Modelagem Geométrica SME0271

Hermite RBF para Curvas Implícitas

Objetivo

- Partir de pontos P e normais N
- Construir uma função implícita cuja a valor vale 0 para cada P e onde o gradiente da função seja igual a N
- Precisamos de uma RBF(gaussiana, multiquadrática, Wendland etc...) e precisamos calcular o gradiente dela e o Hessian.
- $\boldsymbol{\nabla\Psi} = \begin{bmatrix} \frac{\partial\boldsymbol{\Psi}}{\partial x1} & \frac{\partial\boldsymbol{\Psi}}{\partial x2} \end{bmatrix}$ **e** $\mathbf{H\Psi} = \begin{bmatrix} \frac{\partial^2\boldsymbol{\Psi}}{\partial x1^2} & \frac{\partial^2\boldsymbol{\Psi}}{\partial x1 \partial x2} \\ \frac{\partial^2\boldsymbol{\Psi}}{\partial x1 \partial x2} & \frac{\partial^2\boldsymbol{\Psi}}{\partial x2^2} \end{bmatrix}$

Métodos

- Implementação em Matlab de HRBF
- Exemplo 2D com Gaussiano e Wendland
- Exemplo 3D com Gaussiano, Wendland, multiquadrática, multiquadrática inversa e spline
- Partição da unidade em 2D e 3D

Formulação em dimensão 2

- $P = \begin{bmatrix} x1_1 & x2_1 \\ \vdots & \vdots \\ x1_n & x2_n \end{bmatrix}$
- $X1 = \begin{bmatrix} x1_1 & \cdots & x1_1 \\ \vdots & \cdots & \vdots \\ x1_n & \cdots & x1_n \end{bmatrix} - \begin{bmatrix} x1_1 & \cdots & x1_1 \\ \vdots & \cdots & \vdots \\ x1_n & \cdots & x1_n \end{bmatrix}^T$
- $X2 = \begin{bmatrix} x2_1 & \cdots & x2_1 \\ \vdots & \cdots & \vdots \\ x2_n & \cdots & x2_n \end{bmatrix} - \begin{bmatrix} x2_1 & \cdots & x2_1 \\ \vdots & \cdots & \vdots \\ x2_n & \cdots & x2_n \end{bmatrix}^T$
- $X = \sqrt{X1^2 + X2^2}$

Formulação em dimensão 2

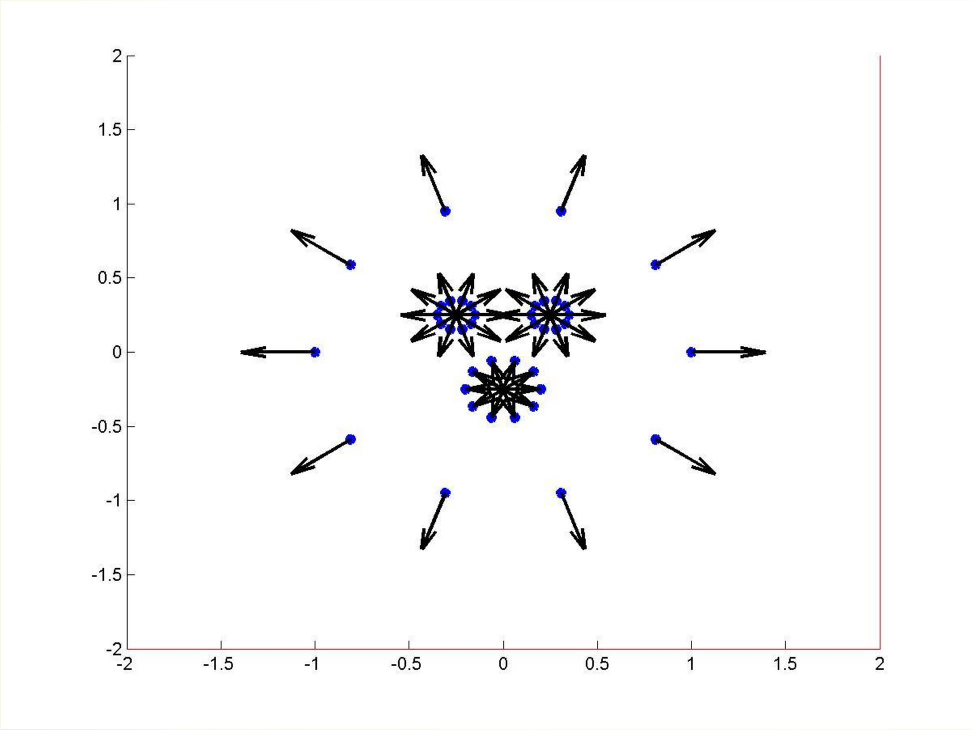
- $A = \begin{bmatrix} \boldsymbol{\Psi}(X) & -\nabla\boldsymbol{\Psi}(X) \\ \nabla\boldsymbol{\Psi}^T(X) & -\mathbf{H}\boldsymbol{\Psi}(\mathbf{X}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}(X) & -\frac{\partial\boldsymbol{\Psi}}{\partial x1}(X) & -\frac{\partial\boldsymbol{\Psi}}{\partial x2}(X) \\ \left(\frac{\partial\boldsymbol{\Psi}}{\partial x1}(X)\right)^T & -\frac{\partial^2\boldsymbol{\Psi}}{\partial x1^2}(X) & -\frac{\partial^2\boldsymbol{\Psi}}{\partial x1\partial x2}(X) \\ \left(\frac{\partial\boldsymbol{\Psi}}{\partial x2}(X)\right)^T & -\left(\frac{\partial^2\boldsymbol{\Psi}}{\partial x1\partial x2}(X)\right)^T & -\frac{\partial^2\boldsymbol{\Psi}}{\partial x2^2}(X) \end{bmatrix}$
- $A\begin{bmatrix} \alpha \\ \beta 1 \\ \beta 2 \end{bmatrix} = \begin{bmatrix} 0 \\ N1 \\ N2 \end{bmatrix} \quad com\ N = \begin{bmatrix} N1_1 & N2_1 \\ \vdots & \vdots \\ N1_n & N2_n \end{bmatrix}$

Formulação em dimensão 2

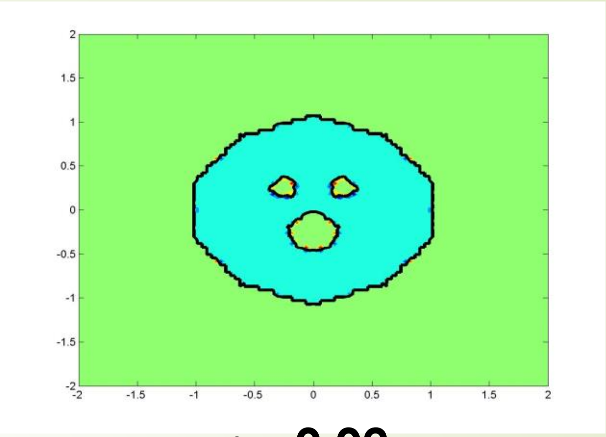
- $U = \begin{bmatrix} U1_1 & U2_1 \\ \vdots & \vdots \\ U1_m & U2_m \end{bmatrix}$
- $V1 = \begin{bmatrix} x1_1 & \cdots & x1_1 \\ \vdots & \cdots & \vdots \\ x1_n & \cdots & x1_n \end{bmatrix} - \begin{bmatrix} U1_1 & \cdots & U1_1 \\ \vdots & \cdots & \vdots \\ U1_m & \cdots & U1_m \end{bmatrix}^T$
- $V2 = \begin{bmatrix} x2_1 & \cdots & x2_1 \\ \vdots & \cdots & \vdots \\ x2_n & \cdots & x2_n \end{bmatrix} - \begin{bmatrix} U2_1 & \cdots & U2_1 \\ \vdots & \cdots & \vdots \\ U2_m & \cdots & U2_m \end{bmatrix}^T$
- $V = \sqrt{V1^2 + V2^2}$
- $f^*(U) = \boldsymbol{\Psi}(V)\alpha - \langle \beta | \nabla \boldsymbol{\Psi}(V) \rangle$

Exemplo 2D : gaussiano

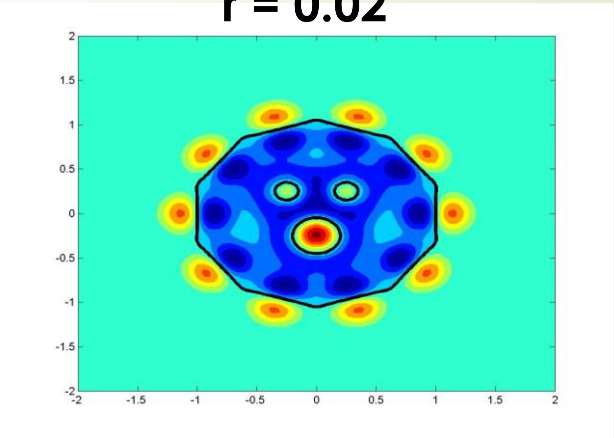
$$\mathbf{\Psi}(\mathrm{x}) = \exp\left({}^{x^2}/_{r^2}\right)$$



