The Elastic Pendulum

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Abstract

In an Elastic Pendulum system with damping and driving force, the initial conditions play a large role in determining the nature of the time evolution of the system. This behavior is nonlinear and can be analyzed by looking at the effects of specific ranges of initial conditions as well as the nature of damping, driving, and gravitational forces at play. Using RK4 and the Linear Multistep (Adams Bashforth), the ranges of the variables where unexpected results occurred were analyzed and compared across the 2 numerical methods.

Introduction

To find the relevant second order differential equations for this system, Lagrangian mechanics were used to derive the equations. From there, non dimensionalization was applied with variables as in Figure 1.1

$$\tau = \frac{t}{T}, T = \sqrt{\frac{m}{k}} \text{ and } \hat{r} = \frac{r}{l}$$

Figure 1.1

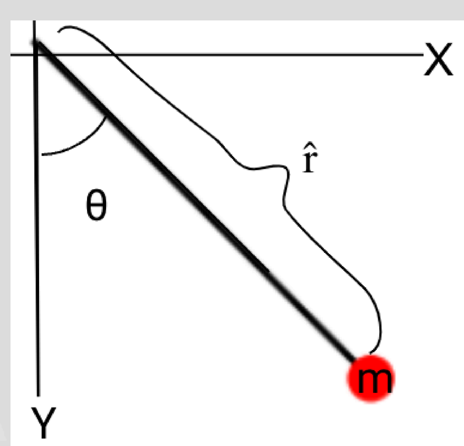
From here, damping and driving forces were added accordingly to produce the equations:

$$\frac{d^2\hat{r}}{d\tau^2} = \hat{r}\left(\frac{d\theta}{d\tau}\right)^2 + (1-\hat{r}) + \gamma cos(\theta) - C_2\frac{d\hat{r}}{d\tau} + F_o cos(\omega_{\hat{r}}\tau)$$

$$\frac{d^2\theta}{d\tau^2} = -(\gamma)\frac{1}{\hat{r}} sin(\theta) - \frac{2}{\hat{r}}\theta' \frac{d\hat{r}}{d\tau} - C_1\frac{d\theta}{d\tau} + F_o cos(\omega_\theta \tau)$$

Figure 1.3

To understand these equations refer to Figure 1.2 displaying the relevance of each of the variables.



$\gamma = \sqrt{\frac{gm}{kl}}$

C_1 = amplitude of damping (angular)

C_2 = amplitude of damping (radial)

F_o = amplitude of driving (angular and radial)

ω_θ = frequency of angular driving

ω_r = frequency of radial driving

Figure 1.3

From here, each variables effect on the time evolution of the pendulum was anlyzed. In the results, the goal was to find ranges of initial conditions, mainly initial angular velocity to figure out where these unique nonlinear results occur. From those results the efficacy of each of the numerical methods was compared in hopes of understanding their strength and weaknesses.

Results

The first variable that caused strange results in my analysis was gamma. In Figures 2.1, 2.2, 2.3, theres no driving or damping and the spring starts at its natural length with θ = ⊓ / 4 and with no initial velocity.

