Vector Calculus - Crib Sheet

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1 Suffix Notation

$\mathbf{a} \cdot \mathbf{b} = a_i b_i$
$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$
$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

2 Vector product identities

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$

3 Grad, Div, Curl and $\boldsymbol{\nabla}$

Del: $\boldsymbol{\nabla} = \mathbf{e}_i \frac{\partial}{\partial x_i} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$
Grad: $(\boldsymbol{\nabla} f)_i = \frac{\partial f}{\partial x_i}$
Div: $\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}$
Curl: $(\boldsymbol{\nabla} \times \mathbf{F})_i = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j}$
Laplacian: $\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x_i \partial x_j}$ or $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$
Vector Laplacian: $\nabla^2 \mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A})$

3.1 Useful identities

$\boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{x}) = \mathbf{a}$
$\boldsymbol{\nabla} r^n = n r^{n-2} \mathbf{r} = n r^{n-1} \hat{\mathbf{r}}$
$\boldsymbol{\nabla} \cdot \mathbf{r} = 3$
$\boldsymbol{\nabla} \times \mathbf{r} = 0$
$\boldsymbol{\nabla} \cdot (r^\alpha \mathbf{r}) = (\alpha + 3) r^\alpha$
$\boldsymbol{\nabla} \times (r^\alpha \mathbf{r}) = 0$

$\boldsymbol{\nabla}(\psi\varphi) = (\boldsymbol{\nabla}\psi)\varphi + \psi(\boldsymbol{\nabla}\varphi)$
$\boldsymbol{\nabla} \cdot (\psi \mathbf{v}) = (\boldsymbol{\nabla}\psi) \cdot \mathbf{v} + \psi \boldsymbol{\nabla} \cdot \mathbf{v}$
$\boldsymbol{\nabla} \times (\psi \mathbf{v}) = (\boldsymbol{\nabla}\psi) \times \mathbf{v} + \psi \boldsymbol{\nabla} \times \mathbf{v}$

$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0$ for any scalar field f
$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$ for any vector field $\mathbf{A}$

4 Coordinate systems

4.1 Plane polars

$$x_1 = \rho\cos\phi, \quad x_2 = \rho\sin\phi$$

$$\mathbf{e}_\rho = \hat{\rho} = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2, \qquad \mathbf{e}_\phi = \hat{\phi} = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2$$

4.2 Cylindrical polars

$$x_1 = \rho\cos\phi, \quad x_2 = \rho\sin\phi, \quad x_3 = z$$

ρ and ϕ are the same as for plane polars, and z as in Cartesian.

4.3 Spherical polars

$$x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta$$

r is the radial distance, θ is the polar angle, ϕ is the azimuthal angle.

5 Volume elements

Cartesian: $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
Cylindrical polars: $\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$
Spherical polars: $\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

6 Area elements

Scalar area element: $\mathrm{d}S = \left|\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\right| \mathrm{d}u\,\mathrm{d}v$
Vector area element: $\mathrm{d}\mathbf{S} = \frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v$

7 Curves, curvature and normals

For a curve $\mathbf{r}(s)$ parametrised by arc length s:
$\mathbf{t}(s) = \frac{d\mathbf{r}}{ds}$ is a unit *tangent* vector
$\mathbf{t}' = \kappa\mathbf{n}$ where $\mathbf{n}(s)$ is the *principal normal* and $\kappa(s)$ is the *curvature*
$a = \frac{1}{\kappa}$ is the *radius of curvature*
$\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is the *binormal*
$\mathbf{b}' = -\tau\mathbf{n}$ where τ is the *torsion*

8 Irrotational and solenoidal

$\mathbf{F}$ is *conservative* or *irrotational* if $\boldsymbol{\nabla} \times \mathbf{F} = 0 \iff \mathbf{F} = \boldsymbol{\nabla} f$ for some *scalar potential* f.
$\mathbf{H}$ is *solenoidal* if $\boldsymbol{\nabla} \cdot \mathbf{H} = 0 \iff \mathbf{H} = \boldsymbol{\nabla} \times \mathbf{A}$ for some *vector potential* $\mathbf{A}$.

