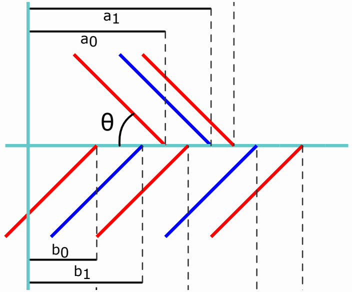
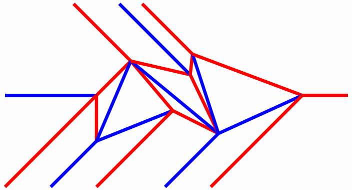
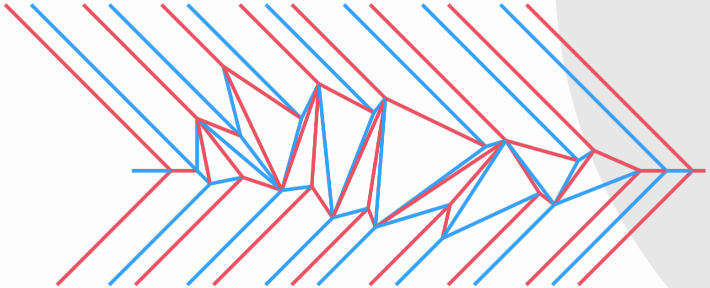
Algorithmic Transitions between Parallel Pleats

