Week 3 Answers

Question i

a

The plot displays a collection of blue dot markers representing data points. The x-axis corresponds to feature 1, the y-axis to feature 2, and the z-axis to the target values. It is clear that the data points lie on a curved plane by moving around the interactive plot. Which means the model should be a non-linear model.

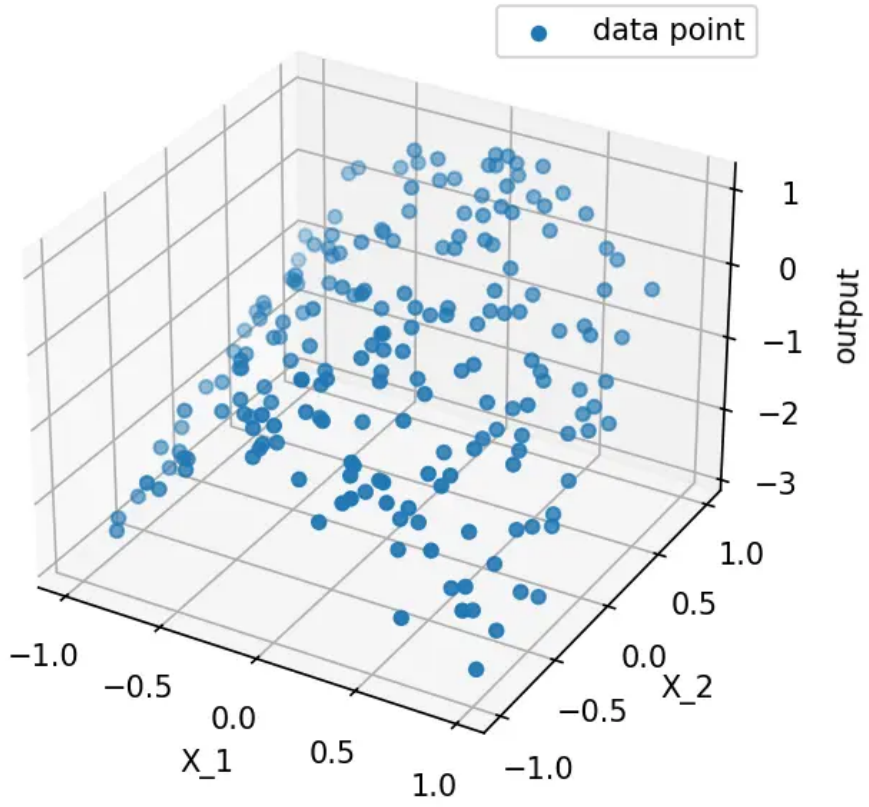


Figure 1: Plot 3D data points

b

In Lasso regression, the parameter $\alpha = \frac{1}{2C}$ controls the strength of the regularization applied to the model. Lasso uses `L1` regularization, which not only reduces the magnitude of coefficients but also has the effect of driving some of them to zero, effectively performing feature selection by ignoring less important features.

The Lasso regression model is trained using values of C in the range `[1,10,100,1000]`, where C represents the inverse of regularization strength. A higher C corresponds to less regularization, allowing the model more flexibility to fit the data, potentially leading to larger coefficients. Conversely, smaller C values apply stronger regularization, leading to smaller coefficients and potentially sparser solutions.

The table below shows the coefficients and intercept of the trained model for each value of C:

Table 1: Coefficients of Lasso Regression with Different C

C	**Bias**	x_1	x_2	x_1^2	x_1x_2	x_2^2	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3
1	-0.625	0	0	0	0	0	0	0	0	0
10	-0.189	0	0.815	-1.401	0	0	0	0	0	0
100	-0.040	0	0.892	-1.882	-0.003	0	0	0	-0.018	0.082
1000	-0.017	0.138	0.856	-1.833	-0.065	-0.044	0	0	-0.092	0.154

x_1^4	$x_1^3x_2$	$x_1^2x_2^2$	$x_1x_2^3$	x_2^4	x_1^5	$x_1^4x_2$	$x_1^3x_2^2$	$x_1^2x_2^3$	$x_1x_2^4$	x_2^5
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
-0.127	0	-0.022	0	0	0	0.025	0.054	0.016	-0.037	0

From the table, we can conclude:

- With $C = 1$,

$$y = -0.625$$

- With $C = 10$,

$$y = -0.189 + 0.815x_2 - 1.401x_1^2$$

- With $C = 100$,

$$y = -0.04 + 0.892x_2 - 1.882x_1^2 - 0.003x_1x_2 - 0.018x_1x_2^2 + 0.082x_2^3$$

- With $C = 1000$,

$$\begin{aligned} y = &-0.017 + 0.138x_1 + 0.856x_2 - 1.833x_1^2 - 0.065x_1x_2 - 0.044x_2^2 \\ &-0.092x_1x_2^2 + 0.154x_2^3 - 0.127x_1^4 - 0.022x_1^2x_2^2 \\ &+0.025x_1^4x_2 + 0.054x_1^3x_2^2 + 0.016x_1^2x_2^3 - 0.037x_1x_2^4 \end{aligned}$$