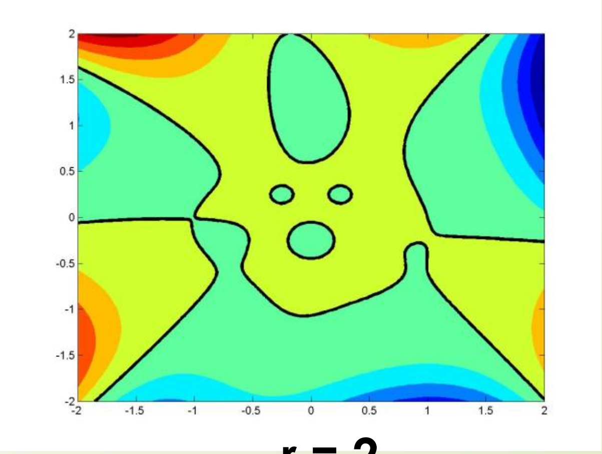
40 pontos



r = 0.02



r = 0.2

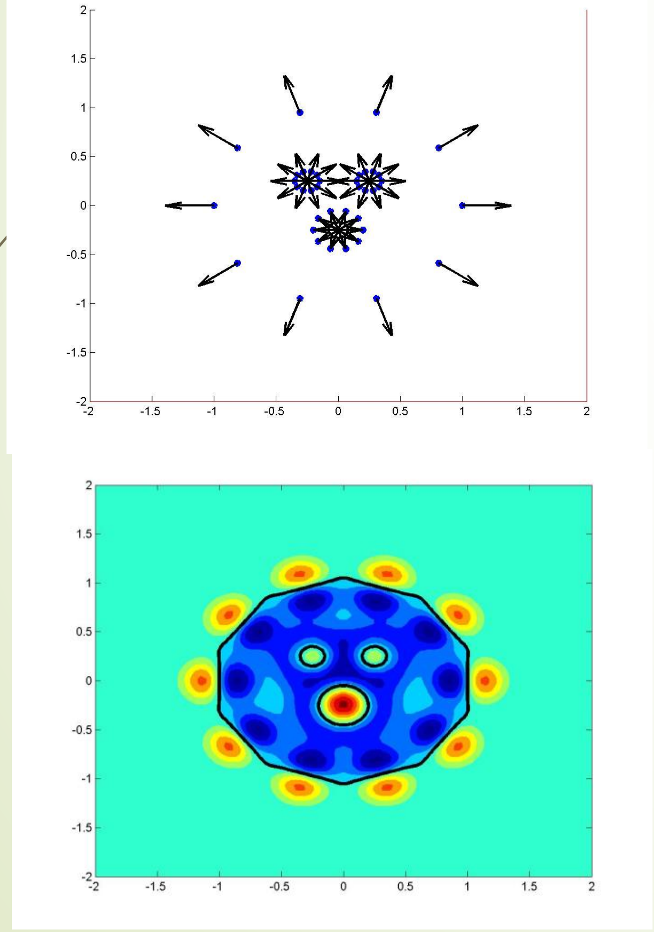


r = 2

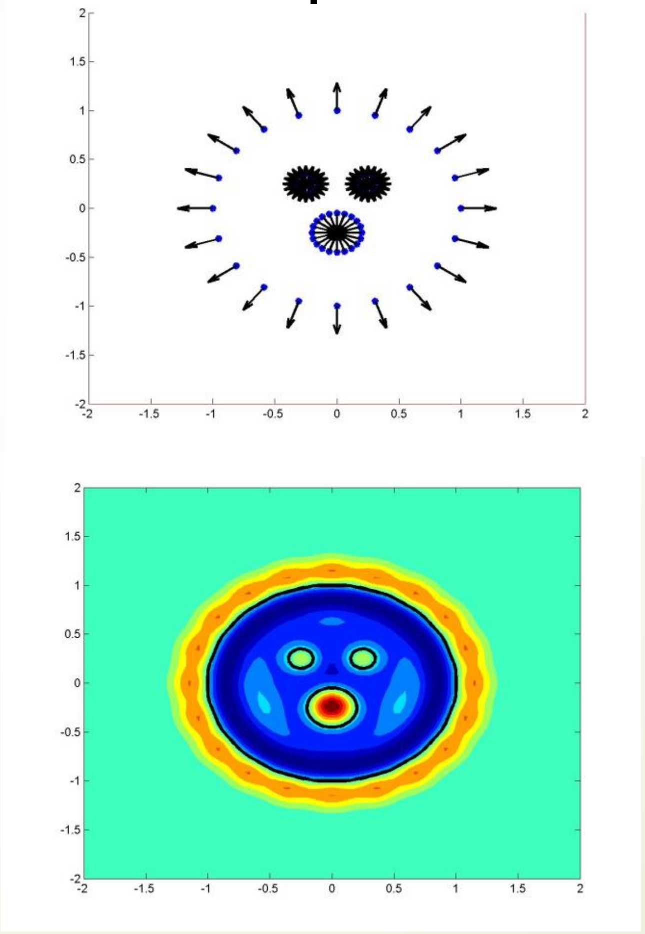
Exemplo 2D : gaussiano

$\boldsymbol{\Psi}(\mathrm{x}) = \exp\left({}^{x^2}/_{r^2}\right)$ com r = 0.2

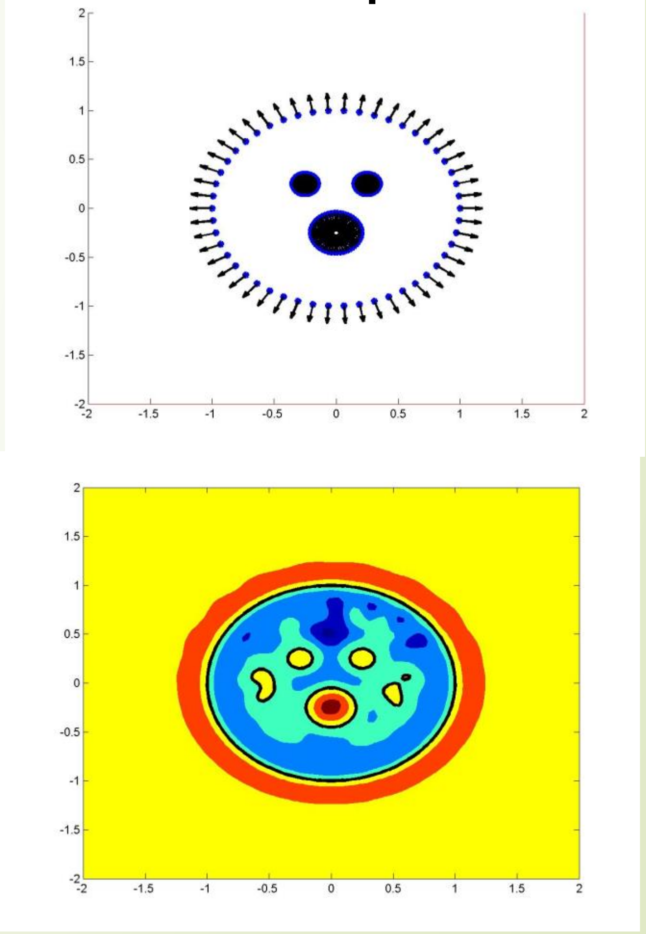
40 pontos



80 pontos



200 pontos



Teoria Polinômio

- $$Poly = \begin{bmatrix} 1 & x1_1 & x2_1 & x1_1 \cdot x2_1 & x1_1^2 & x2_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x1_n & x2_n & x1_n \cdot x2_n & x1_n^2 & x2_n^2 \\ 0 & 1 & 0 & x2_1 & 2 \cdot x1_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & x2_n & 2 \cdot x1_n & 0 \\ 0 & 0 & 1 & x1_1 & 0 & 2 \cdot x2_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & x1_n & 0 & 2 \cdot x2_n \end{bmatrix}$$
- $$\begin{bmatrix} A & Poly \\ Poly^T & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix} \; com \; \lambda = \begin{bmatrix} \alpha \\ \beta 1 \\ \beta 2 \end{bmatrix} \; e \; c = \begin{bmatrix} 0 \\ N1 \\ N2 \end{bmatrix}$$

Teoria
Polinômio

- $\gamma = (Poly^T \cdot A^{-1} \cdot Poly)^{-1} \cdot Poly^T \cdot A^{-1} \cdot c$

- $\lambda = A^{-1} \cdot c \; - (A^{-1} \cdot Poly) \cdot \gamma$

- $f^*(U) = \boldsymbol{\Psi}(V)\alpha - \langle \beta | \nabla \boldsymbol{\Psi}(V) \rangle + \begin{bmatrix} 1 & U1_1 & U2_1 & U1_1 \cdot U2_1 & U1_1^2 & U2_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & U1_m & U2_m & U1_m \cdot U2_m & U1_m^2 & U2_m^2 \end{bmatrix} [\gamma]$