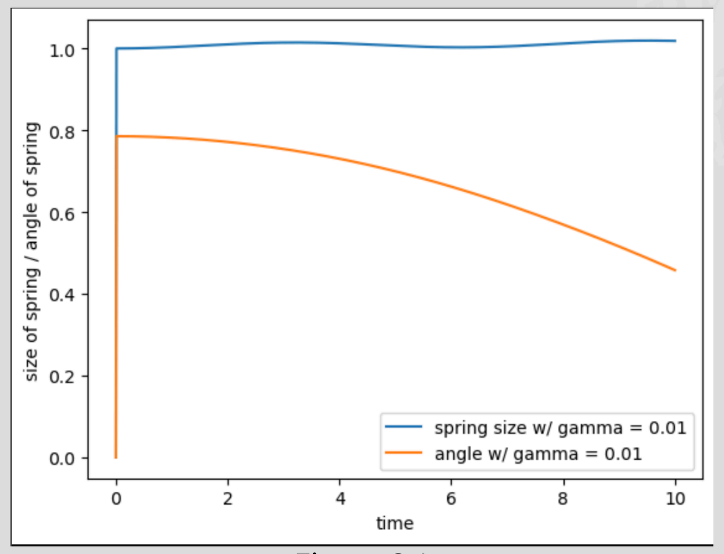


Figure 2.1

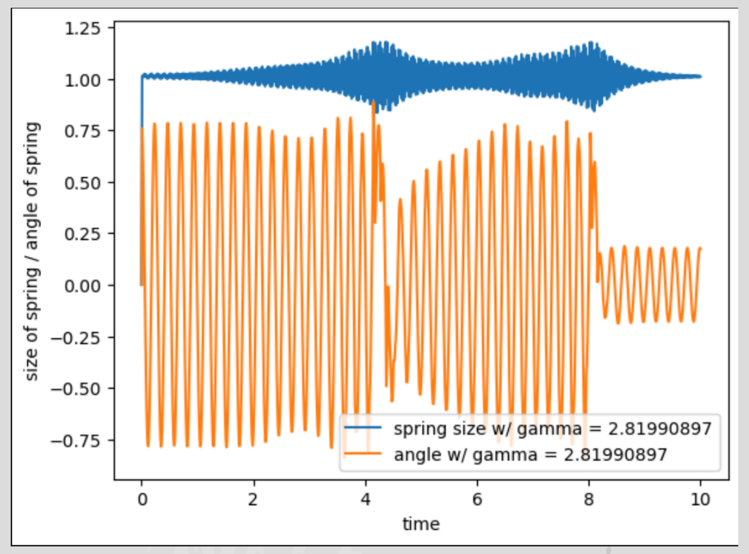


Figure 2.2

From figure 2.1 and 2.2 its important to notice that both the angle and length of the spring are periodic, until gamma approaches 2.81990897, after which, both length and angle explode to >>10e50.

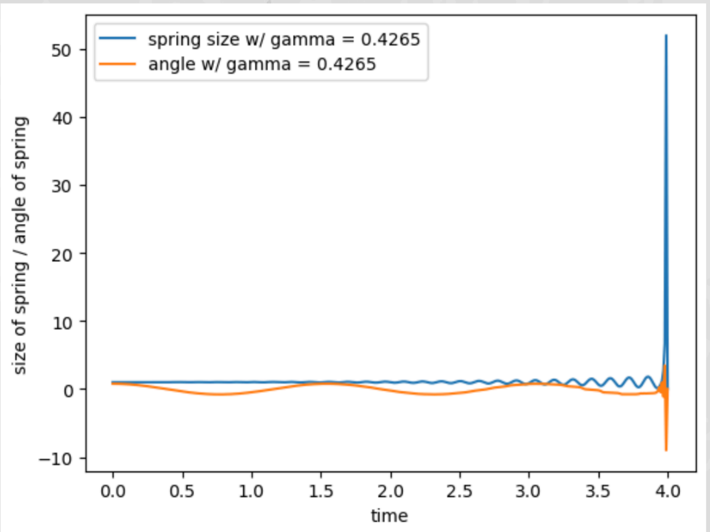


Figure 2.3 adams bashforth

In Figure 2.3, the Adams Bashforth method is used and a much more "realistic" time evolution is present. It is more realistic of a time evolution because when γ the spring is stiff, gravity is powerful and the mass is small, hence the size of the spring should blow up rather than experience chaos before blowing up. This finding leads to the result that, in high energy situations, Adams-Bashforth proves more effective hence it is used from here on.

The next interesting result gathered was the effect of angular driving forces. In figures 3.1, 3.2, 3.3, and 3.4, the initial conditions in these cases are: length of the spring is 1 whereas, angle, and velocity are 0. The gamma value used is .01 throughout and there is no damping.

