

## Week 5 at a glance

**Textbook reading: Section 2.2, 2.1.**

Before Monday, read Theorem 2.20.

Before Wednesday, read Example 2.18 (page 114).

Before Friday, read Figure 3.1.

For Week 6 Monday: Page 165-166 Introduction to Section 3.1.

**We will be learning and practicing to:**

- Clearly and unambiguously communicate computational ideas using appropriate formalism. Translate across levels of abstraction.
  - Describe and use models of computation that don't involve state machines.
    - \* **Use context-free grammars and relate them to languages and pushdown automata.**
  - Use precise notation to formally define the state diagram of a Turing machine
  - Use clear English to describe computations of Turing machines informally.
    - \* **Design a PDA that recognizes a given language.**
  - Give examples of sets that are context-free (and prove that they are).
    - \* **State the definition of the class of context-free languages**
    - \* **Explain the limits of the class of context-free languages**
    - \* **Identify some context-free sets and some non-context-free sets**
- Know, select and apply appropriate computing knowledge and problem-solving techniques. Reason about computation and systems.
  - Describe and prove closure properties of classes of languages under certain operations.
    - \* **Apply a general construction to create a new PDA or CFG from an example one.**
    - \* **Formalize a general construction from an informal description of it.**
    - \* **Use general constructions to prove closure properties of the class of context-free languages.**
    - \* **Use counterexamples to prove non-closure properties of the class of context-free languages.**

## TODO:

Schedule your Test 1 Attempt 1, Test 2 Attempt 1, Test 1 Attempt 2, and Test 2 Attempt 2 times at PrairieTest (<http://us.prairietest.com>)

Review Quiz 5 on PrairieLearn (<http://us.prairielearn.com>), complete by Sunday 11/4/2024

# Monday: Context-free languages

Warmup: Design a CFG to generate the language  $\{a^i b^j \mid j \geq i \geq 0\}$

*Sample derivation:*

Design a PDA to recognize the language  $\{a^i b^j \mid j \geq i \geq 0\}$

**Theorem 2.20:** A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet  $\Sigma$  is called **CFL**.

Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a description. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
  - PDAs can “test for emptiness of stack” without providing details. *How?* We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
  - PDAs can “test for end of input” without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Suppose  $L_1$  and  $L_2$  are context-free languages over  $\Sigma$ . **Goal:**  $L_1 \cup L_2$  is also context-free.

*Approach 1: with PDAs*

Let  $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$  and  $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$  be PDAs with  $L(M_1) = L_1$  and  $L(M_2) = L_2$ .

Define  $M =$

*Approach 2: with CFGs*

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define  $G =$

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*Approach 1: with PDAs*

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*Approach 2: with CFGs*

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define  $G =$

## Wednesday: Context-free and non-context-free languages

### *Summary*

Over a fixed alphabet  $\Sigma$ , a language  $L$  is **regular**

iff it is described by some regular expression

iff it is recognized by some DFA

iff it is recognized by some NFA

Over a fixed alphabet  $\Sigma$ , a language  $L$  is **context-free**

iff it is generated by some CFG

iff it is recognized by some PDA

**Fact:** Every regular language is a context-free language.

**Fact:** There are context-free languages that are not nonregular.

**Fact:** There are countably many regular languages.

**Fact:** There are countably infinitely many context-free languages.

*Consequence:* Most languages are **not** context-free!

## Examples of non-context-free languages

$$\begin{aligned} &\{a^n b^n c^n \mid 0 \leq n, n \in \mathbb{Z}\} \\ &\{a^i b^j c^k \mid 0 \leq i \leq j \leq k, i \in \mathbb{Z}, j \in \mathbb{Z}, k \in \mathbb{Z}\} \\ &\{ww \mid w \in \{0,1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If  $A$  is a context-free language, there is a number  $p$  where, if  $s$  is any string in  $A$  of length at least  $p$ , then  $s$  may be divided into five pieces  $s = uvxyz$  where (1) for each  $i \geq 0$ ,  $uv^i xy^i z \in A$ , (2)  $|uv| > 0$ , (3)  $|vxy| \leq p$ . *We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.*

Recall: A set  $X$  is said to be **closed** under an operation  $OP$  if, for any elements in  $X$ , applying  $OP$  to them gives an element in  $X$ .

| True/False | Closure claim   |
|------------|---|
| True       | The set of integers is closed under multiplication.<br>$\forall x \forall y ( (x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \rightarrow xy \in \mathbb{Z} )$   |
| True       | For each set $A$ , the power set of $A$ is closed under intersection.<br>$\forall A_1 \forall A_2 ( (A_1 \in \mathcal{P}(A) \wedge A_2 \in \mathcal{P}(A)) \rightarrow A_1 \cap A_2 \in \mathcal{P}(A) )$ |
|            | The class of regular languages over $\Sigma$ is closed under complementation.   |
|            | The class of regular languages over $\Sigma$ is closed under union.   |
|            | The class of regular languages over $\Sigma$ is closed under intersection.  |
|            | The class of regular languages over $\Sigma$ is closed under concatenation.   |
|            | The class of regular languages over $\Sigma$ is closed under Kleene star.   |
|            | The class of context-free languages over $\Sigma$ is closed under complementation.  |
|            | The class of context-free languages over $\Sigma$ is closed under union.  |
|            | The class of context-free languages over $\Sigma$ is closed under intersection.   |
|            | The class of context-free languages over $\Sigma$ is closed under concatenation.  |
|            | The class of context-free languages over $\Sigma$ is closed under Kleene star.  |

## Friday: Turing machines

We are ready to introduce a formal model that will capture a notion of general purpose computation.

