

## Monday: Time Complexity

In practice, computers (and Turing machines) don't have infinite tape, and we can't afford to wait unboundedly long for an answer. "Decidable" isn't good enough - we want "Efficiently decidable".

For a given algorithm working on a given input, how long do we need to wait for an answer? How does the running time depend on the input in the worst-case? average-case? We expect to have to spend more time on computations with larger inputs.

A language is **recognizable** if \_\_\_\_\_

A language is **decidable** if \_\_\_\_\_

A language is **efficiently decidable** if \_\_\_\_\_

A function is **computable** if \_\_\_\_\_

A function is **efficiently computable** if \_\_\_\_\_

Definition (Sipser 7.1): For  $M$  a deterministic decider, its **running time** is the function  $f : \mathbb{N} \rightarrow \mathbb{N}$  given by

$$f(n) = \max \text{ number of steps } M \text{ takes before halting, over all inputs of length } n$$

Definition (Sipser 7.7): For each function  $t(n)$ , the **time complexity class**  $TIME(t(n))$ , is defined by

$$TIME(t(n)) = \{L \mid L \text{ is decidable by a Turing machine with running time in } O(t(n))\}$$

An example of an element of  $TIME(1)$  is

An example of an element of  $TIME(n)$  is

Note:  $TIME(1) \subseteq TIME(n) \subseteq TIME(n^2)$

Definition (Sipser 7.12) :  $P$  is the class of languages that are decidable in polynomial time on a deterministic 1-tape Turing machine

$$P = \bigcup_k TIME(n^k)$$

*Compare to exponential time: brute-force search.*

Theorem (Sipser 7.8): Let  $t(n)$  be a function with  $t(n) \geq n$ . Then every  $t(n)$  time deterministic multitape Turing machine has an equivalent  $O(t^2(n))$  time deterministic 1-tape Turing machine.

Definition (Sipser 7.1): For  $M$  a deterministic decider, its **running time** is the function  $f : \mathbb{N} \rightarrow \mathbb{N}$  given by

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Definition (Sipser 7.12) :  $P$  is the class of languages that are decidable in polynomial time on a deterministic 1-tape Turing machine

$$P = \bigcup_k TIME(n^k)$$

Definition (Sipser 7.9): For  $N$  a nondeterministic decider. The **running time** of  $N$  is the function  $f : \mathbb{N} \rightarrow \mathbb{N}$  given by

$$f(n) = \max \text{ number of steps } N \text{ takes on any branch before halting, over all inputs of length } n$$

Definition (Sipser 7.21): For each function  $t(n)$ , the **nondeterministic time complexity class**  $NTIME(t(n))$ , is defined by

$$NTIME(t(n)) = \{L \mid L \text{ is decidable by a nondeterministic Turing machine with running time in } O(t(n))\}$$

$$NP = \bigcup_k NTIME(n^k)$$

**True or False:**  $TIME(n^2) \subseteq NTIME(n^2)$

**True or False:**  $NTIME(n^2) \subseteq TIME(n^2)$

**Every problem in NP is decidable with an exponential-time algorithm**

Nondeterministic approach: guess a possible solution, verify that it works.

Brute-force (worst-case exponential time) approach: iterate over all possible solutions, for each one, check if it works.

## Examples in $P$

*Can't use nondeterminism; Can use multiple tapes; Often need to be "more clever" than naïve / brute force approach*

$$PATH = \{\langle G, s, t \rangle \mid G \text{ is digraph with } n \text{ nodes there is path from } s \text{ to } t\}$$

Use breadth first search to show in  $P$

$$RELPRIME = \{\langle x, y \rangle \mid x \text{ and } y \text{ are relatively prime integers}\}$$

Use Euclidean Algorithm to show in  $P$

$$L(G) = \{w \mid w \text{ is generated by } G\}$$

(where  $G$  is a context-free grammar). Use dynamic programming to show in  $P$ .

