CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

Lecture #5: Phasors

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Notes

Complex Numbers

A complex number is the sum of a *real* part and an *imaginary* part,

$$a + b\,\mathrm{j} \in \mathbb{C} \quad \text{for } a, b \in \mathbb{R}.$$

Imaginary unit j is defined as,

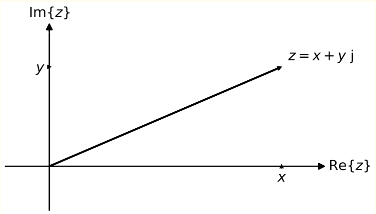
$$\mathrm{j}^2 = -1 \quad \text{so } \mathrm{j} = \pm\sqrt{-1}.$$

(In engineering, j is often used instead of i to avoid confusion with electrical current I.)

They can express what wouldn't be possible otherwise, e.g. roots of $(x+1)^2 = -9$ are at $x = -1 \pm 3\,\mathrm{j}$.

$$\begin{aligned}(-1 \pm 3\mathrm{j} + 1)(-1 \pm 3\mathrm{j} + 1) &= \\ (\pm 3\mathrm{j})^2 = (\pm 3)^2 \mathrm{j}^2 &= \\ (+3)^2 \mathrm{j}^2 \text{ and } (-3)^2 \mathrm{j}^2 &= \\ (9)(-1) &= -9.\end{aligned}$$

They can be used to associate numbers that go together, such as point vector coordinates (x, y).



But some consideration required, e.g compensation for $\mathrm{j}^2 = -1$ to express vector magnitude,

$$\begin{aligned}\text{for } z &= x + y\,\mathrm{j}, \\ \text{complex conjugate } \bar{z} &= x - y\,\mathrm{j}, \\ z\,\bar{z} &= x^2 + y^2, \\ \text{magnitude } |z| &= \sqrt{z\bar{z}}.\end{aligned}$$

Notes

Euler's Formula

Euler's number $\mathrm{e} \approx 2.71828$.

e^x is called the natural exponential function. Also written as $\exp(x)$ and $\exp x$.

One interesting characteristic is that $\mathrm{e}^x = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x$.

Another is that it can be used to express sinusoids,

$$\mathrm{e}^{\mathrm{j}\,\theta} = \cos\theta + \mathrm{j}\,\sin\theta.$$

A Taylor Series expresses the value of any smooth function f at any point b near point a as,

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a)^1 + \frac{f''(a)}{2!}(b-a)^2 + \dots$$

At $a{=}0$, $\frac{\mathrm{d}}{\mathrm{d}a}\mathrm{e}^a{=}\mathrm{e}^a{=}1$, $\frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}a}\mathrm{e}^a{=}\frac{\mathrm{d}}{\mathrm{d}a}\mathrm{e}^a{=}\,1$, etc., so,

$$\exp(b) = \exp(0) + \frac{b^1}{1!} + \frac{b^2}{2!} + \frac{b^3}{3!} + \dots$$

$$\cos\theta = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,\theta^{2n} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

$$\sin\theta = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,\theta^{2n+1}$$
$$= \frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

$$\begin{aligned}\mathrm{e}^{\mathrm{j}\,\theta} = \exp(\mathrm{j}\,\theta) &= 1 + \frac{(\mathrm{j}\,\theta)^1}{1!} + \frac{(\mathrm{j}\,\theta)^2}{2!} + \frac{(\mathrm{j}\,\theta)^3}{3!}\dots\\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\\ &\quad + \mathrm{j}\,(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots)\\ &= \cos\theta + \mathrm{j}\,\sin\theta.\end{aligned}$$

Notes

Phasors

Any sinusoid can be described as $A\,\cos(\omega t + \phi)$. Some terms are *variable*:

- time, $-\infty < t < +\infty$
- temporal frequency, $1/\infty < \nu < \infty$ (remember angular frequency $\omega = 2\pi\,\nu$)

and the others are *constant*:

- amplitude A
- phase ϕ

A *phasor* can be used to encode the constants,

$$A\,\mathrm{e}^{\mathrm{j}\,\phi} = A\,(\cos\phi + \mathrm{j}\,\sin\phi).$$

Variables can be encoded as,

$$\mathrm{e}^{\mathrm{j}\,\omega t} = (\cos\omega t + \mathrm{j}\,\sin\omega t).$$

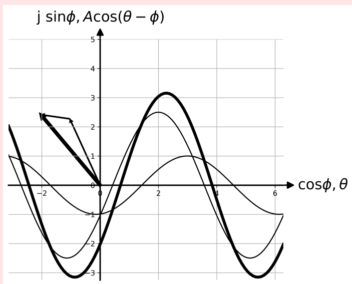
Any sinusoid can be expressed as their product,

$$A\,\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\omega t} = A\,\mathrm{e}^{\mathrm{j}\,(\phi+\omega t)}.$$
$$A\ \mathrm{Re}\left\{\mathrm{e}^{\mathrm{j}\,(\phi+\omega t)}\right\} = A\ \cos(\omega t + \phi).$$

Sum of two sinusoids with the same ang. freq. ω,

$$A_1\ \cos(\omega t + \phi_1) + A_2\ \cos(\omega t + \phi_2) =$$
$$\mathrm{Re}\left\{A_1\,\mathrm{e}^{\mathrm{j}\,(\omega t+\phi_1)} + A_2\,\mathrm{e}^{\mathrm{j}\,(\omega t+\phi_2)}\right\} =$$
$$\mathrm{Re}\left\{(A_1\,\mathrm{e}^{\mathrm{j}\,\phi_1} + A_2\,\mathrm{e}^{\mathrm{j}\,\phi_2})\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\}.$$

Which is the sum of phasors encoding the constants multiplied by an exponential function encoding the variables.



Notes

Phasor calculus

Remember the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))\, g'(x).$$

Let $f(x) = \mathrm{e}^x$, $f(g(x)) = \mathrm{e}^{g(x)}$, $g(x) = yx$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{xy} = \mathrm{e}^{xy}\,\frac{\mathrm{d}}{\mathrm{d}x}xy = \mathrm{e}^{xy}\, y.$$

$$\text{So } \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\omega t} = \omega \mathrm{e}^{\omega t}.$$

The derivative of a phasor is another phasor,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{j}\,\omega t} = \mathrm{j}\,\omega\, \mathrm{e}^{\mathrm{j}\,\omega t} = \omega \mathrm{e}^{\mathrm{j}\,\pi/2}\, \mathrm{e}^{\mathrm{j}\,\omega t}$$

since $\mathrm{j} = 0 + \mathrm{j}\,1 = \cos\frac{\pi}{2} + \mathrm{j}\,\sin\frac{\pi}{2} = \mathrm{e}^{\mathrm{j}\,\pi/2}$.

Differentiate a phasor: multiply by $\mathrm{j}\,\omega = \omega \mathrm{e}^{\mathrm{j}\,\pi/2}$.

Integrate a phasor: multiply by $\frac{1}{\mathrm{j}\,\omega} = \frac{1}{\omega}\mathrm{e}^{-\mathrm{j}\,\pi/2}$.

For example, find the derivative with respect to time of a sinusoid expressed using a phasor,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(A\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\omega t}\right) &= A\mathrm{e}^{\mathrm{j}\,\phi}(\mathrm{j}\,\omega)\mathrm{e}^{\mathrm{j}\,\omega t} \\ &= A\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\pi/2}\omega\mathrm{e}^{\mathrm{j}\,\omega t} \\ &= \omega A\mathrm{e}^{\mathrm{j}\,(\phi+\pi/2)}\mathrm{e}^{\mathrm{j}\,\omega t}.\end{aligned}$$

$$\begin{aligned}\mathrm{Re}\left\{\omega A\mathrm{e}^{\mathrm{j}\,(\phi+\pi/2)}\mathrm{e}^{\mathrm{j}\,\omega t}\right\} &= \omega A\cos(\omega t + \phi + \pi/2) \\ &= \omega A\sin(\omega t + \phi).\end{aligned}$$

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