Elementary Linear Algebra Cheat Sheet

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Contents

Chapter 1

Systems of Linear Equations

1.1 The Vector Space of $m \times n$ Matrices

- $M(m, n)$ is a set of all $m \times n$ matrices, $\mathbb{R}^n$ is the set of all $n \times 1$ matrices.
- C is a **linear combination** of $S = \{A_1, A_2, \ldots, A_k\}$ if $C = b_1 A_1 + b_2 A_2 + \cdots + b_k A_k$ for some scalars b_i
- S is **linearly dependent** if at least one of the A_i is a linear combination of other elements.
- $spanS$ of S is a set of all linear combinations of elements in S.
- **10 Vector Space properties are** (for $X, Y, Z \in \mathcal{V}$ and scalars k, l)
  1. $X + Y \in \mathcal{V}$
  2. $X + Y = Y + X$ (commutativity)
  3. $X + (Y + Z) = (X + Y) + Z$ (associativity)
  4. There is $0 \in \mathcal{V}$ such that $\forall X(X \in \mathcal{V} \wedge X + 0 = X)$ ("Zero element")
  5. For each $X \in \mathcal{V}$, $X \in \mathcal{V}$ exists such that $X + (-X) = 0$
  6. $kX \in \mathcal{V}$
  7. $k(lX) = (kl)X$
  8. $k(X + Y) = kX + kY$
  9. $(k + l)X = kX + lX$
  10. $1X = X$
- A set is a **vector space** if **addition** and **scalar multiplication** is defined so that 10 vector space properties hold.

1.2 Systems

- **Linear system** of equations are in the form

$$
\begin{aligned}
a_{11}x_1 + a_{12} + x_2 + \cdots + a_{1n} + x_n &= b_1 \\
a_{21}x_1 + a_{22} + x_2 + \cdots + a_{2n} + x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2} + x_2 + \cdots + a_{mn} + x_n &= b_m
\end{aligned}
$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

- The set of all column vectors $[x_1, x_2, \ldots, x_n]^t$ of columns of values for the variables that solve the all equations is called the **solution set**
- Two systems of linear equations in the same variables are **equivalent** if they have the same solution set
- To produce equivalent systems of equations that is easier to solve, we use **Gaussian Elimination** on the **augmented matrix** of the system. For example:

$$\begin{cases} x + y + z & = 1 \\ 4x + 3y + 5z & = 7 \\ 2x + y + 3z & = 5 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 3 & 5 & 7 \\ 2 & 1 & 3 & 5 \end{array}\right]$$

$$\xrightarrow[R_3-2R_1]{R_2-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & -1 & 1 & 3 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $z = s$, an arbitrary value. Then $y = -(3 - s) = -3 + s, x = 4 - 2s$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 - 2s \\ -3 + s \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

- The produced solution is in **parametric form**, written as a translation vector ($[4, -3, 0]^t$) plus the spanning vectors ($s[-2, 1, 1]^t$).
- A system of linear equations that has no solution is **inconsistent** (typically, an inconsistent system produces an inconsistent row vector, e.g., $[000|1]$ saying $0 = 1$, after elimination)
- **Rank** of a system is the number of non-zero rows after elimination

1.3 Gaussian Elimination

- **Elementary row operations** are
  (i) Interchanging two rows
  (ii) Adding a multiple of one row onto another
  (iii) Multiplying one row by a non-zero constant

  A system after applying elementary row operations are equivalent to the system before (i.e., shares the same solution set).
- A matrix is in **echelon form** if

a) The first non-zero entry in any non-zero row (**pivot entry**) occurs to the right of the pivot entry in the row directly above it
b) All non-zero rows are grouped together at the bottom

$$\begin{bmatrix} \# & * & * & * & * \\ 0 & \# & * & * & * \\ 0 & 0 & 0 & \# & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\#$ entries denote pivot entries and $*$ entries denote constants in the matrix above.