9 Integral theorems

9.1 Green's Theorem

For smooth functions $P(x, y)$ and $Q(x, y)$,

$$\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A = \int_C P \,\mathrm{d}x + Q \,\mathrm{d}y$$

where A is a bounded region in the xy plane with boundary $C = \partial A$, a piecewise smooth, non-intersecting closed curve, traversed anticlockwise.

9.2 Stokes' Theorem

For a smooth vector field $\mathbf{F}(\mathbf{r})$,

$$\int_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where S is a bounded smooth surface with boundary $C = \partial S$, a piecewise smooth curve, and S and C have compatible orientations (i.e. if $\mathbf{t} \times \mathbf{n}$ points *out* of S).

9.3 Divergence Theorem

For a smooth vector field $\mathbf{F}(\mathbf{r})$,

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F} \,\mathrm{d}V = \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

where V is a bounded volume with boundary $S = \partial V$, a piecewise smooth closed surface with normal $\mathbf{n}$ pointing outwards.

10 General orthogonal curvilinear coordinates

For coordinates u, v, w on $\mathbb{R}^3$ and a smooth function $\mathbf{r}(u, v, w)$, with $\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} \mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v} \mathrm{d}v + \frac{\partial \mathbf{r}}{\partial w} \mathrm{d}w$, have:

$$\frac{\partial \mathbf{r}}{\partial u} = h_u \mathbf{e}_u, \quad \frac{\partial \mathbf{r}}{\partial v} = h_v \mathbf{e}_v \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial w} = h_w \mathbf{e}_w$$

Line element: $\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} \mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v} \mathrm{d}v + \frac{\partial \mathbf{r}}{\partial w} \mathrm{d}w = h_u \mathbf{e}_u \,\mathrm{d}u + h_v \mathbf{e}_v \,\mathrm{d}v + h_w \mathbf{e}_w \,\mathrm{d}w$
Volume element: $\mathrm{d}V = \frac{\partial \mathbf{r}}{\partial u} \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w} \right) \mathrm{d}u \,\mathrm{d}v \,\mathrm{d}w = h_u h_v h_w \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}w$

Grad: $\boldsymbol{\nabla} f = \frac{1}{h_u} \mathbf{e}_u \frac{\partial f}{\partial u} + \frac{1}{h_v} \mathbf{e}_v \frac{\partial f}{\partial v} + \frac{1}{h_w} \mathbf{e}_w \frac{\partial f}{\partial w}$
Div: $\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{h_u h_v h_w} \left(\frac{\partial}{\partial u} (h_v h_w F_u) + \frac{\partial}{\partial v} (h_u h_w F_v) + \frac{\partial}{\partial w} (h_u h_v F_w) \right)$
Curl: $\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \mathbf{e}_u & h_v \mathbf{e}_v & h_w \mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$

11 Laws of Gravitation

The gravitational force on a point mass m at $\mathbf{r}$ is $\mathbf{F}(\mathbf{r}) = m\mathbf{g}(\mathbf{r})$, where $\mathbf{g}$ is the gravitational field. For a closed curve C, $\oint_C \mathbf{g} \cdot \mathrm{d}\mathbf{r} = 0$ - i.e. $\mathbf{g}$ is conservative.