Brandon M. Wong, Erik D. Demaine

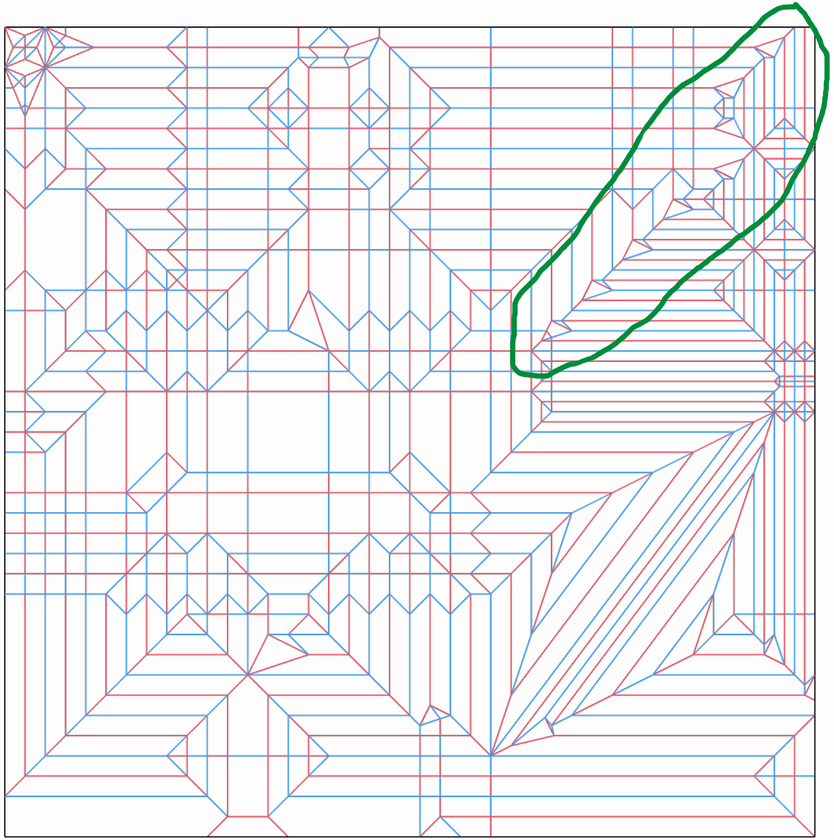






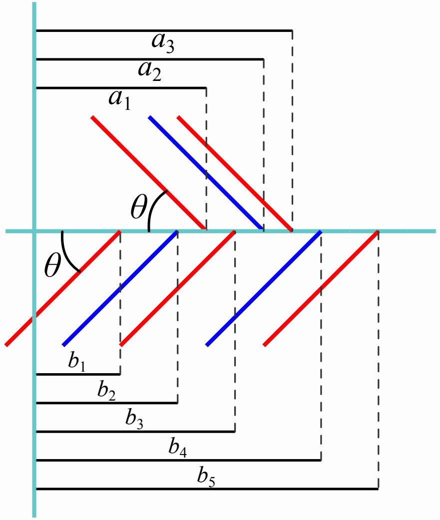
Motivation

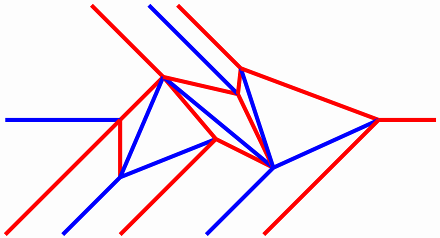
Ridge level shifters



Problem statement

Given two input sets of parallel creases (left), can we construct a flat-foldable crease pattern that merges the two sets (right)?



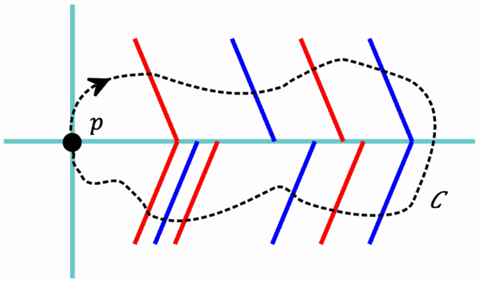


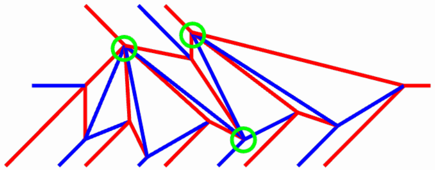
Theorem: a locally flat-foldable transition exists if and only if

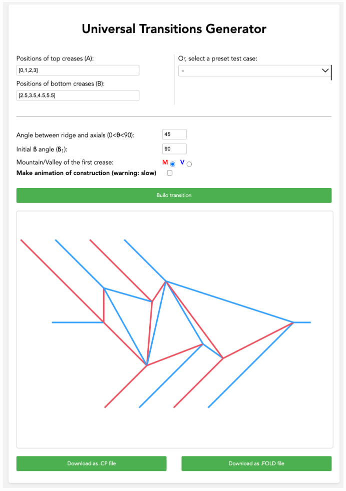
$$a_1 - a_2 + a_3 \ldots a_m = b_1 - b_2 + b_3 \ldots b_n$$

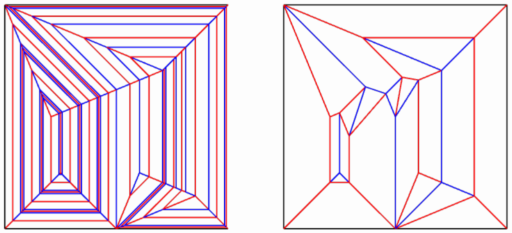
"alternating sum condition"

