Intuitive Overview Of Algebraic Quantum Field Theory

Master's Thesis

Author: Mihail Mladenov

Advisors: Nikolay Nikolov, Dimitar Mladenov

Abstract

The purpose of this work is to make an intuitive[1] overview of ***Algebraic Quantum Field Theory (AQFT)***, requiring minimal knowledge of the subject using only publically available references that are not behind a paywall. Although many of the references themselves might cite references behind a paywall, the combined set of all listed references here, plus the contents of this thesis should be enough to cover all the basics of AQFT "from scratch". While many proofs and results are omitted, references with detailed proofs are supplied as well as clear pathway of how one should approach reading up on this subject.

[1]The word intuitive is subjective and this work is created with the hope that it is intuitive to the average mathematically minded person.

Table of contents

Basic Mathematical Preliminaries

It is impossible to cover all the preliminaries required for understanding AQFT in this work, however as the goal of this document is to be as welcoming to newcomers to the field as possible at least a basic list of the preliminaries will be given, with few references, alongside some brief overview of some important objects used in this work, alongside introduction of a few notational conventions.

Mathematical Logic and Set Theory

In order to express any rigorous mathematical theory, we need a formal language in order to do so. Although expressing everything in terms of this formal language will probably be optimal in terms of the preciseness of the exposition, in practice this often becomes hard for human readability and ease of expression. Because of this, the formal language is usually supplemented, partially replaced or embedded inside human language expressions.

Alongside the formal language, in order to express complex theories we need building blocks. We need some foundational set of objects that we can operate on alongside a list of assumptions about what those objects are and how they operate.

In any case over the years the formal language of ***First Order Logic (FOL)*** and ***Zermelo-Frenkel Set Theory + Choice Axiom (ZFC)*** have proven to be a great pair of formal language and basic set of building blocks for building higher level mathematical theories. These are vast subjects and covering even the basic parts of them in detail is not feasible. One can find an introduction to those subjects freely accessible in **[1]**. Furthermore one can use these books on logic and set theory accessible on the website of the author: **[2, 3]**.

In this paper it is assumed that the reader is already familiar with first order logic, so as meta language here first order logic with standard notation is freely mixed with English.

Set theory will not be covered in depth as well but nevertheless the axioms are stated as well as some important results.

In ZFC we only have one type of "object" which we call ***set*** and only one symbol $\in$ which represents membership of a set, whatever that means (the meaning of the symbol $\in$ is given by the axioms of how it actually works…).

Although set theory can be formulated only by using the membership relation as an only symbol in the theory, in practice it will be impossibly tedious to do so. That's why we will define many other helper symbols and concepts along the way. For convenience alongside the symbol $\in$ we introduce two shorthand notations $\subset$ and $\subseteq$ for the following properties:

$$X \subseteq Y \text{ means } (\forall x)(x \in X \Longrightarrow x \in Y)$$

$$X \subset Y \text{ means } X \subseteq Y \bigwedge X \neq Y$$

Axiom 1. ***(Existence)*** A set exists, or formally

$$\exists S_0$$

Axiom 2. ***(Extensionality)***

$$\forall X \forall Y [\forall x (x \in X \Longleftrightarrow x \in Y) \Longrightarrow X = Y]$$

Informally this tells us that two sets are equal only when they have the same members.

Axiom 3. ***(Comprehension Schema)*** For any formula Φ with free variables $X, Y, a_0, ..., a_n$ we have :

$$\forall X \forall a_0 ... \forall a_n \exists Z \forall y (y \in Z \Longleftrightarrow y \in X \bigwedge \Phi(X, Y, a_0, ..., a_n))$$

Informally this tells us that if we have some logical property Φ and a set X we can extract from X all the members that satisfy this property and form new set with them. Because of the first axiom, such set will be unique. Also note that this is not really a single “axiom” because there are infinitely many statements like that for each formula Φ, that’s why it’s called a “Schema” because it’s a bundle of infinitely many axioms.

Since this axiom schema is so widely used we will use this shorthand notation for the sets generated by it with:

$$\{x \in X \mid \Phi(x, X, a_0, ..., a_n)\}$$

Theorem 1.1.1. ***(Existence of the empty set)*** There is unique set with no members.

Proof. By comprehension let

$$Y = \{x \in S_0 \mid x \neq x\}$$

Because the formula for comprehension is always false, the set Y has no members. $\square$

We will label the empty set with $\emptyset$.

Axiom 4. ***(Pairing)***

$$\forall x \forall y \exists Z \Big(x \in Z \bigwedge y \in Z\Big)$$

This tells us that for any two sets we can form another set that has both of the initial sets as members. If x and y are arbitrary sets and from the pairing axiom we receive a set Z from them, then we can define the ***unordered pair*** $\{x, y\}$ to be

$$\{x, y\} := \Big\{z \in Z \mid z = x \bigvee z = y\Big\}$$

From here it’s easy to see that one can also form a sets with one member $\{x\}$ by using $\{x, x\}$.

Axiom 5. ***(Union)***

$$\forall S \exists U \forall X \forall x (x \in X \bigwedge X \in S \Longrightarrow x \in U$$

This axiom tells us that for each set if this set contains any members which they themselves contain some members, then one can form a set that contains all the inner members. From this set, we can of course extract only those inner members and we will call that set the ***union*** of the first set. In mathematical notation if our initial set is S and the set from the union axiom is U, then we will define and notate the union as

$$\bigcup S = \{x \in U \mid \exists X \in S(x \in X)\}$$

We can also define

$$X \cup Y = \bigcup \{X, Y\}$$

Axiom 6. ***(Power set)***

$$\forall S \exists P \forall X (X \subseteq \Longrightarrow X \in P)$$

Intuitively this axiom tells us that we can form the set of all subsets of a given set. Again if for any set S the set that's given to exist in this power set axiom is P, then we can define the convenient notation $\mathcal{P}(S)$ to mean

$$\mathcal{P}(S) = \{X \in P \mid X \subseteq S\}$$

Axiom 7. ***(Infinity)***

$$\exists S_\infty \big(\emptyset \in S_\infty \bigwedge (\forall x \in S_\infty (x \cup \{x\} \in S_\infty))\big)$$

Intuitively this axiom tells us that certain special set exists (which has so many elements that it corresponds to our intuitive understanding of "infinite").

Axiom 8. ***(Replacement Schema)*** For each formula Φ with free variables $x, y, A, w_0, ...w_n$ we have

$$\forall A \forall w_0, \forall w_n [\forall x \in A \exists ! y (\Phi(x, y, A, w_0, ..., w_n)) \implies$$
$$\implies \exists Y \forall x \in A \exists y \in Y (\Phi(x, y, A, w_0, ..., w_n))]$$

Intuitively this axiom tells us that if we have a certain set X and potentially other "constant" sets $w_0, ..., w_n$ that we carry along and a certain logical property that's encoded into the formula Φ. Then if we have for each member x of X a certain unique element y that makes the property true, this axiom schema tells us that we can find the set, that contains all those y members. In some sense the formula Φ is like a meta version of the notion of function (we will define what intrinsic function is later.) and we can use this axiom schema to "map" the members of sets with meta-language "functions".

Now for the next axiom a convenient notation that we can introduce is

$$X \cap Y := \{x \in X \mid x \in Y\}$$

Axiom 9. ***(Foundation/Regularity)***

$$\forall X (X \neq \emptyset \implies \exists x \in X (x \cap X = \emptyset))$$

At first this axiom might look very strange, but it actually forbids non-intuitive cyclic memberships like a set being contained in itself $X \in X$ or a longer membership chain eventually going ending up being member of the initial set, like $X \in Y_0 \in ... \in Y_n \in X$.

Seeing why the first one contradicts regularity is trivial, to see why the second result holds, consider the set $S = \{X, Y_0, ..., Y_n\}$ (one can form this set by unioning with singletons n times). None of its members is disjoint from the set itself, because each one is contained in some other.

So now before stating the last axiom let's build some more useful objects.

Definition 1.1.1. For two sets x, y we define the ***ordered pair*** (x, y) as

$$(x, y) := \{\{x\}, \{x, y\}\}$$

Definition 1.1.2. If X and Y are sets we call the set $X \times Y$ containing only all the ordered pairs with first element from X and second element from Y the ***set product*** of X and Y.

Theorem 1.1.2. For every two sets X and Y the set $X \times Y$ exists.

Proof sketch. We simply looking how the ordered pairs are encoded and look at the set $\mathcal{P}(\mathcal{P}(X \cup Y))$ and prove that all ordered pairs with first elements from X and second elements from Y are in there. Then we use comprehension to get the unique set that we define as $X \times Y$ as follows

$$X \times Y := \{z \in \mathcal{P}(\mathcal{P}(X \cup Y)) \mid \exists x \in X \exists y \in Y(z = (x, y))\}$$

$\square$

Definition 1.1.3. ***Relation*** is a set of ordered pairs.

It's easy to see that if $(x, y) \in R$ then $x, y \in \bigcup\bigcup R$. Using that for any relation R we can define the following

$$\mathbf{Domain}(\mathrm{R}) = \left\{x \in \bigcup\bigcup R \mid \exists y((x, y) \in R)\right\}$$

$$\mathbf{Range}(\mathrm{R}) = \left\{y \in \bigcup\bigcup R \mid \exists x((x, y) \in R)\right\}$$

For relations we often write xRy instead of $(x, y) \in R$. It's also convenient to introduce the notation of ***restriction*** of R by

$$R \restriction A = R \cap A \times \mathbf{Range}(\mathrm{R})$$

Definition 1.1.4. A ***function*** f is a relation such that

$$\forall x \in \mathbf{Domain}(\mathrm{f}) \exists! y \in \mathbf{Range}(\mathrm{f}) : \quad (x, y) \in f$$

We use the usual notation $f : X \longrightarrow Y$ to mean that f is a function and $\mathbf{Domain}(\mathrm{f}) = X$ and $\mathbf{Range}(\mathrm{f}) \subseteq Y$. In the same setup if $x \in X$, then $f(x)$ means the unique y such that $(x, y) \in f$. The ***set of all functions from X to Y*** we denote by Y^X

Now we can also define other notations about functions. In the same setup if $f : X \longrightarrow Y$ and $X_0 \subseteq X$ and $Y_0 \subseteq Y$ then we define respectively, the image, preimage as

$$f[X_0] := \{f(x) \in \mathbf{Range}(\mathrm{f}) |\ x \in X_0\}$$

$$f^{-1}[Y_0] := \{x \in \mathbf{Domain}(\mathrm{f}) |\ f(x) \in Y_0\}$$

Often times if the domains and ranges of the functions are clear (as it is most of the time), we just use $f(X_0)$ and $f^{-1}(Y_0)$.

Now if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, we define ***function composition*** by

$$g \circ f := \Big\{(x, z) \in \mathbf{Domain}(\mathrm{f}) \times \mathbf{Range}(\mathrm{g}) \ | \\ \exists y \in \mathit{Range}\ (f)\Big((x, y) \in f \bigwedge (y, z) \in g\Big)\Big\}$$

One can easily prove that the composition is again a function **[1, 3]**.

Before stating the last axiom we have to give the following definition

Definition 1.1.5. A function $f : X \longrightarrow \bigcup X$ is called a ***choice function*** on X if

$$\forall S \in X : \quad f(S) \in S$$

Axiom 10. ***(Choice)*** Every non empty set has a choice function.

Now that all axioms are stated here are some important definitions and results that are given without proofs (those can be found at **[1–3]**).

Set difference is the operation

$$X \setminus Y := \{x \in X \mid x \notin Y\}$$

The for a function $f : X \longrightarrow Y$, ***restriction*** of f to Z is the function

$$f \upharpoonright Z := f \cap Z \times Y$$

This is the same concept that was introduced with relations and functions are special case of relations. It can easily be seen that the restriction of a function is also a function.

Now we define three very important and commonly used properties of functions

Definition 1.1.6. A function $f : X \longrightarrow Y$ is ***injective*** if

$$\forall x_0, x_1 \in X : (f(x_0) = f(x_1) \Longrightarrow x_0 = x_1)$$

surjective if

$$\mathbf{Range}(\mathrm{f}) = Y$$

and ***bijective*** if it's both of those

Now we will start defining the very important notion of ordinals.

Definition 1.1.7. A set X is called ***transitive*** when

$$\forall S \in X \forall s \in S : \quad s \in X$$

Definition 1.1.8. An ***ordinal*** is a transitive set of transitive sets

For ordinals α and β we will introduce the notation $a < b$ to mean $\alpha \in \beta$ and $\alpha \leq \beta$ to mean that $\alpha < \beta$ or $\alpha = \beta$. We also introduce the operation $+1$ for ordinals like

$$\alpha + 1 = \alpha \cup \{\alpha\}$$

Theorem 1.1.3.
- $\emptyset$ is an ordinal
- If α is an ordinal then $\alpha \cup \{\alpha\}$ is as well.
- If A is a set of ordinals then $\bigcup A$ is as well.
- Every member of an ordinal is an ordinal
- If α and β are ordinals then $\alpha = \beta$ or $\alpha \in \beta$ or $\beta \in \alpha$.
- $\alpha < \beta \Longleftrightarrow \alpha \subset \beta$ and $\alpha \leq \beta \Longleftrightarrow \alpha \subseteq \beta$

Theorem 1.1.4. If Γ is a nonempty set of ordinals then $\bigcap \Gamma$ is an ordinal, $\bigcap \Gamma \in \Gamma$ and for every $\alpha \in \Gamma$ we have $\bigcap \Gamma \leq \alpha$.

Definition 1.1.9. The first ordinal $\emptyset$ is called ***zero***. An ordinal α is called ***successor ordinal*** if $\alpha = \beta + 1$ for some other ordinal beta. Finally an ordinal is called ***limit ordinal*** if it's not zero or successor ordinal.

Now we define arguably one of the most important limit ordinals.

Definition 1.1.10. Let I be a set assumed to exist from the axiom of infinity. Then we define

$$\omega := \bigcap \Big\{ x \subseteq I \mid \emptyset \in x \bigwedge \forall y \in x (y \cup \{y\} \in x) \Big\}$$

We also call ω the set of ***natural numbers*** and give it alternative name $\mathbb{N}$.

Theorem 1.1.5. The set ω is the first limit ordinal.

Definition 1.1.11. A relation $R \subseteq X \times X$ is called

- ***reflexive*** if $\forall x \in X : (x, x) \in R$.
- ***irreflexive*** if $\forall x \in X : (x, x) \notin R$
- ***transitive*** if $\forall x_0, x_1, x_2 \in X(x_0 R x_1 \bigwedge x_1 R x_2 \Longrightarrow x_0 R x_2)$
- ***total*** if $\forall x, y \in X$ either $x = y$, xRy or yRx.
- ***symmetric*** if $\forall x, y \in X(xRy \Longrightarrow yRx)$

Definition 1.1.12. A set $Y \subseteq X$ has a least element x under the relation $R \subseteq X \times X$ if $\forall y \in Y : (xRy \bigvee x = y)$.

Definition 1.1.13. Relation is called ***equivalence*** relation if it's reflexive, symmetric and transitive.

Definition 1.1.14. A relation is $R \subseteq X \times X$ is a ***well-order*** on X if it's irreflexive, total, transitive and every non-empty subset of X has least element under it. A ***well-ordered set*** is a pair (X, R) with X and R with properties as above.

If the relation is known from the context we often just say that X is well ordered.

Theorem 1.1.6. Every ordinal α is well ordered under the relation.

$$\{(\beta, \gamma) \in \alpha \times \alpha \mid \beta < \gamma\}$$

Definition 1.1.15. Two well-orders $\left(X, \underset{X}{<}\right)$, $\left(Y, \underset{Y}{<}\right)$ are isomorphic if a bijection $f : X \longrightarrow Y$ exists such that $x_0 \underset{X}{<} x_1 \Longleftrightarrow f(x_0) \underset{Y}{<} f(x_1)$ for all $x_0, x_1 \in X$.

Theorem 1.1.7. Every well-order is isomorphic to an ordinal.

Now follows one of the most important results, that basically make us able to do all sorts of constructions inside set theory. These results are stated differently than the versions in **[3]** so we will give explicit proofs.

Theorem 1.1.8. ***(Transfinite induction)*** Let $\Phi(x, w_0, ..., w_n)$ is a meta-language predicate with parameters $w_0, ..., w_n$. If for fixed values of the parameters, $w_0, ..., w_n$ and all ordinals α it holds that

$$(\forall\beta < \alpha : \Phi(\beta, w_0, ..., w_n)) \Longrightarrow \Phi(\alpha, w_0, ..., w_n)$$

then $\Phi(\alpha, w_0, ..., w_n)$ holds for all ordinals α.

Proof. Assume it's not true and let γ be some ordinal for which $\Phi(\alpha, w_0, ..., w_n)$ doesn't hold. Since as an ordinal it's well ordered then every non-empty subset of it must have least element. Now let's look at the set

$$S = \{\beta \subset \gamma \mid \beta \text{ is ordinal for which } \neg\Phi(\beta, w_0, ..., w_n)\}$$

If this set was empty then it would imply that Φ is true for all the ordinals less than γ which would by the assumption of the theorem show that Φ holds for γ, which gives us a contradiction. So this means that S must be non-empty. Now it has a least element α, but since it's a least element then by construction of the set S, for all elements that are less than α we have that Φ holds, which would imply that it also holds for α which is a contradiction so the theorem must be true. □

Definition 1.1.16. A ***formula operation*** G is a mapping in the meta-language of ZFC defined in the following way. Assume that there is a predicate formula $\Phi(x, y)$, such that for every x, there exists unique y such that $\Phi(x, y)$ is true

$$G(x) = y \text{ for which } \Phi(x, y) \text{ holds}$$

Theorem 1.1.9. ***(Transfinite Recursion)*** Let G be a formula operation, then there exist unique formula operation F defined for all ordinals α such that

$$F(\alpha) = G(F \upharpoonright \alpha)$$

The transfinite recursion theorem is an important tool for defining all sorts of things inside ZFC and the transfinite induction theorem is a tool for proving things about them. It's a generalization (to be able to work with potentially uncountable sets) of the intuitive notion of recursive functions (self referential functions who's output on certain input, depends on the previous outputs and inputs of the same function) in computer science. With this it let's us intuitively be able to utilize the power of recursive computation in order to define objects.

Definition 1.1.17. We call two sets A and B ***equipotent*** if there exists a bijection between them.

Definition 1.1.18. A ***cardinal*** κ is an ordinal which is not equipotent with any ordinal $\beta < \alpha$

Using the axiom of choice one can prove the following theorem (proof can be found in **[3]**)

Theorem 1.1.10. ***(Well-ordering principle)*** Every set can be well-ordered (one can find a well-ordering relation on it).

In fact this is equivalent to the axiom of choice in ZF. Now knowing that, the following is true.

Theorem 1.1.11. Every set X is equipotent to some ordinal.

Proof. Using the well ordering principle, the set X can be well-ordered to $(X, <)$ and then one can use **Theorem 1.1.7** to get a bijection to an ordinal. □

It's immediately clear that every set has unique cardinal with which it's equipotent.

Definition 1.1.19. For a set X we define $|X|$ also denoted as $\mathbf{Card}(X)$ to be the unique cardinal equipotent with it, and we call this the ***cardinality***, ***size*** or ***magnitude*** of X.

Theorem 1.1.12. Every natural number $n \in \omega$ is a cardinal, ω is also a cardinal.

Definition 1.1.20. We give the ordinal ω the alternative name $\aleph_0$ and declare it ***the first infinite cardinal***. All the smaller cardinals we call ***finite***. We also call $\aleph_0$ countable and all cardinals bigger than it ***uncountable***.

Here we are not going to bother dealing with other infinite cardinals other than $\aleph_0$. It has been actually shown that with the axioms of ZFC alone it's impossible to answer even basic questions about these infinite cardinals such as $\aleph_1 \stackrel{?}{=} 2^{\aleph}_0$, known as the "Continuum Hypothesis". We have not defined cardinal exponentiation, but basically $2^{\aleph}_0$ is the cardinality of the power set of $\aleph_0$. Nevertheless set theorists have spent enormous amount of work dealing with those cardinals which we will not even attempt to do here because it's not relevant for the physical content of this thesis.

Definition 1.1.21. For two cardinals κ, μ their ***sum*** can be defined to be

$$\kappa + \mu = |(\{\emptyset\} \times \kappa) \cup (\{\emptyset, \{\emptyset\}\} \times \mu)|$$

Here since κ and μ in general will have non-empty intersection, we are "coloring" them by forming a product with two different singletons, so that we know that they will have an empty union.

Theorem 1.1.13. For every two sets A and B, we have

- $A \subseteq B \Longrightarrow |A| \leq |B|$
- $|A| < |\mathcal{P}(A)|$
- $A \cap B = \emptyset \Longrightarrow |A \cup B| = |A| + |B|$

This is the extent to which set theory will be covered. With the material so far, one can easily take the set of natural numbers that we defined $\omega = \mathbb{N}$ and with the help of recursion and induction define the usual operations of ***addition***, ***subtraction***, ***multiplication*** and ***division*** $+, -, \times, /$ and prove their commonly used basic properties. Furthermore one can see in **[3]** constructions for the set of ***integers*** $\mathbb{Z}$, from there the ***rational numbers*** $\mathbb{Q}$ and finaly the ***reals*** $\mathbb{R}$. In the next sections we already assume that the reader has seen these constructions and is familiar with the basic properties of these objects.

Category Theory

Category theory is mathematical theory that is very useful in abstracting away certain common patterns between different mathematical structures. It has a wide range of applications but it is especially common to use it for describing algebraic structures. Since I have chosen the low-level theory for this thesis to be ZFC set theory, the readers who have some knowledge of Category Theory might wonder how does that work with category theory. The reason for this is that in category theory one might often arrive at the situation where one has to work with objects that are "too big" to be properly defined sets inside ZFC.

On the other hand one might have heard that Category Theory can by itself be used as a foundation for mathematics. For the standard category theory usually taught in introductory textbooks, the notion of a set and a foundation is already assumed and this is not the case. However certainly there are extensions that can be used for formalizing mathematics where the notion of set is defined inside the framework of category theory rather than being imported from outside. Such an example (Elementary Theory Of The Category Of Sets) is discussed in **[4]**.

On the other hand, there are various ways one can solve the problem of "too big" collections inside category theory, so it can be formulated rigorously inside ZFC + other axioms (for example ones describing ***Grothendieck Universes***). A discussion of this can be found in **[5]**.

In this paper it is assumed that the problem of how to formally define category theory inside the foundational set theory has already been solved. Similar to high level programmer using a programming language to develop software knowing that there is a compiler that can somehow translate the code into whatever low-level instruction set architecture the CPU uses one can ignore to certain extent the issue of foundations and just work on high level. As with the programmer analogy there are certain situations where the foundations can be important and generally having some awareness of them is good (just like how the programmer actually needs to have some understanding of the lower level machine language and the hardware microarchitecture in order to create high-performance programs). However in general such issues will be beyond the scope and interest of this thesis.

With all that said, here we summerize the basic notions of category theory that are used throughout this paper.

Definition 1.2.1. A ***category*** $\mathcal{C}$ consists of:

- A set of ***objects*** $\mathbf{Obj}(\mathcal{C})$
- For any two objects $X, Y \in \mathbf{Obj}(\mathcal{C})$ a set $\mathbf{Hom}_{\mathcal{C}}(X, Y)$ of ***morphisms***. For $f \in \mathbf{Hom}_{\mathcal{C}}(X, Y)$ we may also write $F : X \longrightarrow Y$.
- For any object X a special morphism $\mathrm{id}_X \in \mathbf{End}_{\mathcal{C}}(X)$ where $\mathbf{End}_{\mathcal{C}}(X) := \mathbf{Hom}_{\mathcal{C}}(X, X)$.
- For every three objects $X, Y, Z \in \mathcal{C}$, a function

$$\circ : \mathbf{Hom}_{\mathcal{C}}(Z, Y) \times \mathbf{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathbf{Hom}_{\mathcal{C}}(X, Z)$$

  such that:
  For all $X, Y, Z, W \in \mathcal{C}$ and $f \in \mathbf{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathbf{Hom}_{\mathcal{C}}(Y, Z)$ and $h \in \mathbf{Hom}_{\mathcal{C}}(Z, W)$:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

$$f \circ \mathrm{id}_X = f \quad \text{and} \quad \mathrm{id}_Y \circ f = f$$

Definition 1.2.2. In a category $\mathcal{C}$ a morphism $f \in \mathbf{Hom}_{\mathcal{C}}(X, Y)$ is called an ***isomorphism*** if there exists a morphism $f^{-1} \in \mathbf{Hom}_{\mathcal{C}}(Y, X)$ such that:

$$f^{-1} \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ f^{-1} = \mathrm{id}_Y$$

If $f \in \mathbf{Hom}_{\mathcal{C}}(X, Y)$ is isomorphism then we write $X \cong Y$.