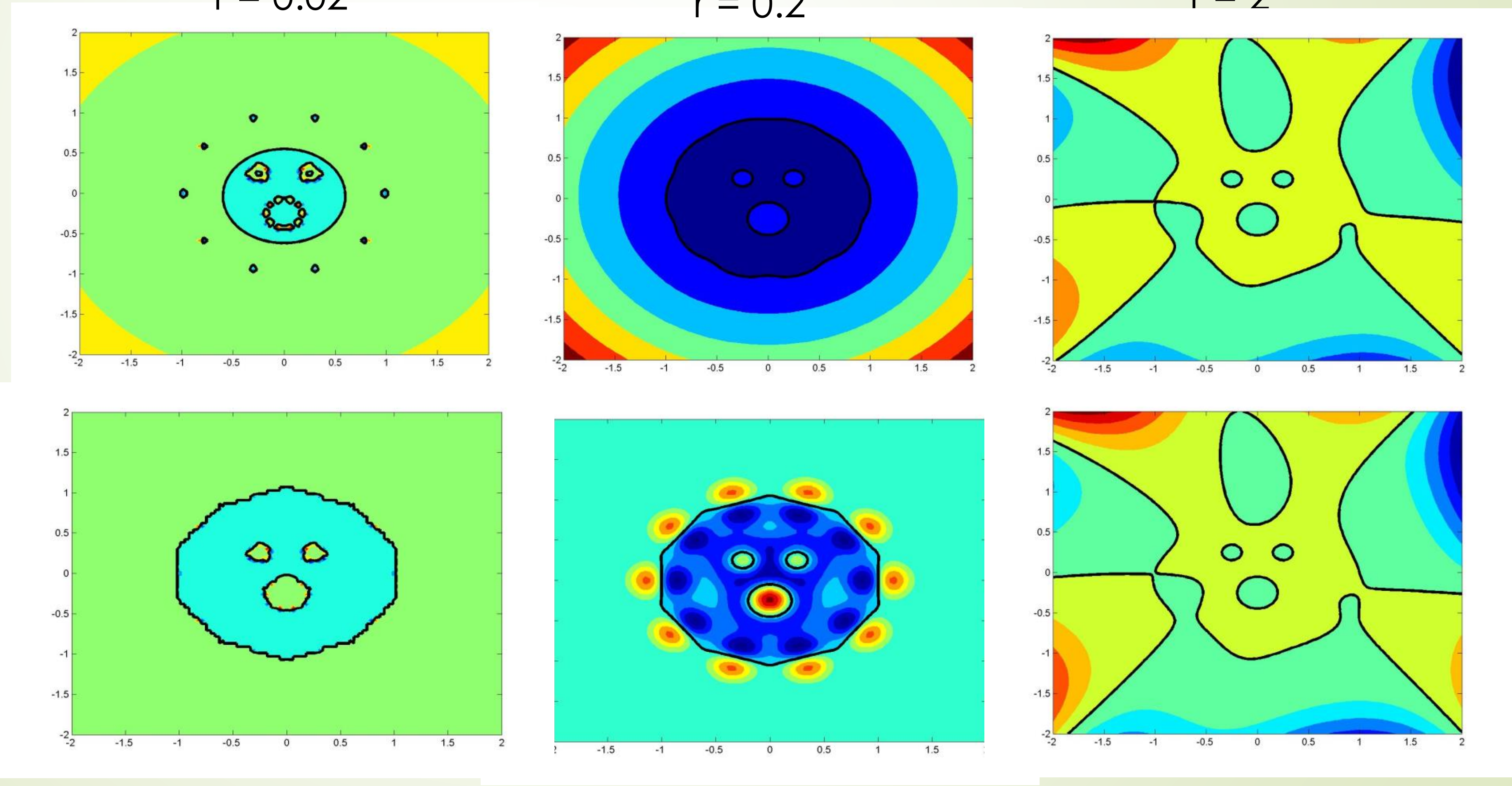
Exemplo 2D : gaussiano

$$\boldsymbol{\Psi}(\mathrm{x}) = \exp\left({}^{x^2}/_{r^2}\right)$$

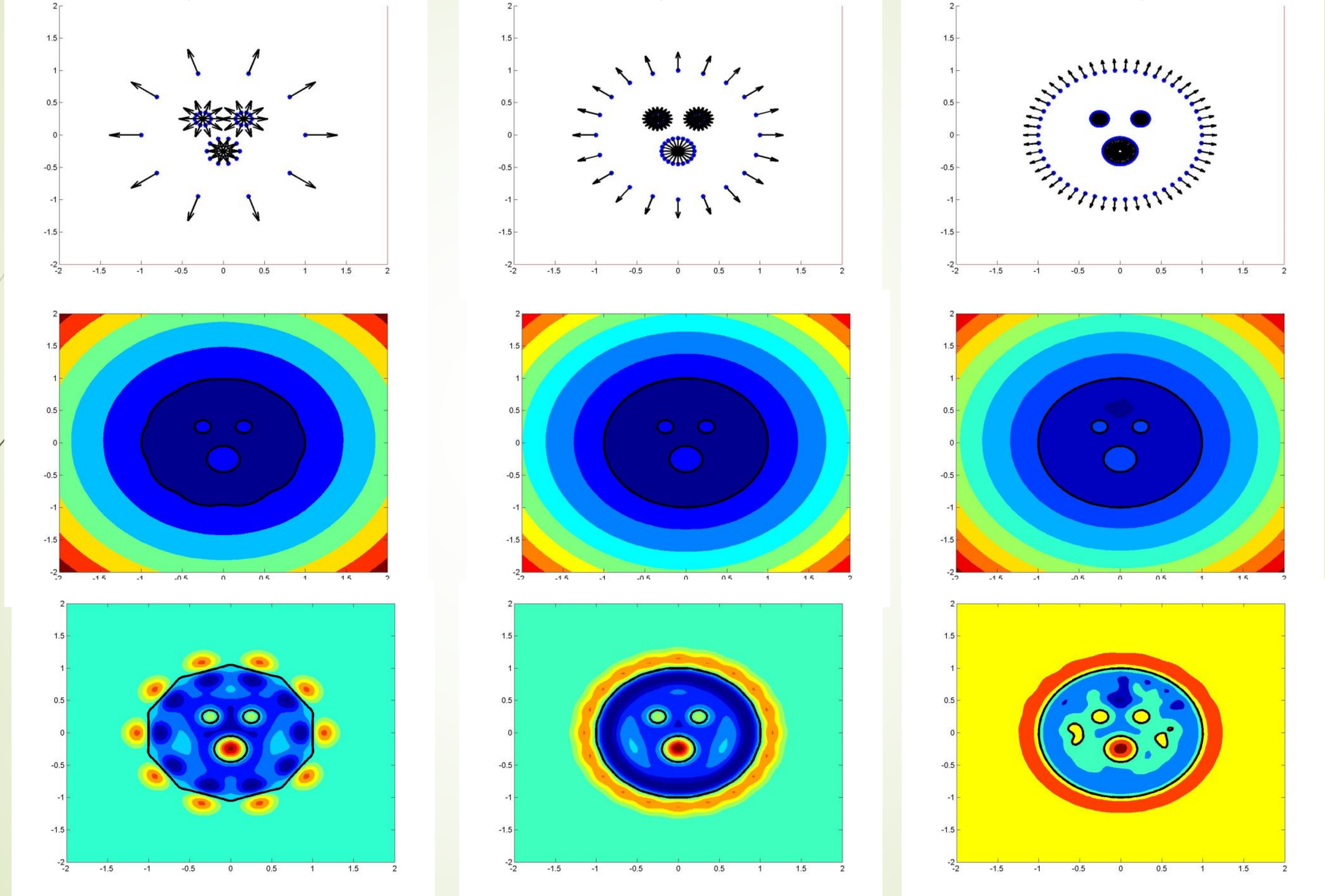
r = 0.02 r = 0.2 r = 2

Com função polinomial

Sem função polinomial



40 pontos | 80 pontos | 200 pontos



Com função polinomial (r = 0.2)

Sem função polinomial (r = 0.2)

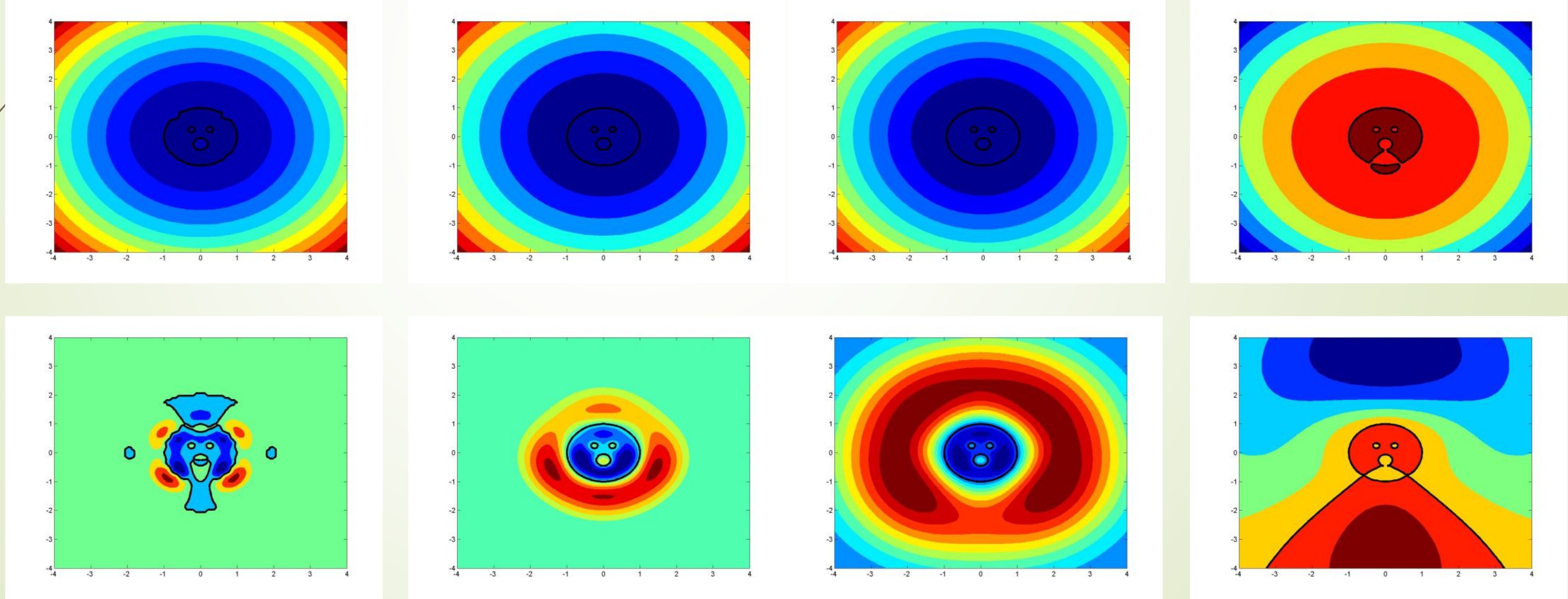
Wendland (80 pontos)

$$\mathbf{\Psi}(\mathrm{x}) = \begin{cases} \left(1 - \frac{x}{r}\right)^4 \left(4\frac{x}{r} + 1\right) para \frac{x}{t} < 1 \\ 0\ senão \end{cases}$$