Table of contents





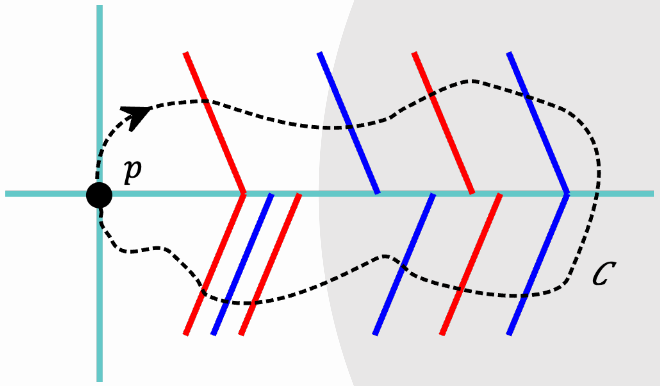




01

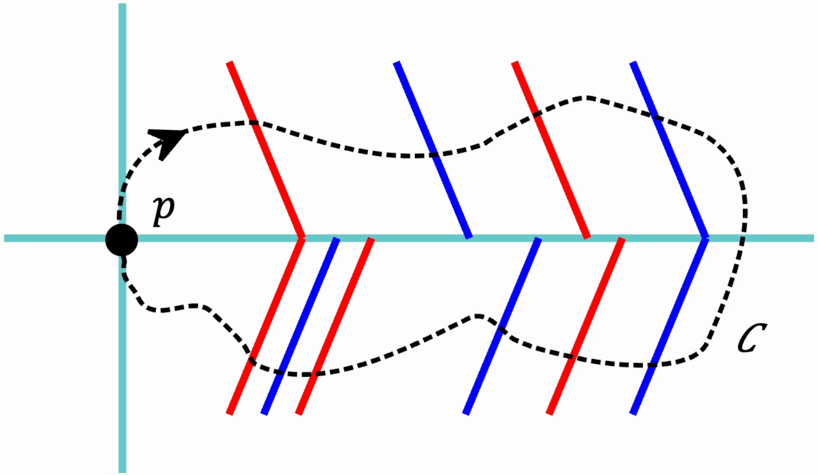
Necessity

A flat foldable transition exists ***only if*** the alternating sum condition is met.



Necessity proof

(Theorem 1) Using generalized Kawasaki's theorem: [1]



Position after reflecting over *i* creases in A: $x_i = (2a_0 - 2a_1 + 2a_2 - \cdots 2a_i)\sin\theta$

If reflecting over A and reflecting over B bring *p* to the same point, then: $(2a_0 - 2a_1 + 2a_2 - \cdots 2a_m)\sin\theta = (2b_0 - 2b_1 + 2b_2 - \cdots 2b_n)\sin\theta$

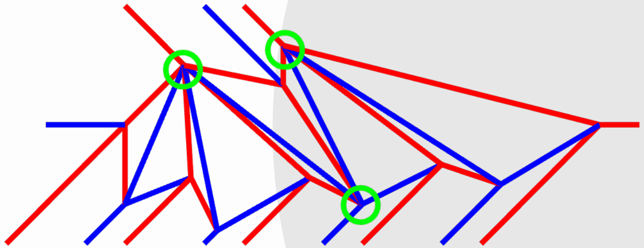
This simplifies to the alternating sums: $a_0 - a_1 + a_2 - a_3 + \cdots a_m = b_0 - b_1 + b_2 - b_3 + \cdots b_n$

[1] Thomas C. Hull. *Origametry: Mathematical Methods in Paper Folding*. Cambridge University Press, 2020.

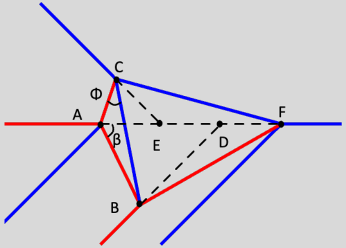
02

Sufficiency

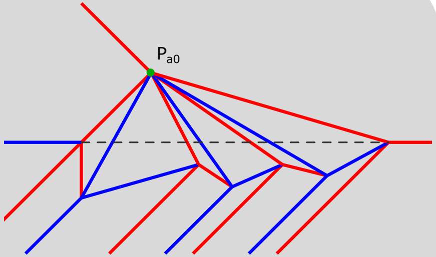
A flat foldable transition always exists ***if*** the alternating sum condition is met.



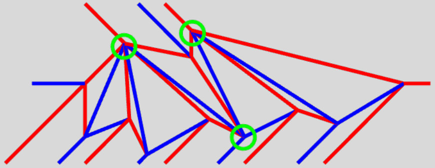
Overview: Sufficiency



1-to-3 case



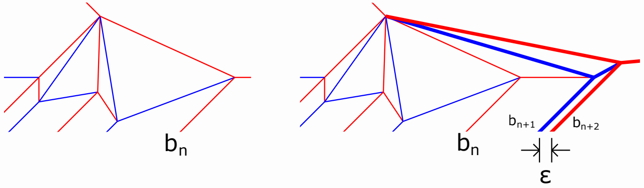
1-to-n case



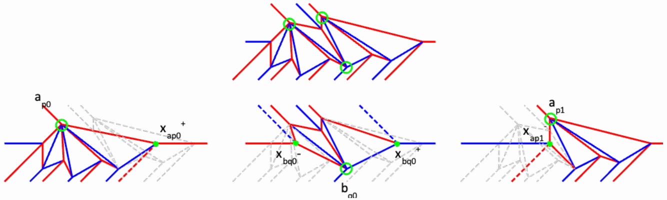
m-to-n case

(n is odd)