Definition 1.2.3. A ***functor*** $\mathcal{F}$ from a category $\mathcal{C}$ to category $\mathcal{D}$ maps objects of $\mathcal{C}$ to objects of $\mathcal{D}$ and morphisms of $\mathcal{C}$ to morphisms of $\mathcal{D}$ in such a way so that:

$$\forall X \in \mathbf{Obj}(\mathcal{C}) : \quad \mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$$

For each $X, Y, Z \in \mathbf{Obj}(\mathcal{C})$ and each $f \in \mathbf{Hom}_{\mathcal{C}}(Y, Z)$ and $g \in \mathbf{Hom}_{\mathcal{C}}(X, Y)$:

$$\mathcal{F}(f) \in \mathbf{Hom}_{\mathcal{C}}(\mathcal{F}(Y), \mathcal{F}(Z))$$

$$\mathcal{F}(g) \in \mathbf{Hom}_{\mathcal{C}}(\mathcal{F}(X), \mathcal{F}(Y))$$

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$

Definition 1.2.4. A ***contravariant functor*** $\mathcal{F}$ from a category $\mathcal{C}$ to category $\mathcal{D}$ maps objects of $\mathcal{C}$ to objects of $\mathcal{D}$ and morphisms of $\mathcal{C}$ to morphisms of $\mathcal{D}$ in such a way so that:

$$\forall X \in \mathbf{Obj}(\mathcal{C}) : \quad \mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$$

For each $X, Y, Z \in \mathbf{Obj}(\mathcal{C})$ and each $f \in \mathbf{Hom}_{\mathcal{C}}(Y, Z)$ and $g \in \mathbf{Hom}_{\mathcal{C}}(X, Y)$:

$$\mathcal{F}(f) \in \mathbf{Hom}_{\mathcal{C}}(\mathcal{F}(Z), \mathcal{F}(Y))$$

$$\mathcal{F}(g) \in \mathbf{Hom}_{\mathcal{C}}(\mathcal{F}(Y), \mathcal{F}(X))$$

$$\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$$

Definition 1.2.5. A functor $\mathcal{F}$ between the categories $\mathcal{C}$ and $\mathcal{D}$ is ***faithful*** if the map

$$\mathcal{F}_{XY} : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$$

$$\mathcal{F}_{XY}(f) := \mathcal{F}(f)$$

is injective. Such functor is called ***full*** if the map is surjective.

Topology

Topology is another important subject that's present in practically all physical theories. Topology is an abstract study of the intuitive notions of "continuity" and "closeness" (in a sense an abstract notion of things being close without actually involving any concrete concepts of distance). The main reference for this chapter can be found in **[6]**. The most important definitions and results are given here (mostly without proofs).

Definition 1.3.1. Given a set X, a system of sets $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a ***topology*** on X if it has the following properties

- $\emptyset, x \in \mathcal{T}$
- $\forall \mathcal{S} \subset \mathcal{T} (\mathcal{S} \neq \emptyset \implies \bigcup \mathcal{S} \in T)$
- $\forall O_0, O_1 \in \mathcal{T} (O_0 \cap O_1 \in \mathcal{T})$

The members of $\mathcal{T}$ are called ***$\mathcal{T}$-open*** sets, or when no confusion arise to what the set $\mathcal{T}$ is in the context, we call them just ***open*** sets. If for some set C, we have that $X \setminus C \in \mathcal{T}$ we call the set C a ***$\mathcal{T}$-closed*** set or just ***closed*** if X and $\mathcal{T}$ are evident form the context.

Definition 1.3.2. A pair $(X, \mathcal{T})$ of set X and a topology on $\mathcal{T}$ is called ***topological space***.

Definition 1.3.3. For a topological space $(X, \mathcal{T})$, we call a system of sets $\mathcal{B} \subseteq \mathcal{T}$ a ***base for*** $\mathcal{T}$ if the set of all unions of members of $\mathcal{B}$ is equal to $\mathcal{T}$. It's also common to say that $\mathcal{B}$ generates $\mathcal{T}$.

Definition 1.3.4. We call a topological space $(X, \mathcal{T})$ ***second countable*** if there exists countable base for it.

Here one might wonder why the concept of "second" countability is introduced before the first. The reason is that the concept of "first" countability can be defined by using the notion of neighbourhood systems which we have not defined yet. On top of that we are not going to focus on this notion in the rest of this paper.

Definition 1.3.5. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{C}$ be the system of all closed sets of the topology.

The relation $\mathcal{N}_{(X,\mathcal{T})} \subseteq X \times \mathcal{P}(X)$ defined by

$$(x, U) \in \mathcal{N}_{(X,\mathcal{T})} \iff \exists O \in \mathcal{T}(x \in O \subset U)$$

is called the ***neighborhood system*** of $(X, \mathcal{T})$ From this the ***open neighborhood system*** and ***closed neighborhood system*** of $(X, \mathcal{T})$ are respectively defined by

$$\mathcal{N}^{\text{open}}_{(X,\mathcal{T})} := \mathcal{N}_{(X,\mathcal{T})} \cap (X \times \mathcal{T})$$

$$\mathcal{N}^{\text{closed}}_{(X,\mathcal{T})} := \mathcal{N}_{(X,\mathcal{T})} \cap (X \times \mathcal{C})$$

Definition 1.3.6. Let $\mathcal{N}_{(X,\mathcal{T})}$ be a neighborhood system, then for a set $A \subseteq X$ we call $\mathcal{N}_{(X,\mathcal{T})}A := \mathbf{Range}\left(\mathcal{N}_{(\mathrm{X},\mathcal{T})} \upharpoonright \mathrm{A}\right)$ the ***neighborhood system of*** A and we call its members ***neighborhoods*** of A. If here $\mathcal{N}_{(X,\mathcal{T})}$ was replaced with the open or closed versions then it would respectively be called the ***open/closed neighborhood system of*** A. Furthermore for a point $x \in X$, we will call the ***neighborhood system of*** x, the neighborhood system of $\{x\}$ and respectively call it's members ***neighborhoods*** of x. We similarly make open/closed versions of this final definition for singletons.

Now we give two definitions that will help us avoid being too overly verbose when defining the next few concepts related to sequences.

Definition 1.3.7. Let X be a set, $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X and Φ a formula. We say that $\Phi(x_n)$ is ***eventually true*** or ***eventually*** $\Phi(x_n)$ for short if there exists $N \in \mathbb{N}$, such that $\Phi(x_n)$ is true for all $n \geq N$.

Definition 1.3.8. Let X be a set, $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X and Φ a formula. We say that $\Phi(x_n)$ is ***frequently true*** or ***frequently*** $\Phi(x_n)$ for short if for every $N \in \mathbb{N}$, there is $N \leq n \in \mathbb{N}$ such that $\Phi(x_n)$ is true.

Definition 1.3.9. Let $(X, \mathcal{T})$ be a topological space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X. A point $x \in X$ is called ***limit point of*** the sequence if $x_n \in U$ for every $U \in \mathcal{N}_{(X,\mathcal{T})}\{x\}$ eventually.

In this case we also say that $\{x_n\}_{n\in\mathbb{N}}$ ***converges*** to x and write $x_n \longrightarrow x$.

The set of all limit points is denoted with $\lim\limits_{n\to\infty} x_n$. If the sequence has a limit point we call it ***convergent***.

If $\lim\limits_{n\to\infty} x_n = \{x\}$, then we also write $\lim\limits_{n\to\infty} x_n = x$ (this should not cause confusion in a given context where $x \in X$).

Definition 1.3.10. Let $(X, \mathcal{T})$ be a topological space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X. A point $x \in X$ is called ***adherence point of*** the sequence if $x_n \in U$ for every $U \in \mathcal{N}_{(X,\mathcal{T})}\{x\}$ frequently.

The set of all adherence points is denoted with $\operatorname*{adh}\limits_{n\to\infty} x_n$.

If $\operatorname*{adh}\limits_{n\to\infty} x_n = \{x\}$, then we also write $\operatorname*{adh}\limits_{n\to\infty} x_n = x$ (this should not cause confusion in a given context where $x \in X$).

For a topological space $(X, \mathcal{T})$, and a point x, we will use the notation

$$\mathcal{T}[x] := \{U_x \in \mathcal{T} \mid x \in U_x\}$$

Given this notation we can now give the following definition

Definition 1.3.11. A function $f : X \longrightarrow Y$ between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is said to be ***continuous at*** $x \in X$ if

$$\forall U_{f(x)} \in \mathcal{T}_Y[f(x)] : \quad f^{-1}\left(U_{f(x)}\right) \in \mathcal{T}_X[x]$$

Definition 1.3.12. A function $f : X \longrightarrow Y$ between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is said to be ***continuous*** if it's continuous at any point of X.

Continuity is a core notion in topology, hence the study of continuous functions is very important. As seen by the definition continuous functions preserve the pre-images of open sets. Now we give a stronger notion.

Definition 1.3.13. A function $f : X \longrightarrow Y$ between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is said to be ***homeomorphism*** if it's bijective and both f and f^{-1} are continuous. In this case we say that the two spaces are ***homeomorphic***.

In this case since homemorphisms are with continuous inverses, they bidirectionally preserve open sense. Since there is one to one correspondence between both points and open sets from the source to the target topological spaces under homeomorphisms, intuitively this means that those spaces are completely equivalent for the purposes of topology (as if we've just renamed their building blocks).

Definition 1.3.14. A function $f : X \longrightarrow Y$ between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is said to be ***homeomorphic embedding*** (or simply ***embedding***) if $f \upharpoonright \mathbf{Range}(\mathrm{f})$ is homeomorphism.

We can think of embeddings as identifying a topological space with a "subspace" of a larger space. To give definition of subspace we use the following result.

Theorem 1.3.1. If $(X, \mathcal{T})$ is topological space and $A \subseteq X$, then the pair

$$(A, \mathcal{T} \upharpoonright A) := (A, \{O \in \mathcal{P}(A) | \ \exists U \in \mathcal{T}(O = U \cap A)\})$$

is also a topological space.

Now that we can restrict the topology to subsets of the original topological space, we can justify the following definition.

Definition 1.3.15. For topological space $(X, \mathcal{T})$ and a subset $A \subseteq X$ we call the topology $\mathcal{T} \restriction A$ ***the relative topology on*** $A$and the space $(A, \mathcal{T} \restriction A)$ a ***topological subspace*** of $(X, \mathcal{T})$.

Now we focus on some results which we state without proof, but proofs can be found in **[6]**.

Theorem 1.3.2. For a function $f : X \longrightarrow Y$ between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, the following notions are equivalent:

- f is continuous at x
- $\forall U_{f(x)} \in \mathcal{N}_{(Y,\mathcal{T}_Y)}\{f(x)\} : \quad U_{f(x)} \in \mathcal{N}_{(X,\mathcal{T}_X)}\{x\}$

Theorem 1.3.3. For a set of topological spaces $\{(X_i, \mathcal{T}_i)\}_{i=0}^{2}$ and $\{f_i : X_i \longrightarrow X_{i+1}\}_{i=0}^{1}$ a set of maps continuous at $x_0 \in X_0$ and $f(x_0) \in X_1$ respectively. The map $f_1 \circ f_0$ is continuous at x.

Definition 1.3.16. If $(X, \mathcal{T})$ is topological space, $\mathcal{C}_{\mathcal{T}}$ the set of all closed sets and $A \subseteq X$ then the ***closure of*** A is the set

$$\mathcal{cl}_{\mathcal{T}}(A) := \bigcap\{C \in \mathcal{C}_{\mathcal{T}} \mid A \subset C\}$$

Definition 1.3.17. If $(X, \mathcal{T})$ is topological space and $A \subseteq X$ then the set A is called ***dense in*** X if $\mathcal{cl}_{\mathcal{T}}(A) = X$.

Definition 1.3.18. A topological space is called ***separable*** if it contains a countable subset that is dense in it.

Now let's give a brief look at the topology that can be induced on spaces on which the concept of distance (or metric) can be defined.

Definition 1.3.19. Let X be a set, a ***pseudo metric*** on X is a map

$$d : X \times X \longrightarrow R_+$$

for which

- $d(x, x) = 0$ for all $x \in X$
- $d(x, y) = d(y, x)$ for all $x, y \in X$
- $d(x, z) < d(x, y) + d(y, z)$ for all $x, y, z \in X$

If on top of that it's true that

- $x = y \iff d(x, y) = 0$ for all $x, y \in X$

then d is called a ***metric***. The pair (X, d) is called a ***metric space***

If d is a pseudo-metric on X, then the following sets, called ***open spheres*** around x with radius r

$$B_{r(x)} := \{y \in X \mid d(x, y) < r\}$$

generate topological base on (X, d) which turns it into topological space. The induced topology is called ***pseudo-metric topology*** on (X, d) or ***metric topology*** if d is a metric **[6]**.

Definition 1.3.20. Let $\{x\}_{k \in \mathbb{N}}$ is a sequence in a metric space (X, d). The sequence is called ***Cauchy sequence*** if for every positive real number ε, there is $n \in \mathbb{N}$ such that for $i, j > n$, we have

$$x_i \in B_\varepsilon(x_j)$$

Definition 1.3.21. A metric space (X, d) is called ***complete*** if every Cauchy sequence is convergent.

Algebra

As it can be seen from the title of this paper, algebra is probably one of the key subjects. The study of algebraic structures is a vast field and it's hard to cover even a small part of that in this document. An introduction to this subject can be seen in this free book published under the GNU Open Document License **[7]**. A more complete reference is **[8]**.

Here we will summarize the definitions of the basic structures that we are going to use.

Definition 1.4.1. A ***binary operation*** on a set S is a function $\triangle$

$$\triangle : S \times S \longrightarrow S$$

Often for every $s_0, s_1 \in S$ the infix notation $s_0 \triangle s_1$ is used to mean $\triangle(s_0, s_1)$.

Definition 1.4.2. A binary operation $\triangle$ on S is said to be ***associative*** when

$$\forall s_0, s_1, s_3 \in S : \qquad s_0 \triangle (s_1 \triangle s_2) = (s_0 \triangle s_1) \triangle s_2$$

Definition 1.4.3. A binary operation $\triangle$ on S is said to be ***commutative*** when

$$\forall s_0, s_1 \in S : \qquad s_0 \triangle s_1 = s_1 \triangle s_0$$

Definition 1.4.4. A binary operation $\triangle$ on S is said to have an ***identity*** element e if

$$\forall s \in S : \qquad s \triangle e = e \triangle s = s$$

Often we might notate an identity of a binary operation on S with 1_S (or 0_S depending on the context) and if no confusion occurs with the set in which this element is in, then the index might be omitted.

Definition 1.4.5. An element $s \in S$ is said to be ***invertible*** under the operation $\triangle$ on S with identity element e if

$$\exists s^{-1} \in S : \qquad s \triangle s^{-1} = s^{-1} \triangle s = e$$

Definition 1.4.6. A binary operation $\otimes$ on a set S is said to be ***distributive*** over another binary operation $\oplus$ on the same set S if

$$\forall s_0, s_1, s_2 \in S : \qquad s_0 \otimes (s_1 \oplus s_2) = s_0 \otimes s_1 \oplus s_0 \otimes s_2$$

$$\forall s_0, s_1, s_2 \in S : \qquad (s_1 \oplus s_2) \otimes s_0 = s_1 \otimes s_0 \oplus s_2 \otimes s_0$$

Definition 1.4.7. A ***magma*** is a structure consisting of a set and a binary operation on that set $(S, \triangle)$.

Definition 1.4.8. A ***semigroup*** is a magma in which the operation is associative .

Definition 1.4.9. A ***monoid*** is a semigroup in which the operation has identity element.

Definition 1.4.10. A ***group*** is a monoid in which every element is invertible.

Definition 1.4.11. A group is said to be ***abelian*** or ***commutative*** if the operation is commutative.

Definition 1.4.12. A ***ring*** is set together with two binary operations on the set $(S, \otimes, \oplus)$ such that it is group with respect to $\oplus$ and a monoid with respect to $\otimes$, $\otimes$ is distributive with respect to $\oplus$.

The group-like operation is often called "addition" and the monoid-like is called "multiplication" or "product". Often times depending on the context the symbol for the multiplication operation can be omitted.

Definition 1.4.13. A ring is a ***commutative*** if its group inducing operation is commutative.

Definition 1.4.14. A ***field*** is a commutative ring $(S, \otimes, \oplus)$ where every element which is not the identity of $\oplus$ is invertible under $\otimes$ and the identities of $\oplus$ and $\otimes$ are not the same.

Definition 1.4.15. If $(R, \otimes_R, \oplus_R)$ is a ring with $\otimes_R$ identity e_R and $(M, \oplus_M)$ is a group, then the function

$$\underset{R\to M}{\otimes} : R \times M \longrightarrow M$$

satisfying for all $r, s \in R$ and $x, y \in M$

$$r \underset{R\to M}{\otimes} (x \oplus_M y) = r \underset{R\to M}{\otimes} x \oplus_M r \underset{R\to M}{\otimes} y$$

$$(r \oplus_R s) \underset{R\to M}{\otimes} x = r \underset{R\to M}{\otimes} x \oplus_M s \underset{R\to M}{\otimes} x$$

$$(r \otimes_R s) \underset{R\to M}{\otimes} x = r \underset{R\to M}{\otimes} \left(s \underset{R\to M}{\otimes} x \right)$$

$$e_R \underset{R\to M}{\otimes} x = x$$

is called ***ring action*** of this ring on the group.

Definition 1.4.16. A ***module*** is a structure consisting of a commutative group a ring and ring action of the ring on the group.

Often times the ring action is called "scalar multiplication".

In all of these cases depending on the context we might say things like S is a group R is a ring if the operations on them that make them into this structure are clear in the context. On top of that we might use the same symbols for operations on different sets in different structures, similar to "overloading" in programming languages. Most of the times "multiplicative" operations are not written at all and the symbols are just concatenated together for briefness.

Definition 1.4.17. A ***vector space*** is a module in which the ring happens to be a field.

In this case the field members are said to be "scalars" and the abelian group members are said to be "vectors".

We often say "V is a vector space over the field F" to mean that in the context there are some operations that make F a field, there is an operation that we call ***vector addition*** that makes V into abelian group and we have a ring (in this case a field) action from F to V which completes the structure of vector space. In all those cases we save a lot of words by these brief expressions. We also use the symbol 0_V for the identity element of the abelian group set and 0_F for the identity of the "additive" operation of F and 1_F for the "multiplicative" operation on F. The multiplicative operation is also called ***scalar multiplication***.

Definition 1.4.18. A set $S \subseteq V$ of vectors from a vector space V over a field F is said to be ***linearly independent*** if for every $k < \aleph_0$ and subsets $\{v_0, ..., v_k\} \subset V$ and $\{\alpha_0, ..., \alpha_k\} \subset F$ for which

$$\sum_{i=0}^{k} \alpha_i v_i = 0_V$$

we have

$$\{\alpha_0, ..., \alpha_k\} = \{0_F\}$$

Definition 1.4.19. A ***(Hamel) basis*** of a vector space V over the field F is a set $B \subseteq V$ such that :

- For every vector $v \in V$ one can find $k < \aleph_0$ and the subsets $\{v_0, ..., v_k\} \subset V$ and $\{\alpha_0, ..., \alpha_k\} \subset F$ such that

$$v = \sum_{i=0}^{k} \alpha_i v_i$$

- The set B is linearly independent.

Theorem 1.4.1. Every vector space has a basis.

Proof sketch. Using a choice function and the transfinite recursion theorem from ZFC (or alternatively Zorn's lemma) construct a maximal linearly independent set. Now assume that some vector outside that set is not expressible as finite linear combination from the set which would imply that the said vector is linearly independent from the others which contradicts the maximality. □

Now it's time to deal with structure preserving maps between algebraic objects.

Definition 1.4.20. Let $(A, \triangle_0, \dots \triangle_m, a_0, \dots, a_k)$ and $(A', \triangle'_0, \dots \triangle'_m, a'_0, \dots, a'_k)$ be two algebraic structures of the same kind, with sets A and A', operations $\triangle_i$ and $\triangle'_i$ and special elements ***constants*** a_i and a'_i.

A function $f : A \longrightarrow A'$ is called a ***homomorphism*** between structures of that kind if

$$\forall i \in m : \quad f\Big(\triangle_i\Big(x_0, \dots, x_{t_i}\Big)\Big) = \triangle'_i\Big(f(x_0), \dots, f\Big(x_{t_i}\Big)\Big)$$

$$\forall i \in k : \quad f(a_i) = a'_i$$

If f is bijective it's called ***isomorphism*** between those structures. If these algebraic structures are isomorphic we often write $A \cong A'$.

In this definition by constants it's meant special objects like identities, etc. In the case of vector spaces the homomorphism between it's underlying Abelian group that preserve scalar multiplication are called ***linear maps***.

Definition 1.4.21. Let A and A' be a sets that alongside some operation have been made into an algebraic structures which have identity elements 0_A and $0_{A'}$. Let $f : A \longrightarrow A'$ is a function. We call the ***kernel of the homomorphism*** the preimage of $0_{A'}$ under f

$$\ker f := f^{-1}(0_{A'})$$

The ***support of*** f is the set

$$\sup f := A \setminus \ker f$$

In the case of vector spaces when we talk about kernels and supports we mean with respect to the identity of the additive group (the zero vector).

Definition 1.4.22. ***Topological vector space*** is a vector space equipped with topology in which the addition and scalar multiplication are both continuous functions.

In most cases, the topology will not be explicitly mentioned and will be clear from the context.

Definition 1.4.23. An absolute value on a field $\mathbb{F}$ is a function

$$|\cdot| : \mathbb{F} \longrightarrow \mathbb{R}_+$$

such that

- $|x| = 0 \iff x = 0_{\mathbb{F}}$
- $|x + y| \leq |x| + |y|$
- $|xy| = |x|\ |y|$

for all $x, y \in \mathbb{F}$.

Definition 1.4.24. Let V be a vector space over the field $\mathbb{F}$ with absolute value $|\cdot|$. A ***seminorm*** on V is a function

$$\|\cdot\| : V \longrightarrow \mathbb{R}$$

such that

- (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$
- (Homogeneity) $\|\lambda x\| = |\lambda|\ \|x\|$

for all $x, y \in V$.

Definition 1.4.25. A ***norm*** on a vector space V is a seminorm $\|\cdot\|$ for which

$$\|x\| = 0 \quad \Longrightarrow \quad x = 0_V$$

for all $x \in V$.

A norm on a vector space induces a distance defined by

$$d(x, y) := \|x - y\|$$

Now using the distance topology, the vector space becomes topological vector space. We say that vector spaces with a norm which makes them a topological vector space are ***normed***.

Definition 1.4.26. In a vector space V over $\mathbb{F}$, a set $C \subseteq V$ is said to be ***convex*** if for all $x, y \in C$ and all $0_{\mathbb{F}} \leq t \leq 1_{\mathbb{F}}$ we have

$$tx + (1_{\mathbb{F}} - t)y \in C$$

Definition 1.4.27. A ***locally convex space*** is a topological vector space for which there is a local neighborhood basis around the zero vector which is made out of convex sets.

Definition 1.4.28. A complete, metrizable, locally convex topological vector space is called a ***Frechet space***.

One can show that a seminorm p induces a topology by system of neighborhoods of around the zero vector generated by the sets $\{x \in \mathbb{V} \mid | \; p(x) \leq \alpha\}_{\alpha \in \mathbb{R}_+}$ **[9]**.

Definition 1.4.29. A topology on a vector space is said to be ***generated by a set of seminorms*** $\{p\}_{\alpha \in A}$ if the open sets in the topology are generated by finite intersections of open sets in the topologies induced by each seminorm.

Theorem 1.4.2. A topological vector space has its topology defined by countable family of seminorms, it's Hausdorff and it's complete with respect to the seminorms, then it's Frechet.

A proof of the statements that lead to this fact can be found in **[9]** (local convexity is equivalent to topology induced by set of seminorms, Hausdorff and generated by countable set of seminorms implies metrizability).

Definition 1.4.30. A complete normed topological vector space is called ***Banach space***.

Clearly every Banach space is also a Frechet space.

Definition 1.4.31. If V_0, V_1, V_2 are vector spaces, a map $V_0 \times V_1 \longrightarrow V_2$ is called ***bilinear*** if fixing each argument induces a linear map for the other argument. We denote the space of all bilinear maps from $V_0 \times V_1$ to V_2 by

$$\mathbf{BiLin}(V_0 \times V_1, V_2)$$

Definition 1.4.32. A vector space A equipped with an associative bilinear map from $\mathbf{BiLin}(A \times A, A)$ is called an ***algebra***.

We often use the bilinear map (which also by our definition is a binary operation) "multiplication" and don't use functional notation but instead write the bilinear map on two elements a and b by concatenating them together like ab.

Definition 1.4.33. A an algebra for which the multiplication has an identity element is called unitial.

Definition 1.4.34. ***Banach algebra*** is an algebra which is also a Banach Space.

Definition 1.4.35. Let $\mathbb{F}$ be a ring with unit $\mathbb{1}_{\mathbb{F}}$. An ***involution for the field*** is a function $* \ \mathbb{F} \longrightarrow \mathbb{F}$ such that

- $(x + y)^* = x^* + y^*$
- $(xy)^* = y^* + x^*$
- $\mathbb{1}_{\mathbb{F}}^* = \mathbb{1}_{\mathbb{F}}$
- $x^{**} = x$

An example of such operation is the conjugation on the field of the complex numbers $\mathbb{C}$.

