CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

Lecture #6: Random Phasor Sums

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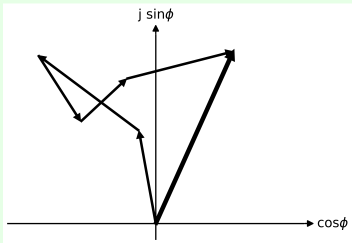
Notes

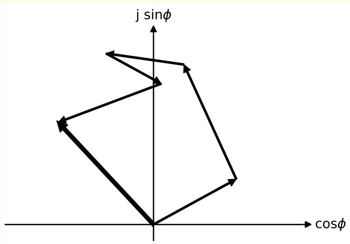
Random walks

Multiple wave phasors can be summed like vectors to find the *resultant* wave phasor at a point in space and time.

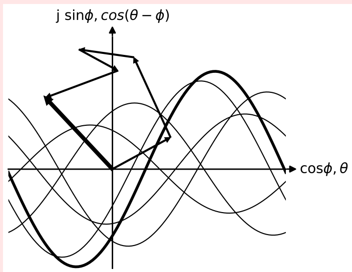
When wave amplitudes are independent and wave phases are independent* their summation is called a "random walk."



Multiple phasors and their resultant sum.



Same amplitudes, different phases, smaller resultant.



Notes

Independence and randomness

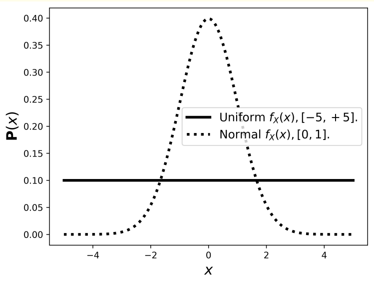
Independence means that one event or value of a random quantity X (e.g. a wave's amplitude) has no effect on another, Y (e.g. its phase,)

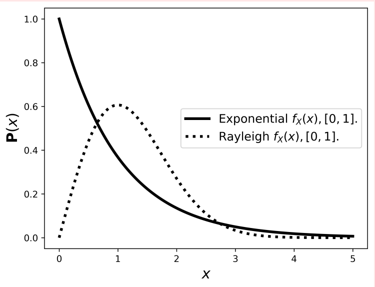
$$\mathbf{P}(X|Y) = \mathbf{P}(X \cap Y)/\mathbf{P}(Y)$$
$$\mathbf{P}(X \cap Y) = \mathbf{P}(X|Y)\mathbf{P}(Y)$$
$$\mathbf{P}(X \cap Y) = \mathbf{P}(X)\mathbf{P}(Y) \text{ i.f.f.}$$
$$\mathbf{P}(X) = \mathbf{P}(X|Y) \text{ and } \mathbf{P}(Y) = \mathbf{P}(Y|X).$$

A random quantity is one whose value depends on the outcome of a random phenomenon.

Its occurance may be known* to follow a particular probability density function f_X, or probability mass function p_X, with discriptive parameters μ, σ, *etc*.

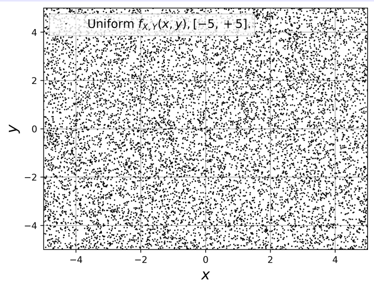
Example PDFs are *Uniform, Normal, Exponential, Poisson, Rayleigh.*

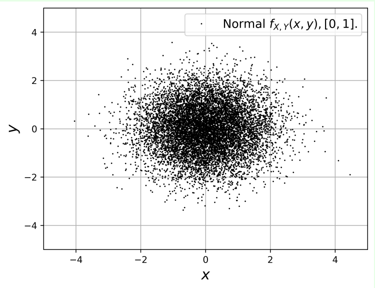


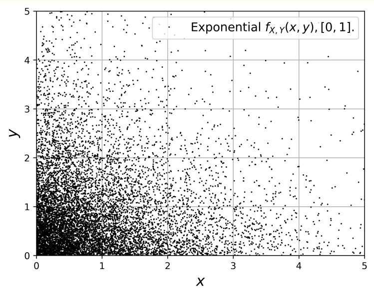


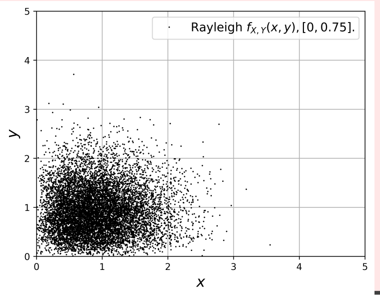
Notes

10,000 values (x, y) chosen randomly









Notes

Descriptive statistics

Expected value or *mean* of a continuous random quantity X with probability density function f_X,

$$\mathbf{E}[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\,\mathrm{d}x.$$

And for a discrete, finite, random quantity X with probability mass function p_X,

$$\mathbf{E}[X] = \sum_{i=1}^{N} x_i\, p_X(x_i).$$

This is the *arithmetic mean* when probability mass function p_X is uniformly $1/N$.