- Every matrix can be reduced into echelon form
- The corresponding variables of the system to the pivot entries are called **pivot variables**. Non-pivot variables are called free variables.
- The set of pivot variables of equivalent systems are the same
- Although the specific echelon form of a given matrix depends on the steps used in reducing the matrix, the final form of the solution does not
- A matrix is in **row-reduced echelon form (RREF or simply "reduced form")** if
  a) It is in echelon form
  b) All pivot entries are 1
  c) All entries above the pivots are 0

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{Echelon form}$$

$$\begin{bmatrix} & 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 2 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & \end{bmatrix} R_1 - R_2 \tag{RREF}$$

- Each $m \times n$ matrix is row equivalent to only one RREF matrix
- **More Unknown Theorem**: A system of linear equation with more unknowns than equations will either have no solution or an infinite number of solutions

1.4 Column Space and Nullspace

- The **coefficient matrix** of a system is the augmented matrix of the system with its last column (vector of constants) deleted
- The **column space** of a matrix is the span of the columns of the matrix
- A system is solvable iff the vector of constants is in the column space of the coefficient matrix
- A subset $\mathcal{W}$ of some vector space $\mathcal{V}$ is a **subspace** of $\mathcal{V}$ if it is **closed under linear combinations** (for $x, y \in \mathcal{W}$ and scalars s and t, $sX + sY \in \mathcal{W}$)
- In geometric sense, subspaces are lines and planes through the origin
- $\mathcal{W} = span(\{A_1, \ldots, A_n\})$ where $A_1, \ldots, A_k \in \mathcal{V}$ is a subspace of $\mathcal{V}$
- **Products** of $m \times n$ matrix and $n \times 1$ matrix is a notation for forming linear combinations of the columns of the coefficient matrix

- **Linearity properties** of matrix multiplication are (where A is $m \times n$ and X, Y are $n \times 1$ matrix)

$$A(X+Y) = AX + AY \qquad \text{(distributive law)}$$
$$A(aX) = aAX \qquad \text{(scalar law)}$$

- $AX = 0$ is a **homogeneous system** corresponding to the system $AX = B$
- The **nullspace** of a matrix A is the solution space for $AX = 0$.
  For example, to find a nullspace of

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 3 & 7 & 3 \end{bmatrix}$$

  $AX = 0$ corresponds to

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 0 \\ 1 & 1 & 3 & 2 & 0 \\ 2 & 3 & 7 & 3 & 0 \end{array}\right]$$

  Compute RREF.

$$\begin{array}{l} \xrightarrow{R_2 - R_1} \\ \xrightarrow{R_3 - 2R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{R_1 + 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

  x_3 and x_4 are free variables, and $x_1 = -2x_3 - 3x_4$, $x_2 = -x_3 + x_4$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - 3x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

  Thus, $null(A)$ is $span\{[-2,-1,1,0]^t, [-3,1,0,1]^t\}$.
- **Trasnlation theorem**: The general solution to $AX = B$ is $X = T + Z$, where T is any particular solution to $AX = B$ and $Z \in null(A)$
- The solution to $AX = B$ is unique iff $null(A) = \emptyset$ (assuming a solution exists for $AX = B$)
- The nullspace of A ($m \times n$ matrix) is a subspace of $\mathbb{R}^n$
- The spanning vectors of the parametric form for the solution of $AX = 0$ (which spans $null(A)$) are linearly independent (regardless of A)
- **Subspace properties** are
  1) If $X, Y \in \mathcal{W}$, then $X + Y \in mathcalW$
  2) $\forall_{X \in \mathcal{W}}$ and some scalars a, $aX \in \mathcal{W}$
  3) $0\ in\mathcal{W}$
- A subspace of a vector space $\mathcal{V}$ is itself a V.S. under addition and scalar multiplication of $\mathcal{V}$

Chapter 2

Linear Independence and Dimension

2.1 Test for Linear Independence

- For $S = \{A_1, A_2, \ldots, A_n\}$, the **dependency equation** is of the form $x_1 A_1 + \ldots + x_n A_n = 0$
- S is linearly independent iff the solution to the dependency equation is $x_1 = x_2 = \ldots = x_n = 0$
- To find dependency among elements, we obtain the **dependency system** by equating entries in the dependency equation and obtain **pivot matrix**. Then we let one of the free variables equal to -1 and all other free variables to 0, expressing the matrix corresponding to the free variable in terms of pivot matrices. For example:

$$S = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -5 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix} \right\}$$

