PROBABILITY SCRIBBLES

MIHAIL MLADENOV

ABSTRACT. This text is intended to condense my own knowledge in probability theory and is not optimized for reading by other people. The main goal is for me to gain understanding of some simple facts in probability theory.

CONTENTS

1. Preface 2
2. Measure Theory Basics 2
2.1. Measurable spaces 2
2.2. Measurable functions 4
2.3. Measure spaces 4
2.4. Integration 4
3. Basic Probability 4
3.1. Probability spaces 4

Date: February 25, 2018.

1. Preface

Throughout this text the ZFC axioms are assumed and first-order predicate logic is used as meta-language. Many results in set theory and topology will be used without providing proof. Throughout this book the formal language is freely mixed with informal/abusive/bad/hand-wavy English and symbolic notation. Although the goal has been to minimize hand-wavy bullshit there's still a lot of crap.

Since this is not intended to be directly read by other people there's no need to clarify a lot of the notation used (which is fairly standard), except in the cases where there's serious danger that I might forget what I meant by using that notation in which case explanation should be provided.

The definition-theorem-proof style is used in writing this text.

2. Measure Theory Basics

2.1. Measurable spaces.

Definition 1. *Let F be a set. A nonempty set $\mathcal{F} \subseteq \mathscr{P}(E)$ is called σ-algebra on F if*

- $(\forall A \in \mathcal{F})(F \setminus B \in \mathcal{F})$
- $\left(\forall A \in \mathcal{F}^{\mathbb{N}}\right)\left(\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{F}\right)$ *

Proposition 1. *If $\mathcal{F}$ is σ-algebra on F, then $\emptyset \in \mathcal{F}$ and $F \in \mathcal{F}$.*

Proof. Since $\mathcal{F}$ is σ-algebra it's nonempty. Now let $A \in \mathcal{F}$ then it's true that $F = A \bigcup (F \setminus A) \in \mathcal{F}$ and $\emptyset = F \setminus F \in \mathcal{F}$. □

Definition 2. *We say that a sequence $A \in \mathcal{F}^{\mathbb{N}}$ is increasing with limit A_∞ (denoted by $A \uparrow A_\infty$) if*

- $(\forall n \in \mathbb{N})(A_n \subseteq A_{n+1})$
- $\bigcup\limits_{n\in\mathbb{N}} A_n = A_\infty$

Definition 3. *If F is a set we denote the set of all σ-algebras over F by $\mathfrak{Sig}(F)$.*

Definition 4. *The ordered pair $\langle F, \mathcal{F}\rangle$ is called measurable space if $\mathcal{F} \in \mathfrak{Sig}(F)$.*

Definition 5. *Let F be a set. A set $\mathcal{F} \subseteq \mathscr{P}(E)$ is called d-system on F if*

- $F \in \mathcal{F}$
- $(\forall A \in \mathcal{F})(\forall B \in \mathcal{F})(A \subseteq B \Longrightarrow B \setminus A \in \mathcal{F})$
- $\left(\forall A \in \mathcal{F}^{\mathbb{N}}\right)(A \uparrow A_\infty \Longrightarrow A_\infty \in \mathcal{F})$

Definition 6. *Let F be a set. A set $\mathcal{F} \subseteq \mathscr{P}(E)$ is called π-system on F if*

$$(\forall A \in \mathcal{F})(\forall B \in \mathcal{F})(A \cap B \in \mathcal{F})$$

Proposition 2. *Let F be a set. A set $\mathcal{F} \subseteq \mathscr{P}(E)$ is σ-algebra on F if and only if it's both π and d system.*

Proof. Assuming that $\mathcal{F}$ is σ-algebra on F, we can immediately infer that it's π system using the countable union property and the complement property of the σ-algebra together with De Morgan's law. It's also immediate that it's d system.

