# Week8 monday

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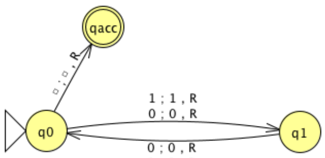
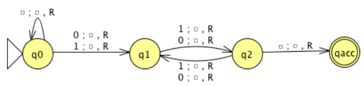
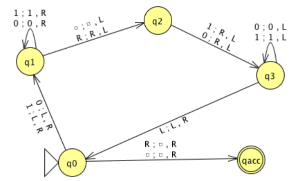
|---|---|---|
| **Acceptance problem** for Turing machines | $A_{TM}$ | $\{\langle M, w\rangle \mid M$ is a Turing machine that accepts input string $w\}$ |
| **Language emptiness testing** for Turing machines | $E_{TM}$ | $\{\langle M\rangle \mid M$ is a Turing machine and $L(M) = \emptyset\}$ |
| **Language equality testing** for Turing machines | $EQ_{TM}$ | $\{\langle M_1, M_2\rangle \mid M_1$ and $M_2$ are Turing machines and $L(M_1) = L(M_2)\}$ |




$M_1$

$M_2$




$M_3$

Example strings in $A_{TM}$

Example strings in $E_{TM}$

Example strings in $EQ_{TM}$

**Theorem**: $A_{TM}$ is Turing-recognizable.

**Strategy**: To prove this theorem, we need to define a Turing machine $R_{ATM}$ such that $L(R_{ATM}) = A_{TM}$.

Define $R_{ATM}$ = "

Proof of correctness:

We will show that $A_{TM}$ is undecidable. *First, let's explore what that means.*

To prove that a computational problem is **decidable**, we find/ build a Turing machine that recognizes the language encoding the computational problem, and that is a decider.

How do we prove a specific problem is **not decidable**?

How would we even find such a computational problem?

*Counting arguments for the existence of an undecidable language:*

- The set of all Turing machines is countably infinite.
- Each recognizable language has at least one Turing machine that recognizes it (by definition), so there can be no more Turing-recognizable languages than there are Turing machines.
- Since there are infinitely many Turing-recognizable languages (think of the singleton sets), there are countably infinitely many Turing-recognizable languages.
- Such the set of Turing-decidable languages is an infinite subset of the set of Turing-recognizable languages, the set of Turing-decidable languages is also countably infinite.

Since there are uncountably many languages (because $\mathcal{P}(\Sigma^*)$ is uncountable), there are uncountably many unrecognizable languages and there are uncountably many undecidable languages.

Thus, there's at least one undecidable language!

**What's a specific example of a language that is unrecognizable or undecidable?**

To prove that a language is undecidable, we need to prove that there is no Turing machine that decides it.

**Key idea**: proof by contradiction relying on self-referential disagreement.

**Theorem**: $A_{TM}$ is not Turing-decidable.

**Proof**: Suppose **towards a contradiction** that there is a Turing machine that decides $A_{TM}$. We call this presumed machine $M_{ATM}$.

By assumption, for every Turing machine $M$ and every string $w$

- If $w \in L(M)$, then the computation of $M_{ATM}$ on $\langle M, w \rangle$ ________________________________
- If $w \notin L(M)$, then the computation of $M_{ATM}$ on $\langle M, w \rangle$ ________________________________

Define a **new** Turing machine using the high-level description:

$D =$" On input $\langle M \rangle$, where $M$ is a Turing machine:

1. Run $M_{ATM}$ on $\langle M, \langle M \rangle \rangle$.
2. If $M_{ATM}$ accepts, reject; if $M_{ATM}$ rejects, accept."

Is $D$ a Turing machine?

Is $D$ a decider?

What is the result of the computation of $D$ on $\langle D \rangle$?

Definition: A language $L$ over an alphabet $\Sigma$ is called **co-recognizable** if its complement, defined as $\Sigma^* \setminus L = \{x \in \Sigma^* \mid x \notin L\}$, is Turing-recognizable.

**Theorem** (Sipser Theorem 4.22): A language is Turing-decidable if and only if both it and its complement are Turing-recognizable.

**Proof, first direction:** Suppose language $L$ is Turing-decidable. WTS that both it and its complement are Turing-recognizable.

**Proof, second direction:** Suppose language $L$ is Turing-recognizable, and so is its complement. WTS that $L$ is Turing-decidable.

Notation: The complement of a set $X$ is denoted with a superscript $c$, $X^c$, or an overline, $\overline{X}$.