Computational Geometry: Delaunay Triangulations and Voronoi Diagrams

Lecture 4

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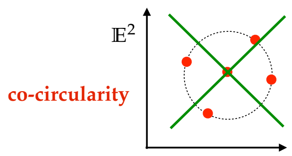
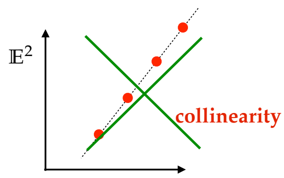
Basic Properties

Unless we explicitly state otherwise, the set P in the following results has at least 3 points, not all of which are collinear, and no four of which can lie in the same circumference.





Basic Properties

Lemma 4.1. Let $\mathcal{T}(P)$ be a triangulation of a point set P in $\mathbb{E}^2$. Let n be the number of points in P, and let n_b be the number of points in P that lie on the boundary of $\text{conv}(P)$. Then,

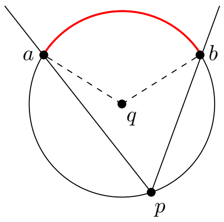
- $\mathcal{T}(P)$ has exactly $3n - n_b - 3$ edges, and
- $\mathcal{T}(P)$ has exactly $2n - n_b - 2$ triangles.

(proof on the board)

Basic Properties

Facts from Elementary Geometry

Theorem 4.2. (Euclid's Elements, Book 3) The measure of an inscribed angle is one-half of that of its intercepted arc.

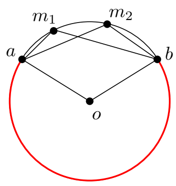


$$\angle apb = \frac{\angle aqb}{2}$$

Basic Properties

Facts from Elementary Geometry

Theorem 4.3. (Apollonius Theorem) Let a and b be two distinct points on a circle. Let m_1 and m_2 be points on the circle lying on the same arc between a and b. Then, we have



$$\angle am_1b = \angle am_2b\,.$$

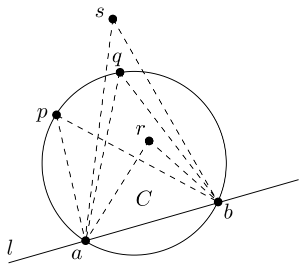
Basic Properties

Corollary 4.4. Let C be a circle, l be a line intersecting C in points a and b, and p, q, r, and s points lying on the same side of l. Suppose that p and q lie on (the boundary of) C, that r lies inside C, and that s lies outside C. Then, we have that

$$\angle arb > \angle apb = \angle aqb > \angle asb\,.$$

Basic Properties

Corollary 4.4.



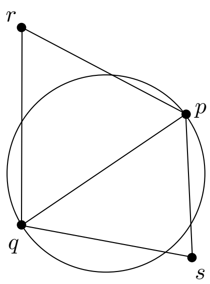
(*proof on the board*)

Basic Properties

Lemma 4.5. A triangulation $\mathcal{T}(P)$ of a nonempty and finite point set, $P \subset \mathbb{E}^2$, is the (unique) Delaunay triangulation, $\mathcal{DT}(P)$, of P if and only if for every interior edge of $\mathcal{T}(P)$, the sum of its opposite angles is strictly less than 180 degrees.

Basic Properties

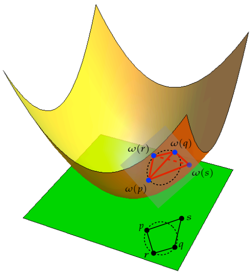
Lemma 4.5.



(*proof on the board*)

Basic Properties

Lemma 4.5.

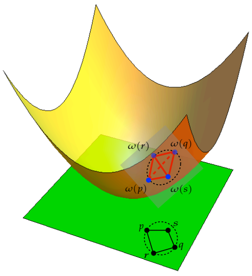


$\angle prq + \angle psq < \pi$

(proof on the board)

Basic Properties

Lemma 4.5.

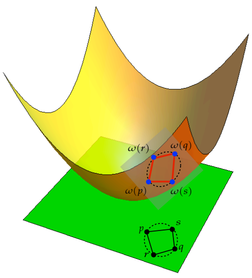


$\angle prq + \angle psq > \pi$

(proof on the board)

Basic Properties

Lemma 4.5.



$\angle prq + \angle psq = \pi$

(proof on the board)

Basic Properties

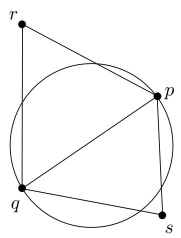
An edge $\overline{pq}$ in a triangulation of P is called *locally Delaunay* if and only if the sum of its opposite angles is less than 180 degrees.

