Week9 monday

Recall definition: A is **mapping reducible to** B means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

Notation: when A is mapping reducible to B, we write $A \leq_m B$.

Theorem (Sipser 5.23): If $A \leq_m B$ and A is undecidable, then B is undecidable.

Last time we proved that $A_{TM} \leq_m HALT_{TM}$ where

$$HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a Turing machine, } w \text{ is a string, and } M \text{ halts on } w\}$$

and since A_{TM} is undecidable, $HALT_{TM}$ is also undecidable. The function witnessing the mapping reduction mapped strings in A_{TM} to strings in $HALT_{TM}$ and strings not in A_{TM} to strings not in $HALT_{TM}$ by changing encoded Turing machines to ones that had identical computations except looped instead of rejecting.

True or False: $\overline{A_{TM}} \leq_m \overline{HALT_{TM}}$

True or False: $HALT_{TM} \leq_m A_{TM}$.

Proof: Need computable function $F : \Sigma^* \to \Sigma^*$ such that $x \in HALT_{TM}$ iff $F(x) \in A_{TM}$. Define

$F =$ " On input x,

1. Type-check whether $x = \langle M, w\rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle \qquad\qquad \rangle$
2. Construct the following machine M'_x:

3. Output $\langle M'_x, w\rangle$."

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in HALT_{TM}$ iff its image is in A_{TM}?

Input string	Output string
$\langle M, w\rangle$ where M halts on w	
$\langle M, w\rangle$ where M does not halt on w	
x not encoding any pair of TM and string	

Theorem (Sipser 5.28): If $A \leq_m B$ and B is recognizable, then A is recognizable.

Proof:

Corollary: If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable.

Strategy:

(i) To prove that a recognizable language R is undecidable, prove that $A_{TM} \leq_m R$.

(ii) To prove that a co-recognizable language U is undecidable, prove that $\overline{A_{TM}} \leq_m U$, i.e. that $A_{TM} \leq_m \overline{U}$.

$$E_{TM} = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) = \emptyset\}$$

Can we find algorithms to recognize

E_{TM} ?

$\overline{E_{TM}}$?

Claim: $A_{TM} \leq_m \overline{E_{TM}}$. *And hence also* $\overline{A_{TM}} \leq_m E_{TM}$

Proof: Need computable function $F : \Sigma^* \to \Sigma^*$ such that $x \in A_{TM}$ iff $F(x) \notin E_{TM}$. Define

$F =$ " On input x,

1. Type-check whether $x = \langle M, w \rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle \quad \rangle$
2. Construct the following machine M'_x:

3. Output $\langle M'_x \rangle$."

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in A_{TM}$ iff its image is **not** in E_{TM} ?

Input string	Output string
$\langle M, w \rangle$ where $w \in L(M)$	
$\langle M, w \rangle$ where $w \notin L(M)$	
x not encoding any pair of TM and string	

Week9 wednesday

Recall: A is **mapping reducible to** B, written $A \leq_m B$, means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

So far:

- A_{TM} is recognizable, undecidable, and not-co-recognizable.
- $\overline{A_{TM}}$ is unrecognizable, undecidable, and co-recognizable.
- $HALT_{TM}$ is recognizable, undecidable, and not-co-recognizable.
- $\overline{HALT_{TM}}$ is unrecognizable, undecidable, and co-recognizable.
- E_{TM} is unrecognizable, undecidable, and co-recognizable.
- $\overline{E_{TM}}$ is recognizable, undecidable, and not-co-recognizable.

$$EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are both Turing machines and } L(M_1) = L(M_2)\}$$

Can we find algorithms to recognize

EQ_{TM} ?

$\overline{EQ_{TM}}$?

Goal: Show that EQ_{TM} is not recognizable and that $\overline{EQ_{TM}}$ is not recognizable.

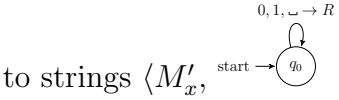
Using Corollary to **Theorem 5.28**: If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable, it's enough to prove that

$\overline{HALT_{TM}} \leq_m EQ_{TM}$ aka $HALT_{TM} \leq_m \overline{EQ_{TM}}$

$\overline{HALT_{TM}} \leq_m \overline{EQ_{TM}}$ aka $HALT_{TM} \leq_m EQ_{TM}$

Need computable function $F_1 : \Sigma^* \to \Sigma^*$ such that $x \in HALT_{TM}$ iff $F_1(x) \notin EQ_{TM}$.

Strategy:

Map strings $\langle M, w\rangle$ to strings $\langle M'_x,$ 

$\rangle$. This image string is not in EQ_{TM} when $L(M'_x) \neq \emptyset$.

We will build M'_x so that $L(M'_x) = \Sigma^*$ when M halts on w and $L(M'_x) = \emptyset$ when M loops on w.

Thus: when $\langle M, w\rangle \in HALT_{TM}$ it gets mapped to a string not in EQ_{TM} and when $\langle M, w\rangle \notin HALT_{TM}$ it gets mapped to a string that is in EQ_{TM}.

Define

$F_1 =$ " On input x,

1. Type-check whether $x = \langle M, w\rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle$ $\rangle$
2. Construct the following machine M'_x:

3. Output $\langle M'_x,$ 

$\rangle$ "

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in HALT_{TM}$ iff its image is **not** in EQ_{TM} ?

Input string	Output string
$\langle M, w\rangle$ where M halts on w	
$\langle M, w\rangle$ where M loops on w	
x not encoding any pair of TM and string	

Conclude: $HALT_{TM} \leq_m \overline{EQ_{TM}}$

Need computable function $F_2 : \Sigma^* \to \Sigma^*$ such that $x \in HALT_{TM}$ iff $F_2(x) \in EQ_{TM}$.

Strategy:
Map strings $\langle M, w \rangle$ to strings $\langle M'_x,$  $\rangle$. This image string is in EQ_{TM} when $L(M'_x) = \Sigma^*$.

We will build M'_x so that $L(M'_x) = \Sigma^*$ when M halts on w and $L(M'_x) = \emptyset$ when M loops on w.

Thus: when $\langle M, w \rangle \in HALT_{TM}$ it gets mapped to a string in EQ_{TM} and when $\langle M, w \rangle \notin HALT_{TM}$ it gets mapped to a string that is not in EQ_{TM}.

Define

$F_2 =$ " On input x,

1. Type-check whether $x = \langle M, w \rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle$ $\rangle$
2. Construct the following machine M'_x:

3. Output $\langle M'_x,$  $\rangle$ "

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in HALT_{TM}$ iff its image is in EQ_{TM} ?

Input string	Output string
$\langle M, w \rangle$ where M halts on w	
$\langle M, w \rangle$ where M loops on w	
x not encoding any pair of TM and string	

Conclude: $HALT_{TM} \leq_m EQ_{TM}$

Two models of computation are called **equally expressive** when every language recognizable with the first model is recognizable with the second, and vice versa.

Church-Turing Thesis (Sipser p. 183): The informal notion of algorithm is formalized completely and correctly by the formal definition of a Turing machine. In other words: all reasonably expressive models of computation are equally expressive with the standard Turing machine.

Some examples of models that are **equally expressive** *with deterministic Turing machines:*

May-stay machines The May-stay machine model is the same as the usual Turing machine model, except that on each transition, the tape head may move L, move R, or Stay.

Formally: $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$$

Claim: Turing machines and May-stay machines are equally expressive. *To prove . . .*

To translate a standard TM to a may-stay machine: never use the direction S!

To translate one of the may-stay machines to standard TM: any time TM would Stay, move right then left.

Multitape Turing machine A multitape Turing macihne with k tapes can be formally representated as $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ where Q is the finite set of states, Σ is the input alphabet with $\textvisiblespace \notin \Sigma$, Γ is the tape alphabet with $\Sigma \subsetneq \Gamma$, $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$ (where k is the number of states)

If M is a standard TM, it is a 1-tape machine.

To translate a k-tape machine to a standard TM: Use a new symbol to separate the contents of each tape and keep track of location of head with special version of each tape symbol. Sipser Theorem 3.13

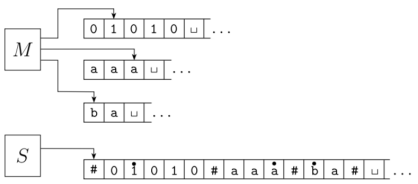


FIGURE 3.14
Representing three tapes with one

Enumerators Enumerators give a different model of computation where a language is **produced, one string at a time**, rather than recognized by accepting (or not) individual strings.

Each enumerator machine has finite state control, unlimited work tape, and a printer. The computation proceeds according to transition function; at any point machine may "send" a string to the printer.

$$E = (Q, \Sigma, \Gamma, \delta, q_0, q_{print})$$

Q is the finite set of states, Σ is the output alphabet, Γ is the tape alphabet ($\Sigma \subsetneq \Gamma, \sqcup \in \Gamma \setminus \Sigma$),

$$\delta : Q \times \Gamma \times \Gamma \to Q \times \Gamma \times \Gamma \times \{L, R\} \times \{L, R\}$$

where in state q, when the working tape is scanning character x and the printer tape is scanning character y, $\delta((q, x, y)) = (q', x', y', d_w, d_p)$ means transition to control state q', write x' on the working tape, write y' on the printer tape, move in direction d_w on the working tape, and move in direction d_p on the printer tape. The computation starts in q_0 and each time the computation enters q_{print} the string from the leftmost edge of the printer tape to the first blank cell is considered to be printed.

The language **enumerated** by E, $L(E)$, is $\{w \in \Sigma^* \mid E$ eventually, at finite time, prints $w\}$.

Theorem 3.21 A language is Turing-recognizable iff some enumerator enumerates it.

Proof, part 1: Assume L is enumerated by some enumerator, E, so $L = L(E)$. We'll use E in a subroutine within a high-level description of a new Turing machine that we will build to recognize L.

Goal: build Turing machine M_E with $L(M_E) = L(E)$.

Define M_E as follows: $M_E =$ "On input w,

1. Run E. For each string x printed by E.
2. Check if $x = w$. If so, accept (and halt); otherwise, continue."

Proof, part 2: Assume L is Turing-recognizable and there is a Turing machine M with $L = L(M)$. We'll use M in a subroutine within a high-level description of an enumerator that we will build to enumerate L.

Goal: build enumerator E_M with $L(E_M) = L(M)$.

Idea: check each string in turn to see if it is in L.

How? Run computation of M on each string. *But*: need to be careful about computations that don't halt.

Recall String order for $\Sigma = \{0, 1\}$: $s_1 = \varepsilon$, $s_2 = 0$, $s_3 = 1$, $s_4 = 00$, $s_5 = 01$, $s_6 = 10$, $s_7 = 11$, $s_8 = 000$, ...

Define E_M as follows: $E_M =$ " *ignore any input.* Repeat the following for $i = 1, 2, 3, \ldots$

1. Run the computations of M on s_1, s_2, ..., s_i for (at most) i steps each
2. For each of these i computations that accept during the (at most) i steps, print out the accepted string."

Nondeterministic Turing machine

At any point in the computation, the nondeterministic machine may proceed according to several possibilities: $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ where

$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

The computation of a nondeterministic Turing machine is a tree with branching when the next step of the computation has multiple possibilities. A nondeterministic Turing machine accepts a string exactly when some branch of the computation tree enters the accept state.

Given a nondeterministic machine, we can use a 3-tape Turing machine to simulate it by doing a breadth-first search of computation tree: one tape is "read-only" input tape, one tape simulates the tape of the nondeterministic computation, and one tape tracks nondeterministic branching. Sipser page 178

Summary

Two models of computation are called **equally expressive** when every language recognizable with the first model is recognizable with the second, and vice versa.

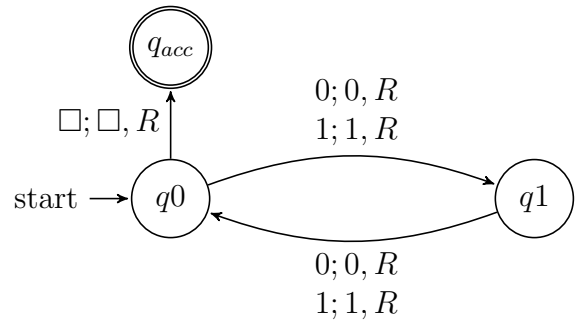
To prove the existence of a Turing machine that decides / recognizes some language, it's enough to construct an example using any of the equally expressive models.

But: some of the **performance** properties of these models are not equivalent.

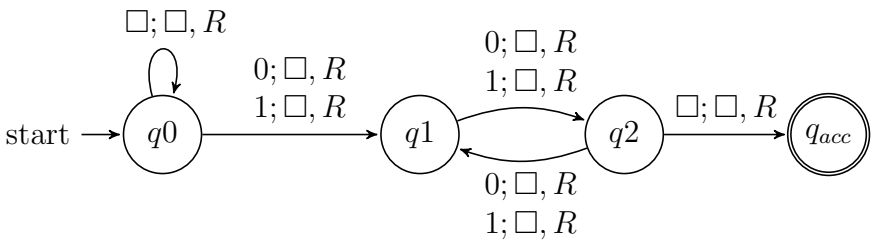
Week8 monday

Acceptance problem for Turing machines A_{TM} $\{\langle M, w\rangle \mid M \text{ is a Turing machine that accepts input string } w\}$
Language emptiness testing for Turing machines E_{TM} $\{\langle M\rangle \mid M \text{ is a Turing machine and } L(M) = \emptyset\}$
Language equality testing for Turing machines EQ_{TM} $\{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are Turing machines and } L(M_1) = L(M_2)\}$

M_1



M_2



Example strings in A_{TM}

Example strings in E_{TM}

Example strings in EQ_{TM}

Theorem: A_{TM} is Turing-recognizable.

Strategy: To prove this theorem, we need to define a Turing machine R_{ATM} such that $L(R_{ATM}) = A_{TM}$.

Define R_{ATM} = “

Proof of correctness:

We will show that A_{TM} is undecidable. *First, let's explore what that means.*

To prove that a computational problem is **decidable**, we find/ build a Turing machine that recognizes the language encoding the computational problem, and that is a decider.

How do we prove a specific problem is **not decidable**?

How would we even find such a computational problem?

Counting arguments for the existence of an undecidable language:

- The set of all Turing machines is countably infinite.
- Each recognizable language has at least one Turing machine that recognizes it (by definition), so there can be no more Turing-recognizable languages than there are Turing machines.
- Since there are infinitely many Turing-recognizable languages (think of the singleton sets), there are countably infinitely many Turing-recognizable languages.
- Such the set of Turing-decidable languages is an infinite subset of the set of Turing-recognizable languages, the set of Turing-decidable languages is also countably infinite.

Since there are uncountably many languages (because $\mathcal{P}(\Sigma^*)$ is uncountable), there are uncountably many unrecognizable languages and there are uncountably many undecidable languages.

Thus, there's at least one undecidable language!

What's a specific example of a language that is unrecognizable or undecidable?

To prove that a language is undecidable, we need to prove that there is no Turing machine that decides it.

Key idea: proof by contradiction relying on self-referential disagreement.

Theorem: A_{TM} is not Turing-decidable.

Proof: Suppose **towards a contradiction** that there is a Turing machine that decides A_{TM}. We call this presumed machine M_{ATM}.

By assumption, for every Turing machine M and every string w

- If $w \in L(M)$, then the computation of M_{ATM} on $\langle M, w\rangle$ _____
- If $w \notin L(M)$, then the computation of M_{ATM} on $\langle M, w\rangle$ _____

Define a **new** Turing machine using the high-level description:

$D =$" On input $\langle M\rangle$, where M is a Turing machine:

1. Run M_{ATM} on $\langle M, \langle M\rangle\rangle$.
2. If M_{ATM} accepts, reject; if M_{ATM} rejects, accept."

Is D a Turing machine?

Is D a decider?

What is the result of the computation of D on $\langle D \rangle$?

Summarizing:

- A_{TM} is recognizable.
- A_{TM} is not decidable.

Recall definition: A language L over an alphabet Σ is called **co-recognizable** if its complement, defined as $\Sigma^* \setminus L = \{x \in \Sigma^* \mid x \notin L\}$, is Turing-recognizable.

and Recall Theorem (Sipser Theorem 4.22): A language is Turing-decidable if and only if both it and its complement are Turing-recognizable.

- A_{TM} is recognizable.
- A_{TM} is not decidable.
- $\overline{A_{TM}}$ is not recognizable.
- $\overline{A_{TM}}$ is not decidable.

Week8 wednesday

Mapping reduction

Motivation: Proving that A_{TM} is undecidable was hard. How can we leverage that work? Can we relate the decidability / undecidability of one problem to another?

> If problem X is **no harder than** problem Y
>
> ... and if Y is easy,
>
> ... then X must be easy too.

> If problem X is **no harder than** problem Y
>
> ... and if X is hard,
>
> ... then Y must be hard too.

"Problem X is no harder than problem Y" means "Can answer questions about membership in X by converting them to questions about membership in Y".

Definition: A is **mapping reducible to** B means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

Notation: when A is mapping reducible to B, we write $A \leq_m B$.

Intuition: $A \leq_m B$ means A is no harder than B, i.e. that the level of difficulty of A is less than or equal the level of difficulty of B.

TODO

1. What is a computable function?
2. How do mapping reductions help establish the computational difficulty of languages?

Computable functions

Definition: A function $f : \Sigma^* \to \Sigma^*$ is a **computable function** means there is some Turing machine such that, for each x, on input x the Turing machine halts with exactly $f(x)$ followed by all blanks on the tape

Examples of computable functions:

The function that maps a string to a string which is one character longer and whose value, when interpreted as a fixed-width binary representation of a nonnegative integer is twice the value of the input string (when interpreted as a fixed-width binary representation of a non-negative integer)

$$f_1 : \Sigma^* \to \Sigma^* \qquad f_1(x) = x0$$

To prove f_1 is computable function, we define a Turing machine computing it.

High-level description

"On input w

1. Append 0 to w.
2. Halt."

Implementation-level description

"On input w

1. Sweep read-write head to the right until find first blank cell.
2. Write 0.
3. Halt."

Formal definition $(\{q0, qacc, qrej\}, \{0, 1\}, \{0, 1, \sqcup\}, \delta, q0, qacc, qrej)$ where δ is specified by the state diagram:

The function that maps a string to the result of repeating the string twice.

$$f_2 : \Sigma^* \to \Sigma^* \qquad f_2(x) = xx$$

The function that maps strings that are not the codes of NFAs to the empty string and that maps strings that code NFAs to the code of a DFA that recognizes the language recognized by the NFA produced by the macro-state construction from Chapter 1.

The function that maps strings that are not the codes of Turing machines to the empty string and that maps strings that code Turing machines to the code of the related Turing machine that acts like the Turing machine coded by the input, except that if this Turing machine coded by the input tries to reject, the new machine will go into a loop.

$$f_4 : \Sigma^* \to \Sigma^* \qquad f_4(x) = \begin{cases} \varepsilon & \text{if } x \text{ is not the code of a TM} \\ \langle (Q \cup \{q_{trap}\}, \Sigma, \Gamma, \delta', q_0, q_{acc}, q_{rej}) \rangle & \text{if } x = \langle (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej}) \rangle \end{cases}$$

where $q_{trap} \notin Q$ and

$$\delta'((q, x)) = \begin{cases} (r, y, d) & \text{if } q \in Q,\ x \in \Gamma,\ \delta((q, x)) = (r, y, d), \text{ and } r \neq q_{rej} \\ (q_{trap}, \textvisiblespace, R) & \text{otherwise} \end{cases}$$

Definition: A is **mapping reducible to** B means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

Making intutition precise ...

Theorem (Sipser 5.22): If $A \leq_m B$ and B is decidable, then A is decidable.

Theorem (Sipser 5.23): If $A \leq_m B$ and A is undecidable, then B is undecidable.

Week8 friday

Recall definition: A is **mapping reducible to** B means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

Notation: when A is mapping reducible to B, we write $A \leq_m B$.

Intuition: $A \leq_m B$ means A is no harder than B, i.e. that the level of difficulty of A is less than or equal the level of difficulty of B.

Example: $A_{TM} \leq_m A_{TM}$

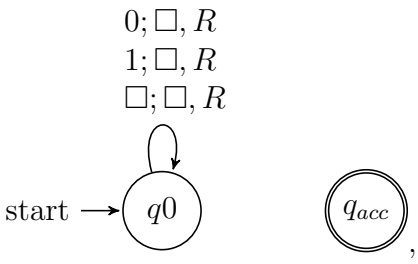
Example: $A_{DFA} \leq_m \{ww \mid w \in \{0,1\}^*\}$

Halting problem

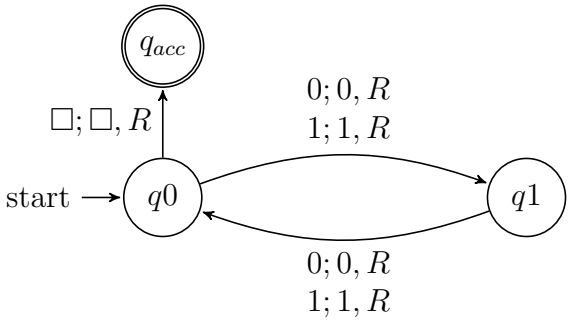
$$HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a Turing machine, } w \text{ is a string, and } M \text{ halts on } w\}$$

Define $F : \Sigma^* \to \Sigma^*$ by

$$F(x) = \begin{cases} const_{out} & \text{if } x \neq \langle M, w\rangle \text{ for any Turing machine } M \text{ and string } w \text{ over the alphabet of } M \\ \langle M'_x, w\rangle & \text{if } x = \langle M, w\rangle \text{ for some Turing machine } M \text{ and string } w \text{ over the alphabet of } M. \end{cases}$$

where $const_{out} = \langle$ 

$, \varepsilon\rangle$ and M'_x is a Turing machine that computes like M except, if the computation of M ever were to go to a reject state, M'_x loops instead.

$F(\langle$ 

$, \varepsilon\rangle) =$

To use this function to prove that $A_{TM} \leq_m HALT_{TM}$, we need two claims:

Claim (1): F is computable

Claim (2): for every x, $x \in A_{TM}$ iff $F(x) \in HALT_{TM}$.