r = 1 r = 2 r = 5 r = 10

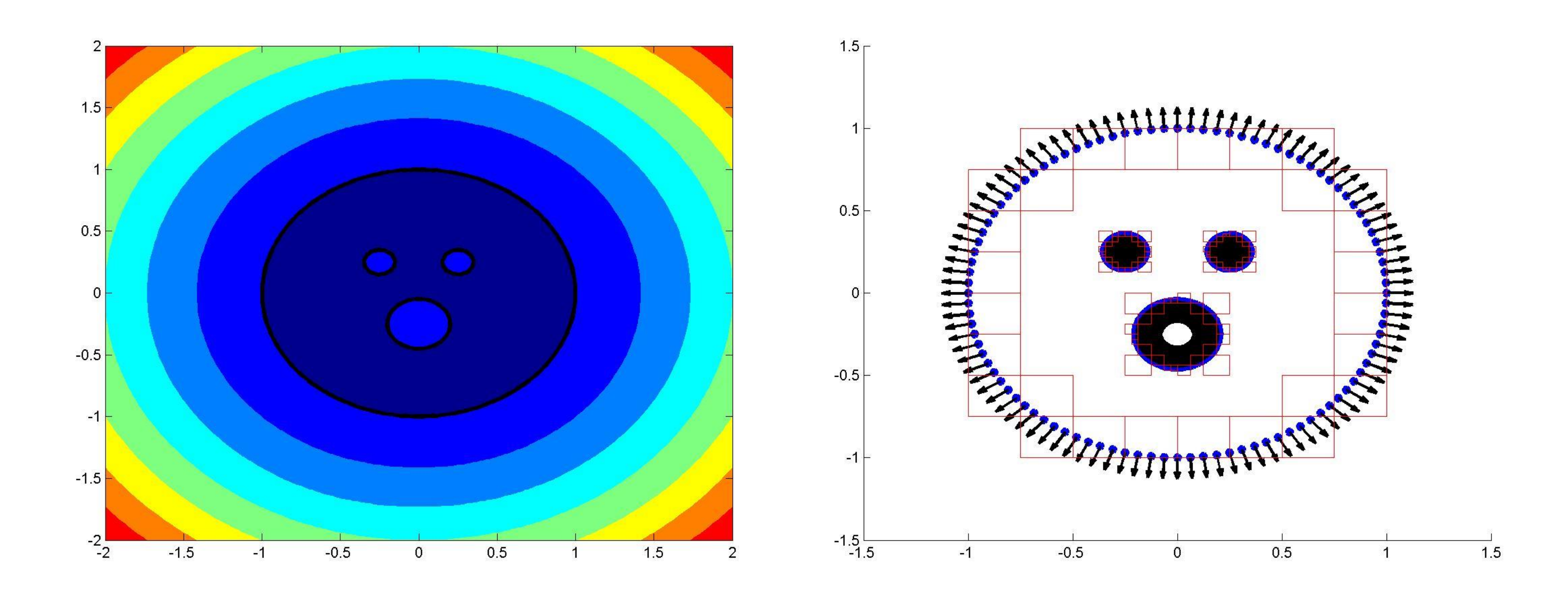
Com função polinomial

Sem função polinomial



Gaussiano com subdivisão dos pontos

- r = 0.2
- 400 pontos totais
- 5 pontos maximais per retângulo



Wendland 3D esfera

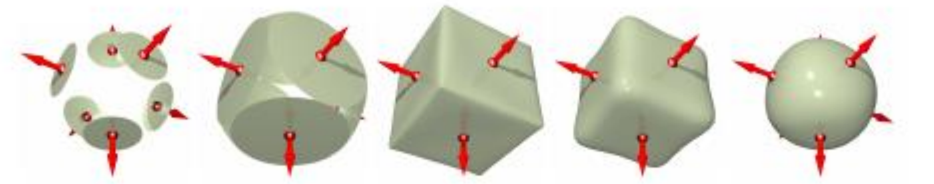
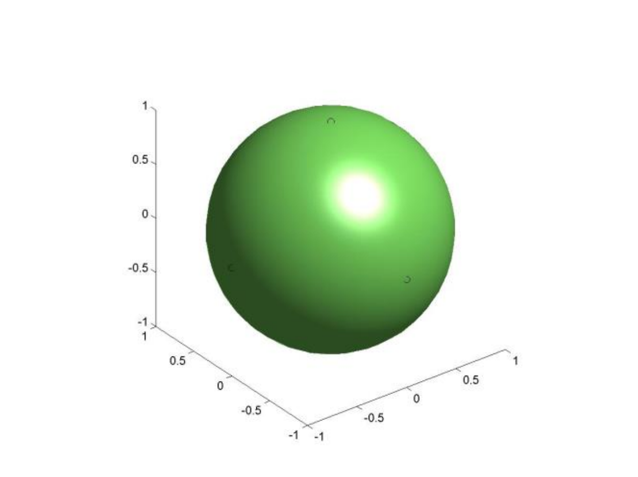
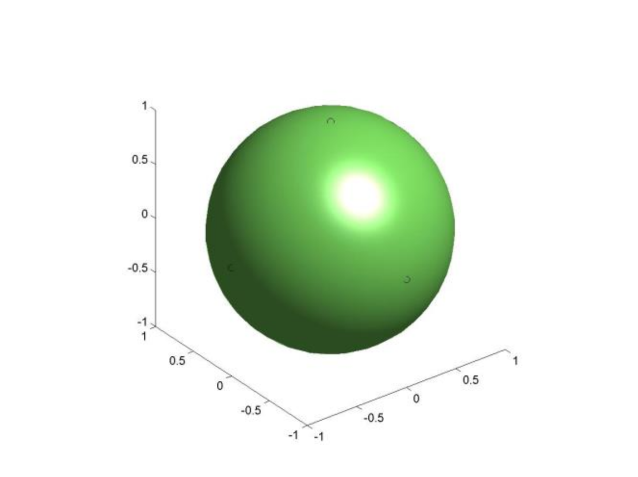


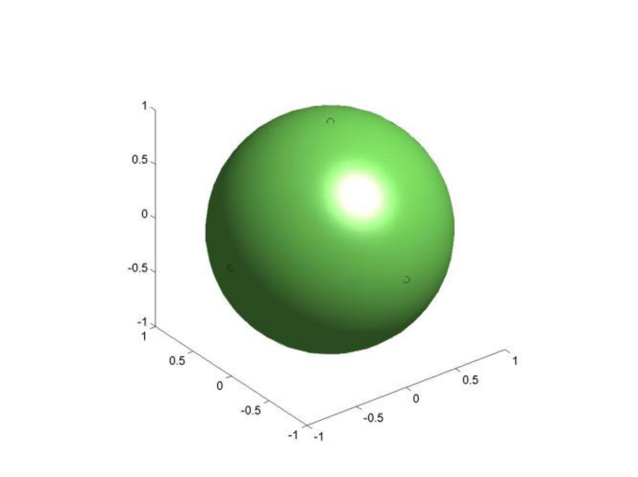
Figure 2: *Varying the radius in* ψ_r *with no augmentation* ($r = 0.5$, 1, 1.5, 2 *and* 4)*: six antipodal points on the sphere.*

r = 1.5 | **r = 2** | **r = 4**

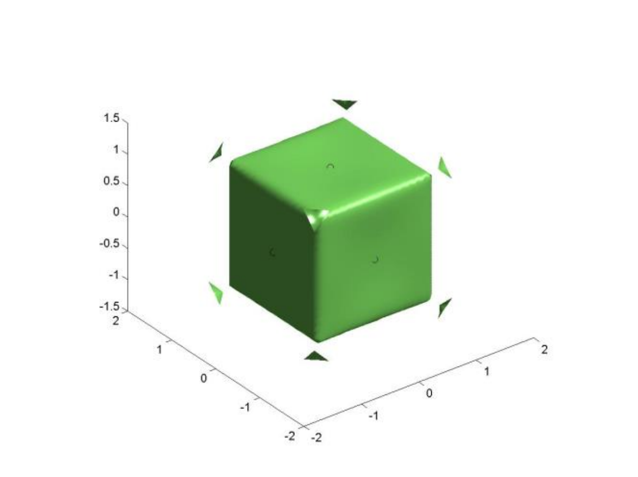
Com função polinomial

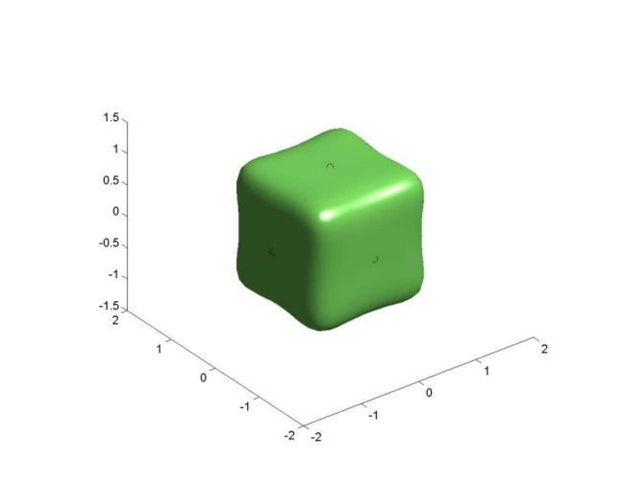


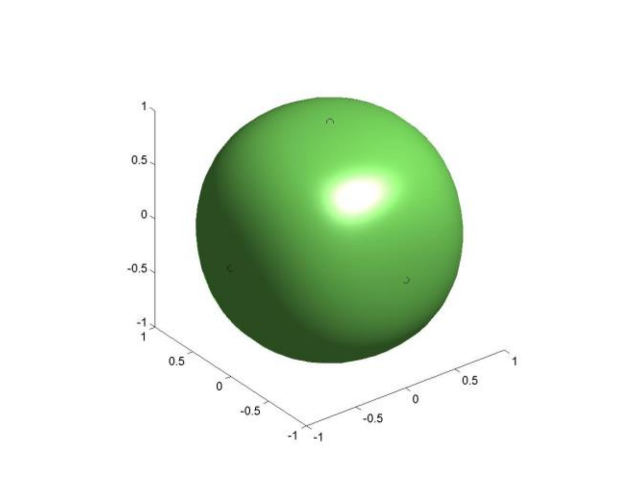




Sem função polinomial

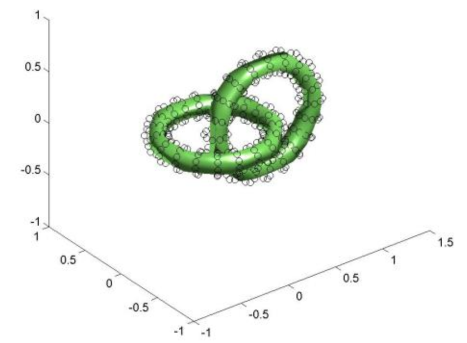




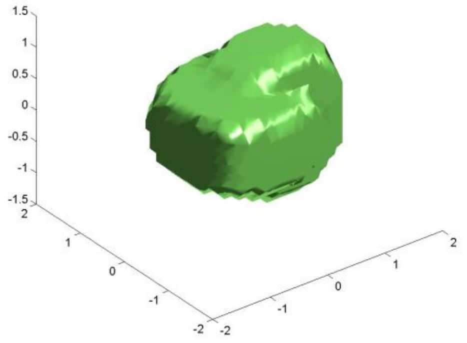


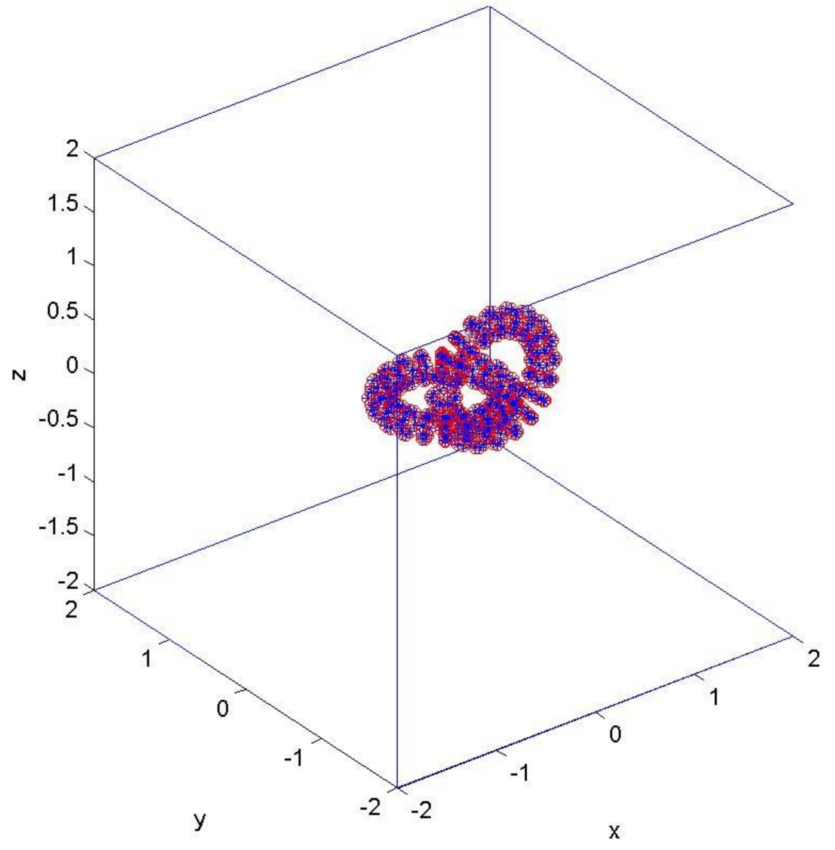
Wendland 3D
dois toros (400 pontos totais)

