Day12

Definition A **pushdown automaton** (PDA) is specified by a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q is the finite set of states, Σ is the input alphabet, Γ is the stack alphabet,

$$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Gamma_\varepsilon)$$

is the transition function, $q_0 \in Q$ is the start state, $F \subseteq Q$ is the set of accept states.

For the PDA state diagrams below, $\Sigma = \{0, 1\}$.

Mathematical description of language	State diagram of PDA recognizing language

$$\Gamma = \{\$, \#\}$$

$$\Gamma = \{\maltese, 1\}$$

$$\{0^i 1^j 0^k \mid i, j, k \geq 0\}$$

Note: alternate notation is to replace ; with $\rightarrow$ on arrow labels.

Corollary: for each language L over Σ, if there is an NFA N with $L(N) = L$ then there is a PDA M with $L(M) = L$

Proof idea: Declare stack alphabet to be $\Gamma = \Sigma$ and then don't use stack at all.

Big picture: PDAs are motivated by wanting to add some memory of unbounded size to NFA. How do we accomplish a similar enhancement of regular expressions to get a syntactic model that is more expressive?

DFA, NFA, PDA: Machines process one input string at a time; the computation of a machine on its input string reads the input from left to right.

Regular expressions: Syntactic descriptions of all strings that match a particular pattern; the language described by a regular expression is built up recursively according to the expression's syntax

Context-free grammars: Rules to produce one string at a time, adding characters from the middle, beginning, or end of the final string as the derivation proceeds.

Day13

Definitions below are on pages 101-102.

Term	Typical symbol or **Notation**	Meaning
Context-free grammar (CFG)	G	$G = (V, \Sigma, R, S)$
The set of **variables**	V	Finite set of symbols that represent phases in production pattern
The set of **terminals**	Σ	Alphabet of symbols of strings generated by CFG $V \cap \Sigma = \emptyset$
The set of **rules**	R	Each rule is $A \to u$ with $A \in V$ and $u \in (V \cup \Sigma)^*$
The **start** variable	S	Usually on left-hand-side of first/ topmost rule
Derivation	$S \Rightarrow \cdots \Rightarrow w$	Sequence of substitutions in a CFG (also written $S \Rightarrow^* w$). At each step, we can apply one rule to one occurrence of a variable in the current string by substituting that occurrence of the variable with the right-hand-side of the rule. The derivation must end when the current string has only terminals (no variables) because then there are no instances of variables to apply a rule to.
Language **generated** by the context-free grammar G	$L(G)$	The set of strings for which there is a derivation in G. Symbolically: $\{w \in \Sigma^* \mid S \Rightarrow^* w\}$ i.e. $\{w \in \Sigma^* \mid \text{there is derivation in } G \text{ that ends in } w\}$
Context-free language		A language that is the language generated by some context-free grammar

Examples of context-free grammars, derivations in those grammars, and the languages generated by those grammars

$G_1 = (\{S\}, \{0\}, R, S)$ with rules

$$S \to 0S$$
$$S \to 0$$

In $L(G_1)$ ...

Not in $L(G_1)$ ...

$G_2 = (\{S\}, \{0, 1\}, R, S)$

$$S \to 0S \mid 1S \mid \varepsilon$$

In $L(G_2)$ ...

Not in $L(G_2)$ ...

$(\{S, T\}, \{0, 1\}, R, S)$ with rules

$$S \to T1T1T1T$$
$$T \to 0T \mid 1T \mid \varepsilon$$

In $L(G_3)$ ...

Not in $L(G_3)$ ...

$G_4 = (\{A, B\}, \{0, 1\}, R, A)$ with rules

$$A \to 0A0 \mid 0A1 \mid 1A0 \mid 1A1 \mid 1$$

In $L(G_4)$ ...

Not in $L(G_4)$ ...

Design a CFG to generate the language $\{a^n b^n \mid n \geq 0\}$

Design a CFG to generate the language $\{a^i b^j \mid j \geq i \geq 0\}$

Design a PDA to recognize the language $\{a^i b^j \mid j \geq i \geq 0\}$

Day14

Theorem 2.20: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet Σ is called **CFL**.

Consequences:

- Quick proof that every regular language is context free

- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier

- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a descripton. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:

  - PDAs can "test for emptiness of stack" without providing details. *How?* We can always push a special end-of-stack symbol, $, at the start, before processing any input, and then use this symbol as a flag.