Definition 1.4.36. Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$ with involution $*_{\mathbb{F}}$. A ***involution on the algebra*** is a map $*_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ such that

- $(x+y)^{*_{\mathcal{A}}} = x^{*_{\mathcal{A}}} + y^{*_{\mathcal{A}}}$
- $(xy)^{*_{\mathcal{A}}} = y^{*_{\mathcal{A}}} + x^{*_{\mathcal{A}}}$
- $x^{*_{\mathcal{A}}*_{\mathcal{A}}} = x$
- $(\lambda x)^{*_{\mathcal{A}}} = \lambda^{*_{\mathbb{F}}} x^{*_{\mathcal{A}}}$

for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{F}$.

Definition 1.4.37. An algebra together with involution is called a ***$*$-algebra***.

Definition 1.4.38. Let $\mathcal{A}$ be a $*$-algebra.

- The element a^* is called ***the adjoint*** of a.
- An element is called positive if it's from the type $a * a$.
- An element a is called self adjoint if $a^* = a$.

Definition 1.4.39. A Banach Algebra $\mathcal{A}$ that is also $*$-algebra is called C^*-algebra if for every $a \in \mathcal{A}$

$$\|a^* a\| = \|a\|^2$$

Theorem 1.4.3. Let $\mathcal{A}$ be a C^* algebra, and $a \in \mathcal{A}$ is a self adjoint element, then a can be expressed as a difference of two orthogonal positive elements

$$a = a_+ - a_-$$

The full proof of this theorem can be found at **[10]**. It will be important later when dealing with states in AQFT.

Definition 1.4.40. Let V be a vector space over field with involution $\mathbb{F}$. A ***positive semi-definite Hermitian product*** is a map

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{F}$$

such that
- $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
- $\langle x, y \rangle = \langle y, x \rangle^*$
- $\langle x, x \rangle \geq 0$

for all $x, y, z \in V$ and all $\alpha, \beta \in \mathbb{F}$.

If $\langle x, x \rangle = 0 \Longrightarrow x = 0_V$ on top of that then the map is called ***inner product***. A vector space with an inner product is called ***inner product space***.

Inner products induce a norm defined by

$$\|x\| := \sqrt{\langle x, x \rangle}$$

It's easy to check that this is a norm (proof in **[11]**). Hence inner product spaces can be regarded as normed vector spaces.

Definition 1.4.41. For a vector space V over the field $\mathbb{F}$, a ***symplectic form*** is a bilinear map ω on V for which:
- $\forall v \in V : \quad \omega(v, v) = 0_V$
- $((\forall v \in V)\ \omega(v, u) = 0) \Longrightarrow u = 0$

The pair (V, ω) is called a ***symplectic vector space***.

Symplectic vector spaces will be important later for defining quantum field theories.

Definition 1.4.42. A complete (with regards to the metric induced by the cannonical norm induced by the inner product) inner product vector space is called a ***Hilbert space***.

Every Hilbert space is also a Banach space by definition.

Hilbert spaces are important because as can be seen in later chapters they can be used to mathematically model the state-spaces of quantum systems.

Definition 1.4.43. Let E and F be vector spaces over a filed $\mathbb{F}$. The pair (V, ι), of vector space V over $\mathbb{F}$ and a bilinear map ι is called ***a tensor product of*** E and F if whenever φ is a bilinear map from $E \times F$ into some $\mathbb{F}$ vector space U then there is a unique linear $\hat{\varphi}$ such that the following diagram commutes

$$\begin{tikzcd} E \times F \arrow[r, "\varphi"] \arrow[d, "\iota"'] & U \\ V \arrow[ur, dashed, "\hat{\varphi}"'] & \end{tikzcd}$$

Theorem 1.4.4. If E and F are vector spaces over $\mathbb{F}$, then a tensor product of them exists and is unique up to a canonical isomorphism, meaning if (V_0, ι_0) and (V_1, ι_1) are two tensor products of them, then there is a unique linear isomorphism $\tau : V_1 \longrightarrow V_0$ such that $\tau\iota_1 = \iota_0$. Any tensor product (V, ι) of E and F is spanned by the image of $E \times F$ under ι.

The proof of this exact theorem can be found in **[8]**. This leads us to the following pseudo-definition.

Definition 1.4.44. For vector spaces E and F over $\mathbb{F}$, by $E \otimes F$ we will denote the vector space from any (might be a particular one depending on the context) tensor product (V, ι) with $E \otimes F := V$ and by $e \otimes f$ we will denote the elements $\iota(e, f)$ for $e \in E$ and $f \in F$, such elements will be called ***pure tensors***.

Definition 1.4.45. We denote the space of linear maps from vector space V to a vector space V', both over the field $\mathbb{F}$ by $\mathbf{Hom}_{\mathbb{F}}(V, V')$.

Here we are overloading the notation from category theory. Usually in the categorical sense if that was the category (for example of vector spaces), then the subscript would be the name of the category itself. However here we just denote the field under which the vector spaces are defined. It's easy to check that $\mathbf{Hom}_{\mathbb{F}}(V, V')$ forms a vector space over $\mathbb{F}$ by the operations

$$f + g := \left\{(x, f(x) + g(x)) \mid x \in \mathbf{Domain}(\mathrm{f}_0) \bigwedge x \in \mathbf{Domain}(\mathrm{f}_1)\right\}$$

$$\lambda f := \{(x, \lambda f(x)) \mid x \in \mathbf{Domain}(\mathrm{f})\}$$

for all $f, g \in \mathbf{Hom}_{\mathbb{F}}(V, V')$ and $\lambda \in \mathbb{F}$. Similar thing can be done for $\mathbf{BiLin}(V \times V', V'')$ (the functions with two arguments are simply functions taking pairs in our definitions in the set theory chapter, so this is essentially the same definition of function addition and scalar multiplication). With those operations it also can be regarded as a vector space and this gives us the ability to state the following theorem.

Theorem 1.4.5. If E, F and V are vector spaces over $\mathbb{F}$, the following isomorphisms exist

$$\mathbf{Hom}_{\mathbb{F}}(E \otimes F, V) \cong \mathbf{BiLin}(E \times F, V)$$

$$\mathbf{Hom}_{\mathbb{F}}(E \otimes F, V) \cong \mathbf{Hom}_{\mathbb{F}}(E, \mathbf{Hom}_{\mathbb{F}}(F, V))$$

The proof of this theorem can be found at **[8]**.

The utility from tensor products comes from the fact that they can encode bilinear (and multilinear if tensor product is done inductively more than once) relationships between vector spaces in a very convenient way. By the defining property of them they can "split" a bilinear relationship into a composition of a trivial bilinear part and a linear part.

Definition 1.4.46. If V is a vector space over a field $\mathbb{F}$, then we define ***the dual space of*** V by V^* to be $\mathbf{Hom}_{\mathbb{F}}(V, \mathbb{F})$. We often call it's elements ***linear functionals***.

Theorem 1.4.6. For a finitely dimensional vector space V with basis $\{v_i\}_{i=0}^k$, the set of functionals

$$\hat{v}_i : V \longrightarrow \mathbb{F}$$

$$\hat{v}_i\left(v_j\right) := \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

forms a basis for V^*. This basis is often called ***the dual basis*** of $\{v_i\}_{i=0}^k$.

Theorem 1.4.7. A finite dimensional vector space V is isomorphic to V^{**}.

One can again see the proof of this in **[8]**.

Theorem 1.4.8. For a finite dimensional vector spaces E and F we have

$$(E \otimes F)^* \cong E^* \otimes F^*$$

Proof sketch. Since $E^* \times F^*$ is spanned by pure tensors one needs to define linear maps on it only on them (they are uniquely extended by linearity). For two functionals $\hat{e} \in E^*$ and $\hat{f} \in \mathbb{F}$ one defines the map

$$I : E^* \otimes F^* \longrightarrow (E \times F)^*$$

$$\left(I\left(\hat{e} \otimes \hat{f}\right)\right)(e \otimes f) := \hat{e}(e)\hat{f}(f)$$

for all $e \in E$, $f \in F$ and $\hat{e} \in E^*$ and $\hat{f} \in F^*$. One then checks that this map is linear isomorphism. $\square$

Now using this property one can show the following isomorphism

Theorem 1.4.9. For finitely dimensional vector spaces E and F over the field $\mathbb{F}$ we have

$$E^* \otimes F \cong \mathbf{Hom}_{\mathbb{F}}(E, F)$$

Proof. Using the previous 3 theorems one has:

$$\begin{gathered}
E^* \otimes F \cong \\
\cong (E^* \otimes F)^{**} = \mathbf{Hom}_{\mathbb{F}}\big((E^* \otimes F)^*, \mathbb{F}\big) \cong \mathbf{Hom}_{\mathbb{F}}(E^{**} \otimes F^*, \mathbb{F}) \cong \\
\cong \mathbf{Hom}_{\mathbb{F}}(E, \mathbf{Hom}_{\mathbb{F}}(F^*, \mathbb{F})) = \mathbf{Hom}_{\mathbb{F}}(E, F^{**}) \cong \\
\cong \mathbf{Hom}_{\mathbb{F}}(E, F)
\end{gathered}$$

□

Now it's time for another important structure on vector spaces capturing anti-symmetric multilinear relationships to be introduced, the so-called exterior product.

Definition 1.4.47. A bilinear map $\omega \in \mathbf{BiLin}(E \times E, V)$ is called ***alternating*** if

$$\omega(e, e) = 0_V$$

Definition 1.4.48. Let E be a vector space over a field $\mathbb{F}$. We say that a pair of a vector space and alternating bilinear map (V, ι) is ***an exterior product of*** E if for every alternating bilinear map ω, there exists unique linear map $\hat{\omega}$ for which the following diagram commutes

$$\begin{tikzcd}
E \times F \arrow[r, "\omega"] \arrow[d, "\iota"'] & U \\
V \arrow[ur, dashed, "\hat{\omega}"'] &
\end{tikzcd}$$

Theorem 1.4.10. If E is a vector space over $\mathbb{F}$, then an exterior product of it exists and is unique up to a canonical isomorphism, meaning if (V_0, ι_0) and (V_1, ι_1) are two exterior products of them, then there is a unique linear isomorphism $\tau : V_1 \longrightarrow V_0$ such that $\tau \circ \iota_1 = \iota_0$. Any exterior product (V, ι) of E is spanned by the image of $E \times E$ under ι.

Definition 1.4.49. For vector space E over $\mathbb{F}$, by $E \wedge E$ we will denote the vector space from any (might be a particular one depending on the context) exterior product (V, ι) with $E \wedge E := V$ and by $e_0 \wedge e_1$ we will denote the elements $\iota(e_0, e_1)$ for $e_0, e_1 \in E$.

Theorem 1.4.11. Let E and F be vector spaces over $\mathbb{F}$, then

$$\mathbf{Hom}_{\mathbb{F}}(E \wedge E, F) \cong$$
$$\cong \{\omega \in \mathbf{BiLin}(E \times E, F) \mid \forall e \in E : \quad \omega(e, e) = 0_F\}$$

For the proof of these theorems the reader of this thesis is advised to consult **[8]**. There one can also read on more of the properties of tensor and exterior products as well as inductively defining multiple products and powers, that we will denote by $\bigotimes^k E$ and $\bigwedge^k E$.

Measure Theory

Measure theory is a mathematical framework that abstracts the intuitive notions of “size” or “volume” or “averaging over things”. As such is central to other fields of mathematics such as Analysis and Probability Theory.

Definition 1.5.1. A ***σ-algebra*** on set X is a set $\mathcal{X} \subset \mathcal{P}(X)$ such that

- $\emptyset, X \in \mathcal{X}$
- If $A \in \mathcal{X}$ then $\mathcal{X} \setminus A \in \mathcal{X}$.
- If $(\forall n \in \mathbb{N}) A_n \in \mathcal{X}$ then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{X}$.

A pair $(X, \mathcal{X})$ of a set and a σ-algebra on it is called a ***measurable space***.

Definition 1.5.2. Let X be a topological space. The ***Borel σ-algebra*** of X denoted by $\mathfrak{B}(X)$ is the smallest σ-algebra on X containing the open sets of X.

Definition 1.5.3. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces. A function $f : X \longrightarrow Y$ is called ***measurable*** if

$$\forall E \in \mathcal{Y} : \quad f^{-1}(E) \in \mathcal{X}$$

Let $\overline{\mathbb{R}}$ is the set $\mathbb{R} \cup \{-\infty, +\infty\}$ for which we define

$$-\infty < x < +\infty$$

for all $x \in \mathbb{R}$. The standard topology on $\mathbb{R}$ can be extended by appending the neighbourhoods of infinity $(x, +\infty]$ and $[-\infty, x)$. The standard operations of addition and multiplications can be extended by

$$\pm\infty \cdot x = \pm\infty$$

$$\pm\infty \pm x = \pm\infty$$

$$\pm\infty + \pm\infty = \pm\infty$$

for all $x \in \mathbb{R}$.

Definition 1.5.4. Let $(X, \mathcal{X})$ be a measurable space. We call a function μ a ***measure*** on $(X, \mathcal{X})$ if

- $\mu : \mathcal{X} \longrightarrow \overline{\mathbb{R}}$
- $\mu(\emptyset) = 0$
- $\mu(E) \geq 0$ for all $E \in \mathcal{X}$
- (σ ***additivity***) If for all $n, m \in \mathbb{N}$ we have $E_n, E_m \in \mathcal{X}$ and $E_n \cap E_m = \emptyset$, then

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$$

The triple $(X, \mathcal{X}, \mu)$ will be referred to as a ***measure space***.

Definition 1.5.5. We call the measure $\mu : \mathcal{X} \longrightarrow [0, 1]$ a ***probability measure*** if $\mu(\emptyset) = 0$ and $\mu(X) = 1$.

Theorem 1.5.1. There exists a unique measure λ called ***Lebesgue measure*** on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ such that

$$\lambda((a, b)) = b - a$$

Proof of this fact can be seen in **[12]**.

Definition 1.5.6. A measure μ on $(X, \mathcal{X})$ is called ***counting measure*** if

$$\mu(E) = \begin{cases} \mathbf{Card}(E) & \text{if } \mathbf{Card}(E) < \aleph_0 \\ +\infty & \text{if } \mathbf{Card}(E) \geq \aleph_0 \end{cases}$$

Counting measures exists for every $(X, \mathcal{X})$, to see this, simply define

$$\mu(E) = \begin{cases} \mathbf{Card}(E) & \text{if } \mathbf{Card}(E) < \aleph_0 \\ +\infty & \text{if } \mathbf{Card}(E) \geq \aleph_0 \end{cases}$$

It follows that $\mu(\emptyset) = 0$ $\mu(E) \geq 0$ for all $E \in \mathcal{X}$. Furthermore sigma additivity follows directly from the definition of addition of cardinal numbers.

If $(X, \mathcal{X}, \mu)$ is a measurable space, we will denote the space of all measurable real valued functions by $\mathfrak{M}(X, \mathcal{X})$ and all positive and negative respectively measurable functions by $\mathfrak{M}^+(X, \mathcal{X})$ and $\mathfrak{M}^-(X, \mathcal{X})$.

Definition 1.5.7. A function $f : X \longrightarrow \mathbb{R}$ is called ***simple*** if

$$\mathbf{Card}(\mathbf{Range}(\mathrm{f})) < \aleph_0$$

.

Definition 1.5.8. For a set E, we define the characteristic function of X by

$$\chi_E : E \longrightarrow \{0, 1\}$$

$$\chi_E(x) := \begin{cases} 1 & \text{if} \quad x \in E \\ 0 & \text{if} \quad x \notin E \end{cases}$$

One can write a simple function $f : X \longrightarrow \mathbb{R}$ as

$$f(x) = \sum_{y \in \mathbf{Range}(\mathrm{f})} y \chi_{f^{-1}(\{y\})}(x)$$

Definition 1.5.9. If $f \in \mathfrak{M}(X, \mathcal{X})$ real-valued simple function, then we define ***the Lebesgue integral of*** f with respect to the measure μ as

$$\int_X f d\mu := \int_X f(x) d\mu(x) := \sum_{y \in \mathbf{Range}(\mathrm{f})} y \mu(f^{-1}(\{y\}))$$

Now for a general measurable function $f : X \longrightarrow \mathbb{R}$, we can break it down to two positive parts f^+ and f^- such that $f = f^+ - f^-$ by using

$$f^+(x) := \max(f(x), 0) \quad \text{and} \quad f^-(x) := \max(-f(x), 0)$$

Definition 1.5.10. Let $f \in \mathfrak{M}^+(X, \mathcal{X})$, then ***the Lebesgue integral of*** f with respect to the measure μ is defined as

$$\int_X f d\mu := \sup\left\{ \int_X f_s \, d\mu \,\middle|\, 0 \leq f_s \leq f \ , \ f_s \text{ is simple} \right\}$$

Definition 1.5.11. Let $f \in \mathfrak{M}(X, \mathcal{X})$, then ***the Lebesgue integral of*** f with respect to the measure μ is defined as

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$$

Definition 1.5.12. A measurable real-valued function is called ***integrable*** if it's Lebesgue integral is real. We denote the set of integrable real valued functions on $(X, \mathcal{X})$ by $\mathfrak{I}(X, \mathcal{X})$.

Theorem 1.5.2. If $f, g \in \mathfrak{I}(X, \mathcal{X})$ and $f + g \in \mathfrak{I}(X, \mathcal{X})$, and $\lambda f \in \mathfrak{I}(X, \mathcal{X})$ for $\lambda \in \mathbb{R}$, then

$$\int_X \lambda f d\mu = \lambda \int_X f d\mu$$

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$$

For the proof of this theorem as well as other important properties of the Lebesgue integral one is advised to read **[12]**.

Analysis

It's hard to define what Analysis is, but one of it's main topics of interest is to study the properties of functions on certain kinds of topological vector spaces, by approximating them locally with linear maps, defining measures on them etc… In this chapter we will state some of the basic definitions and results that we are going to utilize throughout this thesis. The first part will consists of definitions of some of the basic operations such as differentiation (mostly in order to introduce the notation used in this thesis) and then we will focus on some specific properties of Hilbert spaces, that will be utilized in the next chapters about quantum mechanics and later about quantum field theory.

Definition 1.6.1. Let E and F be Banach spaces and let O_E be an open subset of E, a continuous function $f : O_E \longrightarrow F$ is said to be ***differentiable at a point*** $x_0 \in O_E$ if there exists a linear mapping $L \in \mathbf{Hom}(E, F)$ such that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - L(x - x_0)\|}{\|x - x_0\|} = 0$$

The mapping L is called the ***derivative at*** x_0 of f.

One can show that if such linear mapping exists then it's unique which would justify the definition. Furthermore this linear map is continuous **[13]**.

Intuitively what the derivative does is to approximate the change of the potentially complicated function (difference between the images of two arbitrary close points on the domain) with a linear function.

Definition 1.6.2. If $f : E \longrightarrow F$ is a function between Banach spaces which is differentiable at every point of an open subset O_E of E, then we say that f is ***differentiable in*** O_E. We denote the map, that at each point of O_E assigns the derivative of f at that point by Df and call it ***the derivative in*** O_E of f.

We can see that the derivative in O_E is linear map valued function $Df : O_E \longrightarrow \mathbf{Hom}(E, F)$

If the space E is one dimensional Banach space then $\mathbf{Hom}(E, F)$ is isomorphic to F. In this case the derivative in O_E will be F valued after applying this isomorphism (by abuse of notation we won't make distinction between the two). In fact if $E = \mathbb{R}$ then one can show that the derivative at x_0 coincides with the limit

$$\left.\frac{d}{dx}\right|_{x=x_0} f(x) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

For proofs and further details of the above **[13]**.

Now if we have a map $f : E_0 \times E_1 \longrightarrow F$ where E_0, E_1 and F are Banach spaces then at a point $(e_0, e_1) \in E_0 \times E_1$ one can consider the mappings

$$f(\cdot, e_1) : E_0 \longrightarrow F$$

$$x \mapsto f(x, e_1)$$

$$f(e_0, \cdot) : E_1 \longrightarrow F$$

$$x \mapsto f(e_0, x)$$

If those mappings are differentiable at some points x_{E_0} and x_{E_1} respectively, we call their derivatives ***partial derivatives*** of f at $\left(x_{E_0}, e_1\right)$ and $\left(e_0, x_{E_1}\right)$ respectively and we denote them by

$$(\partial_0 f)\left(x_{E_0}, e_1\right) := (Df(\cdot, e_1))\left(x_{E_0}\right)$$

$$(\partial_1 f)\left(e_0, x_{E_1}\right) := (Df(e_0, \cdot))\left(x_{E_1}\right)$$

In general by induction we can do this for k-times set products and in this case we will denote the partial derivatives by ∂_i.

Now let's focus on Hilbert spaces.

Definition 1.6.3. Let $\mathcal{H}$ be a Hilbert space we call a function in $\mathbf{Hom}(\mathcal{H},\mathcal{H})$ or a restriction of such function to a subset of $\mathcal{H}$ a ***operator*** on $\mathcal{H}$.

Definition 1.6.4. Let A be a operator defined on a dense subset of $\mathcal{H}$. The ***adjoint*** operator of A, denoted by A^* is a operator such that

$$\mathbf{Domain}(\mathrm{A}^*) =$$

$$= \left\{\varphi \mid \left(\exists \eta_\varphi \in \mathcal{H}\right)(\forall \psi \in \mathbf{Domain}(\mathrm{A})) \quad \langle A\psi, \varphi\rangle = \langle \psi, \eta_\varphi\rangle\right\}$$

$$\forall \varphi \in \mathbf{Domain}(\mathrm{A}^*) \quad A^*(\varphi) := \eta_\varphi$$

Here it's important to emphasize that in general the adjoint operator might actually have empty domain.

Definition 1.6.5. An operator A on a Hilbert space $\mathcal{H}$ for which we have $A = A^*$ is called ***self-adjoint***. We denote the set of self adjoint operators on $\mathcal{H}$ y $\mathbf{Op}_{\mathrm{SA}}(\mathcal{H})$

Definition 1.6.6. An operator $P \in \mathbf{Op}_{\mathrm{SA}}(\mathcal{H})$ is called an ***orthogonal projection*** if $PP = P$.

Definition 1.6.7. The ***spectrum*** of densely defined operator A on a complex Hilbert space $\mathcal{H}$ is the set

$$\mathbf{Spec}(A) = \mathbb{C} \setminus \left\{z \in \mathbb{C} \mid \exists (A - z\,\mathrm{id}_{\mathcal{H}})^{-1} \in \boldsymbol{\mathscr{BL}}(\mathcal{H})\right\}$$

Now it's time to introduce the concept of projection valued measures which will be important later when formalizing quantum mechanics. They are similar to regular measures, but instead of real numbers they assign orthogonal projections.

Definition 1.6.8. Let $\mathcal{H}$ be a Hilbert space. A map P on the Borel sigma algebra of $\mathbb{R}$ is called a ***projection valued measure*** if

- $P : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathbf{Op}_{\mathrm{SA}}(\mathcal{H})$
- $P(\Omega)$ is an orthogonal projection for all $\Omega \in \mathfrak{B}(\mathbb{R})$.
- $P(\mathbb{R}) = \mathrm{id}_{\mathcal{H}}$
- If $\Omega = \cup_{n\in\mathbb{N}} \Omega_n$ with $\omega_i \cap \omega_j$ for $i \neq j$, then for every $\psi \in \mathcal{H}$

$$P(\Omega)(\psi) = \sum_{n\in\mathbb{N}} P(\Omega_n)(\psi)$$

Similarly to regular measures, one can define integration of integrable functions on $\mathbb{R}$ with respect to the projection valued measure, which gives an operator on the Hilbert space. For details on this see **[14]**. For a Borel set Ω, we might in the future simply notate the projection $P(\Omega)$ by P_Ω in future chapters.

Theorem 1.6.1. ***(Spectral Theorem)*** To every self-adjoint operator A there exists a unique projection valued measure $P^{(A)}$ such that

$$A = \int_R \lambda dP^{(A)}(\lambda)$$

Proof of this theorem can be found in **[14]**.

This allows to define arbitrary integrable function on $\mathbb{R}$ acting on self-adjoint operators of $\mathcal{H}$.

Definition 1.6.9. Let $\mathcal{H}$ is a Hilbert space, $A \in \mathbf{Op}_{\mathrm{SA}}(\mathcal{H})$ and f a integrable function from $\mathbb{R}$ to $\mathbb{R}$. Then we define

$$f(A) := \int_R f(\lambda) dP^{(A)}(\lambda)$$

Definition 1.6.10. An operator U on Hilbert space $\mathcal{H}$ is called ***unitary*** if

$$\langle U(\varphi), U(\psi)\rangle = \langle \varphi, \psi\rangle$$

for all $\varphi, \psi \in \mathcal{H}$.