Com função polinomial
r = 0.6



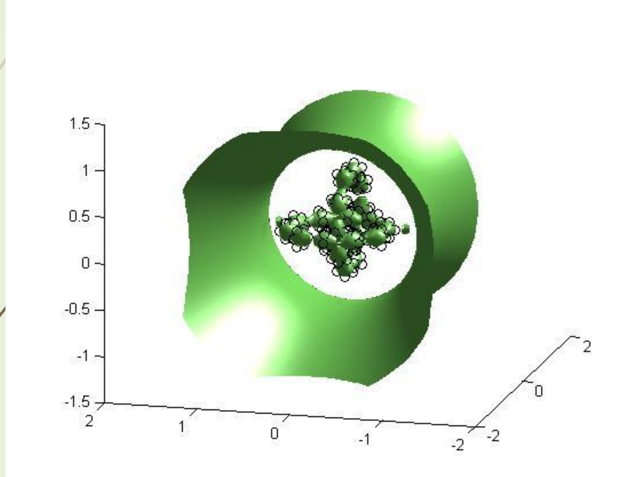
Sem função polinomial
r = 0.6



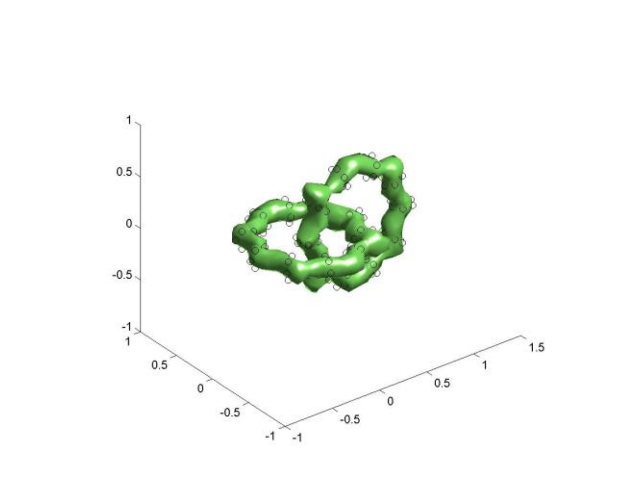


Wendland 3D
dois toros (120 pontos totais)

