HW2CSE105W25: Homework assignment 2

CSE105W25

Due: January 30th at 5pm, via Gradescope

In this assignment,

You will practice designing multiple representations of regular languages and working with general constructions of automata to demonstrate the richness of the class of regular languages.

Resources: To review the topics for this assignment, see the class material from Week 2 and Week 3. We will post frequently asked questions and our answers to them in a pinned Piazza post.

Reading and extra practice problems: Sipser Section 1.1, 1.2, 1.3. Chapter 1 exercises 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19, 1.20, 1.21, 1.22, 1.23. Chapter 1 problem 1.31, 1.36, 1.37.

For all HW assignments: Weekly homework may be done individually or in groups of up to 3 students. You may switch HW partners for different HW assignments. Please ensure your name(s) and PID(s) are clearly visible on the first page of your homework submission and then upload the PDF to Gradescope. If working in a group, submit only one submission per group: one partner uploads the submission through their Gradescope account and then adds the other group member(s) to the Gradescope submission by selecting their name(s) in the "Add Group Members" dialog box. You will need to re-add your group member(s) every time you resubmit a new version of your assignment. Each homework question will be graded either for correctness (including clear and precise explanations and justifications of all answers) or fair effort completeness. On the "graded for correctness" questions, you may only collaborate with CSE 105 students in your group; if your group has questions about a problem, you may ask in drop-in help hours or post a private post (visible only to the Instructors) on Piazza. On the "graded for completeness" questions, you may collaborate with all other CSE 105 students this quarter, and you may make public posts about these questions on Piazza.

All submitted homework for this class must be typed. You can use a word processing editor if you like (Microsoft Word, Open Office, Notepad, Vim, Google Docs, etc.) but you might find it useful to take this opportunity to learn LaTeX. LaTeX is a markup language used widely in computer science and mathematics. The homework assignments are typed using LaTeX and you can use the source files as templates for typesetting your solutions. To generate state diagrams of machines, you can (1) use the LaTex tikzpicture environment (see templates in the class notes), or (2) use the software tools Flap.js or JFLAP described in the class syllabus (and include a screenshot in your PDF), or (3) you can carefully and clearly hand-draw the diagram and take a picture and include it in your PDF. We recommend that you submit early drafts to Gradescope so that in case of any technical difficulties, at least some of your work is present. You may update your submission as many times as you'd like up to the deadline.

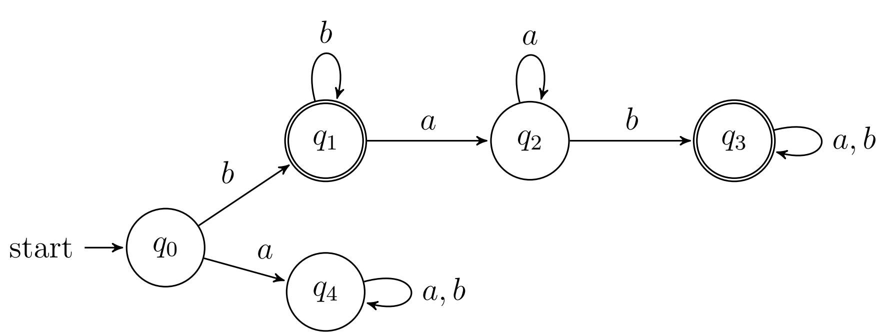
Integrity reminders

- Problems should be solved together, not divided up between the partners. The homework is designed to give you practice with the main concepts and techniques of the course, while getting to know and learn from your classmates.
- On the "graded for correctness" questions, you may only collaborate with CSE 105 students in your group. You may ask questions about the homework in office hours (of the instructor, TAs, and/or tutors) and on Piazza (as private notes viewable only to the Instructors). You *cannot* use any online resources about the course content other than the class material from this quarter – this is primarily to ensure that we all use consistent notation and definitions (aligned with the textbook) and also to protect the learning experience you will have when the 'aha' moments of solving the problem authentically happen.
- Do not share written solutions or partial solutions for homework with other students in the class who are not in your group. Doing so would dilute their learning experience and detract from their success in the class.

You will submit this assignment via Gradescope (https://www.gradescope.com) in the assignment called "hw2CSE105W25".