Definition 1.6.11. Let $\mathcal{H}$ be a Hilbert space, the function

$$U : \mathbb{R} \longrightarrow \mathcal{H}^{\mathcal{H}}$$

is called a ***strongly continuous one-parameter unitary group*** if

- For each $s, t \in \mathbb{R}$ we have that $U(t)$ is unitary operator and

$$U(t+s) = U(t)U(s)$$

- If $\psi \in \mathcal{H}$ and $t \longrightarrow t_0$ then $U(t)(\psi) \longrightarrow U(t_0)(\psi)$

Theorem 1.6.2. Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and let $U(t) = e^{itA}$. Then

- U is strongly continuous one-parameter unitary group.
- For all $\psi \in \mathbf{Domain}(\mathrm{A})$

$$\lim_{t\longrightarrow 0} \frac{U(t)(\psi) - \psi}{t} = iA\psi$$

- If $\lim\limits_{t\longrightarrow 0} \frac{U(t)(\psi)-\psi}{t}$ exists then $\psi \in \mathbf{Domain}(\mathrm{A})$

Theorem 1.6.3. ***(Stone's theorem)*** Let U be a strongly continuous one-parameter unitary group on a hilbert space $\mathcal{H}$. Then, there exists a self adjoint operator A on $\mathcal{H}$ such that

$$U(t) = e^{itA}$$

The proofs of both **Theorem 1.6.3** and **Theorem 1.6.2** can be found in **[11]**.

Theorem 1.6.4. Let A and B be self adjoint operatos on a hilbert space $\mathcal{H}$. Then all the projections in their associated projection valued measures commute if and only if

$$\left[e^{itA}, e^{isB}\right] = 0_{\mathcal{H}}$$

for all $s, t \in \mathbb{R}$.

Proof of this fact can be seen in **[11]**.

Definition 1.6.12. We say that an operator T on a Hilbert space $\mathcal{H}$ is of ***trace class*** if there exists some orthonormal basis $B \subseteq \mathcal{H}$ for which

$$\sum_{\psi \in B} \langle \psi, T\psi \rangle$$

converges to a finite value.

Theorem 1.6.5. If an operator T on Hilbert space is trace class then the sum

$$\sum_{\psi \in B} \langle \psi, T\psi \rangle$$

is independent of orthonormal basis B.

Proof can be seen in **[14]**.

Definition 1.6.13. For a trace class operator T we define the ***trace*** as

$$\mathbf{tr}(T) := \sum_{\psi \in B} \langle \psi, T\psi \rangle$$

The set of all trace class operators on $\mathcal{H}$ with trace equal to one which are ***positive***, meaning $\langle \varphi, T\varphi \rangle \geq 0$ for all $\varphi \in \mathcal{H}$ we denote by $\mathbf{Tr}_1(\mathcal{H})$.

The trace class operators have a lot of other nice properties, so the reader is encouraged to check **[11, 14, 15]** for further details.

Theorem 1.6.6. Let $\mathcal{H}$ be a Hilbert space and A is self adjoint operator. If $\rho \in \mathbf{Tr}_1(\mathcal{H})$ then the map

$$\mu_\rho^{(A)} : E \mapsto \mathbf{tr}\left(\rho P_E^{(A)}\right)$$

is a Borel probability measure on $\mathbb{R}$

For further details see **[15]**.

Differential Geometry

Differential Geometry is the abstract study of generalized smooth spaces that can have properties far different than people's natural intuition for the "flat" Euclidean spaces. In order for this to be done this discipline builds upon tools from various other "lower-level" subjects such as Algebra, Topology and Analysis. In this section we will define the "bare"-minimum subset of differential geometry required to follow the rest of the paper. The main reference for this chapter is **[16]** and most statements will be given without proof. Unless stated otherwise (by providing other reference) the omitted proofs can be found there.

Definition 1.7.1. A paracompact Hausdorff topological space $(X, \mathcal{T})$ is said to be a ***topological manifold*** if it's ***locally homeomorphic*** to Euclidean vector space $\mathbb{R}^n$ equipped with its standard metric topology. By locally homeomorphic its meant that for each point $x \in X$ there exists $O_x \in \mathcal{N}^{\text{open}}_{(X,\mathcal{T})}\{x\}$ such that $O_x \cong \mathbb{R}^n$. The number n is called the ***dimension*** of the topological manifold.

There are various generalizations of topological manifolds to instead be locally homeomorphic to Banach or some other type of topological vector space and allow infinite dimensionality. However for the purposes of this thesis only the current more "down to earth" definition will be of interest.

For the purposes of describing physical space-times, topological manifolds are still way too abstract to be generally used by themselves. That's why much more structure needs to be built on top of them. One of the first steps is to make them "smooth". Intuitively smooth will mean that the manifold instead of being locally homeomorphic to $\mathbb{R}^n$ will be in some sense "locally diffeomorphic" to it. But so far such a notion of "locally diffeomorphic" doesn't make any sense, because diffeomorphism can only be defined between two spaces, where derivatives can be defined (Banach spaces). Instead we will force those local homeomorphisms to overlap "smoothly". So the first step is to build up a theory of these local homeomorphism maps and come up with a sensible notion of smoothness.

From now on when speaking about topological spaces, we will not explicitly write or even mention about the topology. It's assumed to exist and if it's concrete then it should be inferred from the context.

Definition 1.7.2. Let T be a topological manifold of dimension n. A n-dimensional ***coordinate chart*** or simply a ***chart*** is homeomorphic embedding of an open set of T into $\mathbb{R}^n$.

Definition 1.7.3. Let T be a n-dimensional topological manifold. Let $\varphi_0 : U_0 \longrightarrow \mathbb{R}^n$ and $\varphi_1 : U_1 \longrightarrow \mathbb{R}^n$ be two charts.It is said that these charts ***overlap smoothly*** if the functions

$$\varphi_1 \circ \varphi_0^{-1} : \varphi_0[U_0 \cap U_1] \subseteq \mathbb{R}^n \longrightarrow \varphi_1[U_0 \cap U_1] \subseteq \mathbb{R}^n$$

$$\varphi_0 \circ \varphi_1^{-1} : \varphi_1[U_0 \cap U_1] \subseteq \mathbb{R}^n \longrightarrow \varphi_0[U_0 \cap U_1] \subseteq \mathbb{R}^n$$

are smooth with respect to the Banach space structure of $\mathbb{R}^n$

Note here that the smooth overlap property is trivially true in the case the intersection of the domains of the charts is empty.

Definition 1.7.4. We call a set $\mathcal{A}$ a n-dimensional ***smooth atlas*** on a topological manifold X if $\mathcal{A}$ is a collection of n-dimensional charts for which

- Each point in S is contained in the domain of some chart from $\mathcal{A}$
- Any two charts in $\mathcal{A}$ overlap smoothly.

Definition 1.7.5. A smooth atlas is called ***maximal*** if it has all the possible compatible charts with it (which overlap smoothly with the rest).

Theorem 1.7.1. Every smooth atlas is contained in an unique complete atlas.

Definition 1.7.6. A ***smooth manifold*** $(M, \mathcal{A})$ is a pair of topological manifold and a maximal smooth atlas on it.

From now on we will stop carrying the atlases around and when we speak about smooth manifolds, we will just say smooth manifold M and omit the respective maximal atlas which is assumed to exist and is clear from the context.

Definition 1.7.7. A function $f : M \longrightarrow \mathbb{R}$ on a smooth manifold M is said to be ***smooth*** if for every chart φ, the function $f \circ \varphi^{-1}$, which is called ***the coordiante representation of*** f, is smooth. The set of all such smooth functions is notated by

$$\mathcal{C}^\infty(M) := \{f \in \mathbb{R}^M \mid f \ \text{ is smooth on M}\}$$

Theorem 1.7.2. For a smooth manifold M, the set $\mathcal{C}^\infty(M)$ is a commutative ring under the operations of pointwise sum and product $f_0 + f_1$ and $f_0 f_1$ for $f_0, f_1 \in \mathcal{C}^\infty(M)$ defined by

$$\forall p \in M : (f_0 + f_1)(p) = f_0(p) + f_1(p)$$

$$\forall p \in M : (f_0 f_1)(p) = f_0(p) f_1(p)$$

Definition 1.7.8. A function between manifolds $f : M \longrightarrow N$ is called ***smooth*** if for every coordinate chart φ_M on M and chart φ_N on N it's true that $\varphi_N \circ f \circ \varphi_M^{-1}$ is smooth.

Definition 1.7.9. A smooth function between two manifolds

$$f : M \longrightarrow N$$

is called ***diffeomorphism*** if it's bijective and its inverse is smooth. The two manifolds are then said to be ***diffeomorphic***

Now it's time to define one of the most important things about smooth manifolds, their tangent spaces. Those are local vector space structures that allow the usage of the regular tools of linear algebra to be utilized for retrieving information about the manifold locally.

Definition 1.7.10. Let M be a smooth manifold and let $p \in M$. A ***tangent vector at*** p is a function $v : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R}$ for which

- For each $\alpha_0, \alpha_1 \in \mathbb{R}$ and $f_0, f_1 \in \mathcal{C}^\infty(M)$

$$v(\alpha_0 f_0 + \alpha_1 f_1) = \alpha_0 v(f_0) + \alpha_1 v(f_1)$$

- For each $f_0, f_1 \in \mathcal{C}^\infty(M)$

$$v(f_0 f_1) = v(f_0) f_1(p) + f_0(p) v(f_1)$$

Definition 1.7.11. Let M be a smooth manifold and $p \in M$. We call the set

$$T_p M := \left\{ u \in \mathbb{R}^{\mathcal{C}^\infty(M)} \mid u \ \text{ is a tangent vector} \right\}$$

the ***tangent space*** of M at p.

Theorem 1.7.3. The tangent space at each point of a smooth manifold M is a vector space with addition and scalar multiplication defined as

$$\big(\forall p \in M \forall v_0, v_1 \in T_p M \forall f \in \mathcal{C}^\infty(M)\big) :$$
$$(v_0 + v_1)(f) = v_0(f) + v_1(f)$$

$$\big(\forall p \in M \forall \alpha \in \mathbb{R} \forall v \in T_p M \forall f \in \mathcal{C}^\infty(M)\big) :$$
$$(\alpha v)(f) = \alpha v(f)$$

Definition 1.7.12. For a smooth manifold M we call the ***tangent bundle*** the space

$$TM := \bigcup_{p \in M} \{p\} \times T_pM$$

along with the projection $\pi_{TM} : TM \longrightarrow M$ defined as

$$\forall (p, v) \in TM : \quad \pi_{TM}(p, v) := p$$

Clearly the tangent bundle is the combined space of all the tangent spaces around every point inside the manifold. It's easy to demonstrate that the tangent bundle is a manifold on its own.

Theorem 1.7.4. The tangent bundle of a n-dimensional smooth manifold M is a smooth manifold of dimension $2n$.

The above statement is easy to see by observing that for every chart x of a neighbourhood of p, that can be represented as $x(p) = \left(x^0(p), ..., x^{n-1}(p)\right)$ one can construct a chart $\xi : TM \longrightarrow \mathbb{R}^{2n}$ of TM by

$$\xi(p, v) = \left(x(p), v\left(x^0\right), ..., v\left(x^{n-1}\right)\right)$$

It's also not hard to check that those charts cover the whole manifold and overlap smoothly.

Definition 1.7.13. At each point p of a manifold M we call the dual space of the tangent space $T_p^*M := \left(T_pM\right)^*$ the ***cotangent space*** at p. It's members are called ***covectors***.

The space

$$T^*M := \bigcup_{p \in M} T_p^*M$$

along with the projection $\pi_{T^*M} : T^*M \longrightarrow M$, defined by $\pi_{T^*M}(p, \xi) = p$ is called the ***cotangent bundle*** of M.

Similarly to the tangent bundle, the cotangent bundle is also a manifold.

Definition 1.7.14. For a manifold M a ***smooth vector field*** is a derivation on the algebra $\mathcal{C}^\infty(M)$.

Many times we will just call those simply vector fields without using the word smooth if it's clear from the context.

Intuitively vector field is something that at each point of the manifold smoothly assigns a vector from the tangent space at that point. To see how that's the case, for a vector field X and a function f, we can define the tangent vector at p associated with the field X by

$$X_p(f) := (X(f))(p)$$

It's easy to check that because X is derivation X_p will have the necessary properties of a tangent vector.

Definition 1.7.15. If M and N are smooth are a smooth manifolds and $f : M \longrightarrow N$ is smooth map, the differential ***differential*** df of f is a smooth map

$$df : TM \longrightarrow TN$$

such that forall $p \in M$

- $df\big(T_pM\big) \subseteq T_{f(p)}N$
- $(df)_p := df \restriction T_pM : T_pM \longrightarrow T_{f(p)}N$ is linear.

If one has smooth map $f : M \longrightarrow N$, then one can define pointwise the map $(df)_p$ acting on $v \in T_pM$ as

$$(df)_p(v) := \{(g, v(g \circ f)) \mid g \in \mathcal{C}^\infty(N)\}$$

Clearly this is a function and one can prove that the union of all $(df)_p$ gives us a function with exactly the properties required from the differential. Hence for every function the differential is a unique function that satisfies these properties. Intuitively it approximates the smooth mapping at each point with linear transformations of the respective tangent spaces.

Definition 1.7.16. A smooth map between two manifolds M and N is called ***submersion***/***immersion*** if its differential is surjective/injective.

Now it's time to generalize the notion of the tangent and cotangent bundles, this will also give as a different way to characterize vector spaces.The idea is that the same way we can associate at every point of a manifold a vector space which gives us information about how the manifold behaves locally (the tangent space) and we can define vector fields that smoothly give us tangent vectors at each point, we can attach other information in a smooth way. This concept is known as a ***fiber bundle***.

Firstly we will give the definition of a fiber bundle, but then we will focus on a specific kind of fiber bundles known as vector bundles, which will be used throughout this thesis.

Definition 1.7.17. If M is a smooth manifold, then a ***fiber bundle*** of M or a ***fiber bundle with base*** M is an object composed of the following

- A smooth manifold E called a ***total space***
- A smooth manifold F called a ***standard fiber***
- A surjective submersion $\pi : E \longrightarrow M$ called the ***natural projection***
- A collection of ***local trivializations*** which is an open cover of the base $\{U_\alpha\}_{\{\alpha \in A\}}$ together with a set of diffeomorphisms

$$\psi_\alpha : F \times U_\alpha \longrightarrow \pi^{-1}(U_\alpha)$$

such that

$$\forall f \in F \forall p \in M : \quad \pi \circ \psi_\alpha(f, p) = p$$

In this definition one can observe how this generalizes the notion of the tangent bundle which smoothly assigns tangent spaces at each point to something that assigns smoothly at each point an arbitrary manifold. In this case the standard fiber of the tangent bundle is simply the n dimensional vector space isomorphic to each tangent space, the base is the manifold and the total space is the tangent bundle.

Definition 1.7.18. The preimage $\pi^{-1}(\{x\})$ of the natural projection map $\pi : E \longrightarrow M$ of a fiber bundle at $x \in M$ is called the ***fiber*** at x and denoted with E_x.

Definition 1.7.19. A ***vector bundle*** of rank n over manifold M is a fiber bundle with base M, projection π and cover and trivializations $\{(U_\alpha, \psi_\alpha)\}_{\{\alpha \in A\}}$, where the standard fiber is n dimensional vector space $\mathbb{F}^n$ (here $\mathbb{F}$ is either real or complex) and for all $x \in U_\alpha$ the map

$$\mathbb{F}^n \longrightarrow \pi^{-1}(\{x\})$$

$$v \mapsto \psi_\alpha(x, v)$$

is a vector space isomorphism.

The tangent and cotanget bundle are an examples of vector bundles.

For a manifold M, we will often times write that we have fiber bundle $\pi : E \longrightarrow M$ or simply a fiber bundle E, without specifying all the necessary components in case they are clear from the context. This can save a lot of writing. This is similar abuse of notation we use for almost all other structures that we introduce.

Definition 1.7.20. For a manifold M and a fiber bundle $\pi : E \longrightarrow M$, a ***smooth section*** of the bundle is a map

$$\sigma : M \longrightarrow E$$

such that

$$\pi \circ \sigma = \mathrm{id}_M$$

We denote the space of smooth sections of the bundle by $\mathbf{\Gamma}(M, E)$ or $\mathbf{\Gamma}(E)$ if M is clear from the context.

Smooth sections are simply continuous right inverses of the projection. One way one can think of them is that at every point in the manifold they smoothly assign members of the fiber at that point.

Definition 1.7.21. Let M, M' be two smooth manifolds with vector bundles $\pi_E : E \longrightarrow M$ and $\pi_{E'} : E' \longrightarrow M'$ and $f : M \longrightarrow M'$ be a smooth map. A ***vector bundle map covering*** f is a map $\varphi : E \longrightarrow E'$ satisfying

- The following diagram commutes

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi_E"'] & E' \arrow[d, "\pi_{E'}"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

- The fiberwise φ is linear, meaning for all $p \in M$

$$\varphi \upharpoonright E_p \in \mathbf{Hom}\Big(E_p, E'_{f(p)}\Big)$$

Definition 1.7.22. If E and F are two vector bundles on the same manifold M, then we denote the space of bundle maps that cover the identity id_M by

$$\mathbf{BHom}_M(E, F)$$

Furthermore we call such maps ***bundle homomorphisms*** or ***bundle morphisms***.

Definition 1.7.23. Let M be a smooth manifold. A smooth section $g \in \mathbf{\Gamma}(M, T^*M \otimes T^*M)$ which is symmetric and non-degenerate when regarded as bilinear form on the tangent space at each point and has the same index on each point is called ***pseudo-Riemannian metric***. The pair (M, g) is called a ***pseudo-Riemannian*** manifold.

Definition 1.7.24. Let M be a n-dimensional smooth manifold. M is called to be ***orientable*** if there exists nowhere vanishing smooth section $\omega \in \mathbf{\Gamma}\left(M, \bigwedge^k T^*M\right)$. The manifold together with such choice of section is called ***oriented***.

It's clear that if this section not identically zero on any tangent space, then regarding it as alternating map on each tangent space, it will split the space of ordered bases of the tangent space at that point on bases on which the form is positive and such on which it's not. Picking on each tangent space the positive equivalence class coincides with the intuitive intuition of orientation.

Physical Preliminaries

Quantum Mechanics

Here we will summarize a popular mathematical model for the main postulates a quantum system must fulfill.

Definition 2.1.1. We call mathematical model $\mathcal{S}$ of a physical system a ***quantum system*** if it fulfills the following conditions.

- $\mathcal{S}$ is associated with a complex Hilbert space $\mathcal{H}_{\mathcal{S}}$ which models the ***configuration space*** of the physical system.
- Quantities that can be measured in the physical system are modeled with self adjoint operators $A \in \mathbf{Op}_{\mathrm{SA}}(\mathcal{H}_{\mathcal{S}})$ on $\mathcal{H}_{\mathcal{S}}$ and are called ***observables***. The ***set of possible values of outcomes of a measurements of observable*** is $\mathbf{Spec}(A)$.
- A ***state*** of a quantum system at time t is positive trace class operator with unit trace $\rho_t \in \mathbf{Tr}_1(\mathcal{H}_{\mathcal{S}})$.
- For any observalbe A in a state ρ and if $\mu_\rho^{(A)}$ is the probability measure from **Theorem 1.6.6**. Then the ***probability of an outcome of a measurement of*** A in ρ to be in a Borel set $E \subset \mathbf{Spec}(A)$ is

  $$\mu_\rho^{(A)}(E)$$

  The ***mean value of measurements of the observable*** is

  $$\langle A \rangle_\rho := \int_{\mathbf{Spec}(A)} \lambda \ d\mu_\rho^{(A)}(\lambda)$$

  If A with corresponding PVM $P^{(A)}$ is measured at time t to be in the Borel set $E \subset \mathbf{Spec}(A)$, then the system's state is ***collapsed*** to

  $$\rho_{P_E^{(A)}} := \frac{P_E^{(A)} \rho P_E^{(A)}}{\mathbf{tr}\left(\rho P_E^{(A)}\right)}$$

One can see that if ρ_0 and ρ_1 are two states of a quantum system, then a convex combination of them $\lambda\rho_0 + (1-\lambda)\rho_1$ with $\lambda \in [0,1]$ is also state (follows directly from the linearity of the trace on $\mathbf{Tr}_1$).

Definition 2.1.2. We call a state of a quantum system ***pure state*** if it cannot be represented as a convex combination of other states. Non pure states are called ***mixed***.

Theorem 2.1.1. Let ρ be a pure state in $\mathcal{H}$. Then there exists $\psi_\rho \in \mathcal{H}$ with $\|\psi_\rho\| = 1$ such that for the map

$$\langle\psi_\rho, \cdot\rangle\psi_\rho : \mathcal{H} \longrightarrow \mathcal{H}$$

$$\varphi \mapsto \langle\psi_\rho, \varphi\rangle\psi_\rho$$

we have

$$\rho = \langle\psi_\rho, \cdot\rangle\psi_\rho$$

Furthermore let $\psi \in \mathcal{H}$ such that $\|\psi\| = 1$. Then there exists a pure state ρ_ψ such that

$$\rho_\psi = \langle\psi, \cdot\rangle\psi$$

The importance of this theorem is that pure states can be identified with rays in the Hilbert space. Further details can be found at **[15]**.

If for two observables A and B we have that $\left[e^{itA}, e^{isB}\right] \neq 0_{\mathcal{H}}$, then some of their corresponding projections in their associate projection valued measures won't commute (by **Theorem 1.6.4**). In this case from the definition of quantum system looking at the collapse property measuring these observables in different order won't in general give the same result. Such observables are said to be ***incompatible***.

While these are the basic postulates of quantum mechanics, giving a complete overview of this subject with all the detail is beyond the scope of this thesis, so the reader is encouraged to read **[11]** and **[15]** for details.

Relativity

The mathematical modeling in this section mostly follows [16] and the proofs of the main results can be found there. Some of the notations here are slightly different but that notation has already been introduced in the Differential Geometry section of this thesis.

For the purposes of this thesis the space-time (the “container” where everything exists) of interests will generally be a globally hyperbolic Lorentzian manifold (which will be defined later). So in this case only facts about such manifolds will be provided, without considering the theory of General Relativity which describes gravitation in its full generality.

Definition 2.2.1. We call a semi-Riemannian manifold (M, g) a ***Lorentzian manifold*** if $\mathbf{dim}((M, g)) > 1$ and $\mathbf{index}(g) = 1$.

Definition 2.2.2. Let (M, g) be a Lorentzian manifold. We call a tangent vector v_p in T_pM

$$\begin{array}{ccc} \textbf{spacelike} & \text{if} & g_p(v_p, v_p) > 0 \\ \textbf{null/lightlike} & \text{if} & g_p(v_p, v_p) = 0 \\ \textbf{timelike} & \text{if} & g_p(v_p, v_p) < 0 \\ \textbf{casual} & \text{if} & g_p(v_p, v_p) \leq 0 \end{array}$$

This property of the vectors we call their ***causal character***.

Definition 2.2.3. Let (M, g) be a Lorentzian manifold and let γ be a curve. We call the curve ***spacelike***, ***timelike***, ***lightlike*** or ***casual*** if all of it’s tangent vectors have the respective causal character. Similarly for vector field it’s $***\ast$like if it’s $***\ast$like pointwise.

Theorem 2.2.1. Let (M, g) be a Lorentzian manifold and let $p \in M$ be a point. A subspace W of T_pM is timelike if and only if $W^{\perp}$ is spacelike.

Definition 2.2.4. Let (M, g) be a Lorentzian manifold and let $p \in M$ be a point. Let V_τ be the space of all timelike vectors in T_pM. For every $u \in V_\tau$ we define the ***timecone*** of T_pM containing u as

$$C(u) = \left\{v \in V_\tau \mid g_p(u, v) < 0\right\}$$

The ***opposite timecone*** is

$$C(-u) = -C(u) = \left\{c \in V_\tau \mid g_p(u, v) > 0\right\}$$

One important observation in this definition is the fact that since $u^\perp$ is spacelike, then V_τ is the disjoint union of $C(u)$ and $C(-u)$.Let (M, g) be a Lorentzian manifold and let $p \in M$ be a point.

Theorem 2.2.2. Two timelike vectors v and u in some tangent space of a Lorentzian Manifold with a metric g are in the same timecone if and only if $g_p(v, u) < 0$.

Now we can finally define what it means for a Lorentzian manifold to be time-orientable. Having two timecones in each tangent space, there is no intrinsic way to distinguish them. Picking time orientation is to select a special timecone in each tangent space, but we also impose that pick to be done in a smooth way (it doesn't make sense for time-orientation to "suddenly jump" by going "from point to point").

