

Week 5 at a glance

Textbook reading: Section 2.2, 2.1.

Before Monday, read Theorem 2.20.

Before Wednesday, read Example 2.18 (page 114).

Before Friday, read Figure 3.1.

For Week 6 Monday: Page 165-166 Introduction to Section 3.1.

We will be learning and practicing to:

- Clearly and unambiguously communicate computational ideas using appropriate formalism. Translate across levels of abstraction.
 - Describe and use models of computation that don't involve state machines.
 - * **Use context-free grammars and relate them to languages and pushdown automata.**
 - Use precise notation to formally define the state diagram of a Turing machine
 - Use clear English to describe computations of Turing machines informally.
 - * **Design a PDA that recognizes a given language.**
 - Give examples of sets that are context-free (and prove that they are).
 - * **State the definition of the class of context-free languages**
 - * **Explain the limits of the class of context-free languages**
 - * **Identify some context-free sets and some non-context-free sets**
- Know, select and apply appropriate computing knowledge and problem-solving techniques. Reason about computation and systems.
 - Describe and closure properties of classes of languages under certain operations.
 - * **Apply a general construction to create a new PDA or CFG from an example one.**
 - * **Formalize a general construction from an informal description of it.**
 - * **Use general constructions to prove closure properties of the class of context-free languages.**
 - * **Use counterexamples to prove non-closure properties of the class of context-free languages.**

TODO:

Schedule your Test 1 Attempt 1, Test 2 Attempt 1, Test 1 Attempt 2, and Test 2 Attempt 2 times at PrairieTest (<http://us.prairietest.com>)

Review Quiz 5 on PrairieLearn (<http://us.prairielearn.com>), complete by Sunday 11/4/2024

Monday: Context-free languages

Warmup: Design a CFG to generate the language $\{a^i b^j \mid j \geq i \geq 0\}$

Sample derivation:

Design a PDA to recognize the language $\{a^i b^j \mid j \geq i \geq 0\}$

Theorem 2.20: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet Σ is called **CFL**.

Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a description. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
 - PDAs can “test for emptiness of stack” without providing details. *How?* We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
 - PDAs can “test for end of input” without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Suppose L_1 and L_2 are context-free languages over Σ . **Goal:** $L_1 \cup L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

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Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

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Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

Wednesday: Context-free and non-context-free languages

Summary

Over a fixed alphabet Σ , a language L is **regular**

iff it is described by some regular expression

iff it is recognized by some DFA

iff it is recognized by some NFA

Over a fixed alphabet Σ , a language L is **context-free**

iff it is generated by some CFG

iff it is recognized by some PDA

Fact: Every regular language is a context-free language.

Fact: There are context-free languages that are not nonregular.

Fact: There are countably many regular languages.

Fact: There are countably infinitely many context-free languages.

Consequence: Most languages are **not** context-free!

Examples of non-context-free languages

$$\begin{aligned} &\{a^n b^n c^n \mid 0 \leq n, n \in \mathbb{Z}\} \\ &\{a^i b^j c^k \mid 0 \leq i \leq j \leq k, i \in \mathbb{Z}, j \in \mathbb{Z}, k \in \mathbb{Z}\} \\ &\{ww \mid w \in \{0,1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there is a number p where, if s is any string in A of length at least p , then s may be divided into five pieces $s = uvxyz$ where (1) for each $i \geq 0$, $uv^i xy^i z \in A$, (2) $|uv| > 0$, (3) $|vxy| \leq p$. *We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.*

Recall: A set X is said to be **closed** under an operation OP if, for any elements in X , applying OP to them gives an element in X .

| True/False | Closure claim |
|------------|---|
| True | The set of integers is closed under multiplication. $\forall x \forall y ((x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \rightarrow xy \in \mathbb{Z})$ |
| True | For each set A , the power set of A is closed under intersection. $\forall A_1 \forall A_2 ((A_1 \in \mathcal{P}(A) \wedge A_2 \in \mathcal{P}(A)) \rightarrow A_1 \cap A_2 \in \mathcal{P}(A))$ |
| | The class of regular languages over Σ is closed under complementation. |
| | The class of regular languages over Σ is closed under union. |
| | The class of regular languages over Σ is closed under intersection. |
| | The class of regular languages over Σ is closed under concatenation. |
| | The class of regular languages over Σ is closed under Kleene star. |
| | The class of context-free languages over Σ is closed under complementation. |
| | The class of context-free languages over Σ is closed under union. |
| | The class of context-free languages over Σ is closed under intersection. |
| | The class of context-free languages over Σ is closed under concatenation. |
| | The class of context-free languages over Σ is closed under Kleene star. |

Friday: Turing machines

We are ready to introduce a formal model that will capture a notion of general purpose computation.

