Contents

1 Some facts about the system

Consider the dynamical system

$$\dot{x_i} = x_i \left(\prod_{j=1}^{N} \left(\frac{x_j}{x_i} \right)^{v_{ij}} - \prod_{j=1}^{N} \left(\frac{x_j}{x_i} \right)^{w_{ij}} \right), \qquad i = 1, \ldots, N, \tag{1}$$

$$x_i(0) = x_{i0}, \quad - \quad \text{initial conditions.}$$

In the case when $\mathbf{v} = \{v_{ij}\}_{i,j=1,\ldots,N}$ and $\mathbf{w} = \{w_{ij}\}_{i,j=1,\ldots,N}$ are stochastic matrices, meaning $\sum_{j=1}^{N} v_{ij} = \sum_{j=1}^{N} w_{ij} = 1$ the system (1) becomes

$$\dot{x_i} = \prod_{j=1}^{N} (x_j)^{v_{ij}} - \prod_{j=1}^{N} (x_j)^{w_{ij}}, \qquad i = 1, \ldots, N. \tag{2}$$

The above dynamical system (2) has an equilibrium when

$$\prod_{j=1}^{N} (x_j)^{v_{ij}} - \prod_{j=1}^{N} (x_j)^{w_{ij}} = 0, \qquad i = 1, \ldots, N. \tag{3}$$

Taking the logarithm of (3) and denoting $y_i = \lg x_i$, $\Delta_{ij} = v_{ij} - w_{ij}$ the system (3) transforms to the following

$$\sum_{j=1}^{N} \Delta_{ij} y_j = 0, \qquad i = 1, \ldots, N. \tag{4}$$

Therefore, (3) holds if (4) holds. Clearly, (4) holds for $y = (y_1, y_2, \ldots, y_N) = (0, 0, \ldots, 0)$, i.e. the sytem (2) has an equilibrium in $x = (x_1, x_2, \ldots, x_N) = (1, 1, \ldots, 1)$. Other equilibrium points are only possible when the matrix $\mathbf{\Delta} = \{\Delta_{ij}\}_{i,j=1,\ldots,N}$ is singular. Which is true when matrices $\mathbf{v}$ and $\mathbf{w}$ are stochastic, since adding up all the columns in $\mathbf{\Delta}$ will result in a zero column. Note that for a singular matrix $\mathbf{\Delta}$ there are infinitely many equilibriums, because there are infinitely many solutions of the system (4). Let's find those equilibriums. Apart of $x = (0, 0, \ldots, 0)$ and $x = (1, 1, \ldots, 1)$ (3) holds for $x_1 = x_2 = \ldots = x_N$ because then (3) become

$$x^1 - x^1 = 0, \qquad i = 1, \ldots, N.$$

where $x = x_i$, $\forall i = 1, \ldots, N$.

Therefore, the equilibriums of the system (2) constitute a susbpace $U_e \subset \mathbb{R}^N$, $U_e = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N | x_1 = x_2 = \ldots = x_N = x^e \in \mathbb{R}\}$.

Let's look at the nature of those equilibriums. The linearization of the system around a poin $x \in U_e$ has the form

$$\dot{\delta x} = \Delta f(x)\delta x,$$

where $\Delta f(x) = \dfrac{\partial f_i(x)}{\partial x_j}$, $i, j = 1, 2, \ldots, N$.

$$\frac{\partial f_i(x)}{\partial x_j} = v_{ij} x^{v_{ij}-1} \prod_{k=1, k\neq j}^{N} x^{v_{ik}} - w_{ij} x^{w_{ij}-1} \prod_{k=1, k\neq j}^{N} x^{w_{ik}} =$$

$$v_{ij} x^{\sum_{k=1}^{N} v_{ik}-1} - w_{ij} x^{\sum_{k=1}^{N} w_{ik}-1} = v_{ij} - w_{ij} = \Delta_{ij},$$

therefore, $\Delta f(x) = \mathbf{\Delta}$ for all $x \in U_e$.

