

Day13

Definitions below are on pages 101-102.

Term	Typical symbol or Notation	Meaning
Context-free grammar (CFG)	G	$G = (V, \Sigma, R, S)$
The set of variables	V	Finite set of symbols that represent phases in production pattern
The set of terminals	Σ	Alphabet of symbols of strings generated by CFG $V \cap \Sigma = \emptyset$
The set of rules	R	Each rule is $A \rightarrow u$ with $A \in V$ and $u \in (V \cup \Sigma)^*$
The start variable	S	Usually on left-hand-side of first/ topmost rule
Derivation	$S \Rightarrow \dots \Rightarrow w$	Sequence of substitutions in a CFG (also written $S \Rightarrow^* w$). At each step, we can apply one rule to one occurrence of a variable in the current string by substituting that occurrence of the variable with the right-hand-side of the rule. The derivation must end when the current string has only terminals (no variables) because then there are no instances of variables to apply a rule to.
Language generated by the context-free grammar G	$L(G)$	The set of strings for which there is a derivation in G . Symbolically: $\{w \in \Sigma^* \mid S \Rightarrow^* w\}$ i.e. $\{w \in \Sigma^* \mid \text{there is derivation in } G \text{ that ends in } w\}$
Context-free language		A language that is the language generated by some context-free grammar

Examples of context-free grammars, derivations in those grammars, and the languages generated by those grammars

$G_1 = (\{S\}, \{0\}, R, S)$ with rules

$$\begin{aligned} S &\rightarrow 0S \\ S &\rightarrow 0 \end{aligned}$$

In $L(G_1)$...

Not in $L(G_1)$...

$$G_2 = (\{S\}, \{0, 1\}, R, S)$$

$$S \rightarrow 0S \mid 1S \mid \varepsilon$$

In $L(G_2) \dots$

Not in $L(G_2) \dots$

$(\{S, T\}, \{0, 1\}, R, S)$ with rules

$$S \rightarrow T1T1T1T$$

$$T \rightarrow 0T \mid 1T \mid \varepsilon$$

In $L(G_3) \dots$

Not in $L(G_3) \dots$

$G_4 = (\{A, B\}, \{0, 1\}, R, A)$ with rules

$$A \rightarrow 0A0 \mid 0A1 \mid 1A0 \mid 1A1 \mid 1$$

In $L(G_4) \dots$

Not in $L(G_4) \dots$

Design a CFG to generate the language $\{a^n b^n \mid n \geq 0\}$

Design a CFG to generate the language $\{a^i b^j \mid j \geq i \geq 0\}$

Design a PDA to recognize the language $\{a^i b^j \mid j \geq i \geq 0\}$

Day14

Theorem 2.20: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet Σ is called **CFL**.

Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a description. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
 - PDAs can “test for emptiness of stack” without providing details. *How?* We can always push a special end-of-stack symbol, $\$$, at the start, before processing any input, and then use this symbol as a flag.
 - PDAs can “test for end of input” without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Suppose L_1 and L_2 are context-free languages over Σ . **Goal:** $L_1 \cup L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

Suppose L_1 and L_2 are context-free languages over Σ . **Goal:** $L_1 \circ L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

Summary

Over a fixed alphabet Σ , a language L is **regular**

- iff it is described by some regular expression
- iff it is recognized by some DFA
- iff it is recognized by some NFA

Over a fixed alphabet Σ , a language L is **context-free**

- iff it is generated by some CFG
- iff it is recognized by some PDA

Fact: Every regular language is a context-free language.

Fact: There are context-free languages that are nonregular.

Fact: There are countably many regular languages.

Fact: There are countably infinitely many context-free languages.

Consequence: Most languages are **not** context-free!

Examples of non-context-free languages

$$\begin{aligned} &\{a^n b^n c^n \mid 0 \leq n, n \in \mathbb{Z}\} \\ &\{a^i b^j c^k \mid 0 \leq i \leq j \leq k, i \in \mathbb{Z}, j \in \mathbb{Z}, k \in \mathbb{Z}\} \\ &\{ww \mid w \in \{0, 1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there is a number p where, if s is any string in A of length at least p , then s may be divided into five pieces $s = uvxyz$ where (1) for each $i \geq 0$, $uv^i xy^i z \in A$, (2) $|uv| > 0$, (3) $|vxy| \leq p$. *We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.*

Recall: A set X is said to be **closed** under an operation OP if, for any elements in X , applying OP to them gives an element in X .

True/False	Closure claim
True	The set of integers is closed under multiplication. $\forall x \forall y ((x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \rightarrow xy \in \mathbb{Z})$
True	For each set A , the power set of A is closed under intersection. $\forall A_1 \forall A_2 ((A_1 \in \mathcal{P}(A) \wedge A_2 \in \mathcal{P}(A)) \rightarrow A_1 \cap A_2 \in \mathcal{P}(A))$
	The class of regular languages over Σ is closed under complementation.
	The class of regular languages over Σ is closed under union.
	The class of regular languages over Σ is closed under intersection.
	The class of regular languages over Σ is closed under concatenation.
	The class of regular languages over Σ is closed under Kleene star.
	The class of context-free languages over Σ is closed under complementation.
	The class of context-free languages over Σ is closed under union.
	The class of context-free languages over Σ is closed under intersection.
	The class of context-free languages over Σ is closed under concatenation.
	The class of context-free languages over Σ is closed under Kleene star.