- *Similar to DFA, NFA, PDA*: input will be an arbitrary string over a fixed alphabet.
- *Different from NFA, PDA*: machine is deterministic.
- *Different from DFA, NFA, PDA*: read-write head can move both to the left and to the right, and can extend to the right past the original input.
- *Similar to DFA, NFA, PDA*: transition function drives computation one step at a time by moving within a finite set of states, always starting at designated start state.
- *Different from DFA, NFA, PDA*: the special states for rejecting and accepting take effect immediately.

(See more details: Sipser p. 166)

Formally: a Turing machine is  $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$  where  $\delta$  is the **transition function**

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

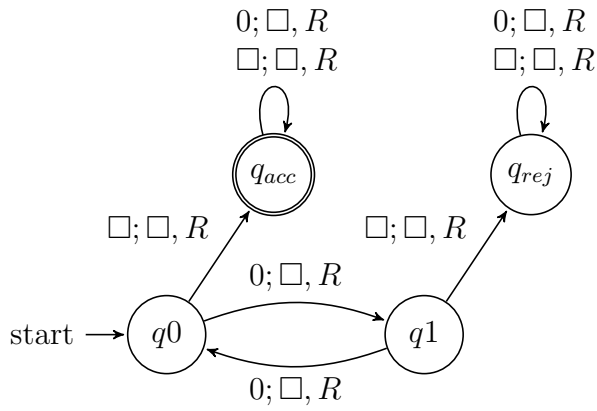
The **computation** of  $M$  on a string  $w$  over  $\Sigma$  is:

- Read/write head starts at leftmost position on tape.
- Input string is written on  $|w|$ -many leftmost cells of tape, rest of the tape cells have the blank symbol. **Tape alphabet** is  $\Gamma$  with  $\sqcup \in \Gamma$  and  $\Sigma \subseteq \Gamma$ . The blank symbol  $\sqcup \notin \Sigma$ .
- Given current state of machine and current symbol being read at the tape head, the machine transitions to next state, writes a symbol to the current position of the tape head (overwriting existing symbol), and moves the tape head L or R (if possible).
- Computation ends **if and when** machine enters either the accept or the reject state. This is called **halting**. Note:  $q_{accept} \neq q_{reject}$ .

The **language recognized by the Turing machine**  $M$ , is  $L(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$ , which is defined as

$$\{w \in \Sigma^* \mid \text{computation of } M \text{ on } w \text{ halts after entering the accept state}\}$$





Formal

definition:

Sample computation:

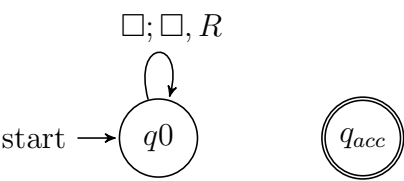
| $q0 \downarrow$ |   |   |   |   |   |   |
|-----------------|---|---|---|---|---|---|
| 0               | 0 | 0 | □ | □ | □ | □ |
|                 |   |   |   |   |   |   |
|                 |   |   |   |   |   |   |
|                 |   |   |   |   |   |   |
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|                 |   |   |   |   |   |   |
|                 |   |   |   |   |   |   |

The language recognized by this machine is ...

**Describing Turing machines** (Sipser p. 185) To define a Turing machine, we could give a

- **Formal definition:** the 7-tuple of parameters including set of states, input alphabet, tape alphabet, transition function, start state, accept state, and reject state; or,
- **Implementation-level definition:** English prose that describes the Turing machine head movements relative to contents of tape, and conditions for accepting / rejecting based on those contents.
- **High-level description:** description of algorithm (precise sequence of instructions), without implementation details of machine. As part of this description, can “call” and run another TM as a subroutine.

Fix  $\Sigma = \{0, 1\}$ ,  $\Gamma = \{0, 1, \sqcup\}$  for the Turing machines with the following state diagrams:



Example of string accepted:

Example of string rejected:

Implementation-level description

High-level description

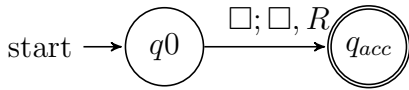


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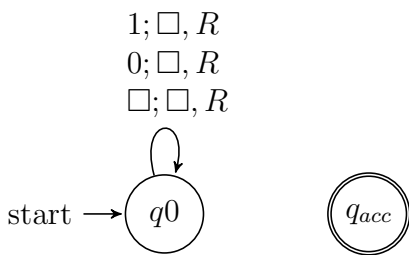


Example of string accepted:

Example of string rejected:

Implementation-level description

High-level description



Example of string accepted:

Example of string rejected:

Implementation-level description

High-level description