Week 9 at a glance

Textbook reading: Section 5.3, Section 5.1, Section 3.2

For Monday, Example 5.26 (page 237).

For Wednesday, Theorem 5.30 (page 238)

For Friday, skim section 3.2.

For Monday of Week 10: Definition 7.1 (page 276)

We will be learning and practicing to:
- Clearly and unambiguously communicate computational ideas using appropriate formalism. Translate across levels of abstraction.

    - Give examples of sets that are regular, context-free, decidable, or recognizable (and prove that they are).

        * **Define and explain computational problems, including A_{**}, E_{**}, EQ_{**}, (for ** DFA or TM) and $HALT_{TM}$**

- Know, select and apply appropriate computing knowledge and problem-solving techniques. Reason about computation and systems.

    - Use mapping reduction to deduce the complexity of a language by comparing to the complexity of another.

        * **Explain what it means for one problem to reduce to another**
        * **Define computable functions, and use them to give mapping reductions between computational problems**
        * **Build and analyze mapping reductions between computational problems**

    - Classify the computational complexity of a set of strings by determining whether it is regular, context-free, decidable, or recognizable.

        * **State, prove, and use theorems relating decidability, recognizability, and co-recognizability.**
        * **Prove that a language is decidable or recognizable by defining and analyzing a Turing machines with appropriate properties.**

    - Describe several variants of Turing machines and informally explain why they are equally expressive.

        * **Define an enumerator**
        * **Define nondeterministic Turing machines**
        * **Use high-level descriptions to define and trace machines (Turing machines and enumerators)**
        * **Apply dovetailing in high-level definitions of machines**

TODO:

Review Quiz 8 on PrairieLearn (http://us.prairielearn.com), due 3/5/2025

Review Quiz 9 on PrairieLearn (http://us.prairielearn.com), due 3/12/2025

Homework 6 submitted via Gradescope (https://www.gradescope.com/), due 3/13/2025

Project submitted via Gradescope (https://www.gradescope.com/), due 3/19/2025

Monday: Mapping reductions and recognizability

Recall definition: A is **mapping reducible to** B means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

Notation: when A is mapping reducible to B, we write $A \leq_m B$.

Theorem (Sipser 5.23): If $A \leq_m B$ and A is undecidable, then B is undecidable.

Last time we proved that $A_{TM} \leq_m HALT_{TM}$ where

$$HALT_{TM} = \{\langle M, w \rangle \mid M \text{ is a Turing machine, } w \text{ is a string, and } M \text{ halts on } w\}$$

and since A_{TM} is undecidable, $HALT_{TM}$ is also undecidable. The function witnessing the mapping reduction mapped strings in A_{TM} to strings in $HALT_{TM}$ and strings not in A_{TM} to strings not in $HALT_{TM}$ by changing encoded Turing machines to ones that had identical computations except looped instead of rejecting.

True or False: $\overline{A_{TM}} \leq_m \overline{HALT_{TM}}$

True or False: $HALT_{TM} \leq_m A_{TM}$.

Proof: Need computable function $F : \Sigma^* \to \Sigma^*$ such that $x \in HALT_{TM}$ iff $F(x) \in A_{TM}$. Define

> $F = $ " On input x,
>
> 1. Type-check whether $x = \langle M, w \rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle \qquad\qquad\qquad \rangle$
> 2. Construct the following machine M'_x:
>
>
>
> 3. Output $\langle M'_x, w \rangle$."

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in HALT_{TM}$ iff its image is in A_{TM}?

Input string	Output string
$\langle M, w \rangle$ where M halts on w	
$\langle M, w \rangle$ where M does not halt on w	
x not encoding any pair of TM and string	

Theorem (Sipser 5.28): If $A \leq_m B$ and B is recognizable, then A is recognizable.

Proof:

Corollary: If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable.

Strategy:

(i) To prove that a recognizable language R is undecidable, prove that $A_{TM} \leq_m R$.

(ii) To prove that a co-recognizable language U is undecidable, prove that $\overline{A_{TM}} \leq_m U$, i.e. that $A_{TM} \leq_m \overline{U}$.

$$E_{TM} = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) = \emptyset\}$$

Can we find algorithms to recognize

E_{TM} ?

$\overline{E_{TM}}$?

Claim: $A_{TM} \leq_m \overline{E_{TM}}$. *And hence also* $\overline{A_{TM}} \leq_m E_{TM}$

Proof: Need computable function $F : \Sigma^* \to \Sigma^*$ such that $x \in A_{TM}$ iff $F(x) \notin E_{TM}$. Define

$F = $ " On input x,

1. Type-check whether $x = \langle M, w \rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle \quad\quad\quad\quad\quad\quad\quad \rangle$

2. Construct the following machine M'_x:

3. Output $\langle M'_x \rangle$."

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in A_{TM}$ iff its image is **not** in E_{TM} ?