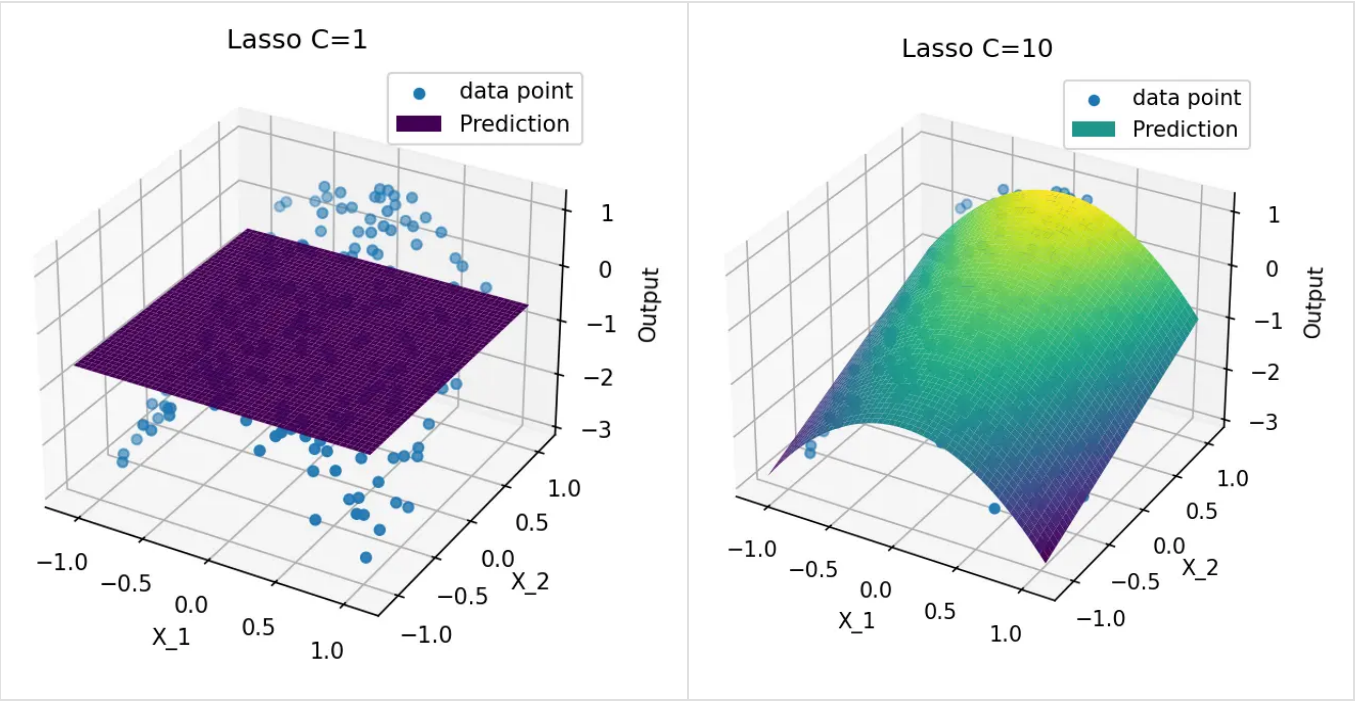
From the feature weights, it can be observed that as C increases, α and bias decrease, resulting in less penalization on misclassified points. Consequently, the feature weights increase as the model focuses on fitting the data points more accurately, leading to a greater number of non-zero weights. This indicates that the model becomes more complex, capturing more details of data, which may increase the risk of overfitting if not managed properly.

c

The figure displays four Lasso prediction plots, each illustrating the scattered original data points and a prediction plane produced by the model for different values of C. The predictions are made on a grid that slightly exceeds the range of the training data, with the grid extending 0.1 beyond each edge of the data range, ensuring a comprehensive visualization of the model's behavior across the input space.

In the first plot (Lasso $C = 1$), the prediction is a flat plane, indicating that the model is underfitting the data, likely due to strong regularization. As C increases to 10, the prediction becomes a curved plane that fits the data much more accurately, showing the model's improved ability to capture the underlying pattern.

In the subsequent plots, as C increases further to 100 and 1000, the prediction plane remains similarly curved with minimal noticeable changes. This suggests that beyond a certain point, increasing C no longer significantly impacts the model's fit, as it has already captured the data's structure effectively.



