Analysis II

Theorems and Corollaries

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1 Uniform convergence

Proposition. Let $f_n, f : E \to \mathbb{R}$. Then $f_n \to f$ uniformly on E if and only if $\sup_{x\in E} |f_n(x) - f(x)| \to 0$.

Theorem (Cauchy criterion for uniform convergence)**.** Let $f_n : E \to \mathbb{R}$. Then (f_n) converges uniformly on E if and only if

$$\forall \epsilon > 0 \quad \exists N = N(\epsilon) \quad \text{such that} \quad n, m \geq N,\ x \in E \implies |f_n(x) - f_m(x)| < \epsilon$$

Theorem. Let $f_n, f : E \to \mathbb{R}$ and suppose that $f_n \to f$ uniformly on E. If $x \in E$ is a point of continuity for each f_n then x is a point of continuity for f.

Corollary. If f_n are continuous on E and $f_n \to f$ uniformly on E then f is continuous on E.

Theorem. Let $f_n, f : [a, b] \to \mathbb{R}$. Suppose that f_n and f are Riemann integrable on $[a, b]$ and that $f_n \to f$ uniformly on $[a, b]$. Then $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.

Theorem. Let $a < b$ and $f_n : [a, b] \to \mathbb{R}$ be differentiable (with one-sided derivatives at the end points). Suppose that

(i) $(f_n(c))$ converges for some $c \in [a, b]$;

(ii) (f_n') converges uniformly on $[a, b]$.

Then $f_n \to f$ uniformly on $[a, b]$ (for some function f) and f is differentiable with $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in [a, b]$.

Theorem (Weierstrass M-test)**.** Let $g_n : E \to \mathbb{R}$ for $n = 0, 1, 2, \ldots$. Suppose there exists $M_n \geq 0$ such that

(i) $|g_n(x)| \leq M_n$ for all $x \in E$ and

(ii) $\sum_{n=0}^\infty M_n$ converges.

Then $\sum_{n=0}^\infty g_n$ converges absolutely uniformly on E.

Theorem (Radius of convergence)**.** Let $\sum_{n=0}^\infty c_n(x - a)^n$ be a power series. Then there exists a unique $R \in [0, \infty]$ (the *radius of convergence*) such that

(i) If $|x-a|<R$ then $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely.

(ii) If $|x-a|>R$ then $\sum_{n=0}^{\infty} c_n(x-a)^n$ diverges.

(iii) If $R>0$ and $0<r<R$ then $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely uniformly on the set $\{x : |x-a| \leq r\}$.

Theorem (Term-wise differentiation of power series)**.** Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series with radius of convergence $R>0$. Then

(i) the *derived series* $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ has radius of convergence R;

(ii) if $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ is the function defined by the power series on $\{x : |x-a|<R\}$ then f is differentiable on $\{x : |x-a|<R\}$ with $f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ for all $x \in \{x : |x-a|<R\}$.

2 Uniform continuity

Theorem. Let $a<b \in \mathbb{R}$. If $f : [a,b] \to \mathbb{R}$ is continuous then f is uniformly continuous.

Theorem. Let $a<b \in \mathbb{R}$ and let $f : [a,b] \to \mathbb{R}$ be a bounded, Riemann integrable function with $|f(x)| \leq M$. Let $g : [-M,M] \to \mathbb{R}$ be continuous. Then the composite function $g \circ f : [a,b] \to \mathbb{R}$ is Riemann-integrable.

Corollary. Let $a<b \in \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous. Then g is integrable.

Theorem (Riemann Criterion for Integrability)**.** Let $f : [a,b] \to \mathbb{R}$. Then f is integrable if and only if $\forall \epsilon > 0$ there exists a partition P of $[a,b]$ such that $U(P,f)-L(P,f)<\epsilon$.

Theorem. Let $f : [a,b] \to [A,B] \subseteq \mathbb{R}$ be integrable and $g : [A,B] \to \mathbb{R}$ be continuous. Then $g \circ f : [a,b] \to \mathbb{R}$ is integrable.

Theorem. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of integrable functions converging uniformly to a function $f : [a,b] \to \mathbb{R}$. Then f is integrable and $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.

