Neil's Probability Corner

Problem 1. *(from game night)* You're given the opportunity to play "Cards with betting", a game where 10 cards are randomly drawn from two shuffled decks (each a standard 52-card deck), one at a time. You can bet on the following:

1. Next card black? (spades or clubs)
2. Next card red? (hearts or diamonds)
3. Next card low? (2-7)
4. Next card high? (9-A)
5. Three-of-a-kind in the 10 cards
6. Four-of-a-kind in the 10 cards
7. Full house (three-of-a-kind and a pair) in the 10 cards
8. 6 or more black cards in the 10 cards
9. 7 or more black cards in the 10 cards
10. 8 or more black cards in the 10 cards
11. 9 or more black cards in the 10 cards
12. 10 black cards in the 10 cards
13. (6+ to 10 but for red cards instead)

You can bet on the first four for every card, but the remaining ones can only be bet on before any cards are revealed. Estimate the probabilities of each one to determine what payout you would need for it to be profitable. Does it ever make sense to bet on two "contradictory" things? Why or why not?

Problem 2. X and Y are independent identically distributed random variables, $X, Y \sim \mathcal{N}(0,1)$, the unit normal distribution. What's the probability that $X > 2Y$ given that X is positive? That is, find $\Pr[X > 2Y \mid X > 0]$. *Hint: there's at least two ways to do this, one of which doesn't need the pdf given below.*

$$\text{Recall that the pdf for } \mathcal{N}(\mu, \sigma^2) \text{ is } f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

Problem 3. You have 100 coins, each with their own probability of flipping heads, p_i, $1 \le i \le 100$. Prove that if the probability of flipping an even number of heads is 50/50, then at least one coin must be fair. That is, $\Pr[\text{even \# of heads}] = 0.5 \implies \exists\, i\,(p_i = 0.5)$

Nash's Auction Design

Problem 4. Recall the first-price auction game ("Dutch auction") from Game Night. There are n bidders, and bidder i has item valuation v_i drawn i.i.d. from the uniform distribution from $[0, 1]$. They each make a bid b_i, where only the highest bidder receives the item. We assume the bidders have quasi-linear utility, i.e. they each want to maximize

$$u_i(v_i) = \begin{cases} v_i - b_i & \text{if } b_i = \max_{j\in[n]} b_j \\ 0 & \text{else} \end{cases}$$

In this game, a Bayes-Nash equilibrium (BNE) is one where every bidder has some strategy $b_i(v_i)$ such that they have no incentive to deviate in expectation. Prove that the BNE here is obtained when $b_i(v_i) = \frac{n-1}{n} \cdot v_i$ for all bidders $i \in [n]$. (Prove that if every other bidder excluding bidder i uses this strategy, then bidder i will maximize their expected utility with this strategy as well.)

Problem 5. Consider the second-price auction game ("English auction"). Once again, we have n bidders and one item that they bid on. Their item valuations v_i don't have to be drawn from a distribution this time. Once again, the highest bidder receives the item, but now they pay the price of the second highest bid, not their bid. Hence, their utility is now:

$$u_i(v_i) = \begin{cases} v_i - max_{j\neq i} b_j & \text{if } b_i = \max_{j\in[n]} b_j \\ 0 & \text{else} \end{cases}$$

Prove that bidding truthfully, i.e. $b_i(v_i) = v_i$, is a dominant strategy (no matter what the other bidders do, this strategy maximizes bidder i's utility).

Non-transitive Dice

Problem 6. Recall the RPS version of the transitive dice game. You each have a set of 4 non-transitive dice and choose one dice to roll. The higher roll wins the game. Now consider a variation where you choose two dice (not necessarily distinct) to roll, and the higher sum wins the game. What is the optimal strategy (how often should you roll each of the combinations)? For reference, dice $A = 004444 \to B = 333333 \to C = 222266 \to D = 111555$ (and D beats A). Below is the 6x6 payoff matrix, the 4 rolls AA, AB, AD, BB were removed as they are dominated by other strategies.

$$\begin{array}{c} \\ AC \\ BB \\ BC \\ CC \\ CD \\ DD \end{array} \begin{array}{c} \begin{array}{cccccc} AC & BB & BC & CC & CD & DD \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 16/27 & 4/9 & 10/27 & 1/2 \\ 1/2 & 1/2 & 2/3 & 4/9 & 1/3 & 1/2 \\ 11/27 & 1/3 & 1/2 & 16/27 & 1/2 & 5/12 \\ 5/9 & 5/9 & 11/27 & 1/2 & 17/27 & 5/9 \\ 17/27 & 2/3 & 1/2 & 10/27 & 1/2 & 5/8 \\ 1/2 & 1/2 & 7/12 & 4/9 & 3/8 & 1/2 \end{pmatrix} \end{array}$$

Problem 7. Consider the same problem as above, but where you must choose two distinct dice to roll. (Hint: this is a deterministic strategy, no large calculation should be needed).