The dependency equation is

$$x \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + y \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + z \begin{bmatrix} -1 & -2 \\ -3 & -5 \end{bmatrix} + w \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} x + y - z - w & 2x + 2y - 2z - 2w \\ x + 2y - 3z & 3x + 4y - 5z - 2w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The dependency system is

$$\begin{cases} x + y - z - w & = 0 \\ 2x + 2y - 2z - 2w & = 0 \\ x + 2y - 3z & = 0 \\ 3x + 4y - 5z - 2w & = 0 \end{cases}$$
$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 2 & 2 & -2 & -2 & 0 \\ 1 & 2 & -3 & 0 & 0 \\ 3 & 4 & -5 & -2 & 0 \end{bmatrix}$$

$$\begin{array}{c} \xrightarrow{R_2 - 2R_1} \\ \xrightarrow{R_3 - R_1} \\ \xrightarrow{R_4 - 3R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_1 - R_3} \\ \\ \\ \xrightarrow{R_4 - R_3} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_3} \\ \xrightarrow{R_2} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, x and y are pivot variables, making

$$\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}, \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$$

pivot matrices, and $x = -z + 2w, y = 2z - w$.

– To express

$$\begin{bmatrix}-1 & -2\\-3 & -5\end{bmatrix}$$

(the matrix corresponding to z) in terms of the pivot matrices, we let $z = -1, w = 0$

$$x = -(-1) + 2(0) = 1, y = 2(-1) - (0) = -2$$

$$1\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix} - 2\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} - \begin{bmatrix}-1 & -2\\-3 & -5\end{bmatrix} + 0\begin{bmatrix}-1 & -2\\0 & -2\end{bmatrix} = 0$$

$$\therefore \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix} - 2\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} = \begin{bmatrix}-1 & -2\\-3 & -5\end{bmatrix}$$

Voila.

– To express

$$\begin{bmatrix}-1 & -2\\0 & -2\end{bmatrix}$$

(the matrix corresponding to w) in terms of the pivot matrices, we let $z = 0, w = -1$

$$x = -(0) + 2(-1) = -2, y = 2(0) - (-1) = 1$$

$$-2\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix} + 1\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} + 0\begin{bmatrix}-1 & -2\\-3 & -5\end{bmatrix} - 1\begin{bmatrix}-1 & -2\\0 & -2\end{bmatrix} = 0$$

$$\therefore -2\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix} + \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} = \begin{bmatrix}-1 & -2\\0 & -2\end{bmatrix}$$

Voila again.

- Set of pivot matrices are linearly independent and non-pivot matrices are linear combinations of the pivot matrices.
- Linearly dependent vectors do not contribute to the span, and you can produce a smaller spanning set by deleting a linearly dependent element of the set.
- A **basis** is a linearly independent subset that spans the vector space.
- The *original* columns corresponding to the pivot variables of a homogeneous system (actually, for the sake of finding pivot variables, any $AX = B, B \in \mathbb{R}$ should suffice) are called **pivot columns** and the pivot columns of a matrix A form a basis for the column space of A. For example, to find a basis for the column space of

$$A = \begin{bmatrix}1 & 2 & -1 & 3\\2 & 2 & -4 & 4\\1 & 3 & 0 & 4\end{bmatrix}$$

The homogeneous system ($AX = 0$) corresponds to the following augmented matrix

$$\left[\begin{array}{cccc|c}1 & 2 & -1 & 3 & 0\\2 & 2 & -4 & 4 & 0\\1 & 3 & 0 & 4 & 0\end{array}\right]$$

$$\xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{cccc|c}1 & 2 & -1 & 3 & 0\\0 & -2 & -2 & -2 & 0\\0 & 1 & 1 & 1 & 0\end{array}\right] \xrightarrow{R_3 + \frac{R_2}{2}} \left[\begin{array}{cccc|c}1 & 2 & -1 & 3 & 0\\0 & -2 & -2 & -2 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right]$$

Since x_1 and x_2 are pivot variables, the corresponding columns A_1 and A_2 form the basis $\{[1,2,1]^t, [2,2,3]^t\}$.

