# Methods - Summary

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# 1 Fourier Series

## 1.1 Fourier coefficients

The $n$th Fourier coefficient of a function $f : [-\pi, \pi] \to \mathbb{R}$ (or $\mathbb{C}$) is

$$\hat{f}_n = \frac{1}{2\pi}\left(e^{in\theta}, f(\theta)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\,\mathrm{d}\theta$$

The Fourier series is given by

$$f(\theta) = \sum_{n\in\mathbb{Z}} e^{in\theta}\hat{f}_n$$

If $f$ is *real-valued*, can write

$$f(\theta) = \hat{f}_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos n\theta f(\theta)\,\mathrm{d}\theta$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin n\theta f(\theta)\,\mathrm{d}\theta$$

If $f$ has a jump discontinuity at $x$ then the Fourier series takes the value $\frac{f(x^-)+f(x^+)}{2}$ at $x$.

## 1.2 Rate of convergence

If the first $(m-1)$ derivatives of $f(\theta)$ are continuous (but the $m$th derivative has finitely many jump discontinuities) then $\hat{f}_k \sim \mathcal{O}\left(\frac{1}{k^{m+1}}\right)$ as $k \to \infty$.

## 1.3 Parseval's identity

$$(f, f) = \int_{-\pi}^{\pi} f^*(\theta) f(\theta)\,\mathrm{d}\theta = \int_{-\pi}^{\pi} |f(\theta)|^2\,\mathrm{d}\theta = 2\pi\sum_{k\in\mathbb{Z}} |\hat{f}_k|^2$$

# 2 Sturm-Liouville Theory

## 2.1 Sturm-Liouville operator

A Sturm-Liouville operator is a second-order differential operator $\mathcal{L}$ that can be written in the form

$$\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) - q(x)$$

If a function $f$ is periodic ($f(a) = f(b)$ and $f'(a) = f'(b)$) or we impose the boundary conditions

$$\alpha_a f(a) + \beta_a f'(a) = 0$$
$$\alpha_b f(b) + \beta_b f'(b) = 0$$

for $\alpha_a, \alpha_b, \beta_a, \beta_b$ constant, then

$$(\mathcal{L}f, g) = (f, \mathcal{L}g) + \left[p(f^{*\prime}g - f^* g')\right]_a^b = (f, \mathcal{L}g)$$

i.e. the S-L operator is *self-adjoint*.

## 2.2 Eigenfunctions

A weight function $w : \Omega \to \mathbb{R}^{\geq 0}$ is a real, non-negative function with at most finitely many isolated zeroes. We can define a new inner product

$$(f, g)_w \equiv \int_a^b f^* g w \, \mathrm{d}x = (f, wg) = (wf, g)$$

A function $y(x)$ is an *eigenfunction* of a SL operator $\mathcal{L}$ with weight $w(x)$ if

$$(\mathcal{L}y)(x) = \lambda w(x) y(x)$$

for some $\lambda \in \mathbb{C}$ (the *eigenvalue*).

### 2.2.1 Properties

(i) The eigenvalues $\lambda_1, \lambda_2, \ldots$ are real and can be ordered such that $\lambda_1 < \lambda_2 < \ldots \to \infty$.

(ii) Each eigenvalue $\lambda_n$ corresponds to a unique eigenfunction $y_n(x)$, the $n$th fundamental solution.

(iii) The normalised eigenfunctions form an orthonormal basis

$$\int_a^b y_n(x) y_m(x) w(x) dx = \delta_{mn}$$

## 2.3 Eigenfunction expansion

Any function $f : \Omega \to \mathbb{C}$ can be expanded in the basis of eigenfunctions as

$$f(x) = \sum_{n=1}^{\infty} f_n y_n(x)$$

where $\{y_1, y_2, \ldots\}$ are a basis of orthogonal eigenfunctions of $\mathcal{L}$ and the coefficients are given by

$$f_n = (y_n, f)_w = \int_\Omega y_n^*(x) f(x) w(x) dx$$

Parseval's Identity is now

$$(f, f)_w = \sum_{n=1}^{\infty} |f_n|^2$$

# 3 Laplace's equation

Let $\Omega \subset \mathbb{R}^n$ be compact, and $\psi : \Omega \to \mathbb{C}$. We say $\psi$ is *harmonic* if it satisfies Laplace's equation $\nabla^2\psi = 0$ throughout the interior of $\Omega$.

