

# HW1CSE105W25: Homework assignment 1

CSE105W25

Due: January 16th at 5pm, via Gradescope

## In this assignment,

You will practice reading and applying the definitions of alphabets, strings, languages, Kleene star, and regular expressions. You will use regular expressions and relate them to languages.

**Resources:** To review the topics for this assignment, see the class material from Weeks 0 and 1 and Review Quiz 1. We will post frequently asked questions and our answers to them in a pinned Piazza post.

**Reading and extra practice problems:** Sipser Section 0, 1.3. Chapter 0 exercises 0.1, 0.2, 0.3, 0.5, 0.6, 0.9. Chapter 1 exercises 1.19, 1.23.

**For all HW assignments:** Weekly homework may be done individually or in groups of up to 3 students. You may switch HW partners for different HW assignments. Please ensure your name(s) and PID(s) are clearly visible on the first page of your homework submission and then upload the PDF to Gradescope. If working in a group, submit only one submission per group: one partner uploads the submission through their Gradescope account and then adds the other group member(s) to the Gradescope submission by selecting their name(s) in the “Add Group Members” dialog box. You will need to re-add your group member(s) every time you resubmit a new version of your assignment. Each homework question will be graded either for correctness (including clear and precise explanations and justifications of all answers) or fair effort completeness. On the “graded for correctness” questions, you may only collaborate with CSE 105 students in your group; if your group has questions about a problem, you may ask in drop-in help hours or post a private post (visible only to the Instructors) on Piazza. On the “graded for completeness” questions, you may collaborate with all other CSE 105 students this quarter, and you may make public posts about these questions on Piazza.

All submitted homework for this class must be typed. You can use a word processing editor if you like (Microsoft Word, Open Office, Notepad, Vim, Google Docs, etc.) but you might find it useful to take this opportunity to learn LaTeX. LaTeX is a markup language used widely in computer science and mathematics. The homework assignments are typed using LaTeX and you can use the source files as templates for typesetting your solutions. To generate state diagrams of

machines, you can (1) use the LaTeX tikzpicture environment (see templates in the class notes), or (2) use the software tools Flap.js or JFLAP described in the class syllabus (and include a screenshot in your PDF), or (3) you can carefully and clearly hand-draw the diagram and take a picture and include it in your PDF. We recommend that you submit early drafts to Gradescope so that in case of any technical difficulties, at least some of your work is present. You may update your submission as many times as you'd like up to the deadline.

## Integrity reminders

- Problems should be solved together, not divided up between the partners. The homework is designed to give you practice with the main concepts and techniques of the course, while getting to know and learn from your classmates.
- On the “graded for correctness” questions, you may only collaborate with CSE 105 students in your group. You may ask questions about the homework in office hours (of the instructor, TAs, and/or tutors) and on Piazza (as private notes viewable only to the Instructors). You *cannot* use any online resources about the course content other than the class material from this quarter – this is primarily to ensure that we all use consistent notation and definitions (aligned with the textbook) and also to protect the learning experience you will have when the ‘aha’ moments of solving the problem authentically happen.
- Do not share written solutions or partial solutions for homework with other students in the class who are not in your group. Doing so would dilute their learning experience and detract from their success in the class.

You will submit this assignment via Gradescope (<https://www.gradescope.com>) in the assignment called “hw1CSE105W25”.

## Assigned questions

### 1. Strings and languages: finding examples and edge cases (12 points):

- (a) (*Graded for completeness*)<sup>1</sup> Give five (different) example alphabets that are meaningful or useful to you in some way. Specify them formally, either with roster notation (which means listing all and only distinct elements between { and } and separated by commas) or with another approach to precisely define all and only the elements of the alphabet.
- (b) (*Graded for completeness*) Give an example of a finite set over an alphabet and an infinite set over an alphabet. You get to choose the alphabet, and you get to choose the sets.

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<sup>1</sup>This means you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for incorrect answers. To demonstrate your honest effort in answering the question, we expect you to include your attempt to answer *each* part of the question. If you get stuck with your attempt, you can still demonstrate your effort by explaining where you got stuck and what you did to try to get unstuck.

The goal is to practice communicating your choices and definitions with clear and precise notation. One habit that will be useful (for this course, and beyond), is to think of your response for each question as a well-formed paragraph: include all the information that is relevant so that your solution is self-contained, and so that each sentence is grammatically constructed.

- (c) (*Graded for correctness*)<sup>2</sup> Define an alphabet  $\Sigma_1$  and an alphabet  $\Sigma_2$  and a language  $L_1$  over  $\Sigma_1$  that is also a language over  $\Sigma_2$  and a language  $L_2$  over  $\Sigma_2$  that is **not** a language over  $\Sigma_1$ . A complete and correct answer will use clear and precise notation (consistent with the textbook and class notes) and will include a description of why the given example  $L_1$  is a language over both  $\Sigma_1$  and  $\Sigma_2$  and a description of why the given example  $L_2$  is a language over  $\Sigma_2$  and not over  $\Sigma_1$ .

## 2. Regular expressions (20 points):

- (a) (*Graded for completeness*) Give three regular expressions that all describe the set of all strings over  $\{a, b\}$  that have odd length. Ungraded bonus challenge: Make the expressions as different as possible!
- (b) (*Graded for completeness*) A friend tells you that each regular expression that has a Kleene star ( $*$ ) describes an infinite language. Are they right? Either help them justify their claim or give a counterexample to disprove it and explain your counterexample.
- (c) (*Graded for correctness*) For this question, the alphabet is  $\{a, b, c\}$ . A friend is trying to design a regular expression that describes the set of all strings over this alphabet that end in  $c$ . Classify each of the following attempts as
- Correct. Explain why.
  - Error Type 1: Incorrect, because (even though each string that is in the language described by the regular expression ends in  $c$ ) there is a string that ends in  $c$  that is not in the language described by the regular expression. Give this example string and explain why it proves we're in this case.
  - Error Type 2: Incorrect, because (even though each string that ends in  $c$  is in the language described by the regular expression), there is a string in the language described by the regular expression that does not end in  $c$ . Give this example string and explain why it proves we're in this case.
  - Error Type 3: Incorrect, because there are two counterexample strings, one which is a string that ends in  $c$  that is not in the language described by the regular expression and one which is in the language described by the regular expression but does not end in  $c$ . Give both example strings and describe why each has the given property.

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<sup>2</sup>This means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

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*Worked example for reference:* Consider the regular expression  $(a \cup b \cup c)^*$ . This regular expression has **Error Type 2** because it describes the set of all strings over  $\{a, b, c\}$ , so even though each string that ends in  $c$  is in this language, there is an example, say  $ab$  that is a string in the language described by the regular expression (because we consider the string formed as a result of the Kleene star operation which has 2 slots and where the first slot matches the  $a$  in  $a \cup b \cup c$  and the second slot matches  $b$  in  $a \cup b \cup c$ ) but does not end in  $c$  (it ends in  $b$ ).

