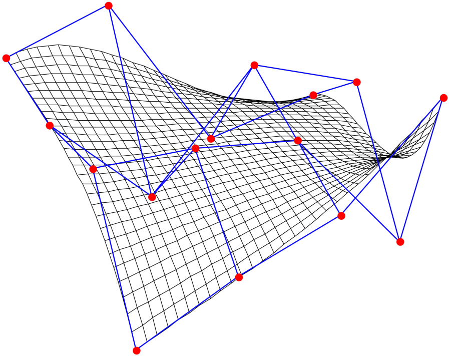


Modelagem Geométrica – SME0271

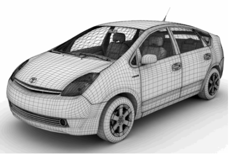
Curvas e Superfícies de Bézier

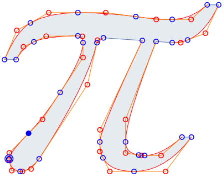
Luiz Otávio Toratti
ICMC-USP

07 de outubro de 2016

Curvas de Bézier

Objetivo: Construir curvas de fácil controle para auxiliar no design e fabricação assistida por computador.



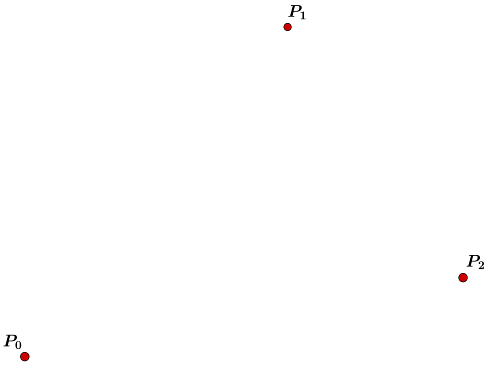


CorelDRAW

AUTOCAD

Bézier Quadrática

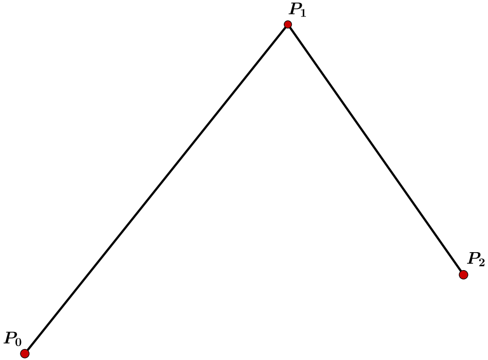
Algoritmo de De Casteljau



Passo 1: Escolha três pontos de controle

Bézier Quadrática

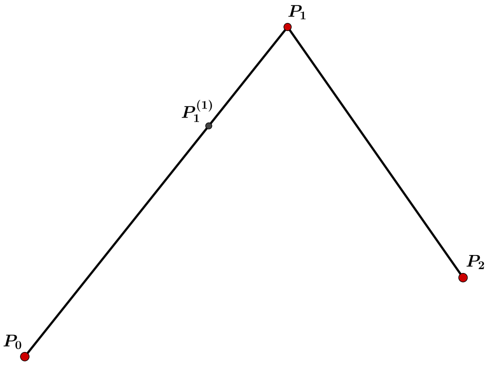
Algoritmo de De Casteljau



Passo 2: Dividir os segmentos na mesma proporção

Bézier Quadrática

