CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

Lecture #2: Wave Equation

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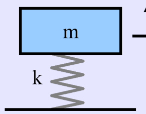
Notes

Simple Harmonic Oscillator



The force F acting on a mass m, sitting on a spring with stiffness k, can be described in different ways:

Newton force equals mass by acceleration, $F = m\,\ddot{y}$.

Hooke force prop. to spring displacement, $F = -k\,y$.

These independent descriptions of F yield a mathematical expression relating system parameters to each other: $m\,\ddot{y} = -k\,y$ so,

$$m\,\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + k\,y(t) = 0.$$

Because of the derivative, this is called a *differential* equation whose solution is a function $y(t)$ that satisfies it.

A solution for position y with respect to time t is,

$$y(t) = y_0 \cos(\omega\,t + \phi)$$

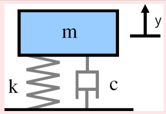
where,

y_0 is the amplitude of initial displacement,

$\omega = \sqrt{k/m}$ is angular frequency,

ϕ is phase which is this case is 0.

Note that $f = \omega/2\pi$ is temporal frequency.



Friction can be modelled by a damper with strength c which applies force $F = -c\,\dot{y}$. This can be equated with the other differential equations, $m\,\ddot{y} = -k\,y = -c\,\dot{y}$, so,

$$m\,\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + c\,\frac{\mathrm{d}y(t)}{\mathrm{d}t} + k\,y(t) = 0.$$

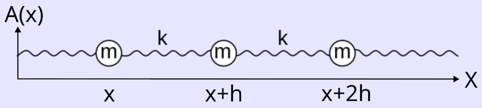
A solution with $\zeta = \frac{c}{2\sqrt{km}}$ is,

$$y(t) = y_0\,\mathrm{e}^{-\zeta\,\omega\,t} \cos\left(\sqrt{1-\zeta^2}\,\omega\,t + \phi\right).$$

Notes

Wave equation development

A linear object such as a string can be approximated as a series of masses m connected by springs of lengths h and spring constants k,



Let $A(x, t)$ be the height of a mass at position x at time t.

The force acting on mass m at position $x + h$ at time t can be described independently:

Newton $F(x + h, t) = m \frac{\partial^2}{\partial t^2} A(x + h, t)$.

Hooke $F(x + h, t) = F(x + 2h, t) - F(x, t) =$

$k[A(x+2h, t)-A(x+h, t)]-k[A(x+h, t)-A(x, t)]$

i.e. the difference between forces acting on the neighbours to which mass m is connected; which are proportional to height differences.

Equating these descriptions of force gives,

$$\frac{\partial^2}{\partial t^2} A(x + h, t) =$$

$$\frac{k}{m} [A(x + 2h, t) - 2A(x + h, t) + A(x, t)].$$

For N masses evenly spaced over total length $L = Nh$ and total mass $M = Nm$ and an average spring constant $K = k/N$, the coefficient on the rhs becomes,

$$\frac{KL^2}{M} \frac{1}{h^2}$$

.

Let $c^2 = \frac{KL^2}{M}$ and consider the continuous system situation where $N \to \infty$ (which implies taking the limit as $h \to 0$,)

$$\boxed{\frac{\partial^2 A(x, t)}{\partial t^2} = c^2 \frac{\partial^2 A(x, t)}{\partial x^2}.}$$

Notes

Taking the limit

$$\lim_{h \to 0} \frac{A(x + 2h, t) - 2A(x + h, t) + A(x, t)}{h^2}$$

This expression is *indeterminite* because for $h = 0$ it becomes $\frac{0}{0}$ which is 0? ∞? 1?

Luckily, we have l'Hospital's Rule which says that, under certain conditions,

$$\lim_{x \to y} \frac{f(x)}{g(x)} = \lim_{x \to y} \frac{f'(x)}{g'(x)} = \lim_{x \to y} \frac{f''(x)}{g''(x)} = \ldots.$$

So the quotient terms can be replaced by their derivates,

$$\frac{\partial}{\partial h}[A(x + 2h, t) - 2A(x + h, t) + A(x, t)] =$$

$$2\, A'(x + 2h, t) - 2\, A'(x + h, t).$$

$$\text{And} \quad \frac{\mathrm{d}}{\mathrm{d}h} h^2 = 2h.$$

$$\lim_{h \to 0} \frac{A(x + 2h, t) - 2\, A(x + h, t) + A(x, t)}{h^2} =$$

$$\lim_{h \to 0} \frac{2\, A'(x + 2h, t) - 2\, A'(x + h, t)}{2h} =$$

$$\lim_{h \to 0} \frac{4\, A''(x + 2h, t) - 2\, A''(x + h, t)}{2} =$$

$$\lim_{h \to 0} [2\, A''(x + 2h, t) - A''(x + h, t)] =$$

$$2 \lim_{h \to 0} A''(x + 2h, t) - \lim_{h \to 0} A''(x + h, t) =$$

$$2\, A''(x, t) - A''(x, t) =$$

$$A''(x, t) = \frac{\partial^2 A(x, t)}{\partial x^2}.$$

Notes

Wave equation solution

A solution for $A(x, t)$ is:

