# Analysis I

Theorems and Corollaries

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## 1 Sequences and convergence

**Theorem.** An ordered field with the least upper bound property has the monotone sequences property.

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**Theorem** (The nestled-intervals property)**.** Let $[a_n, b_n], n \in \mathbb{N}$ be non-empty, closed intervals with $[a_1, b_1] \subset [a_2, b_2] \subset [a_3, b_3] \subset \ldots$. Then $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \emptyset$.

**Theorem** (Bolzano-Weierstrauss Theorem)**.** Every bounded sequence of reals has a convergent subsequence.

**Theorem** (General principle of convergence)**.** Let $F$ be an ordered field with the monotone sequences property. Then every Cauchy sequence in $F$ converges.

**Theorem.** Every ordered field $F$ that satisfies the general principal of convergence and has the Archimedean property has the monotone sequences property.

## 2 Infinite series

**Theorem** (The comparison test)**.** Let $0 \leq a_n \leq b_n$ for every $n$. If $\sum b_n < \infty$ then $\sum a_n < \infty$.

**Theorem** (The generalised comparison test)**.** Let $a_n \geq 0, b_n \geq 0$ for every $n$ and suppose that there are $C, M$ such that $a_n \leq Cb_n$ for every $n \geq M$. Then if $\sum b_n < \infty$, then $\sum a_n < \infty$.

**Theorem** (The ratio test)**.** Let $a_n \geq 0$ for every $n$, and suppose that there is some $\rho$ with $0 < \rho < 1$ and some $n$ such that $\forall n \geq N\ a_{n+1} \leq \rho a_n$. Then $\sum a_n < \infty$.

**Corollary.** Let $a_n \geq 0$ for every $n$ and suppose there is some $\rho$ with $O < \rho < 1$ such that $\frac{a_{n+1}}{a_n} \to \rho$ as $n \to \infty$. Then $\sum a_n < \infty$.

**Theorem** (The alternating series test)**.** Let $(a_n)$ be a sequence with $0 \leq a_n$ for every $n$ and suppose that $(a_n)$ is decreasing and tends to 0. Then $\sum(-1)^{n+1}a_n$ converges.

**Theorem.** If a series converges absolutely, then it converges.

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**Theorem** (General principal of convergence in $\mathbb{C}$)**.** Cauchy sequences in $\mathbb{C}$ converge.

**Theorem** (Complex version of alternating series test)**.** Let $(a_n)$ be a real sequence with $a_1 \geq a_2 \geq \dots$ and $a_n \to 0$. Let $z \in \mathbb{C}$ be a number such that $|Z| = 1, z \neq 1$. Then $\sum a_n z^n$ converges.

# 3 Continuous functions

**Theorem.** Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $x \in A$. Suppose that $f$ is continuous at $x$. Let $(a_n)$ be a sequence in $A$ and suppose that $a_n \to x$. Then $f(a_n) \to f(x)$.

**Theorem.** Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$. Let $x \in A$. Suppose that for every sequence $(a_n)$ in $A$ with $a_n \to x$ we have $f(a_n) \to f(x)$. Then $f$ is continuous at $x$.

**Theorem.** A composition of continuous functions is continuous. More precisely, let $A, B \in \mathbb{R}$ and let $f : A \to B$ and $g : B \to \mathbb{R}$. Suppose that $x \in A$, $f$ is continuous at $x$, and $g$ is continuous at $f(x)$. Then $g \circ f$ is continuous at $x$.

**Theorem** (Continuous induction principle)**.** For each $x \in [a, b]$ let $P(x)$ be a statement about $x$. Suppose that the following conditions hold:

(i) $P(a)$ holds

(ii) For every $x$, if $P(x)$ holds, then there exists $\epsilon > 0$ such that $P(u)$ holds for every $u \in [x, x + \epsilon)$.

(iii) For every $x$, if $P(x)$ does *not* hold, then there exists $\epsilon > 0$ such that for every $u \in (x - \epsilon, x]$ $P(u)$ does not hold.

Then $\forall x \in [a, b]$ $P(x)$ holds.

**Theorem** (The intermediate value theorem)**.** Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) < 0$ and $f(b) > 0$. Then there exists $c \in (a, b)$ such that $f(c) = 0$.

**Corollary.** Let $a < b$, $c < d$ and let $f : [a, b] \to [c, d]$ be an injection such that $f(a) = c$ and $f(b) = d$. Suppose also that $f$ is continuous. Then $f$ is a strictly increasing bijection and its inverse is continuous.

**Theorem.** Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f$ is bounded and attains its bounds.

## 4 Differentiation

**Theorem** (Rolle's theorem)**.** Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Suppose that $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = 0$.

**Theorem** (The mean value theorem)**.** Let $a < b$, let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

**Theorem.** Let $a < b$ and let $f : [a, b] \to [c, d]$ be a function that is continuous on $[a, b]$ and differentiable on $(a, b)$. Suppose also that $f'(x) > 0$ for every $x \in (a, b)$. Then $f$ has an inverse $g : [c, d] \to [a, b]$, and $g$ is continuous on $[c, d]$, differentiable on $(a, b)$ and $g'(y) = \frac{1}{f'(g(y))}$ for every $y \in (c, d)$. (We are also assuming that $f(a) = c, f(b) = d$.)