- The span of the pivot columns and the span of the columns in the reduced form are different (i.e., the columns in the reduced form do not form the basis). However they share the scalars required to produce linearly dependent columns. Formally, for row equivalent matrices A and B, if $B_j = c_1B_1 + \dots c_{j-1}B_{j-1} + c_{j+1}B_{j+1} + \dots + c_nB_n$, then $A_j = c_1A_1 + \dots c_{j-1}A_{j-1} + c_{j+1}A_{j+1} + \dots + c_nA_n$.
For example:
In the previous example, it is clearly the case that span$\{[1,2,1]^t, [2,2,3]^t\} \neq$ span$\{[1,0,0]^t, [2,-2,0]^t\}$. Nonetheless, since

$$A_3 = [-1,-4,0]^t = -3[1,2,1]^t = -3A_1 + A_2 = -3[1,2,1]^t + [2,2,3]^t$$
$$A_4 = [3,4,4]^t = [1,2,1]^t + [2,2,3]^t = A_1 + A_2$$

we know that (let A^r be the reduced form of A)

$$A_3^r = [-1,-2,0]^t = -3[1,0,0]^t = -3A_1^r + A_2^r = -3[1,0,0]^t + [2,-2,0]^t$$
$$A_4^r = [3,-2,0]^t = [1,0,0]^t + [2,-2,0]^t = A_1^r + A_2^r$$

2.2 Dimension

- The **dimension** of a vector space $\mathcal{V}$ is the smallest number of elements necessary to span $\mathcal{V}$
- The **Dimension Theorem** states that any basis for an n-dimensional vector space contains exactly n elements
- Some consequences of the dimension theorem (for an n dimensional V.S. $\mathcal{V}$):
  1) Any set of $\mathcal{V}$ containing more than n elements must be linearly dependent
  2) Any set of n elements of $\mathcal{V}$ that spans $\mathcal{V}$ must be linearly independent and is a basis
  3) Any linear independent set of n elements of $\mathcal{V}$ must span $\mathcal{V}$
- **Standard basis** for $\mathbb{R}^n$ is formed by the columns of the $n \times n$ **identity matrix** I. For example, following is the 3×3 identity matrix and the standard basis for $\mathbb{R}^3$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\{I_1, I_2, I_3\} \text{ where } I_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, I_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, I_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

[Calc 3 trauma warning] These vectors are denoted by i, j, k in $\mathbb{R}^3$.

- Standard basis for $M(m,n)$ is a set of matrices $E_{i,j}$ whose non-zero element is a 1 in the (i,j). The dimension of $M(m,n)$ is mn. For example, $M(2,2)$ is 4-dimensional and its standard basis is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

- Standard basis for $\mathcal{P}_n$ is a set of $1 \cdot x^i$. $\mathcal{P}_n$ has a dimension $n+1$. For example, the general polynomial of degree 3 is

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

and is spanned by the standard basis

$$\{1, x, x^2, x^3\}$$

- There exists V.S¿ that are **infinite-dimensional**. For example, $\mathbb{R}^\infty$ is the set of all vectors in the form $[x_1, x_2, \ldots x_n, \ldots]$ (e.g., $X = [1, 2, \ldots, n, \ldots], Y = [\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots]$). It is a V.S. since it is closed under linear combinations ($X + Y \in \mathbb{R}^\infty \wedge 2X \in \mathbb{R}^\infty$). The standard basis for $\mathbb{R}^\infty$ is in the form $I_3 = [0, 0, 1, 0, 0, \ldots, 0, \ldots]$. Since there are infinitely many independent I_j, $\mathbb{R}^\infty$ is infinite-dimensional.

