Set Theory

* **Sets:-** A well defined collection of Objects.
In other words
A well defined, unordered collection of Distinct Object / elements of same Type.

* **Representation of Sets:-**

(1) Tabular / Roster Method
A set is defined by actually listing it's element.

(2) Set Builder Method / Set Selector Method
A set is defined by some Property which characterizes all it's elements.

→ e.g

Sets	Roster M.	Selector M.
$\mathbb{N}$	$\{1, 2, 3, 4, \ldots\}$	$\{x \mid x \in \mathbb{N}\}$
• Even Integer	$\{\ldots -2, 0, 2, 4 \ldots\}$	$\{2x \mid x \in \mathbb{Z}\}$
• +ve rationals	$\{0.1, 0.2, 0.3, \ldots\}$	$\{x \mid x \in \mathbb{Q}, x > 0\}$ or $\{p/q \mid p, q \in \mathbb{N}\}$

* **Subsets:-** Let A and B be two sets. If each element of set A is an element of set B then 'A is called subset of B' and is denoted by,
$A \subset B$
is also said to be 'A is contained in B'.

For example $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$
Here A is a subset of B, or we can say that B is the superset of A.

→ Every set is a subset of itself.

* **Proper subset:-** If '$A \subset B$' and $A \neq B$ then A is Proper subset of B.
(i.e $A \subset B$ but $B \not\subset A$)

e.g $A = \{1, 3\}$, $B = \{1, 2, 3\}$, $C = \{1, 3, 2\}$
A is Proper subset of C. ~~wheares~~ Whereas B is not a Proper subset of C since $B = C$.

* **Equality of sets:-** If A is a subset of B and B is also a subset of A, then two sets A and B are equal

e.g $A \subseteq B$ and $B \subseteq A$ implies $A = B$

* **Empty set** (or Null set or Void set):-
A set having no elements is called empty set.

e.g $A = \{x ;\ x^2 + 4 = 0,\ x \in \mathbb{R}\}$
$B = \{x ;\ x < 0,\ x \in \mathbb{N}\}$

* **Universal Set:-** If all the sets under consideration will be subsets of a fixed set then fixed set is universal, usually denoted by X.

e.g $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ etc.

* **Ordered Pair:-** Let A and B be two sets. Let $a \in A$, $b \in B$ then (a, b) denotes ordered pair. Where a and b are known as first and second co-ordinate of the ordered pair (a, b).

* **Cartesian Product** :- Let A and B be two sets then $A \times B$ is called Cartesian Product of A and B.

i.e $A \times B$ is a set of all distinct ordered pairs.

$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$

$A = \{1,2,3\}$, $B = \{a,b\}$

then $A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

- If A conteins m element and B conteins n element then $A \times B$ conteins $m.n$ elements.

* **Power Set** :- If S is any set, then set of all subsets of S is called Power set of S and is denoted by $P(S)$.

i.e $S = \{1,2,3\}$

$P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

So if S conteins 3 elements power set of S,
i.e $P(S)$ will contein 2^3, i.e 8 elements.

* **Power set of Power set S** :-

E.g $S = \{1,2\}$

$P(S) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

$P(P(S)) = \{\phi, \{\phi\}, \{\{1\}\}, \{\{2\}\}, \{\{1,2\}\}, \{\phi, \{1\}\},$
$\{\phi, \{2\}\}, \{\phi, \{1,2\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{1,2\}\},$
$\{\{2\}, \{1,2\}\}, \{\phi, \{1\}, \{2\}\}, \{\phi, \{1\}, \{1,2\}\},$
$\{\{1\}, \{2\}, \{1,2\}\}, \{\phi, \{2\}, \{1,2\}\},$
$\{\phi, \{1\}, \{2\}, \{1,2\}\}\}.$

S contein 2 elements.

P(S) contein 2^2, i.e 4 elements.

Then $P(P(S))$ will contein 2^4 i.e 16 elements.

So $P(P(P(S)))$ will contein 2^{16} elements etc.

Example $S = \{1, 2\}$

$P(S) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

1. $1 \in S$ is true.
2. $1 \subset S$ is False. As 1 is an element of S not a Subset of S.
3. $\{1\} \subset S$ is true.
4. $\{1\} \in S$ is false.
5. $\{1\} \in P(S)$ is true.
6. $\phi \subset P(S)$ is true.
7. $\phi \in P(S)$ is true.

Statement 6 and 7 both are correct as ϕ is an element of P(S) also ϕ is subset of every set.

8. $\{\{1\}\} \in P(S)$ is false.
9. $\{\{1\}\} \in P(P(S))$ is true.
10. $\{\{1\}\} \subset P(P(S))$ is false.

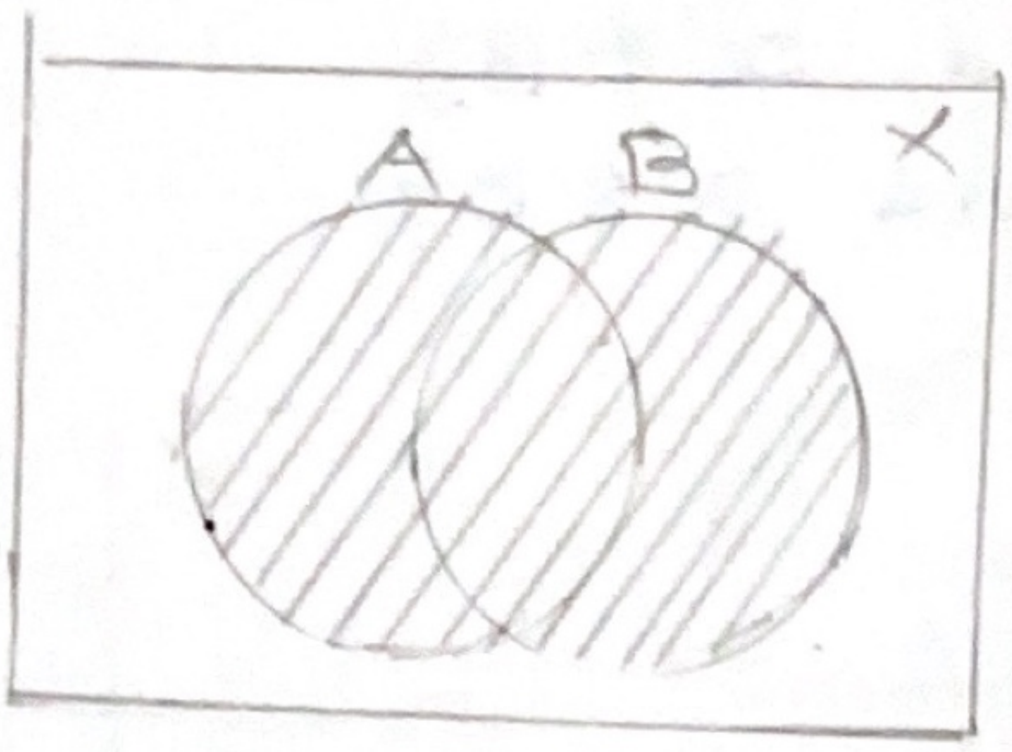
* Remark. When a is an element of S then it is denoted by $a \in S$. When A is a subset of S it is denoted by $A \subset S$. Hence for connection between element and set we use '$\in$' and for connection between two sets we use $\subset$ symbol.

* Venn Diagrams & Operations On sets

The relation between sets can be conveniently illustrated by diagrams called venn diagrams. In that, universal sets represented by large rectangle and subsets by closed curve (generally circles).

* **Union of Two sets:-** The union of set A and B is set of all elements either in A or in B.