Definition 2.2.5. Let (M, g) be a Lorentzian manifold. Furthermore let $\mathfrak{C}_\mathfrak{p} = \left\{C(u) \mid u \in T_pM \bigwedge g_p(u, u) < 0\right\}$ and $\mathfrak{C} = \bigcup_{p\in M} \mathfrak{C}_\mathfrak{p}$. A function $\tau : M \longrightarrow \mathfrak{C}$ is called a ***time orientation of*** the manifold if

- $\forall p \in M : \tau(p) \in \mathfrak{C}_\mathfrak{p}$
- For all $p \in M$ exists a smooth vector field X on some neighborhood U of p such that for each $q \in U$ it holds that $X_q \in \tau(q)$

The manifold is said to be ***time-orientable*** if time orientation exists on it. When one chooses a specific time-orientation it's said that one has ***time-oriented*** the manifold.

As it can clearly be seen from the previous definition, the properties seem to imply that the time-orientation can be factored into a composition of a smooth section of the tangent bundle (which is one possible characterization of the smooth vector fields) that spits only timelike vectors and the map $(p, u_p) \mapsto C(u_p)$. From observing this intuitively it's no surprise that the following theorem holds:

Theorem 2.2.3. A Lorentzian manifold is time-orientable if and only if a smooth timelike vector field exists on it.

Now similarly to timecones we can define causal cones, by also including the causal vectors in the definition.

Definition 2.2.6. A ***causal cone*** for a vector u in a tangent space of Lorentzian manifold with metric g at a point p is defined as

$$|(C)(u) = \{v \in T_pM \mid v \text{ is causal and } g_p(u, v) < 0$$

Definition 2.2.7. We call the time-orientation of a Lorentzian manifold the ***future*** and its negative the ***past***. Causal curves for which all of their tangent vectors are in the future causal cone are called ***future-directed***.

Definition 2.2.8. We define the following ***causality relations*** on time-oriented Lorentzian manifold (M, g). For $p, q \in M$

- $p \rightsquigarrow^+ q$ when there is future directed timelike curve from p to q
- $p \longrightarrow^+ q$ when there is future directed causal curve from p to q
- $p \rightsquigarrow^- q$ when there is past directed timelike curve from p to q
- $p \longrightarrow^- q$ when there is past directed causal curve from p to q

Definition 2.2.9. On time oriented Lorentzian manifold M and a subset X, we define the following sets

$$\mathbf{J}^M_\pm(X) = \{q \in M \mid \exists p \in X : \quad p \longrightarrow^\pm q\}$$

called ***causal future*** and ***causal past*** (depending on $\pm$)

$$\mathbf{I}^M_\pm(X) = \{q \in M \mid \exists p \in X : \quad p \rightsquigarrow^\pm q\}$$

called ***chronological future*** and ***chronological past*** respectively

Definition 2.2.10. For a time oriented Lorentzian manifold M, the ***strong causality condition*** is said to hold on $p \in M$ if for every open neighborhood U of p there exists an open neighbourhood of $V \subset U$ of p such that each causal curve in M with endpoints in V is entirely contained in U. The strong causality condition is said to hold in the entire manifold if it holds at each point.

Intuitively the strong causality condition forbids the existence of causal curves that are almost closed, meaning that if a curve starts arbitrary close to p it cannot end arbitrary close to p.

Definition 2.2.11. A connected time oriented Lorentzian manifold M is called ***globally hyperbolic*** if the strong causality condition holds on it and for all $p, q \in M$ the set $\mathbf{J}^M_+(\{p\}) \cap \mathbf{J}^M_-(\{q\})$ is compact.

One can see that a non empty open subset $O \subset M$ of a globally hyperbolic manifold will be in itself globally hyperbolic if and only if $\mathbf{J}^M_+(\{p\}) \cap \mathbf{J}^M_-(\{q\})$ is compact for all $q, p \in O$, because the strong causality condition on holding on M implies it holding on O.

Definition 2.2.12. A subset S of connected time oriented Lorentzian manifold M is called ***Cauchy surface*** if each inextendible timelike curve in M intersects S exactly once.

One can intuitively think of Cauchy surfaces as “time-slices”.

Theorem 2.2.4. Let M be a connected time oriented Lorentzian manifold M, then the following conditions are equivalent:

- M is globally hyperbolic.
- There exists a Cauchy surface in M.

For proof of each side of this theorem check **[16, 17]**.

Some important intermediate results

In this section we will look at some important results that will be required later when dealing with algebraic quantum field theory. Firstly we look at Weyl systems which will be used to model the algebras of observables in the theories that we create. Then we focus on symplectic spaces and how Weyl systems can naturally arise from them. After that we look at distributions on manifolds and differential operators. The significance of that part will be that for certain space-times and certain differential operators we can essentially create a symplectic space on the space of solutions of such operators. This will naturally give rise to a Weyl system.

Weyl Systems And Symplectic Vector Spaces

Definition 3.1.1. Let (V, ω) is a real symplectic vector space. A ***Weyl system*** of (V, ω) is a pair $(\mathcal{A}, \mathcal{W})$ of a C^*-algebra $\mathcal{A}$ and a map $\mathcal{W} : V \longrightarrow \mathcal{A}$ such that for all vectors $\varphi, \psi \in V$, we have

- $\mathcal{W}(0) = 1_A$
- $\mathcal{W}(-\psi) = (\mathcal{W}(\psi))^*$
- $$\mathcal{W}(\psi)\mathcal{W}(\varphi) = \exp\left(-i\frac{\omega(\psi, \varphi)}{2}\right)\mathcal{W}(\psi + \varphi)$$

Now the goal is to show that for every real symplectic vector space, we can build a Weyl system on it. We use the approach from **[18]**.

Theorem 3.1.1. Let (V, ω) be a real symplectic vector space. There exists Weyl system on it.

Proof. Let $\mathcal{H} := \boldsymbol{L}^2_{\mu_c}(V, \mathbb{C})$ is the Hilbert space of square integrable complex valued functions on V with respect to a counting measure μ_c. This means that $\mathcal{H}$ consists of functions $f : V \longrightarrow \mathbb{C}$ for which

$$\|f\|^2_{\boldsymbol{L}^2_{\mu_c}} := \int_V |f(\varphi)|^2 \; d\mu_c(\varphi) < \infty$$

The inner product of $\mathcal{H}$ is defined as

$$\langle f, g\rangle_{L^2_{\mu_c}} := \int_V \overline{f(\varphi)} g(\varphi) d\mu_c(\varphi)$$

Now let's define the algebra $\mathscr{BL}(\mathcal{H})$ to be the C^* algebra of bounded linear operators on $\mathcal{H}$. We define the map $\mathcal{W} : V \longrightarrow \mathscr{BL}(\mathcal{H})$ by

$$((\mathcal{W}(\varphi))(f))(\psi) := \exp\left(i\frac{\omega(\varphi,\psi)}{2}\right) f(\varphi+\psi)$$

for all $\varphi, \psi \in V$ and all $f \in \mathcal{H}$.

First we need to check that $\mathcal{W}(\varphi)$ is bounded linear operator.

$$\|\mathcal{W}(\varphi)\| := \sup_{f\in\mathcal{H};\ \|f\|_{\boldsymbol{L}^2_{\mu_c}}=1} \|(\mathcal{W}(\varphi))(f)\|_{\boldsymbol{L}^2_{\mu_c}} =$$

$$= \sup_{f\in\mathcal{H};\ \|f\|_{\boldsymbol{L}^2_{\mu_c}}=1} \sqrt{\int_V \left|\exp\left(i\frac{\omega(\varphi,\psi)}{2}\right) f(\varphi+\psi)\right|^2 d\mu_c(\psi)} =$$

$$= \sup_{f\in\mathcal{H};\ \|f\|_{\boldsymbol{L}^2_{\mu_c}}=1} \sqrt{\int_V \left|\exp\left(i\frac{\omega(\varphi,\psi)}{2}\right)\right|^2 |f(\varphi+\psi)|^2 d\mu_c(\psi)} =$$

$$= \sup_{f\in\mathcal{H};\ \|f\|_{\boldsymbol{L}^2_{\mu_c}}=1} \sqrt{\int_V |f(\varphi+\psi)|^2 d\mu_c(\psi)} =$$

$$= \sup_{f\in\mathcal{H};\ \|f\|_{\boldsymbol{L}^2_{\mu_c}}=1} \sqrt{\int_V |f(\chi)|^2 d\mu_c(\chi-\varphi)} =$$

$$= \sup_{f\in\mathcal{H};\ \|f\|_{\boldsymbol{L}^2_{\mu_c}}=1} \|f\|_{\boldsymbol{L}^2_{\mu_c}} = 1$$

This proves that the operator is bounded. Now let's check linearity. Let $f, g \in \mathcal{H}$ and $\psi \in V$.

$$((\mathcal{W}(\varphi))(f+g))(\psi) = \exp\left(i\frac{\omega(\varphi,\psi)}{2}\right)(f+g)(\varphi+\psi) =$$

$$= \exp\left(i\frac{\omega(\varphi,\psi)}{2}\right)(f(\varphi+\psi)+g(\varphi+\psi)) =$$

$$= \exp\left(i\frac{\omega(\varphi,\psi)}{2}\right)f(\varphi+\psi) + \exp\left(i\frac{\omega(\varphi,\psi)}{2}\right)g(\varphi+\psi) =$$

$$= ((\mathcal{W}(\varphi))(f))(\psi) + ((\mathcal{W}(\varphi))(g))(\psi)$$

This demonstrates that the operator is linear so combined with the boundedness we have $\mathcal{W}(\varphi) \in \boldsymbol{\mathscr{BL}}(\mathcal{H})$ for every $\varphi \in V$. Now we need to check the tree properties necessary for this to be a Weyl system.

$$((\mathcal{W}(0_V))(f))(\psi) = \exp\left(i\frac{\omega(0_V,\psi)}{2}\right)f(0_V+\psi) = f(\psi)$$

This implies that $\mathcal{W}(0_V) = \mathrm{id}_{\mathcal{H}} = 1_{\boldsymbol{\mathscr{BL}}(\mathcal{H})}$.

Now we need to check that $\mathcal{W}(-\varphi) = (\mathcal{W}(\varphi))^*$. To do so, we take arbitrary $f, g \in \mathcal{H}$ and compute

$$\langle(\mathcal{W}(-\varphi))(f), g\rangle_{\boldsymbol{L}^2_{\mu_c}} =$$

$$= \int_V \overline{((\mathcal{W}(-\varphi))(f))(\psi)}g(\psi)d\mu_c(\psi) =$$

$$= \int_V \overline{\exp\left(i\frac{\omega(-\varphi,\psi)}{2}\right)f(-\varphi+\psi)}g(\psi))d\mu_c(\psi) =$$

$$= \int_V \overline{\exp\left(i\frac{\omega(-\varphi,\chi+\varphi)}{2}\right)f(\chi)}g(\chi+\varphi))d\mu_c(\chi) =$$

$$= \int_V \overline{\exp\left(i\frac{\omega(-\varphi,\chi)}{2}\right)f(\chi)}g(\chi+\varphi))d\mu_c(\chi) =$$

$$= \int_V \overline{f(\chi)}\exp\left(i\frac{\omega(\varphi,\chi)}{2}\right)g(\varphi+\chi))d\mu_c(\chi) =$$

$$= \int_V \overline{f(\chi)}\exp\left(i\frac{\omega(\varphi,\chi)}{2}\right)g(\varphi+\chi))d\mu_c(\chi) = \langle f, (\mathcal{W}(\varphi))(g)\rangle_{\boldsymbol{L}^2_{\mu_c}}$$

Now remains to check the last property of multiplying to algebra elements together, we take arbitrary $\varphi, \psi, \chi \in V$.

$$\begin{gathered}
((\mathcal{W}(\varphi)\mathcal{W}(\psi))(f))(\chi) = \\
= (\mathcal{W}(\varphi)((\mathcal{W}(\psi))(f)))(\chi) = \\
= \exp\left(i\frac{\omega(\varphi,\chi)}{2}\right)(\mathcal{W}(\psi)(f))(\varphi+\chi) = \\
= \exp\left(i\frac{\omega(\varphi,\chi)}{2}\right)\exp\left(i\frac{\omega(\psi,\varphi+\chi)}{2}\right)f(\psi+\varphi+\chi) = \\
= \exp\left(i\frac{\omega(\varphi,\chi)+\omega(\psi,\varphi+\chi)}{2}\right)f(\psi+\varphi+\chi) = \\
= \exp\left(i\frac{\omega(\varphi,\chi)+\omega(\psi,\varphi)+\omega(\psi,\chi)}{2}\right)f(\psi+\varphi+\chi) = \\
= \exp\left(i\frac{\omega(\psi+\varphi,\chi)+\omega(\psi,\varphi)}{2}\right)f(\psi+\varphi+\chi) = \\
= \exp\left(i\frac{\omega(\psi,\varphi)}{2}\right)\exp\left(i\frac{\omega(\psi+\varphi,\chi)}{2}\right)f(\psi+\varphi+\chi) = \\
= \exp\left(i\frac{\omega(\psi,\varphi)}{2}\right)\exp\left(i\frac{\omega(\psi+\varphi,\chi)}{2}\right)f(\psi+\varphi+\chi) = \\
= \exp\left(-i\frac{\omega(\varphi,\psi)}{2}\right)((\mathcal{W}(\varphi+\psi))(f))(\chi)
\end{gathered}$$

Since this computation was done for an arbitrary χ, we have

$$\mathcal{W}(\varphi)\mathcal{W}(\psi) = \exp\left(-i\frac{\omega(\varphi,\psi)}{2}\right)\mathcal{W}(\varphi+\psi)$$

Now let $\mathcal{A}$ be the C^* subalgebra of $\mathscr{BL}(\mathcal{H})$ generated by the system $\{\mathcal{W}(\varphi)\}_{\varphi\in V}$. Then $(\mathcal{A},\mathcal{W})$ forms the desired Weyl system. □

Theorem 3.1.2. Let (V,ω) is a symplectic space and let $(\mathcal{A},\mathcal{W})$ be the Weyl system from **Theorem 3.1.1**, then $\{\mathcal{W}(\varphi)\}_{\varphi\in V}$ is linearly independent **[18]**.

Definition 3.1.2. Let's define by $\mathcal{Sympl}$ the category of symplectic vector spaces, with morphisms being the symplectic linear maps and composition being function composition.

This is obviously a coherent definition of category, the composition of symplectic linear maps obviously gives another symplectic linear map, the composition is associative and the identity endomorphism is simply the identity map.

Definition 3.1.3. Define the category $\mathcal{AlgC}^*$ with objects unitial C^* algebras and morphisms injective unit-preserving $*$-homeomorphisms, composition of morphisms is simply function composition.

Here again from the compositions of unit-preserving $*$-homeomorphisms being unit preserving $*$-homeomorphisms, the associativity of the composition and the identity maps acting as identity morphisms, one can easily see that this definition is consistent with the rules of category theory.

Theorem 3.1.3. Let (V,ω) be a symplectic vector space. Then if $\left(\mathcal{A}_{(V,\omega)}, \mathcal{W}_{(V,\omega)}\right)$ is the construction of a Weyl system for (V,ω) from the **Theorem 3.1.1**, then

$$\mathcal{CCR} : \mathcal{Sympl} \longrightarrow \mathcal{AlgC}^*$$

defines a functor by

$$\mathcal{CCR}(V,\omega) := \mathcal{A}_{(V,\omega)}$$

and for $F \in \mathbf{Hom}_{\mathcal{Sympl}}((V_0,\omega_0),(V_1,\omega_1))$

$$\mathcal{CCR}(F) : \mathcal{CCR}(V_0,\omega_0) \longrightarrow \mathcal{CCR}(V_1,\omega_1)$$

$$\mathcal{W}_{(V_0,\omega_0)}(\varphi) \mapsto \mathcal{W}_{(V_1,\omega_1)}(F(\varphi))$$

for any $\varphi \in V_0$, where the map is extended by linearity in the algebra.

Proof. Firstly let $F \in \mathbf{Hom}_{\mathcal{Sympl}}((V_0,\omega_0),(V_1,\omega_1))$. We need to show that $\mathcal{CCR}(F)$ is a unit-preserving $*$-homomorphism.

Since $\mathcal{CCR}(V_0, \omega_0)$ is generated from the set $\left\{\mathcal{W}_{(V_0,\omega_0)}(\varphi)\right\}_{\varphi\in V}$ of linearly independent generators the map $\mathcal{CCR}(F)$ is well defined, we only need to check of it's properties on the set of generators. It's also linear because it's extended by linearity by definition.

We know that $\mathcal{CCR}(F)$ preserves the unit because

$$\begin{gathered}\mathcal{CCR}(F)\Big(\mathbb{1}_{A_{(V_0,\omega_0)}}\Big) = \mathcal{CCR}(F)\Big(\mathcal{W}_{(V_0,\omega_0)}\Big(\mathbb{0}_{V_0}\Big)\Big) = \\ = \mathcal{W}_{(V_1,\omega_1)}\Big(F\Big(\mathbb{0}_{V_0}\Big)\Big) = \mathcal{W}_{(V_1,\omega_1)}\Big(\mathbb{0}_{V_1}\Big) = \mathbb{1}_{A_{(V_1,\omega_1)}} = \\ = \mathbb{1}_{\mathcal{CCR}(V_0,\omega_0)}\end{gathered}$$

Now we need to check the multiplicativity of the map. Let $\varphi, \psi \in V_0$. We have

$$\begin{gathered}\mathcal{CCR}(F)\Big(\mathcal{W}_{(V_0,\omega_0)}(\varphi)\cdot\mathcal{W}_{(V_0,\omega_0)}(\psi)\Big) = \\ = \mathcal{CCR}(F)(\mathcal{W}_{(V_0,\omega_0)}\left(\exp\left(i\frac{\omega_0(\varphi,\psi)}{2}\right)\mathcal{W}_{(V_0,\omega_0)}(\varphi+\psi)\right) = \\ = \exp\left(i\frac{\omega_0(\varphi,\psi)}{2}\right)\mathcal{CCR}(F)\Big(\mathcal{W}_{(V_0,\omega_0)}(\varphi+\psi)\Big) = \\ = \exp\left(i\frac{\omega_0(\varphi,\psi)}{2}\right)\mathcal{W}_{(V_1,\omega_1)}(F(\varphi+\psi)) = \\ = \exp\left(i\frac{\omega_1(F(\varphi),F(\psi))}{2}\right)\mathcal{W}_{(V_1,\omega_1)}(F(\varphi)+F(\psi)) = \\ = \mathcal{W}_{(V_1,\omega_1)}(F(\varphi))\cdot\mathcal{W}_{(V_1,\omega_1)}F(\psi) = \\ = \mathcal{CCR}(F)\Big(\mathcal{W}_{(V_0,\omega_0)}(\varphi)\Big)\cdot\mathcal{CCR}(F)\Big(\mathcal{W}_{(V_0,\omega_0)}(\psi)\Big)\end{gathered}$$

Now it remains to show that it preserves the involution.

$$\begin{gathered}\mathcal{CCR}(F)\Big(\mathcal{W}_{(V_0,\omega_0)}(\varphi)^*\Big) = \mathcal{CCR}(F)\Big(\mathcal{W}_{(V_0,\omega_0)}(-\varphi)\Big) = \\ = \mathcal{W}_{(V_1,\omega_1)}(-F(\varphi)) = \Big(\mathcal{W}_{(V_1,\omega_1)}(F(\varphi))\Big)^* = \\ = \Big(\mathcal{CCR}(F)\Big(\mathcal{W}_{(V_0,\omega_0)}(\varphi)\Big)\Big)^*\end{gathered}$$

The injectivity of $\mathcal{CCR}(F)$ follows from the injectivity of F. This proves that the map is of the right type (unit-preserving injective $*$-homomorphism).

It's clear that if F is the identity, then the map $\mathcal{CCR}(F)$ is simply the identity on the algebra. Now let's take two morphisms F_0, F_1 from $\mathcal{Sympl}$ and compute

$$\begin{gathered}
\mathcal{CCR}(F_1 \circ F_0)\Big(\mathcal{W}_{V_0,\omega_0}(\varphi)\Big) = \\
= \mathcal{W}(V_2,\omega_2)((F_1 \circ F_0)(\varphi)) = \mathcal{W}(V_2,\omega_2)(F_1(F_0(\varphi))) = \\
= \mathcal{CCR}(F_1)(\mathcal{W}(V_1,\omega_1)(F_0(\varphi))) = \\
= \mathcal{CCR}(F_1)(\mathcal{CCR}(F_0)(\mathcal{W}(V_0,\omega_0)(\varphi))) = \\
= (\mathcal{CCR}(F_1) \circ \mathcal{CCR}(F_0))(\mathcal{W}(V_0,\omega_0)(\varphi))
\end{gathered}$$

With this we have proven that $\mathcal{CCR}$ is a functor. $\square$

Distributions And Differential Operators On Globally Hyperbolic Manifolds

In this section we will focus on some more specialized results about distributions and differential operators on globally hyperbolic manifolds that will be useful in the next sections for illustrating the construction of particular quantum field theories. Here the main reference where one can find the omitted proofs is **[18]**. A shorter summery can be found in **[19, 20]**.

First we will look at the properties of the space of smooth sections of a real or complex vector bundles on globally hyperbolic Lorentzian Manifolds. But before that let us clarify something that can be confusing.[2]

Let (M, g) be a Lorentzian manifold and let $\pi_E : E \longrightarrow M$ be a $\mathbb{F}$-vector bundle (the field $\mathbb{F}$ is real or complex) on that manifold. First let's not that one way to model the smooth vector fields on a manifold is as a smooth sections $\mathbf{\Gamma}(TM)$ of the tangent bundle. Now assume that we have connection on the bundle E.

$$\overset{E}{\nabla} : \mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(E) \longrightarrow \mathbf{\Gamma}(E)$$

For a fixed $S \in \mathbf{\Gamma}(E)$ one can define the function

$$\overset{E}{\nabla}_{\square} S : \mathbf{\Gamma}(TM) \longrightarrow \mathbf{\Gamma}(E)$$

$$X \mapsto \overset{E}{\nabla}_X S$$

From the properties of connections this function is clearly $\mathcal{C}^\infty(M)$-linear. Now one important thing that we have about this functions is that they are in bijective correspondence with the bundle homomorphism over M. To prove this let's take two $\mathbb{F}$-vector bundles $\pi_E : E \longrightarrow M$ and $\pi_F : F \longrightarrow M$ and a bundle homomorphism $\varphi : E \longrightarrow F$. We need to find a bijection between the bundle maps and the linear maps between the smooth sections of the two bundles:

[2] At least it was really confusing for the author of this thesis before he was able to understand it.

$$※ : \mathbf{BHom}_M(E, F) \longrightarrow \mathbf{Hom}_{\mathcal{C}^\infty(M)}(\mathbf{\Gamma}(E), \mathbf{\Gamma}(F))$$

If we have any bundle morphism $\varphi \in \mathbf{BHom}_M(E, F)$, let's define this map as

$$※ (\varphi) := \left\{(S, \varphi \circ S) \in \mathbf{\Gamma}(E) \times F^M\right\}$$

Clearly $※ (\varphi)$ is a function from $\mathbf{\Gamma}(E)$ to F^M because it maps only one value to each input. However we need to show that the function is of the right type, meaning it sends smooth sections of E to smooth sections of F and that it's linear. First let's check how $\pi_F \circ ((※ (\psi))(S)) : M \longrightarrow M$ works for $S \in \mathbf{\Gamma}(E)$ at a point x.

$$(\pi_F \circ ((※ (\varphi))(S)))(x) = (\pi_F \circ (\varphi \circ S))(x) = \pi_F(\varphi(S(x))) =$$
$$= \pi_F(\varphi(S_x)) = x$$

The last equality follows from the fact that since φ is a bundle morphism, hence it sends fibers from E at x to fibers from F at x.

Now to show that $(※ (\varphi))(S)$ is a smooth section it only remains to demonstrate that it's smooth, however this is trivial because by definition of bundle morphism, we have that φ is smooth and S is also smooth so their composition $\varphi \circ S = (※ (\varphi))(S)$ is also smooth. With this now we have that $※ (\varphi) \in \mathbf{\Gamma}(F)^{\mathbf{\Gamma}(E)}$.

We still have to prove that $※ (\varphi)$ is linear. To do so let's take a $f_0, f_1 \in \mathcal{C}^\infty(M)$ and two sections $S_0, S_1 \in \mathbf{\Gamma}(E)$. We have for all $x \in M$

$$(※ (\varphi)(f_0 S_0 + f_1 S_1))(x) = \varphi((f_0 S_0 + f_1 S_1)(x)) =$$
$$= \varphi(f_0 S_0(x) + f_0 S_1(x)) = f_0 \varphi(S_0(x)) + f_1 \varphi(S_1(x)) =$$
$$= f_0(※ (\varphi)(S_0))(x) + f_1(※ (\varphi(S_1))(x) =$$
$$= (f_0 ※ (\varphi)(S_0) + f_1 ※ (\varphi)(S_1))(x)s$$

This proves that $※$ indeed gives us a function of the right type. Now it remains to show that it's injective and surjective.

To prove injectivity let's take two bundle morphisms $\varphi_0, \varphi_1 : E \longrightarrow F$ that differ at $v_E \in E_x$ for some $x \in X$.