Proposition. Let $f : [a,b] \to \mathbb{R}$ be integrable. Then $||f(x)||$ is integrable and $||\int_a^b f(x)dx|| \leq \int_a^b ||f(x)||dx$.

Theorem (*Lebesgue's criterion for Riemann integrability*)**.** Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if the set of discontinuities of f has measure zero.

Theorem (*Weierstrass approximation theorem*)**.** Let $f : [0,1] \to \mathbb{R}$ be continuous. Then there exists a sequence of polynomials converging to f uniformly on $[0,1]$. In fact, the sequence of *Bernstein polynomials*

$$f_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}$$

converges uniformly to f on $[0,1]$.

3 Normed spaces

Proposition (Young's inequality)**.** Let $a, b \geq 0$ and $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

Proposition (Hölder's inequality)**.** Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $f, g \in C([a, b])$, $\int_a^b |f(x)||g(x)|dx \leq \|f\|_p \|g\|_q$.

Proposition (Minkowski inequality)**.** Let $1 \leq p < \infty$ and $f, g \in C([a, b])$. Then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proposition. Let $x^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right) \in \mathbb{R}^n$ for $k = 1, 2, \dots$. Then $x^{(k)} \to x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ (with respect to $\|\cdot\|_2$) if and only if $x_j^{(k)} \to x_j$ as $k \to \infty$ for each $1 \leq j \leq n$.

Proposition. Suppose that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms on a vector space V. Then

(i) A set $E \subseteq V$ is bounded with respect to $\|\cdot\|$ if and only if E is bounded with respect to $\|\cdot\|'$.

(ii) $x_k \to x$ with respect to $\|\cdot\|$ if and only if $x_k \to x$ with respect to $\|\cdot\|'$.

Theorem (Bolzano-Weierstass for $\mathbb{R}^n$)**.** Any bounded sequence in $\mathbb{R}^n$ (with the Euclidean norm) has a convergent subsequence.

Theorem (*Arzela-Ascoli theorem*)**.** If a sequence (f_k) in $C([0, 1])$ is uniformly bounded and equicontinuous, then it has a uniformly convergent subsequence.

Proposition. Let $(V, \|\cdot\|)$ be a normed space. If $x_n \to x$ then (x_n) is Cauchy.

Proposition. Any Cauchy sequence is bounded.

Theorem. $\mathbb{R}^n$ (with the Euclidean norm) is complete.

Theorem. $C([a, b])$ with the norm $\|\cdot\|_\infty$ is complete.

Proposition. Every open ball is an open set.

Proposition. A subset $E \subseteq V$ is closed if and only if its complement $V \setminus E$ is open.

Theorem. Let $(V, \|\cdot\|)$ be a normed space.

(i) The union of any collection (finite, countable or uncountable) of open subsets is open.

(ii) The intersection of a *finite* number of open sets is open.

(iii) The empty set $\emptyset$ and the whole set V are open.

Corollary. Let $(V, \|\cdot\|)$ be a normed space.

(i) The intersection of any collection of closed subsets is closed.

(ii) The union of a *finite* number of closed sets is closed.

(iii) The empty set $\emptyset$ and the whole set V are closed.

Theorem. Let $E \subseteq V$ and $f : E \to V'$. Then f is continuous at $x \in E$ if and only if $f(x_n) \to f(x)$ for every sequence (x_n) in E with $x_n \to x$.

Proposition. Let $(V, \|\cdot\|)$ be a normed space. Then $\|\cdot\| : V \to \mathbb{R}$ is Lipschitz continuous.

Lemma. (i) If $(V, \|\cdot\|)$ is a normed space and $E \subseteq V$ is sequentially compact, then E is closed and bounded.

(ii) A closed, bounded subset $E \subseteq \mathbb{R}^n$ is sequentially compact.

Theorem. Let $(V, \|\cdot\|)$ be a normed space and let $K \subseteq \mathbb{R}^n$ be closed and bounded. Suppose $f : K \to V$ is continuous (where $\mathbb{R}^n$ has the Euclidean norm). Then

(i) f is uniformly continuous;

(ii) $f(K)$ is sequentially compact;

(iii) $f(K)$ is closed and bounded.