2.3 Row Space and Rank-Nullity Theorem

- The **row space** of an $m * n$ matrix A is the subspace of $M(1, n)$ spanned by rows of A
- Row equivalent matrices have the same row space
- **Nonzero Rows Theorem** states that the nonzero rows of any echelon form of A form a basis for the row space of A¿ For example,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ -1 & -4 & 0 \\ 3 & 4 & 4 \end{bmatrix} \begin{matrix} \xrightarrow{R_2 - 2R_1} \\ \xrightarrow{R_3 + R_1} \\ \xrightarrow{R_4 - 3R_1} \end{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{matrix} \xrightarrow{R_3 - R_2} \\ \xrightarrow{R_4 - R_2} \end{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  Thus, $\{[1, 2, 1]^t, [0, -2, 1]^t\}$ form the basis for the row space(A).
  We may obtain a different basis by using a different echelon form.

$$\xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{R_2}{-2}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  $\{[1, 0, 2]^t, [0, 1, -\frac{1}{2}]^t\}$ is the basis obtained using RREF. This basis is important as each element has 1 in the position where others have 0's.

- **Rank Theorem** states that the row space and column space of a matrix A has the same dimension, and the dimension is the rank of A.
- $rank(A) = rank(A^t)$
- A rank of an $m \times n$ matrix A cannot exceed either m or n
- The set of columns of A is linearly independent iff $rank(A) = n$
- The set of rows of A is linearly independent iff $rank(A) = m$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & -7 \\ 3 & 1 & 4 & 7 & 6 \\ 0 & \pi & -2 & -2 & e \\ -2 & -5 & 17 & 15 & \sqrt{31} \end{bmatrix}$$

  This matrix has the rank at most 4. The set of columns are L.D. because by Dimension Theorem, 5 vectors in $\mathbb{R}^4$ must be L.D. If $rank(A) = 4$, the set of rows are L.I.

- The dimension of the nullspace of A is called the **nullity** of A (denoted by $null(A)$)
- **Rank-Nullity Theorem** states that for an $m \times n$ matrix A,

$$rank(A) + null(A) = n$$

- $AX = B$ is solvable for all $B \in \mathbb{R}^m$ iff $rank(A) = m$ For example, for a 3×4 matrix with the echelon form

$$A = \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$rank(A) = 3 = m$$

We can see that any augmented matrix $AX = B$ for all $B \in \mathbb{R}^3$ will have a solution of $x_1 = b_1, x_2 = b_2, x_4 = b_4$, and x_3 is a free variable. This is because the column space of A contains 3 linearly independent elements of $\mathbb{R}^3$, spanning $\mathbb{R}^3$ as a result.

- $AX = B$ has at most one solution for all $B \in \mathbb{R}^m$ iff $rank(A) = n$.
For a 4×3 matrix with the echelon form

$$A = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$rank(A) = 3 = n$$

We can see that any $B \in \mathbb{R}^4$ with nonzero element in the 4th row will make $AX = B$ inconsistent (no/zero solution). Otherwise, for $B = [b_1, b_2, b_3, 0]^t$, $AX = B$ has exactly one solution: $x_1 = b_1, x_2 = b_2, x_3 = b_3$. This is because by the Rank-Nullity Theorem, A has a nullity of 0, meaning the nullspace of A is $\emptyset$, and by the Translation Theorem, $AX = B$ has a unique solution if one exists.

- An $n \times n$ matrix A with rank n is **nonsingular** and the solution to $AX = B$ both exists and is unique for all $B \in \mathbb{R}^n$. Nonsingular matrix A implies

  (a) the nullspace of A is $\emptyset$
  (b) For each $B \in \mathbb{R}^n$, the system $AX = B$ has at most one solution
  (c) The system $AX = B$ has a solution $\forall B \in \mathbb{R}^n$
  (d) A is row equivalent to the identity matrix

For example, assume a general 3×3 matrix

$$A = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, AX = B \text{ is } \begin{bmatrix} 1 & * & * & b_1 \\ 0 & 1 & * & b_2 \\ 0 & 0 & 1 & b_3 \end{bmatrix}$$

It is trivial to see that

  (a) $AX = B$ has exactly one solution $x_1 = x_2 = x_3 = 0$
  (b) if $B = 0$, then the only solution to $AX = B = 0$ is $x_1 = x_2 = x_3 = 0$, meaning that the nullspace is $\emptyset$

Chapter 3

Linear Transformations

Chapter 4

Determinants

4.1 Definition of the Determinant

- For a 2×2 matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  The factor $\delta = ad - cb$ is called **determinant**.

