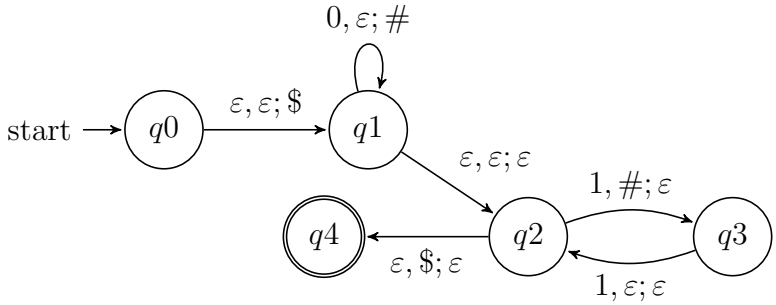
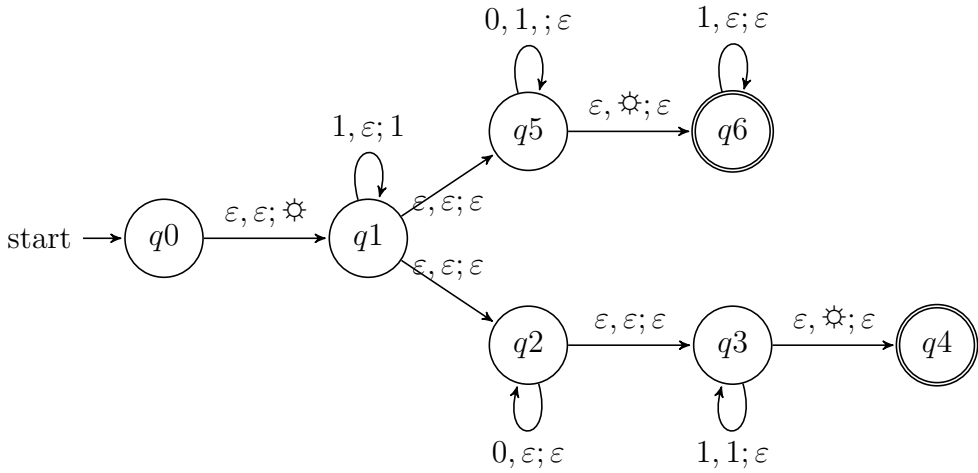
Day12

Definition A **pushdown automaton** (PDA) is specified by a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q is the finite set of states, Σ is the input alphabet, Γ is the stack alphabet,

$$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Gamma_\varepsilon)$$

is the transition function, $q_0 \in Q$ is the start state, $F \subseteq Q$ is the set of accept states.

For the PDA state diagrams below, $\Sigma = \{0, 1\}$.

Mathematical description of language	State diagram of PDA recognizing language
	$\Gamma = \{\$, \#\}$ 
	$\Gamma = \{☼, 1\}$ 

$\{0^i 1^j 0^k \mid i, j, k \geq 0\}$

Note: alternate notation is to replace ; with → on arrow labels.

Corollary: for each language L over Σ, if there is an NFA N with $L(N) = L$ then there is a PDA M with $L(M) = L$

Proof idea: Declare stack alphabet to be $\Gamma = \Sigma$ and then don't use stack at all.

Big picture: PDAs are motivated by wanting to add some memory of unbounded size to NFA. How do we accomplish a similar enhancement of regular expressions to get a syntactic model that is more expressive?

DFA, NFA, PDA: Machines process one input string at a time; the computation of a machine on its input string reads the input from left to right.

Regular expressions: Syntactic descriptions of all strings that match a particular pattern; the language described by a regular expression is built up recursively according to the expression's syntax

Context-free grammars: Rules to produce one string at a time, adding characters from the middle, beginning, or end of the final string as the derivation proceeds.

Day13

Definitions below are on pages 101-102.

Term	Typical symbol or Notation	Meaning
Context-free grammar (CFG)	G	$G = (V, \Sigma, R, S)$
The set of **variables**	V	Finite set of symbols that represent phases in production pattern
The set of **terminals**	Σ	Alphabet of symbols of strings generated by CFG $V \cap \Sigma = \emptyset$
The set of **rules**	R	Each rule is $A \to u$ with $A \in V$ and $u \in (V \cup \Sigma)^*$
The **start** variable	S	Usually on left-hand-side of first/ topmost rule
Derivation	$S \Rightarrow \cdots \Rightarrow w$	Sequence of substitutions in a CFG (also written $S \Rightarrow^* w$). At each step, we can apply one rule to one occurrence of a variable in the current string by substituting that occurrence of the variable with the right-hand-side of the rule. The derivation must end when the current string has only terminals (no variables) because then there are no instances of variables to apply a rule to.
Language **generated** by the context-free grammar G	$L(G)$	The set of strings for which there is a derivation in G. Symbolically: $\{w \in \Sigma^* \mid S \Rightarrow^* w\}$ i.e. $\{w \in \Sigma^* \mid$ there is derivation in G that ends in $w\}$
Context-free language		A language that is the language generated by some context-free grammar

Examples of context-free grammars, derivations in those grammars, and the languages generated by those grammars

$G_1 = (\{S\}, \{0\}, R, S)$ with rules

$$S \to 0S$$
$$S \to 0$$

In $L(G_1)$ ...

Not in $L(G_1)$ ...

$G_2 = (\{S\}, \{0, 1\}, R, S)$

$$S \to 0S \mid 1S \mid \varepsilon$$

In $L(G_2)$ ...

Not in $L(G_2)$ ...

$(\{S, T\}, \{0, 1\}, R, S)$ with rules

$$S \to T1T1T1T$$
$$T \to 0T \mid 1T \mid \varepsilon$$

In $L(G_3)$ ...

Not in $L(G_3)$ ...