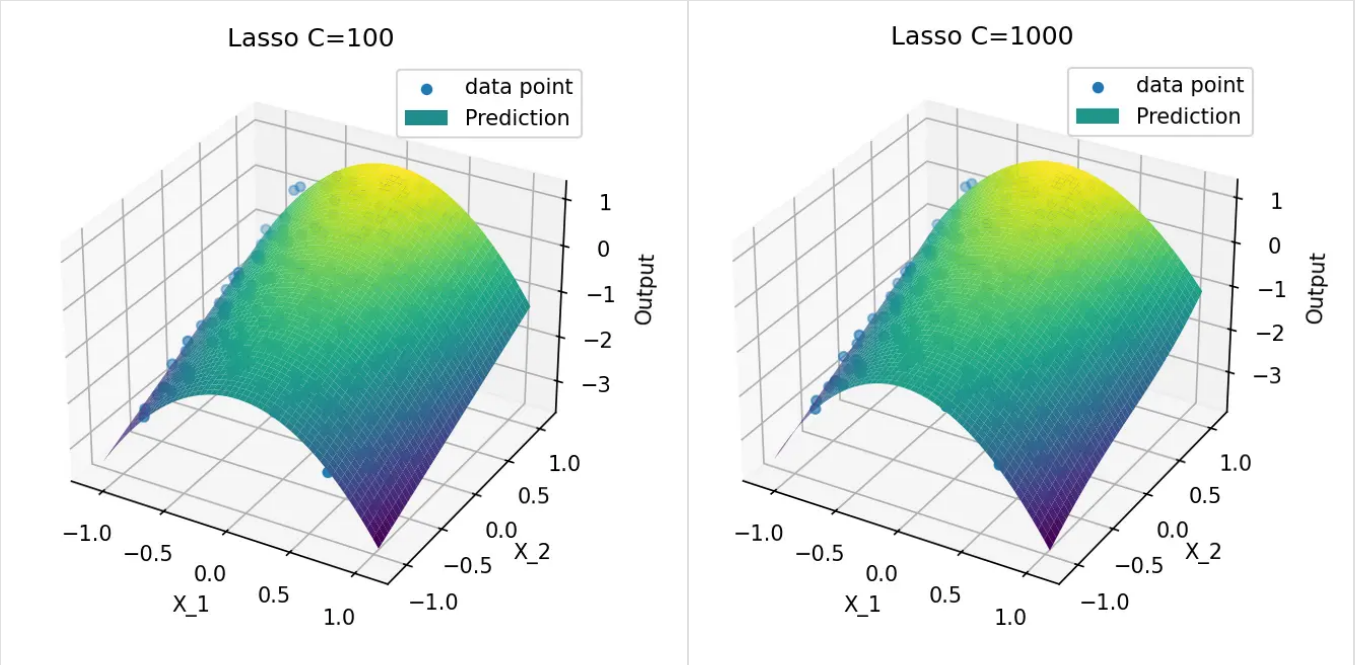


Figure 2: Plots of Lasso Predictions with various C and data points

d

Underfitting occurs when a model is too simple to capture the underlying patterns in the data, leading to poor performance even on the training set. Overfitting happens when a model becomes too complex and fits the training data too closely, capturing noise or irrelevant details, which causes it to perform poorly on unseen data.

The penalty weight parameter C in Lasso regression controls the trade-off between underfitting and overfitting by adjusting the regularization strength. When C is small, strong regularization leads to underfitting, as seen in the top-left plot ($C = 1$), where the prediction is a flat plane. When C is large, the model closely fits the data, risking overfitting, as shown in the bottom-right plot ($C = 1000$). Adjusting C allows balancing between simplicity and model complexity for better generalization.

e

In Ridge regression, $\alpha = \frac{1}{2C}$ controls the strength of `L2` regularization. Smaller C values apply stronger regularization, shrinking the coefficients toward zero and producing simpler models. Larger C values reduce regularization, allowing the model to fit the data more closely with larger coefficients.

The model is trained using C values in the range `[0.001,0.01,1,10]`. Higher C corresponds to less regularization, while lower C leads to stronger regularization. The table below shows the coefficients and intercept for each C, illustrating the model's behavior as C changes.

Table 2: Coefficients of Ridge Regression with Different C

C	Bias	x_1	x_2	x_1^2	x_1x_2	x_2^2	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3
0.001	-0.605	-0.014	0.089	-0.053	0.013	0.009	-0.008	0.025	-0.006	0.058
0.01	-0.466	-0.040	0.358	-0.362	0.046	0.034	-0.014	0.088	-0.015	0.218
1	-0.082	0.040	0.783	-1.388	-0.069	0.020	-0.042	0.039	-0.131	0.311
10	-0.028	0.126	0.762	-1.743	-0.156	-0.059	-0.216	-0.082	-0.317	0.595

x_1^4	$x_1^3x_2$	$x_1^2x_2^2$	$x_1x_2^3$	x_2^4	x_1^5	$x_1^4x_2$	$x_1^3x_2^2$	$x_1^2x_2^3$	$x_1x_2^4$	x_2^5
-0.046	0.009	-0.016	0.009	0.011	-0.004	0.014	-0.004	0.015	-0.003	0.045
-0.310	0.042	-0.113	0.036	0.033	0.005	0.051	-0.004	0.045	-0.007	0.156
-0.532	-0.031	-0.189	0.021	-0.011	-0.029	0.022	0.171	0.015	-0.077	-0.096
-0.204	0.038	-0.079	0.113	0.068	0.038	0.136	0.559	0.107	-0.129	-0.409

From the table, we can conclude:

- With $C = 0.001$,

$$\begin{aligned} y = &-0.605 - 0.014x_1 + 0.089x_2 - 0.053x_1^2 + 0.013x_1x_2 + 0.009x_2^2 - 0.008x_1^3 \\ &+0.025x_1^2x_2 - 0.006x_1x_2^2 + 0.058x_2^3 - 0.046x_1^4 + 0.009x_1^3x_2 - 0.016x_1^2x_2^2 + 0.009x_1x_2^3 \\ &+0.011x_2^4 - 0.004x_1^5 + 0.014x_1^4x_2 - 0.004x_1^3x_2^2 + 0.015x_1^2x_2^3 - 0.003x_1x_2^4 + 0.045x_2^5 \end{aligned}$$

- With $C = 0.01$,

$$\begin{aligned} y = &-0.466 - 0.040x_1 + 0.358x_2 - 0.362x_1^2 + 0.046x_1x_2 + 0.034x_2^2 - 0.014x_1^3 \\ &+0.088x_1^2x_2 - 0.015x_1x_2^2 + 0.218x_2^3 - 0.310x_1^4 + 0.042x_1^3x_2 - 0.113x_1^2x_2^2 + 0.036x_1x_2^3 \\ &+0.033x_2^4 + 0.005x_1^5 + 0.051x_1^4x_2 - 0.004x_1^3x_2^2 + 0.045x_1^2x_2^3 - 0.007x_1x_2^4 + 0.156x_2^5 \end{aligned}$$

- With $C = 1$,

$$\begin{aligned} y = &-0.082 - 0.040x_1 + 0.783x_2 - 1.388x_1^2 - 0.069x_1x_2 + 0.020x_2^2 - 0.042x_1^3 \\ &+0.039x_1^2x_2 - 0.131x_1x_2^2 + 0.311x_2^3 - 0.532x_1^4 - 0.031x_1^3x_2 - 0.189x_1^2x_2^2 + 0.021x_1x_2^3 \\ &-0.011x_2^4 - 0.029x_1^5 + 0.022x_1^4x_2 + 0.171x_1^3x_2^2 + 0.015x_1^2x_2^3 - 0.077x_1x_2^4 - 0.96x_2^5 \end{aligned}$$

- With $C = 10$,

$$\begin{aligned} y = &-0.028 + 0.126x_1 + 0.762x_2 - 1.743x_1^2 - 0.156x_1x_2 - 0.059x_2^2 - 0.216x_1^3 \\ &-0.082x_1^2x_2 - 0.317x_1x_2^2 + 0.594x_2^3 - 0.204x_1^4 + 0.038x_1^3x_2 - 0.079x_1^2x_2^2 + 0.113x_1x_2^3 \\ &+0.068x_2^4 + 0.038x_1^5 + 0.136x_1^4x_2 + 0.559x_1^3x_2^2 + 0.106x_1^2x_2^3 - 0.129x_1x_2^4 - 0.409x_2^5 \end{aligned}$$

From the values of the coefficients and intercepts, we can see that as C increases, α and bias decrease, meaning less penalization is applied to misclassified points. This allows the model to

focus more on fitting the data accurately, leading to larger feature weights. While this reduces error by capturing more underlying patterns, it also increases the risk of overfitting if C becomes too large.

The plots below show Ridge regression predictions for different values of C. Similar to the Lasso regression trends, as C increases, the model transitions from underfitting to overfitting. With smaller C, the prediction is a flat plane, indicating underfitting. As C increases to 1, the prediction curve fits the data more closely. By $C = 10$, the prediction plane stabilizes, as the model has already closely fit the data.

