Computational Geometry: Delaunay Triangulations and Voronoi Diagrams

Lecture 3

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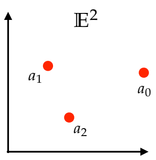
Triangulations

Given a *finite* family, $(a_i)_{i \in I}$, of points in $\mathbb{E}^n$, we say that $(a_i)_{i \in I}$ is *affinely independent* if and only if the family of vectors, $(\boldsymbol{a_i a_j})_{j \in (I - \{i\})}$, is linearly independent for some $i \in I$.



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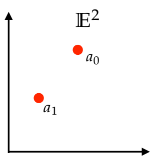

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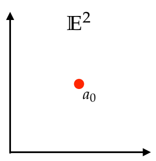

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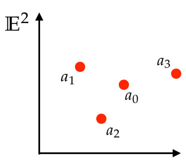

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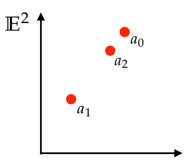
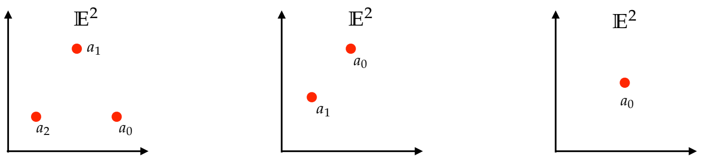


Not A. I.

Triangulations

In $\mathbb{E}^n$, the largest number of affinely independent points is $n+1$.

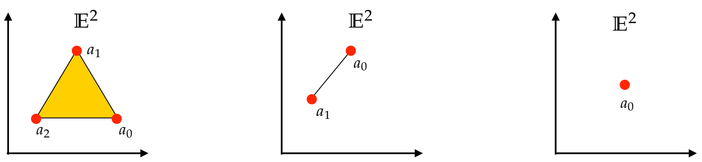
Let $a_0, \ldots, a_d$ be any $d+1$ affinely independent points in $\mathbb{E}^n$.



Triangulations

The *simplex* σ spanned by the points $a_0, \ldots, a_d$ is the convex hull, $\mathrm{conv}(\{a_0, \ldots, a_d\})$, of these points, and is denoted by $[a_0, \ldots, a_d]$.

The points $a_0, \ldots, a_d$ are the *vertices* of σ.



Triangulations

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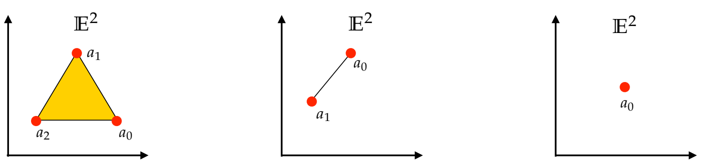
The *dimension*, $\dim(\sigma)$, of σ is d, and σ is called a *d-simplex*.



Triangulations

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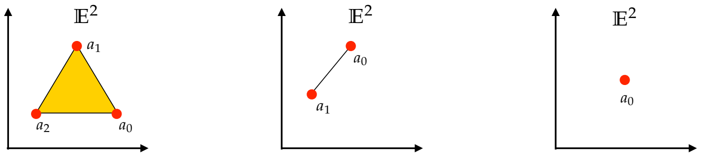
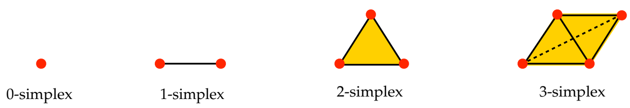
In $\mathbb{E}^n$, we have simplices of dimension $0, 1, \ldots, n$ only.



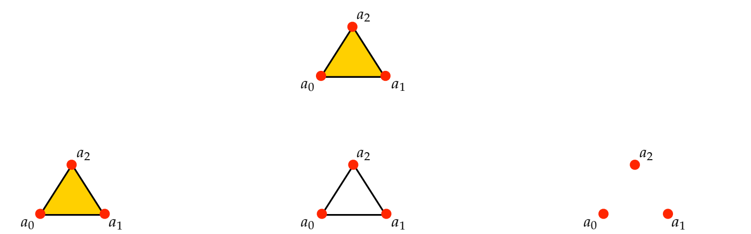
Triangulations

A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron, and so on.