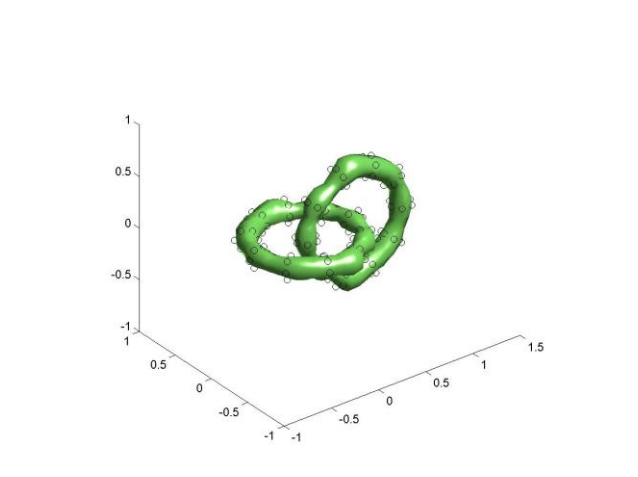
r = 0.3



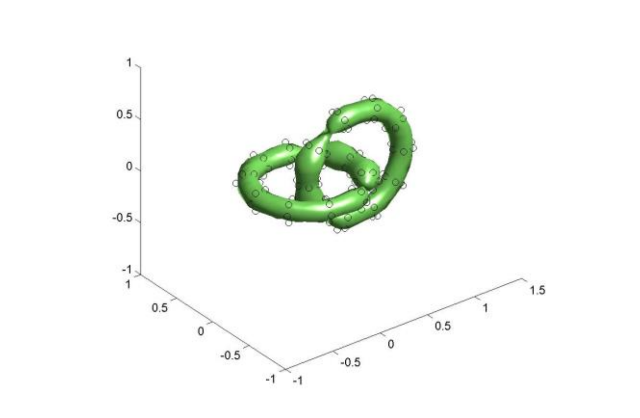
r = 0.4



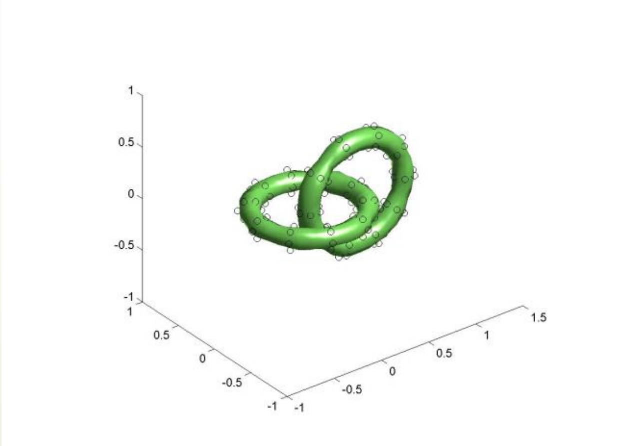
r = 0.6



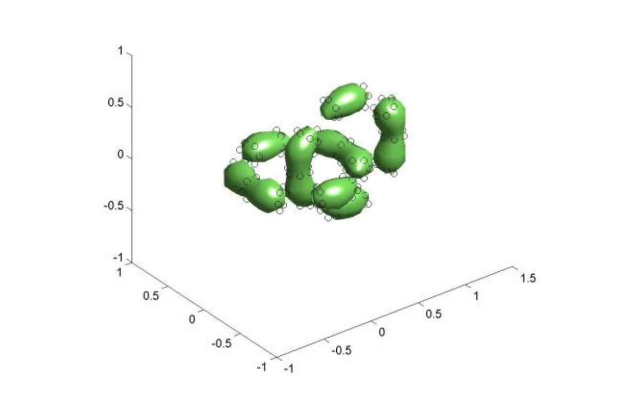
r = 1



r = 1.2

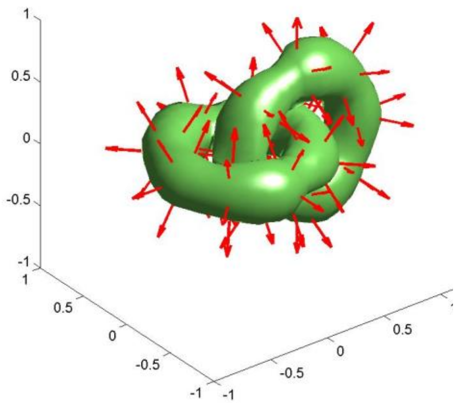


r = 3

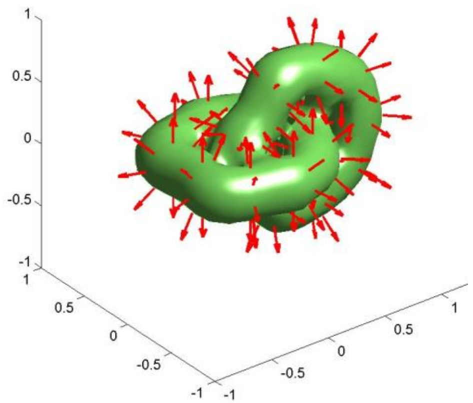


Variação do numero de ponto

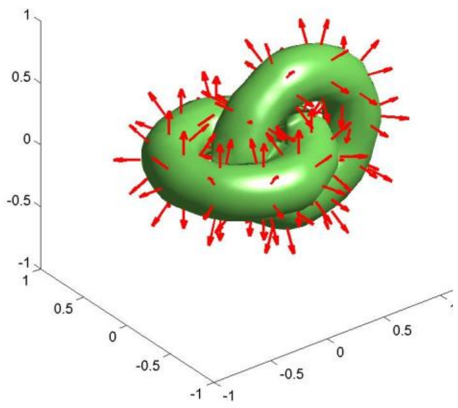
r = 1.3 e 96 pontos



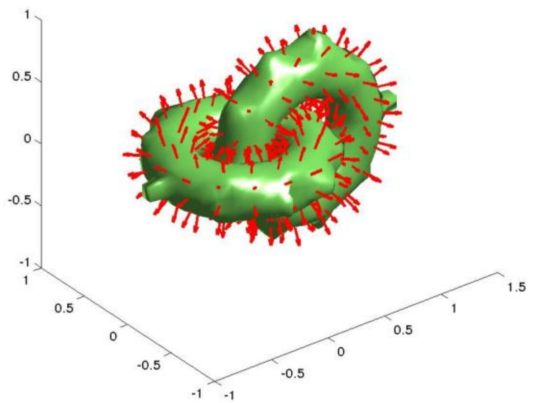
r = 1.1 e 144 pontos



r = 1.1 e 160 pontos



r = 5.3 e 360 pontos



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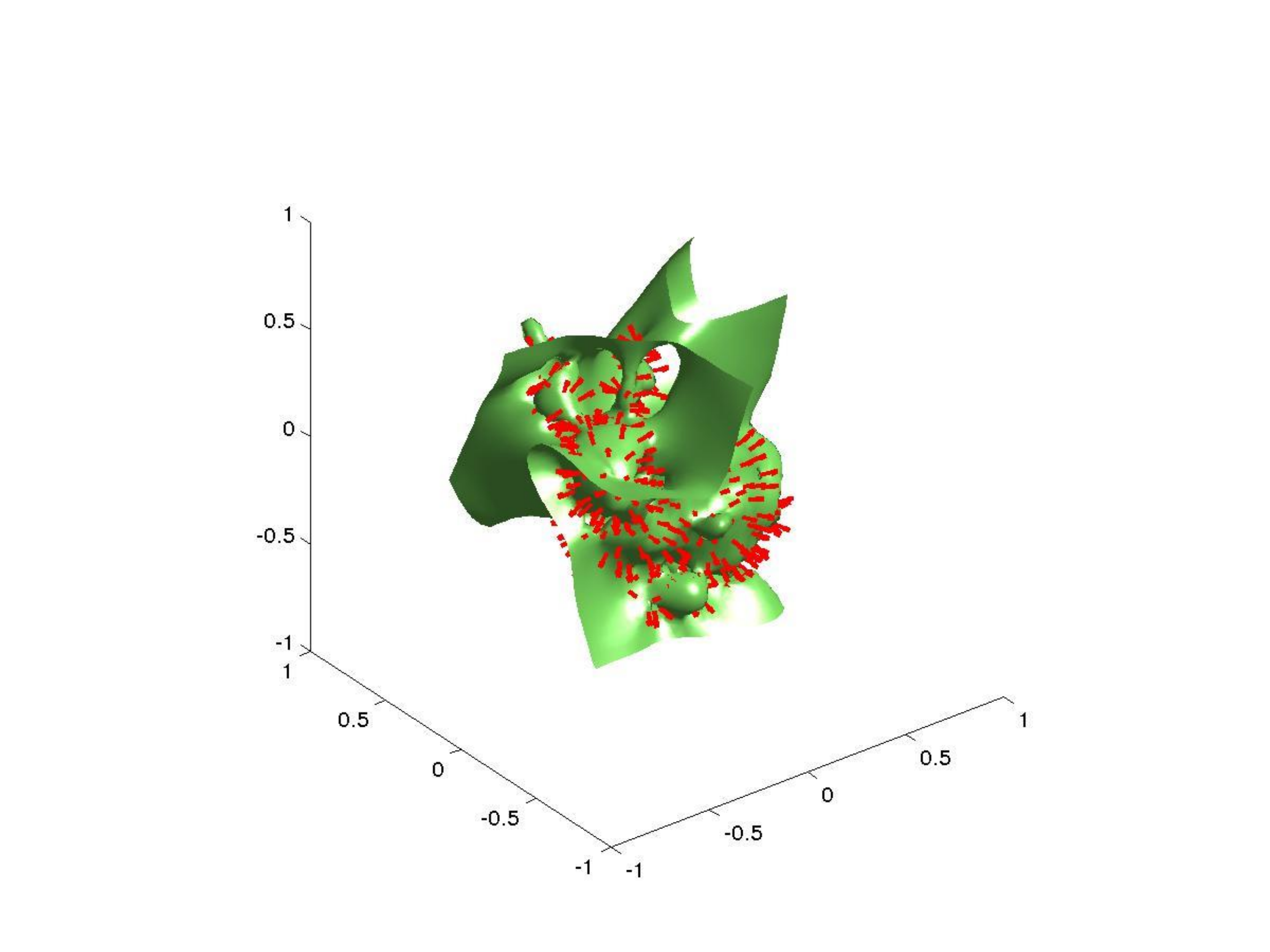
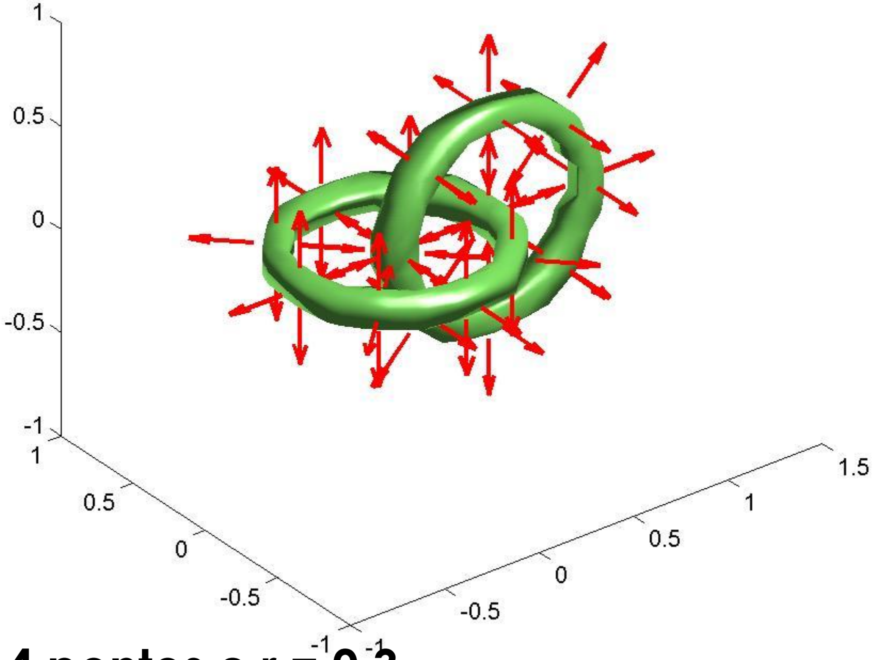
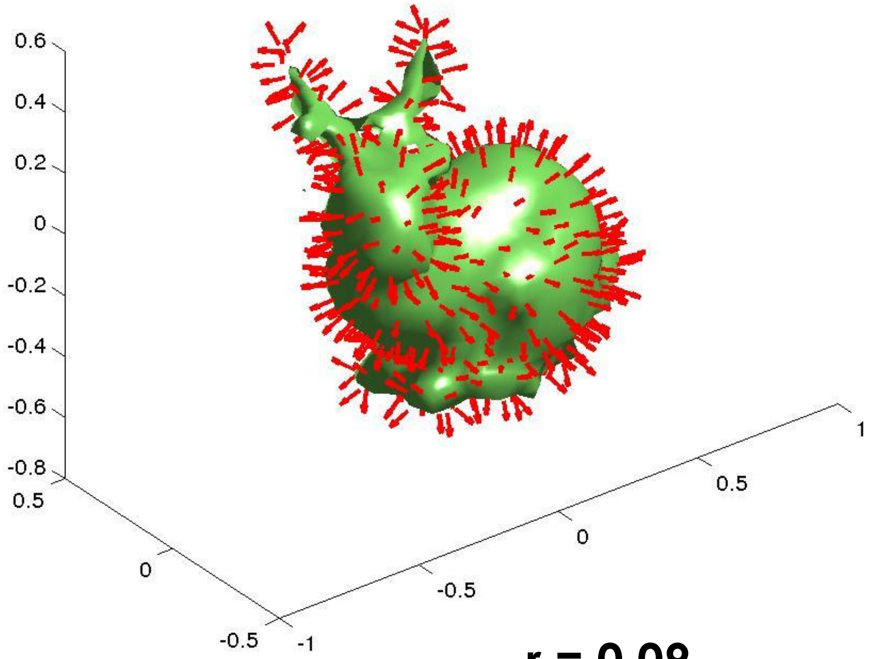
Coelho com Wendland r = 1