  - PDAs can "test for end of input" without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Suppose L_1 and L_2 are context-free languages over Σ. **Goal**: $L_1 \cup L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

Suppose L_1 and L_2 are context-free languages over Σ. **Goal**: $L_1 \circ L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

Summary

Over a fixed alphabet Σ, a language L is **regular**

<div align="center">

iff it is described by some regular expression
iff it is recognized by some DFA
iff it is recognized by some NFA

</div>

Over a fixed alphabet Σ, a language L is **context-free**

<div align="center">

iff it is generated by some CFG
iff it is recognized by some PDA

</div>

Fact: Every regular language is a context-free language.

Fact: There are context-free languages that are nonregular.

Fact: There are countably many regular languages.

Fact: There are countably infinitely many context-free languages.

Consequence: Most languages are **not** context-free!

Examples of non-context-free languages

$$\{a^n b^n c^n \mid 0 \le n, n \in \mathbb{Z}\}$$
$$\{a^i b^j c^k \mid 0 \le i \le j \le k, i \in \mathbb{Z}, j \in \mathbb{Z}, k \in \mathbb{Z}\}$$
$$\{ww \mid w \in \{0,1\}^*\}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there is a number p where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ where (1) for each $i \ge 0$, $uv^i xy^i z \in A$, (2) $|uv| > 0$, (3) $|vxy| \le p$. *We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.*

Recall: A set X is said to be **closed** under an operation OP if, for any elements in X, applying OP to them gives an element in X.

True/False	Closure claim
True	The set of integers is closed under multiplication. $\forall x \forall y\ (\ (x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \to xy \in \mathbb{Z}\ )$
True	For each set A, the power set of A is closed under intersection. $\forall A_1 \forall A_2\ (\ (A_1 \in \mathcal{P}(A) \wedge A_2 \in \mathcal{P}(A) \in \mathbb{Z}) \to A_1 \cap A_2 \in \mathcal{P}(A)\ )$
	The class of regular languages over Σ is closed under complementation.
	The class of regular languages over Σ is closed under union.
	The class of regular languages over Σ is closed under intersection.
	The class of regular languages over Σ is closed under concatenation.
	The class of regular languages over Σ is closed under Kleene star.
	The class of context-free languages over Σ is closed under complementation.
	The class of context-free languages over Σ is closed under union.
	The class of context-free languages over Σ is closed under intersection.
	The class of context-free languages over Σ is closed under concatenation.
	The class of context-free languages over Σ is closed under Kleene star.

Day9

Definition and Theorem: For an alphabet Σ, a language L over Σ is called **regular** exactly when L is recognized by some DFA, which happens exactly when L is recognized by some NFA, and happens exactly when L is described by some regular expression

We saw that: The class of regular languages is closed under complementation, union, intersection, set-wise concatenation, and Kleene star.

Extra practice:

Disprove: There is some alphabet Σ for which there is some language recognized by an NFA but not by any DFA.

Disprove: There is some alphabet Σ for which there is some finite language not described by any regular expression over Σ.

Disprove: If a language is recognized by an NFA then the complement of this language is not recognized by any DFA.

Fix alphabet Σ. Is every language L over Σ regular?

Set	Cardinality
$\{0, 1\}$	
$\{0, 1\}^*$	
$\mathcal{P}(\{0, 1\})$	
The set of all languages over $\{0, 1\}$	
The set of all regular expressions over $\{0, 1\}$	
The set of all regular languages over $\{0, 1\}$	

Strategy: Find an **invariant** property that is true of all regular languages. When analyzing a given language, if the invariant is not true about it, then the language is not regular.

Pumping Lemma (Sipser Theorem 1.70): If A is a regular language, then there is a number p (a *pumping length*) where, if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$ such that

- $|y| > 0$

- for each $i \geq 0$, $xy^i z \in A$

- $|xy| \leq p$.

Proof idea: In DFA, the only memory available is in the states. Automata can only "remember" finitely far in the past and finitely much information, because they can have only finitely many states. If a computation path of a DFA visits the same state more than once, the machine can't tell the difference between the first time and future times it visits this state. Thus, if a DFA accepts one long string, then it must accept (infinitely) many similar strings.

Proof illustration

True or False: A pumping length for $A = \{0, 1\}^*$ is $p = 5$.

True or False: A pumping length for $A = \{0, 1\}^*$ is $p = 2$.

True or False: A pumping length for $A = \{0, 1\}^*$ is $p = 105$.

Restating **Pumping Lemma**: If L is a regular language, then it has a pumping length.