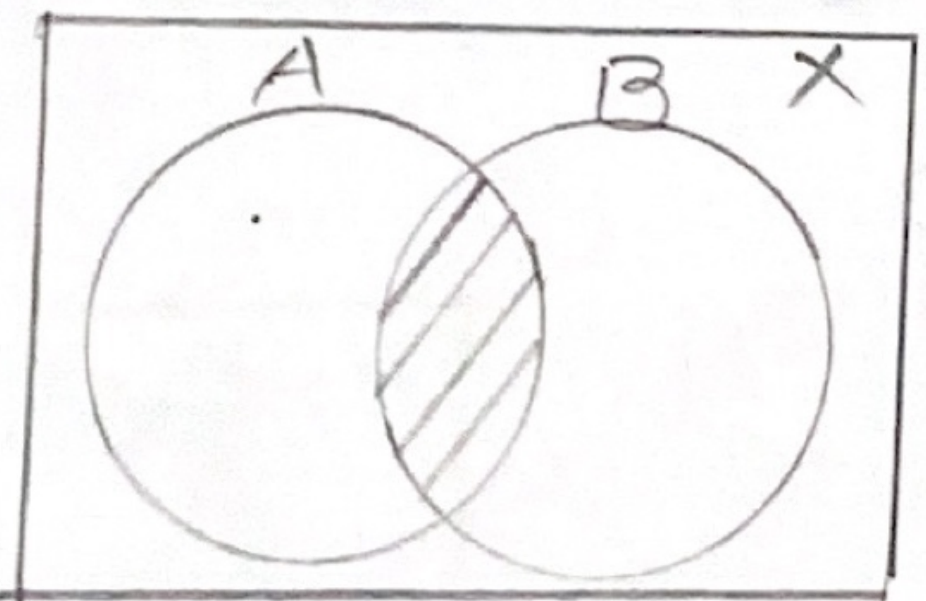
i.e $A \cup B = \{x ; x \in A \text{ or } x \in B\}$



$A \cup B$ ▨

* **Intersection of Two Sets:-** The intersection of A and B is the set of all elements which are in A and also in B. We denoted the intersection of A and B by $A \cap B$.

i.e $A \cap B = \{x ; x \in A \text{ and } x \in B\}$

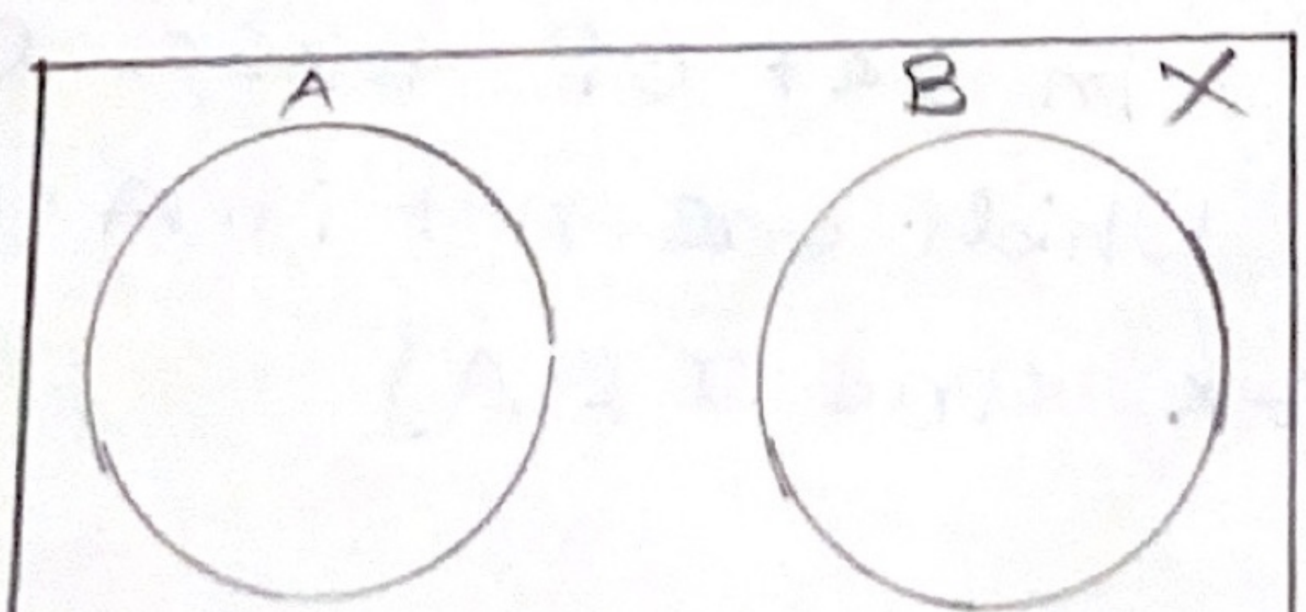


$A \cap B$ ▨

Remark: If $B \subset A$, then $A \cap B = B$

* **Disjoint Sets:-** Set A and set B are called disjoint sets if no common elements in A and B.

i.e $A \cap B = \phi$

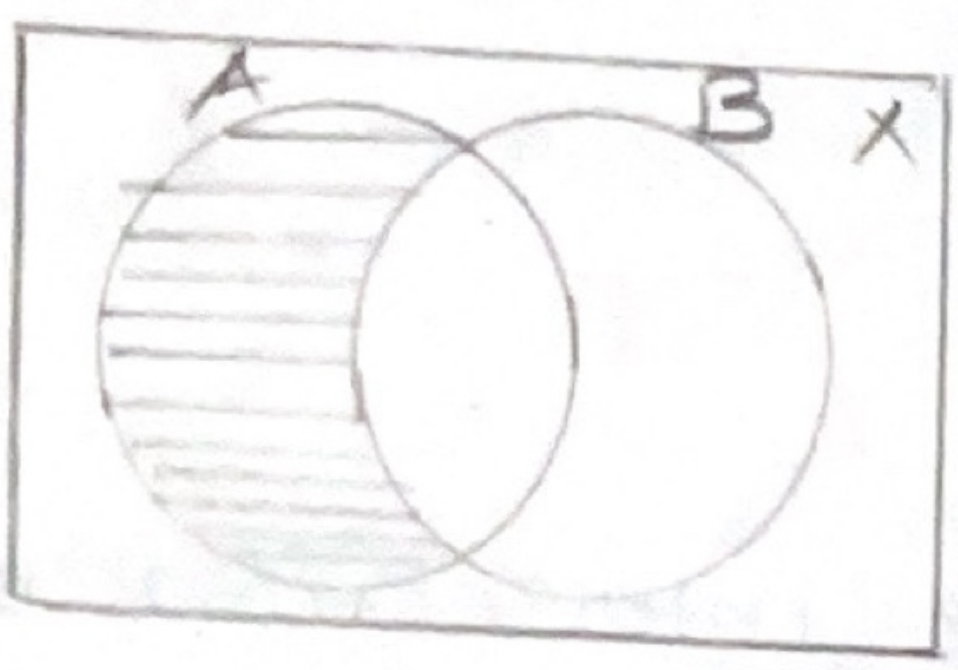


$A \cap B = \phi$

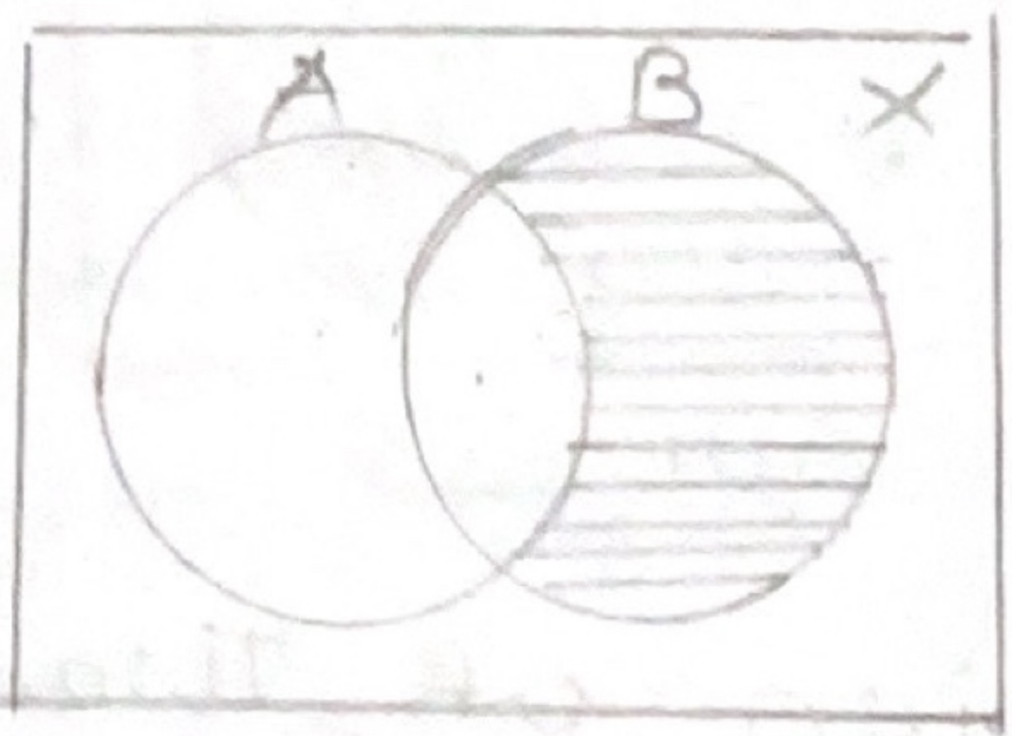
* **Difference of Two sets:-** The difference of A and B is denoted by '$A \sim B$' or '$A - B$' is set of all those elements which are in set A and not in B.

