Definition A **pushdown automaton** (PDA) is specified by a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q is the finite set of states, Σ is the input alphabet, Γ is the stack alphabet,

$$\delta: Q \times \Sigma_{\varepsilon} \times \Gamma_{\varepsilon} \to \mathcal{P}(Q \times \Gamma_{\varepsilon})$$

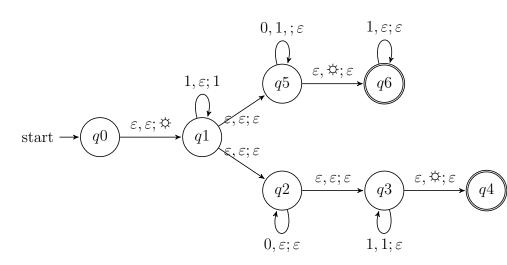
is the transition function, $q_0 \in Q$ is the start state, $F \subseteq Q$ is the set of accept states.

For the PDA state diagrams below, $\Sigma = \{0, 1\}$.

$$\Gamma = \{\$, \#\}$$

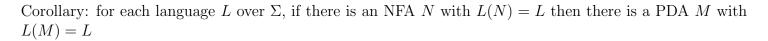


$$\Gamma = \{ \circlearrowleft, 1 \}$$



$$\{0^i 1^j 0^k \mid i, j, k \ge 0\}$$

Note: alternate notation is to replace; with \rightarrow on arrow labels.



Proof idea: Declare stack alphabet to be $\Gamma = \Sigma$ and then don't use stack at all.

Big picture: PDAs are motivated by wanting to add some memory of unbounded size to NFA. How do we accomplish a similar enhancement of regular expressions to get a syntactic model that is more expressive?

DFA, NFA, PDA: Machines process one input string at a time; the computation of a machine on its input string reads the input from left to right.

Regular expressions: Syntactic descriptions of all strings that match a particular pattern; the language described by a regular expression is built up recursively according to the expression's syntax

Context-free grammars: Rules to produce one string at a time, adding characters from the middle, beginning, or end of the final string as the derivation proceeds.

Definitions below are on pages 101-102.

Term	Typical symbol	Meaning
	or Notation	
Context-free grammar (CFG)	G	$G = (V, \Sigma, R, S)$
The set of variables	V	Finite set of symbols that represent phases in pro-
		duction pattern
The set of terminals	Σ	Alphabet of symbols of strings generated by CFG $V \cap \Sigma = \emptyset$
The set of rules	R	Each rule is $A \to u$ with $A \in V$ and $u \in (V \cup \Sigma)^*$
The start variable	S	Usually on left-hand-side of first/ topmost rule
Derivation	$S \Rightarrow \cdots \Rightarrow w$	Sequence of substitutions in a CFG (also written $S \Rightarrow^* w$). At each step, we can apply one rule to one occurrence of a variable in the current string by substituting that occurrence of the variable with the right-hand-side of the rule. The derivation must end when the current string has only terminals (no variables) because then there are no instances of
Language generated by the context-free grammar G	L(G)	variables to apply a rule to. The set of strings for which there is a derivation in G . Symbolically: $\{w \in \Sigma^* \mid S \Rightarrow^* w\}$ i.e.
Context-free language		$\{w \in \Sigma^* \mid \text{there is derivation in } G \text{ that ends in } w\}$ A language that is the language generated by some context-free grammar

Examples of context-free grammars, derivations in those grammars, and the languages generated by those grammars

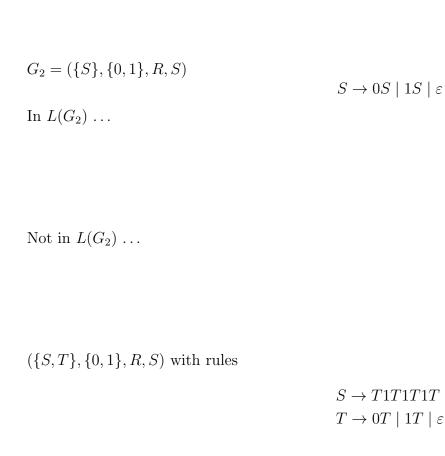
$$G_1 = (\{S\}, \{0\}, R, S)$$
 with rules

$$S \to 0 S$$

$$S \to 0$$

In $L(G_1)$...

Not in $L(G_1)$...



In $L(G_3)$...

Not in $L(G_3)$...