So, a triangulation of P is the Delaunay triangulation, $\mathcal{DT}(P)$, of P if and only if all its edges are locally Delaunay.

How can we decide if an edge is locally Delaunay?

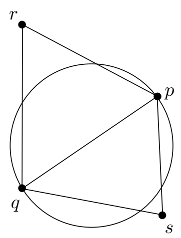
Basic Properties

We can find the angles, $\angle qrp$ and $\angle qsp$, opposite to the edge $\overline{pq}$, and then compute their sum. If the result is less than 180 degrees, the edge is locally Delaunay. Otherwise, it is not.



Basic Properties

However, for a reason that will become clear soon, we decide whether edge $\overline{pq}$ is locally Delaunay using a different test.



Basic Properties

The InCircle Predicate

Assume that the canonical basis, (e_1, e_2), of $\mathbb{E}^2$ has positive orientation. Then, the sequence, p, r, q, of points in $\mathbb{E}^2$ has *positive orientation* if and only if the determinant below is positive:

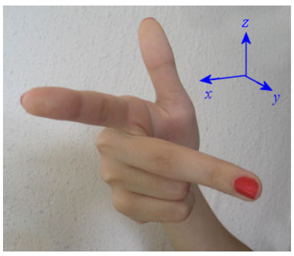
$$\lambda_{\mathbb{R}^2}(\boldsymbol{pr}, \boldsymbol{pq}) = \begin{vmatrix} x_r & y_r & 1 \\ x_q & y_q & 1 \\ x_p & y_p & 1 \end{vmatrix}.$$

Basic Properties

Similarly, if p, r, q, and s is a sequence of points in $\mathbb{E}^3$, then this sequence has *positive orientation* if and only if we have that

$$\lambda_{\mathbb{R}^3}(\boldsymbol{pr}, \boldsymbol{pq}, \boldsymbol{ps}) = \begin{vmatrix} x_r & y_r & z_r & 1 \\ x_q & y_q & z_q & 1 \\ x_s & y_s & z_s & 1 \\ x_p & y_p & z_p & 1 \end{vmatrix} > 0\,.$$

Basic Properties

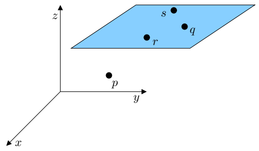


http://www.intmath.com/vectors/6-3-dimensional-space.php

Basic Properties

Whenever $\lambda_{\mathbb{R}^3}(\boldsymbol{pr}, \boldsymbol{pq}, \boldsymbol{ps})$ is positive, we say that the sequence, r, q, and s, is *oriented clockwise with respect to point p*.

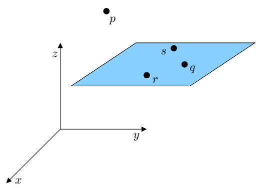
Equivalently, the plane through r, q, and s and whose normal is defined by $\boldsymbol{rq} \times \boldsymbol{rs}$ is seen from "below" by the point p:



Basic Properties

Whenever $\lambda_{\mathbb{R}^3}(\boldsymbol{pr}, \boldsymbol{pq}, \boldsymbol{ps})$ is negative, we say that the sequence, r, q, and s, is *oriented counterclockwise with respect to p*.

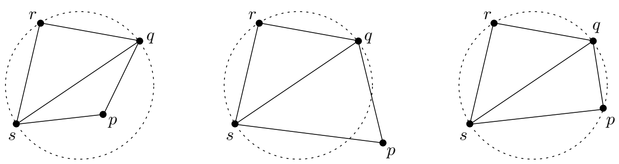
Equivalently, the plane through r, q, and s and whose normal is defined by $\boldsymbol{rq} \times \boldsymbol{rs}$ is seen from "above" by the point p:



Basic Properties

So, the INCIRCLE() predicate is as follows:

Given p, q, r, and s in $\mathbb{E}^2$, our goal is to find out whether the point p is inside the circumference defined by q, r, and s.



Basic Properties

For that, we first compute $\omega(p)$, $\omega(q)$, $\omega(r)$, and $\omega(s)$ in $\mathbb{E}^3$.

