1 Some facts about the system

Consider the dynamical system

$$\dot{x_i} = x_i\phi(x_i)\left(\prod_{j=1}^{N}\left(\frac{x_j}{x_i}\right)^{v_{ij}} - \prod_{j=1}^{N}\left(\frac{x_j}{x_i}\right)^{w_{ij}}\right), \qquad i = 1,\dots,N, \tag{1}$$

$$x_i(0) = x_{i0}, \quad - \quad \text{initial conditions,}$$

where x – the state, $\phi(x) = \sqrt{x(x_{max} - x)}$.

In the case when $\mathbf{v} = \{v_{ij}\}_{i,j=1,\dots,N}$ and $\mathbf{w} = \{w_{ij}\}_{i,j=1,\dots,N}$ are stochastic matrices, meaning $\sum_{j=1}^{N} v_{ij} = \sum_{j=1}^{N} w_{ij} = 1$ the system (1) becomes

$$\dot{x_i} = \phi(x_i)\left(\prod_{j=1}^{N}(x_j)^{v_{ij}} - \prod_{j=1}^{N}(x_j)^{w_{ij}}\right), \qquad i = 1,\dots,N. \tag{2}$$

The above dynamical system (2) has an equilibrium when

$$\phi(x_i)\left(\prod_{j=1}^{N}(x_j)^{v_{ij}} - \prod_{j=1}^{N}(x_j)^{w_{ij}}\right) = 0, \qquad i = 1,\dots,N. \tag{3}$$

Taking the logarithm of (3) and denoting $y_i = \lg x_i$, $\Delta_{ij} = v_{ij} - w_{ij}$ the system (3) transforms to the following

$$\sum_{j=1}^{N}\Delta_{ij}y_j = 0, \qquad i = 1,\dots,N. \tag{4}$$

Therefore, (3) holds if (4) holds. Clearly, (4) holds for $y = (y_1, y_2, \dots, y_N) = (0, 0, \dots, 0)$, i.e. the sytem (2) has an equilibrium in $x = (x_1, x_2, \dots, x_N) = (1, 1, \dots, 1)$. Other equilibrium points are only possible when the matrix $\mathbf{\Delta} = \{\Delta_{ij}\}_{i,j=1,\dots,N}$ is singular. Which is true when matrices $\mathbf{v}$ and $\mathbf{w}$ are stochastic, since adding up all the columns in $\mathbf{\Delta}$ will result in a zero column. Note that for a singular matrix $\mathbf{\Delta}$ there are infinitely many equilibriums, because there are infinitely many solutions of the system (4). Let's find those equilibriums. Apart of $x = (0, 0, \dots, 0)$ and $x = (1, 1, \dots, 1)$ (3) holds for $x_1 = x_2 = \ldots = x_N$ because then (3) become

$$\phi(x)\left(x^1 - x^1\right) = 0, \qquad i = 1,\dots,N.$$

where $x = x_i$, $\forall i = 1,\dots,N$.

Therefore, the equilibriums of the system (2) constitute a susbpace $U_e \subset \mathbb{R}^N$, $U_e = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N | x_1 = x_2 = \ldots = x_N = x^e \in \mathbb{R}\}$.

Let's look at the nature of those equilibriums. The linearization of the system around a poin $x \in U_e$ has the form

$$\dot{\delta x} = \Delta f(x) \delta x,$$

where $\Delta f(x) = \dfrac{\partial f_i(x)}{\partial x_j}$, $i, j = 1, 2, \ldots, N$.

$$\frac{\partial f_i(x)}{\partial x_j} = v_{ij} x^{v_{ij}-1} \prod_{k=1, k \neq j}^{N} x^{v_{ik}} - w_{ij} x^{w_{ij}-1} \prod_{k=1, k \neq j}^{N} x^{w_{ik}} =$$

$$v_{ij} x^{\sum_{k=1}^{N} v_{ik} - 1} - w_{ij} x^{\sum_{k=1}^{N} w_{ik} - 1} = v_{ij} - w_{ij} = \Delta_{ij},$$

therefore, $\Delta f(x) = \mathbf{\Delta}$ for all $x \in U_e$.

Note that the matrix $\mathbf{\Delta}$ always has a zero eignvalue, because the $|\mathbf{\Delta}| = 0$. This implies that the equilibriums $x \in U_e$ are not asymptotically stable.

2 Numerical observations

2.1 Example 1

Consider the case $N = 3$, the time horizon $T = 100$, $x_{max} = 1$,

$$\mathbf{v} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix},$$

$$\mathbf{w} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix}.$$

Here $\mathbf{v}$ and $\mathbf{w}$ are stochastic, then $|\mathbf{\Delta}| = 0$. Therefore the problem has more than one equilibrium. Let us have a look at the graph of the solution with the initial conditions $x_0 = (0.3 \quad 0.6 \quad 0.9)$.