i.e $A - B = \{x;\ x \in A \text{ and } x \notin B\}$

Hence $B - A = \{x;\ x \notin A \text{ and } x \in B\}$



A - B = ▤

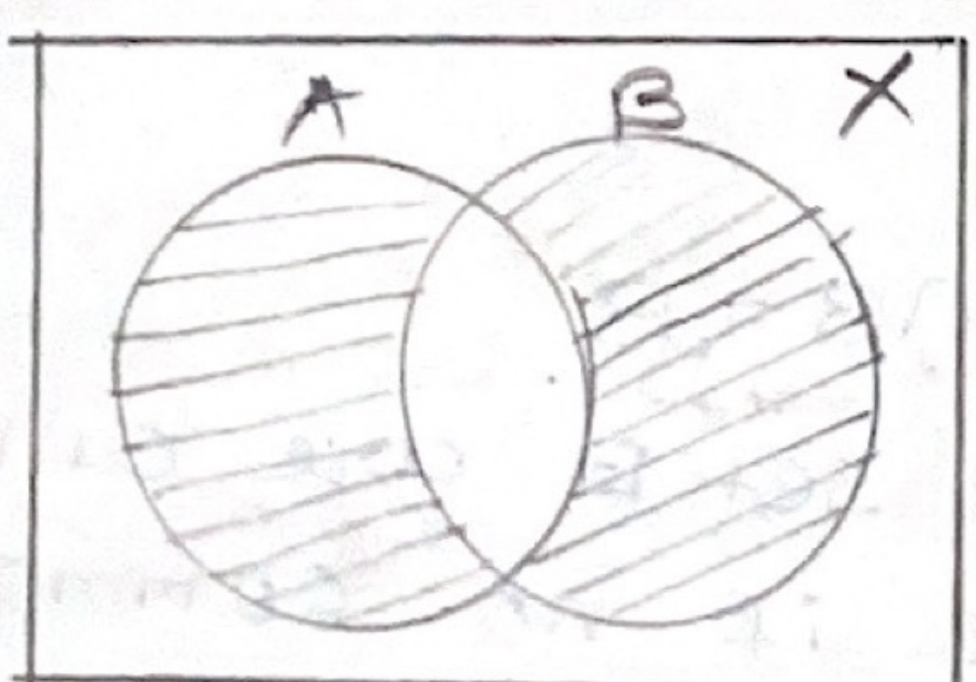


B - A = ▤

* **Symmetric Difference of Two sets:-** Symmetric diff. of set A and B is set of those element which are either in A or in B but not in Both. denoted by $A \oplus B$

i.e $A \oplus B = (A-B) \cup (B-A)$

$A \oplus B = (A \cup B) - (A \cap B)$



$A \oplus B$ = ▤

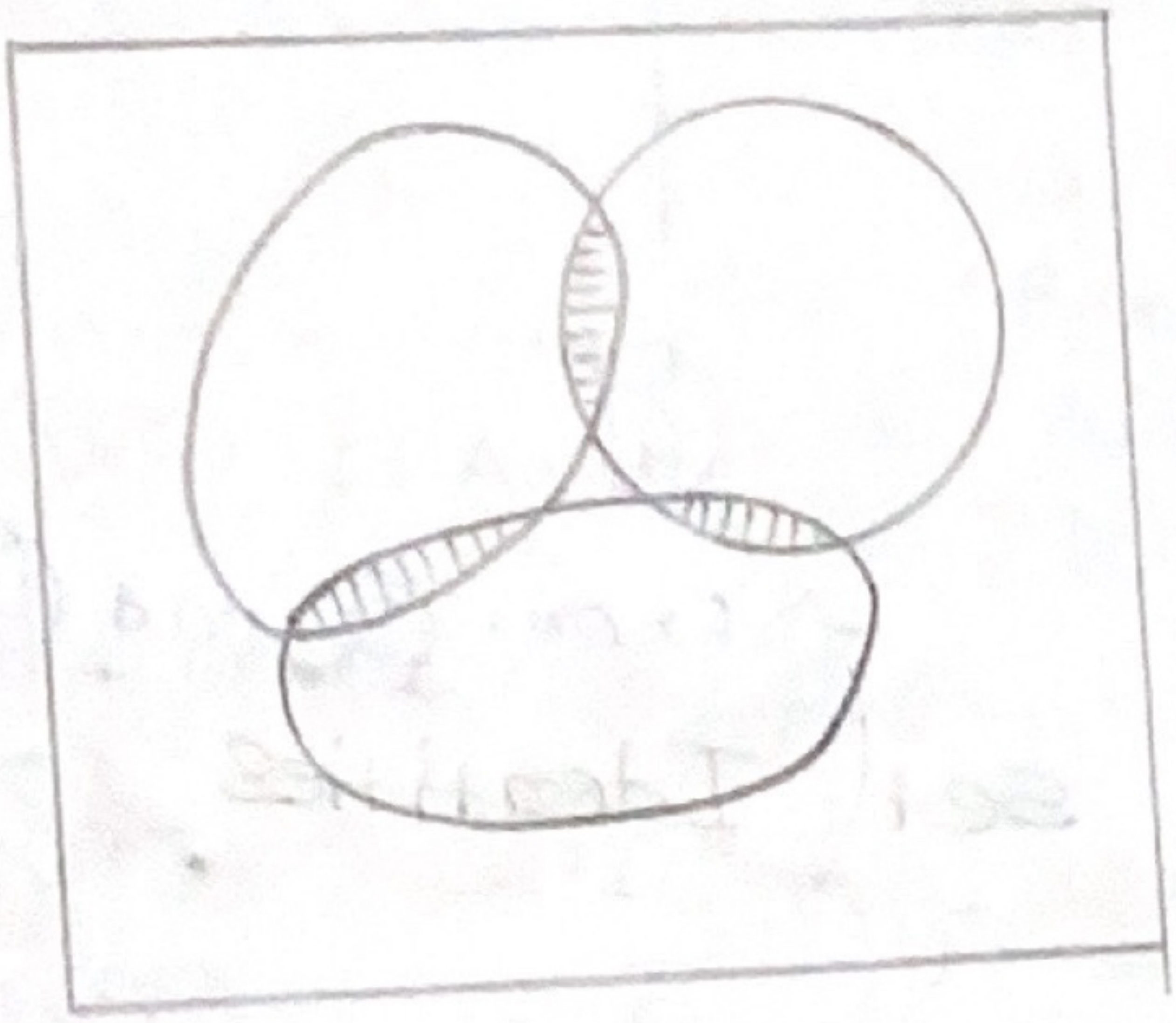
* **Complement of a set:-** Complement of a Set A in set of those element which are not in A (but in X)

i-e $A' = X - A = \{x:\ x \in X \text{ and } x \notin A\}$

Example:-1 Let A, B and C be sets such that

$(A \cap B \cap C) = \phi$, $(A \cap B) \neq \phi$, $(A \cap C) \neq \phi$, $(B \cap C) \neq \phi$.

Show the corresponding venn diagram.



