

Monday

Recall Definition (Sipser 7.1): For M a deterministic decider, its **running time** is the function $f : \mathbb{N} \rightarrow \mathbb{N}$ given by

$$f(n) = \max \text{ number of steps } M \text{ takes before halting, over all inputs of length } n$$

Recall Definition (Sipser 7.7): For each function $t(n)$, the **time complexity class** $TIME(t(n))$, is defined by

$$TIME(t(n)) = \{L \mid L \text{ is decidable by a Turing machine with running time in } O(t(n))\}$$

Recall Definition (Sipser 7.12) : P is the class of languages that are decidable in polynomial time on a deterministic 1-tape Turing machine

$$P = \bigcup_k TIME(n^k)$$

Definition (Sipser 7.9): For N a nondeterministic decider. The **running time** of N is the function $f : \mathbb{N} \rightarrow \mathbb{N}$ given by

$$f(n) = \max \text{ number of steps } N \text{ takes on any branch before halting, over all inputs of length } n$$

Definition (Sipser 7.21): For each function $t(n)$, the **nondeterministic time complexity class** $NTIME(t(n))$, is defined by

$$NTIME(t(n)) = \{L \mid L \text{ is decidable by a nondeterministic Turing machine with running time in } O(t(n))\}$$

$$NP = \bigcup_k NTIME(n^k)$$

True or False: $TIME(n^2) \subseteq NTIME(n^2)$

True or False: $NTIME(n^2) \subseteq DTIME(n^2)$

Examples in P

Can't use nondeterminism; Can use multiple tapes; Often need to be "more clever" than naïve / brute force approach

$$PATH = \{\langle G, s, t \rangle \mid G \text{ is digraph with } n \text{ nodes there is path from } s \text{ to } t\}$$

Use breadth first search to show in P

$$RELPRIME = \{\langle x, y \rangle \mid x \text{ and } y \text{ are relatively prime integers}\}$$

Use Euclidean Algorithm to show in P

$$L(G) = \{w \mid w \text{ is generated by } G\}$$

(where G is a context-free grammar). Use dynamic programming to show in P .

Examples in NP

"Verifiable" i.e. NP, Can be decided by a nondeterministic TM in polynomial time, best known deterministic solution may be brute-force, solution can be verified by a deterministic TM in polynomial time.

$$HAMPATH = \{\langle G, s, t \rangle \mid G \text{ is digraph with } n \text{ nodes, there is path from } s \text{ to } t \text{ that goes through every node exactly once}\}$$

$$VERTEX - COVER = \{\langle G, k \rangle \mid G \text{ is an undirected graph with } n \text{ nodes that has a } k\text{-node vertex cover}\}$$

$$CLIQUE = \{\langle G, k \rangle \mid G \text{ is an undirected graph with } n \text{ nodes that has a } k\text{-clique}\}$$

$$SAT = \{\langle X \rangle \mid X \text{ is a satisfiable Boolean formula with } n \text{ variables}\}$$

Every problem in NP is decidable with an exponential-time algorithm

Nondeterministic approach: guess a possible solution, verify that it works.

Brute-force (worst-case exponential time) approach: iterate over all possible solutions, for each one, check if it works.

| Problems in P | Problems in NP |
|---|--------------------|
| (Membership in any) regular language | Any problem in P |
| (Membership in any) context-free language | |
| A_{DFA} | SAT |
| E_{DFA} | $CLIQUE$ |
| EQ_{DFA} | $VERTEX - COVER$ |
| $PATH$ | $HAMPATH$ |
| $RELPRIME$ | \dots |
| \dots | |

Million-dollar question: Is $P = NP$?

One approach to trying to answer it is to look for *hardest* problems in NP and then (1) if we can show that there are efficient algorithms for them, then we can get efficient algorithms for all problems in NP so $P = NP$, or (2) these problems might be good candidates for showing that there are problems in NP for which there are no efficient algorithms.

Review: Week 10 Monday

Please complete the review quiz questions on Gradescope about complexity (P and NP)

Wednesday

Definition (Sipser 7.29) Language A is **polynomial-time mapping reducible** to language B , written $A \leq_P B$, means there is a polynomial-time computable function $f : \Sigma^* \rightarrow \Sigma^*$ such that for every $x \in \Sigma^*$

$$x \in A \quad \text{iff} \quad f(x) \in B.$$

The function f is called the polynomial time reduction of A to B .