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i. The regular expression is

$$(a \cup b)^* \circ c$$

ii. The regular expression is

$$(a \circ b \circ c)^*$$

iii. The regular expression is

$$a^*c \cup b^*c \cup c^*c$$

### 3. Functions over languages (18 points):

For each language  $L$  over an alphabet  $\Sigma$ , we have the associated sets of strings (also over  $\Sigma$ )

$$L^* = \{w_1 \cdots w_k \mid k \geq 0 \text{ and each } w_i \in L\}$$

and

$$SUBSTRING(L) = \{w \in \Sigma^* \mid \text{there exist } x, y \in \Sigma^* \text{ such that } xwy \in L\}$$

and

$$EXTEND(L) = \{w \in \Sigma^* \mid w = uv \text{ for some strings } u \in L \text{ and } v \in \Sigma^*\}$$

Also, recall the set operations union and intersection: for any sets  $X$  and  $Y$

$$X \cup Y = \{w \mid w \in X \text{ or } w \in Y\}$$

$$X \cap Y = \{w \mid w \in X \text{ and } w \in Y\}$$

(a) (*Graded for completeness*) Specify an example language  $A$  over  $\{0, 1\}$  such that

$$SUBSTRING(A) = EXTEND(A)$$

or explain why there is no such example. A complete solution will include either (1) a precise and clear description of your example language  $A$  and a precise and clear description of the result of computing  $SUBSTRING(A)$ ,  $EXTEND(A)$  (using the given definitions) to justify this description and to justify the set equality, or (2) a sufficiently general and correct argument why there is no such example, referring back to the relevant definitions.

(b) (*Graded for correctness*) Specify an example language  $B$  over  $\{0, 1\}$  such that

$$SUBSTRING(B) \cap EXTEND(B) = \{\varepsilon\}$$

and

$$SUBSTRING(B) \cup EXTEND(B) = \{0, 1\}^*$$

or explain why there is no such example. A complete solution will include either (1) a precise and clear description of your example language  $B$  and a precise and clear description of the result of computing  $SUBSTRING(B)$ ,  $EXTEND(B)$  (using the given definitions) to justify this description and to justify the set equality with  $\{\varepsilon\}$  and  $\{0, 1\}^*$  (respectively), or (2) a sufficiently general and correct argument why there is no such example, referring back to the relevant definitions.

- (c) (*Graded for correctness*) Specify an example **infinite** language  $C$  over  $\{0, 1\}$  such that

$$SUBSTRING(C) \neq \{0, 1\}^*$$

and

$$SUBSTRING(C) = C^*$$

or explain why there is no such example. A complete solution will include either (1) a precise and clear description of your example language  $C$  and a precise and clear description of the result of computing  $SUBSTRING(C)$ ,  $C^*$  (using the given definitions) to justify this description and to justify the set nonequality claims, or (2) a sufficiently general and correct argument why there is no such example, referring back to the relevant definitions.

- (d) (*Graded for correctness*) Specify an example **finite** language  $D$  over  $\{0, 1\}$  such that

$$EXTEND(D) \neq \{0, 1\}^*$$

and

$$EXTEND(D) = D^*$$

or explain why there is no such example. A complete solution will include either (1) a precise and clear description of your example language  $D$  and a precise and clear description of the result of computing  $EXTEND(D)$ ,  $D^*$  (using the given definitions) to justify this description and to justify the set nonequality claims, or (2) a sufficiently general and correct argument why there is no such example, referring back to the relevant definitions.

HW2CSE105W25: Homework assignment 2 Due: January 30th at 5pm, via Gradescope

### **In this assignment,**

You will practice designing multiple representations of regular languages and working with general constructions of automata to demonstrate the richness of the class of regular languages.

**Resources:** To review the topics for this assignment, see the class material from Week 2 and Week 3. We will post frequently asked questions and our answers to them in a pinned Piazza post.

**Reading and extra practice problems:** Sipser Section 1.1, 1.2, 1.3. Chapter 1 exercises 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19, 1.20, 1.21, 1.22, 1.23. Chapter 1 problem 1.31, 1.36, 1.37.

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All submitted homework for this class must be typed. You can use a word processing editor if you like (Microsoft Word, Open Office, Notepad, Vim, Google Docs, etc.) but you might find it useful to take this opportunity to learn LaTeX. LaTeX is a markup language used widely in computer science and mathematics. The homework assignments are typed using LaTeX and you can use the source files as templates for typesetting your solutions. To generate state diagrams of machines, you can (1) use the LaTeX tikzpicture environment (see templates in the class notes), or (2) use the software tools Flap.js or JFLAP described in the class syllabus (and include a screenshot in your PDF), or (3) you can carefully and clearly hand-draw the diagram and take a picture and include it in your PDF. We recommend that you submit early drafts to Gradescope so that in case of any technical difficulties, at least some of your work is present. You may update your submission as many times as you’d like up to the deadline.

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is designed to give you practice with the main concepts and techniques of the course, while getting to know and learn from your classmates.

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### Assigned questions

1. **Finite automata** (10 points): Consider the finite automaton  $M = (Q, \Sigma, \delta, q_0, F)$  whose state diagram is depicted below



- (*Graded for completeness*)<sup>3</sup> Write the formal definition of this automaton. In other words, give the five defining parameters  $Q, \Sigma, \delta, q_0, F$  so that they are consistent with the state diagram of  $M$ .
- (*Graded for correctness*)<sup>4</sup> Give a regular expression  $R$  so that  $L(R) = L(M)$ . In other words, we want a regular expression that describes the language recognized by this finite

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<sup>4</sup>This means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using

automaton. Justify your answer by referring to the definition of the semantics of regular expressions and computations of finite automata. Include an explanation for why each string in  $L(R)$  is accepted by the finite automaton *and* for why each string not in  $L(R)$  is rejected by the finite automaton.

*Ungraded bonus: can you find more than one such regular expression?*

- (c) (*Graded for completeness*) Keeping the same set of states  $Q$ , input alphabet  $\Sigma$ , same start state  $q_0$ , and same transition function  $\delta$ , choose a new set of accepting states  $F_{new1}$  so that the new finite automaton  $M_1 = (Q, \Sigma, \delta, q_0, F_{new1})$  that results recognizes a **proper superset** of  $L(M)$ , or explain why there is no such example. A complete solution will include either (1) a precise and clear description of your choice of  $F_{new1}$  *and* a precise and clear explanation of why every string that is accepted by  $M$  is also accepted by  $M_1$  *and* an example of a string that is accepted by  $M_1$  and is rejected by  $M$ ; or (2) a sufficiently general and correct argument why there is no such example, referring back to the relevant definitions.
- (d) (*Graded for correctness*) Keeping the same set of states  $Q$ , input alphabet  $\Sigma$ , same start state  $q_0$ , and same transition function  $\delta$ , choose a new set of accepting states  $F_{new2}$  so that the new finite automaton  $M_2 = (Q, \Sigma, \delta, q_0, F_{new2})$  that results recognizes a **nonempty proper subset** of  $L(M)$ , or explain why there is no such example. A complete solution will include either (1) a precise and clear description of your choice of  $F_{new2}$  *and* an example string accepted by  $M_2$  *and* a precise and clear explanation of why every string that is accepted by  $M_2$  is also accepted by  $M$  *and* an example of a string that is accepted by  $M$  and is rejected by  $M_2$ ; or (2) a sufficiently general and correct argument why there is no such example, referring back to the relevant definitions.