## 3.1 Boundary conditions

(i) Dirichlet: $\psi|_{\partial\Omega} = f$ where $f : \partial\Omega \to \mathbb{C}$

(ii) Neumann: $\mathbf{n} \cdot \boldsymbol{\nabla}\psi|_{\partial\Omega} = g$ where $g : \partial\Omega \to \mathbb{C}$ and $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

## 3.2 Spherical polars

In spherical polars, we have

$$(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

for $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2}$$

Suppose $\psi$ is harmonic on $\Omega = \{r \leq a\} \subset \mathbb{R}^3$, and assume $\psi$ is independent of $\phi$. Then a separable solution $\psi(r, \theta) = R(r)\Theta(\theta)$ satisfies the radial equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \lambda R$$

and the angular equation

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) = -\lambda\Theta\sin\theta$$

## 3.3 Legendre's Equation

Via the change of variables $x \equiv \cos\theta$, the angular equation above becomes Legendre's equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{d\Theta}{\mathrm{d}x}\right) = -\lambda\Theta$$

which is a second order ODE of SL-type with $p(x) = (1-x^2), q(x) = 0, w(x) = 1$.

Trying a solution of the form $\Theta(x) = \sum_{n=0}^{\infty} a_n x^n$ gives the recurrence relation

$$a_{n+2} = a_n\left(\frac{n(n+1)-\lambda}{(n+1)(n+2)}\right)$$

and hence two linearly independent solutions $\Theta(x) = a_0\Theta_0(x) + a_1\Theta_1(x)$. Imposing the boundary conditions requires that the power series terminates (so that the series does not diverge at $x = \pm1$), hence $\lambda = l(l+1)$ for $l \in \mathbb{Z}_{\geq 0}$.

The Legendre polynomial $P_\ell(x)$ of degree $l$ is conventionally normalised such that $P_\ell(1) = 1$, giving

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \end{aligned}$$

Rodrigues' formula is

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^\ell$$

and the polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} P_\ell(x)P_m(x)\,\mathrm{d}x = \frac{2}{2\ell+1}\delta_{\ell m}$$

### 3.4 Cylindrical coordinates

In cylindrical coordinates we have

$$(x,y,z) = (r\cos\theta, r\sin\theta, z)$$

and

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

### 3.5 Bessel's Equation

The cylindrical radial equation (after rescaling) becomes Bessel's equation

$$x^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + x\frac{\mathrm{d}R}{\mathrm{d}x} + (x^2-n^2)R = 0$$

This second order ODE has two linearly independent solutions $J_n(x)$ and $Y_n(X)$, the Bessel functions of the first and second kind, of order $n$. Note that $Y_n(x) \sim \mathcal{O}(\log x)$ for small $x$, so is not regular at the origin.

## 4 The Heat Equation

The heat equation is

$$\frac{\partial\phi}{\partial t} = K\nabla^2\phi$$

where $\phi(x,t)$ is the temperature at position $x$ and time $t$.

## 5 The Wave Equation

The wave equation is

$$\frac{\partial^2\phi}{\partial t^2} = c^2\nabla^2\phi$$

where $\phi(x,t)$ is the displacement from the equilibrium at position $x$ and time $t$.

## 6 Distributions

Let $D(\mathbb{R}^n)$ be the space of infinitely smooth functions $\phi : \mathbb{R}^n \to \mathbb{R}$ with compact support. A *distribution* is a linear map $T : D(\mathbb{R}^n) \to \mathbb{R}$, so if $\phi \in D(\mathbb{R}^n)$, then $T : \phi \mapsto T[\phi]$.

If $f : \mathbb{R}^n \to \mathbb{R}$ is a 'normal' function, the distribution $T_f$ is defined by

$$T_f[\phi] = (f,\phi) = \int_{\mathbb{R}^\ltimes} f(x)\phi(x)\,\mathrm{d}V$$

The derivative of a generalised function $T$ is given by $T'[\phi] = -T[\phi']$.

## 6.1 Dirac delta function

The Dirac $\delta$ is defined by

$$\delta[\phi] \equiv \phi(0)$$

We often write

$$\delta[\phi] = \int_{-\infty}^{\infty} \delta(x)\phi(x)dx$$

where $\delta(x)$ is called the *Dirac $\delta$-function.*

### 6.1.1 Properties

(i) If $f$ is continuous about the origin

$$g(x)\delta(x) = g(0)\delta(x)$$

(ii)

$$\int_{\mathbb{R}} \delta(x-a)\phi(x)\,\mathrm{d}x = \phi(a)$$

(iii)

$$\int_{\mathbb{R}} \delta(cx)\phi(x)\,\mathrm{d}x = \frac{1}{|c|}\phi(0)$$

(iv) If $f$ has only simple zeroes at $\{x_1, x_2, \dots, x_n\}$ then

$$\delta(f(x))\phi(x)dx = \sum_{i=1}^{n} \frac{\phi(x_i)}{|f'(x_i)|}$$

### 6.1.2 Fourier series

The Fourier series expansion for $\delta(x)$ is

$$\delta(x) = \frac{1}{2L}\sum_{n=-\infty}^{\infty} \exp\left(\frac{in\pi x}{L}\right)$$