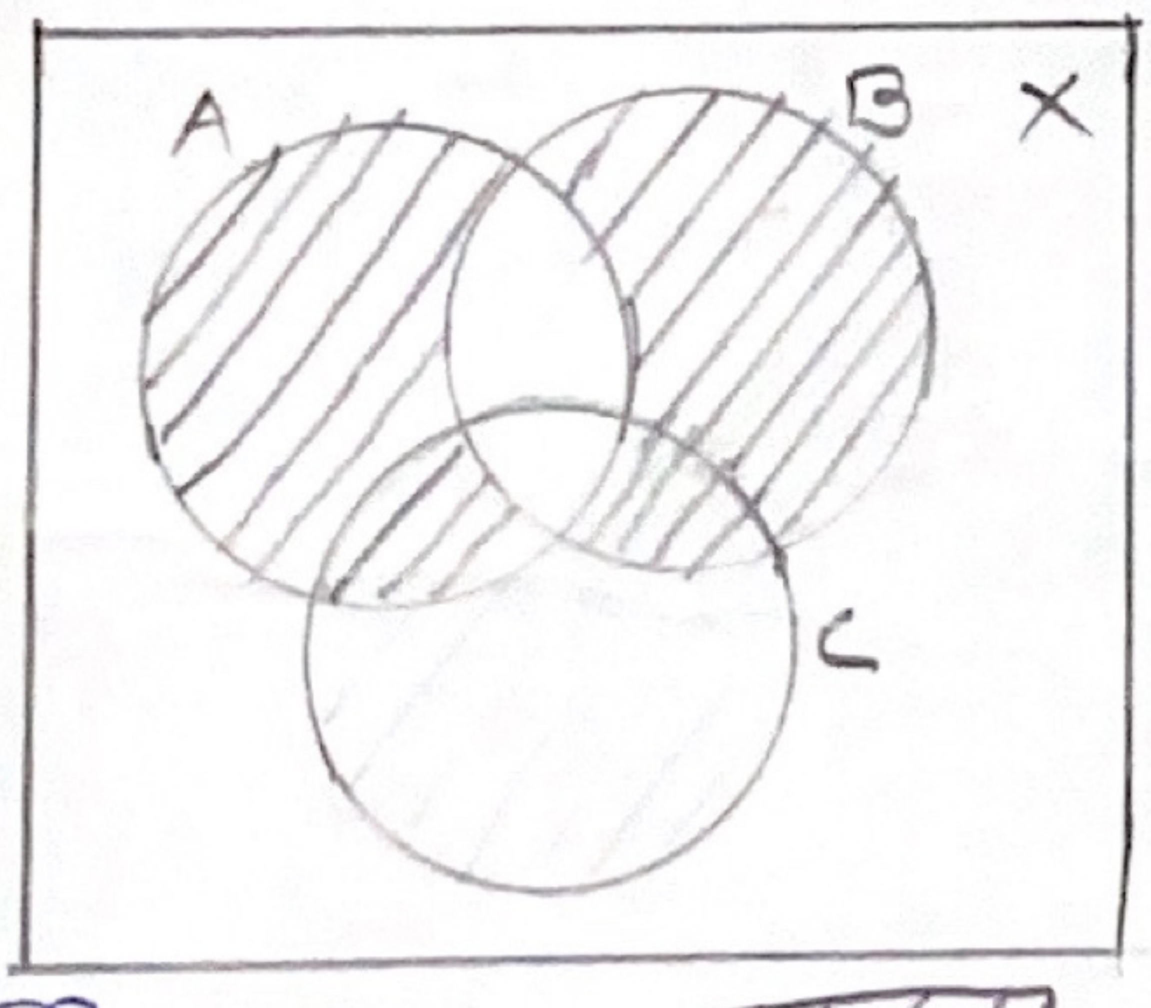
Example:-2 using Venn diagram, prove or disprove.

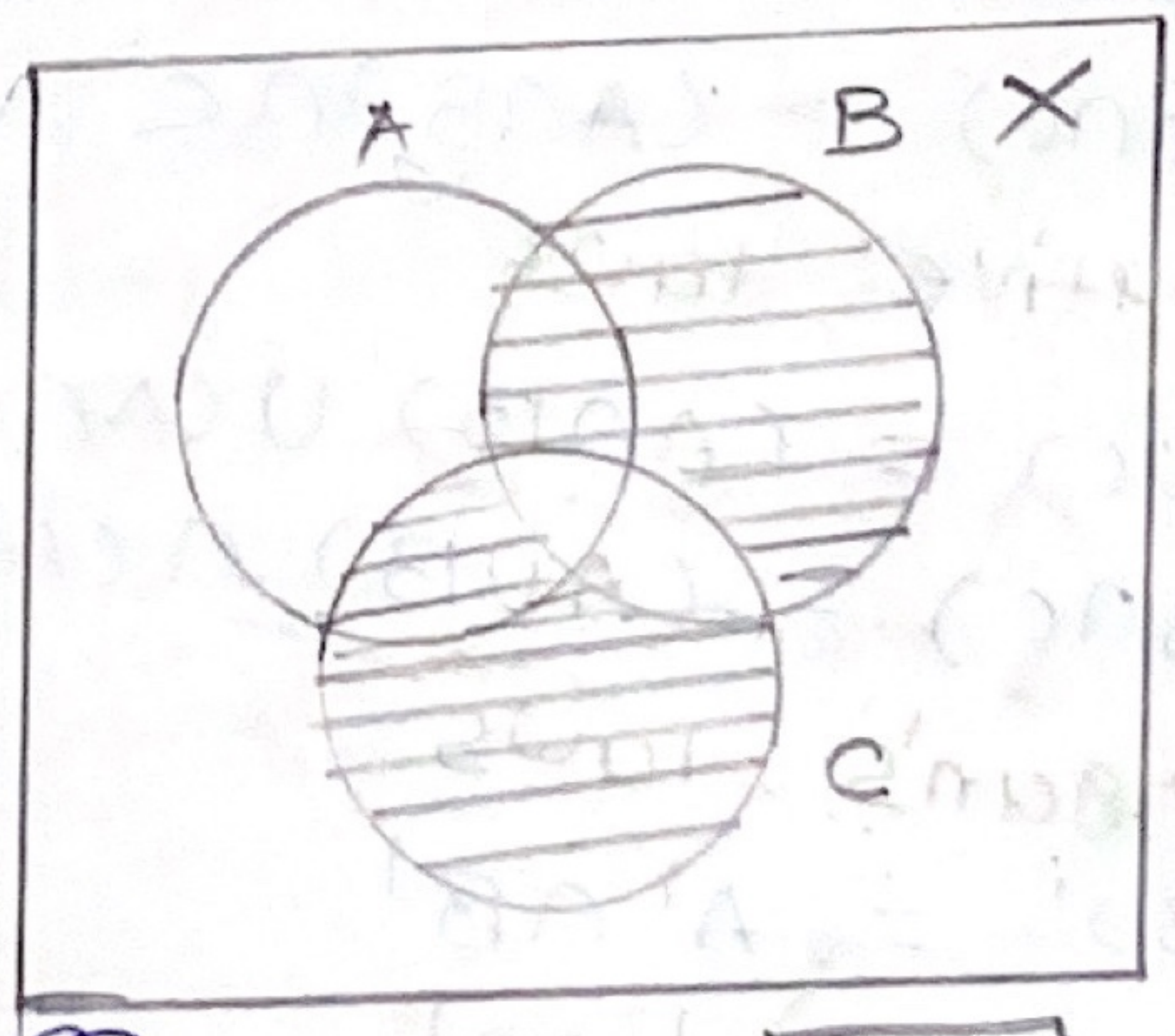
$$A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

Solution:

$A \oplus B = (A \cup B) - (A \cap B)$

$B \oplus C = (B \cup C) - (B \cap C)$



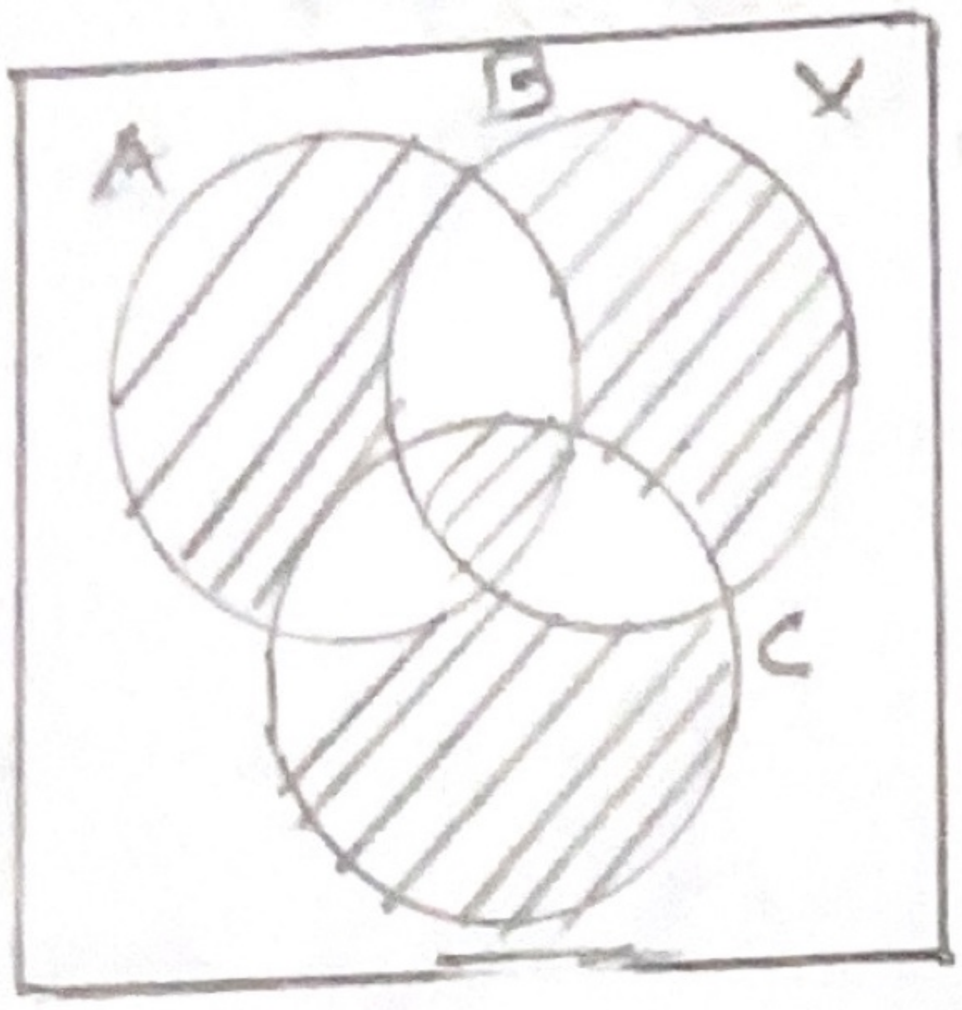


① $A \oplus B$ = ▨

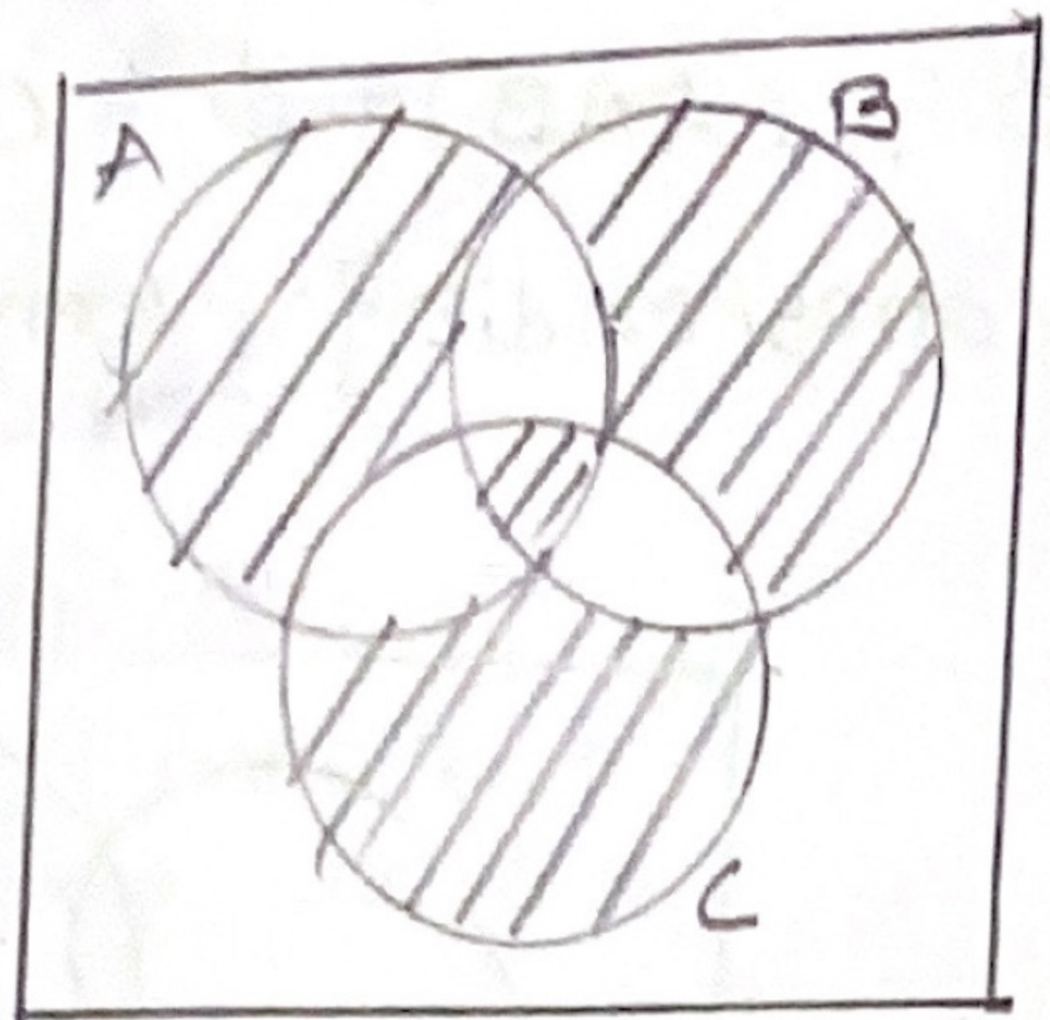
② $B \oplus C$ = ▤

$(A \oplus B) \oplus C$

$((A \cup B) - (A \cap B)) \cup C) - ((\cap C)$



③ $(A \oplus B) \oplus C$



④ $A \oplus (B \oplus C)$

→ From ③ and ④ $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

* Some Basic set Identities.

→ Idempotent laws

i) $A \cup A = A$

ii) $A \cap A = A$

→ Commutative laws

i) $A \cup B = B \cup A$

ii) $A \cap B = B \cap A$

→ Associative laws

i) $A \cup (B \cup C) = (A \cup B) \cup C$

ii) $A \cap (B \cap C) = (A \cap B) \cap C$

→ Distributive laws

i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

→ DeMorgan's laws

i) $(A \cup B)' = A' \cap B'$

ii) $(A \cap B)' = A' \cup B'$

→ Absorption laws

i) $A \cup (A \cap B) = A$

ii) $A \cap (A \cup B) = A$

→ Double complement

$(A')' = A$

* Partitions of a set:-

Let A be a set and $A_1, A_2, A_3, \ldots A_n$ are subsets of A then $A_1, A_2, \ldots A_n$ are called Partitions of A if

(i) $\bigcup_{i=1}^{n} A_i = A$ $\quad A_1 \cup A_2 \cup A_3 \ldots \cup A_n = A$

(ii) $A_i \cap A_j = \phi$ if $i \neq j$ (Pairwise disjoint)

e.g. $A-B$, $A \cap B$ and $B-A$ are Partitions of $A \cup B$ as $(A-B) \cup (B-A) \cup (A \cap B) = A \cup B$ and they are Pairwise disjoint.

* Some Conclusions from set operations

→ $A \cup \phi = A$

→ $A \cap X = A$

→ $A \cup X = X$

→ $A \cap \phi = \phi$

→ $A \cup A' = X$

→ $A \cap A' = \phi$

* Cardinality of Finite Set:-

A number of distinct element in set A is said to be cardinality of finite set.

→ Cardinality of Empty set is 0.

e.g. 1. $A = \{2, 3, 4, 5\}$

$n(A)$ or $|A| = 4$

* Principle of Inclusion and Exclusion:-

→ For Two sets:-

$|A \cup B| = |A| + |B| - |A \cap B|$

→ For Three sets:-

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

→ For Four sets:-

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

⇒ Let $\{A_1, A_2, \dots A_n\}$ be a finite collection of sets. Then

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| + \dots + (-1)^{n-1} |A_1 \cap A_2 \cap A_3 \dots \cap A_n|$$

* Understand a problem:-

→ In a problem a word 'or' means 'union' and a word 'and' means 'intersection'.

→ In a problem a word 'at least' means 'union'.

* Words	Meaning
→ Either in A or in B or in C	$\lvert A \cup B \cup C \rvert$
→ In A but not in B and C	$\lvert A \cup B \cup C \rvert - \lvert B \cup C \rvert$
→ Only in A (For two sets)	$\lvert A \rvert - \lvert A \cap B \rvert$

Example-1 Consider a set of integers from 1 to 250. Find how many of these numbers are divisible by 3 or 5 or 7? Also indicate how many are divisible by 3 or 5 but not by 7.

