1)

1.1)

Soit $x, y \in I$, alors

$$f(x) = f(y) \iff \frac{x}{x+1} = \frac{y}{y+1} \iff \cancel{xy} + x = \cancel{xy} + y$$

$$\iff x = y$$

1.2)

f cont. sur $I \implies f(I) = J$ intervalle

1.3)

f strictement monotone car pour $x, y \in I$ tels que $x < y$ on a

$$x < y \iff \underbrace{xy}_{\geq 0} + x < xy + y \iff x\underbrace{(y+1)}_{\geq 0} < y\underbrace{(x+1)}_{\geq 0}$$

$$\iff \frac{x}{x+1} < \frac{y}{y+1}$$

$$\iff f(x) < f(y)$$

Alors $J = \left[\lim_{x \to 0^+} f(x), \lim_{x \to +\infty} f(x)\right[= [0; 1[$

1.4)

$f : \mathbb{R} \to [0;1[$ bijective car inj. et surj.

1.5)

On a :

$$y = \frac{x}{x+1}$$

$$yx + y = x$$

$$y = x(1-y)$$

$$x = \frac{y}{1-y} \implies \begin{cases} f^{-1} : [0;1[\to \mathbb{R}_+ \\ f^{-1}(y) = \frac{y}{1-y} \end{cases}$$

---

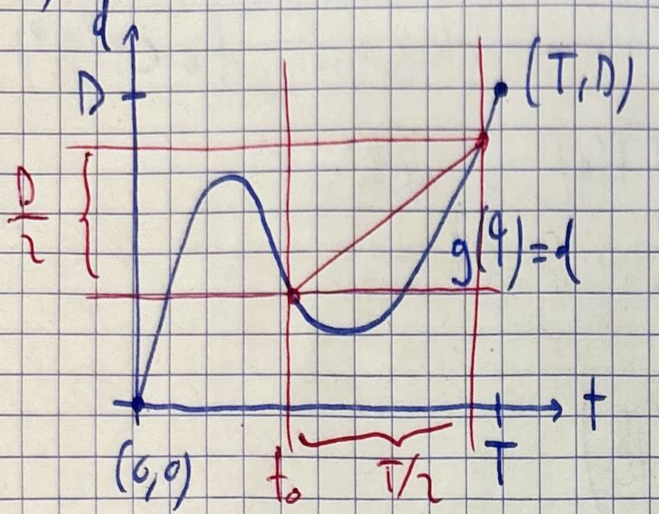
2) 2.1)

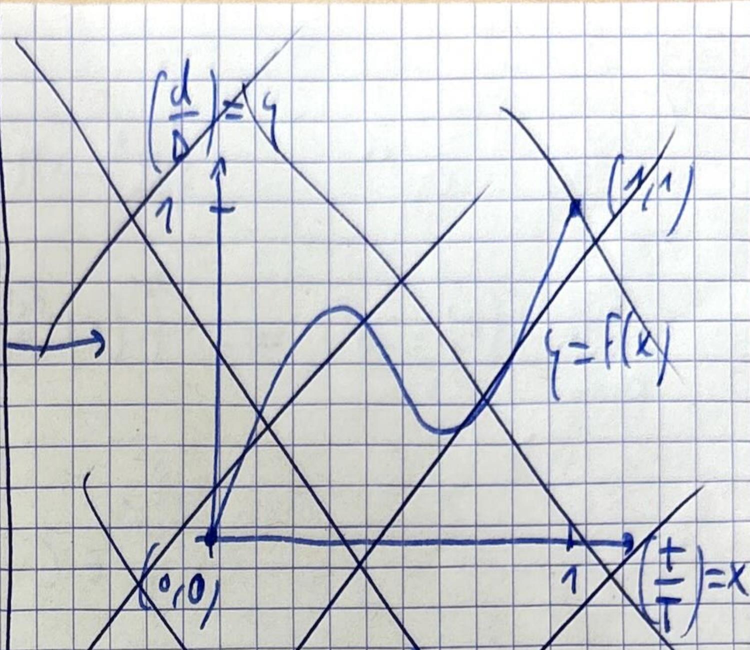
Définir $g : \left[0; \frac{1}{2}\right] \to \mathbb{R}$ comme $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$.

$$\left.\begin{array}{l} g \text{ continue} \\ g(0) = f(0) - f\left(\frac{1}{2}\right) \\ g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) - f(1) \underset{f(0) = f(1)}{=} -g(0) \end{array}\right\} \exists\, x_0 \in [0;1] \text{ t.q. } g(x_0) = 0$$

$$f(x_0) = f\left(x_0 + \frac{1}{2}\right)$$

2.2) Soit $T, D > 0$.





$g : [0;T] \to [0;D]$ continue

$g(0) = 0 \ , \ g(T) = D$

f cont.