More generally, let $Y_n$ be eigenfunctions of a SL operator with weight function $w(x)$. Then $\delta(x-\xi) = \sum_{n\in\mathbb{Z}} c_n Y_n(x)$ with $c_n = \int_a^b Y_n^*(x)\delta(x-\xi)w(x)\,\mathrm{d}x = Y_n^*(\xi)w(\xi)$ so that

$$\begin{aligned}\delta(x-\xi) &= w(\xi)\sum_{n\in\mathbb{Z}} Y_n(x)Y_n^*(\xi) \\ &= w(x)\sum_{n\in\mathbb{Z}} Y_n(x)Y_n^*(\xi)\end{aligned}$$

since $\delta(x-\xi)\frac{w(x)}{w(\xi)} = \delta(x-\xi)$.

# 7 Green's Functions

Let $\mathcal{L} = \alpha(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta(x)\frac{\mathrm{d}}{\mathrm{d}x} + \gamma(x)$ be a general second order differential operator on $x \in [a,b]$. Then Green's function $G(x,\xi)$ of $\mathcal{L}$ is defined as the solution to

$$\mathcal{L}G(x,\xi) = \delta(x-\xi)$$

that obeys $G(a,\xi)=0$ and $G(b,\xi)=0$.

The solution to the forced ordinary differential equation $\mathcal{L}y(x)=f(x)$ for some forcing term $f(x)$ which satisfies $y(a)=y(b)=0$ is given by

$$y(x)=\int_a^b G(x,\xi)f(\xi)\,\mathrm{d}\xi$$

Let $y_1(x)$ and $y_2(x)$ be 2 linearly independent solutions to the homogeneous equation $\mathcal{L}G(x,\xi)=0$ in the regions $a\le x<\xi$ and $\xi<x\le b$ where $y_1(a)=0$ and $y_2(b)=0$. Then

$$G(x,\xi)=\begin{cases} c_1(\xi)y_1(x) & : x\in[a,\xi) \\ c_2(\xi)y_2(x) & : x\in(\xi,b] \end{cases}$$

The condition that $G(x,\xi)$ is continuous gives

$$c_1(\xi)y_1(\xi)=c_2(\xi)y_2(\xi)$$

and the jump condition in the derivative gives

$$c_2(\xi)y_2'(\xi)-c_1(\xi)y_1'(\xi)=\frac{1}{\alpha(\xi)}$$

This determines the coefficients as

$$c_1(\xi)=\frac{y_2(\xi)}{\alpha(\xi)W(\xi)}$$
$$c_2(\xi)=\frac{y_1(\xi)}{\alpha(\xi)W(\xi)}$$

where $W(x)=y_1(x)y_2'(x)-y_2(x)y_1'(x)$ is the *Wronskian* of $y_1$ and $y_w$.

The Green's function is therefore

$$\begin{aligned} G(x,\xi) &= \frac{1}{\alpha(\xi)W(\xi)}\begin{cases} y_2(\xi)y_1(x) & : x\in[a,\xi) \\ y_1(\xi)y_2(x) & : x\in(\xi,b] \end{cases} \\ &= \frac{1}{\alpha(\xi)W(\xi)}\left[\Theta(\xi-x)y_2(\xi)y_1(x)+\Theta(x-\xi)y_1(\xi)y_2(x)\right] \end{aligned}$$

The general solution to the forced problem $\mathcal{L}y=f(x)$ obeying $y(a)=y(b)=0$ is

$$y(x)=y_2(x)\int_a^x \frac{y_1(\xi)}{\alpha(\xi)W(\xi)}f(\xi)\,\mathrm{d}\xi+y_1(x)\int_x^b \frac{y_2(\xi)}{\alpha(\xi)W(\xi)}f(\xi)\,\mathrm{d}\xi$$