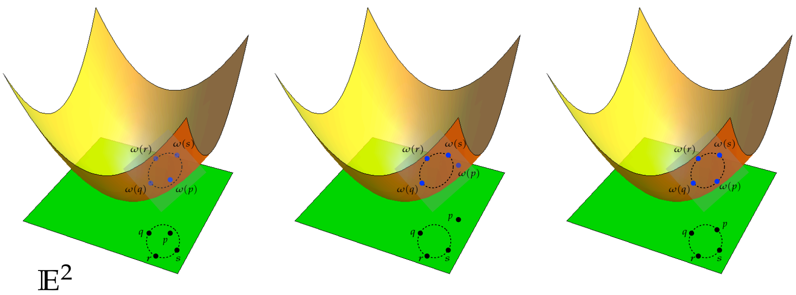
Next, we compute

$$d_1 = \lambda_{\mathbb{R}^2}(\boldsymbol{qr}, \boldsymbol{qs}) \quad \text{and} \quad d_2 = \lambda_{\mathbb{R}^3}(\boldsymbol{\omega(p)\omega(q)}, \boldsymbol{\omega(p)\omega(r)}, \boldsymbol{\omega(p)\omega(s)}).$$

Finally, p lies inside the circumference that passes through q, r, and s if and only if $d_1 \cdot d_2 > 0$. *Can you see why this works?*

Basic Properties

$\mathbb{E}^3$



$\mathbb{E}^2$

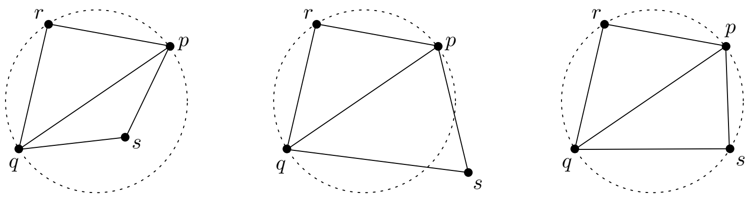
$d_1 > 0$	$d_1 > 0$	$d_1 > 0$
$d_2 > 0$	$d_2 < 0$	$d_2 = 0$

Basic Properties

Lemma 4.6. Let edge $\overline{pq}$ be incident to triangles $\triangle prq$ and $\triangle psq$, and let C be the circumference through p, r, and q. The edge $\overline{pq}$ is not locally Delaunay if and only if the point s lies in the interior or on C. Moreover, if the points p, q, r, and s form a convex quadrilateral and do not lie on a common circumference, then either $\overline{pq}$ or $\overline{rs}$ is locally Delaunay.

Basic Properties

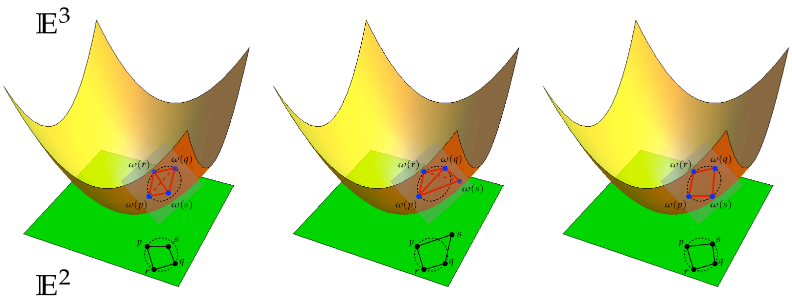
Lemma 4.6.



(*proof on the board*)

Basic Properties

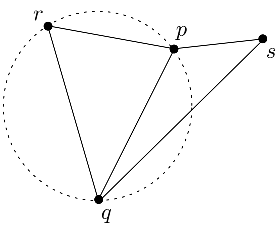
Lemma 4.6.



(*proof on the board*)

Basic Properties

Lemma 4.6.

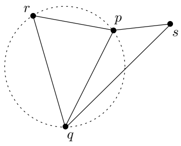


(*proof on the board*)

Basic Properties

Lemma 4.6 suggests us an algorithm for computing the Delaunay triangulation, $\mathcal{DT}(P)$, from *some* triangulation of P.

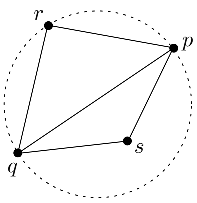
Let $\overline{pq}$ be an interior edge of $\mathcal{T}(P)$ that is incident to triangles $\triangle prq$ and $\triangle psq$. From the proof of Lemma 4.6, we know that $\square prqs$ is convex whenever $\overline{pq}$ is not locally Delaunay.