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- **Laplace Expansion** or **Cofactor Expansion** states that the determinant of any $n \times n$ matrix can be obtained using using determinants of $(n-1) \times (n-1)$ submatrices. Specifically, for every i,

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij}$$

  Where a_{ij} is the entry of ith row and jth column and A_{ij} is the determinant of the submatrix obtained by removing the ith row and jth column.

- We can use Laplace Expansion to find the determinant of 3×3 matrix. Choosing $i = 1$,

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \cdot \begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{vmatrix} - b \cdot \begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & \cancel{e} & f \\ g & \cancel{h} & i \end{vmatrix} + c \cdot \begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{i} \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= aei + bfg + cdh - ceg - bdi - afh \end{aligned}$$

  Choosing $i = 2$ yields the same result

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= -d \cdot \begin{vmatrix} \cancel{a} & b & c \\ \cancel{d} & \cancel{e} & \cancel{f} \\ \cancel{g} & h & i \end{vmatrix} + e \cdot \begin{vmatrix} a & \cancel{b} & c \\ \cancel{d} & \cancel{e} & \cancel{f} \\ g & \cancel{h} & i \end{vmatrix} - f \cdot \begin{vmatrix} a & b & \cancel{c} \\ \cancel{d} & \cancel{e} & \cancel{f} \\ g & h & \cancel{i} \end{vmatrix} \\ &= -d(bi - ch) + e(ai - cg) - f(ah - bg) \\ &= -bdi + cdh + aei - ceg - afh + bfg \\ &= aei + bfg + cdh - ceg - bdi - afh \end{aligned}$$

  Focus on the signs since the values of $(-1)^{i+j}$ differ when we chose to fix $i = 1$ versus $i = 2$.
  Also, you can change the i and j in the summation and the equation still holds since the determinants of A and A^t are identical.

- **Row Interchange Property**: Let A be an $n \times n$ matrix and supposed B is obtained by interchanging two rows of A. Then $\det B = -\det A$
- **Row Scalar Property**: Supposed the $n \times n$ matrix B is obtained from A by multiplying each element in the ith row by some scalar c. Then $\det B = c\det A$
- **Row Additive Property**: Let U, V, and A_i be $1 \times n$ row matrices, where $i = 2, 3, \ldots, n$. Then

$$\det \begin{bmatrix} U+V \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \det \begin{bmatrix} U \\ A_2 \\ \vdots \\ A_n \end{bmatrix} + \det \begin{bmatrix} V \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$$

For example, given that

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix} = -2$$

We can compute

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 5 \end{vmatrix}$$

using the row additive property.

$$[1,0,5] = [1,0,0] + [0,0,5]$$
$$\therefore \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 5 \end{vmatrix} = -2 + (1 \cdot 2 \cdot 5) = 8$$

- The determinant of an upper triangular matrix is the product of the entries on the main diagonal.

$$\begin{aligned} \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} &= a(df - e \cdot 0) - b(0 \cdot f - e \cdot 0) + c(0 \cdot 0 - d \cdot 0) \\ &= adf \end{aligned}$$

4.2 Reduction and Determinants

- Row Interchange Property states that the interchanging two rows must negate the value of the determinant.

$$\begin{bmatrix} 1 & 2 & 3 \\ 69 & 420 & 3232341 \\ 1 & 2 & 3 \end{bmatrix}$$

Exchanging the row 1 and 3 must negate the value of the determinant. Zero is the only real number that equals its own negative. Thus, **any $n \times n$ matrix with two equal rows has a zero determinant**.