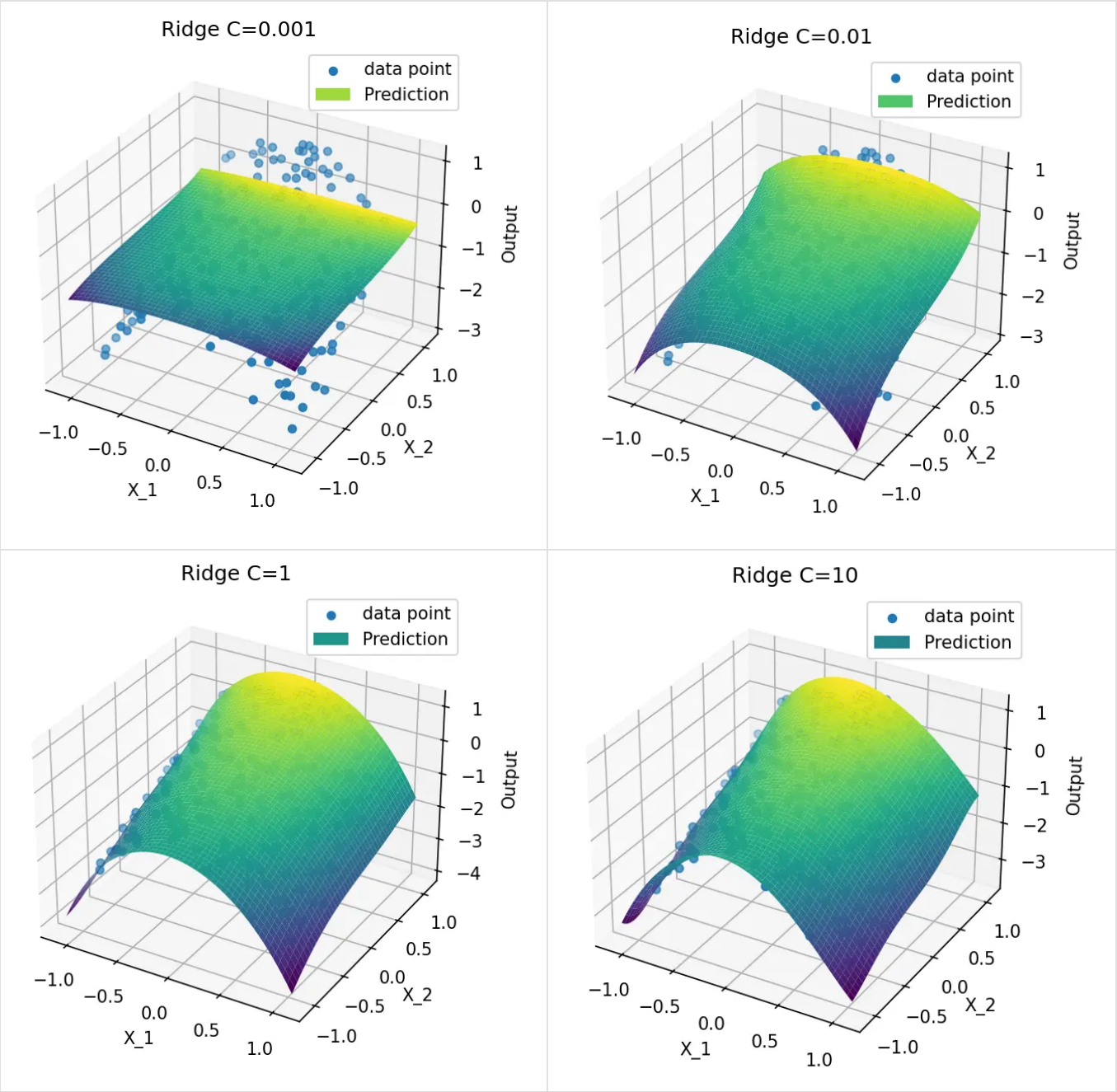


Figure 3: Plots of Ridge Predictions with various C and data points

Question ii

a

Initially, C values `[0.1,1,10,100]` were tested (see top-left plot in Figure 4). The difference in mean squared error (MSE) between $C = 0.1$ and $C = 1$ was negligible, and MSE barely changed beyond $C = 10$. Therefore, $C = 0.1$ was removed, and the upper limit adjusted to $C = 50$.

Next, the range was updated to `[1,5,10,20,50]` (top-right plot). Since MSE remained stable beyond $C = 20$, the upper limit was reduced to $C = 20$ for a closer look at values between 1 and 20.

Finally, the refined range `[1,5,7.5,10,20]` was used (last plot). The MSE stabilized after $C = 7.5$, indicating minimal performance gains from increasing C further.

These adjustments helped to narrow down the optimal C range, where MSE remains stable and increasing C provides no significant benefit.