Triangulations

The convex hull of any *nonempty* (proper) subset of vertices of a simplex σ is also a simplex, called a *(proper) face* of σ.



a 2-face: $[a_0, a_1, a_2]$ 3 proper 1-faces: $[a_0, a_1]$, $[a_0, a_2]$, $[a_1, a_2]$ 3 proper 0-faces: $[a_0]$, $[a_1]$, $[a_2]$

Triangulations

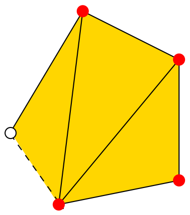
A *simplicial complex*, $\mathcal{K}$, in $\mathbb{E}^n$ is a finite set of simplices in $\mathbb{E}^n$ such that

(1) if $\sigma \in \mathcal{K}$ and $\tau \preceq \sigma$ then $\tau \in \mathcal{K}$, and

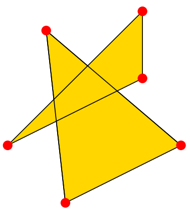
(2) if $\sigma \cap \tau \neq \emptyset$ then $\sigma \cap \tau \preceq \sigma, \tau$, for all $\sigma, \tau \in \mathcal{K}$,

where $a \preceq b$ denotes "a is a (not necessarily proper) face of b".

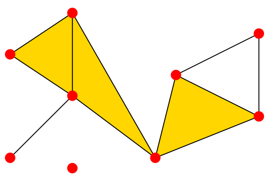
Triangulations



violates (1)



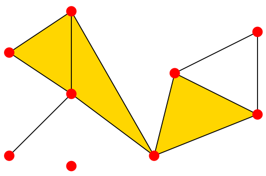
violates (2)



A simplicial complex

Triangulations

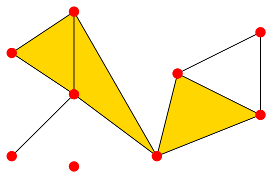
The *dimension*, $\dim(\mathcal{K})$, of a simplicial complex $\mathcal{K}$ is the largest dimension of a simplex in $\mathcal{K}$. We refer to a d-dimensional simplicial complex as simply a *d-(simplicial) complex*.



a 2-complex

Triangulations

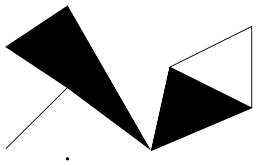
Note that a simplicial complex is a *discrete* object (i.e., a finite collection of simplices). In turn, each simplex is a set of points.



a 2-complex

Triangulations

The (point) set consisting of the union of all points in the simplices of a simplicial complex, $\mathcal{K}$, is called the *underlying space* of $\mathcal{K}$ and denoted by $|\mathcal{K}|$. Note that $\mathcal{K}$ is a *continuous* object.

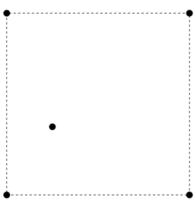


its underlying space

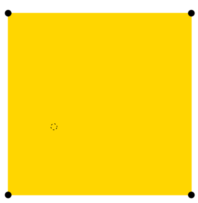
Triangulations

A *triangulation of a nonempty and finite set, P, of points of $\mathbb{E}^n$*, is a simplicial complex, $\mathcal{T}(P)$, such that all vertices of $\mathcal{T}(P)$ are in P and the union of all simplices of $\mathcal{T}(P)$ equals $\text{conv}(P)$.

$\mathbb{E}^2$



5 points in $\mathbb{E}^2$



convex hull of the 5 points

Triangulations

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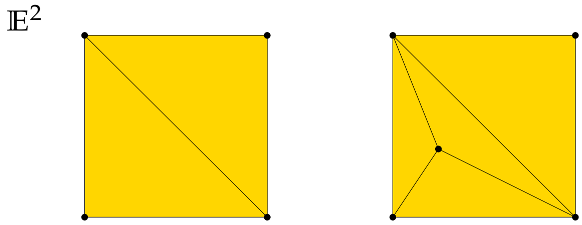
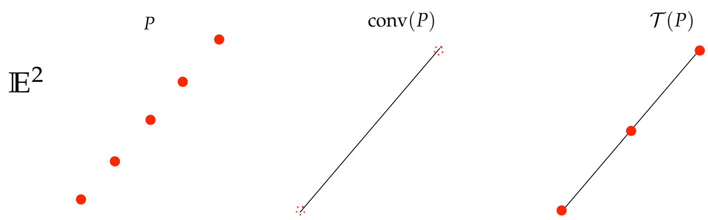