- **In any $n \times n$ matrix A, adding a multiple of one row of A onto a different row does not change the determinant of A.**

Suppose we get a matrix B by adding a scalar multiple of the first row to the second.

$$\det B = \det \begin{bmatrix} A_1 \\ A_2 + cA_1 \\ \vdots \\ A_n \end{bmatrix} = \det \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = c \cdot \det \begin{bmatrix} A_1 \\ A_1 \\ \vdots \\ A_n \end{bmatrix} = \det A + 0 = \det A$$

- So far, we learned how elementary row operations affect the determinant.
  - interchanging two rows: negates the determinant
  - multiplying a row by a nonzero scalar: multiplies the determinant by the scalar
  - adding a scalar multiple of a row to another: does not affect the determinant

  We can use these properties to learn how to use row operations to calculate determinants.
  If we produce an echelon form matrix R by applying row operations to a matrix A, the determinant of A is

$$det(A) = \frac{\prod diag(R)}{d}$$

where d is the product of the scalars multiplied and $(-1)^{\text{number of rows swapped}}$.
For example, to compute

$$\begin{vmatrix} 3 & 6 & 9 & 12 \\ 1 & 2 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{vmatrix}$$

$$= 3 \cdot \xrightarrow{\frac{R_1}{3}} \begin{vmatrix} 3 & 6 & 9 & 12 \\ 1 & 2 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{vmatrix} = 3 \cdot \begin{matrix} \xrightarrow{R_2 - R_1} \\ \xrightarrow{R_3 - 3R_1} \end{matrix} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -3 \\ 0 & -1 & -7 & -11 \\ 0 & 2 & 4 & 2 \end{vmatrix}$$

$$= -3 \cdot \begin{matrix} \xrightarrow{R_3} \\ \xrightarrow{R_2} \end{matrix} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -7 & -11 \\ 0 & 0 & -1 & -3 \\ 0 & 2 & 4 & 2 \end{vmatrix} = -3 \cdot \xrightarrow{R_4 + 2R_2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -7 & -11 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -10 & -20 \end{vmatrix}$$

$$= -3 \cdot \xrightarrow{R_4 - 10R_3} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -7 & -11 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 10 \end{vmatrix}$$

$$= -3(1 \cdot -1 \cdot -1 \cdot 10) = -30$$

We can even further simplify the process by only using he row operation number 3 (adding a scalar multiple of a row to another), though this totally will make the numbers messier.

$$\begin{vmatrix} 2 & -3 & 1 \\ 2 & 0 & 1 \\ 1 & -4 & 5 \end{vmatrix} = \begin{matrix} \xrightarrow{R^2 - R^1} \\ \xrightarrow{R^3 - \frac{R^1}{2}} \end{matrix} \begin{vmatrix} 2 & -3 & 1 \\ 0 & 3 & 0 \\ 0 & -\frac{5}{2} & \frac{9}{2} \end{vmatrix} \xrightarrow{R^3 - (3 \cdot -\frac{2}{5})R^2} = \begin{vmatrix} 2 & -3 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{9}{2} \end{vmatrix}$$

$$\therefore det(A) = 2 \cdot 3 \cdot \frac{9}{2} = 27$$

Fun fact The Laplace expansion recursively takes $O(n!)$ ($O(n^2)$ if sub-determinants are memoized), Gaussian elimination takes $O(n^3)$.

- An $n \times n$ matrix A is invertible iff $\det A \neq 0$

4.2.1 Uniqueness of the Determinant

- **Uniqueness Theorem**: Supposed thatt D is a function that transforms $n \times n$ matrices into numbers such that
  (a) $D(I) = I$
  (b) D satisfies the row interchange, the row scalar, and the row addirivity properties

  Then $D(A) = \det A$ for all $n \times n$ matrices A.
- **Product Theorem** states that for all $n \times n$ matrices A and B, $\det(AB) = (\det(A))(\det(B))$.
- For any $n \times n$ matrix, $\det A = \det A^t$.