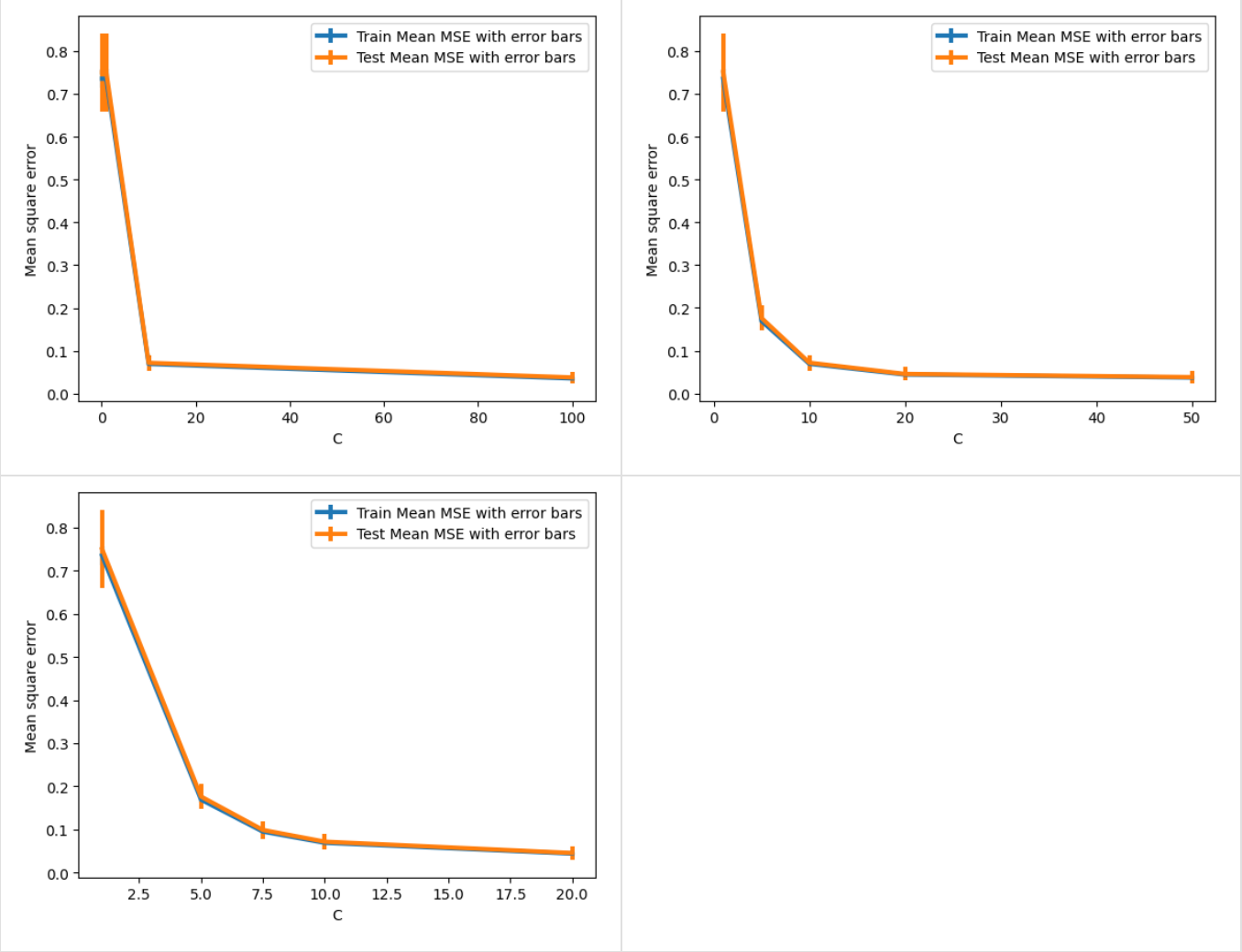


Figure 4: Plots of Lasso Predictions Errorbar

b

Based on the cross-validation data, I recommend using $C = 7.5$. The mean squared error stabilizes beyond this value, indicating that further increases in C do not significantly improve the model's performance. Choosing a higher C risks overfitting, as the model could become too complex and less generalizable to new data. Therefore, $C = 7.5$ strikes a good balance between accuracy and generalization.

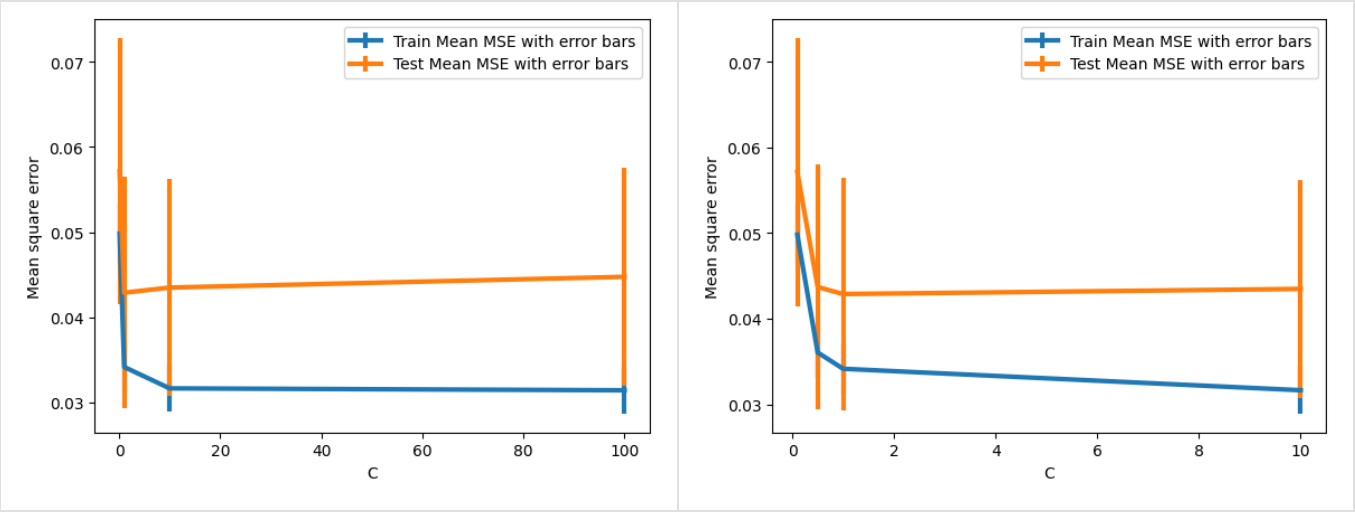
c

Initially, C values `[0.1, 1, 10, 100]` are tested shown in the top-left figure from Figure 5. The mean squared error barely changes beyond $C = 10$. Therefore the upper limit is shrunk to $C = 10$ and explore more value between 0.1 and 1.

Next, the range is updated to `[0.1, 0.5, 1, 10]` in the top-right plot from Figure 5. Since the mean squared error remains stable beyond $C = 1$, the upper limit is reduced to $C = 1$.