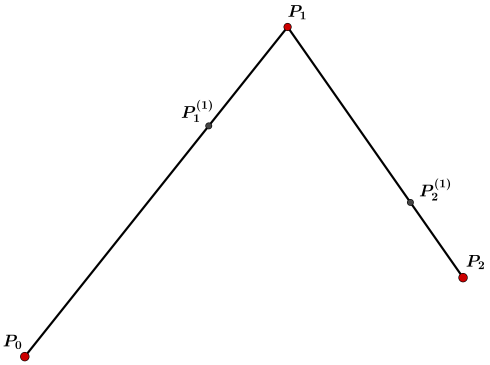
Algoritmo de De Casteljau



Passo 2: Dividir os segmentos na mesma proporção

Bézier Quadrática

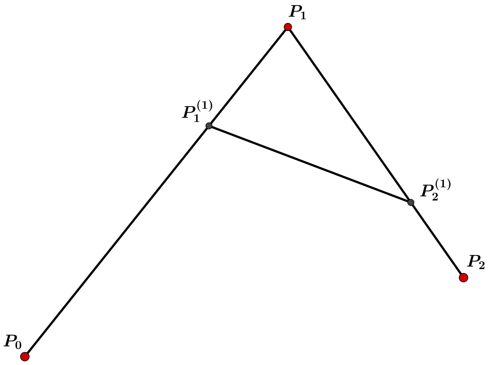
Algoritmo de De Casteljau



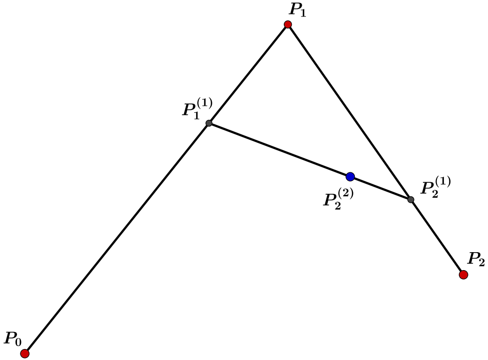
Passo 2: Dividir os segmentos na mesma proporção

Bézier Quadrática

Algoritmo de De Casteljau



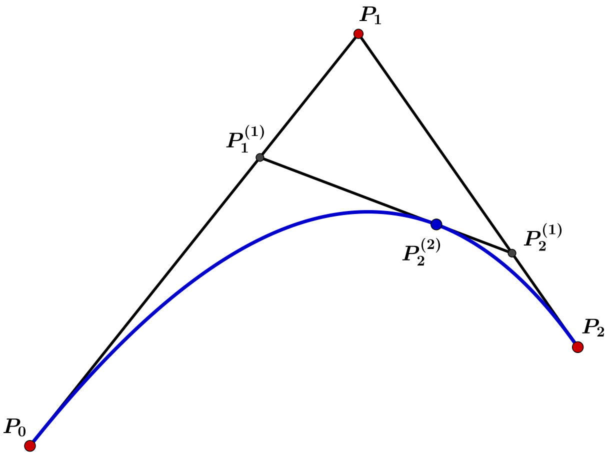
Passo 2: Dividir os segmentos na mesma proporção



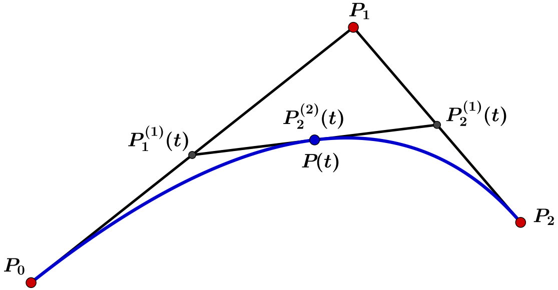
Passo 3: O último ponto obtido é um ponto da curva.

Bézier Quadrática

Algoritmo de De Casteljau

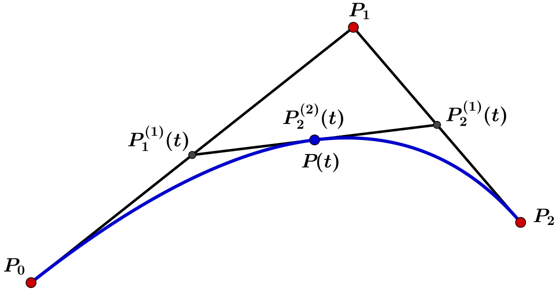


Bézier Quadrática



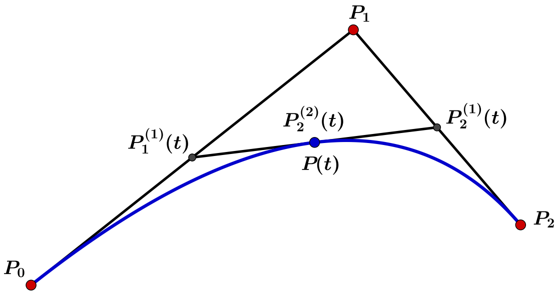
- $\mathsf{P}(t) = \mathsf{P}_2^{(2)}(t)$