The Green's function can be expanded in terms of the eigenfunctions $Y_n$ (with eigenvalues $\lambda_n$) as

$$G(x,\xi)=\sum_{n\in\mathbb{Z}}\frac{1}{\lambda_n}Y_n(x)Y_n^*(\xi)$$

# 8 Fourier Transforms

Let $f:\mathbb{R}\to\mathbb{C}$ and suppose that $|f|$ is integrable over any interval $[a,b]$ with $\int_{\mathbb{R}}|f(x)|\,\mathrm{d}x$ convergent. The *Fourier transform* is defined as

$$\tilde{f}(k)=FT[f]=\int_{-\infty}^{\infty}e^{-ikx}f(x)\,\mathrm{d}x$$

Fourier's inversion theorem says

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,\mathrm{d}x \iff f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)\,\mathrm{d}k$$

Duality for Fourier transform

$$\tilde{\tilde{f}}(x) = FT[\tilde{f}(x)] = \int_{-\infty}^{\infty} e^{-ikx}\tilde{f}(k)\,\mathrm{d}k = 2\pi f(-x)$$

i.e. if we know $g(k) = FT[f(x)]$ then $FT[g(k)] = 2\pi f(-x)$.

## 8.1 Properties

- Linearity: $\widetilde{\lambda f + \mu g} = \lambda\tilde{f} + \mu\tilde{g}$.
- Translation: If $g(x) = f(x-a)$ then $\tilde{g}(k) = e^{-ika}\tilde{f}(k)$.
- Frequency shift: If $g(x) = e^{i\lambda x}f(x)$ then $\tilde{g}(k) = \tilde{f}(k+\lambda)$.
- Rescaling: If $g(x) = f(\lambda x)$ then $\tilde{g}(k) = \frac{1}{|\lambda|}f\left(\frac{k}{|\lambda|}\right)$.
- Multiplication by $x$: If $g(x) = xf(x)$ then $\tilde{g}(k) = i\tilde{f}'(k)$.
- Differentiation: If $g(x) = f'(x)$ then $\tilde{g}(k) = ik\tilde{f}(k)$.

## 8.2 Convolution

If $\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)$ then

$$h(x) = \int_{-\infty}^{\infty} g(y)f(x-y)\,\mathrm{d}y \equiv (g * f)(x) = (f * g)(x)$$

called the *convolution* of $f(x)$ and $g(x)$.

Setting $\tilde{g}(k) = \tilde{f}(k)^*$ gives Parseval's theorem for Fourier transforms

$$\int_{-\infty}^{\infty} |f(y)|^2\,\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2\,\mathrm{d}k$$

## 8.3 Linear differential equations

Suppose input $I(t)$ and output $O(t)$ are related by $\mathcal{L}O(t) = I(t)$. We can Fourier transform both sides of the equation by noting that, writing $\hat{O}(\omega) = FT[O(t)] = \int e^{-i\omega t}O(t)$,

$$\begin{aligned}
FT\left[\frac{dO(t)}{dt}\right] &= i\omega\hat{O}(t)\\
FT\left[\frac{d^2O(t)}{dt^2}\right] &= -\omega^2\hat{O}(t)\\
\vdots\qquad & \qquad\vdots\\
FT\left[\frac{d^nO(t)}{dt^n}\right] &= (i\omega)^n\hat{O}(t)
\end{aligned}$$

We can then write $\hat{O}(\omega) = \hat{R}(\omega)\hat{I}(\omega)$ and solve for the transform function $\hat{R}(\omega)$, which gives the response function as

$$R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{R}(\omega)\,\mathrm{d}\omega$$

The solution is given by

$$O(t) = \int I(u)R(t-u)\,\mathrm{d}u$$

i.e. $R(t-t') = G(t;t')$ *is* the Green's function.

### 8.4 Fourier Transform of Gaussian

Note that

$$\phi(x) = e^{-a^2x^2} \iff \tilde{\phi}(k) = \sqrt{\frac{\pi}{a^2}}e^{-\frac{k^2}{4a^2}} .$$

## 9 Discrete Fourier Transforms

The discrete Fourier transform is a linear map $DFT : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$\tilde{\mathbf{a}}_m = DFT(\mathbf{a})_m = \sum_{j=0}^{N-1} a_i\omega$$

for $m = 0, 1, \ldots, N-1$, where $\omega = e^{-\frac{2\pi i}{N}}$.

$DFT$ is represented by an $N \times N$ matrix with entries

$$(DFT)_{jm} = \omega^{jm} = e^{-\frac{2\pi i}{N}jm}$$

### 9.1 Properties

- $DFT$ is Unitary: $DFT^{-1} = \frac{1}{N}DFT^{\dagger}$
- Parseval's Theorem / Plancherel's Theorem: $\mathbf{a}\cdot\mathbf{a} = \frac{1}{N}\tilde{\mathbf{a}}\cdot\tilde{\mathbf{a}}$
- If $\mathbf{c} = \mathbf{a} * \mathbf{b}$ is the convolution given by

  $$c_k = \sum_{m=0}^{N-1} a_m b_{k-m} \quad k = 0, 1, \ldots, N-1$$

  then $\tilde{\mathbf{c}}_k = \tilde{\mathbf{a}}_k\tilde{\mathbf{b}}_k$.