Solution: Let A, B, C denote the set of integers from 1 to 250 divisible by 3 or 5 or 7.

$$|A| = \left[\frac{250}{3}\right] = 83$$

$$|B| = \left[\frac{250}{5}\right] = 50$$

$$|C| = \left[\frac{250}{7}\right] = 35$$

$$|A \cap B| = \left[\frac{250}{3\times 5}\right] = 16$$

$$|A \cap C| = \left[\frac{250}{3\times 7}\right] = 11$$

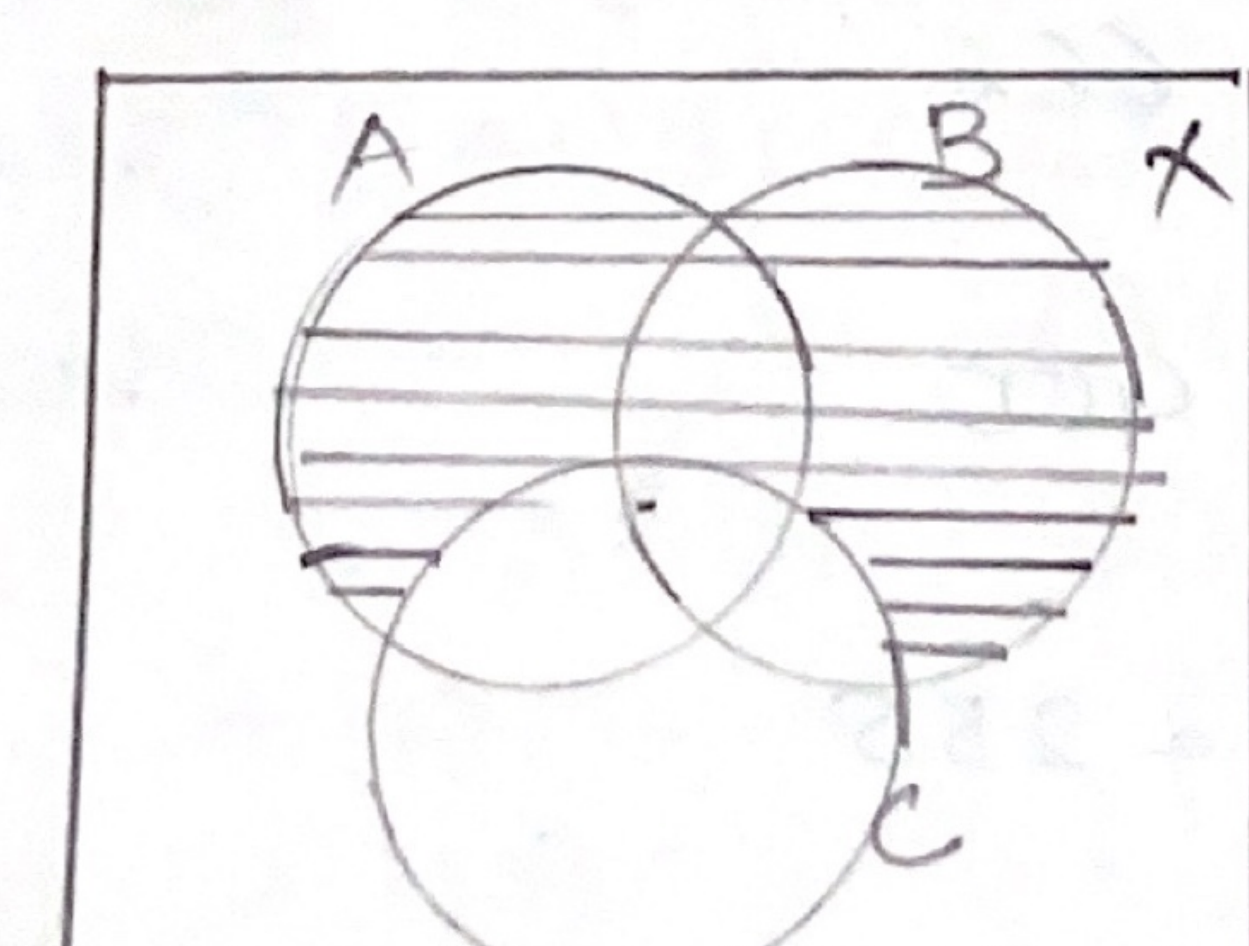
$$|B \cap C| = \left[\frac{250}{5\times 7}\right] = 7$$

$$|A \cap B \cap C| = \left[\frac{250}{3\times 5\times 7}\right] = 2$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 83 + 50 + 35 - 16 - 11 - 7 + 2$$

$$= 135$$



$|A \cup B \cup C| - |C|$

$|A|$ is number of integers divisible by 3.

$|B|$ number of integers divisible by 5.

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 83 + 50 - 16$$
$$= 133 - 16$$
$$= 117$$

Number of element divisible by 3 or 5. but not by 7 is

$|A \cup B \cup C| - |C| = 136 - 35 = 101.$ (0)

Ex: 2 How many integers between 1 to 2000 are divisible by 2, 3, 5 or 7.

Solution: Suppose set A denotes the number of integers between 1 to 2000 divisible by 2.

set B denotes of integers between 1 to 2000 divisible by 3.

set C denotes of integers between 1 to 2000 divisible by 5.

set D is the number of integers between 1 to 2000 divisible by 7.

$$|A| = \left[\frac{2000}{2}\right] = 1000$$

$$|B| = \left[\frac{2000}{3}\right] = 666$$

$$|C| = \left[\frac{2000}{5}\right] = 400$$

$$|D| = \left[\frac{2000}{7}\right] = 285$$

$|A \cap B| = \left[\frac{2000}{2\times 3}\right] = 333$

$|A \cap C| = \left[\frac{2000}{2\times 5}\right] = 200$

$|B \cap C| = \left[\frac{2000}{3\times 5}\right] = 133$

$|A \cap D| = \left[\frac{2000}{2\times 7}\right] = 142$

$|B \cap D| = \left[\frac{2000}{3\times 7}\right] = 95$

$|C \cap D| = \left[\frac{2000}{5\times 7}\right] = 57$

$|A \cap B \cap C| = \left[\frac{2000}{2\times 3\times 5}\right] = 66$

$|A \cap B \cap D| = \left[\frac{2000}{2\times 3\times 7}\right] = 47$

$|A \cap C \cap D| = \left[\frac{2000}{2\times 5\times 7}\right] = 28$

$|B \cap C \cap D| = \left[\frac{2000}{3\times 5\times 7}\right] = 19$

$|A \cap B \cap C \cap D| = \left[\frac{2000}{2\times 3\times 5\times 7}\right] = 9$

Number of elements divisible by 2 or 3 or 5 or 7 are $|A \cup B \cup C \cup D|$

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C|$
$- |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$
$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|$
$+ |B \cap C \cap D| - |A \cap B \cap C \cap D|$

$\therefore |A \cup B \cup C \cup D| = 1000 + 666 + 400 + 285$

$- 333 - 200 - 142 - 133 - 95 - 57$

$+ 66 + 47 + 28 + 19 - 9$

$= 2351 - 960 + 160 - 9$

$= 1542$

Ex:-3 Among 100 students, 32 study mathematics, 20 study physics, 45 study biology, 15 study mathematics and biology, 7 study mathematics and physics, 10 study physics and biology and 30 do not study any of the three subjects.

a) Find the number of students studying all three subjects.

b) Find the number of students studying exactly one of the three subjects.

Solution: Let A, B, C denotes the set of students studying mathematics, physics and biology respectively.