two triangulations of the same point set

Triangulations

In our definition we do assume that the affine hull of P has dimension n. So, a triangulation of P may contain no simplex of dimension n, such as the example below for $n = 2$:



Triangulations

Note that not all points of P need to be vertices of $\mathcal{T}(P)$, except for the extremes points of $\mathrm{conv}(P)$, which are always in $\mathcal{T}(P)$.

Whenever all points in P are vertices of $\mathcal{T}(P)$, we call $\mathcal{T}(P)$ a *full-triangulation*. This is the type of triangulation we will study.

From now on, we drop the word "full" and refer to the term "triangulation" as a full-triangulation of the given point set.

Triangulations

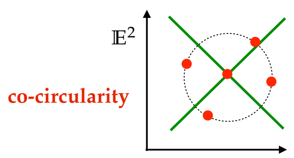
Theorem 3.1. Every nonempty and finite set, $P \subset \mathbb{E}^2$, admits a triangulation, which partitions the convex hull of P.

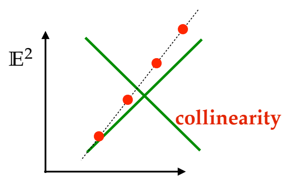
(*proof discussed in the end of the lecture*)

The Delaunay Triangulation

Let P be a nonempty and finite set of points in $\mathbb{E}^2$.

For the time being, let us assume that (1) not all points of P are collinear, and (2) no four points of P lie in the circumference defined by 3 of them. Observe that "(1) $\Rightarrow$ $|P| \geq 3$".





The Delaunay Triangulation

The Lifting Procedure

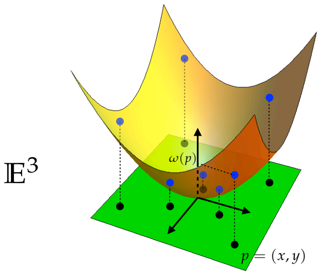
Let $\omega : \mathbb{E}^2 \to \mathbb{R}$ be the function defined as

$$\omega(p) = x^2 + y^2 \, ,$$

for every $p = (x, y) \in \mathbb{E}^2$.

The Delaunay Triangulation

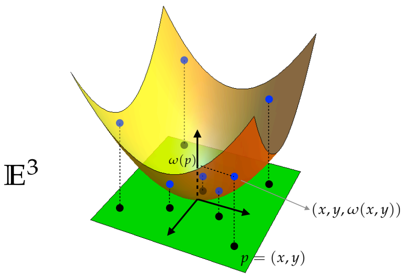
Note that ω can be seen as a *height* function that lifts the point $p = (x, y)$ to the paraboloid of equation $z = x^2 + y^2$ in $\mathbb{E}^3$.



The Delaunay Triangulation

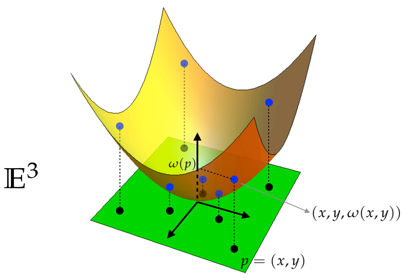
Let

$$P^\omega = \{(x, y, \omega(x, y)) \in \mathbb{E}^3 \mid (x, y) \in P\}.$$



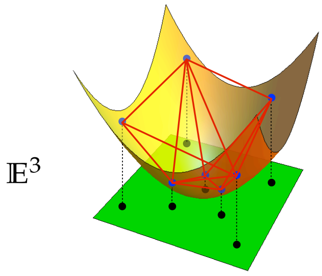
The Delaunay Triangulation

Note that P is the orthogonal projection of P^ω onto the xy-plane.