- *Similar to DFA, NFA, PDA*: input will be an arbitrary string over a fixed alphabet.
- *Different from NFA, PDA*: machine is deterministic.
- *Different from DFA, NFA, PDA*: read-write head can move both to the left and to the right, and can extend to the right past the original input.
- *Similar to DFA, NFA, PDA*: transition function drives computation one step at a time by moving within a finite set of states, always starting at designated start state.
- *Different from DFA, NFA, PDA*: the special states for rejecting and accepting take effect immediately.

(See more details: Sipser p. 166)

Formally: a Turing machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where δ is the **transition function**

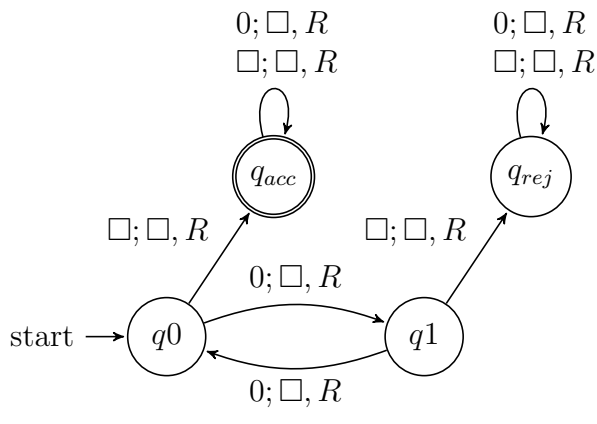
$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

The **computation** of M on a string w over Σ is:

- Read/write head starts at leftmost position on tape.
- Input string is written on $|w|$ -many leftmost cells of tape, rest of the tape cells have the blank symbol. **Tape alphabet** is Γ with $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$. The blank symbol $\sqcup \notin \Sigma$.
- Given current state of machine and current symbol being read at the tape head, the machine transitions to next state, writes a symbol to the current position of the tape head (overwriting existing symbol), and moves the tape head L or R (if possible).
- Computation ends **if and when** machine enters either the accept or the reject state. This is called **halting**. Note: $q_{accept} \neq q_{reject}$.

The **language recognized by the Turing machine** M , is $L(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$, which is defined as

$$\{w \in \Sigma^* \mid \text{computation of } M \text{ on } w \text{ halts after entering the accept state}\}$$



Formal

definition:

Sample computation:

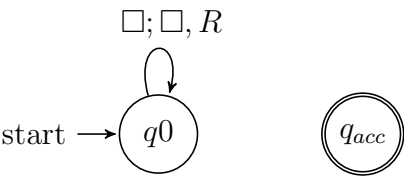
| $q0 \downarrow$ | | | | | | |
|-----------------|---|---|---|---|---|---|
| 0 | 0 | 0 | □ | □ | □ | □ |
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The language recognized by this machine is ...

Describing Turing machines (Sipser p. 185) To define a Turing machine, we could give a

- **Formal definition:** the 7-tuple of parameters including set of states, input alphabet, tape alphabet, transition function, start state, accept state, and reject state; or,
- **Implementation-level definition:** English prose that describes the Turing machine head movements relative to contents of tape, and conditions for accepting / rejecting based on those contents.
- **High-level description:** description of algorithm (precise sequence of instructions), without implementation details of machine. As part of this description, can “call” and run another TM as a subroutine.

Fix $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \sqcup\}$ for the Turing machines with the following state diagrams:



Example of string accepted:

Example of string rejected:

Implementation-level description

High-level description



Example of string accepted:

Example of string rejected:

Implementation-level description

High-level description

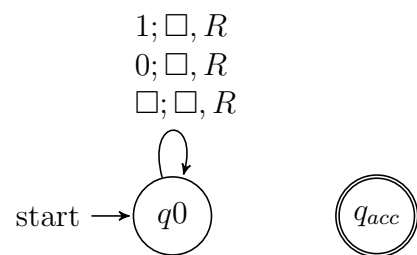


Example of string accepted:

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Implementation-level description

High-level description



Example of string accepted:

Example of string rejected:

Implementation-level description

High-level description