Assigned questions

1. **Finite automata** (10 points): Consider the finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ whose state diagram is depicted below



(a) (*Graded for completeness*) [1] Write the formal definition of this automaton. In other words, give the five defining parameters Q, Σ, δ, q_0, F so that they are consistent with the state diagram of M.

(b) (*Graded for correctness*) [2] Give a regular expression R so that $L(R) = L(M)$. In other words, we want a regular expression that describes the language recognized by this finite automaton. Justify your answer by referring to the definition of the semantics of regular expressions and computations of finite automata. Include an explanation for why each string in $L(R)$ is accepted by the finite automaton *and* for why each string not in $L(R)$ is rejected by the finite automaton.

Ungraded bonus: can you find more than one such regular expression?

(c) (*Graded for completeness*) Keeping the same set of states Q, input alphabet Σ, same start state q_0, and same transition function δ, choose a new set of accepting states F_{new1} so that the new finite automaton $M_1 = (Q, \Sigma, \delta, q_0, F_{new1})$ that results recognizes a **proper superset** of $L(M)$, or explain why there is no such example. A complete solution will include either (1) a precise and clear description of your choice of F_{new1} *and* a precise and clear explanation of why every string that is accepted by M is also accepted by M_1 *and* an example of a string that is accepted by M_1 and is rejected by M; or (2) a sufficiently general and correct argument why there is no such example, referring back to the relevant definitions.

(d) (*Graded for correctness*) Keeping the same set of states Q, input alphabet Σ, same start state q_0, and same transition function δ, choose a new set of accepting states F_{new2} so that the new finite automaton $M_2 = (Q, \Sigma, \delta, q_0, F_{new2})$ that results recognizes a **nonempty proper subset** of $L(M)$, or explain why there is no such example. A complete solution will include either (1) a precise and clear description of your choice of F_{new2} *and* an example string accepted by M_2 *and* a precise and clear explanation of why every string that is accepted by M_2 is also accepted by M *and* an example of a string that is accepted by M and is rejected by M_2; or (2) a sufficiently general and correct argument why there is no such example, referring back to the relevant definitions.

2. **Automata design** (12 points): As background to this question, recall that integers can be represented using base b expansions, for any convenient choice of base b. The precise definition is: for b an integer greater than 1 and n a positive integer, the **base b expansion of n** is defined to be

$$(a_{k-1} \cdots a_1 a_0)_b$$

[1]This means you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for incorrect answers. To demonstrate your honest effort in answering the question, we expect you to include your attempt to answer *each* part of the question. If you get stuck with your attempt, you can still demonstrate your effort by explaining where you got stuck and what you did to try to get unstuck.

[2]This means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

where k is a positive integer, $a_0, a_1, \ldots, a_{k-1}$ are nonnegative integers less than b, $a_{k-1} \neq 0$, and

$$n = \sum_{i=0}^{k-1} a_i b^i$$

Notice: *The base b expansion of a positive integer n is a string over the alphabet $\{x \in \mathbb{Z} \mid 0 \leq x < b\}$ whose leftmost character is nonzero.*

An important property of base b expansions of integers is that, for each integer b greater than 1, each positive integer $n = (a_{k-1} \cdots a_1 a_0)_b$, and each nonnegative integer a less than b,

$$bn + a = (a_{k-1} \cdots a_1 a_0 a)_b$$

In other words, shifting the base b expansion to the left results in multiplying the integer value by the base. In this question we'll explore building deterministic finite automata that recognize languages that correspond to useful sets of integers.

(a) (*Graded for completeness*) Design a DFA that recognizes the set of binary (base 2) expansions of positive integers that are powers of 2. A complete solution will include the state diagram of your DFA and a brief justification of your construction by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

Hints: (1) A power of 2 is an integer x that can be written as 2^y for some nonnegative integer y, (2) the DFA should accept the strings 100, 10 and 100000 and should reject the strings 010, 1101, and ε (can you see why?).

(b) (*Graded for correctness*) Design a DFA that recognizes the set of binary (base 2) expansions of positive integers that are less than 10. Your DFA must use **fewer than ten states**. A complete solution will include the state diagram of your DFA and a brief justification of your construction by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

(c) (*Graded for completeness*) Find a positive integer B greater than 1 so that there is a DFA that recognizes the set of base B expansions of positive integers that are less than 10 and it uses as few states as possible. A complete solution will include the state diagram of your DFA and a brief justification of your choice of base.