The Delaunay Triangulation

Consider the convex hull, $\mathrm{conv}(P^\omega)$, of P^ω. Denote it by $\mathcal{P}$.



The Delaunay Triangulation

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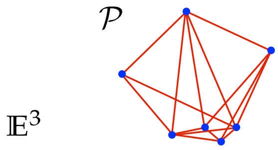
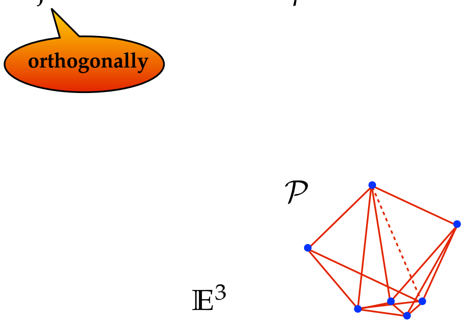


The Delaunay Triangulation

Project the *lower envelope* of $\mathcal{P}$ onto the xy-plane.



The Delaunay Triangulation

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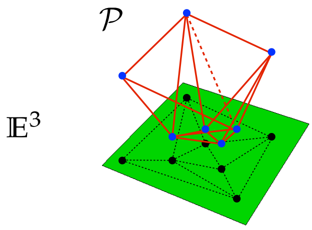
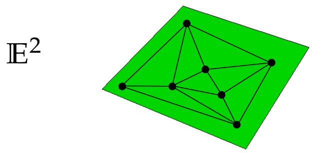


The Delaunay Triangulation

If the result of the projection of the lower envelope of $\mathcal{P}$ is a triangulation, then we call it the *Delaunay triangulation* of P.

We denote the Delaunay triangulation of P by $\mathcal{DT}(P)$.

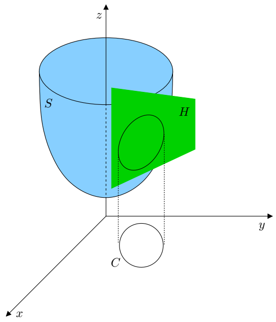


The Delaunay Triangulation

Proposition 3.2. Let $S \subset \mathbb{E}^3$ be the paraboloid given by the equation $z = x^2 + y^2$, and let $H \subset \mathbb{E}^3$ be a non-vertical hyperplane, i.e., one whose normal vector has non-zero last coordinate. Let C be the projection of $H \cap S$ into $\mathbb{E}^2$ obtained by dropping the last coordinate of all points in $H \cap S$. Then, C is either empty, a single point, or a circumference.

The Delaunay Triangulation

Proposition 3.2.



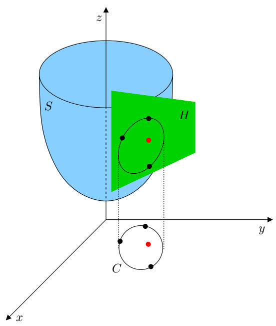
(proof on the board)

The Delaunay Triangulation

Lemma 3.3. Let $Q \subset P$ be any subset of P with 3 affinely independent points. Then, Q^ω corresponds to the vertex set of a lower facet of the polytope, $\mathcal{P}$, if and only if all points in Q lie on a circle and all points of $P - Q$ are outside it.

The Delaunay Triangulation

Lemma 3.3.



(proof on the board)

The Delaunay Triangulation

By hypothesis, no four points of P lie in the same circumference.

So, Lemma 3.3 implies that all facets of the lower envelope of $\mathcal{P}$ are triangles in $\mathbb{E}^3$, and so are their projections onto $\mathbb{E}^2$.

By hypothesis, not all points of P are collinear. This means that the proper faces of the lower envelope of $\mathcal{P}$ are triangles.

The Delaunay Triangulation

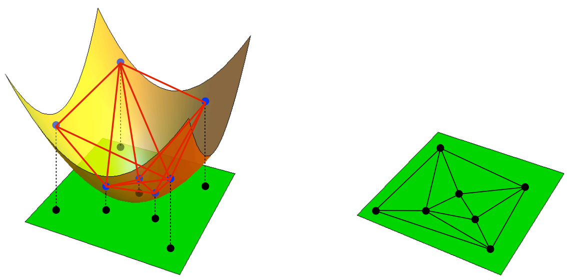
What can we conclude from the previous remarks?

