Day13

Definitions below are on pages 101-102.

Term	Typical symbol or Notation	Meaning
Context-free grammar (CFG)	G	$G = (V, \Sigma, R, S)$
The set of variables	$\stackrel{ ext{C}}{V}$	Finite set of symbols that represent phases in pro-
1110 500 51 101 101 105	·	duction pattern
The set of <b>terminals</b>	$\Sigma$	Alphabet of symbols of strings generated by CFG $V \cap \Sigma = \emptyset$
The set of <b>rules</b>	R	Each rule is $A \to u$ with $A \in V$ and $u \in (V \cup \Sigma)^*$
The <b>start</b> variable	S	Usually on left-hand-side of first/ topmost rule
Derivation	$S \Rightarrow \cdots \Rightarrow w$	Sequence of substitutions in a CFG (also written $S \Rightarrow^* w$ ). At each step, we can apply one rule to one occurrence of a variable in the current string by substituting that occurrence of the variable with the right-hand-side of the rule. The derivation must end when the current string has only terminals (no variables) because then there are no instances of
Language <b>generated</b> by the context-free grammar $G$	L(G)	variables to apply a rule to. The set of strings for which there is a derivation in $G$ . Symbolically: $\{w \in \Sigma^* \mid S \Rightarrow^* w\}$ i.e. $\{w \in \Sigma^* \mid \text{there is derivation in } G \text{ that ends in } w\}$
Context-free language		A language that is the language generated by some context-free grammar

Examples of context-free grammars, derivations in those grammars, and the languages generated by those grammars

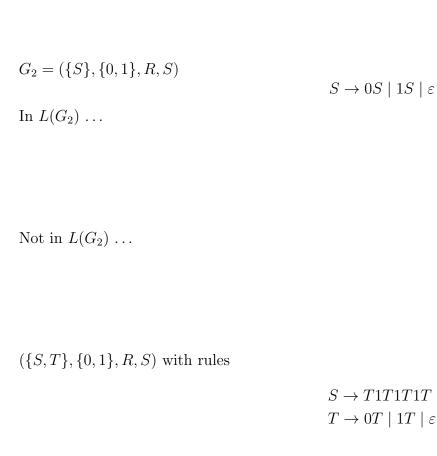
$$G_1 = (\{S\}, \{0\}, R, S)$$
 with rules

$$S \to 0S$$

$$S \to 0$$

In  $L(G_1)$  ...

Not in  $L(G_1)$  ...



In  $L(G_3)$  ...

Not in  $L(G_3)$  ...

 $G_4 = (\{A, B\}, \{0, 1\}, R, A)$  with rules

 $A \to 0A0 \mid 0A1 \mid 1A0 \mid 1A1 \mid 1$ 

In  $L(G_4)$  ...

Not in  $L(G_4)$  ...



## Day14

**Theorem 2.20**: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet  $\Sigma$  is called **CFL**.

## Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a descripton. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
  - PDAs can "test for emptiness of stack" without providing details. How? We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
  - PDAs can "test for end of input" without providing details. How? We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Suppose  $L_1$  and  $L_2$  are context-free languages over  $\Sigma$ . Goal:  $L_1 \cup L_2$  is also context-free.

Approach 1: with PDAs

Let  $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$  and  $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$  be PDAs with  $L(M_1) = L_1$  and  $L(M_2) = L_2$ .

Define M =

 $Approach\ 2:\ with\ CFGs$ 

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define G =

Suppose  $L_1$  and  $L_2$  are context-free languages over  $\Sigma$ . Goal:  $L_1 \circ L_2$  is also context-free.

Approach 1: with PDAs

Let  $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$  and  $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$  be PDAs with  $L(M_1) = L_1$  and  $L(M_2) = L_2$ .

Define M =

 $Approach\ 2:\ with\ CFGs$ 

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define G =

Summary
Over a fixed alphabet $\Sigma$ , a language $L$ is <b>regular</b>
iff it is described by some regular expression iff it is recognized by some DFA iff it is recognized by some NFA
Over a fixed alphabet $\Sigma$ , a language $L$ is <b>context-free</b>
iff it is generated by some CFG iff it is recognized by some PDA
Fact: Every regular language is a context-free language.
Fact: There are context-free languages that are nonregular.
Fact: There are countably many regular languages.
Fact: There are countably infinitely many context-free languages.

 ${\it Consequence} . \ {\it Most languages are } \ {\bf not} \ {\it context-free!}$ 

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## Examples of non-context-free languages

$$\begin{aligned} &\{a^nb^nc^n\mid 0\leq n, n\in\mathbb{Z}\}\\ &\{a^ib^jc^k\mid 0\leq i\leq j\leq k, i\in\mathbb{Z}, j\in\mathbb{Z}, k\in\mathbb{Z}\}\\ &\{ww\mid w\in\{0,1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there is a number p where, if s is any string in A of length at least p, then s may be divided into five pieces s = uvxyz where (1) for each  $i \ge 0$ ,  $uv^ixy^iz \in A$ , (2) |uv| > 0, (3)  $|vxy| \le p$ . We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.

Recall: A set X is said to be **closed** under an operation OP if, for any elements in X, applying OP to them gives an element in X.

True/False	Closure claim		
True	The set of integers is closed under multiplication.		
	$\forall x \forall y  ( (x \in \mathbb{Z} \land y \in \mathbb{Z}) \to xy \in \mathbb{Z} )$		
True	For each set $A$ , the power set of $A$ is closed under intersection.		
	$\forall A_1 \forall A_2 ( (A_1 \in \mathcal{P}(A) \land A_2 \in \mathcal{P}(A) \in \mathbb{Z}) \to A_1 \cap A_2 \in \mathcal{P}(A) )$		
	The class of regular languages over $\Sigma$ is closed under complementation.		
	The class of regular languages over $\Sigma$ is closed under union.		
	The class of regular languages over $\Sigma$ is closed under intersection.		
	The class of regular languages over $\Sigma$ is closed under concatenation.		
	The class of regular languages over $\Sigma$ is closed under Kleene star.		
	The class of context-free languages over $\Sigma$ is closed under complementation.		
	The class of context-free languages over $\Sigma$ is closed under union.		
	The class of context-free languages over $\Sigma$ is closed under intersection.		
	The class of context-free languages over $\Sigma$ is closed under concatenation.		
	The class of context-free languages over $\Sigma$ is closed under Kleene star.		