Week 5 at a glance

Textbook reading: Section 2.2, 2.1.

Before Monday, read Theorem 2.20.

Before Wednesday, read Example 2.18 (page 114).

Before Friday, read Figure 3.1.

For Week 6 Monday: Page 165-166 Introduction to Section 3.1.

We will be learning and practicing to:

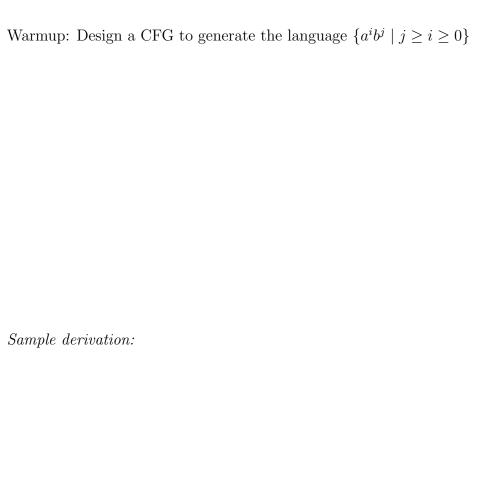
- Clearly and unambiguously communicate computational ideas using appropriate formalism. Translate across levels of abstraction.
 - Describe and use models of computation that don't involve state machines.
 - * Use context-free grammars and relate them to languages and pushdown automata.
 - Use precise notation to formally define the state diagram of a Turing machine
 - Use clear English to describe computations of Turing machines informally.
 - * Design a PDA that recognizes a given language.
 - Give examples of sets that are context-free (and prove that they are).
 - * State the definition of the class of context-free languages
 - * Explain the limits of the class of context-free languages
 - * Identify some context-free sets and some non-context-free sets
- Know, select and apply appropriate computing knowledge and problem-solving techniques. Reason about computation and systems.
 - Describe and prove closure properties of classes of languages under certain operations.
 - * Apply a general construction to create a new PDA or CFG from an example one.
 - * Formalize a general construction from an informal description of it.
 - * Use general constructions to prove closure properties of the class of context-free languages.
 - * Use counterexamples to prove non-closure properties of the class of context-free languages.

TODO:

Schedule your Test 1 Attempt 1, Test 2 Attempt 1, Test 1 Attempt 2, and Test 2 Attempt 2 times at PrairieTest (http://us.prairietest.com)

Review Quiz 5 on PrairieLearn (http://us.prairielearn.com), complete by Sunday 11/4/2024

Monday: Context-free languages



Theorem 2.20: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet Σ is called **CFL**.

Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a descripton. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
 - PDAs can "test for emptiness of stack" without providing details. *How?* We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
 - PDAs can "test for end of input" without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Suppose L_1 and L_2 are context-free languages over Σ . Goal: $L_1 \cup L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define M =

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define G =

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Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define M =

 $Approach\ 2:\ with\ CFGs$

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define G =

Wednesday: Context-free and non-context-free languages

| Wednesday. Convent free and non convent free languages | | | | | |
|--|--|--|--|--|--|
| Summary | | | | | |
| Over a fixed alphabet Σ , a language L is regular | | | | | |
| iff it is described by some regular expression iff it is recognized by some DFA iff it is recognized by some NFA | | | | | |
| Over a fixed alphabet Σ , a language L is context-free | | | | | |
| iff it is generated by some CFG iff it is recognized by some PDA | | | | | |
| | | | | | |
| Fact: Every regular language is a context-free language. | | | | | |
| Fact: There are context-free languages that are not nonregular. | | | | | |
| Fact: There are countably many regular languages. | | | | | |
| Fact: There are countably infinitely many context-free languages. | | | | | |
| Consequence: Most languages are not context-free! | | | | | |

Examples of non-context-free languages

$$\begin{aligned} &\{a^nb^nc^n\mid 0\leq n, n\in\mathbb{Z}\}\\ &\{a^ib^jc^k\mid 0\leq i\leq j\leq k, i\in\mathbb{Z}, j\in\mathbb{Z}, k\in\mathbb{Z}\}\\ &\{ww\mid w\in\{0,1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there is a number p where, if s is any string in A of length at least p, then s may be divided into five pieces s = uvxyz where (1) for each $i \ge 0$, $uv^ixy^iz \in A$, (2) |uv| > 0, (3) $|vxy| \le p$. We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.