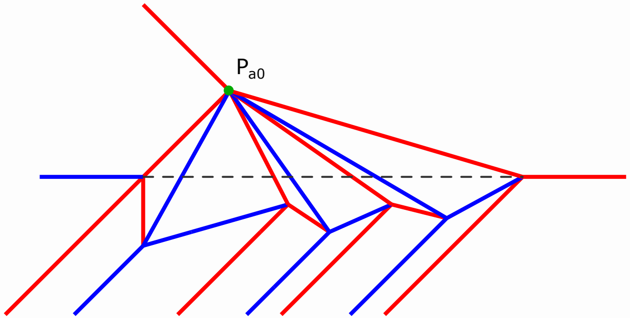
Inductively add creases to the bottom set

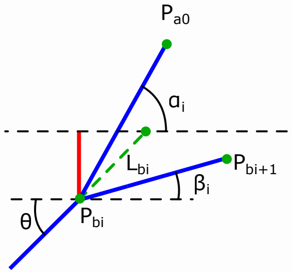


Inductively add on "premade" 1-to-n cases



Notation and constitutive equations





$$\beta_i = -\beta_{i-1} + \alpha_i - 180^\circ + \theta$$

$$\alpha_1 - \alpha_2 + \alpha_3 - \cdots - \alpha_{n-1} + \alpha_n = 180^\circ - \theta$$

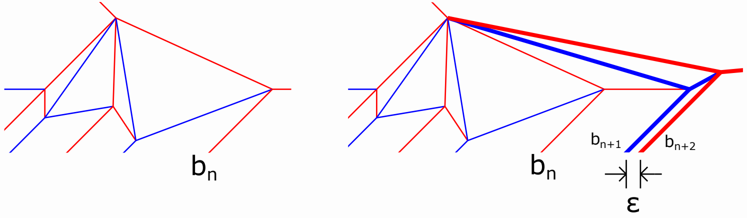
$$L_{b_i} = L_{b_{i-1}} - \frac{\sin \beta_{i-1}}{\sin(\theta - \beta_{i-1})}(b_i - b_{i-1})$$

$$\alpha_i = \arctan\left(\frac{(L_{a_1} + L_{b_i})\sin\theta}{a_1 - b_i - (L_{a_1} - L_{b_i})\cos\theta}\right)$$

$$L_{a_1} = \frac{\sin(\theta + |\beta_1|)}{\sin|\beta_1|}(a_1 - b_1).$$

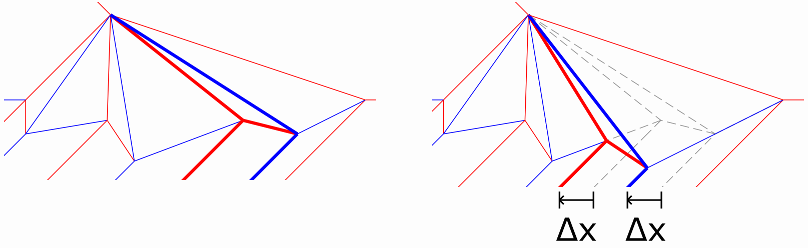
1-to-(n+2) inductive step

Given a flat foldable 1-to-n transition, we prove that a 1-to-(n+2) transition also exists



(Lemma 4) We can add two creases at the end that are infinitesimally close together

- Flat foldability is preserved
- The alternating sum is preserved



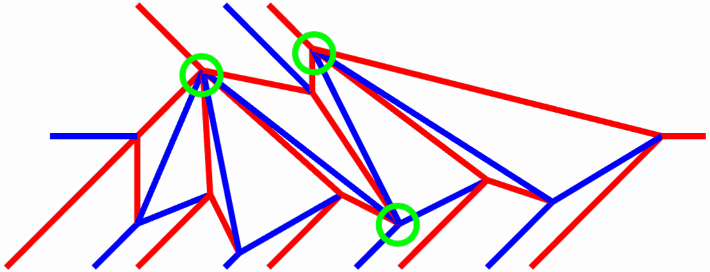
(Lemma 5) We can move the second and third to last creases together some equal amount