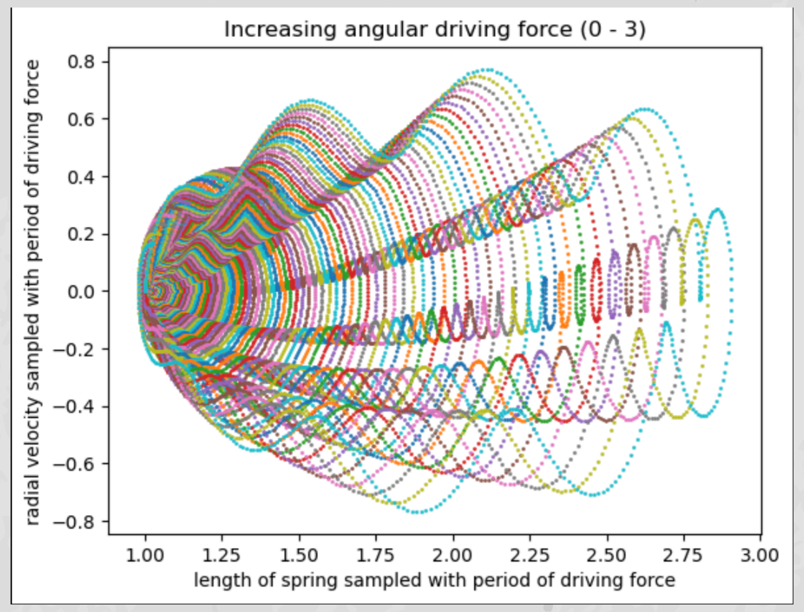


Figure 3.1

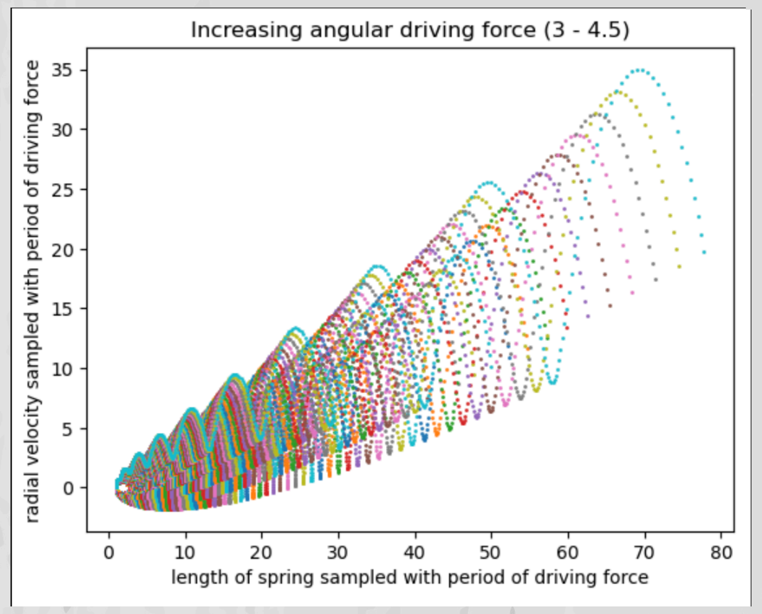


Figure 3.2

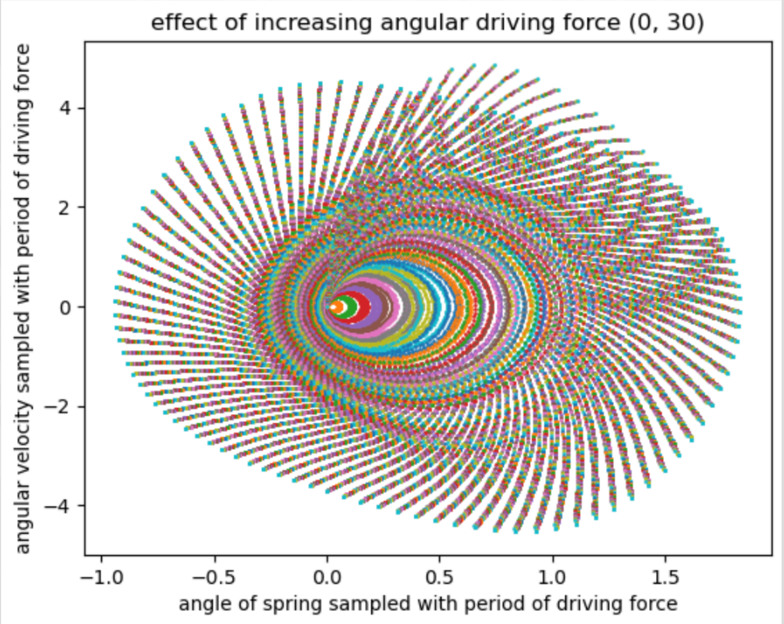


Figure 3.3

In Figure 3.1, as the amplitude of the angular driving force increases, the absolute value of the distance from (1, 0) increases exponentially. It increases exponentially because in Figure 3.2 once the driving force passes ~3, both length and radial velocity begin to soar into large values. In Figure 3.3, as the driving force increases, the angle of the spring experiences some phase shift and the angular velocities' absolute value begins to increase.

The Last analysis of initial conditions is the effect of angular momentum on the time evolution and final position of the spring. For Figures 4.1, 4.2, and 4.3 the amplitude of the angular/radial driving force is set to 1 with angular/radial driving frequency ⊓, and length of the spring 1 (natural length) whereas, angle, and velocity are 0. The gamma value used is .01 throughout and there is no damping.