Coelho com gaussiana



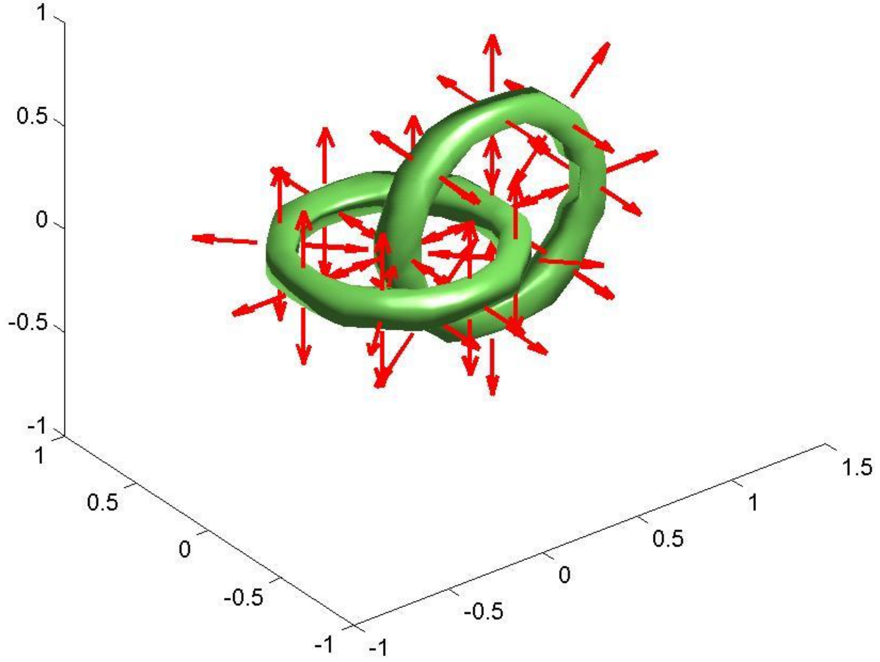
64 pontos e r = 0.3



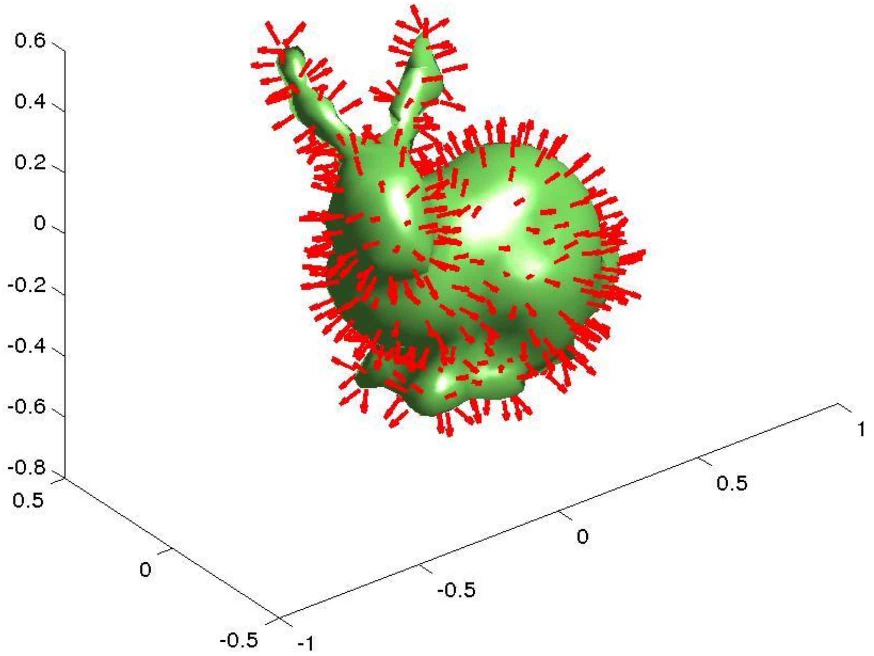
r = 0.09

Multiquadrática

$$\Psi(x) = \sqrt{1 + \left(\frac{x}{r}\right)^2}$$



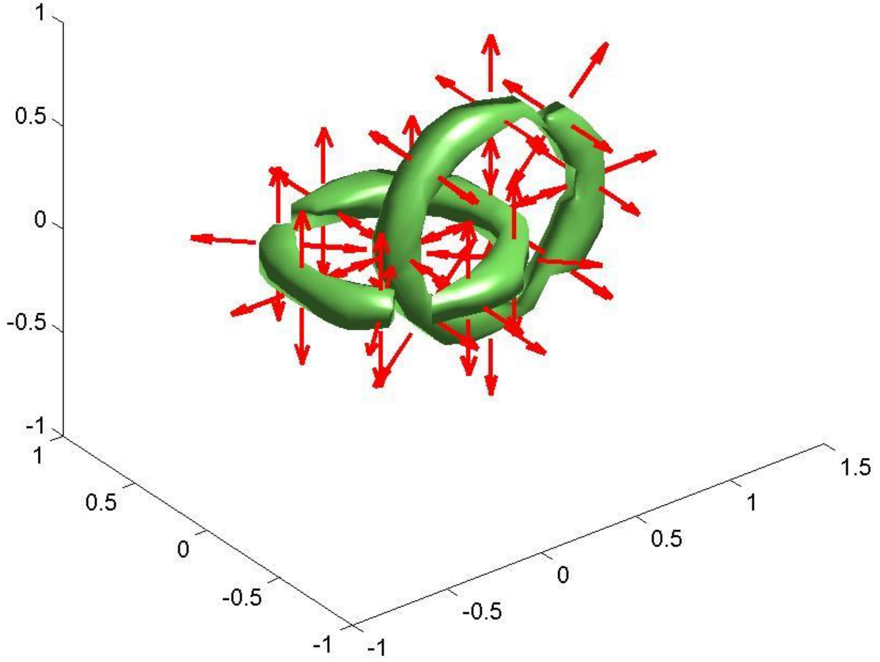
64 pontos e r = 0.5



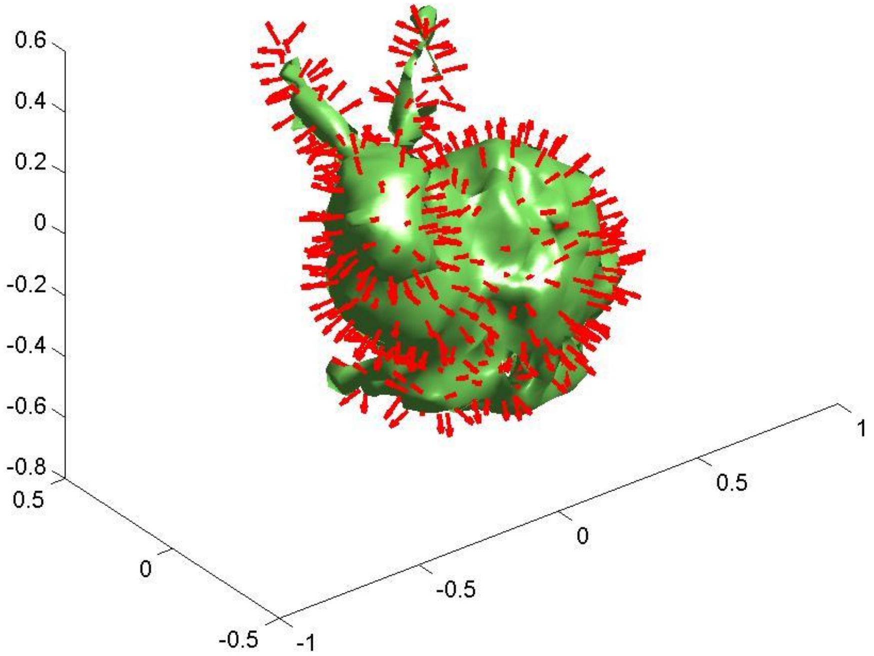
r = 0.002

Multiquadrática inversa

$$\Psi(x) = \frac{1}{\sqrt{1 + \left(\frac{x}{r}\right)^2}}$$



64 pontos e r = 0.7

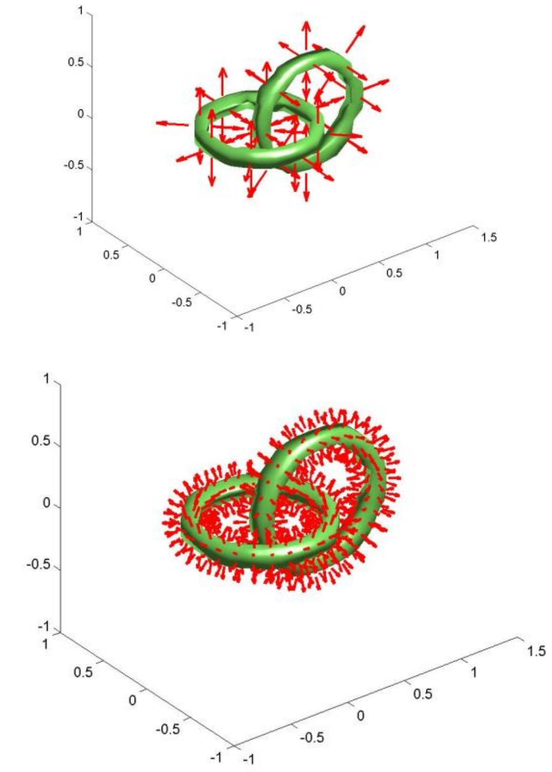


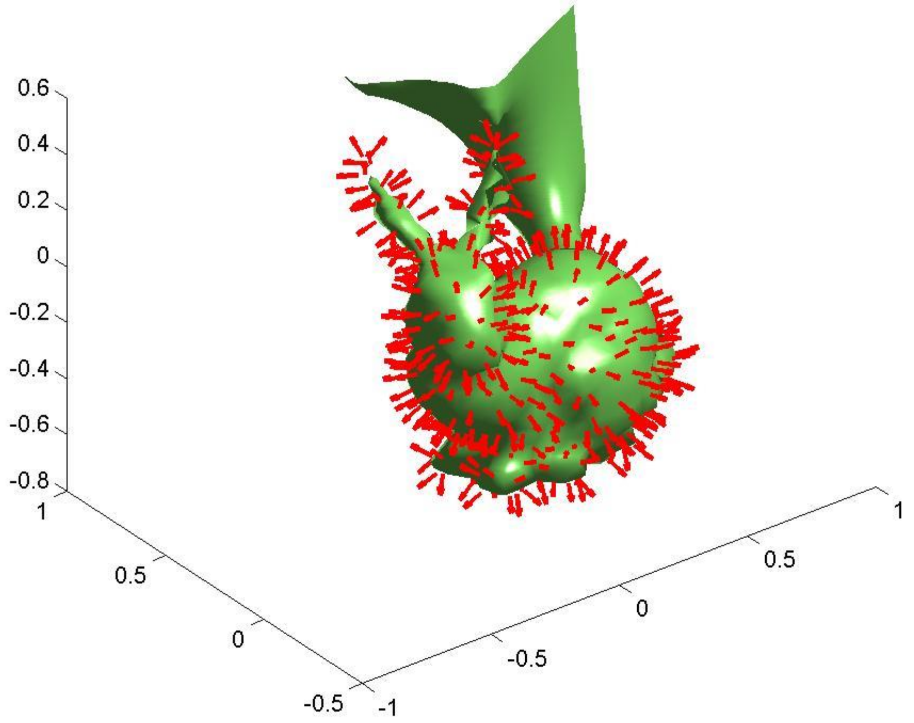
r = 1.1

Spline poli-harmônica
$\boldsymbol{\Psi}(x) = x^4 \ln(x)$

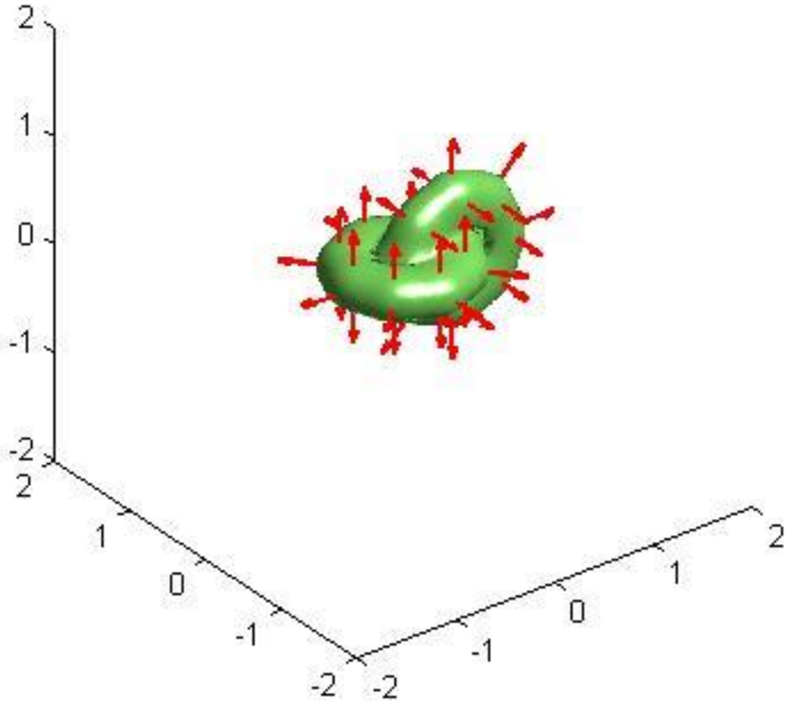
64 pontos

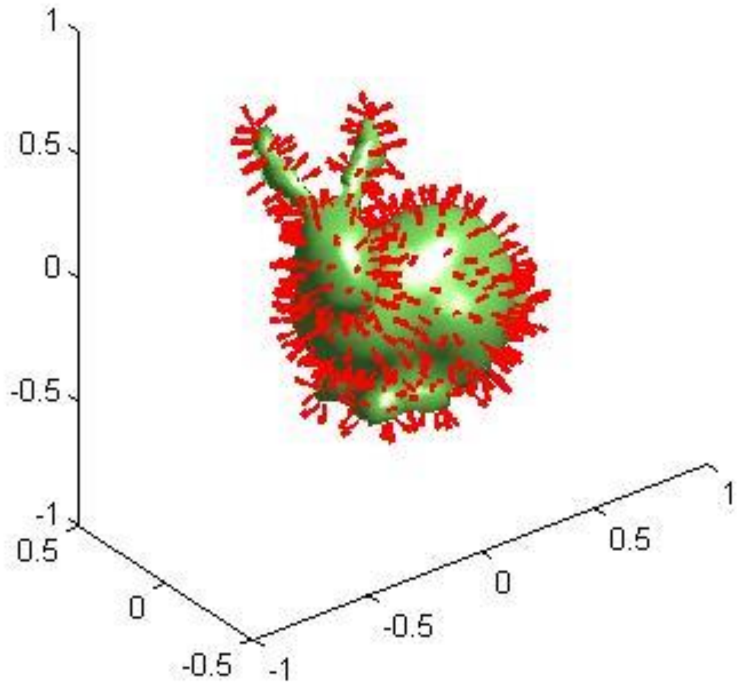
600 pontos



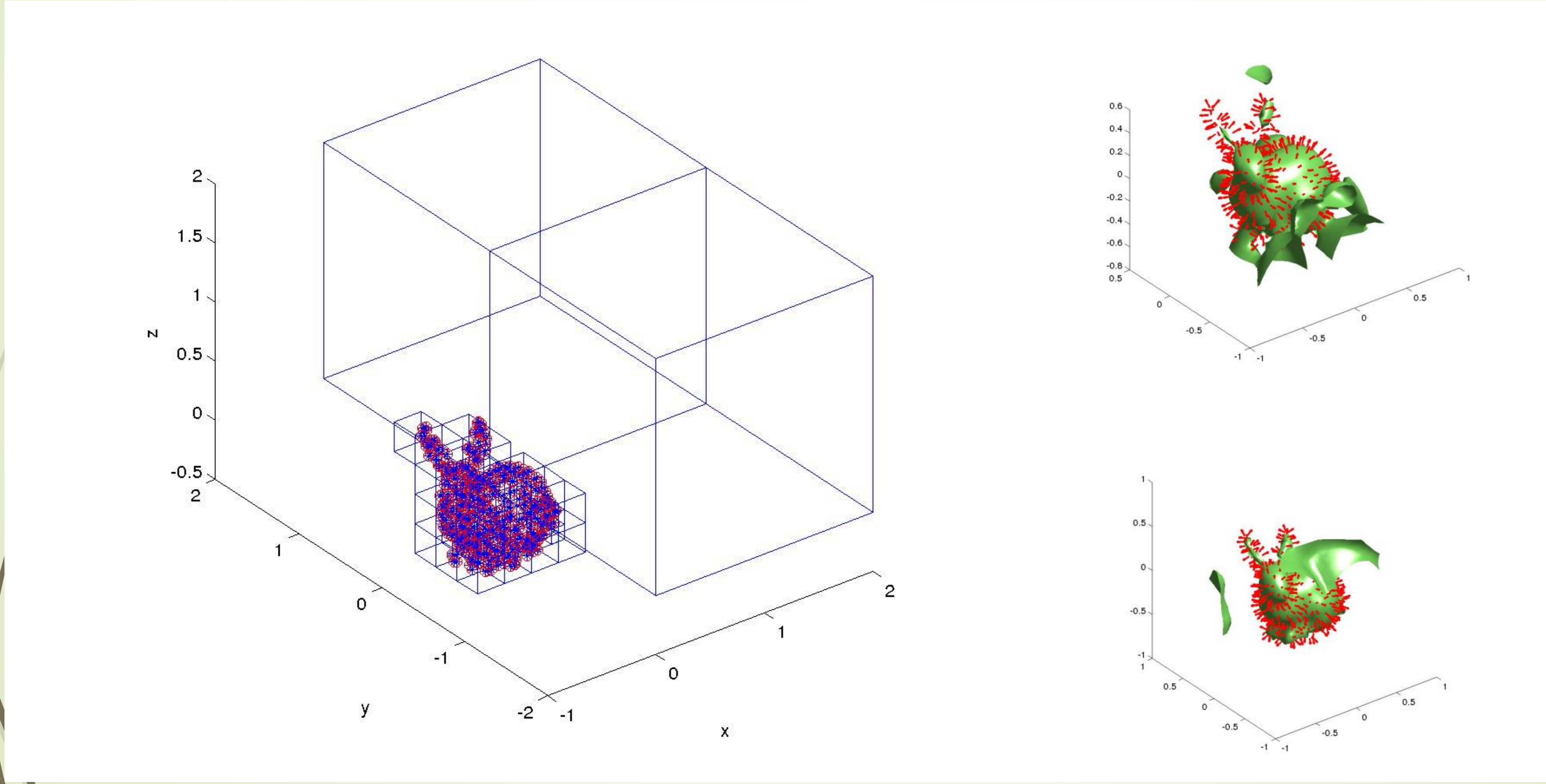