 $G_4 = (\{A, B\}, \{0, 1\}, R, A)$ with rules

 $A \to 0A0 \mid 0A1 \mid 1A0 \mid 1A1 \mid 1$

In $L(G_4)$...

Not in $L(G_4)$...



Theorem 2.20: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet Σ is called **CFL**.

Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a descripton. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
 - PDAs can "test for emptiness of stack" without providing details. How? We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
 - PDAs can "test for end of input" without providing details. How? We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Suppose L_1 and L_2 are context-free languages over Σ . Goal: $L_1 \cup L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define M =

 $Approach\ 2:\ with\ CFGs$

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define G =

Suppose L_1 and L_2 are context-free languages over Σ . Goal: $L_1 \circ L_2$ is also context-free.

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Define M =

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define G =

Summary

Over a fixed alphabet Σ , a language L is **regular**

iff it is described by some regular expression iff it is recognized by some DFA iff it is recognized by some NFA

Over a fixed alphabet Σ , a language L is **context-free**

iff it is generated by some CFG iff it is recognized by some PDA

Fact: Every regular language is a context-free language.

Fact: There are context-free languages that are not nonregular.

Fact: There are countably many regular languages.

Fact: There are countably infinitely many context-free languages.

Consequence: Most languages are **not** context-free!

Examples of non-context-free languages

$$\begin{aligned} &\{a^nb^nc^n\mid 0\leq n, n\in\mathbb{Z}\}\\ &\{a^ib^jc^k\mid 0\leq i\leq j\leq k, i\in\mathbb{Z}, j\in\mathbb{Z}, k\in\mathbb{Z}\}\\ &\{ww\mid w\in\{0,1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there is a number p where, if s is any string in A of length at least p, then s may be divided into five pieces s = uvxyz where (1) for each $i \ge 0$, $uv^ixy^iz \in A$, (2) |uv| > 0, (3) $|vxy| \le p$. We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.

Recall: A set X is said to be **closed** under an operation OP if, for any elements in X, applying OP to them gives an element in X.

True/False	Closure claim				
True	The set of integers is closed under multiplication.				
	$\forall x \forall y ((x \in \mathbb{Z} \land y \in \mathbb{Z}) \to xy \in \mathbb{Z})$				
True	For each set A , the power set of A is closed under intersection.				
	$\forall A_1 \forall A_2 ((A_1 \in \mathcal{P}(A) \land A_2 \in \mathcal{P}(A) \in \mathbb{Z}) \to A_1 \cap A_2 \in \mathcal{P}(A))$				
	The class of regular languages over Σ is closed under complementation.				
	The class of regular languages over Σ is closed under union.				
	The class of regular languages over Σ is closed under intersection.				
	The class of regular languages over Σ is closed under concatenation.				
	The class of regular languages over Σ is closed under Kleene star.				
	The class of context-free languages over Σ is closed under complementation.				
	The class of context-free languages over Σ is closed under union.				
	The class of context-free languages over Σ is closed under intersection.				
	The class of context-free languages over Σ is closed under concatenation.				
	The class of context-free languages over Σ is closed under Kleene star.				

Definition and Theorem: For an alphabet Σ , a language L over Σ is called **regular** exactly when L is recognized by some DFA, which happens exactly when L is recognized by some NFA, and happens exactly when L is described by some regular expression

We saw that: The class of regular languages is closed under complementation, union, intersection, set-wise concatenation, and Kleene star.

Extra practice:

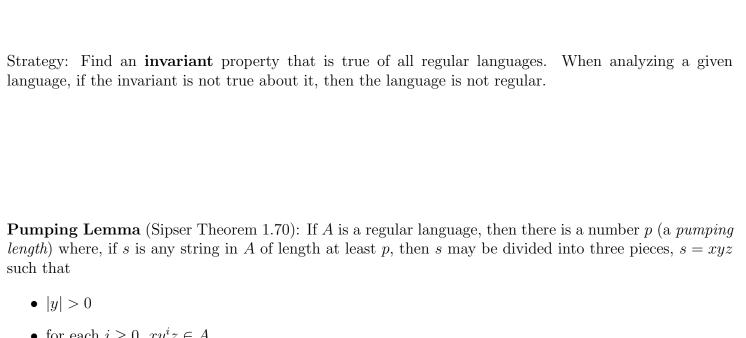
Disprove: There is some alphabet Σ for which there is some language recognized by an NFA but not by any DFA.

Disprove: There is some alphabet Σ for which there is some finite language not described by any regular expression over Σ .

Disprove: If a language is recognized by an NFA then the complement of this language is not recognized by any DFA.