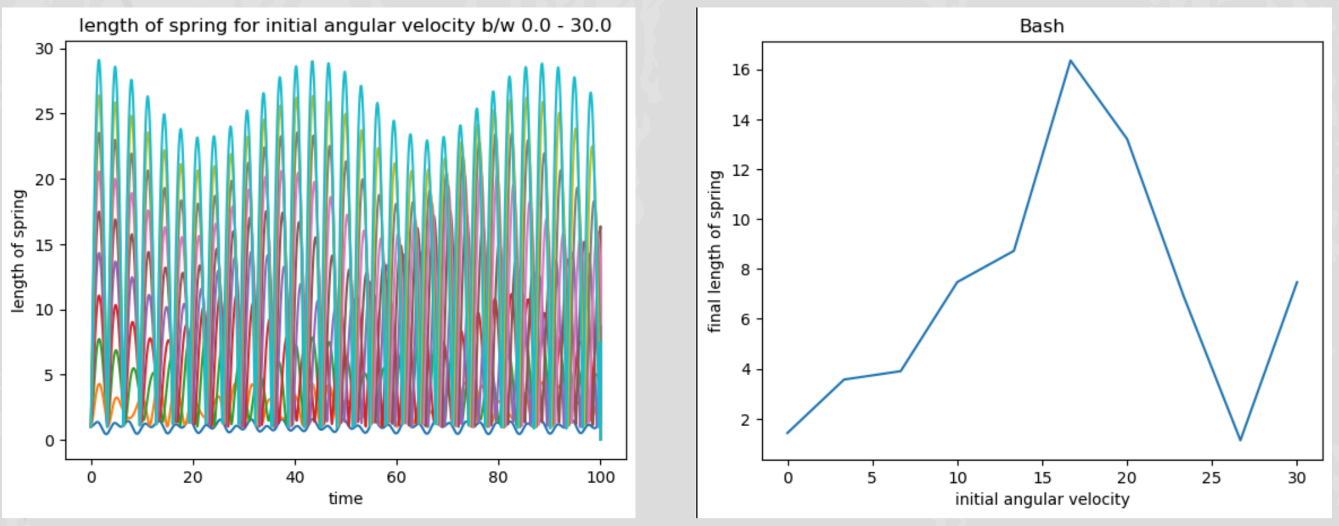


Figure 4.1 : Radial Driving Force

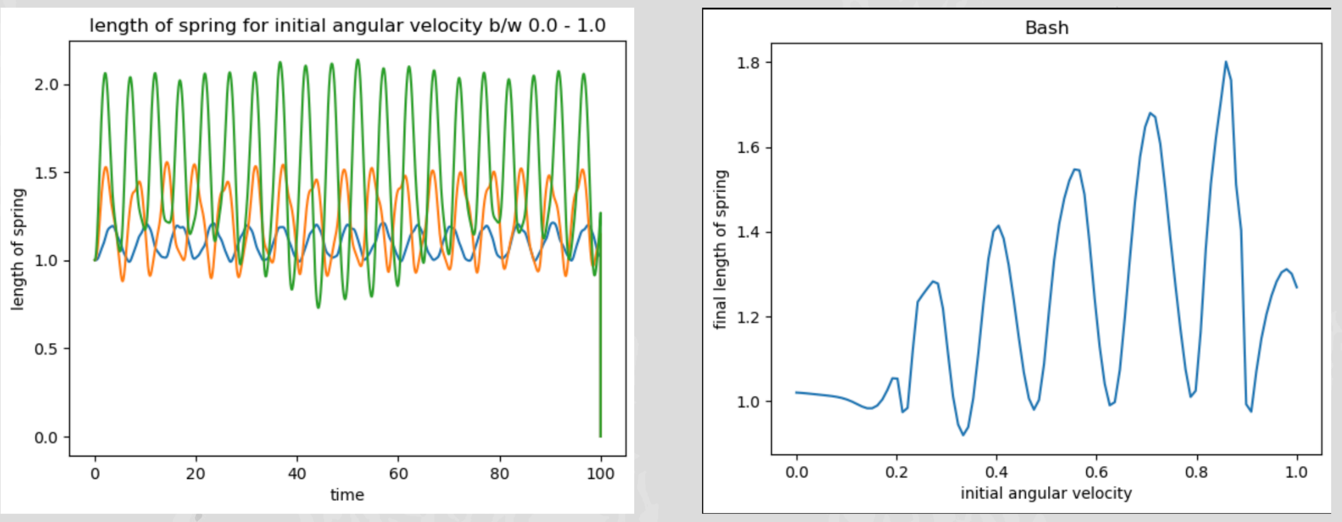


Figure 4.2 : Angular Driving Force