Note that the matrix $\mathbf{\Delta}$ has always a zero eignvalue, because $|\mathbf{\Delta}| = 0$ therefore, the linearization fails to determine the stability of the equilibrium point. We can only tell the unstability of the equilibrium i.e. when there exists an eigenvalue of $\mathbf{\Delta}$ with the real larger than 0. Nevertheless, in our simulations whenever $Re\lambda_i \leq 0$ for all i and for some i $Re\lambda_i = 0$ the solutions seems always to converge asymptotically to some equilibrium in U_e (see Example 1). The "purely "stable equilibrium, wich is not asymptotically stable was observed when all the eigenvalues of $\mathbf{\Delta}$ had zero real parts (see Example 3).

2 Numerical observations

2.1 Example 1. Converging pattern

Consider the case $N = 3$, the time horizon $T = 100$,

$$\mathbf{v} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix},$$

$$\boldsymbol{\Delta} = \begin{pmatrix} -0.5 & 0 & 0.5 \\ 0.5 & -0.5 & 0 \\ 0 & 0.5 & -0.5 \end{pmatrix}.$$

The real parts of egenvalues of $\boldsymbol{\Delta}$ are less or equal zero, namely $Re\lambda_1 = -0.75$, $Re\lambda_2 = -0.75$, and $Re\lambda_3 = 0$. Let us have a look at the graph of the solution with the initial conditions $x_0 = (0.3 \quad 0.6 \quad 0.9)$.

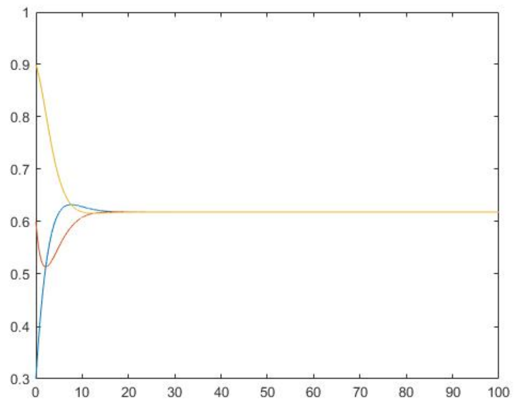

Figure 1: $x_0 = (0.3 \quad 0.6 \quad 0.9)$, $x^e = 0.6184178227652$

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The equilibrium $U_e \ni x^e = 0.6184178227652$.

If we change the initial conditions slightly the equilibrium point also changes. For example for the initial conditions $x_0 = (0.35 \quad 0.6 \quad 0.9)$ the solutions converge to a different equilibrium $U_e \ni x^e = 0.635643409191$ In (Fig. 2) the

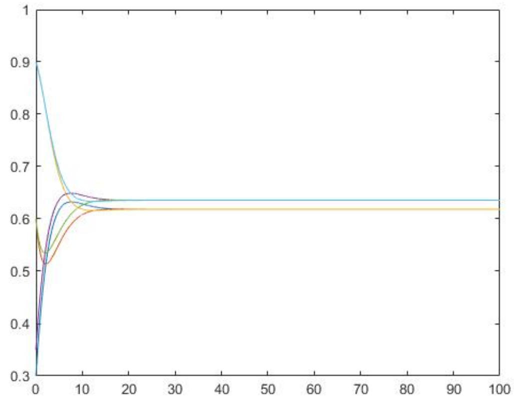

Figure 2: $x_0 = (0.35 \quad 0.6 \quad 0.9)$, $x^e = 0.635643409191$

two solutions are plotted together for comparison. In this case it looks like the solutions converge assymptotically.

For the four dimensional case the same behaviour was observed.

2.2 Example 2. Diverging pattern and the cutoff

In case $\mathbf{\Delta}$ has eigenvalues with positive real parts the equilibriums U_e are not stable and the solutions diverge. Consider for example $N = 3$, $T = 10$ and the following matrices

$$\mathbf{v} = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix},$$

$$\mathbf{\Delta} = \begin{pmatrix} 0 & -0.5 & 0.5 \\ 0 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0 \end{pmatrix}.$$

The real parts of egenvalues of $\mathbf{\Delta}$ are $Re\lambda_1 = 0.25$, $Re\lambda_2 = 0.25$, and $Re\lambda_3 = 0$ therefore, the equilibriums U_e are not stable. The solutions for the initial condition $x_0 = (0.3 \quad 0.6 \quad 0.9)$ can be seen on the figure below.