$f(0) = 0 \ , \ f(1) = 1$

$$\boxed{g(t) = D \cdot f\left(\frac{t}{T}\right)}$$

$$\boxed{g\left(t_0 + \frac{T}{2}\right) - g(t_0) = \frac{D}{2}}$$

$h(t) = g\left(t + \frac{T}{2}\right) - g(t) - \frac{D}{2}$ définie en $\left[0, \frac{T}{2}\right]$

h continue

$$\left.\begin{aligned} h(0) &= g\left(\tfrac{T}{2}\right) - g(0) - \tfrac{D}{2} = g\left(\tfrac{T}{2}\right) - \tfrac{D}{2} \\ h\left(\tfrac{T}{2}\right) &= g(T) - g\left(\tfrac{T}{2}\right) - \tfrac{D}{2} = \tfrac{D}{2} - g\left(\tfrac{T}{2}\right) = -h(0) \end{aligned}\right\} \ \exists\, t_0 \in \left[0; \tfrac{T}{2}\right] \text{ t.q. } h(t_0) = 0$$

3)

$f(x) = e^{-\frac{1}{x^2}}$ $\quad f : \mathbb{R}^* \longrightarrow \mathbb{R}$ $\quad$ clairement $f \in C^0(\mathbb{R}^*)$

$\lim_{x \to 0} f(x) = 0 \implies \tilde{f}(x) \overset{\text{déf.}}{=} \begin{cases} f(x) & \text{si } x \in \mathbb{R}^* \\ 0 & \text{si } x = 0 \end{cases}$

$\tilde{f} \in C^0(\mathbb{R})$

$f(x) = \left(x\,|\ln(x)| + 1\right)^{\ln(x)}$ $\quad f : \mathbb{R}_+^* \longrightarrow \mathbb{R}$

clairement $f \in C^0(\mathbb{R}_+^*)$

$\ln\left(\lim_{x \to 0^+} f(x)\right) = \ln(\ell)$

$\lim_{x \to 0^+} \left[\ln(x) \ln\left(x\,|\ln(x)| + 1\right)\right] = \ln(\ell)$

$\lim_{x \to 0^+} \ln\left[x\left(x \ln(x) + 1\right)\right] = \ln(\ell)$

$\ln(\ell) \overset{\text{B.H.}}{=} \lim_{x \to 0^+} \dfrac{\dfrac{|\ln(x)| + \frac{\cancel{x}}{\ln(x)} \cdot \frac{1}{\cancel{x}}}{x\,|\ln(x)| + 1}}{-\dfrac{1}{\ln(x)^2} \cdot \dfrac{1}{x}}$

$\ln(\ell) = \lim_{x \to 0^+} - \dfrac{(\ln(x) + 1)\, x \ln(x)^2}{x \ln(x) + 1}$

$$\ln(\ell) = \lim_{y\to+\infty} -y \ln\left[e^{-y}|-y|+1\right]$$

$$e^{-y} = x$$

$$= \lim_{y\to+\infty} -\frac{\ln\left(e^{-y}y+1\right)}{\frac{1}{y}} \overset{\text{D.H.}}{=} \lim_{y\to+\infty} \frac{\frac{1}{e^{-y}y+1}\left(-e^{-y}y+e^{-y}\right)}{\frac{1}{y^2}}$$

$$= \lim_{y\to+\infty} \frac{\cancel{e^{-y}}(1-y)y^2}{\cancel{e^{-y}}(y+e^{y})} = 0 = \ln(\ell) \implies \ell = 1$$

$$\tilde{f}(x) = \begin{cases} f(x) & \text{si } x>0 \\ 1 & \text{si } x=0 \end{cases} \qquad \tilde{f} \in C^0(\mathbb{R}_+)$$

5

$$g_\alpha(x) = x^\alpha$$

$$g_{\alpha\in\mathbb{N}^*} : \mathbb{R} \to \mathbb{R}$$

$$g_0 : \mathbb{R}^* \to \mathbb{R} \qquad x^0 = x^{n-n} = \frac{x^n}{x^n}$$

$$g_{\alpha\in\mathbb{Z}^*_-} : \mathbb{R}^* \to \mathbb{R} \qquad x^{-n} = \frac{1}{x^n}$$

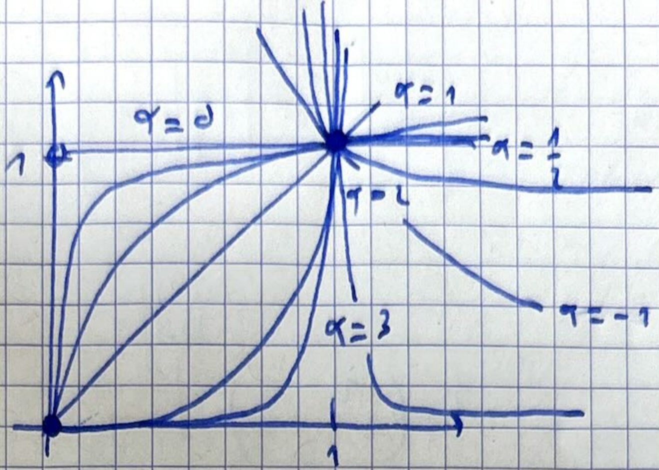
$$x^{\pm p/q} = \sqrt[q]{x^{\pm p}} = \left(\sqrt[q]{x}\right)^{\pm p}$$