Finally, the last plot has the refined range `[1, 5, 7.5, 10, 20]`. It can be observed that the mean squared error stabilizes after $C = 0.5$, suggesting that further increases in C beyond this value yield minimal changes in performance.

As a result, $C = 0.5$ is recommended as it effectively balances fitting the training data without a high risk of overfitting. This value provides a stable and efficient model performance while maintaining generalization capabilities.



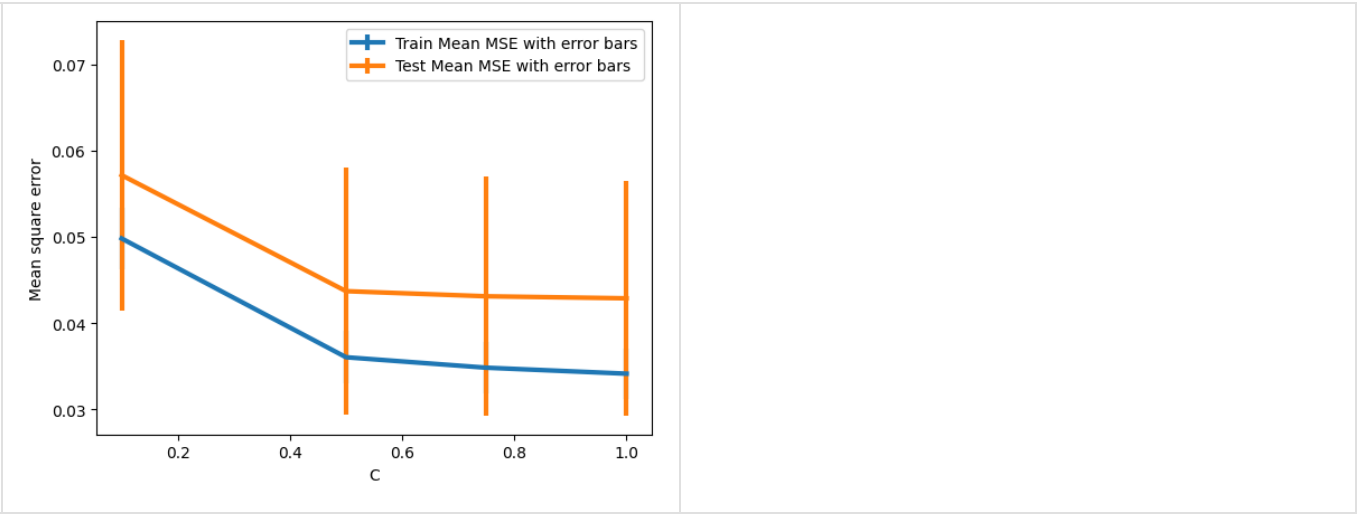


Figure 5: Plots of Ridge Predictions Errorbar

Appendix

```
import numpy as np
import pandas as pd
from sklearn.linear_model import Lasso, Ridge
from sklearn.model_selection import train_test_split
from sklearn.metrics import mean_squared_error
import matplotlib.pyplot as plt
from sklearn.preprocessing import PolynomialFeatures
from sklearn.model_selection import KFold


# id:6--12-6
df = pd.read_csv("week3.csv",sep=',')
x1=df.iloc[:,0]
x2=df.iloc[:,1]
X=np.column_stack((x1,x2))
y=df.iloc[:,2]

#(i)
#(a) plot 3d scatter
fig = plt.figure()
ax = fig.add_subplot(111, projection = '3d')
ax.scatter(X[:, 0], X[:, 1], y)
ax.set_xlabel('X_1')
ax.set_ylabel('X_2')
ax.set_zlabel('Target Value')
plt.show()

#(b) (c)
poly = PolynomialFeatures(5)
```

```
Xpoly5 = poly.fit_transform(X)
# split training and test data
Xpoly5_train, Xpoly5_test, y_train, y_test = train_test_split(Xpoly5, y,
test_size = 0.2, random_state=1)

## get all combinations of the two features with the power of 5
# feature_names = poly.get_feature_names_out(input_features=['x1', 'x2'])
# # Print the feature names
# for name in feature_names:
#     print(name)

# create grid little bigger than the range of original features
x1_grid = np.linspace(X[:, 0].min() - 0.1,  X[:, 0].max() + 0.1)
x2_grid = np.linspace(X[:, 1].min() - 0.1,  X[:, 1].max() + 0.1)
x1_grid, x2_grid = np.meshgrid(x1_grid, x2_grid)
print("feature 0 extented range:{}-{}".format(x1_grid.min(), x1_grid.max()))
print("feature 1 extented range:{}-{}".format(x2_grid.min(), x2_grid.max()))
# create polynomial features for the grid data so that
# it can be tested by a model trained with polynomial features up to the power
of 5
x_grid_test = np.c_[x1_grid.ravel(), x2_grid.ravel()]
x_grid_test = poly.fit_transform(x_grid_test)

Crange = [1,10,100,1000]
for C in Crange:
    model = Lasso(alpha = 1/(2*C), fit_intercept=True)
    model.fit(Xpoly5_train, y_train)
    ypred = model.predict(Xpoly5_test)

    # (b) report the parameters
    print("lasso C = {}: Coefficients: {} Intercept: {}".format(C,
model.coef_, model.intercept_))
    print("Mean Squared Error: ", mean_squared_error(y_test, ypred))

    fig = plt.figure()
    ax = fig.add_subplot(111, projection = '3d')
    ax.scatter(X[:, 0], X[:, 1], y)

    # (c) draw Xtest and ypred on a 3D surface
    ypred = model.predict(x_grid_test)
    ypred = ypred.reshape(x1_grid.shape)

    # Plot surface
    ax.plot_surface(x1_grid, x2_grid, ypred, cmap='viridis', edgecolor='none')

    # Labels
```

```
    ax.set_xlabel('X_1')
    ax.set_ylabel('X_2')
    ax.set_zlabel('Output')
    ax.set_title('C={}'.format(C))

    # Show plot
    plt.show()

# (e)
Crange = [0.001, 0.01, 1, 10]
for C in Crange:
    model = Ridge(alpha=1/(2*C))
    model.fit(Xpoly5_train, y_train)
    ypred = model.predict(Xpoly5_test)

    # (b) report the parameters