- Flat foldability is preserved
- The alternating sum is preserved

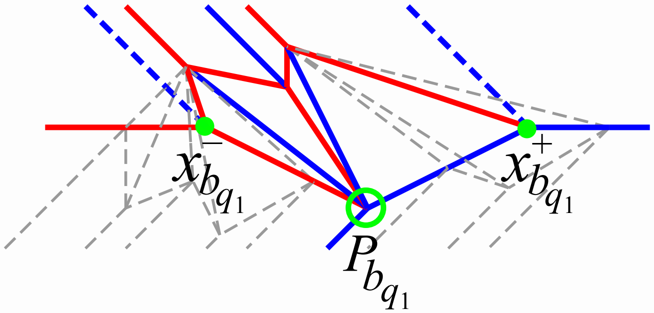
Therefore, for any 1-to-n case that satisfies the alternating sum condition, a flat foldable transition exists.

m-to-n transition: pivots

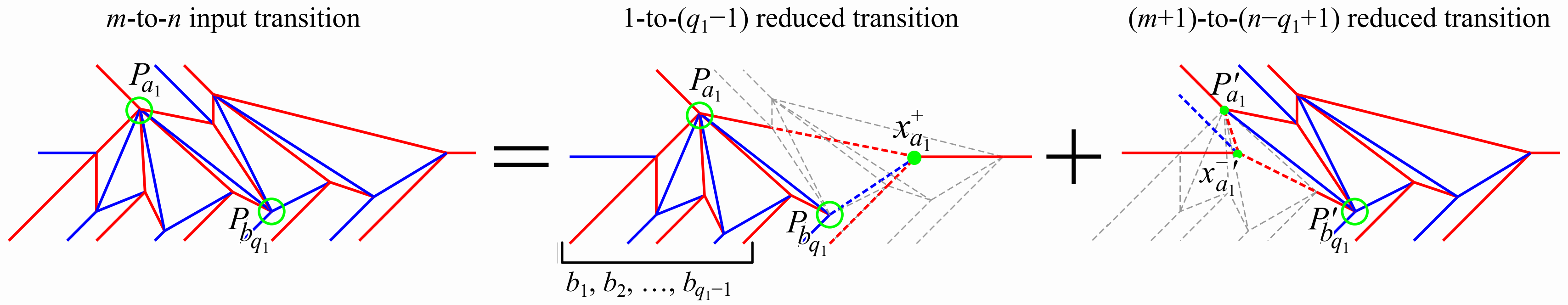
Pivots: points that connect to more than one points from the opposite set