2. **Automata design** (12 points): As background to this question, recall that integers can be represented using base  $b$  expansions, for any convenient choice of base  $b$ . The precise definition is: for  $b$  an integer greater than 1 and  $n$  a positive integer, the **base  $b$  expansion of  $n$**  is defined to be

$$(a_{k-1} \cdots a_1 a_0)_b$$

where  $k$  is a positive integer,  $a_0, a_1, \dots, a_{k-1}$  are nonnegative integers less than  $b$ ,  $a_{k-1} \neq 0$ , and

$$n = \sum_{i=0}^{k-1} a_i b^i$$

Notice: *The base  $b$  expansion of a positive integer  $n$  is a string over the alphabet  $\{x \in \mathbb{Z} \mid 0 \leq x < b\}$  whose leftmost character is nonzero.*

An important property of base  $b$  expansions of integers is that, for each integer  $b$  greater than 1, each positive integer  $n = (a_{k-1} \cdots a_1 a_0)_b$ , and each nonnegative integer  $a$  less than  $b$ ,

$$bn + a = (a_{k-1} \cdots a_1 a_0 a)_b$$

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mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.



In other words, shifting the base  $b$  expansion to the left results in multiplying the integer value by the base. In this question we'll explore building deterministic finite automata that recognize languages that correspond to useful sets of integers.

- (a) (*Graded for completeness*) Design a DFA that recognizes the set of binary (base 2) expansions of positive integers that are powers of 2. A complete solution will include the state diagram of your DFA and a brief justification of your construction by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

*Hints:* (1) A power of 2 is an integer  $x$  that can be written as  $2^y$  for some nonnegative integer  $y$ , (2) the DFA should accept the strings 100, 10 and 100000 and should reject the strings 010, 1101, and  $\varepsilon$  (can you see why?).

- (b) (*Graded for correctness*) Design a DFA that recognizes the set of binary (base 2) expansions of positive integers that are less than 10. Your DFA must use **fewer than ten states**. A complete solution will include the state diagram of your DFA and a brief justification of your construction by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

- (c) (*Graded for completeness*) Find a positive integer  $B$  greater than 1 so that there is a DFA that recognizes the set of base  $B$  expansions of positive integers that are less than 10 and it uses as few states as possible. A complete solution will include the state diagram of your DFA and a brief justification of your choice of base.

*Hint: sometimes rewriting the defining membership condition for a set in different ways helps us find alternate representations of that set.*

### 3. Nondeterminism (15 points): For this question, the alphabet is $\{a, b, c\}$ .

- (a) (*Graded for completeness*) Design a NFA that recognizes the language

$$L_1 = \{w \in \{a, b, c\}^* \mid w \text{ starts with } a \text{ and ends with } a\}$$

A complete solution will include the state diagram of your NFA and a brief justification of your construction that explains the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

- (b) (*Graded for correctness*) Design a NFA that recognizes the language

$$L_2 = \{w \in \{a, b, c\}^* \mid w \text{ has no consecutive repeated characters}\}$$

For example, the empty string,  $a$ ,  $bac$ , and  $abca$  are each elements of this language but  $aa$  and  $abb$  and  $abbc$  are not elements of this language.

A complete solution will include the state diagram of your NFA and a brief justification of your construction that explains the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

(c) (*Graded for completeness*) Consider the language

$$L_1 \cup L_2 = \{w \in \{a, b, c\}^* \mid w \text{ starts with } a \text{ and ends with } a \\ \text{or has no consecutive repeated characters}\}$$

Give at least two representations of this language among the following:

- A regular expression that describes  $L_1 \cup L_2$
- A DFA that recognizes  $L_1 \cup L_2$
- A NFA that recognizes  $L_1 \cup L_2$

You can design your automata directly or use the constructions from class and chapter 1 in the book to build these automata from automata for the simpler languages.

A complete solution will include at least two of the representations as well as a brief justification of each construction.

(d) (*Graded for completeness*) Consider the language

$$L_1 \cap L_2 = \{w \in \{a, b, c\}^* \mid w \text{ starts with } a \text{ and ends with } a \\ \text{and has no consecutive repeated characters}\}$$

Give at least two representations of this language among the following:

- A regular expression that describes  $L_1 \cap L_2$
- A DFA that recognizes  $L_1 \cap L_2$
- A NFA that recognizes  $L_1 \cap L_2$

You can design your automata directly or use the constructions from class and chapter 1 in the book to build these automata from automata for the simpler languages.

A complete solution will include at least two of the representations as well as a brief justification of each construction.

**4. General constructions** (13 points): In this question, you'll practice working with formal general constructions for automata and translating between state diagrams and formal definitions.

Recall the definitions of operations we've talked about that produce new languages from old: for each language  $L$  over an alphabet  $\Sigma$ , we have the associated sets of strings (also over  $\Sigma$ )

$$L^* = \{w_1 \cdots w_k \mid k \geq 0 \text{ and each } w_i \in L\}$$

and

$$SUBSTRING(L) = \{w \in \Sigma^* \mid \text{there exist } x, y \in \Sigma^* \text{ such that } xwy \in L\}$$

and

$$EXTEND(L) = \{w \in \Sigma^* \mid w = uv \text{ for some strings } u \in L \text{ and } v \in \Sigma^*\}$$

Also, recall the set operations union and intersection: for any sets  $X$  and  $Y$

$$X \cup Y = \{w \mid w \in X \text{ or } w \in Y\}$$

$$X \cap Y = \{w \mid w \in X \text{ and } w \in Y\}$$

Let  $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$  and  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$  be DFA.

For simplicity, assume that  $Q_1 \cap Q_2 = \emptyset$  and that  $q_0 \notin Q_1 \cup Q_2$ .

Consider the following definitions of new automata parameterized by these DFA:

- The NFA  $N_\alpha = (Q_1 \cup Q_2 \cup \{q_0\}, \Sigma, \delta_\alpha, q_0, F_1 \cup F_2)$  with the transition function given by

$$\delta_\alpha((q, x)) = \begin{cases} \{q_1, q_2\} & \text{if } q = q_0, x = \varepsilon \\ \emptyset & \text{if } q = q_0, x \in \Sigma \\ \{\delta_1((q, x))\} & \text{if } q \in Q_1, x \in \Sigma \\ \{\delta_2((q, x))\} & \text{if } q \in Q_2, x \in \Sigma \\ \emptyset & \text{if } q \in Q_1 \cup Q_2, x = \varepsilon \end{cases}$$

- The NFA  $N_\beta = (Q_1 \times Q_2, \Sigma, \delta_\beta, (q_1, q_2), F_1 \times F_2)$  with the transition function given by

$$\delta_\beta(((r, s), x)) = \{(\delta_1((r, x)), \delta_2((s, x)))\}$$

and

$$\delta_\beta(((r, s), \varepsilon)) = \emptyset$$

for  $r \in Q_1, s \in Q_2, x \in \Sigma$ .