Then let's pick a section $S \in \mathbf{\Gamma}(E)$ for which $S(x) = v_E$. Now we have

$$(\divideontimes(\varphi_0)(S))(x) = \varphi_0(S(x)) = \varphi_0(v_E) \neq$$
$$\neq \varphi_1(v_E) = \varphi_1(S(x)) = (\divideontimes(\varphi_1)(S))(x)$$

For the surjectivity let's pick a function $\Phi \in \mathbf{Hom}_{\mathcal{C}^\infty(M)}(E, F)$. We have to find a bundle morphism φ, such that $\divideontimes(\varphi) = \Phi$.

Before constructing this map let's first pick any $x \in M$ and any $v \in E_x$. We first need to find a section S_v in $\mathbf{\Gamma}(E)$ such that $S_{v(x)} = v$. To show that such a section exists let's first pick a local trivialization $\rho : U \times \mathbb{F}^n \longrightarrow \pi_E^{-1}(U)$ for a neighborhood $U \in \mathcal{N}^{\text{open}}\{x\}$. Now since M is smooth manifold there exists a bump function $b : U \longrightarrow \mathbb{F}$ which is strictly zero outside of U and one at x. Furthermore let $\rho(x, v_{\mathbb{F}}) = v$. Now we can construct a local section $s_v : U \longrightarrow E$ by

$$\forall p \in U : \quad s_v(p) := b\rho(p, v_{\mathbb{F}})$$

Now from s_v, we can construct global smooth section S by defining

$$S_{v(p)} = \begin{cases} s_{v(p)} & \text{if } \ p \in U \\ 0 & \text{if } \ p \notin U \end{cases}$$

It is clearly the right type because it's smooth and composing it with π gives us the identity.

Now that we have the needed section S_v for every v we can construct the map $\varphi : E \longrightarrow F$ by

$$\varphi(v) = (\Phi(S_v))(x)$$

To show that the value doesn't depend on the choice of section S_v we will use the $\mathcal{C}^\infty(M)$-linearity of Φ. If we pick two such sections S_v and S_v' using the linearity we can check that

$$(\Phi(S_v))(x) - (\Phi(S_v'))(x) = (\Phi(S_v) - \Phi(S_v'))(x) = ((\Phi(S_v - S_v'))(x)$$

Now let's pick a neighborhood U around x where we have a local frame $s_0, ..., s_n$ and pick a bump function supported on U for which $f(p) = 1$.

From this we have

$$f(S_v - S'_v) = \sum_{k=0}^{n} c_k s_k$$

for some unique smooth functions c_k on M. Because at x $S_v - S'_v$ is zero, then $c_{k(p)} = 0$. Now using the $\mathcal{C}^\infty(M)$-linearity for Φ we can calculate

$$(\Phi(f(S_v - S'_v)))(x) = \left(\Phi\left(\sum_{k=0}^{n} c_k s_k\right)\right)(x) = \sum_{k=0}^{n} c_{k(x)}(\Phi(s_k)(x)) = 0$$

Hence φ is well defined, its linearity, smoothness and the fact that it sends fibers of E at x to fibers of F at x follow from the properties of Φ (one can use local trivialization as above). Now we can easily check that $※(\varphi) = \Phi$, hence ※ is surjective.

There is another bijective correspondence one can consider. Namely the correspondence between the bundle morphisms from E to F and the smooth sections of the $\mathbf{Hom}_{\mathbb{F}}(E, F)$ bundle. We will need to define a bijective map

$$※ : \mathbf{\Gamma}(\mathbf{Hom}_{\mathbb{F}}(E, F)) \longrightarrow \mathbf{BHom}_M(E, F)$$

Let's define for a section $S \in \mathbf{\Gamma}(\mathbf{Hom}_{\mathbb{F}}(E, F))$

$$※(S) := S \circ \pi_E$$

We need to check that $※(S) \in \mathbf{BHom}_M(E, F)$. It's clear that the map $※(S)$ is smooth because it's composition of smooth maps. Furthermore it's clear that it's linear fiberwise because the way S gives a linear map for each point, so this is satisfied.

It's also clear that the map ※ is injective, because if we have two sections S, S' that differ at some point (meaning they give functions that differ at some vector inside the fiber at that point), then $※(S)$ and $※(S')$ would differ at some vector in the fiber at that point.

Now to prove surjectivity, we need to find a section $S \in \mathbf{\Gamma}(\mathbf{Hom}_{\mathbb{F}}(E, F))$ that will give rise to a specific bundle morphism φ. We can define

$$S(x) = \varphi \upharpoonleft (\pi_E^{-1}(\{x\}))$$

Clearly at every point x, S assigns linear function, by the definition of bundle morphism. The only thing we need to check in order for S to be a section is to check the smoothness. For any point in the manifold one can pick a local trivializations around that point $\rho_E : U_E \times \mathbb{F}^m \longrightarrow \pi_E^{-1}(U_E)$ and $\rho_F : U_F \times \mathbb{F}^n \longrightarrow \pi_F^{-1}(U_F)$. Now we can look at the map $\hat{S}(x) := \rho_F^{-1} \circ S(x) \circ \rho_E^{-1} : \{x\} \times \mathbb{F}^m \longrightarrow \{x\} \times \mathbb{F}^n$. Firstly from the two trivializations we can pick a local smooth frames for F and E by mapping some basis of F^n and F^m at each point with the trivialization. Let those local frames be $e_0, ...e_{m-1}$ and $f_0, ..., f_{n-1}$. We can express every section from each bundle in terms of linear combination of the local frame with unique smooth coefficients, where the coefficients α_k are smooth. Let's now fix $A_{ij}(x)$ such that

$$(S(x))\big(e_j(x)\big) = \sum_{i=0}^{n-1} A_{ij}(x) f_i(x)$$

Here A_{ij} are unique and smooth by construction. It's clear that different pointwise linear map would give a rise to different A_{ij} and they determine its action completely (one can recover the map from smooth coefficients A_{ij}), hence there is bijective correspondence between them (these are actually pointwise the ordinary "matrix components" of the linear map at each point).

So now let's pick a chart of M, $\psi_{U_E \cap U_F}$ of $U_E \cap U_F$. The map

$$(x, S) \mapsto \big(\psi(x), A_{00}(x)...A_{0,n-1}(x)...A_{m-1,n-1}(x)\big) \in F^{pmn}$$

is a bijective smooth map (because the components are smooth), hence diffeomorphism and chart of $\mathbf{Hom}_{\mathbb{F}}(E, F)$. Since in this chart S has smooth coefficients it's smooth at that point and since the point is arbitrary it is smooth in general.

Now using the fact that $\mathbf{Hom}_{\mathbb{F}}(E, F)$ is isomorphic to $E^* \otimes F$ (the tensor product being done fiberwise) we can easily construct a bijection

$$\# : \mathbf{\Gamma}(\mathbf{Hom}_{\mathbb{F}}(E, F)) \longrightarrow \mathbf{\Gamma}(E^* \otimes F)$$

Now since we have demonstrated that both ※ and ⨳ are bijections, we can for every map from $\mathbf{Hom}_{\mathcal{C}^\infty(M)}(\mathbf{\Gamma}(E), \mathbf{\Gamma}(F))$ associate a map in $\mathbf{\Gamma}(E^\star \otimes F)$ using $\# \circ \text{※}^{-1} \circ \text{⨳}^{-1}$.

Now going back to the original context, if we have a Levi-Civita connection $\overset{TM}{\nabla}$ on TM and connection $\overset{E}{\nabla}$ we can induce connection $\overset{(T^*M)^{\otimes k} \otimes E}{\nabla}$ on $(T^*M)^{\otimes k} \otimes T * M \otimes E$.

So using the above bijections one can define for a smooth section $S \in \mathbf{\Gamma}(E)$, the following expression

$$\nabla^k_\square S = \begin{cases} \overset{E}{\nabla}_\square S \quad \text{if} \quad k = 1 \\ \overset{(T^*M)^{\otimes k} \otimes E}{\nabla_\square} \#\big(\text{⨳}^{-1}\left(\text{※}^{-1}\big(\big(\nabla^{k-1}_\square S\big)\big)\right)\big) \quad \text{if} \quad k > 1 \end{cases}$$

Clearly $\nabla^k_\square S \in \mathbf{\Gamma}\Big((T^*M)^{\otimes k-1} \otimes T^*M \otimes E\Big)$. Now having this context let's state the following theorem.

Theorem 3.2.1. Let M be globally hyperbolic Lorentzian manifold, let E be a vector bundle and let ∇ be a connection on that bundle. For any compact set $K \subset M$ and any $n \in \mathbb{N}$ and $S \in \mathbf{\Gamma}(E)$

$$\|S\|_{K,n,\nabla,\,|\cdot|} := \max_{0 \le k \le n} \max_{x \in K} \Big|\big(\nabla^k_\square S\big)(x)\Big|$$

defines a set of seminorms on $\mathbf{\Gamma}(E)$. Furthermore due to compactness different choices of ∇ and $|\cdot|$ give rise to equivalent seminorms, so one can select a family of seminorms $\|\cdot\|_{K,n}$. Furthermore chosing a countable sequence $K_0 \subset ... \subset K_l \subset ...$ of compact sets covering M such that K_i is contained in the interior of K_{i+1} gives rise to an equivalent countable family of seminorms $\|\cdot\|_{K_i,n}$.

With this family of seminorms $\mathbf{\Gamma}(E)$ turns into a Frechet space. A sequence of sections converges in this space if and only if the sections and all their higher derivatives converge locally uniformly. **[18, 19]**.

Now with the space of smooth sections being a topological vector space, one can define the notion of continuous linear functionals on it.

For brevity throughout this section it's assumed (unless stated otherwise) that we work in globally hyperbolic manifold M with the space of smooth sections on finite dimensional vector bundles (over a field $\mathbb{F}$ which is either the complex or real numbers) being defined structure as above.

Definition 3.2.1. We denote the space of ***smooth sections of*** E ***with compact support*** by $\mathbf{\Gamma}_c(E)$.

Note that the space of compactly supported smooth sections is a closed subspace **[19]** of the space of all smooth sections, so it's a Frechet space as well.

Definition 3.2.2. A $\mathbb{F}$-linear map $F : \mathbf{\Gamma}_c(E^*) \longrightarrow W$ is called ***distribution in*** E ***with values in*** W if it's continuous in the sense that for all convergent sequences $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathbf{\Gamma}_c(E^*)$ with $\lim_{n\longrightarrow\infty}\varphi_n = \varphi$ its true that

$$\lim_{n\longrightarrow\infty} F[\varphi_n] = F[\varphi]$$

The space of distributions in E with values in W by $\boldsymbol{\mathcal{D}}(E, W)$. For the special case of the topological dual of $\mathbf{\Gamma}_c(E^*)$ we use the notation $\boldsymbol{\mathcal{D}}(E) := \boldsymbol{\mathcal{D}}(E, \mathbb{F})$.

We call the space $\mathbf{\Gamma}_c(E^*)$ the space of ***test sections in*** E^*.

An important example of a distribution is the ***delta-distribution***. For a point $x \in M$ and a bundle E, one can define $\delta_x \in \boldsymbol{\mathcal{D}}(E)$ to be

$$\forall\varphi \in \mathbf{\Gamma}_c(E^*) : \quad \delta_x[\varphi] = \varphi(x)$$

Furthermore any locally integrable smooth section S of E, can be interpreted as distribution in E by defining

$$\forall\varphi \in \mathbf{\Gamma}_c(E^*) : \quad f[\varphi] = \int_M \varphi(f)dV$$

Now before going any further we will focus on linear differential operators on sections.

Definition 3.2.3. Let $\pi_E : E \longrightarrow M$ and $\pi_F : F \longrightarrow M$ be a $\mathbb{F}$-vector bundles of rank l and p over a n-dimensional manifold M. A ***linear differential operator of order at most*** k from E to F is a $\mathbb{F}$-linear map

$$D : \mathbf{\Gamma}(E) \longrightarrow \mathbf{\Gamma}(F)$$

with the following properties:

For every point p in M there exists open neighborhood U of p where both E and F are trivialized with the maps $\rho_E : U \times \mathbb{F}^l \longrightarrow \pi_E^{-1}(U)$ and $\rho_F : U \times \mathbb{F}^p \longrightarrow \pi_F^{-1}(U)$ and there are smooth maps $A_\alpha : U \longrightarrow \mathbf{Hom}_{\mathbb{F}}\left(\mathbb{F}^l, \mathbb{F}^p\right)$ such that for any section $s \in \mathbf{\Gamma}(E)$ in a chart x of U, we have

$$\rho_F^{-1} \circ D(s) \circ x^{-1} = \sum_{|\alpha| \leq k} \left(A_\alpha \circ x^{-1}\right)\left(\partial_\alpha^{|\alpha|}\left(\rho_E^{-1} \circ s \circ x^{-1}\right)\right)$$

Where α is a multi-index $\alpha = \{\alpha_0, ..., \alpha_r\} \in \mathbb{N}^{r+1}$ with

$$|\alpha| := \sum_{i=0}^{r} \alpha_i \leq k$$

$$\partial_\alpha^{|\alpha|} := \partial_0^{\alpha_0} \partial_1^{\alpha_1} \ldots \partial_r^{\alpha_r}$$

We say that the linear differential operator is of ***order*** k if it's of order at most k but not of order at most $k - 1$.

We call the function

$$\hat{D} : \left\{\rho_E^{-1} \circ s \circ x^{-1} \mid s \in \mathbf{\Gamma}(E)\right\} \longrightarrow \left(\mathbb{F}^p\right)^{\mathbb{F}^n}$$

$$\rho_E^{-1} \circ s \circ x^{-1} \mapsto \rho_F^{-1} \circ D(s) \circ x^{-1}$$

the ***coordinate representation of the operator*** with respect to the given chart and trivializations with ***coefficients*** A_α.

This definition simply tells us that linear differential operator of order at most k is simply one who's action on a section point-wise expressed in coordinates (using charts and trivializations) can be described with linear point-wise functions acting on partial derivatives of order at most k. The multiplicative operation between $A_\alpha \circ x^{-1} : \mathbb{F}^n \longrightarrow \mathbf{Hom}_{\mathbb{F}}\big(\mathbb{F}^l, \mathbb{F}^p\big)$ and $\rho_E^{-1} \circ s \circ x^{-1} : \mathbb{F}^n \longrightarrow \mathbb{F}^l$ is simply point-wise application of the function linear function when evaluating $A_\alpha \circ x^{-1}$ at a point on the resulting $\mathbb{F}^l$ vector from the partial derivative of the section at a point of $\mathbb{F}^n$.

Definition 3.2.4. Let $\pi_E : E \longrightarrow M$ and $\pi_F : F \longrightarrow M$ be a $\mathbb{F}$-vector bundles of rank l and p over a n-dimensional manifold M and D is a linear differential operator of order k from E to F.

Let D be coordinate represented locally using the trivializations ρ_E, ρ_F and chart x around a neighbourhood of a point p with coefficients A_α as in the previous definition.

We define the ***principal symbol of*** D to be a map

$$\hat{\boldsymbol{\sigma}}_D : T^*M \longrightarrow \mathbf{Hom}_{\mathbb{F}}(E, F)$$

Such that for every $\xi = \sum_{k=0}^{n-1} \xi_i dx^i \in T_p^*M$

$$\hat{\boldsymbol{\sigma}}_D(\xi) := \sum_{|(\alpha_0,\ldots,\alpha_r)|=k} \left(\xi_0^{\alpha_0}\ldots\xi_r^{\alpha_r}\right)\Big(\rho_F(p,\cdot) \circ \mathrm{A}_{\alpha_0,\ldots,\alpha_r}(p) \circ \rho_E^{-1}(p)\Big)$$

The above definition is provided locally and it's not immediately clear that it's independent of the choice of local representation (chart and trivializations). However it can be proven to be so **[18, 21]**. The principle symbol tells us about the properties of the "highest order partial derivatives" in it.

Definition 3.2.5. For a Lorentzian manifold (M, g) and a linear or complex vector bundle $\pi_E : E \longrightarrow M$, we call the linear differential operator $P : \boldsymbol{\Gamma}(E) \longrightarrow \boldsymbol{\Gamma}(E)$ of order 2 ***normally hyperbolic*** if

$$\forall \xi \in T^*M : \quad \hat{\boldsymbol{\sigma}}_P(\xi) = g^\sharp(\xi, \xi)\mathrm{id}_E$$

Definition 3.2.6. Let M be time oriented Lorentzian manifold. Let P be a normally hyperbolic differential operator acting on $\mathbf{\Gamma}(E)$ for a vector bundle E over M. Linear maps G_+ and G_- from $\mathbf{\Gamma}_c(E)$ to $\mathbf{\Gamma}(E)$ satisfying

- $P \circ G_\pm = \mathrm{id}_{\mathbf{\Gamma}_c(E)}$
- $G_\pm \circ P = \mathrm{id}_{\mathbf{\Gamma}_c(E)}$
- $\forall \varphi \in \mathbf{\Gamma}_c(E): \quad \sup(G_\pm \varphi) \subseteq \mathbf{J}^M_\pm(\sup(\varphi))$

Are called ***advanced Green's operator for*** P and ***retarded Green's operator for*** P respectively.

Theorem 3.2.2. Let M be a globally hyperbolic Lorentzian manifold and P a normally hyperbolic differential operator acting on sections in a vector bundle E over M. Then there exist unique advanced and retarded Green's operators $G_\pm : \mathbf{\Gamma}_c(E) \longrightarrow \mathbf{\Gamma}_c(E)$ for P. **[18]**

This result is crucial for the incoming construction of AQFTs in the following chapters. The difference between the advanced and retarded Green's operators will let us define the necessary structure on the space of solutions of this operator so then we can use this structure to define the algebra of observables associated with our QFT.

Definition 3.2.7. Let E be a real vector bundle over the manifold M with volume form V and E is equipped with a nondegenerate inner product $\langle , \rangle$. We say that differential operator E is ***formally self adjoint*** with respect to the inner product if

$$\int_M \langle P(\psi), \varphi \rangle dV = \int_M \langle \psi, P(\varphi) \rangle dV$$

for all sections $\psi, \varphi \in \mathbf{\Gamma}_c(E)$

A very important operator for the study of globally hyperbolic operators will be the d'Alembert operator which is the "prototype" of such operators, we will see that those operators are nothing but the d'Alambert operator plus a $\mathcal{C}^\infty(M)$ linear map between smooth sections.

Definition 3.2.8. Let (M, g) be a manifold equipped with Levi-Civita connection on T^*M and a connection $\overset{E}{\nabla}$ on E. We define the ***d'Alembert operator*** to be the map defined by

$$\square^{\overset{E}{\nabla}} : \mathbf{\Gamma}(E) \longrightarrow \mathbf{\Gamma}(E)$$

$$\left(\square^{\overset{E}{\nabla}}(S)\right)(p) := \left(g_p^\sharp \otimes \mathrm{id}_E\right)\left((\nabla_\square^2 S)(p)\right)$$

For all $S \in \mathbf{\Gamma}(E)$ and $p \in M$.

Note that in the previous definition $\nabla_\square^2 S \in \mathbf{\Gamma}(T^*M \otimes T^*M)$ as defined in the previous section. Now it's time for important property of normally hyperbolic operators, which will help us classify them.

Theorem 3.2.3. Let E be a vector bundle over Lorentzian manifold M and let P be a normally hyperbolic operator on $\mathbf{\Gamma}(E)$, then there exists a unique connection $\overset{E}{\nabla}$ on E and unique endomorphism field $B \in \mathbf{\Gamma}(\mathbf{Hom}_{\mathbb{F}}(E, E))$ such that

$$P = \square^{\overset{E}{\nabla}} + ※\left(※(B)\right)$$

The full proof of this can be seen in **[18]** (keep in mind that there the correspondence $※ \circ ※$ is used implicitly (In order to add the two terms together they have to be of the same type, so the authors used the correspondence between smooth endomorphism fields and $\mathcal{C}^\infty(M)$-linear operators on sections).

Theorem 3.2.4. Let E be a vector bundle over Lorentzian manifold M, then the operator $\square^{\overset{E}{\nabla}}$ is formally self adjoint.

Proof can be seen in **[18]**. From this a normally hyperbolic opeartor $P = \square^{\overset{E}{\nabla}} + B$, is formally self adjoint exactly when B is self-adjoint.

Theorem 3.2.5. Let E be a vector bundle with a inner product over globally hyperbolic manifold M, and P be a formally self adjoint normally hyperbolic opeartor on $\mathbf{\Gamma}(E)$ with Green's operators $G_{\pm}$, then

$$\int_M \langle G_{\pm}\varphi, \psi\rangle dV = \int_M \langle \varphi, G_{\mp}\psi\rangle dV$$

for all $\varphi, \psi \in \mathbf{\Gamma}_c(E)$

Proof. From $PG_{\pm} = \mathrm{id}_{\mathbf{\Gamma}_c(M,E)}$ we have

$$\int_M \langle G_{\pm}\varphi, \psi\rangle dV =$$
$$= \int_M \langle G_{\pm}\varphi, PG_{\mp}\psi\rangle dV = \int_M \langle PG_{\pm}\varphi, G_{\mp}\psi\rangle dV =$$
$$= \int_M \langle \varphi, G_{\mp}\psi\rangle dV$$

□

Theorem 3.2.6. Let E be a real vector bundle with a inner product over globally hyperbolic manifold M, and P be a formally self adjoint normally hyperbolic opeartor on $\mathbf{\Gamma}(E)$ with Green's operators $G_{\pm}$, then if we define $G := G_+ - G_-$, then the space

$$\mathbf{Sym}(M, E, P) := \mathbf{\Gamma}_c(M, E)/\ker(G)$$

with symplectic form $\hat{\omega}$ defined by

$$\hat{\omega}(\varphi + \ker(G), \psi + \ker(G)) := \omega(\varphi, \psi)$$

where

$$\omega(\varphi, \psi) := \int_M \langle G\varphi, \psi\rangle dV$$

Proof. First it's easy to see that because of the linearity of the integral G and the inner product that ω linear. Furthermore from **Proof** we have that ω is alternating. However it's degenerate because G has non-trivial kernel.

Now let's check that $\hat{\omega}$ is well defined. If $\varphi + \ker(G) = \varphi' + \ker(G)$ and $\psi + \ker(G) = \psi + \ker(G)$, we have

$$\begin{gathered}\hat{\omega}(\varphi + \ker(G), \psi + \ker(G)) = \\ = \omega(\varphi, \psi) = \omega(\varphi' + \varphi - \varphi', \psi' + \psi - \psi') = \\ = \omega(\varphi', \psi') + \omega(\varphi', \psi - \psi') + \omega(\varphi - \varphi', \psi') + \omega(\varphi - \varphi', \psi - \psi') = \\ = \omega(\varphi', \psi') = \\ = \hat{\omega}(\varphi' + \ker(G), \psi' + \ker(G))\end{gathered}$$

The equality before the last is true because $\psi - \psi' \in \ker(G)$ and $\varphi - \varphi' \in \ker(G)$.

The form $\hat{\omega}$ keeps the linearity and anti-symmetry from ω but now it's non-degenerate, because if $\omega(\varphi, \psi)$ is zero for all $\psi \in \mathbf{\Gamma}_c(M, E)$, then φ must be in the kernel of G. $\square$

Algebraic Quantum Field Theory

AQFT As A Functor

Here we are ready to give abstract definition of what it means for some theory to be classified as AQFT. This approach to defining AQFT is due to **[22]**.

Definition 4.1.1. Define the category $\mathcal{ManGH}$ with objects all the globally hyperbolic, oriented and time oriented four dimensional Lorenzian manifolds and morphisms $\xi \in \mathbf{Hom}_{\mathcal{ManGH}}((M_0, g_0), (M_1, g_1))$ that are isometric embeddings which preserve orientation and time orientation and have the property that if $\gamma : [a, b] \longrightarrow M_1$ is a casual curve and $\gamma(a), \gamma(b) \in \xi(M_0)$, then the whole curve must lie in the image $\xi(M_0)$.

Now one can easily see that this is coherent definition of a category. First it's necessary to check that the composition of morphisms between objects is the morphism of required kind between the required objects, that is for $g \in \mathbf{Hom}_{\mathcal{ManGH}}((M_0, g_0), (M_1, g_1))$ and $f \in \mathbf{Hom}_{\mathcal{ManGH}}((M_1, g_1), (M_2, g_2))$ we should have that $f \circ g \in \mathbf{Hom}_{\mathcal{ManGH}}((M_0, g_1), (M_2, g_2))$. This is easy to check because composition of injective functions is injective, composition of smooth functions is smooth, composition of metric-preserving, time orientation preserving and orientation preserving maps is respectively metric-preserving, time orientation preserving and orientation preserving. The last property of the whole causal curves being mapped in case their end points are being mapped will also be clearly preserved under composition. The associativity is result of the associativity of compositions of functions. It's clear that the identity maps are identity morphisms in the category.

