

## Day9

**Definition and Theorem:** For an alphabet  $\Sigma$ , a language  $L$  over  $\Sigma$  is called **regular** exactly when  $L$  is recognized by some DFA, which happens exactly when  $L$  is recognized by some NFA, and happens exactly when  $L$  is described by some regular expression

**We saw that:** The class of regular languages is closed under complementation, union, intersection, set-wise concatenation, and Kleene star.

*Extra practice:*

**Disprove:** There is some alphabet  $\Sigma$  for which there is some language recognized by an NFA but not by any DFA.

**Disprove:** There is some alphabet  $\Sigma$  for which there is some finite language not described by any regular expression over  $\Sigma$ .

**Disprove:** If a language is recognized by an NFA then the complement of this language is not recognized by any DFA.

**Fix alphabet  $\Sigma$ . Is every language  $L$  over  $\Sigma$  regular?**

| Set  | Cardinality |
|--|-------------|
| $\{0, 1\}$   |             |
| $\{0, 1\}^*$                                       |             |
| $\mathcal{P}(\{0, 1\})$                            |             |
| The set of all languages over $\{0, 1\}$           |             |
| The set of all regular expressions over $\{0, 1\}$ |             |
| The set of all regular languages over $\{0, 1\}$   |             |

Strategy: Find an **invariant** property that is true of all regular languages. When analyzing a given language, if the invariant is not true about it, then the language is not regular.

**Pumping Lemma** (Sipser Theorem 1.70): If  $A$  is a regular language, then there is a number  $p$  (a *pumping length*) where, if  $s$  is any string in  $A$  of length at least  $p$ , then  $s$  may be divided into three pieces,  $s = xyz$  such that

- $|y| > 0$
- for each  $i \geq 0$ ,  $xy^iz \in A$
- $|xy| \leq p$ .

**Proof idea:** In DFA, the only memory available is in the states. Automata can only “remember” finitely far in the past and finitely much information, because they can have only finitely many states. If a computation path of a DFA visits the same state more than once, the machine can’t tell the difference between the first time and future times it visits this state. Thus, if a DFA accepts one long string, then it must accept (infinitely) many similar strings.

**Proof illustration**

**True or False:** A pumping length for  $A = \{0, 1\}^*$  is  $p = 5$ .

**True or False:** A pumping length for  $A = \{0, 1\}^*$  is  $p = 2$ .

**True or False:** A pumping length for  $A = \{0, 1\}^*$  is  $p = 105$ .

Restating **Pumping Lemma:** If  $L$  is a regular language, then it has a pumping length.

**Contrapositive:** If  $L$  has no pumping length, then it is nonregular.

The Pumping Lemma *cannot* be used to prove that a language *is* regular.

The Pumping Lemma **can** be used to prove that a language *is not* regular.

*Extra practice:* Exercise 1.49 in the book.

**Proof strategy:** To prove that a language  $L$  is **not** regular,

- Consider an arbitrary positive integer  $p$
- Prove that  $p$  is not a pumping length for  $L$
- Conclude that  $L$  does not have *any* pumping length, and therefore it is not regular.

**Negation:** A positive integer  $p$  is **not a pumping length** of a language  $L$  over  $\Sigma$  iff

$$\exists s \left( |s| \geq p \wedge s \in L \wedge \forall x \forall y \forall z \left( (s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \exists i (i \geq 0 \wedge xy^i z \notin L) \right) \right)$$

# Day10

**Proof strategy:** To prove that a language  $L$  is **not** regular,

- Consider an arbitrary positive integer  $p$
- Prove that  $p$  is not a pumping length for  $L$ . A positive integer  $p$  is **not a pumping length** of a language  $L$  over  $\Sigma$  iff

$$\exists s \left( |s| \geq p \wedge s \in L \wedge \forall x \forall y \forall z \left( (s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \exists i (i \geq 0 \wedge xy^i z \notin L) \right) \right)$$

*Informally:*

- Conclude that  $L$  does not have *any* pumping length, and therefore it is not regular.

**Example:**  $\Sigma = \{0, 1\}$ ,  $L = \{0^n 1^n \mid n \geq 0\}$ .

Fix  $p$  an arbitrary positive integer. List strings that are in  $L$  and have length greater than or equal to  $p$ :

Pick  $s =$

Suppose  $s = xyz$  with  $|xy| \leq p$  and  $|y| > 0$ .

Then when  $i =$  ,  $xy^i z =$

**Example:**  $\Sigma = \{0, 1\}$ ,  $L = \{ww^{\mathcal{R}} \mid w \in \{0, 1\}^*\}$ . Remember that the reverse of a string  $w$  is denoted  $w^{\mathcal{R}}$  and means to write  $w$  in the opposite order, if  $w = w_1 \cdots w_n$  then  $w^{\mathcal{R}} = w_n \cdots w_1$ . Note:  $\varepsilon^{\mathcal{R}} = \varepsilon$ .

Fix  $p$  an arbitrary positive integer. List strings that are in  $L$  and have length greater than or equal to  $p$ :

Pick  $s =$

Suppose  $s = xyz$  with  $|xy| \leq p$  and  $|y| > 0$ .

Then when  $i =$  ,  $xy^iz =$

**Example:**  $\Sigma = \{0, 1\}$ ,  $L = \{0^j1^k \mid j \geq k \geq 0\}$ .

Fix  $p$  an arbitrary positive integer. List strings that are in  $L$  and have length greater than or equal to  $p$ :

Pick  $s =$

Suppose  $s = xyz$  with  $|xy| \leq p$  and  $|y| > 0$ .

Then when  $i =$  ,  $xy^iz =$

**Example:**  $\Sigma = \{0, 1\}$ ,  $L = \{0^n1^m0^n \mid m, n \geq 0\}$ .

Fix  $p$  an arbitrary positive integer. List strings that are in  $L$  and have length greater than or equal to  $p$ :

Pick  $s =$

Suppose  $s = xyz$  with  $|xy| \leq p$  and  $|y| > 0$ .

Then when  $i =$  ,  $xy^iz =$

*Extra practice:*

| Language                                  | $s \in L$ | $s \notin L$ | Is the language regular or nonregular? |
|---|-----------|--------------|--|
| $\{a^n b^n \mid 0 \leq n \leq 5\}$        |           |              |  |
| $\{b^n a^n \mid n \geq 2\}$               |           |              |  |
| $\{a^m b^n \mid 0 \leq m \leq n\}$        |           |              |  |
| $\{a^m b^n \mid m \geq n + 3, n \geq 0\}$ |           |              |  |
| $\{b^m a^n \mid m \geq 1, n \geq 3\}$     |           |              |  |
| $\{w \in \{a, b\}^* \mid w = w^R\}$       |           |              |  |
| $\{ww^R \mid w \in \{a, b\}^*\}$          |           |              |  |