- The NFA  $N_\gamma = (Q_1 \cup \{q_0\}, \Sigma, \delta_\gamma, q_0, \{q \in Q_1 \mid \exists w \in \Sigma^* (\delta_1^*((q, w)) \in F_1)\})$ , and

$$\delta_\gamma((q, a)) = \begin{cases} \{\delta_1((q, a))\} & \text{if } q \in Q_1, a \in \Sigma \\ \{q' \in Q_1 \mid \exists w \in \Sigma^* (\delta_1^*((q_1, w)) = q')\} & \text{if } q = q_0, a = \varepsilon \\ \emptyset & \text{if } q = q_0, a \in \Sigma \\ \emptyset & \text{if } q \in Q_1, a = \varepsilon \end{cases}$$

*Hint: the notation  $\delta_1^*$  refers to the iterated transition function.*

- (a) (*Graded for correctness*) Illustrate the construction of  $N_\alpha$  by defining a specific pair of example DFAs  $M_1$  and  $M_2$  and applying the construction above to create the new NFA  $N_\alpha$ . Your example DFA should

- Have the same input alphabet as each other,
- Each have exactly three states (all reachable from the respective start state),
- Accept at least one string and reject at least one string,
- Recognize different languages from one another, and
- Not have any states labelled  $q_0$ , and
- Not share any state labels.

Apply the construction above to create the new NFA. A complete submission will include the state diagrams of your example DFA  $M_1$  and  $M_2$  and the state diagram of the NFA  $N_\alpha$  resulting from this construction and a precise and clear description of  $L(M_1)$  and  $L(M_2)$  and  $L(N_\alpha)$ , justified by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the language.

- (b) (*Graded for correctness*) Illustrate the construction of  $N_\beta$  by defining a specific pair of example DFAs  $M_1$  and  $M_2$  and applying the construction above to create the new NFA  $N_\beta$ . Your example DFA should

- Have the same input alphabet as each other,
- Each have exactly two states (all reachable from the respective start state),
- Accept at least one string and reject at least one string,
- Recognize different languages from one another, and
- Not have any states labelled  $q_0$ , and
- Not share any state labels.

Apply the construction above to create the new NFA. A complete submission will include the state diagrams of your example DFA  $M_1$  and  $M_2$  and the state diagram of the NFA  $N_\beta$  resulting from this construction and a precise and clear description of  $L(M_1)$  and  $L(M_2)$  and  $L(N_\beta)$ , justified by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the language.

- (c) (*Graded for correctness*) Illustrate the construction of  $N_\gamma$  by defining a specific example DFA  $M_1$  and applying the construction above to create the new NFA  $N_\gamma$ . Your example DFA should

- Have exactly four states (all reachable from the respective start state),
- Accept at least one string and reject at least one string,
- Not have any states labelled  $q_0$ .

Apply the construction above to create the new NFA. A complete submission will include the state diagram of your example DFA  $M_1$  and the state diagram of the NFA  $N_\gamma$  resulting from this construction and a precise and clear description of  $L(M_1)$  and  $L(N_\gamma)$ , justified by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the language.

- (d) (*Graded for completeness*) If possible, associate each construction above with one of the operations whose definitions we recalled at the start of the question. For example, is it the case that (for all choices of DFA  $M_1$  and  $M_2$ )  $L(N_\alpha) = L(M_1) \cup L(M_2)$ ? or  $L(N_\alpha) = L(M_1) \cap L(M_2)$ ? etc.

A complete solution will consider each of the constructions  $N_\alpha, N_\beta, N_\gamma$  in turn, and for each, either name the operation that's associated with the construction (and explain why) or explain why none of the operations mentioned is associated with the construction.

HW3CSE105W25: Homework assignment 3 Due: February 6th at 5pm, via Gradescope

**In this assignment,**

You will demonstrate the richness of the class of regular languages, as well as its boundaries, and explore push-down automata and their design.

**Resources:** To review the topics for this assignment, see the class material from Week 3 and Week 4. We will post frequently asked questions and our answers to them in a pinned Piazza post.

**Reading and extra practice problems:** Sipser Chapter 1 and Section 2.2. Chapter 1 exercises 1.28, 1.29, 1.30. Chapter 1 problem 1.53, 1.54, 1.55. Chapter 2 exercises 2.7, 2.10.

**For all HW assignments:** Weekly homework may be done individually or in groups of up to 3 students. You may switch HW partners for different HW assignments. Please ensure your name(s) and PID(s) are clearly visible on the first page of your homework submission and then upload the PDF to Gradescope. If working in a group, submit only one submission per group: one partner uploads the submission through their Gradescope account and then adds the other group member(s) to the Gradescope submission by selecting their name(s) in the “Add Group Members” dialog box. You will need to re-add your group member(s) every time you resubmit a new version of your assignment. Each homework question will be graded either for correctness (including clear and precise explanations and justifications of all answers) or fair effort completeness. On the “graded for correctness” questions, you may only collaborate with CSE 105 students in your group; if your group has questions about a problem, you may ask in drop-in help hours or post a private post (visible only to the Instructors) on Piazza. On the “graded for completeness” questions, you may collaborate with all other CSE 105 students this quarter, and you may make public posts about these questions on Piazza.

All submitted homework for this class must be typed. You can use a word processing editor if you like (Microsoft Word, Open Office, Notepad, Vim, Google Docs, etc.) but you might find it useful to take this opportunity to learn LaTeX. LaTeX is a markup language used widely in computer science and mathematics. The homework assignments are typed using LaTeX and you can use the source files as templates for typesetting your solutions. To generate state diagrams of machines, you can (1) use the LaTeX tikzpicture environment (see templates in the class notes), or (2) use the software tools Flap.js or JFLAP described in the class syllabus (and include a screenshot in your PDF), or (3) you can carefully and clearly hand-draw the diagram and take a picture and include it in your PDF. We recommend that you submit early drafts to Gradescope so that in case of any technical difficulties, at least some of your work is present. You may update your submission as many times as you’d like up to the deadline.

**Integrity reminders**

- Problems should be solved together, not divided up between the partners. The homework is designed to give you practice with the main concepts and techniques of the course, while

getting to know and learn from your classmates.

- On the “graded for correctness” questions, you may only collaborate with CSE 105 students in your group. You may ask questions about the homework in office hours (of the instructor, TAs, and/or tutors) and on Piazza (as private notes viewable only to the Instructors). You *cannot* use any online resources about the course content other than the class material from this quarter – this is primarily to ensure that we all use consistent notation and definitions (aligned with the textbook) and also to protect the learning experience you will have when the ‘aha’ moments of solving the problem authentically happen.
- Do not share written solutions or partial solutions for homework with other students in the class who are not in your group. Doing so would dilute their learning experience and detract from their success in the class.

You will submit this assignment via Gradescope (<https://www.gradescope.com>) in the assignment called “hw3CSE105W25”.

### Assigned questions

1. **Static analysis** (10 points): In software engineering, static analysis is an approach to debugging and testing where the properties of a piece of code are inferred without actually running it. In the context of finite automata, we can think of static analysis as the process of inferring properties of the language recognized by a finite automaton from properties of the graph underlying its state diagram. The Pumping Lemma is one example of static analysis. In this question, you’ll explore other examples of how properties of the graph underlying the state diagram of a machine can give us information about the language recognized by the machine.