The projection of the lower envelope of $\mathcal{P}$ is a set of triangles.

It turns out that — *with a little bit of an effort* — we can also show that this set of triangles, along with their edges and vertices, is a triangulation of P. This implies that the edges and vertices of the triangles must be in the lower envelope of $\mathcal{P}$.

The Delaunay Triangulation

By definition, the triangulation resulting from the projection of the lower envelope is the Delaunay triangulation, $\mathcal{DT}(P)$.



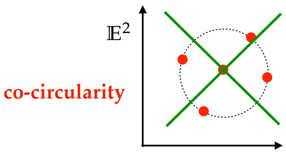
The Delaunay Triangulation

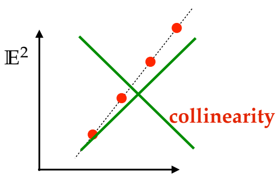
Lemma 3.4. Let P be a nonempty and finite set of points of $\mathbb{E}^2$. If no four points of P lie in the same circumference and not all points of P lie in the same line, then the Delaunay triangulation, $\mathcal{DT}(P)$, of P exists. Furthermore, it is unique.

(proof on the board)

The Delaunay Triangulation

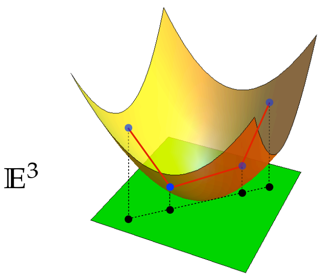
What if one these two assumptions does not hold?





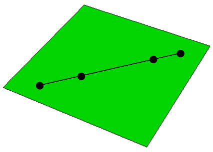
The Delaunay Triangulation

If all points of P are collinear, then $\mathcal{P}$ is 2-dimensional and its lower envelope is a polygonal chain containing all points of P.



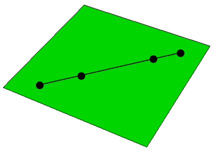
The Delaunay Triangulation

The projection of the lower envelope is also a polygonal chain containing all points of P, which is a triangulation as well.



The Delaunay Triangulation

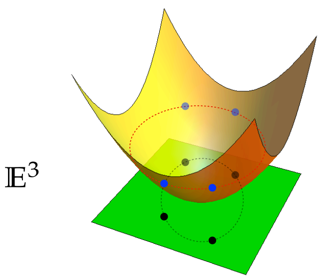
From our definition of Delaunay triangulation, this "degenerate" triangulation is also the Delaunay triangulation of P.



The Delaunay Triangulation

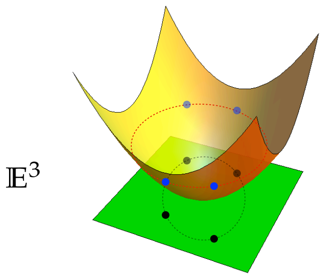
Let Q be a subset of P containing *at least* four points.

Suppose that all points in Q lie in the same circumference, C.



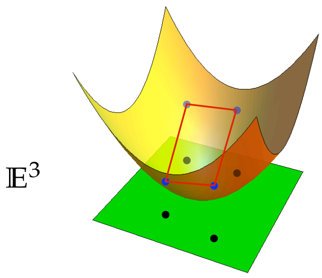
The Delaunay Triangulation

The points in Q cannot be all collinear. Otherwise, they would not lie in the same circumference (assuming "finite" radius).



The Delaunay Triangulation

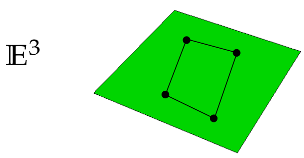
If there is no point of P inside C, then Lemma 3.3 also tells us that Q is exactly the vertex set of a lower envelope facet of $\mathcal{P}$.



The Delaunay Triangulation

The projection of this facet is a convex set with ≥ 4 vertices!

So, the projection of the lower envelope is not a triangulation.



The Delaunay Triangulation

We call the resulting projection the *Delaunay subdivision* of P.