Input string	Output string
$\langle M, w \rangle$ where $w \in L(M)$	
$\langle M, w \rangle$ where $w \notin L(M)$	
x not encoding any pair of TM and string	

Wednesday: More mapping reductions

Recall: A is **mapping reducible to** B, written $A \leq_m B$, means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

So far:

- A_{TM} is recognizable, undecidable, and not-co-recognizable.

- $\overline{A_{TM}}$ is unrecognizable, undecidable, and co-recognizable.

- $HALT_{TM}$ is recognizable, undecidable, and not-co-recognizable.

- $\overline{HALT_{TM}}$ is unrecognizable, undecidable, and co-recognizable.

- E_{TM} is unrecognizable, undecidable, and co-recognizable.

- $\overline{E_{TM}}$ is recognizable, undecidable, and not-co-recognizable.

$$EQ_{TM} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are both Turing machines and } L(M_1) = L(M_2)\}$$

Can we find algorithms to recognize

EQ_{TM} ?

$\overline{EQ_{TM}}$?

Goal: Show that EQ_{TM} is not recognizable and that $\overline{EQ_{TM}}$ is not recognizable.

Using Corollary to **Theorem 5.28**: If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable, it's enough to prove that

$\overline{HALT_{TM}} \leq_m EQ_{TM}$ aka $HALT_{TM} \leq_m \overline{EQ_{TM}}$

$\overline{HALT_{TM}} \leq_m \overline{EQ_{TM}}$ aka $HALT_{TM} \leq_m EQ_{TM}$

Need computable function $F_1 : \Sigma^* \to \Sigma^*$ such that $x \in HALT_{TM}$ iff $F_1(x) \notin EQ_{TM}$.

Strategy:

Map strings $\langle M, w \rangle$ to strings $\langle M'_x,$ [diagram: start $\to q_0$ with self-loop $0,1,\_\to R$, and q_{acc}] $\rangle$. This image string is not in EQ_{TM} when $L(M'_x) \neq \emptyset$.
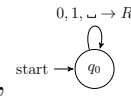
We will build M'_x so that $L(M'_x) = \Sigma^*$ when M halts on w and $L(M'_x) = \emptyset$ when M loops on w.

Thus: when $\langle M, w \rangle \in HALT_{TM}$ it gets mapped to a string not in EQ_{TM} and when $\langle M, w \rangle \notin HALT_{TM}$ it gets mapped to a string that is in EQ_{TM}.

Define

$F_1 = $ " On input x,

1. Type-check whether $x = \langle M, w \rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle \qquad\qquad\qquad \rangle$

2. Construct the following machine M'_x:

3. Output $\langle M'_x,$ [diagram: start $\to q_0$ with self-loop $0,1,\_\to R$, and q_{acc}] $\rangle$ "

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in HALT_{TM}$ iff its image is **not** in EQ_{TM} ?

Input string	Output string
$\langle M, w \rangle$ where M halts on w	
$\langle M, w \rangle$ where M loops on w	
x not encoding any pair of TM and string	

Conclude: $HALT_{TM} \leq_m \overline{EQ_{TM}}$

Need computable function $F_2 : \Sigma^* \to \Sigma^*$ such that $x \in HALT_{TM}$ iff $F_2(x) \in EQ_{TM}$.

Strategy:
Map strings $\langle M, w \rangle$ to strings $\langle M'_x, \,^{start \to \overset{q_0}{\bigcirc}} \rangle$. This image string is in EQ_{TM} when $L(M'_x) = \Sigma^*$.

We will build M'_x so that $L(M'_x) = \Sigma^*$ when M halts on w and $L(M'_x) = \emptyset$ when M loops on w.

Thus: when $\langle M, w \rangle \in HALT_{TM}$ it gets mapped to a string in EQ_{TM} and when $\langle M, w \rangle \notin HALT_{TM}$ it gets mapped to a string that is not in EQ_{TM}.

Define

$F_2 = $ " On input x,

1. Type-check whether $x = \langle M, w \rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle \qquad\qquad\qquad\qquad \rangle$

2. Construct the following machine M'_x:

3. Output $\langle M'_x, \,^{start \to \overset{q_0}{\circledcirc}} \rangle$ "

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in HALT_{TM}$ iff its image is in EQ_{TM} ?

Input string	Output string
$\langle M, w \rangle$ where M halts on w	
$\langle M, w \rangle$ where M loops on w	
x not encoding any pair of TM and string	

Conclude: $HALT_{TM} \leq_m EQ_{TM}$

Friday: Other models of computation

Two models of computation are called **equally expressive** when every language recognizable with the first model is recognizable with the second, and vice versa.

True / False: NFAs and PDAs are equally expressive.

True / False: Regular expressions and CFGs are equally expressive.

Church-Turing Thesis (Sipser p. 183): The informal notion of algorithm is formalized completely and correctly by the formal definition of a Turing machine. In other words: all reasonably expressive models of computation are equally expressive with the standard Turing machine.