- (a) (*Graded for completeness*)<sup>5</sup> Suppose you are given an NFA  $N_0$  over an alphabet  $\Sigma$  and each accepting state in  $N_0$  is not reachable from the start state of  $N_0$ . What can you conclude about the language of the NFA?
- (b) (*Graded for correctness*)<sup>6</sup> Prove or disprove: For any alphabet  $\Sigma$  and any DFA  $M$  over  $\Sigma$ , if every state in  $M$  is accepting then  $L(M) = \Sigma^*$ . A complete answer will clearly indicate whether the statement is true or false and then will justify with a complete and correct argument.

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<sup>5</sup>This means you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for incorrect answers. To demonstrate your honest effort in answering the question, we expect you to include your attempt to answer *each* part of the question. If you get stuck with your attempt, you can still demonstrate your effort by explaining where you got stuck and what you did to try to get unstuck.

<sup>6</sup>This means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

- (c) (*Graded for correctness*) Prove or disprove: For any alphabet  $\Sigma$  and any NFA  $N$  over  $\Sigma$ , if every state in  $N$  is accepting then  $L(N) = \Sigma^*$ . A complete answer will clearly indicate whether the statement is true or false and then will justify with a complete and correct argument.

## 2. Multiple representations (12 points):

- (a) Consider the language  $A_1 = \{uw \mid u \text{ and } w \text{ are strings over } \{0,1\} \text{ and have the same length}\}$  and the following argument.

“Proof” that  $A_1$  is not regular using the Pumping Lemma: Let  $p$  be an arbitrary positive integer. We will show that  $p$  is not a pumping length for  $A_1$ .

Choose  $s$  to be the string  $1^p 0^p$ , which is in  $A_1$  because we can choose  $u = 1^p$  and  $w = 0^p$  which each have length  $p$ . Since  $s$  is in  $A_1$  and has length greater than or equal to  $p$ , if  $p$  were to be a pumping length for  $A_1$ ,  $s$  ought to be pump’able. That is, there should be a way of dividing  $s$  into parts  $x, y, z$  where  $s = xyz$ ,  $|y| > 0$ ,  $|xy| \leq p$ , and for each  $i \geq 0$ ,  $xy^i z \in A_1$ . Suppose  $x, y, z$  are such that  $s = xyz$ ,  $|y| > 0$  and  $|xy| \leq p$ . Since the first  $p$  letters of  $s$  are all 1 and  $|xy| \leq p$ , we know that  $x$  and  $y$  are made up of all 1s. If we let  $i = 2$ , we get a string  $xy^2 z$  that is not in  $A_1$  because repeating  $y$  twice adds 1s to  $u$  but not to  $w$ , and strings in  $A_1$  are required to have  $u$  and  $w$  be the same length. Thus,  $s$  is not pumpable (even though it should have been if  $p$  were to be a pumping length) and so  $p$  is not a pumping length for  $A_1$ . Since  $p$  was arbitrary, we have demonstrated that  $A_1$  has no pumping length. By the Pumping Lemma, this implies that  $A_1$  is nonregular.

- i. (*Graded for completeness*) Find the (first and/or most significant) logical error in the “proof” above and describe why it’s wrong.
  - ii. (*Graded for completeness*) Prove that the set  $A_1$  is actually regular (by finding a regular expression that describes it or a DFA/NFA that recognizes it, and justifying why) **or** fix the proof so that it is logically sound.
- (b) Consider the language  $A_2 = \{u1w \mid u \text{ and } w \text{ are strings over } \{0,1\} \text{ and have the same length}\}$  and the following argument.

“Proof” that  $A_2$  is not regular using the Pumping Lemma: Let  $p$  be an arbitrary positive integer. We will show that  $p$  is not a pumping length for  $A_2$ .

Choose  $s$  to be the string  $1^{p+1} 0^p$ , which is in  $A_2$  because we can choose  $u = 1^p$  and  $w = 0^p$  which each have length  $p$ . Since  $s$  is in  $A_2$  and has length greater than or equal to  $p$ , if  $p$  were to be a pumping length for  $A_2$ ,  $s$  ought to be pump’able. That is, there should be a way of dividing  $s$  into parts  $x, y, z$  where  $s = xyz$ ,  $|y| > 0$ ,  $|xy| \leq p$ , and for each  $i \geq 0$ ,  $xy^i z \in A_2$ . When  $x = \varepsilon$  and  $y = 1^{p+1}$  and  $z = 0^p$ , we have satisfied that  $s = xyz$ ,  $|y| > 0$  (because  $p$  is positive) and  $|xy| \leq p$ . If we let  $i = 0$ , we get the string  $xy^0 z = 0^p$  that is not in  $A_2$  because its middle symbol is a 0, not a 1. Thus,  $s$  is not pumpable (even though it should have been if  $p$  were to be a pumping length) and so  $p$  is not a pumping length for  $A_2$ . Since  $p$  was arbitrary, we have demonstrated that  $A_2$  has no pumping length. By the Pumping Lemma, this implies that  $A_2$  is nonregular.

- i. (*Graded for completeness*) Find the (first and/or most significant) logical error in the “proof” above and describe why it’s wrong.
- ii. (*Graded for completeness*) Prove that the set  $A_2$  is actually regular (by finding a regular expression that describes it or a DFA/NFA that recognizes it, and justifying why) **or** fix the proof so that it is logically sound.

### 3. Pumping (10 points):

- (a) (*Graded for correctness*) Give an example of a language over the alphabet  $\{a, b\}$  that has cardinality 3 and for which 5 is a pumping length and 4 is not a pumping length. Is this language regular? A complete solution will give (1) a clear and precise description of the language, (2) a justification for why 5 is a pumping length, (3) a justification for why 4 is not a pumping length, (4) a correct and justified answer to whether the language is regular.
- (b) (*Graded for completeness*) In class and in the reading so far, we’ve seen the following examples of nonregular sets:

$$\begin{array}{lll}
 \{0^n 1^n \mid n \geq 0\} & \{0^n 1^m \mid 0 \leq m \leq n\} & \{0^n 1^m 0^n \mid n, m \geq 0\} \\
 \{0^n 1^n \mid n \geq 2\} & \{0^i 1^{2i} \mid 0 \leq i\} & \{w \in \{0, 1\}^* \mid w = w^R\} \\
 \{0^n 1^m \mid 0 \leq n \leq m\} & \{0^i 1^{i+1} \mid 0 \leq i\} & \{ww^R \mid w \in \{0, 1\}^*\}
 \end{array}$$

Modify one of these sets in some way and use the Pumping Lemma to prove that the resulting set is still nonregular.

**4. Regular and nonregular languages (12 points):** In Week 2’s review quiz, we saw the definition that a set  $X$  is said to be **closed under an operation** if, for any elements in  $X$ , applying to them gives an element in  $X$ . For example, the set of integers is closed under multiplication because if we take any two integers, their product is also an integer .