Corollary. If $K \subseteq \mathbb{R}^n$ is closed and bounded, and $f : K \to \mathbb{R}$ is continuous then f attains its sup and inf, i.e. there exists $x_1, x_2 \in K$ such that $f(x_1) = \sup_{x \in K} f(x)$ and $f(x_2) = \inf_{x\ \in K} f(x)$.

Theorem. Any two norms on $\mathbb{R}^n$ are Lipschitz equivalent.

Corollary. Any two norms on a finite dimensional vector space are equivalent.

4 Metric spaces

Theorem. Let (X, d) be a metric space.

(i) The union of any collection of open sets is open.

(ii) The intersection of a *finite* collection of open sets is open.

(iii) The empty set $\emptyset$ and the whole set X are both open.

Theorem. A subset $E \subseteq X$ is closed if and only if its complement $X \setminus E$ is open.

Theorem. Let (X, d) be a metric space.

(i) The intersection of any collection of closed sets is closed.

(ii) The union of a *finite* collection of closed sets is closed.

(iii) The empty set $\emptyset$ and the whole set X are both closed.

Proposition. A singleton $\{x\} \subseteq X$ is a closed subset (regardless of the metric).

Corollary. Any finite set is closed (regardless of the metric).

Proposition. Let (X, d) be a metric space. Then $x_n \to x$ if and only if for any neighbourhood V of x in X, $x_n \in V$ for all but finitely many n.

Theorem. Let (X, d) and (X', d') be metric spaces and $f : X \to X'$ be a mapping. The following are equivalent:

(i) f is continuous.

(ii) $x_n, x \in X$ for $n \in \mathbb{N}$ with $x_n \to x \implies f(x_n) \to f(x)$

(iii) For any open set V in X', $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is open in X.

(iv) For any closed set F in X', $f^{-1}(F)$ is closed in X.

Corollary. Let (X, d), (X', d') and (X'', d'') be metric spaces. Suppose $f : X \to X'$ and $g : X' \to X''$ are continuous. Then $g \circ f : X \to X''$ is continuous.

Theorem. Let (X, d) and (X', d') be metric spaces. If $f : X \to X'$ is a *uniformly* continuous map and (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is a Cauchy sequence in X'.

Proposition. If (x_n) is convergent in X then (x_n) is Cauchy.

Proposition. If (x_n) is Cauchy and has a subsequence (x_{n_j}) converging to $x \in X$ then (x_n) converges to x.

Proposition. Let (X, d) be a metric space and $Y \subseteq X$.

(i) If Y with the subspace metric $d|_{Y \times Y}$ is complete then Y is closed in X.

(ii) If (X, d) is complete then $(Y, d|_{Y \times Y})$ is complete if and only if Y is closed in X.

Proposition. If (X, d) is sequentially compact then X is complete and X is bounded.

Theorem. A metric space (X, d) is sequentially compact if and only if it is compact, i.e. it has the 'Heine-Borel property'.

Theorem. A metric space (X, d) is complete and totally bounded if and only if X is sequentially compact.

5 Contraction mapping theorem

Theorem (Contraction mapping theorem)**.** If f is a contraction mapping on a complete metric space X, then f has a unique fixed point in X, i.e. there exists a unique $x \in X$ with $f(x) = x$.

Theorem. Let $f : X \to X$ be a mapping on a complete metric space X. Suppose $f^{(m)} = f \circ f \circ \ldots \circ f : X \to X$ is a contraction. Then f has a unique fixed point.

Theorem. Let $F : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and $x_0 \in \mathbb{R}^n$. Suppose that there exists $k \geq 0$ such that $\forall t \in [a, b]$ and all $x, y \in \mathbb{R}^n$,

$$\|F(t, x) - F(t, y)\|_2 \leq k\|x - y\|_2.$$

Then there exists a unique continuously differentiable function $f : [a, b] \to \mathbb{R}^n$ that satisfies

$$\begin{aligned} \frac{df}{dt} &= F(t, f(t)) \\ f(a) &= x_0. \end{aligned}$$

Theorem (Picard-Lindelöf existence theorem for ODEs)**.** Let $F : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and $x_0 \in \mathbb{R}^n$. Suppose that there exists $R, k \geq 0$ such that $\forall t \in [a, b]$ and all $x, y \in D_R(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq R\}$,

$$\|F(t, x) - F(t, y)\|_2 \leq k\|x - y\|_2.$$

Then

(i) There exists $\epsilon \in (0, b - a]$ and a unique continuously differentiable function $f : [a, a + \epsilon] \to \mathbb{R}^n$ that satisfies

$$\begin{aligned} \frac{df}{dt} &= F(t, f(t)) \quad \forall t \in [a, a + \epsilon] \\ f(a) &= x_0. \end{aligned}$$