Theorem (Sipser 7.31): If $A \leq_P B$ and $B \in P$ then $A \in P$.

Proof:

Definition (Sipser 7.34; based in Stephen Cook and Leonid Levin's work in the 1970s): A language B is **NP-complete** means (1) B is in NP **and** (2) every language A in NP is polynomial time reducible to B .

Theorem (Sipser 7.35): If B is NP-complete and $B \in P$ then $P = NP$.

Proof:

3SAT: A literal is a Boolean variable (e.g. x) or a negated Boolean variable (e.g. \bar{x}). A Boolean formula is a **3cnf-formula** if it is a Boolean formula in conjunctive normal form (a conjunction of disjunctive clauses of literals) and each clause has three literals.

$$3SAT = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$$

Example strings in $3SAT$

Example strings not in $3SAT$

Cook-Levin Theorem: $3SAT$ is NP -complete.

Are there other NP -complete problems? To prove that X is NP -complete

- *From scratch:* prove X is in NP and that all NP problems are polynomial-time reducible to X .
- *Using reduction:* prove X is in NP and that a known-to-be NP -complete problem is polynomial-time reducible to X .

CLIQUE: A k -**clique** in an undirected graph is a maximally connected subgraph with k nodes.

$$CLIQUE = \{\langle G, k \rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$$

Example strings in $CLIQUE$

Example strings not in $CLIQUE$

Theorem (Sipser 7.32):

$$3SAT \leq_P CLIQUE$$

Given a Boolean formula in conjunctive normal form with k clauses and three literals per clause, we will map it to a graph so that the graph has a clique if the original formula is satisfiable and the graph does not have a clique if the original formula is not satisfiable.

The graph has $3k$ vertices (one for each literal in each clause) and an edge between all vertices except

- vertices for two literals in the same clause
- vertices for literals that are negations of one another

Example: $(x \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee y \vee z) \wedge (x \vee y \vee z)$

Review: Week 10 Wednesday

Please complete the review quiz questions on Gradescope about complexity (NP -completeness)

Friday

| Model of Computation | Class of Languages |
|---|--|
| Deterministic finite automata: formal definition, how to design for a given language, how to describe language of a machine? Nondeterministic finite automata: formal definition, how to design for a given language, how to describe language of a machine? Regular expressions: formal definition, how to design for a given language, how to describe language of expression? <i>Also:</i> converting between different models. | Class of regular languages: what are the closure properties of this class? which languages are not in the class? using pumping lemma to prove nonregularity. |
| Push-down automata: formal definition, how to design for a given language, how to describe language of a machine? Context-free grammars: formal definition, how to design for a given language, how to describe language of a grammar? | Class of context-free languages: what are the closure properties of this class? which languages are not in the class? |
| Turing machines that always halt in polynomial time Nondeterministic Turing machines that always halt in polynomial time | P NP |
| Deciders (Turing machines that always halt): formal definition, how to design for a given language, how to describe language of a machine? | Class of decidable languages: what are the closure properties of this class? which languages are not in the class? using diagonalization and mapping reduction to show undecidability |
| Turing machines formal definition, how to design for a given language, how to describe language of a machine? | Class of recognizable languages: what are the closure properties of this class? which languages are not in the class? using closure and mapping reduction to show unrecognizability |

Given a language, prove it is regular

Strategy 1: construct DFA recognizing the language and prove it works.

Strategy 2: construct NFA recognizing the language and prove it works.

Strategy 3: construct regular expression recognizing the language and prove it works.

“Prove it works” means ...

Example: $L = \{w \in \{0,1\}^* \mid w \text{ has odd number of 1s or starts with } 0\}$

Using NFA

Using regular expressions

Example: Select all and only the options that result in a true statement: “To show a language A is not regular, we can...”

- a. Show A is finite
- b. Show there is a CFG generating A
- c. Show A has no pumping length
- d. Show A is undecidable

Example: What is the language generated by the CFG with rules

$$S \rightarrow aSb \mid bY \mid Ya$$

$$Y \rightarrow bY \mid Ya \mid \varepsilon$$

Example: Prove that the language $T = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) \text{ is infinite}\}$ is undecidable.

Example: Prove that the class of decidable languages is closed under concatenation.



Review: Week 10 Friday

Please complete the review quiz questions on Gradescope giving feedback on the quarter. Have a great summer!