Basic Properties

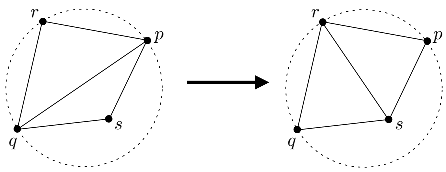
Suppose that $\overline{pq}$ is not locally Delaunay.

So, $\square prqs$ is convex. Since (*by hypothesis*) no four points of P are co-circular, Lemma 4.6 tells us that $\overline{rs}$ is locally Delaunay.



Basic Properties

The convexity of $\square prqs$ allows us to "flip" the edge $\overline{pq}$ of $\mathcal{T}(P)$ to replace it with edge $\overline{rs}$, which is locally Delaunay.



So, we can go on testing each edge of $\mathcal{T}(P)$ and flipping those that are not locally Delaunay *until all edges pass the test*.

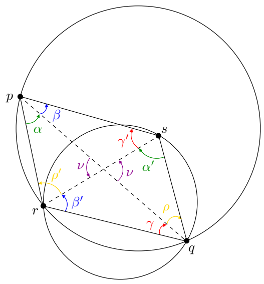
Basic Properties

Algorithm SwapEdges($\mathcal{T}(P)$)

while $\mathcal{T}$ contains a locally non-Delaunay edge $\overline{pq}$ **do**
    let $\triangle prq$ and $\triangle psq$ be the triangles incident to $\overline{pq}$
    remove $\triangle prq$ and $\triangle psq$ from $\mathcal{T}(P)$
    add $\triangle rps$ and $\triangle rqs$ to $\mathcal{T}(P)$
endwhile
return $\mathcal{T}(P)$

Basic Properties

Lemma 4.7. SwapEdges() always terminates and yields $\mathcal{DT}(P)$.

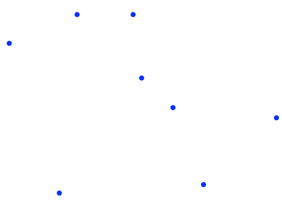


(*proof on the board*)

Basic Properties

How can we obtain "a" triangulation, $\mathcal{T}(P)$, at first place?

Have you heard of *Graham scan*?



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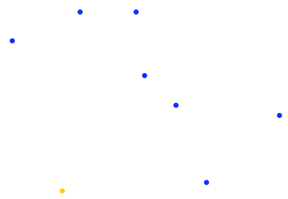

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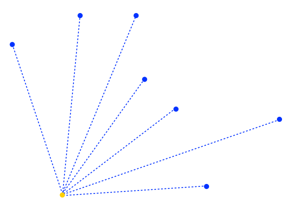

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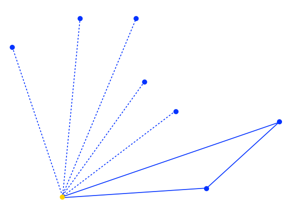

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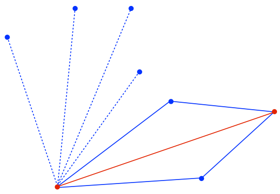

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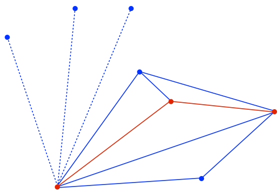

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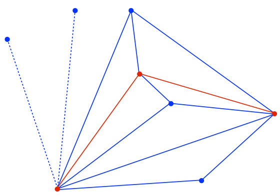

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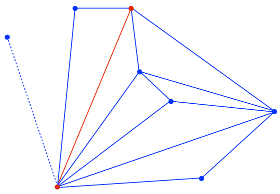

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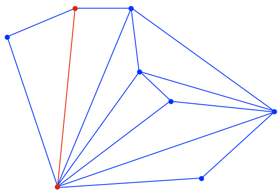

Basic Properties

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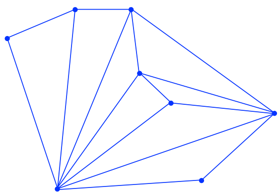
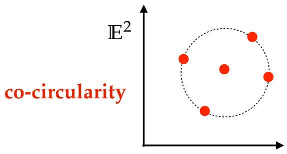


Basic Properties

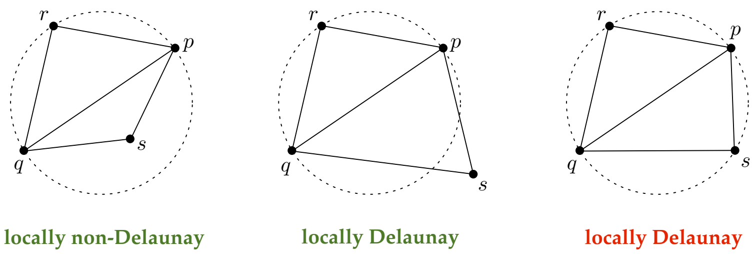
What if four points of P lie in the same circumference?