Hint: sometimes rewriting the defining membership condition for a set in different ways helps us find alternate representations of that set.

3. **Nondeterminism** (15 points): For this question, the alphabet is $\{a, b, c\}$.

(a) (*Graded for completeness*) Design a NFA that recognizes the language

$$L_1 = \{w \in \{a, b, c\}^* \mid w \text{ starts with } a \textbf{ and } \text{ends with } a\}$$

A complete solution will include the state diagram of your NFA and a brief justification of your construction that explains the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

(b) (*Graded for correctness*) Design a NFA that recognizes the language

$$L_2 = \{w \in \{a, b, c\}^* \mid w \text{ has no consecutive repeated characters}\}$$

For example, the empty string, a, bac, and $abca$ are each elements of this language but aa and abb and $abbc$ are not elements of this language.

A complete solution will include the state diagram of your NFA and a brief justification of your construction that explains the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

(c) (*Graded for completeness*) Consider the language

$$\begin{aligned} L_1 \cup L_2 = \{w \in \{a, b, c\}^* \mid\ & w \text{ starts with } a \text{ and ends with } a \\ & \textbf{or} \text{ has no consecutive repeated characters}\} \end{aligned}$$

Give at least two representations of this language among the following:

- A regular expression that describes $L_1 \cup L_2$
- A DFA that recognizes $L_1 \cup L_2$
- A NFA that recognizes $L_1 \cup L_2$

You can design your automata directly or use the constructions from class and chapter 1 in the book to build these automata from automata for the simpler languages.

A complete solution will include at least two of the representations as well as a brief justification of each construction.

(d) (*Graded for completeness*) Consider the language

$$\begin{aligned} L_1 \cap L_2 = \{w \in \{a, b, c\}^* \mid\ & w \text{ starts with } a \text{ and ends with } a \\ & \textbf{and} \text{ has no consecutive repeated characters}\} \end{aligned}$$

Give at least two representations of this language among the following:

- A regular expression that describes $L_1 \cap L_2$
- A DFA that recognizes $L_1 \cap L_2$
- A NFA that recognizes $L_1 \cap L_2$

You can design your automata directly or use the constructions from class and chapter 1 in the book to build these automata from automata for the simpler languages.

A complete solution will include at least two of the representations as well as a brief justification of each construction.

4. **General constructions** (13 points): In this question, you'll practice working with formal general constructions for automata and translating between state diagrams and formal definitions.

Recall the definitions of operations we've talked about that produce new languages from old: for each language L over an alphabet Σ, we have the associated sets of strings (also over Σ)

$$L^* = \{w_1 \cdots w_k \mid k \geq 0 \text{ and each } w_i \in L\}$$

and

$$SUBSTRING(L) = \{w \in \Sigma^* \mid \text{there exist } x, y \in \Sigma^* \text{ such that } xwy \in L\}$$

and

$$EXTEND(L) = \{w \in \Sigma^* \mid w = uv \text{ for some strings } u \in L \text{ and } v \in \Sigma^*\}$$

Also, recall the set operations union and intersection: for any sets X and Y

$$X \cup Y = \{w \mid w \in X \text{ or } w \in Y\}$$

$$X \cap Y = \{w \mid w \in X \text{ and } w \in Y\}$$

Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be DFA.

For simplicity, assume that $Q_1 \cap Q_2 = \emptyset$ and that $q_0 \notin Q_1 \cup Q_2$.