(ii) If $\sup_{[a,b]\times D_R(x_0)} \|F\| \leq \frac{R}{b-a}$ then we can take $\epsilon = b - a$ above, so that there exists a unique continuously differentiable function on the whole interval $[a, b]$ satisfying the ODE system.

Theorem (Cauchy-Peano)**.** Let $x_0 \in \mathbb{R}^n$. Suppose $F : U \to \mathbb{R}^n$ is continuous for some open set $U \in \mathbb{R} \times \mathbb{R}^n$ containing $(0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Then there exists $\epsilon > 0$ such that the ODE system

$$\begin{aligned} \frac{df}{dt} &= F(t, f(t)) \quad \forall t \in [a, a + \epsilon] \\ f(0) &= x_0. \end{aligned}$$

has a continuously differentiable solution in $[0, \epsilon]$.

6 Differentiation in $\mathbb{R}^n$

Proposition. Let $U \subseteq \mathbb{R}^n$ be open, $a \in U$ and $f : U \to \mathbb{R}^m$. Suppose that there exist linear maps $A_1, A_2 : \mathbb{R}^n \to \mathbb{R}^m$ such that $\lim_{h\to 0} \frac{f(a+h)-f(a)-A_i(h)}{\|h\|} = 0$ for $i = 1, 2$. Then $A_1 = A_2$, i.e. the derivative of f is well-defined (when it exists).

Proposition. If $U \subseteq \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^m$ is differentiable at $a \in U$, then f is continuous at a.

Proposition. If $U \subseteq \mathbb{R}^n$ is open and $f = (f_1, f_2, \ldots, f_m) : U \to \mathbb{R}^m$ then f is differentiable at $a \in U$ if and only if $f_j : U \to \mathbb{R}$ is differentiable at a for $j = 1, 2, \ldots, m$.

Moreover, if we write $Df(a)$ as an $m \times n$ real matrix then $Df_j(a)$ is the linear map $\mathbb{R}^n \to \mathbb{R}$ corresponding to the jth row of $Df(a)$.

Proposition. If $U \subseteq \mathbb{R}^n$ is open, $f_1, f_2 : U \to \mathbb{R}^m$ are differentiable at $a \in U$ and $c_1, c_2 \in \mathbb{R}$ are constants, then $c_1f_1 + c_2f_2 : U \to \mathbb{R}^m$ is differentiable at a with $D(c_1f_1 + c_2f_2)(a) = c_1Df_1(a) + c_2Df_2(a)$.

Proposition. If $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then A is differentiable at every point $a \in \mathbb{R}^n$ with $DA(a) = A$.

Theorem (The chain rule)**.** Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open. Let $f : U \to \mathbb{R}^m$ and $g : V \to \mathbb{R}^p$. Let $a \in U$ with $f(a) \in V$. Suppose that f is differentiable at a and g is differentiable at $f(a)$. Then $g \circ f : U \to \mathbb{R}^p$ is differentiable at a with $D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$.

Proposition. Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and let $\|\cdot\|$ be the operator norm.

(i) $\|A\| < \infty$

(ii) $\|\cdot\|$ is a norm on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

(iii) $\|A\| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|A\|}{x}$

(iv) $\|Ax\| \leq \|A\|\|x\|$

(v) Let $b \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$, and write $BA = B \circ A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$. Then $\|BA\| \leq \|B\|\|A\|$.

(vi) Fix bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ and identify $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ with the space of real $m \times n$ matrices. Then the norm

$$\|A\|' = \left(\sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} A_{ij}^2 \right)^{\frac{1}{2}}$$

is Lipshitz equivalent to the operator norm.