Recall the definitions of operations we’ve talked about that produce new languages from old: for each language  $L$  over an alphabet  $\Sigma$ , we have the associated sets of strings (also over  $\Sigma$ )

$$L^* = \{w_1 \cdots w_k \mid k \geq 0 \text{ and each } w_i \in L\}$$

and

$$\text{SUBSTRING}(L) = \{w \in \Sigma^* \mid \text{there exist } x, y \in \Sigma^* \text{ such that } xwy \in L\}$$

and

$$\text{EXTEND}(L) = \{w \in \Sigma^* \mid w = uv \text{ for some strings } u \in L \text{ and } v \in \Sigma^*\}$$

Also, recall the set operations union and intersection: for any sets  $X$  and  $Y$

$$X \cup Y = \{w \mid w \in X \text{ or } w \in Y\}$$

$$X \cap Y = \{w \mid w \in X \text{ and } w \in Y\}$$



- (a) (*Graded for correctness*) Use the general constructions that we developed to prove the closure of the class of regular languages under various operations to produce the state diagram of a NFA that recognizes the language

$$( SUBSTRING(\{0, 01, 111\}) )^*$$

*Hint: Question 4 from Homework 2 might be helpful.*

For full credit, submit (1) a state diagram of an NFA that recognizes  $\{0, 01, 111\}$ , (2) a state diagram of an NFA that recognizes  $SUBSTRING(\{0, 01, 111\})$ , and (3) a state diagram of an NFA that recognizes  $( SUBSTRING(\{0, 01, 111\}) )^*$ , and a brief justification of each state diagram that references the language being recognized or the general constructions being used.

- (b) (*Graded for completeness*) Prove that the class of nonregular languages over  $\{0, 1\}$  is not closed under the  $SUBSTRING$  operation by giving an example language  $A$  that is nonregular but for which  $SUBSTRING(A)$  is regular. A complete solution will give (1) a clear and precise description of the language, (2) a justification for why it is nonregular (either by proving this directly or by referring to specific examples from the class or textbook), (3) a description of the result of applying the  $SUBSTRING$  operation to the language, and (4) a justification for why this resulting language is regular.
- (c) (*Graded for correctness*) Prove that the class of nonregular languages over  $\{0, 1\}$  is not closed under the  $EXTEND$  operation by giving an example language  $B$  that is nonregular but for which  $EXTEND(B)$  is regular. A complete solution will give (1) a clear and precise description of the language, (2) a justification for why it is nonregular (either by proving this directly or by referring to specific examples from the class or textbook), (3) a description of the result of applying the  $EXTEND$  operation to the language, and (4) a justification for why this resulting language is regular.
- (d) (*Graded for completeness*) Prove that the class of nonregular languages over  $\{0, 1\}$  is not closed under the Kleene star operation by giving an example language  $C$  that is nonregular but for which  $C^*$  is regular. A complete solution will give (1) a clear and precise description of the language, (2) a justification for why it is nonregular (either by proving this directly or by referring to specific examples from the class or textbook), (3) a description of the result of applying the Kleene star operation to the language, and (4) a justification for why this resulting language is regular.

**5. Regular and nonregular languages and Push-down automata (PDA)** (6 points): On page 7 of the week 4 notes, we have the following list of languages over the alphabet  $\{a, b\}$

$$\begin{aligned} &\{a^n b^n \mid 0 \leq n \leq 5\} \quad \{b^n a^n \mid n \geq 2\} \quad \{a^m b^n \mid 0 \leq m \leq n\} \\ &\{a^m b^n \mid m \geq n + 3, n \geq 0\} \quad \{b^m a^n \mid m \geq 1, n \geq 3\} \\ &\{w \in \{a, b\}^* \mid w = w^R\} \quad \{ww^R \mid w \in \{a, b\}^*\} \end{aligned}$$

- (a) (*Graded for completeness*) Pick one of the regular languages and design a regular expression that describes it and a DFA that recognizes it. Briefly justify your regular expression

by connecting the subexpressions of it to the intended language and referencing relevant definitions. Briefly justify your DFA design by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

- (b) (*Graded for completeness*) Pick one of the nonregular languages and design a PDA that recognizes it. Draw the state diagram of your PDA. Briefly justify your design by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

## HW4CSE105W25: Homework assignment 4 Due: February 20th at 5pm, via Gradescope

### In this assignment,

You will work with context-free languages and their representations. You will also practice analyzing, designing, and working with Turing machines. You will explore recognizable and decidable languages.

**Resources:** To review the topics for this assignment, see the class material from Weeks 5 and 6. We will post frequently asked questions and our answers to them in a pinned Piazza post.

**Reading and extra practice problems:** Sipser Chapters 2 and 3. Chapter 2 exercises 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.9, 2.11, 2.12, 2.13, 2.16, 2.17. Chapter 3 exercises 3.1, 3.2, 3.5, 3.8.

**For all HW assignments:** Weekly homework may be done individually or in groups of up to 3 students. You may switch HW partners for different HW assignments. Please ensure your name(s) and PID(s) are clearly visible on the first page of your homework submission and then upload the PDF to Gradescope. If working in a group, submit only one submission per group: one partner uploads the submission through their Gradescope account and then adds the other group member(s) to the Gradescope submission by selecting their name(s) in the “Add Group Members” dialog box. You will need to re-add your group member(s) every time you resubmit a new version of your assignment. Each homework question will be graded either for correctness (including clear and precise explanations and justifications of all answers) or fair effort completeness. On the “graded for correctness” questions, you may only collaborate with CSE 105 students in your group; if your group has questions about a problem, you may ask in drop-in help hours or post a private post (visible only to the Instructors) on Piazza. On the “graded for completeness” questions, you may collaborate with all other CSE 105 students this quarter, and you may make public posts about these questions on Piazza.

All submitted homework for this class must be typed. You can use a word processing editor if you like (Microsoft Word, Open Office, Notepad, Vim, Google Docs, etc.) but you might find it useful to take this opportunity to learn LaTeX. LaTeX is a markup language used widely in computer science and mathematics. The homework assignments are typed using LaTeX and you can use the source files as templates for typesetting your solutions. To generate state diagrams of machines, you can (1) use the LaTeX tikzpicture environment (see templates in the class notes), or (2) use the software tools Flap.js or JFLAP described in the class syllabus (and include a screenshot in your PDF), or (3) you can carefully and clearly hand-draw the diagram and take a picture and include it in your PDF. We recommend that you submit early drafts to Gradescope so that in case of any technical difficulties, at least some of your work is present. You may update your submission as many times as you’d like up to the deadline.

### Integrity reminders

- Problems should be solved together, not divided up between the partners. The homework is designed to give you practice with the main concepts and techniques of the course, while

getting to know and learn from your classmates.

- On the “graded for correctness” questions, you may only collaborate with CSE 105 students in your group. You may ask questions about the homework in office hours (of the instructor, TAs, and/or tutors) and on Piazza (as private notes viewable only to the Instructors). You *cannot* use any online resources about the course content other than the class material from this quarter – this is primarily to ensure that we all use consistent notation and definitions (aligned with the textbook) and also to protect the learning experience you will have when the ‘aha’ moments of solving the problem authentically happen.
- Do not share written solutions or partial solutions for homework with other students in the class who are not in your group. Doing so would dilute their learning experience and detract from their success in the class.

You will submit this assignment via Gradescope (<https://www.gradescope.com>) in the assignment called “hw4CSE105W25”.