And $|X| = 100$

$|A| = 32$

$|B| = 20$

$|C| = 45$

$|A \cap C| = 15$

$|A \cap B| = 7$

$|B \cap C| = 10$

$|A' \cap B' \cap C'| = 30$

$|A' \cap B' \cap C'| = 100 - |A \cup B \cup C|$

$|A \cup B \cup C| = 100 - 30$

$= 70$

4) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C|$
$- |B \cap C| + |A \cap B \cap C|$

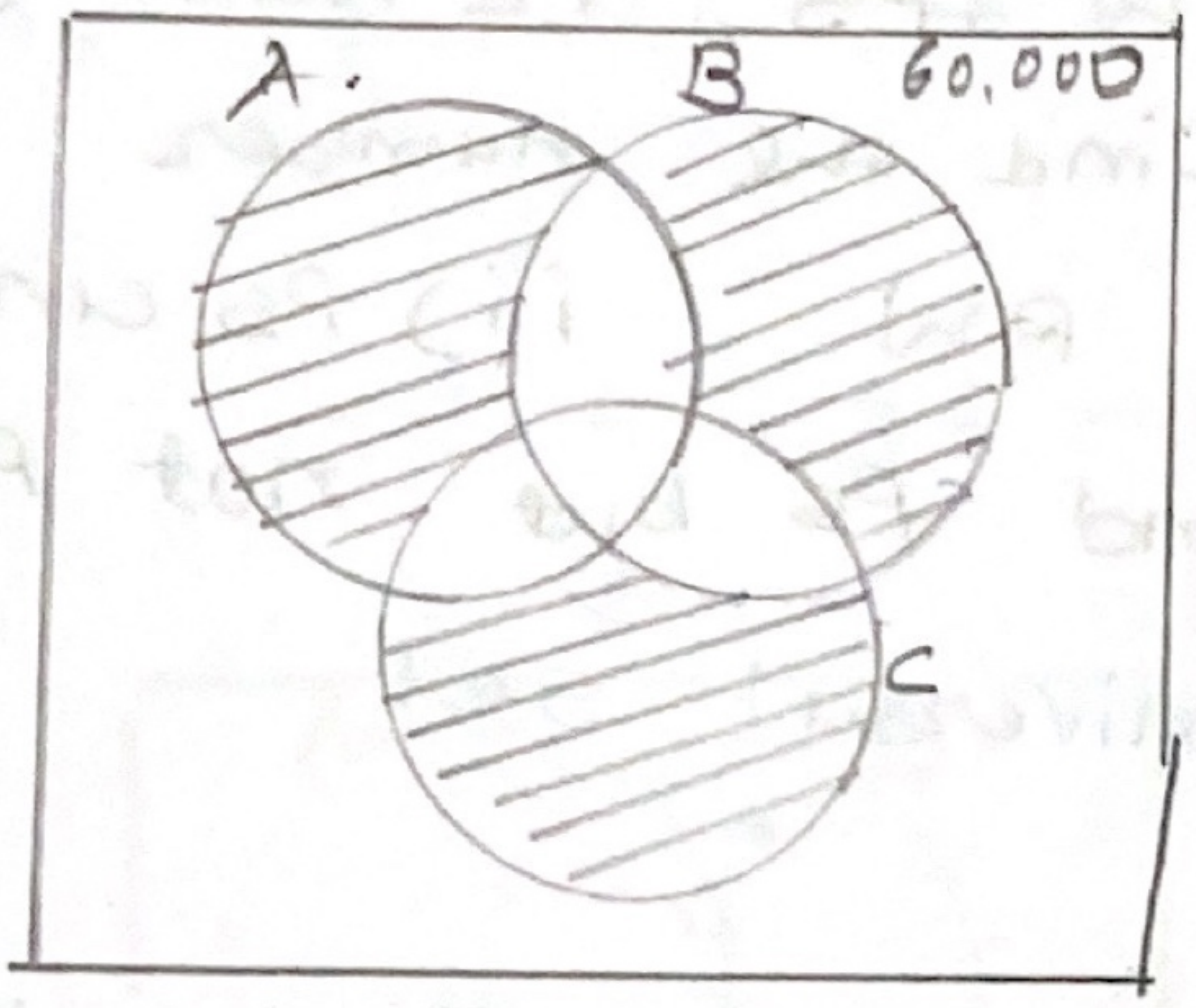
$70 = 32 + 20 + 45 - 7 - 15 - 10 + |A \cap B \cap C|$

$= 97 - 32 + |A \cap B \cap C|$

$70 - 65 = |A \cap B \cap C|$

$|A \cap B \cap C| = 5$

5 Students study all 3 ~~Subjects~~ Subject.



Number of students studying only mathematics is

$|A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$= 32 - 7 - 15 + 5$

$= 15$

Number of students studying only physics is

$|B| - |B \cap A| - |B \cap C| + |A \cap B \cap C|$

$= 20 - 7 - 10 + 5$

$= 8$

Number of students studying only biology is

$|C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 45 - 15 - 10 + 5$

$= 25$

∴ Number of students studying exactly one subject.

$= 15 + 8 + 25 = 48$

Ex:-4 In the survey of 100 new cars, it is found that 60 had Air conditioner (AC), 48 had Power Stearing (PS), 44 had Power windows (PW), 36 had AC + PW, 20 had PS + AC, 16 had PW + PS, 12 had all the three features. Find the number of cars that had i) only PW ii) PS and PW but not AC. iii) AC and PS but not PW.

Solution: $X \rightarrow$ universal set

$|X| = 100$

Let A denotes the number of AC in the cars,

$|A| = 60$

S denotes the number of PS in the cars,

$|S| = 48$

and W denotes the number of PW in the cars.

$|W| = 44$

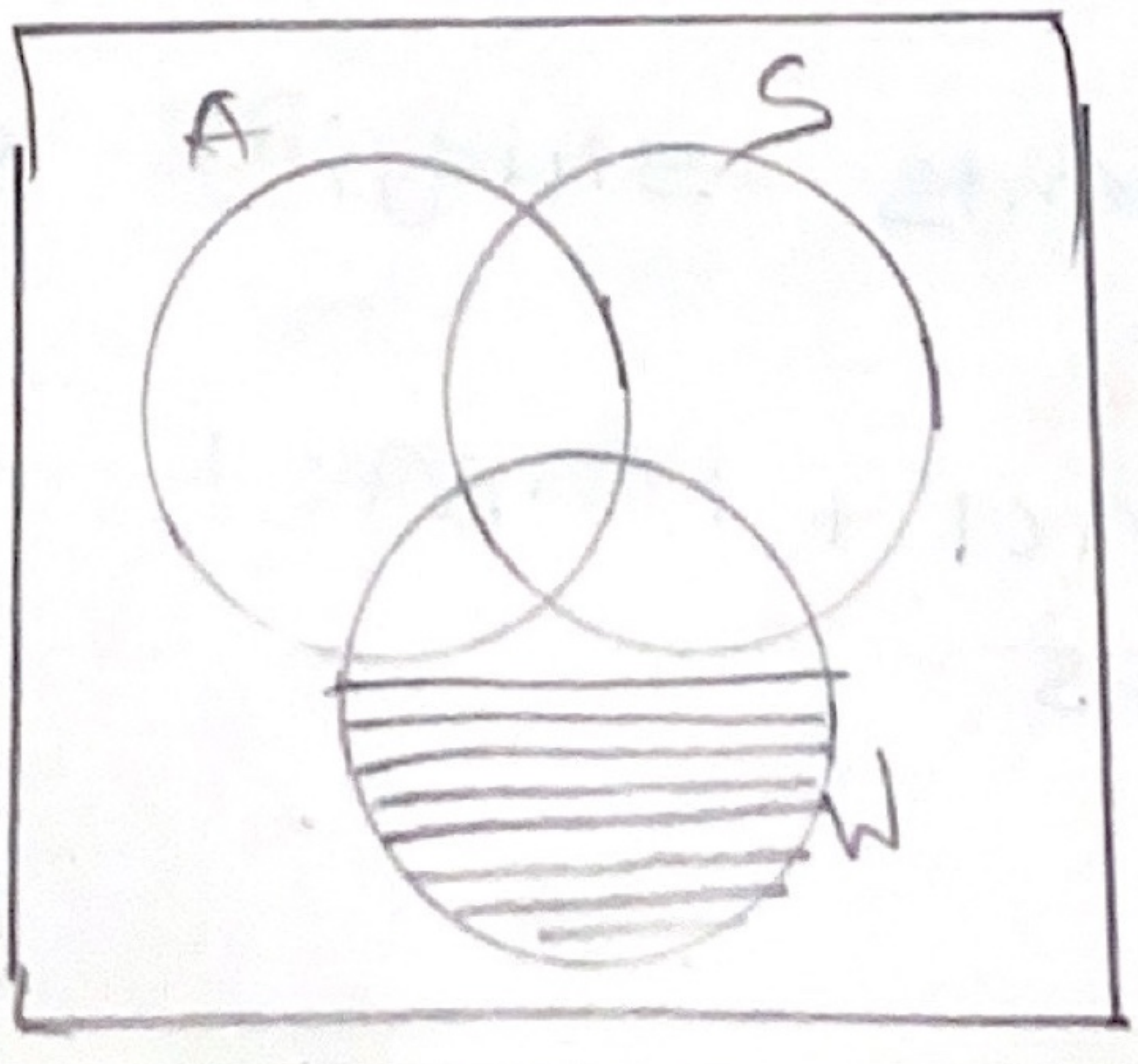
$|A \cap W| = 36$

$|A \cap S| = 20$

$|S \cap W| = 16$

$|A \cap S \cap W| = 12$

i)