## Examples in $NP$

*"Verifiable" i.e. NP, Can be decided by a nondeterministic TM in polynomial time, best known deterministic solution may be brute-force, solution can be verified by a deterministic TM in polynomial time.*

$$HAMPATH = \{\langle G, s, t \rangle \mid G \text{ is digraph with } n \text{ nodes, there is path from } s \text{ to } t \text{ that goes through every node exactly once}\}$$

$$VERTEX - COVER = \{\langle G, k \rangle \mid G \text{ is an undirected graph with } n \text{ nodes that has a } k\text{-node vertex cover}\}$$

$$CLIQUE = \{\langle G, k \rangle \mid G \text{ is an undirected graph with } n \text{ nodes that has a } k\text{-clique}\}$$

$$SAT = \{\langle X \rangle \mid X \text{ is a satisfiable Boolean formula with } n \text{ variables}\}$$

Problems in $P$	Problems in $NP$
(Membership in any) regular language	Any problem in $P$
(Membership in any) context-free language	
$A_{DFA}$	$SAT$
$E_{DFA}$	$CLIQUE$
$EQ_{DFA}$	$VERTEX - COVER$
$PATH$	$HAMPATH$
$RELPRIME$	$\dots$
$\dots$	

Notice:  $NP \subseteq \{L \mid L \text{ is decidable}\}$  so  $A_{TM} \notin NP$

Million-dollar question: Is  $P = NP$ ?

One approach to trying to answer it is to look for *hardest* problems in  $NP$  and then (1) if we can show that there are efficient algorithms for them, then we can get efficient algorithms for all problems in  $NP$  so  $P = NP$ , or (2) these problems might be good candidates for showing that there are problems in  $NP$  for which there are no efficient algorithms.

## Wednesday: P vs. NP

Definition (Sipser 7.29) Language  $A$  is **polynomial-time mapping reducible** to language  $B$ , written  $A \leq_P B$ , means there is a polynomial-time computable function  $f : \Sigma^* \rightarrow \Sigma^*$  such that for every  $x \in \Sigma^*$

$$x \in A \quad \text{iff} \quad f(x) \in B.$$

The function  $f$  is called the polynomial time reduction of  $A$  to  $B$ .

**Theorem** (Sipser 7.31): If  $A \leq_P B$  and  $B \in P$  then  $A \in P$ .

Proof:

Definition (Sipser 7.34; based in Stephen Cook and Leonid Levin's work in the 1970s): A language  $B$  is **NP-complete** means (1)  $B$  is in NP **and** (2) every language  $A$  in  $NP$  is polynomial time reducible to  $B$ .

**Theorem** (Sipser 7.35): If  $B$  is NP-complete and  $B \in P$  then  $P = NP$ .

Proof:

**3SAT:** A literal is a Boolean variable (e.g.  $x$ ) or a negated Boolean variable (e.g.  $\bar{x}$ ). A Boolean formula is a **3cnf-formula** if it is a Boolean formula in conjunctive normal form (a conjunction of disjunctive clauses of literals) and each clause has three literals.

$$3SAT = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$$

Example string in  $3SAT$

$$\langle (x \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee y \vee z) \wedge (x \vee y \vee z) \rangle$$

Example string not in  $3SAT$

$$\langle (x \vee y \vee z) \wedge (x \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee z) \wedge (x \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee y \vee z) \wedge (\bar{x} \vee y \vee \bar{z}) \wedge (\bar{x} \vee \bar{y} \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z}) \rangle$$

**Cook-Levin Theorem:**  $3SAT$  is  $NP$ -complete.

*Are there other  $NP$ -complete problems?* To prove that  $X$  is  $NP$ -complete

- *From scratch:* prove  $X$  is in  $NP$  and that all  $NP$  problems are polynomial-time reducible to  $X$ .
- *Using reduction:* prove  $X$  is in  $NP$  and that a known-to-be  $NP$ -complete problem is polynomial-time reducible to  $X$ .

**CLIQUE:** A  $k$ -**clique** in an undirected graph is a maximally connected subgraph with  $k$  nodes.

$$CLIQUE = \{\langle G, k \rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$$

Example string in *CLIQUE*

Example string not in *CLIQUE*

Theorem (Sipser 7.32):

$$3SAT \leq_P CLIQUE$$

Given a Boolean formula in conjunctive normal form with  $k$  clauses and three literals per clause, we will map it to a graph so that the graph has a clique if the original formula is satisfiable and the graph does not have a clique if the original formula is not satisfiable.