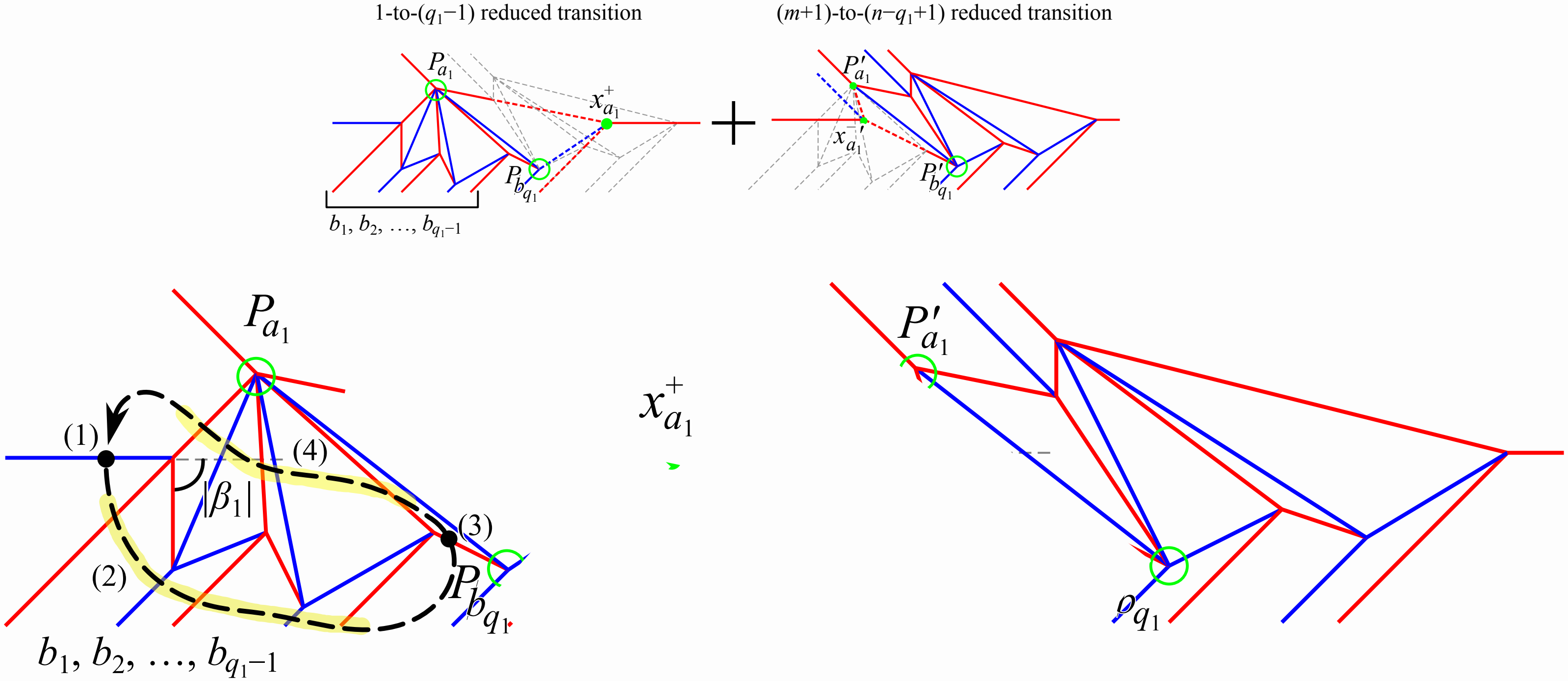
Each pivot is *locally* the 1 of a 1-to-n transition