4.2.2 Volume

//TODO

4.3 A Formula For Inverses

- **Cramer's Rule** states that if A is invertible, then for $AX = Y$ where $X, Y \in \mathbb{R}^n$, $X = [x_1, x_2, \dots, x_n]^t$ where

$$x_i = \frac{\det[A_1, \dots, A_{i-1}, Y, A_{i+1}, \dots, A_n]}{\det A}$$

For example, to find the solution of

$$\begin{cases} 5x_1 + 2x_2 + x_3 = y_1 \\ 2x_1 + 2x_2 2x_3 = y_2 \\ 2x_1 + x_2 + x_3 = y_3 \end{cases}$$

Find the determinant of the augmented matrix first

$$\begin{vmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 2 \begin{matrix} \\ \xrightarrow{\frac{R_2}{2}} \\ \\ \end{matrix} \begin{vmatrix} 5 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 2 \begin{matrix} \xrightarrow{R_1 - 5R_2} \\ \\ \xrightarrow{R_3 - 2R_2} \end{matrix} \begin{vmatrix} 0 & -3 & -4 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{vmatrix} = -2 \begin{matrix} \\ \\ \xrightarrow{-R_3} \end{matrix} \begin{vmatrix} 0 & -3 & -4 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2 \begin{matrix} \xrightarrow{R_1 + 3R_3} \\ \xrightarrow{R_2 - R_3} \\ \\ \end{matrix} \begin{vmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \begin{matrix} \xrightarrow{R_2} \\ \xrightarrow{R_1} \\ \\ \end{matrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = -2 \begin{matrix} \\ \xrightarrow{R_3} \\ \xrightarrow{R_2} \end{matrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix} = 2$$

According to Cramer's rule,

$$x_1 = \frac{\begin{vmatrix} y_1 & 2 & 1 \\ y_2 & 2 & 2 \\ y_3 & 1 & 1 \end{vmatrix}}{2} = \frac{2y_1 + 4y_3 + y_2 - 2y_3 - 2y_2 - 2y_1}{2} = \frac{-y_2 + 2y_3}{2}$$

$$x_2 = \frac{\begin{vmatrix} 5 & y_1 & 1 \\ 2 & y_2 & 2 \\ 2 & y_3 & 1 \end{vmatrix}}{2} = \frac{5y_2 + 4y_1 + 2y_3 - 2y_2 - 2y_1 - 10y_3}{2} = \frac{2y_1 + 3y_2 - 8y_3}{2}$$

$$x_3 = \frac{\begin{vmatrix} 5 & 2 & y_1 \\ 2 & 2 & y_2 \\ 2 & 1 & y_3 \end{vmatrix}}{2} = \frac{10y_3 + 4y_2 + 2y_1 - 4y_1 - 4y_3 - 5y_2}{2} = \frac{-2y_1 - y_2 + 6y_3}{2}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -y_2 + 2y_3 \\ 2y_1 + 3y_2 - 8y_3 \\ -2y_1 - y_2 + 6y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -1 & 2 \\ 2 & 3 & -8 \\ -2 & -1 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

In fact, the matrix being multiplied to Y is A^{-1}

- **Cramer's Rule for finding inverses**: Consider the elements of Laplace Expansion $(-1)^{i+j}a_{ij}A_{ij}$
  - A_{ij} is called a **minor** of the elemenet at (i, j) and is the determinant of the smaller $(n-1)\times(n-1)$ sub-matrix resulting from deleting row i and column j
  - A **cofactor** $C_{ij} = (-1)^{i+j}A_{ij}$
  - A **cofactor matrix** C is the matrix of cofactors. For example for A

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$
$$C = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

  - An **adjugate** (or adjoint) is equal to C^t

$$C^t = \begin{pmatrix} +A_{11} & -A_{21} & +A_{31} \\ -A_{12} & +A_{22} & -A_{32} \\ +A_{13} & -A_{23} & +A_{33} \end{pmatrix}$$

Cramer's rule uses the adjugate to find the inverse, specifically,

$$A^{-1} = \frac{1}{\det A}C^t$$

So, for

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$
$$C^t = \begin{bmatrix} +(3-0) & -(0+1) & +(0+1) \\ -(15-0) & +(6+0) & -(0+5) \\ +(5-0) & -(2-0) & +(2-0) \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

We already found that $\det A = 1$, so $A^{-1} = \frac{1}{1}C^t = C^t$.