Final Project Proposal

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1 Proposal

1. The problem I am interested is the elastic pendulum. I will analyze and understand the role of the parameters δ (controlling the amount of damping), α (controlling the stiffness of the spring), β (controls the amount of non linearity in the restoring force), γ (controls the amplitude of the periodic restoring force), and ω (controls the angular frequency of the periodic driving force) as they apply to each unique scenario of the elastic pendulum system.

   This problem is interesting for a handful of reasons. One reason why this problem interests me is because it serves as an extension in some ways, to systems that we have analyzed previously (pendulum and even the SHO). Another interesting factor is the chaotic nature of the system. This becomes very interesting when we identify the parameter ranges where chaos occurs. Going further we can calculate Lyapunov exponents to quantify the degree of chaos. As system parameters change, the elastic pendulum can also undergo bifurcations. This becomes even more interesting once we observe transitions between different dynamical regimes in bifurcation diagrams. Lastly, this system is interesting because of the way it compares to simple systems that we have analyzed in class and physical examples in engineering (e.g crane operations) and geophysics (e.g modeling of seismic waves).

2. In order to turn this system into a form that is suitable to be plugged in and analyzed by the computer we have to follow a handful of steps. Its important to not that the workflow here can be modified to include other things like drag and friction.
   To start we can use Lagrangian mechanics to analyze the system. The Lagrangian is equal to the kinetic energy minus to Potential energy.

$$L = K - U$$

   Using this equation, we can follow up by using the Euler-Lagrange equation to find the equations of motion.

$$\frac{\delta L}{\delta q} - \frac{d}{dt}(\frac{\delta L}{\delta q'}) = 0$$

   Note that q is a position coordinate and q' is its first derivative with respect to time.
   We know that the kinetic energy of the system is:

$$K = \frac{1}{2}m(r'(t)^2) + \frac{1}{2}m(r(t)^2)(\theta'(t)^2)$$

   Note that r(t) is the length of the spring as a function of time, and $\theta(t)$ is the angle that the spring makes with the y axis as a function of time.

$$U = \frac{1}{2}k(l - r)^2 - mgrcos(\theta)$$

   Note that l is the length of our spring at equilibrium and k is the spring constant. Another important note is that the system has gravitational potential energy and elastic potential energy. Taking these equations, we can now plug them back into Lagrangian.

$$L = \frac{1}{2}m(r'^2) + \frac{1}{2}m(r^2)(\theta'(t)^2) - \frac{1}{2}k(l - r)^2 + mgrcos(\theta)$$

Next, taking derivatives allows us to plug into the Euler Lagrange equation to find the equations of motion.

$$\frac{\delta L}{\delta r} = mr(\theta')^2 + k(l - r) + mgcos(\theta)$$

$$\frac{\delta L}{\delta r'} = mr'$$

$$\frac{d}{dt}(\frac{\delta L}{\delta r'}) = mr''$$

Next, we can take the derivative of the Langrangian with respect to θ

$$\frac{\delta L}{\delta \theta} = -mgrsin(\theta)$$

$$\frac{\delta L}{\delta \theta'} = m(r)^2\theta'$$

$$\frac{d}{dt}(\frac{\delta L}{\delta \theta'}) = mrr'\theta' + m(r)^2\theta''$$

Lastly, We can use both of these derivatives to plug into our original Euler Lagrange equation

$$mr(\theta')^2 + k(l - r) + mgcos(\theta) - mr'' = 0$$

$$-mgrsin(\theta) - mr(t)r'\theta' - mr^2\theta'' = 0$$

If we simply this so that we have the second derivatives on the left, we are left with:

$$\theta'' = \frac{-g}{r}sin(\theta) - \frac{2r'}{r}\theta$$

$$r'' = r\theta'^2 + \frac{k}{m}(l - r) + gcos(\theta)$$

From this point, we can manipulate the second order differential equation however we may need based on our numerical method of choice. We can also note that this equation falls into the form of the Duffing equation which is why we defined the parameters the way we did in the introduction to this problem.

3. This problem is appropriate for numerical analysis because despite its many differences, the form of the second order differential equations is similar to ones we have analyzed in class.

   In terms of what numerical method I aim to use to analyze this system, there are a few candidates which we can analyze the system with and compare to each other. Within the candidates, a clear distinction has to be made between methods that require initial conditions and methods that do not. In my case, I will use initial conditions for the methods that require them.
   Firstly as a baseline, I will use the Runge Kutta method which we have used in class as a control result to compare the others to. Secondly, I plan to use the Linear Multistep method (Adams-Bashforth-Moulton method). In using the Linear Multistep method I plan to test its efficacy against the Runge Kutta method because the two are somewhat similar. As the second new numerical method, I plan to use the Fourier Spectral Method which I will also compare to the other methods. As the third and final new numerical method, I will use the Frobenius method which to my knowledge is uncommonly used for systems like mine but is a general solution technique for the Duffing Equation. Lastly, I hope to test out two more numerical methods and depending on their efficacy I will either replace the other methods with or just keep as a sidenote. Those two methods are the Harmonic Balance method and the Finite Difference method.

4. I plan to write the code for the Runge Kutta method for this specific system in the next week to confirm that the form of the system agrees with the method which I already know and have previously used. Along with that, I plan to also write the code for the Linear Multistep method and compare the results from my Runge Kutta analysis to make sure the two results are in some sort of agreement.

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