Now assume that $\mathcal{F}$ is π and d system. We have that $F \in \mathcal{F}$ and since

$$(\forall A \in \mathcal{F})(\forall B \in \mathcal{F})(A \subseteq B \Longrightarrow B \setminus A \in \mathcal{F})$$

we have that

$$(\forall B \in \mathcal{F})(F \setminus B \in \mathcal{F})$$

*We will often write A_n instead of $A(n)$ for $n \in \mathbb{N}$

Now let $S \in \mathcal{F}^{\mathbb{N}}$ be a sequence. Since a π system is closed under intersection and furthermore we've proven that $\mathcal{F}$ is closed under complements it's also closed under finite number of unions using De Morgan's law and trivial induction. Now we construct the sequence $\tilde{S} \in \mathcal{F}^{\mathbb{N}}$

$$\tilde{S}_n := \bigcup_{k \in (n+1)} S_k$$

Now we have that $(\forall n \in \mathbb{N}) \left(\tilde{S}_n \subseteq \tilde{S}_{n+1} \right)$ and so it follows that $\bigcup_{n \in \mathbb{N}} \tilde{S}_n \in \mathcal{F}$. On the other hand we have

$$\mathcal{F} \ni \bigcup_{n \in \mathbb{N}} \tilde{S}_n = \bigcup_{n \in \mathbb{N}} S_n$$

From the things proven so far we can conclude that $\mathcal{F}$ is σ-algebra. $\square$

Proposition 3. *Let F be a set. Let $\mathfrak{F} \subseteq \mathscr{P}(\mathscr{P}(F))$ be collection of σ-algebras over F. Then the intersection of this collection $\bigcap \mathfrak{F}$ is again σ-algebra.*

Proof. Let $A \in \bigcap \mathfrak{F}$. From this we can conclude that $(\forall \mathcal{F} \in \mathfrak{F})(A \in \mathcal{F})$ and since all of them are σ-algebras we have $(\forall \mathcal{F} \in \mathfrak{F})(F \setminus A \in \mathcal{F})$ from which we immediately conclude that $F \setminus A \in \bigcap \mathfrak{F}$.

Now let $S \in (\bigcap \mathfrak{F})^{\mathbb{N}}$ be a sequence in $\bigcap \mathfrak{F}$. Since for every $\mathcal{F} \in \mathfrak{F}$ we have that $\bigcap \mathfrak{F} \subseteq \mathcal{F}$ it follows that S is sequence in every $\mathcal{F}$. From this we conclude that $\bigcup_{n \in \mathbb{N}} S_n \in \mathcal{F}$ for every $\mathcal{F}$ since it's σ-algebra. But then $\bigcup_{n \in \mathbb{N}} S_n \in \bigcap \mathfrak{F}$. $\square$

Definition 7. *Let F be a set and $\mathcal{C} \subseteq \mathscr{P}(F)$. We define the σ-algebra generated by $\mathcal{C}$, denoted by $\sigma\mathcal{C}$ to be the set*

$$\sigma\mathcal{C} := \bigcap \{\mathcal{F} \in \mathfrak{Sig}(F) \mid \mathcal{C} \subseteq \mathcal{F}\}$$

Lemma 1. *Let $\mathcal{D}$ be a d-system on F and pick $D \in \mathcal{D}$. The set*

$$\mathcal{D}_D := \{X \in \mathcal{D} \mid X \cap D \in \mathcal{D}\}$$

is also d-system on F.

Proof. Since $\mathcal{D}$ is a d-sysyem $F \in \mathcal{D}$ and since $F \cap D = D \in \mathcal{D}$ then $F \in \mathcal{D}_D$.