$G_4 = (\{A, B\}, \{0, 1\}, R, A)$ with rules

$$A \to 0A0 \mid 0A1 \mid 1A0 \mid 1A1 \mid 1$$

In $L(G_4)$ ...

Not in $L(G_4)$ ...

Design a CFG to generate the language $\{a^n b^n \mid n \geq 0\}$

Design a CFG to generate the language $\{a^i b^j \mid j \geq i \geq 0\}$

Design a PDA to recognize the language $\{a^i b^j \mid j \geq i \geq 0\}$

Day14

Theorem 2.20: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet Σ is called **CFL**.

Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a descripton. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
  - PDAs can "test for emptiness of stack" without providing details. *How?* We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
  - PDAs can "test for end of input" without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Suppose L_1 and L_2 are context-free languages over Σ. **Goal**: $L_1 \cup L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

Suppose L_1 and L_2 are context-free languages over Σ. **Goal**: $L_1 \circ L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

Summary

Over a fixed alphabet Σ, a language L is **regular**

iff it is described by some regular expression
iff it is recognized by some DFA
iff it is recognized by some NFA

Over a fixed alphabet Σ, a language L is **context-free**

iff it is generated by some CFG
iff it is recognized by some PDA

Fact: Every regular language is a context-free language.

Fact: There are context-free languages that are nonregular.

Fact: There are countably many regular languages.

Fact: There are countably infinitely many context-free languages.

Consequence: Most languages are **not** context-free!

Examples of non-context-free languages

$$\{a^n b^n c^n \mid 0 \leq n, n \in \mathbb{Z}\}$$
$$\{a^i b^j c^k \mid 0 \leq i \leq j \leq k, i \in \mathbb{Z}, j \in \mathbb{Z}, k \in \mathbb{Z}\}$$
$$\{ww \mid w \in \{0,1\}^*\}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there is a number p where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ where (1) for each $i \geq 0$, $uv^i xy^i z \in A$, (2) $|uv| > 0$, (3) $|vxy| \leq p$. *We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.*

Recall: A set X is said to be **closed** under an operation OP if, for any elements in X, applying OP to them gives an element in X.

True/False	Closure claim
True	The set of integers is closed under multiplication. $\forall x \forall y \, (\, (x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \rightarrow xy \in \mathbb{Z} \,)$
True	For each set A, the power set of A is closed under intersection. $\forall A_1 \forall A_2 \, (\, (A_1 \in \mathcal{P}(A) \wedge A_2 \in \mathcal{P}(A) \in \mathbb{Z}) \rightarrow A_1 \cap A_2 \in \mathcal{P}(A) \,)$
	The class of regular languages over Σ is closed under complementation.
	The class of regular languages over Σ is closed under union.
	The class of regular languages over Σ is closed under intersection.
	The class of regular languages over Σ is closed under concatenation.
	The class of regular languages over Σ is closed under Kleene star.
	The class of context-free languages over Σ is closed under complementation.
	The class of context-free languages over Σ is closed under union.
	The class of context-free languages over Σ is closed under intersection.
	The class of context-free languages over Σ is closed under concatenation.
	The class of context-free languages over Σ is closed under Kleene star.

Day11

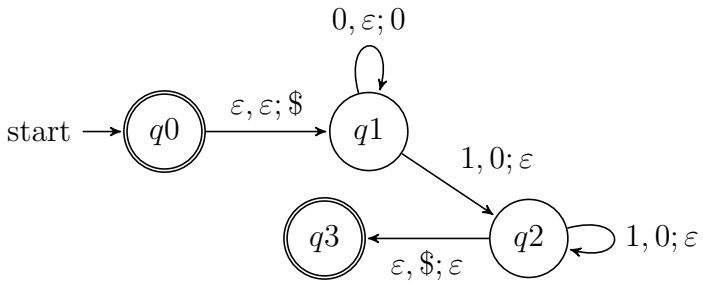
Regular sets are not the end of the story

- Many nice / simple / important sets are not regular
- Limitation of the finite-state automaton model: Can't "count", Can only remember finitely far into the past, Can't backtrack, Must make decisions in "real-time"
- We know actual computers are more powerful than this model...

The **next** model of computation. Idea: allow some memory of unbounded size. How?

- To generalize regular expressions: **context-free grammars**
- To generalize NFA: **Pushdown automata**, which is like an NFA with access to a stack: Number of states is fixed, number of entries in stack is unbounded. At each step (1) Transition to new state based on current state, letter read, and top letter of stack, then (2) (Possibly) push or pop a letter to (or from) top of stack. Accept a string iff there is some sequence of states and some sequence of stack contents which helps the PDA processes the entire input string and ends in an accepting state.

Is there a PDA that recognizes the nonregular language $\{0^n1^n \mid n \geq 0\}$?



The PDA with state diagram above can be informally described as:

> Read symbols from the input. As each 0 is read, push it onto the stack. As soon as 1s are seen, pop a 0 off the stack for each 1 read. If the stack becomes empty and we are at the end of the input string, accept the input. If the stack becomes empty and there are 1s left to read, or if 1s are finished while the stack still contains 0s, or if any 0s appear in the string following 1s, reject the input.

Trace a computation of this PDA on the input string 01.

Extra practice: Trace the computations of this PDA on the input string 011.

A PDA recognizing the set { } can be informally described as:

> Read symbols from the input. As each 0 is read, push it onto the stack. As soon as 1s are seen, pop a 0 off the stack for each 1 read. If the stack becomes empty and there is exactly one 1 left to read, read that 1 and accept the input. If the stack becomes empty and there are either zero or more than one 1s left to read, or if the 1s are finished while the stack still contains 0s, or if any 0s appear in the input following 1s, reject the input.

Modify the state diagram below to get a PDA that implements this description: