

## Week 9 at a glance

**Textbook reading:** Section 5.3, Section 5.1, Section 3.2

For Monday, Example 5.26 (page 237).

For Wednesday, Theorem 5.30 (page 238)

For Friday, skim section 3.2.

*For Monday of Week 10:* Definition 7.1 (page 276)

**We will be learning and practicing to:**

- Clearly and unambiguously communicate computational ideas using appropriate formalism. Translate across levels of abstraction.
  - Give examples of sets that are regular, context-free, decidable, or recognizable (and prove that they are).
    - \* **Define and explain computational problems, including  $A_{**}$ ,  $E_{**}$ ,  $EQ_{**}$ , (for \*\* DFA or TM) and  $HALT_{TM}$**
- Know, select and apply appropriate computing knowledge and problem-solving techniques. Reason about computation and systems.
  - Use mapping reduction to deduce the complexity of a language by comparing to the complexity of another.
    - \* **Explain what it means for one problem to reduce to another**
    - \* **Define computable functions, and use them to give mapping reductions between computational problems**
    - \* **Build and analyze mapping reductions between computational problems**
  - Classify the computational complexity of a set of strings by determining whether it is regular, context-free, decidable, or recognizable.
    - \* **State, prove, and use theorems relating decidability, recognizability, and co-recognizability.**
    - \* **Prove that a language is decidable or recognizable by defining and analyzing a Turing machines with appropriate properties.**
  - Describe several variants of Turing machines and informally explain why they are equally expressive.
    - \* **Define an enumerator**
    - \* **Define nondeterministic Turing machines**
    - \* **Use high-level descriptions to define and trace machines (Turing machines and enumerators)**
    - \* **Apply dovetailing in high-level definitions of machines**

**TODO:**

Review Quiz 8 on PrairieLearn (<http://us.prairielearn.com>), due 3/5/2025

Review Quiz 9 on PrairieLearn (<http://us.prairielearn.com>), due 3/12/2025

Homework 6 submitted via Gradescope (<https://www.gradescope.com/>), due 3/13/2025

Project submitted via Gradescope (<https://www.gradescope.com/>), due 3/19/2025

# Monday: Mapping reductions and recognizability

Recall definition:  $A$  is **mapping reducible to**  $B$  means there is a computable function  $f : \Sigma^* \rightarrow \Sigma^*$  such that *for all* strings  $x$  in  $\Sigma^*$ ,

$$x \in A \quad \text{if and only if} \quad f(x) \in B.$$

Notation: when  $A$  is mapping reducible to  $B$ , we write  $A \leq_m B$ .

**Theorem** (Sipser 5.23): If  $A \leq_m B$  and  $A$  is undecidable, then  $B$  is undecidable.

*Last time* we proved that  $A_{TM} \leq_m HALT_{TM}$  where

$$HALT_{TM} = \{ \langle M, w \rangle \mid M \text{ is a Turing machine, } w \text{ is a string, and } M \text{ halts on } w \}$$

and since  $A_{TM}$  is undecidable,  $HALT_{TM}$  is also undecidable. The function witnessing the mapping reduction mapped strings in  $A_{TM}$  to strings in  $HALT_{TM}$  and strings not in  $A_{TM}$  to strings not in  $HALT_{TM}$  by changing encoded Turing machines to ones that had identical computations except looped instead of rejecting.

True or False:  $\overline{A_{TM}} \leq_m \overline{HALT_{TM}}$

True or False:  $HALT_{TM} \leq_m A_{TM}$ .

**Proof:** Need computable function  $F : \Sigma^* \rightarrow \Sigma^*$  such that  $x \in HALT_{TM}$  iff  $F(x) \in A_{TM}$ . Define

$F =$  “ On input  $x$ ,

1. Type-check whether  $x = \langle M, w \rangle$  for some TM  $M$  and string  $w$ . If so, move to step 2; if not, output  $\langle \quad \quad \quad \rangle$
2. Construct the following machine  $M'_x$ :

3. Output  $\langle M'_x, w \rangle$ .”

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property  $x \in HALT_{TM}$  iff its image is in  $A_{TM}$ ?

Input string	Output string
$\langle M, w \rangle$ where $M$ halts on $w$	
$\langle M, w \rangle$ where $M$ does not halt on $w$	
$x$ not encoding any pair of TM and string	

**Theorem** (Sipser 5.28): If  $A \leq_m B$  and  $B$  is recognizable, then  $A$  is recognizable.

**Proof:**

**Corollary:** If  $A \leq_m B$  and  $A$  is unrecognizable, then  $B$  is unrecognizable.

*Strategy:*

- (i) To prove that a recognizable language  $R$  is undecidable, prove that  $A_{TM} \leq_m R$ .
- (ii) To prove that a co-recognizable language  $U$  is undecidable, prove that  $\overline{A_{TM}} \leq_m U$ , i.e. that  $A_{TM} \leq_m \overline{U}$ .

$$E_{TM} = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) = \emptyset\}$$

Can we find algorithms to recognize

$E_{TM}$  ?

$\overline{E_{TM}}$  ?

**Claim:**  $A_{TM} \leq_m \overline{E_{TM}}$ . And hence also  $\overline{A_{TM}} \leq_m E_{TM}$

**Proof:** Need computable function  $F : \Sigma^* \rightarrow \Sigma^*$  such that  $x \in A_{TM}$  iff  $F(x) \notin E_{TM}$ . Define

$F =$  “ On input  $x$ ,

1. Type-check whether  $x = \langle M, w \rangle$  for some TM  $M$  and string  $w$ . If so, move to step 2; if not, output  $\langle \quad \quad \quad \rangle$
2. Construct the following machine  $M'_x$ :

3. Output  $\langle M'_x \rangle$ .”

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property  $x \in A_{TM}$  iff its image is **not** in  $E_{TM}$  ?

Input string	Output string
$\langle M, w \rangle$ where $w \in L(M)$	
$\langle M, w \rangle$ where $w \notin L(M)$	
$x$ not encoding any pair of TM and string	

## Wednesday: More mapping reductions

Recall:  $A$  is **mapping reducible to**  $B$ , written  $A \leq_m B$ , means there is a computable function  $f : \Sigma^* \rightarrow \Sigma^*$  such that *for all* strings  $x$  in  $\Sigma^*$ ,

$$x \in A \quad \text{if and only if} \quad f(x) \in B.$$

So far:

- $A_{TM}$  is recognizable, undecidable, and not-co-recognizable.
- $\overline{A_{TM}}$  is unrecognizable, undecidable, and co-recognizable.
- $HALT_{TM}$  is recognizable, undecidable, and not-co-recognizable.
- $\overline{HALT_{TM}}$  is unrecognizable, undecidable, and co-recognizable.
- $E_{TM}$  is unrecognizable, undecidable, and co-recognizable.
- $\overline{E_{TM}}$  is recognizable, undecidable, and not-co-recognizable.

$$EQ_{TM} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are both Turing machines and } L(M_1) = L(M_2)\}$$

Can we find algorithms to recognize

$EQ_{TM}$  ?

$\overline{EQ_{TM}}$  ?

*Goal:* Show that  $EQ_{TM}$  is not recognizable and that  $\overline{EQ_{TM}}$  is not recognizable.

Using Corollary to **Theorem 5.28**: If  $A \leq_m B$  and  $A$  is unrecognizable, then  $B$  is unrecognizable, it's enough to prove that

$$\begin{array}{ll} \overline{HALT_{TM}} \leq_m EQ_{TM} & \text{aka } HALT_{TM} \leq_m \overline{EQ_{TM}} \\ \overline{HALT_{TM}} \leq_m \overline{EQ_{TM}} & \text{aka } HALT_{TM} \leq_m EQ_{TM} \end{array}$$

Need computable function  $F_1 : \Sigma^* \rightarrow \Sigma^*$  such that  $x \in HALT_{TM}$  iff  $F_1(x) \notin EQ_{TM}$ .

*Strategy:*

Map strings  $\langle M, w \rangle$  to strings  $\langle M'_x,$    $\rangle$ . This image string is not in  $EQ_{TM}$  when  $L(M'_x) \neq \emptyset$ .

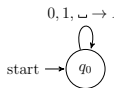

We will build  $M'_x$  so that  $L(M'_x) = \Sigma^*$  when  $M$  halts on  $w$  and  $L(M'_x) = \emptyset$  when  $M$  loops on  $w$ .

Thus: when  $\langle M, w \rangle \in HALT_{TM}$  it gets mapped to a string not in  $EQ_{TM}$  and when  $\langle M, w \rangle \notin HALT_{TM}$  it gets mapped to a string that is in  $EQ_{TM}$ .

Define

$F_1 =$  “ On input  $x$ ,

1. Type-check whether  $x = \langle M, w \rangle$  for some TM  $M$  and string  $w$ . If so, move to step 2; if not, output  $\langle$
2. Construct the following machine  $M'_x$ :

3. Output  $\langle M'_x,$    $\rangle$ ”

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property  $x \in HALT_{TM}$  iff its image is **not** in  $EQ_{TM}$  ?

Input string	Output string
$\langle M, w \rangle$ where $M$ halts on $w$	
$\langle M, w \rangle$ where $M$ loops on $w$	
$x$ not encoding any pair of TM and string	

Conclude:  $HALT_{TM} \leq_m \overline{EQ_{TM}}$

Need computable function  $F_2 : \Sigma^* \rightarrow \Sigma^*$  such that  $x \in HALT_{TM}$  iff  $F_2(x) \in EQ_{TM}$ .

*Strategy:*

Map strings  $\langle M, w \rangle$  to strings  $\langle M'_x, \text{start} \rightarrow \text{q}_0 \rangle$ . This image string is in  $EQ_{TM}$  when  $L(M'_x) = \Sigma^*$ .

We will build  $M'_x$  so that  $L(M'_x) = \Sigma^*$  when  $M$  halts on  $w$  and  $L(M'_x) = \emptyset$  when  $M$  loops on  $w$ .

Thus: when  $\langle M, w \rangle \in HALT_{TM}$  it gets mapped to a string in  $EQ_{TM}$  and when  $\langle M, w \rangle \notin HALT_{TM}$  it gets mapped to a string that is not in  $EQ_{TM}$ .

Define

$F_2 =$  “ On input  $x$ ,

1. Type-check whether  $x = \langle M, w \rangle$  for some TM  $M$  and string  $w$ . If so, move to step 2; if not, output  $\langle \text{start} \rightarrow \text{q}_0 \rangle$
2. Construct the following machine  $M'_x$ :

3. Output  $\langle M'_x, \text{start} \rightarrow \text{q}_0 \rangle$  ”

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property  $x \in HALT_{TM}$  iff its image is in  $EQ_{TM}$  ?

Input string	Output string
$\langle M, w \rangle$ where $M$ halts on $w$	
$\langle M, w \rangle$ where $M$ loops on $w$	
$x$ not encoding any pair of TM and string	

Conclude:  $HALT_{TM} \leq_m EQ_{TM}$



## Friday: Other models of computation

Two models of computation are called **equally expressive** when every language recognizable with the first model is recognizable with the second, and vice versa.

True / False: NFAs and PDAs are equally expressive.

True / False: Regular expressions and CFGs are equally expressive.

**Church-Turing Thesis** (Sipser p. 183): The informal notion of algorithm is formalized completely and correctly by the formal definition of a Turing machine. In other words: all reasonably expressive models of computation are equally expressive with the standard Turing machine.

*Some examples of models that are **equally expressive** with deterministic Turing machines:*

**May-stay machines** The May-stay machine model is the same as the usual Turing machine model, except that on each transition, the tape head may move L, move R, or Stay.

Formally:  $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$  where

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

**Claim:** Turing machines and May-stay machines are equally expressive. *To prove ...*

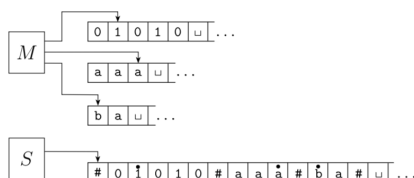
To translate a standard TM to a may-stay machine: never use the direction *S*!

To translate one of the may-stay machines to standard TM: any time TM would Stay, move right then left.

**Multitape Turing machine** A multitape Turing machine with  $k$  tapes can be formally represented as  $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$  where  $Q$  is the finite set of states,  $\Sigma$  is the input alphabet with  $\sqcup \notin \Sigma$ ,  $\Gamma$  is the tape alphabet with  $\Sigma \subsetneq \Gamma$ ,  $\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R\}^k$  (where  $k$  is the number of states)

If  $M$  is a standard TM, it is a 1-tape machine.

To translate a  $k$ -tape machine to a standard TM: Use a new symbol to separate the contents of each tape and keep track of location of head with special version of each tape symbol. Sipser Theorem 3.13



**FIGURE 3.14**  
Representing three tapes with one

**Enumerators** Enumerators give a different model of computation where a language is **produced, one string at a time**, rather than recognized by accepting (or not) individual strings.

Each enumerator machine has finite state control, unlimited work tape, and a printer. The computation proceeds according to transition function; at any point machine may “send” a string to the printer.

$$E = (Q, \Sigma, \Gamma, \delta, q_0, q_{print})$$

$Q$  is the finite set of states,  $\Sigma$  is the output alphabet,  $\Gamma$  is the tape alphabet ( $\Sigma \subsetneq \Gamma, \sqcup \in \Gamma \setminus \Sigma$ ),

$$\delta : Q \times \Gamma \times \Gamma \rightarrow Q \times \Gamma \times \Gamma \times \{L, R\} \times \{L, R\}$$

where in state  $q$ , when the working tape is scanning character  $x$  and the printer tape is scanning character  $y$ ,  $\delta((q, x, y)) = (q', x', y', d_w, d_p)$  means transition to control state  $q'$ , write  $x'$  on the working tape, write  $y'$  on the printer tape, move in direction  $d_w$  on the working tape, and move in direction  $d_p$  on the printer tape. The computation starts in  $q_0$  and each time the computation enters  $q_{print}$  the string from the leftmost edge of the printer tape to the first blank cell is considered to be printed.

The language **enumerated** by  $E$ ,  $L(E)$ , is  $\{w \in \Sigma^* \mid E \text{ eventually, at finite time, prints } w\}$ .

**Theorem 3.21** A language is Turing-recognizable iff some enumerator enumerates it.

**Proof, part 1:** Assume  $L$  is enumerated by some enumerator,  $E$ , so  $L = L(E)$ . We'll use  $E$  in a subroutine within a high-level description of a new Turing machine that we will build to recognize  $L$ .

**Goal:** build Turing machine  $M_E$  with  $L(M_E) = L(E)$ .

Define  $M_E$  as follows:  $M_E =$  “On input  $w$ ,

1. Run  $E$ . For each string  $x$  printed by  $E$ .
2. Check if  $x = w$ . If so, accept (and halt); otherwise, continue.”

**Proof, part 2:** Assume  $L$  is Turing-recognizable and there is a Turing machine  $M$  with  $L = L(M)$ . We'll use  $M$  in a subroutine within a high-level description of an enumerator that we will build to enumerate  $L$ .

**Goal:** build enumerator  $E_M$  with  $L(E_M) = L(M)$ .

**Idea:** check each string in turn to see if it is in  $L$ .

*How?* Run computation of  $M$  on each string. *But:* need to be careful about computations that don't halt.

*Recall* String order for  $\Sigma = \{0, 1\}$ :  $s_1 = \varepsilon$ ,  $s_2 = 0$ ,  $s_3 = 1$ ,  $s_4 = 00$ ,  $s_5 = 01$ ,  $s_6 = 10$ ,  $s_7 = 11$ ,  $s_8 = 000$ , ...

Define  $E_M$  as follows:  $E_M =$  “*ignore any input*. Repeat the following for  $i = 1, 2, 3, \dots$

1. Run the computations of  $M$  on  $s_1, s_2, \dots, s_i$  for (at most)  $i$  steps each
2. For each of these  $i$  computations that accept during the (at most)  $i$  steps, print out the accepted string.”

## Nondeterministic Turing machine

At any point in the computation, the nondeterministic machine may proceed according to several possibilities:  $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$  where

$$\delta : Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

The computation of a nondeterministic Turing machine is a tree with branching when the next step of the computation has multiple possibilities. A nondeterministic Turing machine accepts a string exactly when some branch of the computation tree enters the accept state.

Given a nondeterministic machine, we can use a 3-tape Turing machine to simulate it by doing a breadth-first search of computation tree: one tape is “read-only” input tape, one tape simulates the tape of the nondeterministic computation, and one tape tracks nondeterministic branching. Sipser page 178

## Summary

Two models of computation are called **equally expressive** when every language recognizable with the first model is recognizable with the second, and vice versa.

To prove the existence of a Turing machine that decides / recognizes some language, it’s enough to construct an example using any of the equally expressive models.

But: some of the **performance** properties of these models are not equivalent.