Definition 4.1.2. An ***algebraic quantum field theory*** is a functor $\mathcal{Q}$ between the categories $\mathcal{ManGH}$ and $\mathcal{AlgC}^*$ with the properties:

- ***(Causality)*** For every two morphisms

$$\varphi_0 \in \mathbf{Hom}_{\mathcal{ManGH}}((M_0, g_0), (M, g))$$

$$\varphi_1 \in \mathbf{Hom}_{\mathcal{ManGH}}((M_1, g_1), (M, g))$$

such that the sets $\varphi_0(M_0)$ and $\varphi_1(M_1)$ are casually separated in (M, g), then it's true that:

$$[\mathcal{Q}(\varphi_0)(\mathcal{Q}((M_0, g_0))), \mathcal{Q}(\varphi_1)(\mathcal{Q}((M_1, g_1)))] = \{0\}$$

- ***(Time slicibility)*** For every $\varphi \in \mathbf{Hom}_{\mathcal{ManGH}}((M_0, g_0), (M_1, g_1))$ for which $\varphi(M_0)$ contains a Cauchy surface for M_1 we have

$$\mathcal{Q}(\varphi)(\mathcal{Q}((M_0, g_0))) = \mathcal{Q}((M_1, g_1))$$

Before going any further and proving statements about such objects (quantum field theories), giving constructions of them, etc, it's time to step back and build some intuition about this definition.

Firstly the category $\mathcal{ManGH}$ contains all the possible space-times where our notion of quantum fields (which we have not defined yet) might exist. Since the morphisms in this category are embeddings then we can think of a functor from this category to another category as object that gives us enough information so that to each region of certain space time we can assign certain other structure. Now from ordinary quantum mechanics we know that observables there are self adjoint operators on some Hilbert space and they naturally induce a C^* algebra. From here one can already guess that the intention of the second category is to represent some abstract form of the algebra of observables.

Now from the normal properties of functors to preserve compositions of morphisms, and since the morphisms in both cases are embeddings (hence we can think of them mapping subsets to larger spaces), this tells us that for smaller regions that are subsets of larger regions of spacetime, the algebra of observables assigned to the smaller subsets is a subset of the algebra of observables of the larger subset. So far this is very natural and intuitive. In the larger regions one can observe the things in the smaller regions as well as potentially other things.

The causality property simply tells us that for casually separated regions in some larger space the respective mapped algebras commute element-wise (physically this can be interpreted in the sense that measurements in a region shouldn't affect measurements happening in regions that are casually separated from it).

The third property tells us that the algebra of observables on a globally hyperbolic space-times is determined by the algebra of observables in neighbourhood of a Cauchy surface.

Connection with the classic Haag-Kastler approach

Originally the Haag-Kastler "axiomatic" approach was invented in order to be able to formalize in a mathematically precise manner how the algebras of quantum observables should behave in different regions of spacetime. While the original approach was for Minkowski spacetime, those assumptions about local algebras of quantum observables can be further generalized for more general globally hyperbolic Lorentzian spacetimes **[22]**. In this chapter we will state the conditions of the generalized Haag-Kastler framework, and describe how the functorial formalization from the previous section gives rise to local algebras that satisfy the above mentioned assumptions.

Definition 4.2.1. Let (M, g) be a globally hyperbolic spacetime. Let $\mathfrak{O}_M$ be the set of open, relatively compact, casually compatible and globally hyperbolic subsets of M as well as $\emptyset$ and M itself. If $O \in \mathfrak{O}_M$ then the observables of a quantum system in M that can be measured within that spacetime region are said to satisfy the conditions of the ***Haag-Kastler framework*** if their behavior is dictated by the map

$$\mathfrak{O}_M \ni O \mapsto \mathcal{A}(O)$$

such that $\mathcal{A}(O)$ is C^*-algebra for every O and

- $(\forall O_0, O_1 \in \mathfrak{O}_M) \quad O_0 \subseteq O_1 \implies \mathcal{A}(O_0) \subseteq \mathcal{A}(O_1)$
- If $O_0, O_1 \in \mathfrak{O}_M$ are causally separated, then
$$[\mathcal{A}(O_0), \mathcal{A}(O_1)] = \{0\}$$
- If $O_0 \subseteq O_1$ and they contain a common Cauchy surface then
$$\mathcal{A}(O_0) = \mathcal{A}(O_1)$$
- The algebras $\mathcal{A}(O)$ have common unit
- If there exists a group G of isometric diffeomorphisms
$$\kappa : M \longrightarrow M$$

preserving orientation and time orientation, then there is a representation

$$G \ni \kappa \mapsto \alpha_\kappa$$

by C^*-algebra automorphisms $\alpha_\kappa : \mathcal{A}(M) \longrightarrow \mathcal{A}(M)$ such that

$$\alpha_\kappa(\mathcal{A}(O)) = \mathcal{A}(\kappa(O))$$

Theorem 4.2.1. Let the functor $\boldsymbol{\mathcal{Q}}$ be a quantum field theory and let $(M, g) \in \mathbf{Obj}(\boldsymbol{\mathcal{M}\mathit{an}\mathcal{GH}})$. Then the mapping

$$\mathcal{A} : \mathfrak{O}_M \longrightarrow \boldsymbol{\mathcal{Q}}(M)$$

$$O \mapsto \boldsymbol{\mathcal{Q}}(\mathrm{id}_M \upharpoonleft O)(\boldsymbol{\mathcal{Q}}(O))$$

satisfies the conditions of the Haag-Kastler framework

Proof. Trivially the mapping $\mathcal{A}$ is of the required type and is well defined, because all the subsets in $\mathfrak{O}_M$ are also objects in $\boldsymbol{\mathcal{M}\mathit{an}\mathcal{GH}}$. Now we will try to prove the other properties.

- Taking two sets $O_0, O_1 \in \mathfrak{O}_M$ and $O_0 \subseteq O_1$. Considering the fact that $\mathrm{id}_M \upharpoonleft O_0 = (\mathrm{id}_M \upharpoonleft O_1) \circ \Big(\mathrm{id}_{O_1} \upharpoonleft O_0\Big)$, we get

$$\begin{aligned} &\mathcal{A}(O_0) = \\ &= \boldsymbol{\mathcal{Q}}(\mathrm{id}_M \upharpoonleft O_0)(\boldsymbol{\mathcal{Q}}(O_0)) = \\ &= \boldsymbol{\mathcal{Q}}\Big((\mathrm{id}_M \upharpoonleft O_1) \circ \Big(\mathrm{id}_{O_1} \upharpoonleft O_0\Big)\Big)(\boldsymbol{\mathcal{Q}}(O_0)) = \\ &= \boldsymbol{\mathcal{Q}}(\mathrm{id}_M \upharpoonleft O_1)\Big(\boldsymbol{\mathcal{Q}}\Big(\mathrm{id}_{O_1} \upharpoonleft O_0\Big)(\boldsymbol{\mathcal{Q}}(O_0))\Big) \subseteq \\ &\subseteq \boldsymbol{\mathcal{Q}}(\mathrm{id}_M \upharpoonleft O_1)(\boldsymbol{\mathcal{Q}}(O_1)) = \\ &= \mathcal{A}(O_1) \end{aligned}$$

- Now let $O_0, O_1 \in \mathfrak{O}_M$ are causally separated. Using the morphisms $\mathrm{id}_M \upharpoonleft O_0$ and $\mathrm{id}_M \upharpoonleft O_1$ and the causality property of the $\boldsymbol{\mathcal{Q}}$ functor we have

$$[\mathcal{A}(O_0), \mathcal{A}(O_1)] = [\boldsymbol{\mathcal{Q}}(\mathrm{id}_M \upharpoonleft O_0)\boldsymbol{\mathcal{Q}}(O_0), \boldsymbol{\mathcal{Q}}(\mathrm{id}_M \upharpoonleft O_1)\boldsymbol{\mathcal{Q}}(O_1)] = \{0\}$$

- Now lets look at $O_0, O_1 \in \mathfrak{O}_M$ where O_0 contains a Cauchy surface for O_1 (they share a common Cauchy surface). Now from the time slicibility property of $\mathcal{Q}$ and the morphism $\mathrm{id}_{O_1} \upharpoonleft O_0$, we have

$$\mathcal{Q}\left(\mathrm{id}_{O_1} \upharpoonleft O_0\right)(\mathcal{Q}(O_0)) = \mathcal{Q}(O_1)$$

But now taking the image of $\mathcal{Q}(\mathrm{id}_M \upharpoonleft O_1)$ on both sides we have

$$\begin{gathered}\mathcal{A}(O_1) = \\ = \mathcal{Q}(\mathrm{id}_M \upharpoonleft O_1)(\mathcal{Q}(O_1)) = \\ = \mathcal{Q}(\mathrm{id}_M \upharpoonleft O_1)(\mathcal{Q}\left(\mathrm{id}_{O_1} \upharpoonleft O_0\right)(\mathcal{Q}(O_0)) = \\ = \mathcal{Q}\Big((\mathrm{id}_M \upharpoonleft O_1) \circ \mathcal{Q}\left(\mathrm{id}_{O_1} \upharpoonleft O_1\right)\Big)(\mathcal{Q}(O_0)) = \\ = \mathcal{Q}(\mathrm{id}_M \upharpoonleft O_0)\ \mathcal{Q}(O_0) = \\ = \mathcal{A}(O_0)\end{gathered}$$

- To check that the algebras have common unit lets take arbitrary $O \in \mathfrak{O}_M$. From the fact that

$$\mathcal{A}(M) = \mathcal{Q}(\mathrm{id}_M)\ \mathcal{Q}(M) = \mathrm{id}_{\mathcal{Q}(M)}\ \mathcal{Q}(M) = \mathcal{Q}(M)$$

And $\mathcal{A}(O) = \mathcal{Q}(\mathrm{id}_M \upharpoonleft O)(\mathcal{Q}(O))$, since $\mathcal{Q}(\mathrm{id}_M \upharpoonleft O)$ is by definition of $\mathcal{Q}$ a unit-preserving $*$-homomorphism into $\mathcal{Q}(M)$, we have that $\mathcal{A}(O)$ is C^* subalgebra with the same unit.
- Now finally let G be group of isometric diffeomorphisms preserving orientation and time-orientation. Let $\kappa \in G$, denoting by $\alpha_\kappa := \mathcal{Q}(\kappa)$, we have

$$\begin{gathered}\alpha_\kappa(\mathcal{A}(O)) = \\ = \mathcal{Q}(\kappa)(\mathcal{Q}(\mathrm{id}_M \upharpoonleft O)(\mathcal{Q}(O))) = \\ = \mathcal{Q}(\kappa \circ (\mathrm{id}_M \upharpoonleft O))(\mathcal{Q}(O)) = \\ = \mathcal{Q}((\mathrm{id}_M \upharpoonleft \kappa(O)) \circ (\kappa \upharpoonleft O))(\mathcal{Q}(O)) = \\ = \mathcal{Q}((\mathrm{id}_M \upharpoonleft \kappa(O)) \circ (\kappa \upharpoonleft O))(\mathcal{Q}(O)) = \\ = \mathcal{Q}(\mathrm{id}_M \upharpoonleft \kappa(O))(\mathcal{Q}(\kappa \upharpoonleft O)(\mathcal{Q}(O))) = \\ = \mathcal{Q}(\mathrm{id}_M \upharpoonleft \kappa(O))(\mathcal{Q}(\kappa(O))) = \\ = \mathcal{A}(\kappa(O))\end{gathered}$$

Finally, to check the representation property that for $\kappa_0, \kappa_1 \in G$, we have

$$\alpha_{\kappa_0 \kappa_1} = \mathcal{Q}(\kappa_0 \circ \kappa_1) = \mathcal{Q}(\kappa_0) \circ \mathcal{Q}(\kappa_1) = \alpha_{\kappa_0} \circ \alpha_{\kappa_1}$$

because of the functorial properties of $\mathcal{Q}$. $\square$

This theorem demonstrates that the initial definition of what a quantum field theory is supposed to be is simply framing the standard Haag-Kastler algebraic QFT framework into categorical language.

Construction

Now it's time to make an actual construction of a QFT functor as described in the previous chapters.

With the definitions of the previous chapter, we could of course construct a functor of the specified kind for a particular formally self adjoint normally hyperbolic linear differential operator acting on the trivial line bundle on globally hyperbolic manifolds, (for example the operator $\Box + m^2 + \xi R$) on $\mathcal{C}^\infty(M)$ for fixed constants m and ξ, where R is the scalar curvature of the metric in a fixed object of $\mathcal{ManGH}$ (this would've generated a QFT for the so called ***Klein Gordon*** scalar field). However we generally want to formalize this for more general normally hyperbolic operators, not necessarily acting on trivial line bundles.

Because of this we will supplement the category $\mathcal{ManGH}$ we are working with to a category $\mathcal{ManGHBund}$, by changing the objects instead of being oriented and time oriented globally hyperbolic Lorentzian manifolds (M, g) to be $((M, g), E, P)$ where E is a real vector bundle with non-degenerate inner product and P normally hyperbolic formally self adjoint differential operator on $\mathbf{\Gamma}(E)$. The morphisms will be pairs of maps (φ, F), where φ is defined the same way as the morphisms in $\mathcal{ManGH}$ and F is a bundle isomorphism over φ which is fiberwise an insometry and preserves normally hyperbolic operators.

Namely F being a bundle morphism, the following diagram should commute.

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d] & E' \arrow[d] \\ M \arrow[r, "\varphi"] & M' \end{tikzcd}$$

The condition of preserving normally hyperbolic operators can be expressed with the following diagram commuting

$$
\begin{tikzcd}
\mathbf{\Gamma}_c(M, E) \arrow[r, "P"] \arrow[d, "\mathbf{ext}"] & \mathbf{\Gamma}_c(M, E) \arrow[d, "\mathbf{ext}"] \\
\mathbf{\Gamma}_c(M', E') \arrow[r, "P'"] & \mathbf{\Gamma}_c(M', E')
\end{tikzcd}
$$

Here $\mathbf{ext} \in \mathbf{\Gamma}_c(M', E')^{\mathbf{\Gamma}_c(M,E)}$ is defined as

$$\forall S \in \mathbf{\Gamma}_c(M, E) \quad \mathbf{ext}(S) := F \circ S \circ \varphi^{-1} \cup \mathbf{0}_{M' \setminus f(M)}$$

It's easy to check that $\mathscr{ManGHBund}$ gives a coherent definition of a category, furthermore defining a QFT as a functor from $\mathscr{ManGHBund}$ to $\mathscr{AlgC}^*$ as we did in the previous chapters works the same way.

The first step for constructing a functor of the required type is to construct a functor from $\mathscr{ManGHBund}$ to $\mathscr{Sympl}$.

Theorem 4.3.1. There exists a covariant functor

$$\mathscr{MS} : \mathscr{ManGHBund} \longrightarrow \mathscr{Sympl}$$

defined by

$$\mathscr{MS}((M, g), E, P) := \mathbf{Sym}(M, E, P)$$

for $((M, g), E, P) \in \mathscr{ManGHBund}$ and

$$\mathscr{MS}(\varphi, F) : \mathbf{Sym}(M, E, P) \longrightarrow \mathbf{Sym}(M', E', P')$$

$$S + \ker(G) \mapsto \mathbf{ext}(S) + \ker(G')$$

for $(\varphi, F) \in \mathbf{Hom}_{\mathscr{ManGHBund}}(((M, g), E, P), ((M', g'), E', P'))$

Proof. First let $((M,g),E,P)$ and $((M',g'),E',P')$ be two objects in $\mathcal{ManGHBund}$. Let $G_\pm, G'_\pm$ be respectively the Green's operators for P and P' and let G and G' be $G_+ - G_-$ and $G'_+ - G'_-$ respectively. Now let (φ, F) be a morphism between the two objects of $\mathcal{ManGHBund}$.

Before proving the functorial properties of $\mathcal{MS}$ let's first demonstrate that

$$\mathbf{res} \circ G'_\pm \circ \mathbf{ext} = G_\pm$$

where

$$\forall S \in \mathbf{\Gamma}_c(M',E') : \quad \mathbf{res}(S) := F^{-1} \circ S \circ \varphi$$

Firstly because the image of M under φ is causally compatible, we have

$$\begin{gathered}
\sup\big((\mathbf{res} \circ G'_\pm \circ \mathbf{ext})(S)\big) = \\
= \sup\big(\mathbf{res}((G'_\pm \circ \mathbf{ext})(S))\big) = \\
= \sup\big(F^{-1} \circ((G'_\pm \circ \mathbf{ext})(S)) \circ \varphi\big) = \\
= \varphi^{-1}\big(\sup\big(F^{-1} \circ((G'_\pm \circ \mathbf{ext})(S))\big)\big) = \\
= \varphi^{-1}\big(\sup((G'_\pm \circ \mathbf{ext})(S))\big) \subseteq \\
\subseteq \varphi^{-1}\Big(\mathbf{J}^{M'}_\pm(\sup(\mathbf{ext}(S)))\Big) = \\
= \varphi^{-1}\Big(\mathbf{J}^{M'}_\pm\big(\sup\big(F \circ S \circ \varphi^{-1}\big)\big)\Big) = \\
= \varphi^{-1}\Big(\mathbf{J}^{M'}_\pm(\varphi(\sup(S)))\Big) = \\
= \mathbf{J}^{M'}_\pm(\sup(S))
\end{gathered}$$

By construction of the category morphisms $P' \circ \mathbf{ext} = \mathbf{ext} \circ P$. Furthermore $\mathbf{res} \circ \mathbf{ext} = \mathrm{id}_{\mathbf{\Gamma}(M,E)}$. Let

$$\mathbf{\Gamma}_{\varphi(M)}(M',E') := \{S' \in \mathbf{\Gamma}(M',E') \mid \sup(S') \subseteq \varphi(M)\}$$

then we have

$$\mathbf{ext} \circ \mathbf{res} = \mathrm{id}_{\mathbf{\Gamma}(\varphi(M),M')}$$

Now we can check

$$P \circ \mathbf{res} = \mathbf{res} \circ \mathbf{ext} \circ P \circ \mathbf{res} = \mathbf{res} \circ P' \circ \mathbf{ext} \circ \mathbf{res}$$

From this we have that

$$P \circ \mathbf{res} \restriction \mathbf{\Gamma}_{\varphi(M)}(M', E') = \mathbf{res} \circ P' \restriction \mathbf{\Gamma}_{\varphi(M)}(M', E')$$

Now we can check

$$\begin{gathered} P \circ \mathbf{res} \circ G'_\pm \circ \mathbf{ext} = \\ = \mathbf{res} \circ P' \circ G'_\pm \circ \mathbf{ext} = \mathbf{res} \circ \mathrm{id}_{\mathbf{\Gamma}_c(M', E')} \circ \mathbf{ext} = \\ = \mathrm{id}_{\mathbf{\Gamma}_c(M,E)} \end{gathered}$$

The first equality holds, because $G'_\pm \circ \mathbf{ext}$ is supported on a subset of $\varphi(M)$ and we can use the commutation of P and $\mathbf{res}$.

Furthermore we check

$$\begin{gathered} \mathbf{res} \circ G'_\pm \circ \mathbf{ext} \circ P = \\ = \mathbf{res} \circ G'_\pm \circ P' \circ \mathbf{ext} = \mathbf{res} \circ \mathrm{id}_{\mathbf{\Gamma}_c(M', E')} \circ \mathbf{ext} = \\ = \mathrm{id}_{\mathbf{\Gamma}_c(M,E)} \end{gathered}$$

With all these properties we can see that $\mathbf{res} \circ G'_\pm \circ \mathbf{ext}$ are Green's hyperbolic operators for P, and from their uniqueness we finally get

$$\mathbf{res} \circ G'_\pm \circ \mathbf{ext} = G_\pm$$

Now let's prove that $\mathscr{MS}$ is a functor.

Firstly we observe that the map $\mathbf{ext}$ is linear on smooth sections, to prove that it induces linear map $\mathscr{MS}(\varphi, F)$ on the quotient spaces, we have to show that sections from the kernel of G are sent to sections of the kernel of G'.

From the chapter on Green's operators, we know that if $S \in \ker(G)$, then $S = P(S_0)$ for some $S_0 \in \mathbf{\Gamma}_c(M, E)$. Now

$$\mathbf{ext}(S) = \mathbf{ext}(P(S_0)) = P'(\mathbf{ext}(S_0)) \in P'^{(\mathbf{\Gamma}_c(M', E'))} = \ker(G')$$

From this the map $\mathscr{MS}(\varphi, F)$ is linear on $\frac{\mathbf{\Gamma}_c(M,E)}{\ker(G)}$ and $\frac{\mathbf{\Gamma}_c(M',E')}{\ker(G')}$.

Now to check that the map is symplectomorphism, take $S_0, S_1 \in \mathbf{\Gamma}_c(M, E)$ and compute

$$\int_{M'} \langle G'(\mathbf{ext}(S_0)), \mathbf{ext}(S_1)\rangle dV =$$

$$= \int_M \Big\langle \mathbf{res}\big(G'^{(\mathbf{ext}(S_0))}\big), \mathbf{res}(\mathbf{ext}(S_1))\Big\rangle dV =$$

$$= \int_M \langle G(S_0), S_1\rangle dV$$

From this the linear map $\mathscr{MS}(\varphi, F)$ preserves the symplectic product so it is of the right type.

For $(\mathrm{id}_M, \mathrm{id}_S)$ the $\mathbf{ext}$ map acts as identity, so obviously

$$\mathscr{MS}(\mathrm{id}_M, \mathrm{id}_S) = \mathrm{id}_{\mathbf{Sym}(M,E,P)}$$

The covariant functiorial property

$$\mathscr{MS}((\varphi_1, F_1) \circ (\varphi_0, F_0)) = \mathscr{MS}(\varphi_1, F_1) \circ \mathscr{MS}(\varphi_0, F_0)$$

holds trivially, because of the way $\mathbf{ext}$ composes. □

Definition 4.3.1. We construct the functor $\mathscr{Q}$ by composing $\mathscr{MS}$ with the functor carrying a symplectic vector space to it's associated Weyl Algebra

$$\mathscr{Q} := \mathscr{CCR} \circ \mathscr{MS} : \mathscr{ManGHBund} \longrightarrow \mathscr{AlgC}^*$$

Now it remains to demonstrate the causality and time-slicibility properties of this functor.

Theorem 4.3.2. The functor $\mathscr{Q} := \mathscr{CCR} \circ \mathscr{MS}$ satisfies the causality property, namely for morphisms $(\varphi_0, F_0), (\varphi_1, F_0)$ from $\mathcal{M}_0 := ((M_0, g_0), E_0, P_0)$ and $\mathcal{M}_1 := ((M_1, g_1), E_1, P_1)$ respectively to $\mathcal{M} := ((M, g), E, P)$, such that $\varphi_0(M_0)$ and $\varphi_1(M_1)$ are causally disjoint

$$[\mathscr{Q}(\varphi_0, F_1)(\mathscr{Q}(\mathcal{M}_0)), \mathscr{Q}(\varphi_1, F_1)(\mathscr{Q}(\mathcal{M}_1))] = \{0\}$$

Proof. Let ω be the symplectic form of $\mathbf{Sym}(M, E, P)$ and $S_0 \in \mathbf{\Gamma}_c(M_0, E_0)$ and $S_1 \in \mathbf{\Gamma}_c(M_1, E_1)$. Since $\varphi_0(M_0)$ and $\varphi_1(M_1)$ are causally disjoint, then $S_0^{\mathbf{ext}} := \mathbf{ext}_{\varphi_0,F_0}(S_0)$ and $S_1^{\mathbf{ext}} := \mathbf{ext}_{\varphi_1,F_1}(S_1)$ are with disjoint support. Now

$$\omega\Big(\mathbf{ext}_{\varphi_0,F_0}(S_0) + \ker(G), \mathbf{ext}_{\varphi_1,F_1}(S_1) + \ker(G)\Big) :=$$
$$= \int_M \langle G(S_0^{\mathbf{ext}}), S_1^{\mathbf{ext}} \rangle = 0$$

So now if $\mathcal{W}$ sends us from the symplectic space to the Weyl algebra associated with M, we have

$$\mathcal{W}\big(S_0^{\mathbf{ext}} + \ker(G)\big) \cdot \mathcal{W}\big(S_1^{\mathbf{ext}} + \ker(G)\big) =$$
$$= e^{-i\frac{\omega\left(S_0^{\mathbf{ext}} + \ker(G), S_1^{\mathbf{ext}} + \ker(G)\right)}{2}} \mathcal{W}\big(S_0^{\mathbf{ext}} + S_1^{\mathbf{ext}} + \ker(G)\big) =$$
$$= e^0 \mathcal{W}\big(S_0^{\mathbf{ext}} + S_1^{\mathbf{ext}} + \ker(G)\big) =$$
$$= \mathcal{W}\big(S_0^{\mathbf{ext}} + S_1^{\mathbf{ext}} + \ker(G)\big)$$

This equality directly implies that the commutator of the mapped algebra elements is 0. Since the entire algebras mapped with $\mathcal{W}$ are generated from elements of this kind, we just proved the theorem $\square$

Theorem 4.3.3. The functor $\mathscr{Q} := \mathscr{CCR} \circ \mathscr{MS}$ satisfies the time-slice property, namely for morphisms $(\varphi_0, F_0), (\varphi_1, F_0)$ from $\mathcal{M}_0 := ((M_0, g_0), E_0, P_0)$ and $\mathcal{M}_1 := ((M_1, g_1), E_1, P_1)$ respectively to $\mathcal{M} := ((M, g), E, P)$, such that $\mathcal{M}_0 \subseteq \mathcal{M}_1$ and they contain a common Cauchy surface, then

$$\mathscr{Q}(\varphi_0, F_0)\ \mathscr{Q}(\mathcal{M}_0) = \mathscr{Q}(\varphi_1, F_1)\ \mathscr{Q}(\mathcal{M}_1)$$

The complete proof of this statement can be seen in **[18]**.