Number of cars that had only power windows $= |W \cap A' \cap S'|$

$$|W \cap A' \cap S'| = W \cap (A \cup S)'$$
$$= W - (A \cup S)$$

$$|W - (A \cup S)| = |W| - [(W \cap A) \cup (W \cap S)]$$

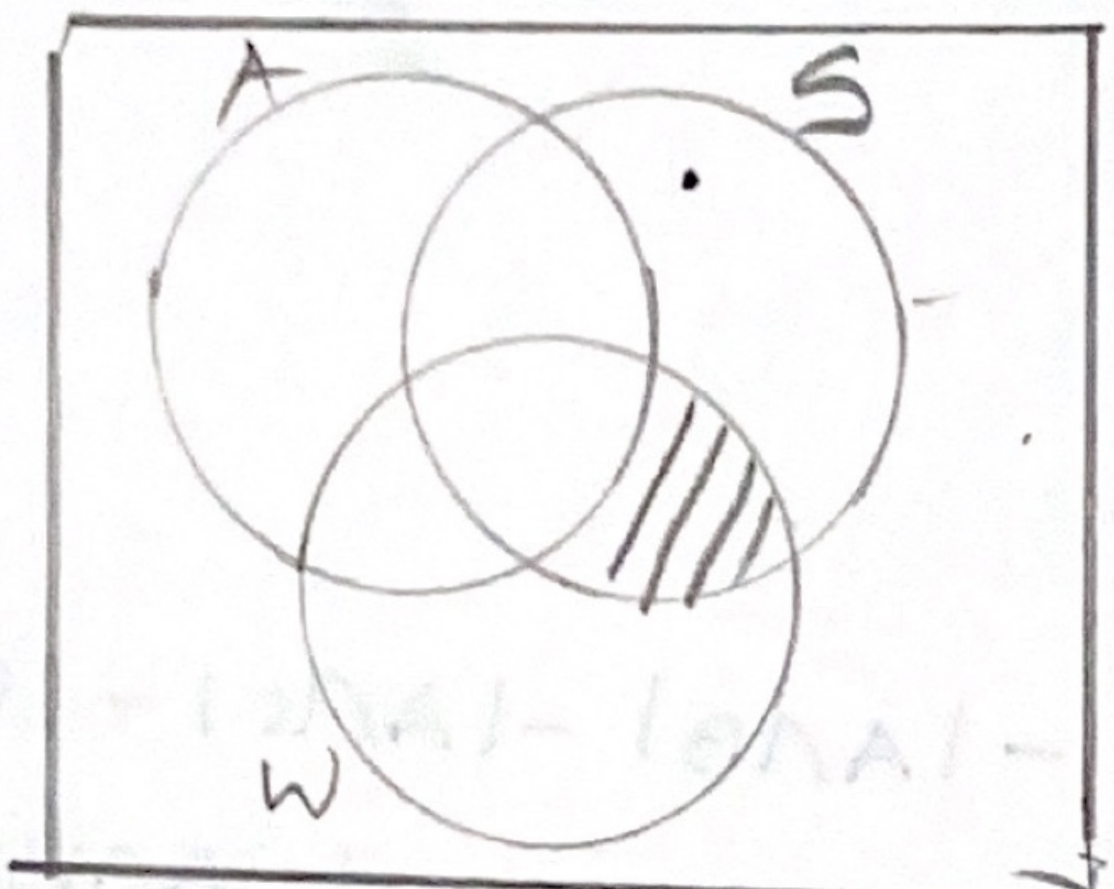
Now

$$|(W \cap A) \cup (W \cap S)| = |W \cap A| + |W \cap S| - |W \cap A \cap S|$$

$$\therefore |W \cap A' \cap S'| = |W| - |W \cap A| - |W \cap S| + |W \cap S \cap A|$$

$$= 44 - 36 - 16 + 12$$
$$= 4$$

ii)

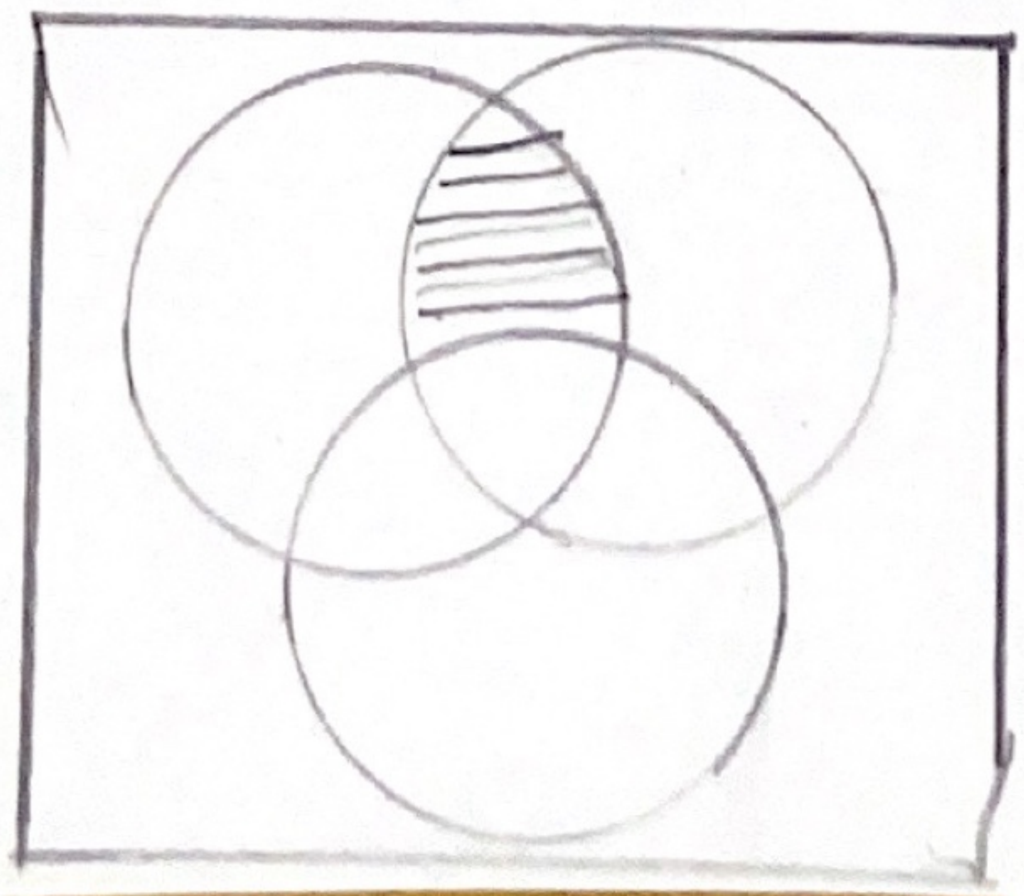


Number of cars that had PS and PW but not AC.

$$|S \cap W \cap A'| = |S \cap W| - |A \cap S \cap W|$$
$$= 16 - 12 = 4$$

iii) Number of cars that had AC and PS but not PW.

$$|A \cap S \cap W'| = |A \cap S| - |A \cap S \cap W|$$
$$= 20 - 12 = 8$$



Ex:-5 It is known that at the university 60 percent of the professors play tennis, 50 % of them play bridge. 70 % jog, 20 % play tennis and bridge, 30 % play tennis and jog, and 40 % play bridge and jog. If some one claimed that 20 % of the professors jog and play bridge and tennis, would you believe this claim? Why?

Solution: Let A, B, C, denotes the number of professors play tennis, bridge and jog respectively.

$|A| = 60$

$|B| = 50$

$|C| = 70$

$|A \cap B| = 20$

$|A \cap C| = 30$

$|B \cap C| = 40$

$|A \cap B \cap C| = 20$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 60 + 50 + 70 - 20 - 30 - 40 + 20$$

$$= 110$$

Which is not possible as $|A \cup B \cup C| \subset X$ and the number of element in $|A \cup B \cup C|$ cannot exceed number of elements in the universal set X.