The algorithm still works if we change the following definition:

Basic Properties

An edge $\overline{pq}$ in a triangulation of P is called *locally Delaunay* if and only if the sum of its opposite angles is *at most* 180 degrees.



Basic Properties

So, when p, q, r, and s lie in the same circumference, SwapEdges() will not flip the edge linking two of them in $\mathcal{T}(P)$.

The resulting triangulation will contain locally Delaunay edges only (according to the new definition of Delaunay edge).

Recall that the resulting triangulation is not unique. It is not the Delaunay triangulation either (as one doesn't exist).

Basic Properties

The algorithm we just saw is *too slow* to be of any practical use.

But, the idea of flipping edges is crucial for the "practical" Delaunay triangulation algorithm we will see in the next lecture.

Basic Properties

Theorem 4.8. Every triangulation of a nonempty and finite point set, $P \subset \mathbb{E}^2$, can be made into a Delaunay triangulation by at most $\binom{|P|}{2}$ flips on locally non-Delaunay edges.

(*proof on the board*)

Basic Properties

Can you come up with a point set, $P \subset \mathbb{E}^2$, and some triangulation, $\mathcal{T}(P)$, of P such that $\Omega(|P|^2)$ edge flips are required to make $\mathcal{T}(P)$ into the Delaunay triangulation of P?

Basic Properties

Corollary 4.9. Every triangulation of a nonempty and finite point set, $P \subset \mathbb{E}^2$, can be made into another triangulation of the same point set by a sequence of at most $3 \cdot \binom{|P|}{2}$ flips.

(*proof on the board*)

Basic Properties

Why is the Delaunay triangulation so "special"?

Theorem 4.10. Delaunay triangulations in $\mathbb{E}^2$ are optimal planar triangulations with respect to the following assertions (which hold for any choice of a *fixed* point set, P, in $\mathbb{E}^2$):

1. They minimize the maximum circumradius of triangles in the triangulation.
2. For all real powers, $m > 0$, they minimize the functional $\sum_{\sigma \in \mathcal{T}(P)} R(\sigma)^m$, where $R(\sigma)$ is the circumradius of σ, and the sum is over all triangles of the triangulation.
3. They maximize the minimum angle in a triangulation.

Basic Properties

To prove Theorem 4.10, we need another result.

However, before we state this result, let us discuss why the assertions in Theorem 4.10 are important. Do you have an idea?

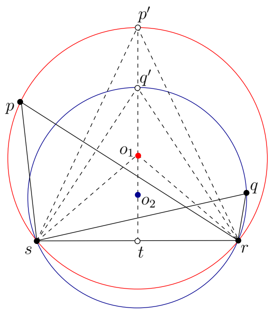
Basic Properties

Lemma 4.11. Given a convex quadrilateral, $\square pqrs$, as in the figure below, with vertex p exterior to (or lying on) the circumference, $C(q, r, s)$, through q, r, and s, then the following holds:

$$\max\{R(\triangle prs), R(\triangle pqr)\} \geq \max\{R(\triangle qrs), R(\triangle pqs)\} .$$

Basic Properties

Lemma 4.11.



(*proof on the board*)

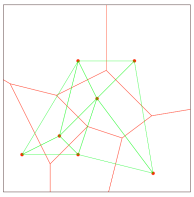
Basic Properties

Recall Lemma 3.3 (from the previous lecture):

Lemma 3.3. Let $Q \subset P$ be any subset of 3 affinely independent points of P. Then, Q^{ω} corresponds to the vertex set of a lower facet of the polytope, $\mathcal{P}$, if and only if the circumference defined by the points in Q contains no point of $P - Q$.

Basic Properties

Lemma 4.12. The dual graph of the Voronoi diagram, $\mathrm{Vor}(P)$, of a nonempty and finite set, $P \subset \mathbb{E}^2$, consists precisely of the edges and vertices of the Delaunay subdivision of P.



(*proof on the board*)