States

In this chapter we will describe how one can recover the state space of the system just from the algebra of observables.

Definition 4.4.1. A ***state on*** a C^* algebra $\mathcal{A}$ is a linear functional $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ such that

- $\forall a \in \mathcal{A} : \quad \omega(a^* a) \geq 0$
- $\omega(1_{\mathcal{A}}) = 1$

Theorem 4.4.1. ***(GNS Representation Construction)*** For every state ω on a C^* algebra A, the map

$$\hat{\omega} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{C}$$

$$(a, b) \mapsto \omega(b^* a)$$

is a positive semi-definite Hermitian product. Furthermore $\hat{\omega}$ induces a inner product $\langle \cdot, \cdot \rangle_\omega$ on $\hat{\mathcal{A}}_\omega := \mathcal{A}/\{a \in \mathcal{A} \mid \hat{\omega}(a, a) = 0\}$ defined by

$$\langle [a]_\omega, [b]_\omega \rangle_\omega := \hat{\omega}(a, b)$$

The Hilbert space completion $\mathcal{H}_\omega$ of $\hat{\mathcal{A}}_\omega$ gives representation of A with the action

$$\pi_\omega : \mathcal{A} \longrightarrow \mathbf{Hom}_{\mathbb{C}}(\mathcal{H}_\omega, \mathcal{H}_\omega)$$

$$\pi_\omega(a)([b]_\omega) := [ab]_\omega$$

where

$$[x]_\omega := x + \{a \in \mathcal{A} \mid \hat{\omega}(a, a) = \mathbb{0}_{\mathcal{A}}\}$$

Proof. Firstly let's check that $\hat{\omega}$ gives a Hermitian product. Positive semi-definiteness follows directly from the definition and the fact that ω is positive linear functional. Let's check the linearity on the first argument.

$$\hat{\omega}(\alpha a + \beta b, c) =$$
$$= \omega(c^*(\alpha a + \beta b)) = \omega(\alpha c^* a + \beta c^* b) = \alpha \omega(c^* a) + \beta \omega(c^* b) =$$
$$= \alpha \hat{\omega}(a, c) + \beta \hat{\omega}(b, c)$$

Before checking for conjugate symmetry, let's first prove few intermediate statements. One should observe that every element from the algebra a can be written as

$$a = \frac{a + a^*}{2} + \frac{a - a^*}{2} = \frac{a + a^*}{2} + i\frac{(-ia) + (-ia)^*}{2} = a_0 + ia_1$$

where clearly a_0 and a_1 are self adjoint. Now from this it's clear that if ω is real on self-adjoint elements then ω would be a $*$-homomorphism so $\omega(a)^* = \omega(a^*)$.

Since from **Theorem 1.4.3** every self adjoint element $x \in \mathcal{A}$ can be expressed as sum of two positive elements $x = x_+ - x_-$, applying ω, we can see that $\omega(x) \in \mathbb{R}$.

Now checking for conjugate symmetry of $\hat{\omega}$ is trivial since

$$\hat{\omega}(a, b) = \omega(b^* a) = \omega\big((a^* b)^*\big) = \omega(a^* b)^* = \hat{\omega}(b, a)^*$$

Now let's look at the map $\langle \cdot, \cdot \rangle_\omega$ on $\hat{\mathcal{A}}_\omega$. We need to check that it's well defined. Let $[a]_\omega = [a']_\omega$ and $[b]_\omega = [b']_\omega$. From this we have by the definition of the equivalence classes that

$$\hat{\omega}(a - a', a - a') = \mathbb{0}_\mathcal{A}$$

$$\hat{\omega}(b - b', b - b') = \mathbb{0}_\mathcal{A}$$

From this

$$\langle [a]_\omega, [b]_\omega \rangle_\omega =$$
$$= \hat{\omega}(a, b) = \hat{\omega}(a' + a - a', b' + b - b') =$$
$$= \hat{\omega}(a', b' + b - b') + \hat{\omega}(a - a', b' + b - b') =$$
$$= \hat{\omega}(a', b') + \hat{\omega}(a', b - b') + \hat{\omega}(a - a', b' + b - b') = \hat{\omega}(a', b') =$$
$$= \langle [a']_\omega, [b']_\omega \rangle_\omega$$

The equality before the last one follows from the fact that if for some element x we have that $\hat{\omega}(x,x) = \mathbb{0}_{\mathcal{A}}$ then for all other elements y we have $\hat{\omega}(x,y)$ is also $\mathbb{0}_{\mathcal{A}}$. This follows directly from the Cauchy-Schwarz inequality for positive semi-definite Hermitian forms because $\hat{\omega}(x,y)^2 \leq \hat{\omega}(x,x)\hat{\omega}(y,y)$.

Now since $\langle \cdot, \cdot \rangle_\omega$ is well defined, and the Hermitian product properties follow from the properties of $\hat{\omega}$ we only need to check that it's positive definite. But this is trivial since if $[a]_\omega \neq [0_{\mathcal{A}}]_\omega$ then $\langle a, a \rangle_\omega$ which is simply $\hat{\omega}(a,a)$ must not be zero, and we know that it's non-negative.

Now let's check that $\pi_{\omega(a)}$ is a well defined function. Let $[b]_\omega = [b']_\omega$. Then

$$\pi_\omega(a)([b]_\omega) =$$
$$= [ab]_\omega = [a(b' + b - b')]_\omega = [ab']_\omega + [a(b - b')]_\omega = [ab']_\omega =$$
$$= \pi_\omega(a)\big([b']_\omega\big)$$

The fact that $\pi_\omega(a)$ is linear and algebra morphism is trivially checked, now let's check that it's a $*$-morphism.

$$\langle \pi_\omega(a)([x]_\omega), [y]_\omega \rangle_\omega =$$
$$= \langle ([ax]_\omega), [y]_\omega \rangle_\omega = \hat{\omega}(ax, y) = \omega(y^* ax) = \omega\big((a^* y)^* x\big) =$$
$$= \hat{\omega}(x, a^* y) = \Big\langle [x]_\omega, [a^* y]_\omega \Big\rangle_\omega =$$
$$= \langle [x]_\omega, \pi_\omega(a^*)([y]_\omega) \rangle_\omega$$

Hence we see that $\pi_\omega(a^*) = (\pi_\omega(a))^*$. □

Here the name "GNS" comes from the names of the mathematicians Gelfand, Naimark and Segal. Outside of the above discussed properties, one can see for the construction above that the unit vector $[\mathbb{1}_{\mathcal{A}}]_\omega$ which is often called ***"vacuum vector"*** can be used to recover the state meaning

$$\Big\langle \pi_{\omega(a)} [\mathbb{1}_{\mathcal{A}}]_\omega, [\mathbb{1}_{\mathcal{A}}]_\omega \Big\rangle_\omega = \omega(a)$$

It's clear that by construction $\pi_\omega(\mathcal{A})[\mathbb{1}_{\mathcal{A}}]_\omega = \hat{A}_\omega$ and is dense in $\mathcal{H}_\omega$. One can also easily prove that the operators $\pi_\omega(a)$ are bounded (details in **[10]**). This leads us to the following definition.

Definition 4.4.2. Let $\mathcal{A}$ be a unitial C^* algebra and ω be a state on it. Then a triple $(\mathcal{H}_\omega, \pi_\omega, \psi_\omega)$ is called a ***GNS Representation*** or ***GNS triple*** for $\mathcal{A}$ and ω if

- $\mathcal{H}_\omega$ is a Hilbert space.
- $\pi_\omega : \mathcal{A} \longrightarrow \boldsymbol{\mathscr{BL}}(\mathcal{H}_\omega)$ is a representation of $\mathcal{A}$.
- The set $\pi_\omega(\mathcal{A})\psi_\omega$ is dense in $\mathcal{H}_\omega$.
- $\langle \pi_\omega(a)\psi_\omega, \psi_\omega \rangle$ for every $a \in \mathcal{A}$

Theorem 4.4.2. For a unitial C^* algebra $\mathcal{A}$ and state ω on it, the GNS triples are unitarily equivalent, meaning for two triples $(\mathcal{H}_\omega, \pi_\omega, \psi_\omega)$ and $(\mathcal{H}'_\omega, \pi'_\omega, \psi'_\omega)$ there exists a unitary operator

$$U : \mathcal{H}_\omega \longrightarrow \mathcal{H}'_\omega$$

$$\forall a \in \mathcal{A} : \quad \pi'_\omega(a) = U\pi_\omega(a)U^{-1}$$

$$\psi'_\omega = U\psi_\omega$$

Proof sketch. Let the dense domains of the two Hilbert spaces, generated by the vacuum vectors be $\mathcal{D}_\omega := \pi_\omega(\mathcal{A})\psi_\omega$ and $\mathcal{D}'_\omega := \pi'_\omega(\mathcal{A})\psi'_\omega$. One then defines the map

$$U : \mathcal{D}_\omega \longrightarrow \mathcal{D}'_\omega$$

$$\forall \pi_{\omega(a)}\psi_\omega \in \mathcal{D}_\omega : \quad U((\pi_\omega(a))(\psi_\omega)) = (\pi'_\omega(a))(\psi'_\omega)$$

One straightforwardly computes that the map is well-defined (assigns a single value to each $\varphi \in \mathcal{D}_\omega$) then that it's linear, bijective and preserves the inner product. Then this map from the respective dense domains can be extended to a unitary map on the entire hilbert spaces $\mathcal{H}_\omega$ and $\mathcal{H}'_\omega$. The other two properties of this map are clear from the definition of it. $\square$

Quantum fields

Now finally it's time to define what fields can be in this framework. Intuitively speaking fields should be something that can be measured in certain bounded region of space time (hence in our framework for them there should be corresponding observable for any such bounded region where they can be measured). One way to model this is to use a operator valued distributions. For every test section with compact support (whos domain will correspond to the space-time region where the field can be measured) they should return a observable.

Definition 4.5.1. Let $\mathcal{Q} : \mathcal{CCR} \circ \mathcal{MS}$ be a quantum field theory and let $((M, g), E, P) \in \mathbf{Obj}(\mathcal{ManGHBund})$. Let ω be a state on $\mathcal{Q}((M, g), E, P)$ and let $(\mathcal{H}_\omega, \pi_\omega, \psi_\omega)$ be a GNS triple for it. A ***quantum field corresponding to*** ω is a distribution

$$\Phi_\omega : \mathbf{\Gamma}_c(M, E) \longrightarrow \mathcal{H}_\omega$$

such that

$$e^{it\Phi_\omega(f)} = \pi_\omega(\mathcal{W}(tf + \ker G))$$

where t is real, $\mathcal{W}$ is the Weyl map for $\mathcal{MS}((M, g), E, P)$ and G is the corresponding difference of advanced and retarded Green's operators for P.

Notice that here representations are used instead of the "raw" C^* algebra elements. The reason for this is the discontinuity of the Weyl system map $\mathcal{W}$ which makes the construction of C^*-algebra valued distribution with the required properties problematic. Next we give a construction of quantum fields for corresponding to certain states that have nice enough properties so that the construction works. Before that let's make some definitions which describe the states for which that's the case.

Definition 4.5.2. Let $\mathcal{Q} : \mathcal{CCR} \circ \mathcal{MS}$ be a quantum field theory and let $((M,g),E,P) \in \mathbf{Obj}(\mathcal{ManGHBund})$ and $\mathcal{W}$ be the Weyl map for $\mathcal{MS}((M,g),E,P)$. Let ω be a state on $\mathcal{Q}((M,g),E,P)$ and let $(\mathcal{H}_\omega, \pi_\omega, \psi_\omega)$ be a GNS triple for it. The state ω is called ***regular*** if the map

$$t \to (\pi_\omega(\mathcal{W}(tf + \ker G)))(h)$$

is continuous for each fixed $h \in \mathcal{H}_\omega$.

The state is called ***strongly regular*** if on top of being regular, the following is satisfied.

- For any $f \in \mathbf{\Gamma}_c(M,E)$ there is a dense subspace $\mathcal{D}_\omega \subset \mathcal{H}_\omega$ contained in the domain of the map

$$h \mapsto -i\frac{d}{dt}\bigg|_{t=0} (\pi_\omega(\mathcal{W}(tf + \ker G)))(h)$$

and the image of this map on the set $\mathcal{D}_\omega$ is contained in $\mathcal{D}_\omega$

- The map from $\mathbf{\Gamma}_c(M,E)$ to $\mathcal{H}_\omega$

$$f \mapsto -i\frac{d}{dt}\bigg|_{t=0} (\pi_\omega(\mathcal{W}(tf + \ker G)))(h)$$

is continuous for every fixed $h \in \mathcal{D}_\omega$.

One might wonder if the derivative in question even exists in the case of regular state because if not, then this definition would make no sense. One can check that this is the case. One can check that if the symplectic form of $\mathcal{MS}((M,g),E,P)$ is Ω then

$$\pi_\omega(\mathcal{W}((t+s)f + \ker G)) = \pi_\omega(\mathcal{W}(tf + sf + \ker G)) =$$
$$= \pi_\omega\left(\exp\left(i\frac{\Omega(tf + \ker G, sf + \ker G)}{2}\right)\mathcal{W}(tf + \ker G)\mathcal{W}(sf + \ker G)\right) =$$
$$= \pi_\omega(\mathcal{W}(tf + \ker G)\mathcal{W}(sf + \ker G)) =$$
$$= \pi_\omega(\mathcal{W}(tf + \ker G))\pi_\omega(\mathcal{W}(sf + \ker G))$$

Here we have used the fact that $\Omega(tf+\ker G, sf+\ker G)=0$ because the form is anti-symmetric. From the above it follows that

$$t \mapsto \pi_\omega(\mathcal{W}(tf+\ker G))$$

is strongly continuous one-parameter group on $\mathcal{H}_\omega$, so according to **Theorem 1.6.3** and **Theorem 1.6.2** the definition makes sense.

Theorem 4.5.1. Let $\mathscr{Q} : \mathscr{CCR} \circ \mathscr{MS}$ be a quantum field theory and let $((M,g),E,P) \in \mathbf{Obj}(\mathscr{ManGHBund})$. Let ω be a strongly regular state on $\mathscr{Q}((M,g),E,P)$ and let $(\mathcal{H}_\omega, \pi_\omega, \psi_\omega)$ be a GNS triple for it. The Quantum field corresponding to ω exists and is defined by

$$\Phi_\omega : \mathbf{\Gamma}_c(M,E) \longrightarrow \mathcal{H}_\omega$$

$$(\Phi_\omega(f))(h) := -i\frac{d}{dt}\bigg|_{t=0} (\pi_\omega(\mathcal{W}(tf+\ker G)))(h)$$

for all $f \in \mathbf{\Gamma}_c(M,E)$ and $h \in \mathcal{H}_\omega$.

Proof. Let $\mathcal{D}_\omega$ be the dense domain of the map by the assumption of strong regularity of ω and let Ω is the symplectic form of $\mathscr{MS}((M,g),E,P)$ and let's denote $[f]_G$ instead of $f+\ker(G)$ for equivalence classes on that space. Take two arbitrary $f,g \in \mathbf{\Gamma}_c(M,E)$ and $\alpha,\beta \in \mathbb{R}$, then compute

$$\Phi_\omega(\alpha f+\beta g)(h) :=$$

$$:= -i\frac{d}{dt}\bigg|_{t=0} \pi_{\omega(\mathcal{W}(t[\alpha f+\beta g]_G))}(h)) =$$

$$= -i\frac{d}{dt}\bigg|_{t=0} \left(e^{i\alpha\beta t^2\frac{\Omega([f]_G,[g]_G)}{2}}\pi_\omega(\mathcal{W}(\alpha t[f]_G))\pi_\omega(\mathcal{W}(\beta t[g]_G))(h)\right) =$$

$$= -i\pi_\omega(\mathcal{W}([0]_G))\frac{d}{dt}\bigg|_{t=0} \pi_\omega(\mathcal{W}(\alpha t[f]_G))(h) -$$

$$-i\pi_\omega(\mathcal{W}([0]_G))\frac{d}{dt}\bigg|_{t=0} \pi_\omega(\mathcal{W}(\beta t[g]_G))(h) =$$

$$= \alpha\Phi_\omega(f)(h) + \beta\Phi_\omega(g)(h)$$

This shows that Φ_ω depends linearly on smoothly supported sections, hence it's a $\mathcal{H}_\omega$ valued distribution.

The fact that $e^{it\Phi_\omega(f)} = \pi_\omega(\mathcal{W}(t[f]_G))$ follows directly from **Theorem 1.6.3** and **Theorem 1.6.2**. □

Theorem 4.5.2. Let Φ_ω be a quantum field corresponding to ω on $((M, g), E, P)$ then

- $\Phi_\omega(P(f)) = \mathbb{0}_{\mathcal{D}_\omega^{\mathcal{D}_\omega}}$
- Φ_ω satisfies the ***cannonical commutation relations***

$$[\Phi_\omega(f), \Phi_\omega(g)] = i\left(\int_M \langle Gf, g\rangle dV\right)\mathrm{id}_{\mathcal{D}_\omega}$$

for all $f, g \in \mathbf{\Gamma}_c(M, E)$ and $h \in \mathcal{D}_\omega$

Proof. Since $G_\pm P = \mathrm{id}_{\mathbf{\Gamma}_c(M,E)}$, we have that

$$GP := (G_+ - G_-)P = \mathbb{0}_{\mathbf{\Gamma}_c(M,E)^{\mathbf{\Gamma}_c(M,E)}}$$

From this we conclude that $P(f) \in \ker(G)$ for any $f \in \mathbf{\Gamma}_c(M, E)$. Now we compute for $h \in \mathcal{D}_\omega$

$$\Phi_\omega(P(f))(h) :=$$
$$:= -i\frac{d}{dt}\bigg|_{t=0} \pi_\omega(\mathcal{W}(t[P(f)]_G)(h) = -i\frac{d}{dt}\bigg|_{t=0} \pi_\omega(\mathcal{W}(t[0]_G)(h) = -i\frac{d}{dt}\bigg|_{t=0} h =$$
$$= \mathbb{0}_{\mathcal{H}}$$

from here we have that $\Phi_\omega(P(f)) = \mathbb{0}_{\mathcal{D}_\omega^{\mathcal{D}_\omega}}$.

Now from the definition of $\mathcal{W}$ from the Weyl system we have for $[f]_G, [g]_G \in \mathbf{Sym}(M, E, P)$

$$\mathcal{W}([f + g]_G) = e^{i\frac{\Omega([f]_G,[g]_G)}{2}}\mathcal{W}([f]_G)\mathcal{W}([g]_G)$$

$$\mathcal{W}([f + g]_G) = e^{i\frac{\Omega([g]_G,[f]_G)}{2}}\mathcal{W}([g]_G)\mathcal{W}([f]_G)$$

From these two equalities we can see that

$$\mathcal{W}([f]_G)\mathcal{W}([g]_G) = e^{-i\Omega([f]_G,[g]_G)}\mathcal{W}([g]_G)\mathcal{W}([f]_G)$$

Now using this and the representation properties of π_ω we see that

$$\Phi_\omega(f)\Phi_\omega(g)(h) :=$$

$$:= \left(-i\frac{d}{dt}\bigg|_{t=0}\pi_\omega(\mathcal{W}(t[f]_G))\right)\left(-i\frac{d}{ds}\bigg|_{s=0}\pi_\omega(\mathcal{W}(s[g]_G))\right)(h) =$$

$$= -\frac{\partial^2}{\partial t\partial s}\bigg|_{t=s=0}\pi_\omega(\mathcal{W}(t[f]_G)\mathcal{W}(s[g]_G))(h) =$$

$$= -\frac{\partial^2}{\partial t\partial s}\bigg|_{t=s=0}\pi_\omega\big(e^{-i\Omega(t[f]_G,s[g]_G)}\mathcal{W}(s[g]_G)\mathcal{W}(t[f]_G)\big)(h) =$$

$$= -\frac{\partial^2}{\partial t\partial s}\bigg|_{t=s=0}\big(e^{-its\Omega([f]_G,[g]_G)}\pi_\omega(\mathcal{W}(s[g]_G)\mathcal{W}(t[f]_G))(h)\big) =$$

$$= -\frac{\partial}{\partial t}\bigg|_{t=0}((-it\Omega([f]_G,[g]_G)\mathcal{W}(t[f]_G) + i\Phi_\omega(g)\mathcal{W}(t[f]_G))(h)) =$$

$$= i\Omega([f]_G,[g]_G)h + \Phi_\omega(g)\Phi_\omega(f)(h) =$$

$$= i\left(\int_M \langle Gf, g\rangle dV\right)h + \Phi_\omega(g)\Phi_\omega(f)(h)$$

But this equality implies exactly

$$[\Phi_\omega(f), \Phi_\omega(g)] = i\left(\int_M \langle Gf, g\rangle dV\right)\mathrm{id}_{\mathcal{D}_\omega}$$

□

Notice that as a consequence of the canonical commutation relations, if f and g have causally disjoint supports, then $[\Phi_\omega(f), \Phi_\omega(g)] = 0$, because $\sup Gf \subseteq \mathbf{J}_-^M(\sup(f)) \cup \mathbf{J}_+^M(\sup(f))$ which has zero intersection with the support of g. This is consistent with the properties we've required from observables to have in this AQFT framework.

References

1. Felix Nagel: Set theory and topology - Part I, https://arxiv.org/pdf/1306.2934
2. J. Donald Monk: Mathematical Logic, https://euclid.colorado.edu/~monkd/monk29.pdf
3. J. Donald Monk: Notes On Set Theory, https://euclid.colorado.edu/~monkd/full.pdf
4. Tom Leinster: Rethinking set theory, https://arxiv.org/pdf/1212.6543
5. Daniel Murfet: Foundations for Category Theory, http://therisingsea.org/notes/FoundationsForCategoryTheory.pdf
6. Felix Nagel: Set theory and topology - Part II, https://arxiv.org/pdf/1306.6926
7. Thomas W. Judson: Abstract Algebra - Theory and Applications, https://judsonbooks.org/aata-files/aata-20240706-sage.pdf
8. Anthony W. Knapp: Basic Algebra, https://www.math.stonybrook.edu/~aknapp/download/b2-alg-coverandinside.pdf
9. Nicolas Bourbaki: Elements Of Mathematics - Topological Vector Spaces, https://ia600507.us.archive.org/29/items/elements_of_mathematics/05_Topological_Vector_Spaces.pdf
10. Karen R. Strung: An introduction to C^*-algebras and the classification programm, https://strung.me/karen/CStarIntroDraft.pdf
11. Michael Reed, B. S.: Methods of Modern Mathematical Physics I, https://tinyurl.com/43v6dpjt
12. Robert G. Bartle: The Elements Of Integration And Lebesgue Measure, https://tinyurl.com/bartlemeasure
13. J. Dieudonne: Foundations Of Modern Analysis, https://ia601400.us.archive.org/28/items/in.ernet.dli.2015.160723/2015.160723.Foundations-Of-Modern-Analysis_text.pdf

14. Gerald Teschl: Mathematical Methods in Quantum Mechanics, https://www.mat.univie.ac.at/~gerald/ftp/book-schroe/schroe.pdf
15. Valter Moretti: Mathematical Foundations Of Quantum mechanics, https://arxiv.org/pdf/1508.06951
16. Barrett O' Neill: Semi Reimannian Geometry With Applications To Relativity, https://tinyurl.com/58th8pr7
17. Antonio N Bernal, M. S.: On Smooth Cauchy hypersurfaces and Geroch's splitting theorem, https://arxiv.org/pdf/gr-qc/0306108
18. Christian Bar, F. P., Nicolas Ginoux: Wave Equations on Lorentzian Manifolds and Quantization, https://arxiv.org/pdf/0806.1036
19. Christian Bar: Green-Hyperbolic Operators On Globally Hyperbolic Spacetimes, https://arxiv.org/pdf/1310.0738
20. Christian Bar, N. G.: Classical And Quantum Fields On Lorentzian Manifolds, https://arxiv.org/pdf/1104.1158
21. Liviu I. Nicolaescu: Lectures On The Geometry Of Manifolds, https://www3.nd.edu/~lnicolae/Lectures.pdf
22. Romeo Brunetti, R. V., Klaus Fredengahen: The Generally Covariant Locality Principle – A New Paradigm for Local Quantum Field Theory, https://arxiv.org/pdf/math-ph/0112041