$$R\cos(kx - \omega t) + (1 - R)\cos(kx + \omega t)$$

ω is angular frequency $2\pi\,\nu$ in rad s^{-1}
k is the wave number $2\pi/\lambda$ in rad m^{-1}
$|R| \leq 1$ specifies direction of travel.

Which is the superposition of two sinusoidal waves travelling in opposite directions. Non-sinusoidal solutions are possible too.

Notes

d'Alembert's solution development

Start with "transformation of variables," let $\xi \equiv x - c_s t$ and $\eta \equiv x + c_s t$,

$$x = \frac{1}{2}(\xi - \eta) \text{ and } t = \frac{1}{2c_s}(\xi + \eta).$$

By the Chain Rule the first derivatives are,

$$\frac{\partial A}{\partial x} = \frac{\partial A}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial A}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial A}{\partial \xi} + \frac{\partial A}{\partial \eta}$$
$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial A}{\partial \eta}\frac{\partial \eta}{\partial t} = -c_s\frac{\partial A}{\partial \xi} + c_s\frac{\partial A}{\partial \eta}$$

and the second derivatives are,

$$\begin{aligned}
\frac{\partial^2 A}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial \xi} + \frac{\partial A}{\partial \eta}\right) \\
&= (\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta})(\frac{\partial A}{\partial \xi} + \frac{\partial A}{\partial \eta}) = \frac{\partial^2 A}{\partial \xi^2} + 2\frac{\partial^2 A}{\partial \xi \partial \eta} + \frac{\partial^2 A}{\partial \eta^2} \\
\frac{\partial^2 A}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t}\right) = \frac{\partial}{\partial t}\left(-c_s\frac{\partial A}{\partial \xi} + c_s\frac{\partial A}{\partial \eta}\right) \\
&= (-c_s\frac{\partial}{\partial \xi} + c_s\frac{\partial}{\partial \eta})(-c_s\frac{\partial A}{\partial \xi} + c_s\frac{\partial A}{\partial \eta}) = c_s^2\frac{\partial^2 A}{\partial \xi^2} - 2c_s^2\frac{\partial^2 A}{\partial \xi \partial \eta} + c_s^2\frac{\partial^2 A}{\partial \eta^2}.
\end{aligned}$$

Notes

So the wave equation $\frac{\partial^2 A}{\partial x^2} - \frac{1}{c_s^2}\frac{\partial^2 A}{\partial t^2} = 0$ becomes

$$(\frac{\partial^2 A}{\partial \xi^2} + 2\frac{\partial^2 A}{\partial \xi \partial \eta} + \frac{\partial^2 A}{\partial \eta^2}) - \frac{1}{c_s^2}(c_s^2\frac{\partial^2 A}{\partial \xi^2} - 2c_s^2\frac{\partial^2 A}{\partial \xi \partial \eta} + c_s^2\frac{\partial^2 A}{\partial \eta^2}) = 0$$

which is $\frac{\partial^2 A}{\partial \xi \partial \eta} = \frac{\partial}{\partial \xi} = 0.$

for which any solution is known to have the form:

$$p(\xi, \eta) = f(\eta) + g(\xi) = f(x + c_s t) + g(x - c_s t)$$

Notes

Chain Rule(s) of differentiation

Rules that specify derivatives of compositions of functions, e.g. for $f(g(x))$.

Leibnitz notation often used: e.g. let $u \equiv g(x)$ and $y \equiv f(u)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = f'(g(x))\, g'(x).$$

Let $x \equiv g(t)$ and $y \equiv h(t)$ and $z \equiv f(x, y)$,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Let $x \equiv g(s, t)$ and $y \equiv h(s, t)$ and $z \equiv f(x, y)$,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}x} + \frac{\partial z}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}x}$$

Notes

Easier approaches

* To solve differential equations, or tricky mathematics in general, you can use a symbolic mathematics-oriented system like Mathematica (e.g. through a Trinity site license version or through the free Wolfram Alpha web interface.)

* The ChatGPT interface to the WolframGPT makes it very easy to use (although perhaps it is available only with a non-free subscription.)

* Using ChatGPT for mathematics without the support of WolframGPT is probably not a good idea!

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