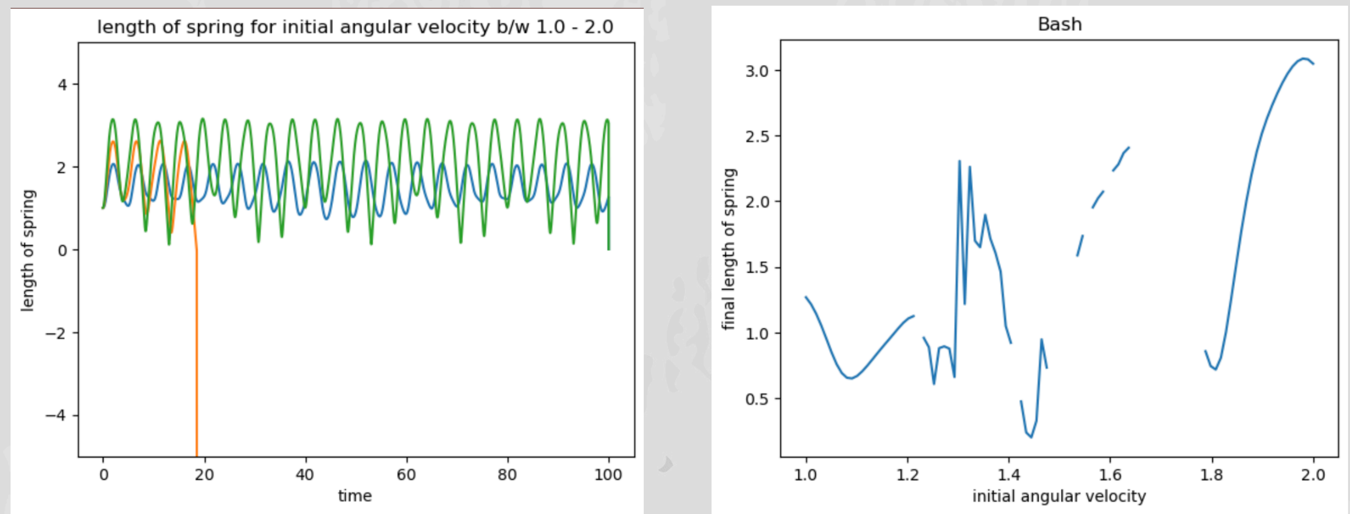


Figure 4.3 : Angular Driving Force

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