Consider the following definitions of new automata parameterized by these DFA:

- The NFA $N_\alpha = (Q_1 \cup Q_2 \cup \{q_0\}, \Sigma, \delta_\alpha, q_0, F_1 \cup F_2)$ with the transition function given by

$$\delta_\alpha(\ (q,x)\ ) = \begin{cases} \{q_1, q_2\} & \text{if } q = q_0,\ x = \varepsilon \\ \emptyset & \text{if } q = q_0,\ x \in \Sigma \\ \{\delta_1(\ (q,x)\ )\} & \text{if } q \in Q_1,\ x \in \Sigma \\ \{\delta_2(\ (q,x)\ )\} & \text{if } q \in Q_2,\ x \in \Sigma \\ \emptyset & \text{if } q \in Q_1 \cup Q_2,\ x = \varepsilon \end{cases}$$

- The NFA $N_\beta = (Q_1 \times Q_2, \Sigma, \delta_\beta, (q_1, q_2), F_1 \times F_2)$ with the transition function given by

$$\delta_\beta(\ (\ (r,s)\ ,x\ )\ ) = \{(\ \delta_1(\ (r,x)\ ), \delta_2(\ (s,x)\ )\ )\}$$

  and

$$\delta_\beta(\ (\ (r,s)\ ,\varepsilon\ )\ ) = \emptyset$$

  for $r \in Q_1, s \in Q_2, x \in \Sigma$.

- The NFA $N_\gamma = (Q_1 \cup \{q_0\}, \Sigma, \delta_\gamma, q_0, \{q \in Q_1 \mid \exists w \in \Sigma^* (\delta_1^*((q,w)) \in F_1)\})$, and

$$\delta_\gamma((q,a)) = \begin{cases} \{\delta_1((q,a))\} & \text{if } q \in Q_1,\ a \in \Sigma \\ \{q' \in Q_1 \mid \exists w \in \Sigma^* (\delta_1^*((q_1,w)) = q')\} & \text{if } q = q_0,\ a = \varepsilon \\ \emptyset & \text{if } q = q_0,\ a \in \Sigma \\ \emptyset & \text{if } q \in Q_1,\ a = \varepsilon \end{cases}$$

  Hint: the notation δ_1^* refers to the iterated transition function.

(a) (*Graded for correctness*) Illustrate the construction of N_α by defining a specific pair of example DFAs M_1 and M_2 and applying the construction above to create the new NFA N_α. Your example DFA should

- Have the same input alphabet as each other,
- Each have exactly three states (all reachable from the respective start state),
- Accept at least one string and reject at least one string,
- Recognize different languages from one another, and
- Not have any states labelled q_0, and
- Not share any state labels.

Apply the construction above to create the new NFA. A complete submission will include the state diagrams of your example DFA M_1 and M_2 and the state diagram of the NFA N_α resulting from this construction and a precise and clear description of $L(M_1)$ and $L(M_2)$ and $L(N_\alpha)$, justified by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the language.

(b) (*Graded for correctness*) Illustrate the construction of N_β by defining a specific pair of example DFAs M_1 and M_2 and applying the construction above to create the new NFA N_β. Your example DFA should

- Have the same input alphabet as each other,
- Each have exactly two states (all reachable from the respective start state),
- Accept at least one string and reject at least one string,
- Recognize different languages from one another, and
- Not have any states labelled q_0, and
- Not share any state labels.

Apply the construction above to create the new NFA. A complete submission will include the state diagrams of your example DFA M_1 and M_2 and the state diagram of the NFA N_β resulting from this construction and a precise and clear description of $L(M_1)$ and $L(M_2)$ and $L(N_\beta)$, justified by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the language.

(c) (*Graded for correctness*) Illustrate the construction of N_γ by defining a specific example DFA M_1 and applying the construction above to create the new NFA N_γ. Your example DFA should

- Have exactly four states (all reachable from the respective start state),
- Accept at least one string and reject at least one string,
- Not have any states labelled q_0.

Apply the construction above to create the new NFA. A complete submission will include the state diagram of your example DFA M_1 and the state diagram of the NFA N_γ resulting from this construction and a precise and clear description of $L(M_1)$ and $L(N_\gamma)$, justified by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the language.

(d) (*Graded for completeness*) If possible, associate each construction above with one of the operations whose definitions we recalled at the start of the question. For example, is it the case that (for all choices of DFA M_1 and M_2) $L(N_\alpha) = L(M_1) \cup L(M_2)$? or $L(N_\alpha) = L(M_1) \cap L(M_2)$? etc.

A complete solution will consider each of the constructions $N_\alpha, N_\beta, N_\gamma$ in turn, and for each, either name the operation that's associated with the construction (and explain why) or explain why none of the operations mentioned is associated with the construction.