Recall: A set X is said to be **closed** under an operation OP if, for any elements in X, applying OP to them gives an element in X.

| True/False | Closure claim | | | | |
|------------|--|--|--|--|--|
| True | The set of integers is closed under multiplication. | | | | |
| | $\forall x \forall y ((x \in \mathbb{Z} \land y \in \mathbb{Z}) \to xy \in \mathbb{Z})$ | | | | |
| True | For each set A , the power set of A is closed under intersection. | | | | |
| | $\forall A_1 \forall A_2 ((A_1 \in \mathcal{P}(A) \land A_2 \in \mathcal{P}(A) \in \mathbb{Z}) \to A_1 \cap A_2 \in \mathcal{P}(A))$ | | | | |
| | The class of regular languages over Σ is closed under complementation. | | | | |
| | The class of regular languages over Σ is closed under union. | | | | |
| | The class of regular languages over Σ is closed under intersection. | | | | |
| | The class of regular languages over Σ is closed under concatenation. | | | | |
| | The class of regular languages over Σ is closed under Kleene star. | | | | |
| | The class of context-free languages over Σ is closed under complementation. | | | | |
| | The class of context-free languages over Σ is closed under union. | | | | |
| | The class of context-free languages over Σ is closed under intersection. | | | | |
| | The class of context-free languages over Σ is closed under concatenation. | | | | |
| | The class of context-free languages over Σ is closed under Kleene star. | | | | |

Friday: Turing machines

We are ready to introduce a formal model that will capture a notion of general purpose computation.

- Similar to DFA, NFA, PDA: input will be an arbitrary string over a fixed alphabet.
- Different from NFA, PDA: machine is deterministic.
- Different from DFA, NFA, PDA: read-write head can move both to the left and to the right, and can extend to the right past the original input.
- Similar to DFA, NFA, PDA: transition function drives computation one step at a time by moving within a finite set of states, always starting at designated start state.
- Different from DFA, NFA, PDA: the special states for rejecting and accepting take effect immediately.

(See more details: Sipser p. 166)

Formally: a Turing machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where δ is the **transition function**

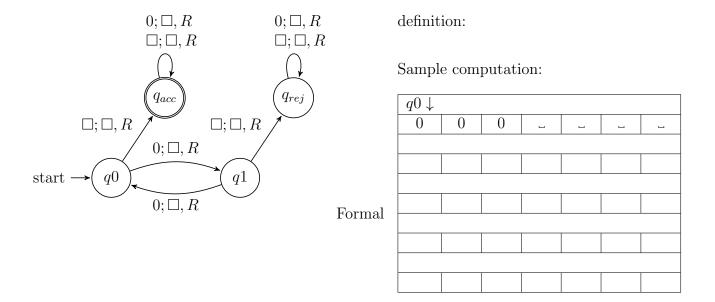
$$\delta: Q \times \Gamma \to Q \times \Gamma \times \{L,R\}$$

The **computation** of M on a string w over Σ is:

- Read/write head starts at leftmost position on tape.
- Input string is written on |w|-many leftmost cells of tape, rest of the tape cells have the blank symbol. **Tape alphabet** is Γ with $\bot \in \Gamma$ and $\Sigma \subseteq \Gamma$. The blank symbol $\bot \notin \Sigma$.
- Given current state of machine and current symbol being read at the tape head, the machine transitions to next state, writes a symbol to the current position of the tape head (overwriting existing symbol), and moves the tape head L or R (if possible).
- Computation ends if and when machine enters either the accept or the reject state. This is called halting. Note: $q_{accept} \neq q_{reject}$.

The language recognized by the Turing machine M, is $L(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$, which is defined as

 $\{w \in \Sigma^* \mid \text{computation of } M \text{ on } w \text{ halts after entering the accept state}\}$



The language recognized by this machine is ...

Describing Turing machines (Sipser p. 185) To define a Turing machine, we could give a

- Formal definition: the 7-tuple of parameters including set of states, input alphabet, tape alphabet, transition function, start state, accept state, and reject state; or,
- Implementation-level definition: English prose that describes the Turing machine head movements relative to contents of tape, and conditions for accepting / rejecting based on those contents.
- **High-level description**: description of algorithm (precise sequence of instructions), without implementation details of machine. As part of this description, can "call" and run another TM as a subroutine.

Fix $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \bot\}$ for the Turing machines with the following state diagrams:



Example of string accepted: Example of string rejected:

Implementation-level description

High-level description



Example of string accepted: Example of string rejected:

Implementation-level description

High-level description



| Example | of | string | accepted: |
|---------|----|--------|-----------|
| Example | of | string | rejected: |

Implementation-level description

High-level description



Example of string accepted: Example of string rejected:

Implementation-level description

High-level description