## Assigned questions

1. **Regular and nonregular languages and context-free grammars (CFG)** (6 points):  
On page 7 of the week 4 notes, we have the following list of languages over the alphabet  $\{a, b\}$

$$\begin{aligned} &\{a^n b^n \mid 0 \leq n \leq 5\} \quad \{b^n a^n \mid n \geq 2\} \quad \{a^m b^n \mid 0 \leq m \leq n\} \\ &\{a^m b^n \mid m \geq n + 3, n \geq 0\} \quad \{b^m a^n \mid m \geq 1, n \geq 3\} \\ &\{w \in \{a, b\}^* \mid w = w^R\} \quad \{ww^R \mid w \in \{a, b\}^*\} \end{aligned}$$

- (a) (*Graded for completeness*)<sup>7</sup> Pick one of the regular languages and design a context-free grammar that generates it. Briefly justify your grammar by describing the role of each of the rules and connecting it to the intended language and referencing relevant definitions.
- (b) (*Graded for completeness*) Pick one of the nonregular languages and design a context-free grammar that generates it. Briefly justify your grammar by describing the role of each of the rules and connecting it to the intended language and referencing relevant definitions.

2. **General constructions for context-free languages** (15 points):

In class in week 5, we described several general constructions with PDAs and CFGs, leaving their details to homework. In this question, we’ll fill in these details. The first constructions help us prove that the class of regular languages is a subset of the class of context-free languages. The other construction allows us to make simplifying assumptions about PDAs recognizing languages.

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<sup>7</sup>This means you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for incorrect answers. To demonstrate your honest effort in answering the question, we expect you to include your attempt to answer *each* part of the question. If you get stuck with your attempt, you can still demonstrate your effort by explaining where you got stuck and what you did to try to get unstuck.

- (a) (*Graded for completeness*) When we first introduced PDAs we observed that any NFA can be transformed to a PDA by not using the stack of the PDA at all. Suppose a friend gives you the following construction to formalize this transformation:

Given a NFA  $N = (Q, \Sigma, \delta_N, q_0, F)$  we define a PDA  $M$  with  $L(M) = L(N)$  by letting  $M = (Q, \Sigma, \Sigma, \delta, q_0, F)$  where  $\delta((q, a, b)) = \delta_N(q, a)$  for each  $q \in Q$ ,  $a \in \Sigma_\epsilon$  and  $b \in \Sigma_\epsilon$ .

For each of the six defining parameters for the PDA, explain whether it's defined correctly or not. If it is not defined correctly, explain why not and give a new definition for this parameter that corrects the mistake.

- (b) (*Graded for correctness*)<sup>8</sup> In the book on page 107, the top paragraph describes a procedure for converting DFAs to CFGs:

You can convert any DFA into an equivalent CFG as follows. Make a variable  $R_i$  for each state  $q_i$  of the DFA. Add the rule  $R_i \rightarrow aR_j$  to the CFG if  $\delta(q_i, a) = q_j$  is a transition in the DFA. Add the rule  $R_i \rightarrow \epsilon$  if  $q_i$  is an accept state of the DFA. Make  $R_0$  the start variable of the grammar, where  $q_0$  is the start state of the machine. Verify on your own that the resulting CFG generates the same language that the DFA recognizes.

Use this construction to get a context-free grammar generating the language

$$\{w \in \{a, b\}^* \mid w \text{ has at least one } a \text{ and does not end in } bb\}$$

by (1) designing a DFA that recognizes this language and then (2) applying the construction from the book to convert the DFA to an equivalent CFG. A complete and correct submission will include the state diagram of the DFA, a brief justification of why it recognizes the language, and then the complete and precise definition of the CFG that results from applying the construction from the book to this DFA. *Ungraded bonus: take a sample string in the language and see how the computation of the DFA on this string translates to a derivation in your grammar.*

- (c) Let  $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$  be a PDA and let  $q_{new}, r_{new}, s_{new}$  be three fresh state labels (i.e.  $Q_1 \cap \{q_{new}, r_{new}, s_{new}\} = \emptyset$ ) and let  $\#$  be a fresh stack symbol (i.e.  $\# \notin \Gamma_1$ ). We define the PDA  $M_2$  as

$$(Q_2, \Sigma, \Gamma_2, \delta_2, q_{new}, \{s_{new}\})$$

with  $Q_2 = Q_1 \cup \{q_{new}, r_{new}, s_{new}\}$  and  $\Gamma_2 = \Gamma_1 \cup \{\#\}$  and  $\delta_2 : Q_2 \times \Sigma_\epsilon \times \Gamma_{2\epsilon} \rightarrow \mathcal{P}(Q_2 \times \Gamma_{2\epsilon})$

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<sup>8</sup>This means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

given by

$$\delta_2((q, a, b)) = \begin{cases} \{(q_1, \#)\} & \text{if } q = q_{new}, a = \varepsilon, b = \varepsilon \\ \delta_1((q, a, b)) & \text{if } q \in Q_1 \setminus F_1, a \in \Sigma_\varepsilon, b \in \Gamma_{1\varepsilon} \\ \delta_1((q, a, b)) & \text{if } q \in F_1, a \in \Sigma, b \in \Gamma_{1\varepsilon} \\ \delta_1((q, a, b)) & \text{if } q \in F_1, a = \varepsilon, b \in \Gamma_1 \\ \delta_1((q, a, b)) \cup \{(r_{new}, \varepsilon)\} & \text{if } q \in F_1, a = \varepsilon, b = \varepsilon \\ \{(r_{new}, \varepsilon)\} & \text{if } q = r_{new}, a = \varepsilon, b \in \Gamma_1 \\ \{(s_{new}, \varepsilon)\} & \text{if } q = r_{new}, a = \varepsilon, b = \# \\ \emptyset & \text{otherwise} \end{cases}$$

for each  $q \in Q_2$ ,  $a \in \Sigma_\varepsilon$ , and  $b \in \Gamma_{2\varepsilon}$ .

In this question, we'll apply this construction for a specific PDA and use this example to extrapolate the effect of this construction.

- i. (*Graded for correctness*) Consider the PDA  $M_1$  with input alphabet  $\{0, 1\}$  and stack alphabet  $\{0, 1, \$\}$  whose state diagram is



Draw the state diagram for the PDA  $M_2$  that results from applying the construction to  $M_1$ . Also, give an example string of length 5 that is accepted by both  $M_1$  and  $M_2$  and justify your choice by describing an accepting computation for each of the PDAs on your input string.