Now let $X, Y \in \mathcal{D}_D$ and $X \subseteq Y$. Since $X \cap D$ and $Y \cap D$ are in $\mathcal{D}$ we have that $(Y \cap D) \setminus (X \cap D) \in \mathcal{D}$. Now let $z \in (Y \cap D) \setminus (X \cap D)$. We have that $z \in Y \cap D$ and $z \notin X \cap D$. From here $z \in Y$ and $z \in D$ and $z \notin X$ hence $z \in (Y \setminus X) \cap D$ and so $(Y \cap D) \setminus (X \cap D) \subseteq (Y \setminus X) \cap D$. If $w \in (Y \setminus X) \cap D$ then $w \in D$ and $w \in Y$ and $w \notin X$. From here follows that $w \in (Y \cap D) \setminus X$ and so $w \in (Y \cap D) \setminus (X \cap D)$. This proves that $(Y \setminus X) \cap D \subseteq (Y \cap D) \setminus (X \cap D)$. From these inclusions we can conclude that $(Y \setminus X) \cap D = (Y \cap D) \setminus (X \cap D)$. Now $(Y \setminus X) \cap D \in \mathcal{D}$ and so $Y \setminus X \in \mathcal{D}_D$.

Let $A \in \mathcal{D}_D^{\mathbb{N}}$ be a sequence and $A \uparrow A_\infty$. Since for all n it's true that $A_n \in \mathcal{D}_D$ we have that $A_n \cap D \in \mathcal{D}$. Now since $\mathcal{D}$ is d-system

$$A_\infty \cap D = \left(\bigcup_{k \in \mathbb{N}} A_n \right) \cap D = \bigcup_{k \in \mathbb{N}} (A_n \cap D) \in \mathcal{D}$$

From this we conclude that $A_\infty \in \mathcal{D}_D$. $\square$

Theorem 1 (Monotone class theorem)**.** *If $\mathcal{D}$ is a d-system on F and $\mathcal{C} \subseteq \mathcal{D}$ is π-system on F, then $\sigma\mathcal{C} \subseteq \mathcal{D}$.*

Proof. Let $\mathcal{D}_* \subseteq \mathcal{D}$ be the intersection of all d-systems containing $\mathcal{C}$.

If $C \in \mathcal{C}$ we define

$$\mathcal{D}'_* := \{D \in \mathcal{D}_* \mid D \cap C \in \mathcal{D}_*\}$$

By the previous lemma $\mathcal{D}'_*$ is a d-system. Now if $X \in \mathcal{C} \subseteq \mathcal{D}_*$ since $\mathcal{C}$ is a π-system we have that $X \cap C \in \mathcal{C} \subseteq \mathcal{D}_*$ which demonstrates that $X \in \mathcal{D}'_*$. Since $\mathcal{D}'_*$ is a d-system containing $\mathcal{C}$ it must contain the intersection of all such systems namely $\mathcal{D}_*$, but since it is itself contained in $\mathcal{D}_*$ they must coincide. This demonstrates that for every $C \in \mathcal{C}$ and for every $D \in \mathcal{D}_*$ it's true that $C \cap D \in \mathcal{D}_*$.

Now let's pick $X \in \mathcal{D}_*$ and define the set

$$\mathcal{D}''_* := \{D \in \mathcal{D}_* \mid D \cap X \in \mathcal{D}_*\}$$

This set is a d-system by the previous lemma. Furthermore it contains $\mathcal{C}$ since for every $C \in \mathcal{C}$ and for every $D \in \mathcal{D}_*$ it's true that $C \cap D \in \mathcal{D}_*$. From this we can conclude that $\mathcal{D}_* \subseteq \mathcal{D}''_*$ but since $\mathcal{D}''_* \subseteq \mathcal{D}_*$ by construction these sets coincide. But now this means that for every $D, X \in \mathcal{D}_*$ we have $D \cap X \in \mathcal{D}_*$ which means that $\mathcal{D}_*$ is a π-system and since it's d-system at the same time it's σ-algebra. □

2.2. **Measurable functions.**

Definition 8. *If $\langle E, \mathcal{E}\rangle$ and $\langle F, \mathcal{F}\rangle$ are measurable spaces, then a map $f : E \longrightarrow F$ is said to be measurable with respect to $\mathcal{E}$ and $\mathcal{F}$ if*

$$(\forall X \in \mathcal{F})\left(f^{-1}[X] \in \mathcal{F}\right)$$

Proposition 4. *Let $\langle E, \mathcal{E}\rangle$, $\langle F, \mathcal{F}\rangle$ and $\langle G, \mathcal{G}\rangle$ be measurable spaces. If the functions $f : E \longrightarrow F$ and $g : F \longrightarrow G$ are measurable then $g \circ f$ is measurable as well.*