Fix alphabet Σ . Is every language L over Σ regular?

Set	Cardinality
$\{0,1\}$	
$\{0,1\}^*$	
$\mathcal{P}(\{0,1\})$	
The set of all languages over $\{0,1\}$	
The set of all regular expressions over $\{0,1\}$	
The set of all regular languages over $\{0,1\}$	



- for each $i \ge 0$, $xy^iz \in A$
- $|xy| \leq p$.

Proof idea: In DFA, the only memory available is in the states. Automata can only "remember" finitely far in the past and finitely much information, because they can have only finitely many states. If a computation path of a DFA visits the same state more than once, the machine can't tell the difference between the first time and future times it visits this state. Thus, if a DFA accepts one long string, then it must accept (infinitely) many similar strings.

Proof illustration

True or False: A pumping length for $A = \{0, 1\}^*$ is p = 5.

True or False: A pumping length for $A = \{0, 1\}^*$ is p = 2.

True or False: A pumping length for $A = \{0, 1\}^*$ is p = 105.

Restating **Pumping Lemma**: If L is a regular language, then it has a pumping length.

Contrapositive: If L has no pumping length, then it is nonregular.

The Pumping Lemma cannot be used to prove that a language is regular.

The Pumping Lemma can be used to prove that a language is not regular.

Extra practice: Exercise 1.49 in the book.

Proof strategy: To prove that a language L is **not** regular,

- ullet Consider an arbitrary positive integer p
- ullet Prove that p is not a pumping length for L
- \bullet Conclude that L does not have any pumping length, and therefore it is not regular.

Negation: A positive integer p is **not a pumping length** of a language L over Σ iff

$$\exists s \ \big(\ |s| \geq p \land s \in L \land \forall x \forall y \forall z \ \big(\ (s = xyz \land |y| > 0 \land |xy| \leq p \) \rightarrow \exists i (i \geq 0 \land xy^iz \not\in L) \big) \ \big)$$

Proof strategy: To prove that a language L is **not** regular,

- \bullet Consider an arbitrary positive integer p
- Prove that p is not a pumping length for L. A positive integer p is **not a pumping length** of a language L over Σ iff

$$\exists s \ \big(\ |s| \ge p \land s \in L \land \forall x \forall y \forall z \ \big(\ (s = xyz \land |y| > 0 \land |xy| \le p \) \rightarrow \exists i (i \ge 0 \land xy^iz \notin L) \big) \ \big)$$

Informally:

• Conclude that L does not have any pumping length, and therefore it is not regular.

Example: $\Sigma = \{0, 1\}, L = \{0^n 1^n \mid n \ge 0\}.$

Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p:

Pick s =

Suppose s = xyz with $|xy| \le p$ and |y| > 0.

Then when i =, $xy^iz =$

Example: $\Sigma = \{0, 1\}$, $L = \{ww^{\mathcal{R}} \mid w \in \{0, 1\}^*\}$. Remember that the reverse of a string w is denoted $w^{\mathcal{R}}$ and means to write w in the opposite order, if $w = w_1 \cdots w_n$ then $w^{\mathcal{R}} = w_n \cdots w_1$. Note: $\varepsilon^{\mathcal{R}} = \varepsilon$. Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p: Pick s =Suppose s = xyz with $|xy| \le p$ and |y| > 0. $, xy^iz =$ Then when i =**Example**: $\Sigma = \{0, 1\}, L = \{0^j 1^k \mid j \ge k \ge 0\}.$ Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p: Pick s =Suppose s = xyz with $|xy| \le p$ and |y| > 0. $, xy^iz =$ Then when i =**Example**: $\Sigma = \{0, 1\}, L = \{0^n 1^m 0^n \mid m, n \ge 0\}.$ Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p: Pick s =Suppose s = xyz with $|xy| \le p$ and |y| > 0.

 $, xy^iz =$

Then when i =

Extra practice:

Language	$s \in L$	$s \notin L$	Is the language regular or nonregular?
$\{a^nb^n\mid 0\leq n\leq 5\}$			
$\{b^na^n\mid n\geq 2\}$			
$\{a^mb^n\mid 0\leq m\leq n\}$			
$\{a^mb^n\mid m\geq n+3, n\geq 0\}$			
$\{b^ma^n\mid m\geq 1, n\geq 3\}$			
$\{w \in \{a, b\}^* \mid w = w^{\mathcal{R}}\}$			
$\{ww^{\mathcal{R}} \mid w \in \{a, b\}^*\}$			