The graph has  $3k$  vertices (one for each literal in each clause) and an edge between all vertices except

- vertices for two literals in the same clause
- vertices for literals that are negations of one another

Example:  $(x \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee y \vee z) \wedge (x \vee y \vee z)$

## Friday: Review

Model of Computation	Class of Languages
<b>Deterministic finite automata:</b> formal definition, how to design for a given language, how to describe language of a machine? <b>Nondeterministic finite automata:</b> formal definition, how to design for a given language, how to describe language of a machine? <b>Regular expressions:</b> formal definition, how to design for a given language, how to describe language of expression? <i>Also:</i> converting between different models.	<b>Class of regular languages:</b> what are the closure properties of this class? which languages are not in the class? using <b>pumping lemma</b> to prove nonregularity.
<b>Push-down automata:</b> formal definition, how to design for a given language, how to describe language of a machine? <b>Context-free grammars:</b> formal definition, how to design for a given language, how to describe language of a grammar?	<b>Class of context-free languages:</b> what are the closure properties of this class? which languages are not in the class?
Turing machines that always halt in polynomial time  Nondeterministic Turing machines that always halt in polynomial time	$P$  $NP$
<b>Deciders</b> (Turing machines that always halt): formal definition, how to design for a given language, how to describe language of a machine?	<b>Class of decidable languages:</b> what are the closure properties of this class? which languages are not in the class? using diagonalization and mapping reduction to show undecidability
<b>Turing machines</b> formal definition, how to design for a given language, how to describe language of a machine?	<b>Class of recognizable languages:</b> what are the closure properties of this class? which languages are not in the class? using closure and mapping reduction to show unrecognizability

**Given a language, prove it is regular**

*Strategy 1:* construct DFA recognizing the language and prove it works.

*Strategy 2:* construct NFA recognizing the language and prove it works.

*Strategy 3:* construct regular expression recognizing the language and prove it works.

*“Prove it works” means ...*

**Example:**  $L = \{w \in \{0,1\}^* \mid w \text{ has odd number of 1s or starts with } 0\}$

Using NFA

Using regular expressions



**Example:** Select all and only the options that result in a true statement: “To show a language  $A$  is not regular, we can...”

- a. Show  $A$  is finite
- b. Show there is a CFG generating  $A$
- c. Show  $A$  has no pumping length
- d. Show  $A$  is undecidable

**Example:** What is the language generated by the CFG with rules

$$S \rightarrow aSb \mid bY \mid Ya$$

$$Y \rightarrow bY \mid Ya \mid \varepsilon$$

**Example:** Prove that the language  $T = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) \text{ is infinite}\}$  is undecidable.

**Example:** Prove that the class of decidable languages is closed under concatenation.



## Week 10 at a glance

### Textbook reading: Chapter 7

*For Monday:* Definition 7.1 (page 276)

*For Wednesday:* Definition 7.7 (page 279)

### Make sure you can:

- Classify the computational complexity of a set of strings by determining whether it is decidable or undecidable and recognizable or unrecognizable.
  - Distinguish between computability and complexity
  - Articulate motivating questions of complexity
  - Define NP-completeness
  - Give examples of PTIME-decidable, NPTIME-decidable, and NP-complete problems
- Use mapping reduction to deduce the complexity of a language by comparing to the complexity of another.
  - Distinguish between computability and complexity
  - Articulate motivating questions of complexity
  - Use appropriate reduction (e.g. mapping, Turing, polynomial-time) to deduce the complexity of a language by comparing to the complexity of another.
  - Use polynomial-time reduction to prove NP-completeness

## TODO:

Student Evaluations of Teaching forms: Evaluations are open for completion anytime BEFORE 8AM on Saturday, March 16. Access your SETs from the Evaluations site

<https://academicaffairs.ucsd.edu/Modules/Evals>

You will separately evaluate each of your listed instructors for each enrolled course.

**\*\*NEW\*\* WINTER 2024 SET INCENTIVE LOTTERY:** In Winter 2024, students who complete all of their student evaluation forms for their undergraduate course will be entered into a lottery to win one of 5 \$100 Visa gift cards! To be entered into the lottery, students must complete at least one instructor evaluation for EACH of their undergraduate courses. They will be automatically entered when they have completed an instructor evaluation for all of their undergraduate courses.

Review quizzes based on class material each day; review quiz for Friday includes opportunity for feedback for course.

Homework assignment 5 due Thursday.