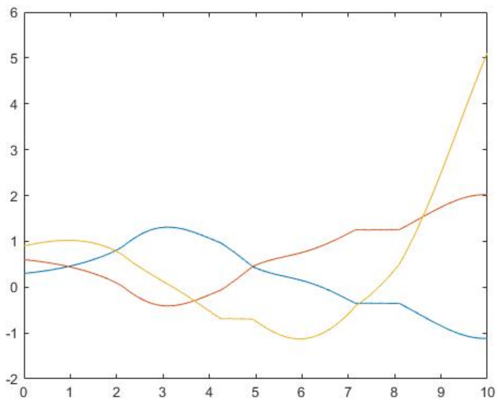

Figure 3: Solutions diverge

At this point it is usefull to consider the cutoff factor $\phi(x) = x(x_{max} - x)$ in (1) to make the solutions stay within the required corridor. The solutions with the cutoff for $T = 100$, $x_{max} = 1$ are plotted at the following figure

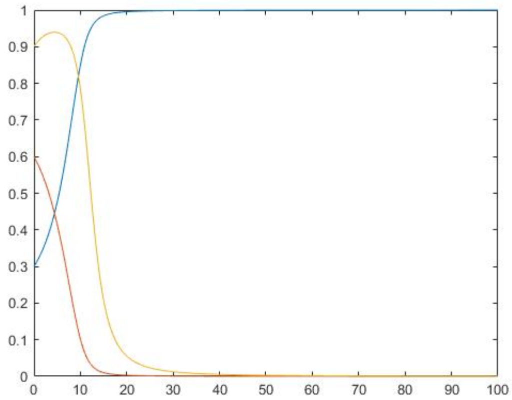

Figure 4: Solutions remain within the [0, 1] corridor.

2.3 Example 3. Oscilating pattern

For $N = 3$, $T = 100$ and the matrices

$$\mathbf{v} = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix},$$

$$\boldsymbol{\Delta} = \begin{pmatrix} 0 & -0.5 & 0.5 \\ 0.5 & 0 & -0.5 \\ -0.5 & 0.5 & 0 \end{pmatrix}$$

all the real parts of eigenvalues of $\boldsymbol{\Delta}$ are zeros. The solutions remain bounded within some corridor and oscilate infinitely, which exibits a stable behaviour of the equilibrium. The solutions for the initial condition $x_0 = (0.3 \quad 0.6 \quad 0.9)$ can be seen on the figure below.

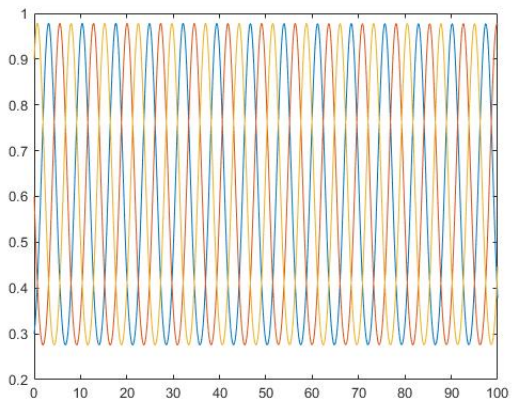

Figure 5: Oscilatory behavior

2.4 Example 4. Diverging-converging pattern

We can also consider matrices $\mathbf{v}$ and $\mathbf{w}$ that vary with time. For example we can construct them so that they switch with time from “bad ”to “good ”matrices. By “bad ”$\mathbf{v}$ and $\mathbf{w}$ we mean such matrices that $\mathbf{\Delta}$ has positive real parts of eignvalues, and “good ”matrices are such that $\mathbf{\Delta}$ has all real parts of eigenvalues less or equal zero.