Bézier Quadrática



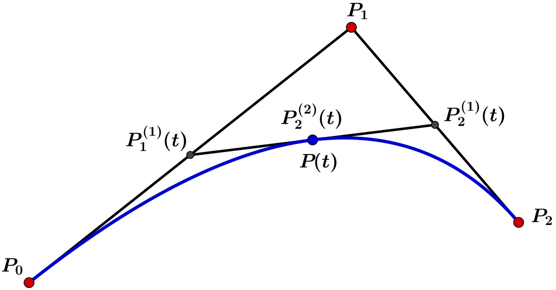
- $\mathsf{P}(t) = \mathsf{P}_2^{(2)}(t)$
- $\mathsf{P}_2^{(2)}(t) = (1 - t)\mathsf{P}_1^{(1)} + t\mathsf{P}_2^{(1)}$

Bézier Quadrática



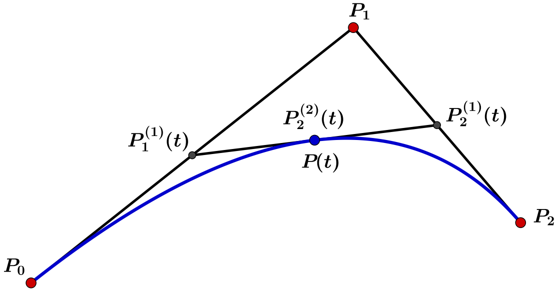
- $\mathbf{P}(t) = \mathbf{P}_2^{(2)}(t)$
- $\mathbf{P}_2^{(2)}(t) = (1-t)\mathbf{P}_1^{(1)} + t\mathbf{P}_2^{(1)}$
- $\mathbf{P}_1^{(1)}(t) = (1-t)\mathbf{P}_0 + t\mathbf{P}_1$

Bézier Quadrática



- $\mathsf{P}(t) = \mathsf{P}_2^{(2)}(t)$
- $\mathsf{P}_2^{(2)}(t) = (1-t)\mathsf{P}_1^{(1)} + t\mathsf{P}_2^{(1)}$
- $\mathsf{P}_1^{(1)}(t) = (1-t)\mathsf{P}_0 + t\mathsf{P}_1$
- $\mathsf{P}_2^{(1)}(t) = (1-t)\mathsf{P}_1 + t\mathsf{P}_2$

Bézier Quadrática



- $\mathbf{P}(t) = \mathbf{P}_{\mathbf{2}}^{(\mathbf{2})}(t)$
- $\mathbf{P}_{\mathbf{i}}^{(\mathbf{j})}(t) = \begin{cases} (1-t)\mathbf{P}_{\mathbf{i-1}}^{(\mathbf{j-1})}(t) + t\mathbf{P}_{\mathbf{i}}^{(\mathbf{j-1})}(t), & \text{se} \quad j = 1, 2 \\ \mathbf{P}_{\mathbf{i}}, & \text{se} \quad j = 0. \end{cases}$

Bézier Quadrática

Forma explícita

$$\mathsf{P}(t) \quad = \quad (1-t)\mathsf{P}_{\mathbf{1}}^{(\mathbf{1})}(t) + t\mathsf{P}_{\mathbf{2}}^{(\mathbf{1})}(t)$$

Bézier Quadrática

Forma explícita

$$
\begin{aligned}
\mathsf{P}(t) &= (1-t)\mathsf{P}_{\mathbf{1}}^{(\mathbf{1})}(t) + t\mathsf{P}_{\mathbf{2}}^{(\mathbf{1})}(t) \\
&= (1-t)[(1-t)\mathsf{P}_{\mathbf{0}} + t\mathsf{P}_{\mathbf{1}}] + t[(1-t)\mathsf{P}_{\mathbf{1}} + t\mathsf{P}_{\mathbf{2}}]
\end{aligned}
$$

Bézier Quadrática

Forma explícita

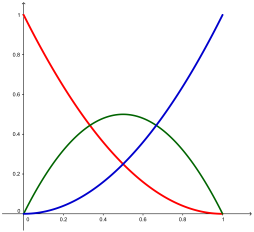
$$
\begin{aligned}
\mathbf{P}(t) &= (1-t)\mathbf{P}_{\mathbf{1}}^{(\mathbf{1})}(t) + t\mathbf{P}_{\mathbf{2}}^{(\mathbf{1})}(t) \\
&= (1-t)[(1-t)\mathbf{P}_{\mathbf{0}} + t\mathbf{P}_{\mathbf{1}}] + t[(1-t)\mathbf{P}_{\mathbf{1}} + t\mathbf{P}_{\mathbf{2}}] \\
&= \color{red}{(1-t)^2}\mathbf{P}_{\mathbf{0}} + \color{green}{2t(1-t)}\mathbf{P}_{\mathbf{1}} + \color{blue}{t^2}\mathbf{P}_{\mathbf{2}}
\end{aligned}
$$