Some examples of models that are **equally expressive** *with deterministic Turing machines:*

$\boxed{\textbf{May-stay} \text{ machines}}$ The May-stay machine model is the same as the usual Turing machine model, except that on each transition, the tape head may move L, move R, or Stay.

Formally: $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$$

Claim: Turing machines and May-stay machines are equally expressive. *To prove ...*

To translate a standard TM to a may-stay machine: never use the direction S!

To translate one of the may-stay machines to standard TM: any time TM would Stay, move right then left.

$\boxed{\textbf{Multitape Turing machine}}$ A multitape Turing machine with k tapes can be formally represented as $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ where Q is the finite set of states, Σ is the input alphabet with $\sqcup \notin \Sigma$, Γ is the tape alphabet with $\Sigma \subsetneq \Gamma$, $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$ (where k is the number of states)

If M is a standard TM, it is a 1-tape machine.

To translate a k-tape machine to a standard TM: Use a new symbol to separate the contents of each tape and keep track of location of head with special version of each tape symbol. Sipser Theorem 3.13
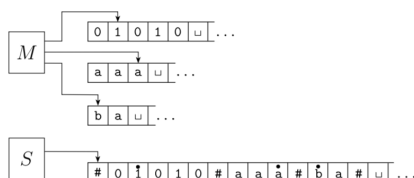
FIGURE **3.14**
Representing three tapes with one

$\boxed{\textbf{Enumerators}}$ Enumerators give a different model of computation where a language is **produced, one string at a time**, rather than recognized by accepting (or not) individual strings.

Each enumerator machine has finite state control, unlimited work tape, and a printer. The computation proceeds according to transition function; at any point machine may "send" a string to the printer.

$$E = (Q, \Sigma, \Gamma, \delta, q_0, q_{print})$$

Q is the finite set of states, Σ is the output alphabet, Γ is the tape alphabet ($\Sigma \subsetneq \Gamma$, $\sqcup \in \Gamma \setminus \Sigma$),

$$\delta : Q \times \Gamma \times \Gamma \to Q \times \Gamma \times \Gamma \times \{L, R\} \times \{L, R\}$$

where in state q, when the working tape is scanning character x and the printer tape is scanning character y, $\delta((q, x, y)) = (q', x', y', d_w, d_p)$ means transition to control state q', write x' on the working tape, write y' on the printer tape, move in direction d_w on the working tape, and move in direction d_p on the printer tape. The computation starts in q_0 and each time the computation enters q_{print} the string from the leftmost edge of the printer tape to the first blank cell is considered to be printed.

The language **enumerated** by E, $L(E)$, is $\{w \in \Sigma^* \mid E \text{ eventually, at finite time, prints } w\}$.

Theorem 3.21 A language is Turing-recognizable iff some enumerator enumerates it.

Proof, part 1: Assume L is enumerated by some enumerator, E, so $L = L(E)$. We'll use E in a subroutine within a high-level description of a new Turing machine that we will build to recognize L.

Goal: build Turing machine M_E with $L(M_E) = L(E)$.

Define M_E as follows: $M_E =$ "On input w,

1. Run E. For each string x printed by E.

2.   Check if $x = w$. If so, accept (and halt); otherwise, continue."

Proof, part 2: Assume L is Turing-recognizable and there is a Turing machine M with $L = L(M)$. We'll use M in a subroutine within a high-level description of an enumerator that we will build to enumerate L.

Goal: build enumerator E_M with $L(E_M) = L(M)$.

Idea: check each string in turn to see if it is in L.

How? Run computation of M on each string. *But*: need to be careful about computations that don't halt.

Recall String order for $\Sigma = \{0, 1\}$: $s_1 = \varepsilon$, $s_2 = 0$, $s_3 = 1$, $s_4 = 00$, $s_5 = 01$, $s_6 = 10$, $s_7 = 11$, $s_8 = 000$, ...

Define E_M as follows: $E_M =$ " *ignore any input.* Repeat the following for $i = 1, 2, 3, \ldots$

1. Run the computations of M on s_1, s_2, ..., s_i for (at most) i steps each

2. For each of these i computations that accept during the (at most) i steps, print out the accepted string."

At any point in the computation, the nondeterministic machine may proceed according to several possibilities: $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ where

$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

The computation of a nondeterministic Turing machine is a tree with branching when the next step of the computation has multiple possibilities. A nondeterministic Turing machine accepts a string exactly when some branch of the computation tree enters the accept state.

Given a nondeterministic machine, we can use a 3-tape Turing machine to simulate it by doing a breadth-first search of computation tree: one tape is "read-only" input tape, one tape simulates the tape of the nondeterministic computation, and one tape tracks nondeterministic branching. Sipser page 178

Summary

Two models of computation are called **equally expressive** when every language recognizable with the first model is recognizable with the second, and vice versa.

To prove the existence of a Turing machine that decides / recognizes some language, it's enough to construct an example using any of the equally expressive models.

But: some of the **performance** properties of these models are not equivalent.