Two-dimensional convex sets that are not triangles can always be triangulated (*it is a well-known result in mathematics*).

So, we can *always* obtain a triangulation of P from the Delaunay subdivision by triangulating those 2D convex sets.

The Delaunay Triangulation

In practice, the resulting triangulation is usually called a Delaunay triangulation. But, conceptually, this is not quite right!

However, keep in mind that there is more than one way of "refining" a Delaunay subdivision to obtain a triangulation.

So, the triangulation (whatever we call it) is not unique!

The Delaunay Triangulation

Let us summarize all facts we have learned so far...

Let P be *any* subset of points of $\mathbb{E}^2$.

If no four points of P lie in the same circumference, then we know that the Delaunay triangulation, $\mathcal{DT}(P)$, of P exists.

Furthermore, this triangulation contains no triangles if the points of P are all collinear. Note that the converse also holds.

The Delaunay Triangulation

If four points of P lie in the same circumference and *this circumference contains no other point of P in its interior*, then $\mathcal{DT}(P)$ does not exist, but the Delaunay subdivision of P does!

From any Delaunay subdivision of P, we can obtain a triangulation of P by triangulating convex sets with ≥ 4 vertices.

The Delaunay Triangulation

However, if for every circumference defined by (at least) four points of P on it, there is always at least one point of P inside it, then the Delaunay triangulation, $\mathcal{DT}(P)$, of P exists.

Why?

Because the lifting of the points on the circumference to the paraboloid doesn't define a facet of the lower envelope of $\mathcal{P}$.

The Delaunay Triangulation

What can we conclude from all these facts?

If P is nonempty and finite set of points of $\mathbb{E}^2$ such that no four points of P define a circumference whose interior is empty of points of P, then $\mathcal{DT}(P)$ always exists and is unique.

Otherwise, we get the Delaunay subdivision, which can always be refined to yield a triangulation of P in a non-unique way.

The Delaunay Triangulation

We have a proof for Theorem 3.1:

Theorem 3.1. Every nonempty and finite set, $P \subset \mathbb{E}^2$, admits a triangulation, which partitions the convex hull of P.

The assertion of Theorem 3.1 also holds in $\mathbb{E}^n$.

We can also show its veracity by first proving the existence of Delaunay triangulations and subdivisions of $P \subset \mathbb{E}^n$, for $n \geq 3$.

The Delaunay Triangulation

We defined the Delaunay triangulation in $\mathbb{E}^2$ only, although our definition of triangulation holds in $\mathbb{E}^n$, for any $n \in \mathbb{N}$.

The lifting procedure can be extended to $\mathbb{E}^n$, for $n = 1$ and $n \geq 3$.

So, our definition of $\mathcal{DT}(P)$ is the same for $P \subset \mathbb{E}^n$, with $n \in \mathbb{N}$.

The Delaunay Triangulation

If all points in $P \subset \mathbb{E}^n$ lie in the same hyperplane in $\mathbb{E}^n$, then $\mathcal{DT}(P)$ has no simplex of dimension n (like we saw for $n = 2$).

If $n + 2$ points in $P \subset \mathbb{E}^n$ lie in the same sphere in $\mathbb{E}^n$ and this sphere contains no point of P in its interior, then the projection of the lower envelope of $\text{conv}(P)$ is not a Delaunay triangulation, but the Delaunay subdivision of P.

The Delaunay Triangulation

However, we can always refine the Delaunay subdivision of P to obtain a Delaunay triangulation of P, *which is not unique*.

We rely on the fact that any n-dimensional polytope, which is the projection of a n-dimensional, lower facet of $\text{conv}(P)$ containing more than $n+1$ vertices, can be triangulated.

The Delaunay Triangulation

Consider Lemma 3.3 again:

Lemma 3.3. Let $Q \subset P$ be any subset of P with 3 affinely independent points. Then, Q^{ω} corresponds to the vertex set of a lower facet of the polytope, $\mathcal{P}$, if and only if all points in Q lie on a circle and all points of $P - Q$ are outside it.

This lemma gives us an algorithm for computing $\mathcal{DT}(P)$.

Can you describe it?