Proposition. Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$. If f is differentiable at $a \in U$ then for any u, $D_u f(a)$ exists and is given by $D_u f(a) = Df(a)u$.

Proposition. Let $U \subseteq \mathbb{R}^n$ be open and $f = (f_1, f_2, \ldots, f_m) : U \to \mathbb{R}^m$. If f is differentiable at $a \in U$ then the partial derivatives $D_j f_i(a)$ exist for $1 \leq i \leq m$, $1 \leq j \leq n$ and the matrix for $Df(a)$ with respect to the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ is $\left(\frac{\partial f_i(x)}{\partial x_j}\right)$.

Theorem. Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$. Suppose there exists an open ball $B_r(a) \subset U$ such that

(i) $D_j f_i(x)$ exists for every $x \in B_r(a)$, for $1 \leq i \leq m, 1 \leq j \leq n$.

(ii) $D_j f_i$ are continuous at a, for $1 \leq i \leq m, 1 \leq j \leq n$.

Then f is differentiable at a with

$$Df(a) = (D_j f_i(a))_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

Corollary. Let $U \subseteq \mathbb{R}^n$ be open. Then $f : U \to \mathbb{R}^m$ is continuously differentiable on U if and only if $D_j f_i(x)$ exists for each x and $D_j f_i$ are continuous on U for $1 \leq i \leq m$ and $1 \leq j \leq n$.

6.1 Mean value inequalities

Theorem. Let $f : [a, b] \to \mathbb{R}^m$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose there exists $M \geq 0$ such that $\|f'(t)\| \leq M$ for all $t \in (a, b)$. Then $\|f(b) - f(a)\| \leq M(b - a)$.

Theorem. Let $f : B_r(a) \to \mathbb{R}^m$ be differentiable at every point in $B_r(a)$. Suppose there exists $M \geq 0$ such that $\|Df(x)\| \leq M$ for all $x \in B_r(a)$. If $b \in B_r(a)$ then $\|f(b) - f(a)\| \leq M\|b - a\|$.

Corollary. If $f : B_r(a) \to \mathbb{R}^m$ is differentiable with $Df(x) = 0$ for all $x \in B_r(a)$ then f is constant.

Theorem. Let $U \subseteq \mathbb{R}^n$ be open and path connected. If $f : U \to \mathbb{R}^m$ is differentiable with $Df \equiv 0$ on U then f is constant on U.

6.2 Inverse function theorem

Theorem (Inverse function theorem)**.** Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ continuously differentiable. Let $a \in U$ be such that $Df(a)$ is invertible (as a linear map $\mathbb{R}^n \to \mathbb{R}^n$). Then there exist open sets V and W such that $f|_V : V \to W$ is a bijection with continuously differentiable inverse.

Theorem (Implicit function theorem)**.** Let $U \subseteq \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ continuously differentiable. Let $S = \{(x, y) \in U : f(x, y) = 0\}$. Suppose $(a, b) \in S$ and that $D_y f$ is invertible at (a, b). Then there exist open subsets V of U containing (a, b) and W of $\mathbb{R}^m$ containing a, and a continuously differentiable map $g : W \to \mathbb{R}^n$ such that $S \cap V = \operatorname{graph} g$, where $\operatorname{graph} g = \{(x, g(x)) : x \in W\}$.

6.3 Higher derivatives

Theorem. Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$. Suppose there exists $a \in U$ and $\rho > 0$ such that $B_\rho(a) \subseteq U$ and $D_{ij}f(x)$ and $D_{ji}f(x)$ exist for all $x \in B_\rho(a)$ and are continuous at $x = a$. Then $D_{ij}f(a) = D_{ji}f(a)$.

Theorem (Taylor's Theorem)**.** Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ differentiable k times in U. Then for any $a \in U$ and $h \in \mathbb{R}$ such that $[a, a+h] = \{a + th : 0 \leq t \leq 1\} \subseteq U$,

$$f(a+h) = f(a) + Df(a)[h] + \frac{1}{2!}D^2f(a)[h]^2 + \ldots + \frac{1}{(k-1)!}D^{k-1}f(a)[h]^{k-1} + \frac{1}{k!}D^k(b)[h]^k$$

for some $b \in [a, a+h]$.