Contrapositive: If L has no pumping length, then it is nonregular.

The Pumping Lemma *cannot* be used to prove that a language *is* regular.

The Pumping Lemma **can** be used to prove that a language *is not* regular.

Extra practice: Exercise 1.49 in the book.

Proof strategy: To prove that a language L is **not** regular,

- Consider an arbitrary positive integer p
- Prove that p is not a pumping length for L
- Conclude that L does not have *any* pumping length, and therefore it is not regular.

Negation: A positive integer p is **not a pumping length** of a language L over Σ iff

$$\exists s \left(\, |s| \geq p \wedge s \in L \wedge \forall x \forall y \forall z \left(\, (s = xyz \wedge |y| > 0 \wedge |xy| \leq p \,) \rightarrow \exists i (i \geq 0 \wedge xy^i z \notin L)) \, \right) \right)$$

Day10

Proof strategy: To prove that a language L is **not** regular,

- Consider an arbitrary positive integer p

- Prove that p is not a pumping length for L. A positive integer p is **not a pumping length** of a language L over Σ iff

$$\exists s \left(|s| \geq p \wedge s \in L \wedge \forall x \forall y \forall z \left((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \exists i (i \geq 0 \wedge xy^i z \notin L) \right) \right)$$

  Informally:

- Conclude that L does not have *any* pumping length, and therefore it is not regular.

Example: $\Sigma = \{0, 1\}$, $L = \{0^n 1^n \mid n \geq 0\}$.

Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p:

Pick $s =$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

<div style="border:1px solid black; height:80px;"></div>

Then when $i =$, $xy^i z =$

Example: $\Sigma = \{0, 1\}$, $L = \{ww^{\mathcal{R}} \mid w \in \{0, 1\}^*\}$. Remember that the reverse of a string w is denoted $w^{\mathcal{R}}$ and means to write w in the opposite order, if $w = w_1 \cdots w_n$ then $w^{\mathcal{R}} = w_n \cdots w_1$. Note: $\varepsilon^{\mathcal{R}} = \varepsilon$.

Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p:

Pick $s =$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

Then when $i =$, $xy^i z =$

Example: $\Sigma = \{0, 1\}$, $L = \{0^j 1^k \mid j \geq k \geq 0\}$.

Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p:

Pick $s =$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

Then when $i =$, $xy^i z =$

Example: $\Sigma = \{0, 1\}$, $L = \{0^n 1^m 0^n \mid m, n \geq 0\}$.

Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p:

Pick $s =$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

Then when $i =$, $xy^i z =$

Extra practice:

Language	$s \in L$	$s \notin L$	Is the language regular or nonregular?
$\{a^n b^n \mid 0 \leq n \leq 5\}$			
$\{b^n a^n \mid n \geq 2\}$			
$\{a^m b^n \mid 0 \leq m \leq n\}$			
$\{a^m b^n \mid m \geq n + 3, n \geq 0\}$			
$\{b^m a^n \mid m \geq 1, n \geq 3\}$			
$\{w \in \{a, b\}^* \mid w = w^{\mathcal{R}}\}$			
$\{w w^{\mathcal{R}} \mid w \in \{a, b\}^*\}$			

Day15

We are ready to introduce a formal model that will capture a notion of general purpose computation.

- *Similar to DFA, NFA, PDA*: input will be an arbitrary string over a fixed alphabet.

- *Different from NFA, PDA*: machine is deterministic.

- *Different from DFA, NFA, PDA*: read-write head can move both to the left and to the right, and can extend to the right past the original input.

- *Similar to DFA, NFA, PDA*: transition function drives computation one step at a time by moving within a finite set of states, always starting at designated start state.

- *Different from DFA, NFA, PDA*: the special states for rejecting and accepting take effect immediately.

(See more details: Sipser p. 166)

Formally: a Turing machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where δ is the **transition function**

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$$

The **computation** of M on a string w over Σ is:

- Read/write head starts at leftmost position on tape.
- Input string is written on $|w|$-many leftmost cells of tape, rest of the tape cells have the blank symbol. **Tape alphabet** is Γ with $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$. The blank symbol $\sqcup \notin \Sigma$.
- Given current state of machine and current symbol being read at the tape head, the machine transitions to next state, writes a symbol to the current position of the tape head (overwriting existing symbol), and moves the tape head L or R (if possible).
- Computation ends **if and when** machine enters either the accept or the reject state. This is called **halting**. Note: $q_{accept} \neq q_{reject}$.