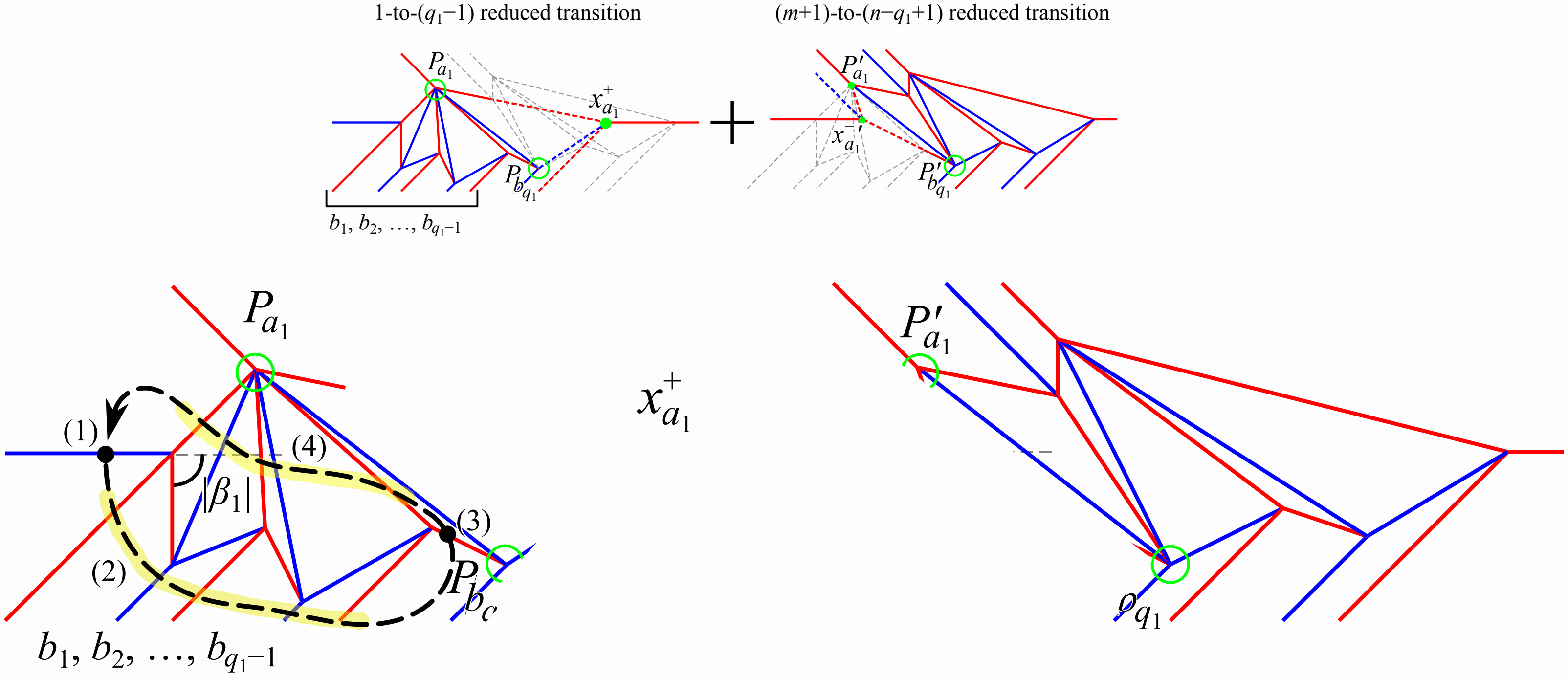
m-to-n transition: overview



m-to-n transition: merge operation



m-to-n transition: merge operation



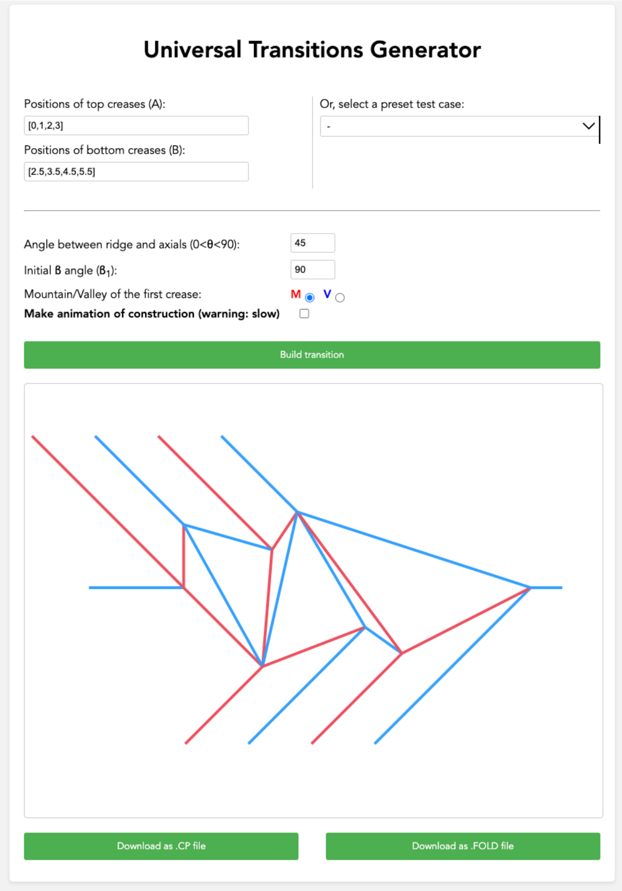
03

Implementation

https://github.com/theplantpsychologist/transitions- implementation

Demonstration

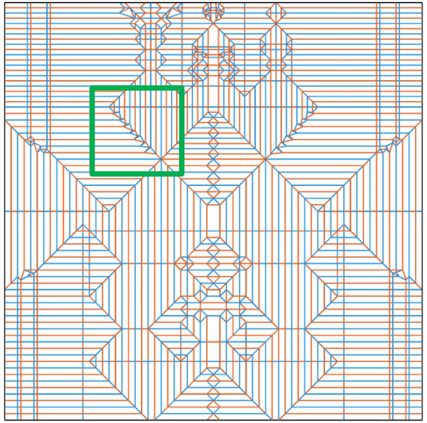
Scan here for github (link to access the app is in the ReadMe)