Bézier Quadrática

Forma explícita

$$
\begin{aligned}
\mathbf{P}(t) &= (1-t)\mathbf{P}_{\mathbf{1}}^{(\mathbf{1})}(t) + t\mathbf{P}_{\mathbf{2}}^{(\mathbf{1})}(t) \\
&= (1-t)[(1-t)\mathbf{P}_{\mathbf{0}} + t\mathbf{P}_{\mathbf{1}}] + t[(1-t)\mathbf{P}_{\mathbf{1}} + t\mathbf{P}_{\mathbf{2}}] \\
&= (1-t)^2\mathbf{P}_{\mathbf{0}} + 2t(1-t)\mathbf{P}_{\mathbf{1}} + t^2\mathbf{P}_{\mathbf{2}}
\end{aligned}
$$

- Atenção nos coeficientes!



Bézier Quadrática

Forma matricial

$$\mathsf{P}(t) \quad = \quad (1-t)^2\mathsf{P}_{\mathbf{0}} + 2t(1-t)\mathsf{P}_{\mathbf{1}} + t^2\mathsf{P}_{\mathbf{2}}$$

Bézier Quadrática

Forma matricial

$$\begin{aligned}
\mathsf{P}(t) &= (1-t)^2\mathsf{P}_{\mathbf{0}} + 2t(1-t)\mathsf{P}_{\mathbf{1}} + t^2\mathsf{P}_{\mathbf{2}} \\
&= \begin{bmatrix}(1-t)^2 & 2t(1-t) & t^2\end{bmatrix} \begin{bmatrix}\mathsf{P}_{\mathbf{0}} \\ \mathsf{P}_{\mathbf{1}} \\ \mathsf{P}_{\mathbf{2}}\end{bmatrix}
\end{aligned}$$

Bézier Quadrática

Forma matricial

$$
\begin{aligned}
\mathsf{P}(t) &= (1-t)^2\mathsf{P}_\mathbf{0} + 2t(1-t)\mathsf{P}_\mathbf{1} + t^2\mathsf{P}_\mathbf{2} \\
&= \begin{bmatrix}(1-t)^2 & 2t(1-t) & t^2\end{bmatrix} \begin{bmatrix}\mathsf{P}_\mathbf{0} \\ \mathsf{P}_\mathbf{1} \\ \mathsf{P}_\mathbf{2}\end{bmatrix} \\
&= \begin{bmatrix}1 & t & t^2\end{bmatrix} \begin{bmatrix}1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1\end{bmatrix} \begin{bmatrix}\mathsf{P}_\mathbf{0} \\ \mathsf{P}_\mathbf{1} \\ \mathsf{P}_\mathbf{2}\end{bmatrix}
\end{aligned}
$$