- ii. (*Graded for completeness*) Compare  $L(M_1)$  and  $L(M_2)$ . Are these sets equal? Does your answer depend on the specific choice of  $M_1$ ? Why or why not?
- iii. (*Graded for completeness*) Consider the PDA  $N$  with input alphabet  $\{0, 1\}$  and stack alphabet  $\{0, 1\}$  whose state diagram is



Remember that the definition of set-wise concatenation is: for languages  $L_1, L_2$  over the alphabet  $\Sigma$ , we have the associated set of strings

$$L_1 \circ L_2 = \{w \in \Sigma^* \mid w = uv \text{ for some strings } u \in L_1 \text{ and } v \in L_2\}$$

In class, we discussed how extrapolating the construction that we used to prove that the class of regular languages is closed under set-wise concatenation by drawing spontaneous transitions from the accepting states in the first machine to the start state of the second machine doesn't work. Use the example of  $M_1$  and  $N$  to prove this by showing that

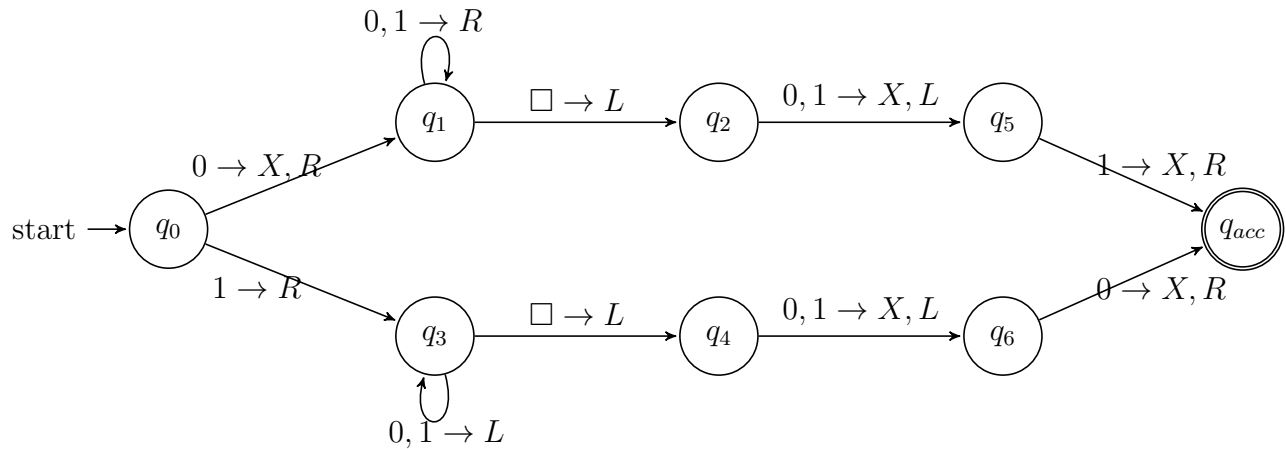
$$L(M_1) \circ L(N)$$

is **not** the language recognized by the machine results from taking the two machines  $M_1$  and  $N$ , setting the start state of  $M_1$  to be the start state of the new machine, setting the set of accepting states of  $N$  to be the set of accepting states of the new machine, and drawing spontaneous arrows from the accepting states of  $M_1$  to the start state of  $N$ . Then, describe the language recognized by the machine that results from taking the two machines  $M_2$  and  $N$ , setting the start state of  $M_2$  to be the start state of the new machine, setting the set of accepting states of  $N$  to be the set of accepting states of the new machine, and drawing spontaneous arrows from the accepting states of  $M_2$  to the start state of  $N$ . Use this description to explain why we used the construction of  $M_2$  from  $M_1$  and how this construction could be used in a proof of the closure of the class of context-free languages under set-wise concatenation.

A complete response will give an example string that witnesses that  $L(M_1) \circ L(N)$  is not equal to the language recognized by the PDA resulting from the wrong construction (described above) **and** the state diagram of the PDA that results from applying that construction to  $M_2$  and  $N$  (instead of  $M_1$ ), with a brief justification about why that approach works.

### 3. Turing machines (9 points):

Consider the Turing machine  $T$  over the input alphabet  $\Sigma = \{0, 1\}$  with the state diagram below (the tape alphabet is  $\Gamma = \{0, 1, X, \square\}$ ). Convention: we do not include the node for the reject state  $q_{rej}$  and any missing transitions in the state diagram have value  $(q_{rej}, \square, R)$



- (a) (*Graded for correctness*) Specify an example string  $w_1$  of length 4 over  $\Sigma$  that is **accepted** by this Turing machine, or explain why there is no such example. A complete solution will include either (1) a precise and clear description of your example string and a precise and clear description of the accepting computation of the Turing machine on this string or (2) a sufficiently general and correct argument why there is no such example, referring back to the relevant definitions.

To describe a computation of a Turing machine, include the contents of the tape, the state of the machine, and the location of the read/write head at each step in the computation.

*Hint:* In class we've drawn pictures to represent the configuration of the machine at each step in a computation. You may do so or you may choose to describe these configurations in words.

- (b) (*Graded for correctness*) Specify an example string  $w_2$  of length 3 over  $\Sigma$  that is **rejected** by this Turing machine or explain why there is no such example. A complete solution will include either (1) a precise and clear description of your example string and a precise and clear description of the rejecting computation of the Turing machine on this string or (2) a sufficiently general and correct argument why there is no such example, referring back to the relevant definitions.
- (c) (*Graded for correctness*) Specify an example string  $w_3$  of length 3 over  $\Sigma$  on which the computation of this Turing machine is **never halts** or explain why there is no such example. A complete solution will include either (1) a precise and clear description of your example string and a precise and clear description of the looping (non-halting) computation of the Turing machine on this string or (2) a sufficiently general and correct argument why there is no such example, referring back to the relevant definitions.

4. **Implementation-level descriptions of deciders and recognizers** (12 points): Consider the language

$$\{a^i b^j \mid i \geq 0, j > 1\}$$

over the alphabet  $\{a, b\}$ .

- (a) (*Graded for correctness*) Give an example of a Turing machine that **decides** this language. A complete solution will include **both** a state diagram and an implementation-level description of this Turing machine, along with a brief explanation of why it recognizes this language, and why it is a decider.
- (b) (*Graded for correctness*) Give an example of a Turing machine that **recognizes but does not decide** this language. A complete solution will include **both** a state diagram and an implementation-level description of this Turing machine, along with a brief explanation of why it recognizes this language, and why it is not a decider.

5. **Classifying languages** (8 points): Our first example of a more complicated Turing machine was of a Turing machine that recognized the language  $\{w\#w \mid w \in \{0, 1\}^*\}$  (Figure 3.10 in the textbook), which we know is not context-free. Let's call that Turing machine  $M_0$ . The language

$$L = \{ww \mid w \in \{0, 1\}^*\}$$

is also not context-free.

- (a) (*Graded for correctness*) Choose an example string of length 3 in  $L$  that is in **not** in  $\{w\#w \mid w \in \{0, 1\}^*\}$  and describe the computation of the Turing machine  $M_0$  on your example string. Include the contents of the tape, the state of the machine, and the location of the read/write head at each step in the computation.



- (b) (*Graded for completeness*) Explain why the Turing machine from the textbook and class that recognized  $\{w\#w \mid w \in \{0,1\}^*\}$  does not recognize  $\{ww \mid w \in \{0,1\}^*\}$ . Use your example to explain why  $M_0$  doesn't recognize  $L$ .
- (c) (*Graded for completeness*) Explain how you would change  $M_0$  to get a new Turing machine that does recognize  $L$ . Describe this new Turing machine using both an implementation-level definition and a state diagram of the Turing machine. You may use all our usual conventions for state diagrams of Turing machines (we do not include the node for the reject state  $q_{rej}$  and any missing transitions in the state diagram have value  $(q_{rej}, \square, R)$ ;  $b \rightarrow R$  label means  $b \rightarrow b, R$  ).