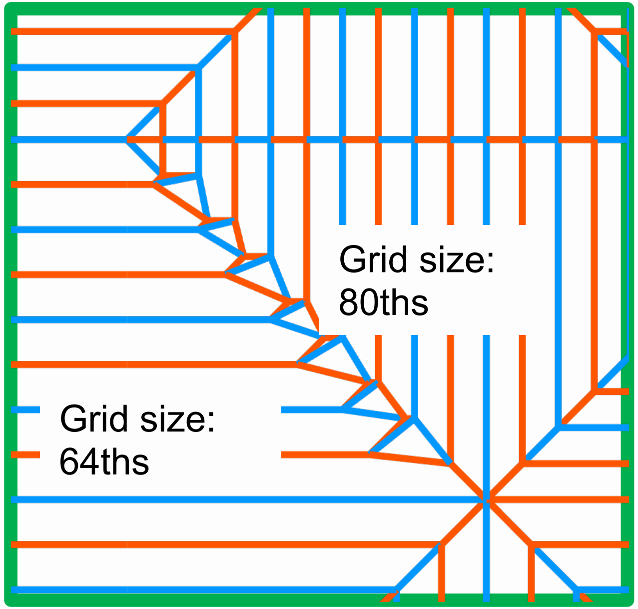


04

Applications

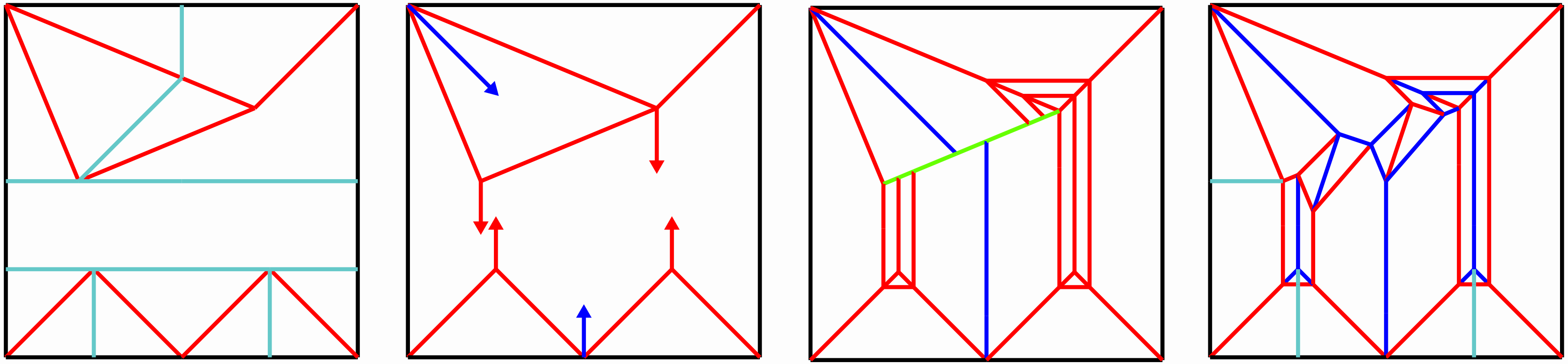
Weird box pleating

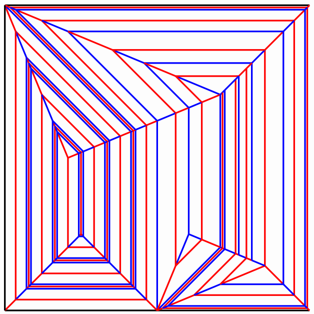




A: [0, 0.8, 1.6, 2.4, 3.2, 4, 4.8, 5.6, 6.4, 7.2, 8]
B: [0, 1, 2, 3, 4, 5, 6, 7, 8]

Terminating dense bouncing





Terminating dense bouncing

- Conjecture: for any axial polygon, any ridge can serve as the transition ridge, the axial creases will eventually satisfy the alternating sum condition around that ridge
- Proof is left as an open problem