of A.

Definition (Elementary matrices)**.** The following three types of matrices are known as *elementary matrices.*

1. (Swapping rows i and j)

$$S_{ij}^n=\begin{pmatrix}1&&&&&&&&&\\&\ddots&&&&&&&&\\&&1&&&&&&&\\&&&0&&&&1&&\\&&&&1&&&&&\\&&&&&\ddots&&&&\\&&&&&&1&&&\\&&&1&&&&0&&\\&&&&&&&&1&\\&&&&&&&&&\ddots&\\&&&&&&&&&&1\end{pmatrix}$$

2. (Adding λ times row j to row i)

$$E_{ij}^n(\lambda)=\begin{pmatrix}1&&&&\\&\ddots&&&\\&&\ddots&\lambda&&\\&&&\ddots&&\\&&&&\ddots&\\&&&&&1\end{pmatrix}$$

3. (Multiplying row i by λ)

$$T_i^n(\lambda)=\begin{pmatrix}1&&&&&\\&\ddots&&&&\\&&1&&&\\&&&\lambda&&\\&&&&1&\\&&&&&\ddots&\\&&&&&&1\end{pmatrix}$$

Definition (Determinant)**.** If $A \in M_n(\mathbb{F})$ the *determinant* of A is

$$\det A = \sum_{\sigma \in S_n} \epsilon(\sigma) \left(\prod_{i=1}^{n} A_{i\sigma(i)} \right)$$

Definition (Volume form)**.** A *volume form* d on $\mathbb{F}^n$ is a function

$$d: \mathbb{F}^n \times \ldots \times \mathbb{F}^n \to \mathbb{F}$$
$$(v_1, \ldots, v_n) \mapsto d(v_1, \ldots, v_n)$$

such that
1. d is *multilinear*; (i.e. for $1 \leqslant i \leqslant n$,
   $d(v_1, \ldots, \lambda v_i + \mu v_i', \ldots, v_n) = \lambda d(v_1, \ldots, v_i, \ldots, v_n) + \mu d(v_1, \ldots, v_i', \ldots, v_n)$)
2. d is alternating. (i.e. if $v_i = v_j$ then $d(v_1, \ldots, v_n) = 0$)

Definition. For $A \in M_n(\mathbb{F})$, define $\hat{A}_{ij}$ to be the element of $M_{n-1}(\mathbb{F})$ obtained by deleting the ith row and jth column of A.

Definition (Adjugate)**.** The *adjugate* of $A \in M_n(\mathbb{F})$ is $(\operatorname{adj} A)_{ij} = (-1)^{i+j} \det \hat{A}_{ji}$.

Definition (Endomorphism)**.** Suppose V is a finite dimensional vector space over $\mathbb{F}$. An *endomorphism* of V is a linear map $\alpha: V \to V$. We write $\operatorname{End}(V)$ to denote the vector space of endomorphisms of V.

Definition (Similar)**.** We say $A, B \in M_n(\mathbb{F})$ are *similar* (or *conjugate*) if there exists $P \in M_n(\mathbb{F})$ invertible such that $B = P^{-1}AP$. (Equivalently, if they represent the same linear map.)

Definition (Trace)**.** For $A \in M_n(\mathbb{F})$ the *trace* of A is $\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}$.

Definition (Trace and determinant of endomorphism)**.** For $\alpha \in \operatorname{End}(V)$ and $\langle e_1, \ldots, e_n \rangle$ a basis for V, if A is the matrix representing α then
- the *trace* of α is $\operatorname{tr}\alpha = \operatorname{tr} A$;
- the *determinant* of α is $\det \alpha = \det A$.

Definition (Eigenvalues, eigenvectors & eigenspaces)**.** Let $\alpha \in \operatorname{End}(V)$.
1. $\lambda \in \mathbb{F}$ is an *eigenvalue* of α if there exists $v \in V \setminus 0$ such that $\alpha v = \lambda v$
2. $v \in V$ is a (λ-)*eigenvector* of α if $\alpha v = \lambda v$ for some $\lambda \in \mathbb{F}$
3. For $\lambda \in \mathbb{F}$, the λ-*eigenspace* of α is $E_\alpha(\lambda) = E(\lambda) = \{\lambda\text{-eigenvectors for } \alpha\} = \ker(\alpha - \lambda \iota)$

Definition (Diagonalisable)**.** We say $\alpha \in \operatorname{End}(V)$ is *diagonalisable* if there exists a basis of V such that the corresponding matrix is diagonal.

Definition (Polynomial)**.** A *polynomial* function $f: \mathbb{F} \to \mathbb{F}$ is of the form $f(t) = a_m t^m + \ldots + a_1 t + a_0$ for some $m \geqslant 0$ and $a_0, \ldots, a_m \in \mathbb{F}$. The largest n such that $a_n \neq 0$ is the *degree* of f, written $\deg f$. (By convention, $\deg 0 = -\infty$.)