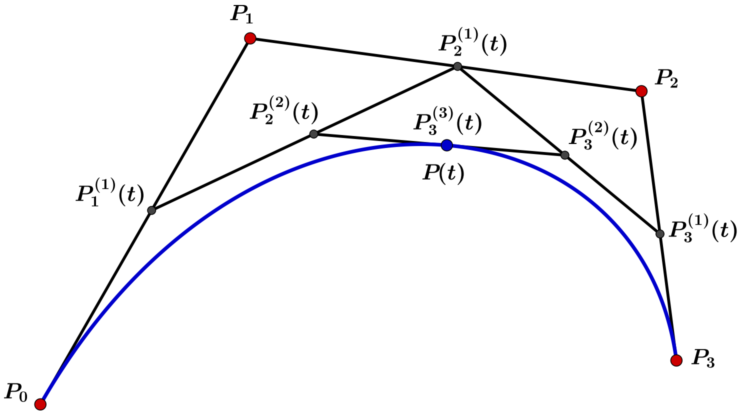
Bézier Quadrática

Forma matricial

$$
\begin{aligned}
\mathbf{P}(t) &= (1-t)^2\mathbf{P_0} + 2t(1-t)\mathbf{P_1} + t^2\mathbf{P_2} \\
&= \begin{bmatrix} (1-t)^2 & 2t(1-t) & t^2 \end{bmatrix} \begin{bmatrix} \mathbf{P_0} \\ \mathbf{P_1} \\ \mathbf{P_2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P_0} \\ \mathbf{P_1} \\ \mathbf{P_2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & t & t^2 \end{bmatrix} M \begin{bmatrix} \mathbf{P_0} \\ \mathbf{P_1} \\ \mathbf{P_2} \end{bmatrix}
\end{aligned}
$$

Bézier Cúbica

Algoritmo de De Casteljau



Bézier Cúbica

De forma análoga obtemos:

- Forma explícita

$$\mathbf{P}(t) = (1-t)^3\mathbf{P_0} + 3t(1-t)^2\mathbf{P_1} + 3t^2(1-t)\mathbf{P_2} + t^3\mathbf{P_3}$$

Bézier Cúbica

De forma análoga obtemos:

- Forma explícita

$$\mathbf{P}(t) = (1-t)^3\mathbf{P_0} + 3t(1-t)^2\mathbf{P_1} + 3t^2(1-t)\mathbf{P_2} + t^3\mathbf{P_3}$$

- Forma matricial

$$\mathbf{P}(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \begin{bmatrix} \mathbf{P_0} \\ \mathbf{P_1} \\ \mathbf{P_2} \\ \mathbf{P_3} \end{bmatrix}$$

onde $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$

Exercícios

GeoGebra

Construa a Bézier cúbica.

MATLAB

Implemente a forma explícita.

Curvas de Bézier (Geral)

Definição (Forma recursiva)

Dados $n+1$ pontos de controle $\{\mathbf{P_0}, \mathbf{P_1}, ..., \mathbf{P_n}\}$, definimos

$$\mathbf{P}(t) = \mathbf{P_n^{(n)}}(t)$$

onde

$$\mathbf{P_i^{(j)}}(t) = \begin{cases} (1-t)\mathbf{P_{i-1}^{(j-1)}}(t) + t\mathbf{P_i^{(j-1)}}(t), & \text{se} \quad j > 0, \\ \mathbf{P_i}, & \text{se} \quad j = 0 \end{cases}$$

e $t \in [0, 1]$.

Curvas de Bézier (Geral)

Teorema (Forma explícita)

Dados $n+1$ pontos de controle $\{\mathbf{P_0}, \mathbf{P_1}, ..., \mathbf{P_n}\}$, então

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P_i} B_{i,n}(t)$$

onde

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

são *Polinômios de Bernstein de grau* n, e $t \in [0,1]$.

Curvas de Bézier (Geral)

Teorema (Forma explícita)

Dados $n+1$ pontos de controle $\{\mathbf{P_0}, \mathbf{P_1}, \ldots, \mathbf{P_n}\}$, então

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P_i} B_{i,n}(t)$$

onde

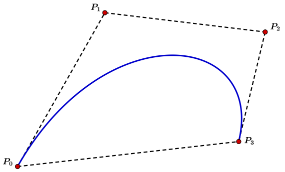
$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

são *Polinômios de Bernstein de grau* n, e $t \in [0, 1]$.

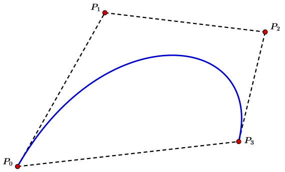
OBS:

- Alterar um ponto de controle muda toda a curva (Controle global);
- O grau do polinômio cresce com o número de pontos de controle.

- A curva está contida no fecho convexo do polígono de controle;

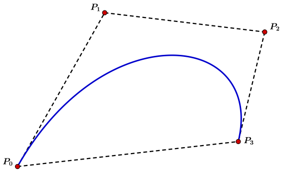


- A curva está contida no fecho convexo do polígono de controle;



- A curva tangencia o polígono de controle no primeiro e último vértice;

- A curva está contida no fecho convexo do polígono de controle;



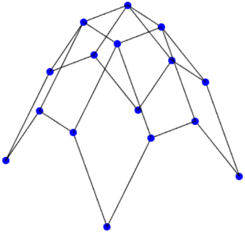
- A curva tangencia o polígono de controle no primeiro e último vértice;
- A curva é invariante sob transformações afins:
  - Translação;
  - Escala;
  - Rotação;
  - etc.