RBF (shift = 0.001)





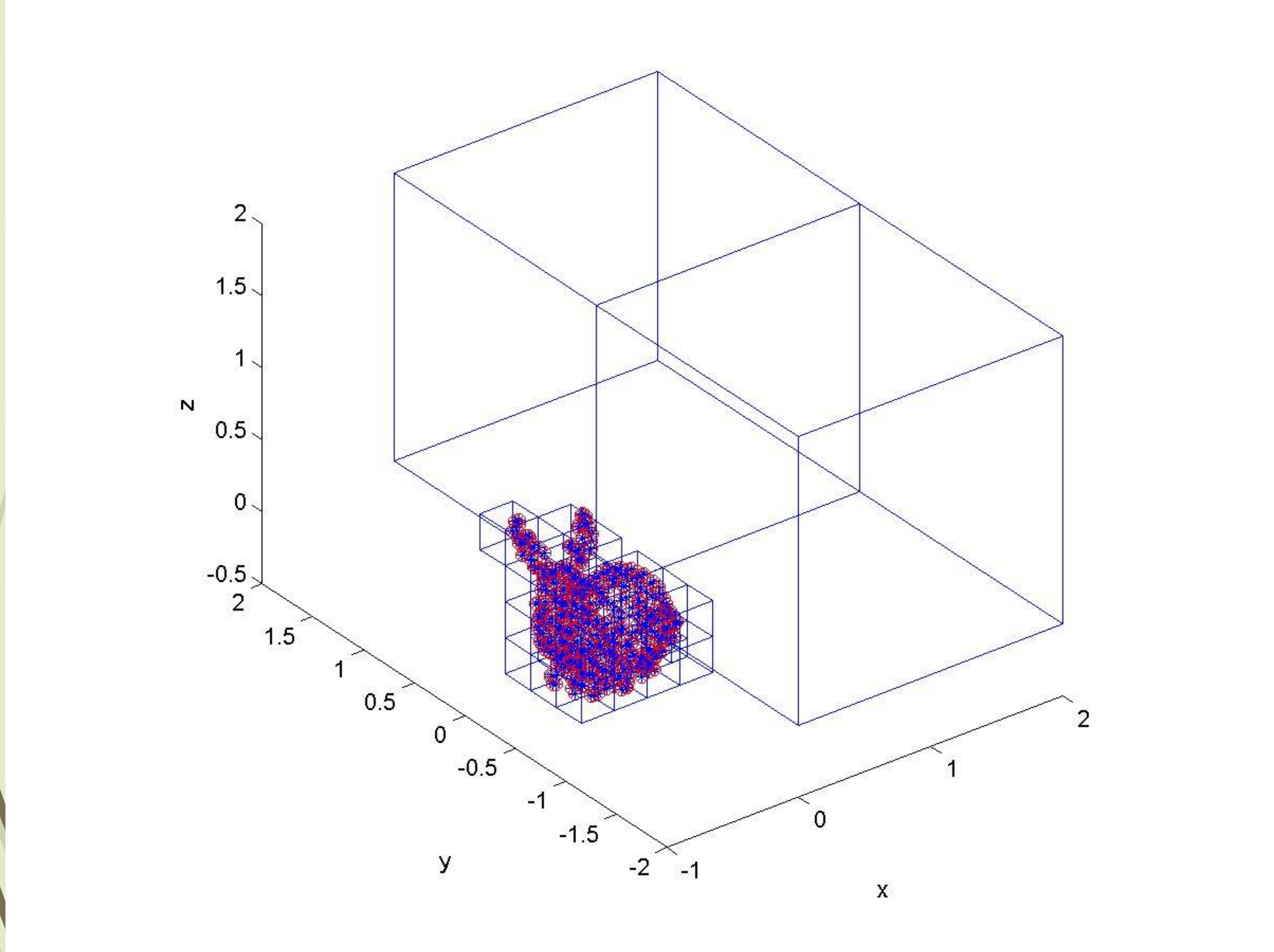
Divisão em caixa para multiquadrática 20 pontos per caixa, r = 0.002

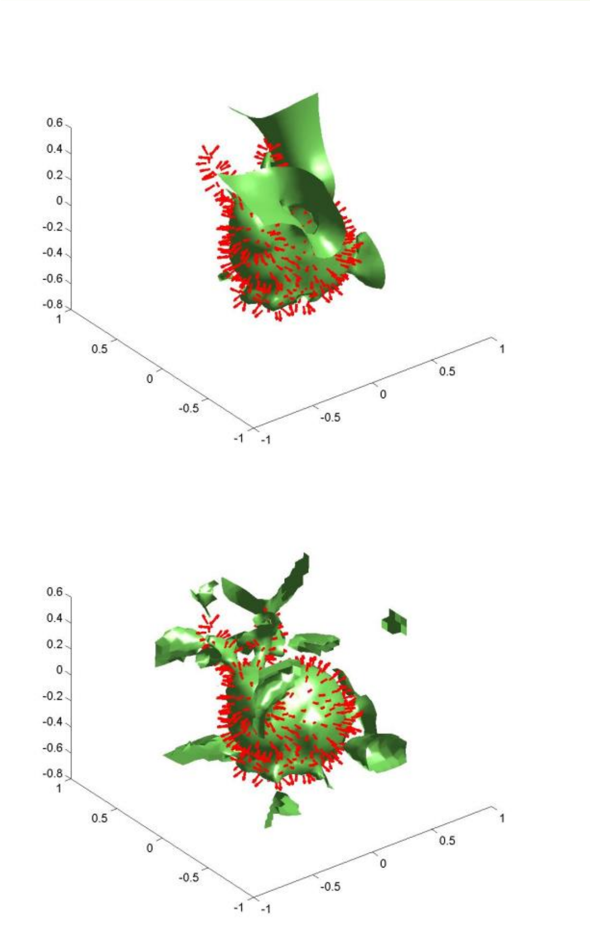


Com função polinomial

Sem função polinomial

Divisão em caixa para gaussiana
20 pontos por caixa, r = 0.1





Com função polinomial

Sem função polinomial

Conclusão

- HRBF melhor que RBF em alguma caso mas sensibilidade ao r muito grande
- Wendland permite bom resultados mas a escolha do r é difícil
- Multiquadrática tem os melhor resultados e é robusto com o r
- Divisão em caixa com polinômio dá resultados com artefacts

Referencia

- Hermite Radial Basis Functions Implicits de I. Macêdo1 , J. P. Gois2 and L. Velho1
- Notas de aula