    print("Ridge C = {}: Coefficients: {} Intercept: {}".format(C,
model.coef_, model.intercept_))
    print("Mean Squared Error: ", mean_squared_error(y_test, ypred))

    fig = plt.figure()
    ax = fig.add_subplot(111, projection = '3d')
    ax.scatter(X[:, 0], X[:, 1], y)

    # (c) draw Xtest and ypred on a 3D surface
    ypred = model.predict(x_grid_test)
    ypred = ypred.reshape(x1_grid.shape)

    # Plot surface
    ax.plot_surface(x1_grid, x2_grid, ypred, cmap='viridis', edgecolor='none')

    # Labels
    ax.set_xlabel('Feature 1')
    ax.set_ylabel('Feature 2')
    ax.set_zlabel('Predicted Value')
    ax.set_title('C={}'.format(C))

    # Show plot
    plt.show()

# (ii)
# (a) (b)
kf = KFold(n_splits=5)
def drawErrorBarLasso(Crange):
    train_mean_error=[]; train_std_error=[]
    test_mean_error=[]; test_std_error=[]
```

```
    for C in Crange:
        model = Lasso(alpha=1/(2*C))
        temp_train = []
        temp_test = []

        # iterate the 5-fold
        for train, test in kf.split(Xpoly5):
            model.fit(Xpoly5[train], y[train])

            ypred_train = model.predict(Xpoly5[train])
            temp_train.append(mean_squared_error(y[train], ypred_train))

            ypred_test = model.predict(Xpoly5[test])
            temp_test.append(mean_squared_error(y[test], ypred_test))

        train_mean_error.append(np.array(temp_train).mean())
        train_std_error.append(np.array(temp_train).std())
        test_mean_error.append(np.array(temp_test).mean())
        test_std_error.append(np.array(temp_test).std())

    plt.errorbar(Crange, train_mean_error, yerr=train_std_error, linewidth=3,
label='Train Mean MSE with error bars')
    plt.errorbar(Crange, test_mean_error, yerr=test_std_error, linewidth=3,
label='Test Mean MSE with error bars')
    plt.xlabel('C')
    plt.ylabel('Mean square error')
    plt.legend(loc='best')
    plt.show()

drawErrorBarLasso([0.1, 1, 10, 100])
drawErrorBarLasso([1, 5, 10, 20, 50])
drawErrorBarLasso([1, 5, 7.5, 10, 20])

# (c)
# Crange = [0.1, 0.5, 1.5, 2, 2.5, 3, 10]
def drawErrorBarRidge(Crange):
    train_mean_error=[]; train_std_error=[]
    test_mean_error=[]; test_std_error=[]
    for C in Crange:
        model = Ridge(alpha=1/(2*C))
        temp_train = []
        temp_test = []

        # iterate the 5-fold
        for train, test in kf.split(Xpoly5):
            model.fit(Xpoly5[train], y[train])
```

```
            ypred_train = model.predict(Xpoly5[train])
            temp_train.append(mean_squared_error(y[train], ypred_train))

            ypred_test = model.predict(Xpoly5[test])
            temp_test.append(mean_squared_error(y[test], ypred_test))

        train_mean_error.append(np.array(temp_train).mean())
        train_std_error.append(np.array(temp_train).std())
        test_mean_error.append(np.array(temp_test).mean())
        test_std_error.append(np.array(temp_test).std())

    plt.errorbar(Crange, train_mean_error, yerr=train_std_error, linewidth=3,
label='Train Mean MSE with error bars')
    plt.errorbar(Crange, test_mean_error, yerr=test_std_error, linewidth=3,
label='Test Mean MSE with error bars')
    plt.xlabel('C')
    plt.ylabel('Mean square error')
    plt.legend(loc='best')
    plt.show()

drawErrorBarRidge([0.1, 1, 10, 100])
drawErrorBarRidge([0.1, 0.5, 1, 10])
drawErrorBarRidge([0.1, 0.5, 0.75, 1])
```