Superfícies de Bézier

Extensão das curvas de Bézier

Dados $(n+1)(m+1)$ pontos de controle. A Superfície de Bézier é dada pelo produto tensorial de curvas de Bézier:

$$\mathbf{P}(u,v) = \sum_{j=0}^{m} \sum_{i=0}^{n} \mathbf{P_{i,j}} B_{i,n}(u) B_{j,m}(v)$$



Superfícies de Bézier

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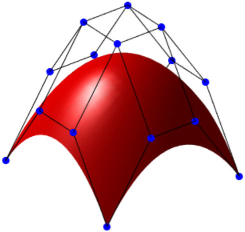

Superfícies de Bézier

Extensão das curvas de Bézier

Dados $(n+1)(m+1)$ pontos de controle. A Superfície de Bézier é dada pelo produto tensorial de curvas de Bézier:

$$\mathbf{P}(u, v) = \sum_{j=0}^{m} \sum_{i=0}^{n} \mathbf{P_{i,j}} B_{i,n}(u) B_{j,m}(v)$$

Propriedades

- Os pontos de controle $\mathbf{P_{0,0}}, \mathbf{P_{0,m}}, \mathbf{P_{n,0}}, \mathbf{P_{n,m}}$ estão não superfície;
- A superfície está contida no fecho convexo dos pontos de controle;
- As curvas de aresta são curvas de Bézier.

Superfícies Bicúbicas de Bezier

Forma matricial

$$\mathbf{P}(u,v) = \sum_{j=0}^{3}\sum_{i=0}^{3}\mathbf{P_{i,j}}B_{i,3}(u)B_{j,3}(v)$$

Superfícies Bicúbicas de Bezier

Forma matricial

$$\begin{aligned}\mathbf{P}(u,v) &= \sum_{j=0}^{3}\sum_{i=0}^{3}\mathbf{P_{i,j}}B_{i,3}(u)B_{j,3}(v)\\ &= \sum_{j=0}^{3}\begin{bmatrix}1 & u & u^2 & u^3\end{bmatrix} M \begin{bmatrix}\mathbf{P_{0,j}}\\ \mathbf{P_{1,j}}\\ \mathbf{P_{2,j}}\\ \mathbf{P_{3,j}}\end{bmatrix} B_{j,3}(v)\end{aligned}$$

Superfícies Bicúbicas de Bezier

Forma matricial

$$
\begin{aligned}
\mathbf{P}(u,v) &= \sum_{j=0}^{3}\sum_{i=0}^{3}\mathbf{P_{i,j}}B_{i,3}(u)B_{j,3}(v) \\
&= \sum_{j=0}^{3}\begin{bmatrix}1 & u & u^2 & u^3\end{bmatrix} M \begin{bmatrix}\mathbf{P_{0,j}}\\ \mathbf{P_{1,j}}\\ \mathbf{P_{2,j}}\\ \mathbf{P_{3,j}}\end{bmatrix} B_{j,3}(v) \\
&= \begin{bmatrix}1 & u & u^2 & u^3\end{bmatrix} M \begin{bmatrix}\mathbf{P_{0,0}} & \mathbf{P_{0,1}} & \mathbf{P_{0,2}} & \mathbf{P_{0,3}}\\ \mathbf{P_{1,0}} & \mathbf{P_{1,1}} & \mathbf{P_{1,2}} & \mathbf{P_{1,3}}\\ \mathbf{P_{2,0}} & \mathbf{P_{2,1}} & \mathbf{P_{2,2}} & \mathbf{P_{2,3}}\\ \mathbf{P_{3,0}} & \mathbf{P_{3,1}} & \mathbf{P_{3,2}} & \mathbf{P_{3,3}}\end{bmatrix} M^T \begin{bmatrix}1\\ v\\ v^2\\ v^3\end{bmatrix}
\end{aligned}
$$

Exercício

MATLAB

Implemente a forma matricial da Bézier bicúbica.