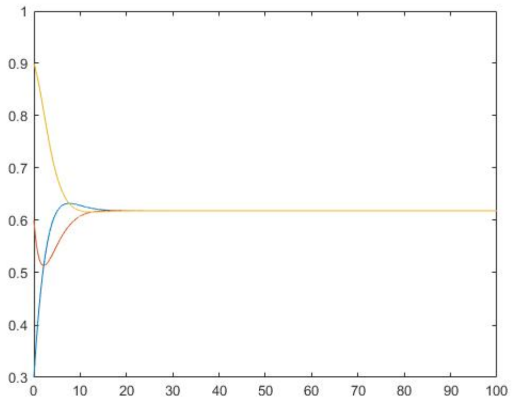

Figure 1: $x_0 = (0.3 \quad 0.6 \quad 0.9)$, $x^e = 0.6184178227652$

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The equilibrium in this case is from the space $U_e \ni x^e = 0.6184178227652$.

If we change the initial conditions slightly the equilibrium point also changes. For example for the initial conditions $x_0 = (0.35 \quad 0.6 \quad 0.9)$ the solutions converge to a different equilibrium $x^e = 0.635643409191$

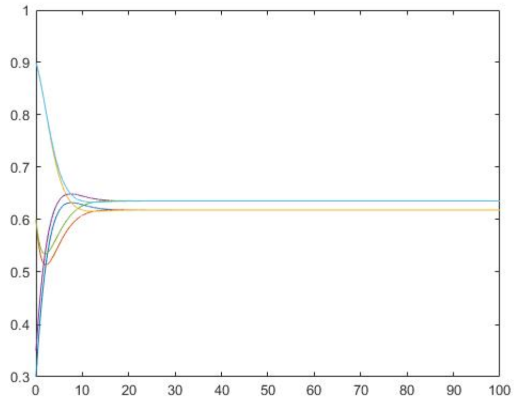

Figure 2: $x_0 = (0.35 \quad 0.6 \quad 0.9)$, $x^e = 0.635643409191$

In (Fig. 2) the two solutions are plotted together for comparison.

2.2 Example 2

To picture how the equilibrium space looks like, consider a two dimensional case $N = 2$, and let us plot the phase portrait of the system (2), for the time horizon $T = 10$, $x_{max} = 1.$ and

$$\mathbf{v} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix},$$

$$\mathbf{w} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

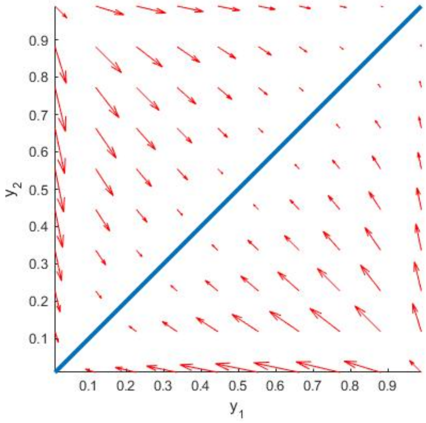

Figure 3: Phase portrait

Here in (Fig. 3) the thick bisectrice is the equilibrium space U_e, and and the red arrows show where the trajectories tend with time. The trajectories along with the phase portrait are plotted on the following (Fig. 4) where the blue

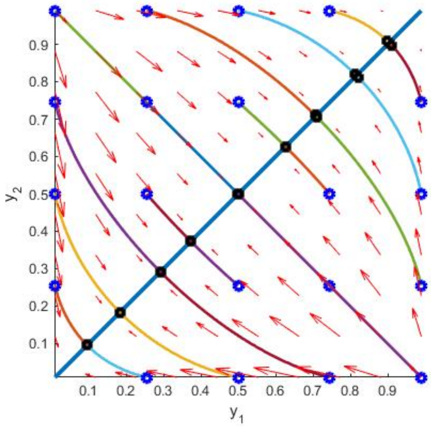

Figure 4: Phase portrait along with the trajectories

circles denote the beginning of the trajectories and the dark squares denote the end of the trajectories.

The trajectory portrait in the $3 - dim$ case in the first example is pictured on the following figure

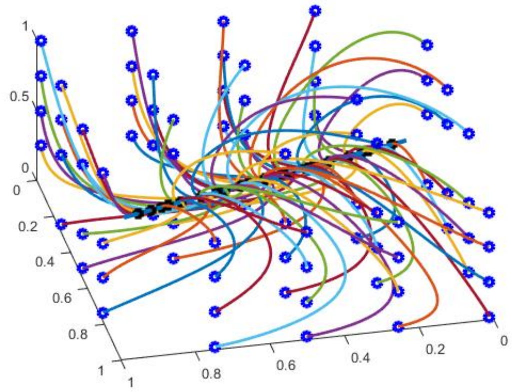

Figure 5: Trajectory portrait.

As can be seen all trajectories converge to the bisectrice of the first cuboid.

2.3 Example 3

The matrices $\mathbf{v}$ and $\mathbf{w}$ can also depend on time. Consider the case $N = 3$, the time horizon $T = 300$, $x_{max} = 1$,

$$\mathbf{v} = \lambda(t) \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix} + (1 - \lambda(t)) \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix}$$

Here $\mathbf{v}$ and $\mathbf{w}$ are stochastic, then $|\mathbf{\Delta}| = 0$. Therefore the problem has more than one equilibrium. Let us have a look at the graph of the solution with the initial conditions $x_0 = (0.3 \quad 0.6 \quad 0.9)$.