Proof. Let's pick $X \in \mathcal{G}$. We have that $g^{-1}[X] \in \mathcal{F}$ since g is measurable. Then $f^{-1}[g^{-1}[X]] \in \mathcal{E}$ because f is measurable. Now we can conclude that $g \circ f$ is measurable by noticing that

$$(g \circ f)^{-1}[X] = f^{-1}[g^{-1}[X]] \in \mathcal{E}$$

□

Proposition 5. *Let $\langle E, \mathcal{E}\rangle$ and $\langle F, \mathcal{F}\rangle$ be measurable spaces and let $\mathcal{F}_0$ be such that $\sigma\mathcal{F}_0 = \mathcal{F}$. Under these circumstances a function $f : E \longrightarrow F$ is measurable if and only if*

$$(\forall X \in \mathcal{F}_0)\left(f^{-1}[X] \in \mathcal{E}\right)$$

Proof. It's immediate that if f is measurable than the condition is satisfied as a special case.

Now let's assume that f satisfies the condition. We define the set

$$\mathcal{F}_* := \left\{X \in \mathcal{F} \mid f^{-1}[X] \in \mathcal{E}\right\}$$

The goal is to show that $\mathcal{F}_*$ is σ-algebra, because since it contains $\mathcal{F}_0$ then it contains $\sigma\mathcal{F}_0 = \mathcal{F}$ and and since it is contained in $\mathcal{F}$ they coincide, proving the proposition.

Firstly let $X \in \mathcal{F}_*$. Since $f^{-1}[X] \in \mathcal{E}$ and $\mathcal{E}$ is σ-algebra can conclude that the following holds

$$f^{-1}[F \setminus X] = f^{-1}[F] \setminus f^{-1}[X] = E \setminus f^{-1}[X] \in \mathcal{E}$$

which implies that $F \setminus X \in \mathcal{F}_*$.

Now let's pick a sequence $S \in \mathcal{F}_*^{\mathbb{N}}$ then

$$f^{-1}\left[\bigcup_{n \in \mathbb{N}} S_n\right] = \bigcup_{n \in \mathbb{N}} f^{-1}[S_n] \in \mathcal{E}$$

since $(n \in \mathbb{N})\left(f^{-1}[S_n] \in \mathcal{E}\right)$ and $\mathcal{E}$ is σ-algebra. From this we can conclude that $\bigcup\limits_{n\in\mathbb{N}} S_n \in \mathcal{F}_*$ which demonstrates that $\mathcal{F}_*$ is σ-algebra. $\square$

Definition 9. *Let $\langle T, \mathcal{O}\rangle$ be a topological space, then $\sigma\mathcal{O}$ is called the Borel σ algebra of T with respect to $\mathcal{O}$ and we denote it by $\mathscr{B}_{\mathcal{O}}(T)$.*

By abuse of notation we will often write $\mathscr{B}(T)$ and infer the topology from the context.

Proposition 6. *Let $\langle T, \mathcal{O}\rangle$ and $\langle \tilde{T}, \tilde{\mathcal{O}}\rangle$ be topological spaces. If $f : T \longrightarrow \tilde{T}$ is continuous with respect to $\mathcal{O}$ and $\tilde{\mathcal{O}}$ then it's measurable with respect to $\mathscr{B}(T)$ and $\mathscr{B}\left(\tilde{T}\right)$.*

Proof. Since f is continuous for every open set $O \in \tilde{\mathcal{O}}$ we have

$$f^{-1}[O] \in \mathcal{O} \subseteq \sigma\mathcal{O} = \mathscr{B}(T)$$

which proves the result by using the previous proposition. $\square$

2.3. **Measure spaces.**

2.4. **Integration.**

3. Basic Probability

3.1. **Probability spaces.**