## 10 Method of Characteristics

Consider the PDE

$$\alpha(x,y)u_x + \beta(x,y)u_y = f(x,y,u)$$

with initial data along the curve $B$ as $u(x^B(t), y^B(t)) = h(t)$. The characteristic curves $(x^C(s,t), y^C(s,t))$ (where $t$ is a label of the curve, and $s$ the parametrisation along the curve) satisfy

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \alpha(x,y) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \beta(x,y)$$

with initial conditions $x^C(0,t) = x^B(t)$ and $y^C(0,t) = y^B(t)$. Then by the chain rule, $\frac{\mathrm{d}u}{\mathrm{d}s} = f(x,y,u)$. Solving this gives $u(s,t)$, and if we can invert the change of variables to give $s(x,y)$ and $t(x,y)$, this gives $u$ as a function of $x$ and $y$.

## 10.1 Second Order PDEs

Consider the PDE

$$\mathcal{L}u = A_{ij}(\mathbf{x})\partial_i\partial_j u + B_i(\mathbf{x})\partial_i u + C(\mathbf{x})u = \gamma(\mathbf{x})$$

with $A_{ij}(\mathbf{x})$ real-valued and symmetric. $\mathcal{L}$ is

$$\left\{\begin{array}{c}\text{elliptic}\\ \text{parabolic}\\ \text{hyperbolic}\end{array}\right\} \text{ if the eigenvalues of } A_{ij} \left\{\begin{array}{c}\text{are all the same sign}\\ \text{include at least one zero}\\ \text{are of mixed sign}\end{array}\right\}.$$

For a second order operator, we can write $\mathcal{L}u = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu$. Then $\mathcal{L}$ is

$$\left\{\begin{array}{ll}\text{elliptic if} & b^2 - ac < 0\\ \text{parabolic if} & b^2 - ac = 0\\ \text{hyperbolic if} & b^2 - ac > 0\end{array}\right\}.$$

The characteristic curves are given by solving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{-b \pm \sqrt{b^2 - ac}}{a}$$

giving solutions of the form $f_\pm(x, y) = \text{constant}$. Performing a change of variables $\xi = f_+$ and $\eta = f_-$ reduces the equation to canonical form.

# 11 d'Alembert's solution of the wave equation in $\mathbb{R}^{1,1}$

Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

with $u(x, 0) = \phi(x)$ and $\partial_t u(x, 0) = \psi(x)$. Applying the method of characteristics gives

$$u(x, t) = f(x - ct) + g(x + ct)$$

with the initial conditions giving

$$f(x) = g(x) = \phi(x) \quad \text{and} \quad -cf'(x) + cg'(x) = \psi(x)$$

leading to the general solution

$$u(x, t) = \frac{1}{2}\big(\phi(x - ct) + \phi(x + ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,\mathrm{d}y$$

# 12 Green's Functions for PDEs

Free space Green's function for the Laplacian in 2d is

$$G_2(r; r_0) = \frac{1}{2\pi}\log|\mathbf{r} - \mathbf{r}_0|$$

and in 3d is

$$G_3(r; r_0) = \frac{1}{4\pi}\frac{1}{|\mathbf{r} - \mathbf{r}_0|}\ .$$

# 13 Green's Identities

- First Identity

$$\int_\Omega \left(\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi + \phi\nabla^2\psi\right) \mathrm{d}V = \int_{\partial\Omega} (\phi\boldsymbol{\nabla}\psi \cdot \mathbf{n})\, \mathrm{d}S$$

- Second Identity

$$\int_\Omega \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) \mathrm{d}V = \int_{\partial\Omega} (\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi) \cdot \mathbf{n}\, \mathrm{d}S$$

- Third Identity

$$\phi(\mathbf{r}_0) = \int_\Omega G(\mathbf{r}, \mathbf{r}_0)\nabla^2\phi\, \mathrm{d}V + \int_{\partial\Omega} (\phi\boldsymbol{\nabla}G - G\boldsymbol{\nabla}\phi) \cdot \mathbf{n}\, \mathrm{d}S$$

# 14 Method of Images

For problems on semi-infinite domains, construct a Green's function such that $G = 0$ (if Dirichlet) or $\frac{\partial G}{\partial n} = 0$ (if Neumann) on the boundary, and apply Green's 3rd Identity.