11.1 Gauss' Law

For a volume V with mass distribution ρ, total mass M, closed surface $S = \partial V$, and G Newton's gravitational constant:
Integral form: $\int_S \mathbf{g} \cdot \mathrm{d}\mathbf{S} = -4\pi GM$
Differential form: $\boldsymbol{\nabla} \cdot \mathbf{g} = -4\pi G\rho$
Writing $\mathbf{g} = -\boldsymbol{\nabla}\varphi$, where $\varphi(\mathbf{r})$ is the gravitational potential,
Poisson's equation form: $\nabla^2\varphi = 4\pi G\rho$

12 Poisson's and Laplace's Equation

Poisson's: $\nabla^2\varphi = -\rho$
Gravity: $\rho \mapsto -4\pi G\rho$
Electrostatics: $\rho \mapsto \rho/\epsilon_0$
Laplace's: $\rho \mapsto 0$

Dirichlet condition: specify φ
Neumann condition: specify $\frac{\partial\varphi}{\partial n} = \mathbf{n} \cdot \boldsymbol{\nabla}\varphi$

13 Green's identities

Green's First Identity: $\int_S (u\boldsymbol{\nabla}v) \cdot \mathrm{d}\mathbf{S} = \int_V (\boldsymbol{\nabla}u \cdot \boldsymbol{\nabla}v + u\nabla^2 v)\,\mathrm{d}V$
Green's Second Identity: $\int_S (u\boldsymbol{\nabla}v - v\boldsymbol{\nabla}u) \cdot \mathrm{d}\mathbf{S} = \int_V (u\nabla^2 v - v\nabla^2 u)\,\mathrm{d}V$

14 Maxwell's Equations

$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$
$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$
$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$
$\boldsymbol{\nabla} \times \mathbf{B} - \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$
$\mathbf{E}$ electric field, $\mathbf{B}$ magnetic field, $\rho(\mathbf{r}, t)$ charge density, $\mathbf{j}$ current density, μ_0 permeability of free space (magnetic), ϵ_0 permittivity of free space (electric)

15 Tensors

Transformation rule: $T'_{ij\ldots k} = R_{ip}R_{jq}\ldots R_{kr}T_{pq\ldots r}$
Symmetric: $T_{ijp\ldots q} = T_{jip\ldots q}$
Anti-Symmetric: $T_{ijp\ldots q} = -T_{jip\ldots q}$
Invariant under R if $T'_{ij\ldots k} = T_{ij\ldots k}$. Isotropic if invariant under all rotations.

15.1 Tensor Divergence Theorem

For $T_{ij\dots kl}(\mathbf{x})$ a smooth tensor field,

$$\int_S T_{ij\dots kl} n_l \, \mathrm{d}S = \int_V \frac{\partial}{\partial x_l}(T_{ij\dots kl}) \, \mathrm{d}V$$

where V is a volume bounded by a smooth surface $S = \partial V$, and n_l is the outward pointing normal.

15.2 Decomposition of rank 2 tensor

$T_{ij} = P_{ij} + \epsilon_{ijk} B_k + \frac{1}{3} Q \delta_{ij}$ where $Q = T_{kk}$, $B_k = \frac{1}{2} \epsilon ijk T_{ij}$, and P_{ij} is traceless symmetric.

15.3 Inertia Tensor

$I_{ij} = \sum_\alpha m_\alpha (|\mathbf{r}_\alpha|^2 \delta ij - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j$
$I_{ik} = \int_V \rho(\mathbf{r})(x_k x_k \delta_{ij} - x_i x_k) \, \mathrm{d}V$

15.4 Isotropic Tensors

Rank 1: 0
Rank 2: $T_{ij} = \alpha \delta_{ij}$, scalar α
Rank 3: $T_{ijk} = \beta \epsilon_{ijk}$
Rank 4: $T_{ijk} = \alpha \delta ij \delta kl + \beta \delta_{ik} \delta_{jl} + \gamma \delta il \delta jk$
Rank n: Combinations of δ_{ij} and ϵ_{ijk}

If $T_{ij\dots k} = \int_V f(\mathbf{x}) x_i x_j \dots x_k \, \mathrm{d}V$ and for a rotation R_{ij}, if $f(\mathbf{x}') = f(\mathbf{x})$ and $V' = V$, then $T_{ij\dots k}$ is invariant under R_{ij}.