5

4)

$$g_\alpha(x) = x^\alpha$$

$$g : \mathbb{R}_+^* \longrightarrow \mathbb{R}$$



$$x^{\frac{1}{2}} = \sqrt{x} \quad \text{vs.} \quad x^{\frac{1}{3}} = \sqrt[3]{x}$$

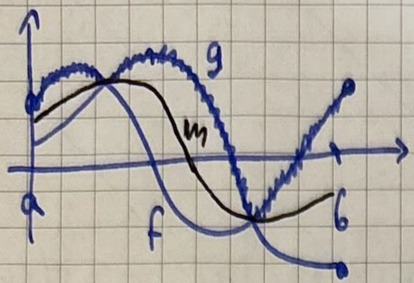
$$\boxed{\cancel{x^\alpha \sin\left(\frac{1}{x}\right)}}$$

Si $\alpha \leq 0$: $\lim\limits_{x \to 0^+} x^\alpha \sin\left(\frac{1}{x}\right)$ <u>n'existe pas</u>

Si $\boxed{\alpha > 0}$: $\lim\limits_{x \to 0^+} x^\alpha \sin\left(\frac{1}{x}\right) = 0$

On a alors $\tilde{g}(x) = \begin{cases} g(x) & \text{si } x \in \mathbb{R}_+^* \\ 0 & \text{si } x = 0 \end{cases}$

5)



$I = [a;b]$

$$m(x) = \frac{f(x)+g(x)}{2}$$

$$\left.\begin{aligned} \max(f,g)(x) &= m(x) + \frac{|f(x)-g(x)|}{2} \\ \min(f,g)(x) &= m(x) - \frac{|f(x)-g(x)|}{2} \end{aligned}\right\} \in C^0(I)$$

---

6)

$[a;b] \ni c$ point fixe de $f \iff f(c) = c$

$$g(x) \overset{\text{def}}{=} f(x) - x \implies g(c) = 0$$

6.1) $f([a;b]) = \underline{\mathrm{Im}_f \subset [a;b]} \implies a < f(x) < b \quad \forall x \in I$

f est <u>bornée</u>!

$$g(a) = f(a) - a > 0$$

$$g(b) = f(b) - b < 0$$

$$g \in C^0(I) \implies \forall y \in [g(b); g(a)] \ \exists c \in I \text{ t.q. } y = g(c)$$

Choisir $y = 0$.

~~6.2)~~

$\cancel{[a;b] \subset f([a;b]) = Im_f}$

$\cancel{f \text{ continue} \Longrightarrow \forall y \in Im_f \ \exists x \in [a;b] \text{ t.q. } y = f(x)}$

$\cancel{\text{Choisir } y \in \underline{[a;b] \subset Im_f}}$

6.2)

$[a;b] \subset f([a;b]) = Im_f$

$f \text{ continue} \overset{\text{Weierstrass}}{\Longrightarrow} \exists \alpha, \beta \in [a;b] \text{ t.q. } \forall x \in [a;b] : f(\alpha) \leq f(x) \leq f(\beta)$

$\Longrightarrow Im_f \text{ intervalle} \Longrightarrow Im_f = [f(\alpha); f(\beta)]$

$[a;b] \subset [f(\alpha); f(\beta)] \Longrightarrow f(\alpha) < a < b < f(\beta)$

$\alpha, \beta \in [a;b] \Longrightarrow a \leq \alpha \leq b$

et $a \leq \beta \leq b$

$g(\alpha) = f(\alpha) - \alpha < a - \alpha \leq 0$

$g(\beta) = f(\beta) - \beta > b - \beta \geq 0$

$g \in C^0(I) \Longrightarrow \exists c \in I \text{ t.q. } g(c) = 0$

7)

- f est <u>injective</u> car, pour $x, y \in \mathbb{R}$,

$$x \neq y \Rightarrow |x-y| > 0 \Rightarrow |f(x)-f(y)| > 0$$

(car $a|x-y|>0$)

$$\Rightarrow f(x) \neq f(y)$$

f <u>cont.</u> ⊕ f inj. $\Rightarrow$ f <u>strictement monotone</u>

$\left.\begin{array}{l}\bullet\ f \text{ continue} \\ \bullet\ f \text{ strictement monotone}\end{array}\right\} \Rightarrow$ <u>f bij.</u>