Definition (Multiplicity)**.** A root $\lambda \in \mathbb{F}$ of $f \in \mathbb{F}[t]$ is a root of *multiplicity* k if $(t-\lambda)^k$ is a factor of $f(t)$ but $(t-\lambda)^{k+1}$ is not.

Definition (Minimal polynomial)**.** The *minimal polynomial* of $\alpha \in \operatorname{End}(V)$ is the non-zero monic polynomial $m_\alpha(t)$ of least degree such that $m_\alpha(\alpha) = 0$.

Definition (Triangulable)**.** The linear map $\alpha \in \operatorname{End}(V)$ is *triangulable* if there is a basis for V such that the corresponding matrix is upper triangular.

Definition (Characteristic polynomial)**.** The *characteristic polynomial* of $\alpha \in \operatorname{End}(V)$ is $\chi_\alpha(t) = \det(t\iota - \alpha)$.

Definition (Multiplicities of eigenvalues)**.** Let $\alpha \in \mathrm{End}(V)$ and λ an eigenvalue of α.
1. the *algebraic multiplicity* a_λ of λ is the multiplicity of λ as a root of $\chi_\alpha(t)$.
2. the *geometric multiplicity* g_λ of λ is $\dim E_\alpha(\lambda)$.
3. the number c_λ is the multiplicity of λ as a root of $m_\alpha(t)$.

Definition (Jordan Normal Form)**.** A matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ is in *Jordan Normal Form* if it is a block diagonal matrix

$$A = \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & ... & 0 \\ 0 & J_{n_2}(\lambda_2) & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & J_{n_k}(\lambda_k) \end{pmatrix}$$

where $k \geqslant 1$, $n_1,...,n_k \in \mathbb{N}$ such that $\sum_{i=1}^k = n$ and $\lambda_1,...,\lambda_k \in \mathbb{C}$ (not necessarily distinct), and the *Jordan blocks* $J_m(\lambda) \in \mathrm{Mat}_m(\mathbb{C})$ have the form

$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & ... & 0 & 0 \\ 0 & \lambda & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & ... & 0 & \lambda \end{pmatrix}$$

Definition (Nilpotent)**.** The linear map $\alpha \in \mathrm{End}(V)$ is *nilpotent* if there exists some $k \geqslant 0$ such that $\alpha^k = 0$.

Definition (Dual space)**.** Let V be a vector space over $\mathbb{F}$. The *dual space* of V is the vector space

$$V^* = \mathcal{L}(V,\mathbb{F}) = \{\alpha : V \to \mathbb{F} \text{ is linear}\}$$

under pointwise addition and scalar multiplication.

Definition (Dual basis)**.** If V is a finite dimensional space over $\mathbb{F}$ with basis $\langle e_1,...e_n\rangle$ then the basis $\langle \epsilon_1,...,\epsilon_n\rangle$ of $V*$ given by $\epsilon_i(e_j) = \delta_{ij}$ is the *dual basis*.

Definition (Annihilator)**.** If $U \subset V$ then the *annihilator* of U is $U^\circ = \{\theta \in V^* | \theta(u) = 0 \forall u \in U\}$.
If $W \subset V^*$ then the *annihilator* of W is $W^\circ = \{v \in V | \theta(v) = 0 \forall \theta \in W\}$.

Definition (Dual map)**.** Let V and W be vector spaces of $\mathbb{F}$ and $\alpha : V \to W$ a linear map. The *dual map* to α is the map $\alpha^* : W^* \to V^*$ given by $\theta \mapsto \theta\alpha$.

Definition (Bilinear form)**.** The map $\psi : V \times W \to \mathbb{F}$ is a *bilinear form* if it is linear in both arguments.
If $\langle e_1,...,e_n\rangle$ is a basis for V and $\langle f_1,...,f_m\rangle$ a basis for W then the matrix A representing ψ with respect to these bases is given by $A_{ij} = \psi(e_i,f_j)$.

Definition (Degenerate)**.** A bilinear form $\psi : V \times W \to \mathbb{F}$ is *degenerate* if there exists some $v \in V \setminus \{0\}$ such that $\psi(v,-) = 0 \in W^*$ or there exists some $w \in W \setminus \{0\}$ such that $\psi(0,w) = 0 \in V^*$. Otherwise ψ is *non-degenerate*.

Definition (Symmetric)**.** A bilinear form $\phi : V \times V \to \mathbb{F}$ is *symmetric* if $\phi(v_1,v_2) = \phi(v_2,v_1)$ for all $v_1,v_2 \in V$.

Definition (Congruent)**.** Square matrices A and B are *congruent* if there exists an invertible matrix P such that $B = P^T AP$. (Equivalently, if they represent the same bilinear form.)