**Theorem** (One-dimensional inverse function theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a function that is continuously differentiable on $(a, b)$ (and continuous on $[a, b]$). Let $x \in (a, b)$ and suppose that $f'(x) \neq 0$. Then there is an interval $[x - \epsilon, x + \epsilon]$ on which $f$ is invertible with continuously differentiable inverse.

**Theorem** (Higher-order Rolle's theorem)**.** Let $x \in \mathbb{R}$, let $h > 0$ and let $f$ be a function that is $n$-times differentiable on an open interval that contains $[x, x + h]$. Suppose that

$$f(x) = f'(x) = f''(x) = \ldots = f^{(n-1)}(x) = f(x + h) = 0$$

Then there exists $\theta \in (0, 1)$ such that $f^{(n)}(x + \theta h) = 0$.

**Corollary** (Higher-order mean value theorem)**.** Let $f$ be $n$-times differentiable on an open interval containing $[x, x + h]$. Let $p$ be a polynomial of degree $\leq n$ such that $p^{(k)}(x) = f^{(k)}(x)$ for $k = 0, 1, \ldots, n - 1$ and $p(x + h) = f(x + h)$. Then there exists $\theta \in (0, 1)$ such that $f^{(n)}(x + \theta h) = p^{(n)}(x + \theta h)$.

**Theorem** (Taylor's Theorem)**.** Let $f$ be $n$-times differentiable on an open interval containing $[x, x + h]$. Then there exists $\theta \in (0, 1)$ such that

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{h^n}{n!}f^{(n)}(x + \theta h)$$

## 5 Power series

**Theorem.** Let $\sum_{r=0}^{\infty} u_r$ and $\sum_{s=0}^{\infty} v_s$ be two absolutely convergent series, and let $(w_n)$ be the convolution of $(u_r)$ and $(v_s)$. Then $\sum_{n=0}^{\infty} w_n = (\sum_{r=0}^{\infty} u_r) (\sum_{s=0}^{\infty} v_s)$.

**Corollary.** Let $(a_r), (b_s)$ be sequences and let $(c_n)$ be their convolution. Let $z$ be within the radii of convergence of $\sum_{r=0}^{\infty} a_r z^r$ and $\sum_{s=0}^{\infty} b_s z^s$. Then

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{s=0}^{\infty} b_s z^s\right) = \sum_{n=0}^{\infty} c_n z^n$$

**Theorem.** Let $\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $R$ and let $|z| < R$. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is differentiable at $z$ with derivative $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$.

# 6 Riemann integration

**Theorem.** Let $f : [a, b] \to \mathbb{R}$ be a monotone function. Then $f$ is integrable.

**Theorem.** Let $a < b$. Then every continuous function $f : [a, b] \to \mathbb{R}$ is uniformly continuous.

**Theorem.** Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f$ is integrable on $[a, b]$.

**Theorem.** Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be bounded, and continuous except at finitely many points. Then $f$ is Riemann integrable.

**Theorem** (Fundamental Theorem of Calculus - i)**.** Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$. Then $F$ is differentiable and $F'(x) = f(x)$.

**Theorem** (Fundamental Theorem of Calculus - ii)**.** Let $f : [a, b] \to \mathbb{R}$ be integrable and suppose that $F : [a, b] \to \mathbb{R}$ is such that $F'(x) = f(x)$. Then $\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$.

**Theorem** (Integration by parts)**.** Let $a < b$ and let $f, g : [a, b] \to \mathbb{R}$ be continuously differentiable. Then $\int_a^b f(x)g'(x)\,\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,\mathrm{d}x$.

**Theorem** (Taylor's Theorem with the integral form of the remainder)**.** Let $f$ be $n$-times continuously differentiable on $[x, x + h]$. Then

$$f(x+h) = f(x) + hf'(x) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x) + \int_x^{x+h} \frac{f^{(n)}(t)(x+h-t)^{n-1}}{(n-1)!}\,\mathrm{d}t$$

**Theorem.** Let $a < b$, $c < d$ and let $\theta : [c, d] \to [a, b]$ be a continuously differentiable function with $\theta(c) = a$, $\theta(d) = b$. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $\int_a^b f(t)\,\mathrm{d}t = \int_c^d f(\theta(s))\theta'(s)\,\mathrm{d}s$.

**Theorem** (The integral test)**.** Let $f : [1, \infty) \to \mathbb{R}$ be a decreasing function taking non-negative values. Then for every $N \in \mathbb{R}$,

$$\sum_{n=2}^{N} f(n) \leq \int_1^N f(x)\,\mathrm{d}x \leq \sum_{n=1}^{N-1} f(n)$$

In particular, $\sum_{n=1}^{\infty} f(n) < \infty$ if and only if $\int_1^{\infty} f(x)\,\mathrm{d}x < \infty$.

# 7 Odds and Ends

**Theorem** (Binomial Expansion)**.** Let $x$ be a real number with $|x| < 1$. Then

$$(1+x)^a = 1 + ax + \binom{a}{2}x^2 + \binom{a}{3}x^3 + \ldots$$

where $\binom{a}{b}$ is defined to be $\frac{a(a-1)(a-2)\ldots(a-k+1)}{k!}$.

**Theorem** (Heine-Borel theorem)**.** Let $\mathcal{U}$ be a collection of open intervals that covers a closed interval $[a, b]$. Then $\mathcal{U}$ has a finite subcover.

**Corollary.** A continuous function $f : [a, b] \to \mathbb{R}$ is bounded.