The **language recognized by the Turing machine** M, is $L(M) = \{w \in \Sigma^* \mid w$ is accepted by $M\}$, which is defined as

$$\{w \in \Sigma^* \mid \text{computation of } M \text{ on } w \text{ halts after entering the accept state}\}$$

State diagram

$0; \square, R$
$\square; \square, R$

$0; \square, R$
$\square; \square, R$

q_{acc}

q_{rej}

$\square; \square, R$

$\square; \square, R$

$0; \square, R$

start $\longrightarrow$ $q0$

$q1$

$0; \square, R$

Formal definition:

Sample computation:

$q0 \downarrow$						
0	0	0	␣	␣	␣	␣

The language recognized by this machine is . . .

Describing Turing machines (Sipser p. 185) To define a Turing machine, we could give a

- **Formal definition**: the 7-tuple of parameters including set of states, input alphabet, tape alphabet, transition function, start state, accept state, and reject state; or,

- **Implementation-level definition**: English prose that describes the Turing machine head movements relative to contents of tape, and conditions for accepting / rejecting based on those contents.

- **High-level description**: description of algorithm (precise sequence of instructions), without implementation details of machine. As part of this description, can "call" and run another TM as a subroutine.

Fix $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \text{\textvisiblespace}\}$ for the Turing machines with the following state diagrams:

$$\square; \square, R$$

start $\longrightarrow$ ($q0$)   ((q_{acc}))

Example of string accepted:
Example of string rejected:

Implementation-level description

High-level description

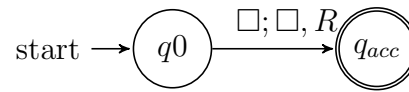start $\longrightarrow$ (q_{rej})   ((q_{acc}))

Example of string accepted:
Example of string rejected:

Implementation-level description

High-level description

Example of string accepted:
Example of string rejected:

Implementation-level description

High-level description

Example of string accepted:
Example of string rejected:

Implementation-level description

High-level description

Day16

Conventions in state diagram of TM: $b \to R$ label means $b \to b, R$ and all arrows missing from diagram represent transitions with output $(q_{reject}, \llcorner, R)$

Computation on input string $01\#01$

q_1 ↓						
0	1	#	0	1	⌴	⌴

Implementation level description of this machine:

Zig-zag across tape to corresponding positions on either side of # to check whether the characters in these positions agree. If they do not, or if there is no #, reject. If they do, cross them off.

Once all symbols to the left of the # are crossed off, check for any un-crossed-off symbols to the right of #; if there are any, reject; if there aren't, accept.

The language recognized by this machine is

$$\{w\#w \mid w \in \{0,1\}^*\}$$

High-level description of this machine is

Extra practice

Computation on input string 01#1

$q_1 \downarrow$						
0	1	#	1	⊔	⊔	⊔

A language L is **recognized by** a Turing machine M means

A Turing machine M **recognizes** a language L means

A Turing machine M is a **decider** means

A language L is **decided by** a Turing machine M means

A Turing machine M **decides** a language L means

Fix $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \square\}$ for the Turing machines with the following state diagrams:

$\square; \square, R$

start $\rightarrow$ $q0$ q_{acc}

Decider? Yes / No

start $\rightarrow$ q_{rej} q_{acc}

Decider? Yes / No

start $\rightarrow$ $q0$ $\xrightarrow{\square; \square, R}$ q_{acc}

Decider? Yes / No

$0; \square, R$
$1; \square, R$
$\square; \square, R$

start $\rightarrow$ $q0$ q_{acc}

Decider? Yes / No

Day17

A **Turing-recognizable** language is a set of strings that is the language recognized by some Turing machine. We also say that such languages are recognizable.

A **Turing-decidable** language is a set of strings that is the language recognized by some decider. We also say that such languages are decidable.

An **unrecognizable** language is a language that is not Turing-recognizable.

An **undecidable** language is a language that is not Turing-decidable.

True or **False**: Any decidable language is also recognizable.

True or **False**: Any recognizable language is also decidable.

True or **False**: Any undecidable language is also unrecognizable.

True or **False**: Any unrecognizable language is also undecidable.

True or **False**: The class of Turing-decidable languages is closed under complementation.

Using formal definition:

Using high-level description:

Church-Turing Thesis (Sipser p. 183): The informal notion of algorithm is formalized completely and correctly by the formal definition of a Turing machine. In other words: all reasonably expressive models of computation are equally expressive with the standard Turing machine.