Definition (Quadratic form)**.** If $\phi : V \times V \to \mathbb{F}$ is a bilinear form then the map $V \to \mathbb{F}$ given by $v \mapsto \phi(v,v)$ is a *quadratic form* on V.

Definition (Positive / negative definite)**.** A symmetric bilinear form ϕ on a real vector space V is

1. *positive definite* if $\phi(v,v)>0$ for all $v\in V\setminus\{0\}$
2. *positive semi-definite* if $\phi(v,v)\geqslant 0$ for all $v\in V$
3. *negative definite* if $\phi(v,v)<0$ for all $v\in V\setminus\{0\}$
4. *negative semi-definite* if $\phi(v,v)\leqslant 0$ for all $v\in V$

Definition (Sesquilinear form)**.** Let V and W be $\mathbb{C}$-vector spaces. Then a *sesquilinear form* is a function $\phi:V\times W\to\mathbb{C}$ such that

$$\phi(\lambda_1 v_1+\lambda_2 v_2,w)=\bar{\lambda_1}\phi(v_1,w)+\bar{\lambda_2}\phi(v_2,w)$$
$$\phi(v,\mu_1 w_1+\mu_2 w_2)=\mu_1\phi(v,w_1)+\mu_2\phi(v,w_2)$$

for all $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{C}$, $v,v_1,v_2\in V$ and $w,w_1,w_2\in W$.

The matrix A representing ϕ with respect to bases $\langle v_1,\ldots,v_m\rangle$ and $\langle w_1,\ldots,w_n\rangle$ is $A_{ij}=\phi(v_i,w_j)$.

Definition (Hermitian form)**.** A sesquilinear form $\phi:V\times V\to\mathbb{C}$ is *Hermitian* if $\phi(x,y)=\phi(\bar{y},x)$ for all $x,y\in V$.

Definition (Hermitian)**.** A matrix A is Hermitian if $A=\bar{A}^T$.

Definition (Inner product)**.** Let V be a vector space over $\mathbb{F}$. An *inner product* on V is a positive definite symmetric/Hermitian form ϕ on V.

Definition (Orthogonality)**.** If V be an inner product space then $v,w\in V$ are *orthogonal* if $(v,w)=0$. A set $\{v_i\,|\,i\in I\}$ is *orthonormal* if $(v_i,v_j)=\delta_{ij}$ for $i,j\in I$. An *orthonormal basis* is a basis that is orthonormal.

Definition (Orthogonal internal direct sum)**.** Let V be an inner product space and V_1,V_2 subspaces of V. Then V is the *orthogonal (internal) direct sum* of V_1 and V_2, written $V=V_1\perp V_2$ if
1. $V=V_1+V_2$
2. $V_1\cap V_2=0$
3. $(v_1,v_w)=0$ for all $v_1\in V_1$ and $v_2\in V_2$

Definition (Orthogonal complement)**.** If $W\subset V$ is a subspace of an inner product space V then the *orthogonal complement* of W in V is

$$W^\perp=\{v\in V\,|\,(w,v)=0 \text{ for all } w\in W\}$$

Definition (Orthogonal external direct sum)**.** The *orthogonal (external) direct sum* of two inner product spaces V_1 and V_2 is the vector space direct sum $V_1\oplus V_2$ with the inner product

$$((v_1,v_2),(w_1,w_2))=(v_1,w_1)+(v_2,w_2)$$

for $v_1,w_1\in V_1$ and $v_2,w_2\in V_2$.

Definition (Projection map)**.** Suppose that $V=U\oplus W$. Then the *projection map* Π onto W is such that $\Pi(u+w)=w$ for $u\in U$ and $w\in W$. If $U=W^\perp$ then it is the *orthogonal projection.*

Definition (Adjoint)**.** If V and W are finite dimensional inner product spaces and $\alpha:V\to W$ a linear map, then the *adjoint* of α is the linear map $\alpha^*:W\to V$ such that $(\alpha(v),w)=(v,\alpha^*(w))$ for all $v\in V$, $w\in W$.

Definition (Self-adjoint)**.** If V is an inner product space, then $\alpha\in\mathrm{End}(V)$ is *self-adjoint* if $\alpha=\alpha^*$.

Definition (Orthogonal)**.** If V is a real inner product space, then $\alpha\in\mathrm{End}(V)$ is *orthogonal* if $(\alpha(v_1),\alpha(v_2))=(v_1,v_2)$ for all $v_1,v_2\in V$.

Definition (Unitary)**.** If V is a complex inner product space, then $\alpha\in\mathrm{End}(V)$ is *unitary* if $(\alpha(v_1),\alpha(v_2))=(v_1,v_2)$ for all $v_1,v_2\in V$.