$$\begin{aligned}\mathbf{E}[X] &= \textstyle\sum_{i=1}^{N} x_i\, 1/N \\ &= 1/N \textstyle\sum_{i=1}^{N} x_i \\ &= (x_1 + x_2 + \dots x_N)/N\end{aligned}$$

Linearity of expectation,

$$\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y].$$

If $Y = aX + b$ for $a, b \in \mathbb{R}$,

$$\mathbf{E}[Y] = a\mathbf{E}[X] + b.$$

If X, Y independent,

$$\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y].$$

Variance is mean of squared distances between X and its mean,

$$\begin{aligned}\sigma_X^2 &= \mathbf{E}[(X - \mathbf{E}[X])^2] \\ &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 \\ &= \mathbf{E}[X^2] \text{ when } \mathbf{E}[X] = 0.\end{aligned}$$

Standard deviation $\sigma_X = \sqrt{\sigma_X^2}$.

Notes

Random phasor sum

Defined as a weighted sum of random phasors:

$$\begin{aligned}\frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n e^{\mathrm{j}\,\phi_n} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \boldsymbol{a}_n = A e^{\mathrm{j}\,\theta} \\ &= \boldsymbol{A} \quad \text{(the "resultant.")}\end{aligned}$$

$$\begin{aligned}\mathbf{E}[\mathrm{Re}\{\boldsymbol{A}\}] &= \mathbf{E}[1/\sqrt{N} \sum_{n=1}^{N} a_n \cos\phi_n] \\ &= 1/\sqrt{N} \sum \mathbf{E}[a_n \cos\phi_n] \\ &= 1/\sqrt{N} \sum \mathbf{E}[a_n]\mathbf{E}[\cos\phi_n] \\ &= 0.\end{aligned}$$

Similarly, $\mathbf{E}[\mathrm{Im}\{\boldsymbol{A}\}] = 0$.

$$\begin{aligned}\sigma^2_{\mathrm{Re}\{\boldsymbol{A}\}} &= \mathbf{E}[\mathrm{Re}\{\boldsymbol{A}\}^2]. \\ \mathrm{Re}\{\boldsymbol{A}\}^2 &= 1/\sqrt{N}(a_1\cos\phi_1 + a_2\cos\phi_2 + \dots)\times \\ &\quad\ 1/\sqrt{N}(a_1\cos\phi_1 + a_2\cos\phi_2 + \dots) \\ &= 1/N \textstyle\sum_n \sum_m a_n a_m \cos\phi_n \cos\phi_m. \\ \mathbf{E}[\mathrm{Re}\{\boldsymbol{A}\}^2] &= 1/N \textstyle\sum_n \sum_m \mathbf{E}[a_n a_m]\times \\ &\quad\ \mathbf{E}[\cos\phi_n \cos\phi_m] \\ &= 1/N \textstyle\sum_n \mathbf{E}[a_n^2]\mathbf{E}[\cos^2\phi_n]\end{aligned}$$

(since for $n \neq m$, $\mathbf{E}[\cos\phi_n \cos\phi_m]$
$= \mathbf{E}[\cos\phi_n]\mathbf{E}[\cos\phi_m] = 0$)

$$= 1/N \textstyle\sum_n \mathbf{E}[a_n^2] \times \mathbf{E}[1/2 + 1/2 \cos 2\phi_n]$$

(since $\cos^2\phi = (1 + \cos 2\phi)/2$)

$$= 1/N \textstyle\sum_n \mathbf{E}[a_n^2]/2.$$

Similarly, $\sigma^2_{\mathrm{Im}\{\boldsymbol{A}\}} = 1/N \sum_n \mathbf{E}[a_n^2]/2$.

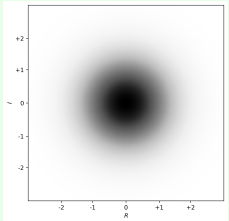
Notes

Large numbers

Central Limit Theorem says that the probability density of the *sum* of N independent, identically-distributed, random quantities approaches Normal as $N \to \infty$,

$$f_{R,I}(r,i) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{r^2+i^2}{2\sigma^2}\right\}$$

Where $R = \mathrm{Re}\{\boldsymbol{A}\}$ and $I = \mathrm{Im}\{\boldsymbol{A}\}$ and $\sigma^2 = \sigma_R^2 = \sigma_I^2$.



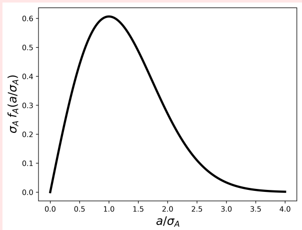
Normal $f_{R,I}(r,i)$, $[0,1]$.

Through transformation of variables,* marginal statistics for A and θ are found as Rayleigh and Uniform respectively,

$$f_A(a) = {}^{a}\!/\!{}_{\sigma^2}\exp\left\{\frac{a^2}{2\sigma^2}\right\}$$

$$f_\theta(\phi) = {}^{1}\!/\!{}_{2\pi}$$

$$\mathbf{E}[A] = \sqrt{{}^{\pi}\!/\!{}_{2}}\;\sigma,\;\; \sigma_A = (2 - {}^{\pi}\!/\!{}_{2})\sigma^2$$



Rayleigh $f_A(a)$, $[0,1]$ (scaled by σ_A.)

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