Consider the following matrices $\mathbf{v}(t) = \lambda(t)\mathbf{v_g} + (1 - \lambda(t))\mathbf{v_b}$ and $\mathbf{w}(t) = \lambda(t)\mathbf{w_g} + (1 - \lambda(t))\mathbf{w_b}$, where $\mathbf{v}_g$, $\mathbf{w}_g$, and $\mathbf{v}_b$, $\mathbf{w}_b$ are good and bad matrices respectively,

$$\lambda(t) = \frac{t}{\gamma T}, \qquad 0 < t \leq T, \quad 0 < \gamma < 1.$$

Then, we have that the bad matrices switch smoothly to the good matrices in time γT. For example, choose the good matrices from the Example 1 and bad matrices from the Example 2. For $N = 3$, $T = 300$, $x_0 = (0.3 \quad 0.6 \quad 0.9)$ and cutoff term $\phi(x) = x(x_{max} - x)$, $x_{max} = 1$ the solutions are plotted at the following figure

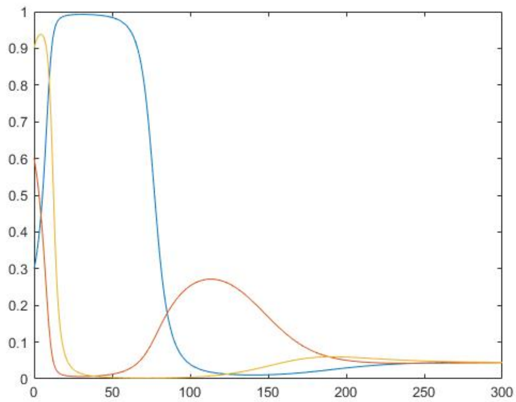

Figure 6: Trajectories switch from diverging to converging.

As matrices switch from bad to good ones the behaviour of the trajectories switches from diverging to converging.

2.5 Example 5. Equilibrium space visualization

To picture how the equilibrium space looks like, consider a two dimensional case $N = 2$, and let us plot the phase portrait of the system (2), for the time horizon $T = 10.$ and

$$\mathbf{v} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix},$$

$$\mathbf{\Delta} = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}.$$

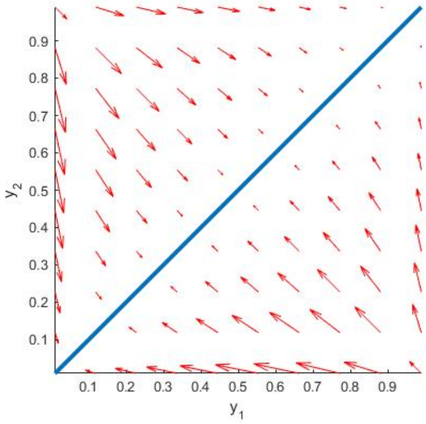

Figure 7: Phase portrait

Here in (Fig. 3) the thick bisectrice is the equilibrium space U_e, and and the red arrows show where the trajectories tend with time. The trajectories along with the phase portrait are plotted on the following (Fig. 4) where the

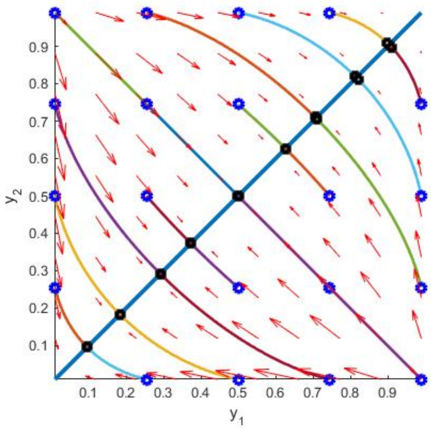

Figure 8: Phase portrait along with the trajedivergectories

blue circles denote the beginning of the trajectories and the dark squares denote the end of the trajectories. The trajectory portrait in the $3-dim$ case in the first example is pictured on the following figure As can be seen all trajectories

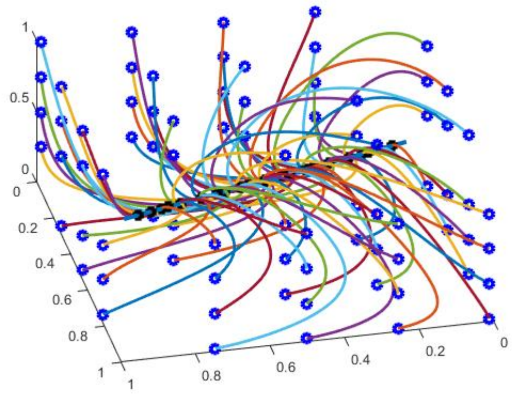

Figure 9: Trajectory portrait.

converge to the bisectrice of the first cuboid.