Relations

* **Relations:** Relations are defined from Cartesian Product.

Ex: $A = \{1,2\}$, $B = \{a,b,c\}$

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

-> Let A and B be two non-empty sets, then a relation R from A to B is a subset of $(A \times B)$ $\quad R \subseteq A \times B$

Ex: $A = \{1,2\}$ and $B = \{1,2,3\}$

$R_1 = \{(1,1), (1,3), (2,1), (2,2)\}$

* **Binary Relation:**

A Binary Relation 'R' from A to B is a set of ordered pairs where first element is from Set A and the second element is from set B.

Ex: If $A = \{1,2,5\}$, $B = \{2,4\}$

then $A \times B = \{(1,2), (1,4), (2,2), (2,4), (5,2), (5,4)\}$

(i) Relation $x < y$

$R = \{(1,2), (1,4), (2,4)\}$

(ii) Relation $x = y$

$R = \{(2,2)\}$

Note: Suppose R is a relation from A to B

(1) R is a set of ordered pair (a,b) where $a \in A$, $b \in B$

(2) Every such ordered pair ~~(a,b)~~ is written as aRb and read as 'a is related to b' by R

(3) R is called Binary Relation

* Domain and Range

Let R be a relation from A to B then the elements of A which are in relation are said to be domain and the element of B which occur in R constitute in Range of R.

Ex: If $R = \{(1,p), (1,q), (2,q)\}$ be a relation then Domain $= \{1, 2\}$ and Range $= \{p, q\}$

* Matrix Representation of Relations:-

Let $A = \{a_1, a_2, \ldots a_m\}$ and $B = \{b_1, b_2, \ldots b_n\}$ are finite sets containing m and n elements respectively and let R be a relation from A to B. Then R can be represented by the mn matrix.

$$M_R = [M_{ij}] \text{ where } M_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

The Matrix M_R is called the matrix of R.

Ex: Let $A = \{1, 3, 4\}$,

$R = \{(1,1), (1,3), (3,3), (4,4)\}$

Solution:

$$M_R = \begin{array}{c} \\ 1 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Ex: $A = \{1, 2, 3, 4\}$

$B = \{1, 4, 6, 8, 9\}$

aRb if and only if $b = a^2$ then find the relation matrix M_R.

Solution: $R = \{(1,1), (2,4), (3,9)\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 4 & 6 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad 4 \times 5$$

Ex: $A = \{1, 2, 3, 4, 6\} = B$

aRb if and only if $a|b$ then find the relation matrix M_R.

Solution:

$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,4),$
$(2,6), (3,6), (2,2), (3,3), (4,4), (6,6)\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad 5 \times 5$$

Ex: $A = \{1, 2, 3, 5, 6, 10, 15, 30\} = B$,
aRb iff $a|b$. Find relation Matrix.

Solution:

$R = \{(1,1), (1,2), (1,3), (1,5), (1,6), (1,10), (1,15), (1,30), (2,2), (2,6), (2,10), (2,30), (3,3), (3,6), (3,15), (3,30), (5,5), (5,10), (5,15), (5,30), (6,6), (6,30), (10,10), (10,30), (15,15), (15,30), (30,30)\}$.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 5 \\ 6 \\ 10 \\ 15 \\ 30 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 5 & 6 & 10 & 15 & 30 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

* Graphical Representation of a Relation

(Digraph)

-> A graph consist set of vertices and set of edges.

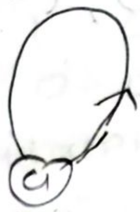
-> Each element of the set is represented by a point; These points are called nodes or vertices.

-> An arc (arc) is drawn from each point to its related point. These arcs are called edges.

-> All arcs with an arrow are called directed arcs.

-> The resulting graphical representation of R is called a directed graph or digraph of R.

aRa



aRb

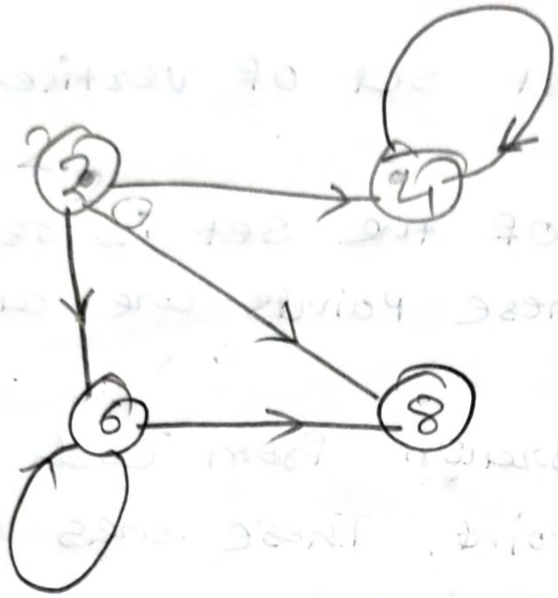
bRa

aRb ∧ bRa

Ex: Let $A = \{2,4,6\}$ and $B = \{4,6,8\}$, $R \subseteq A \times B$

$R = \{(2,4), (2,6), (2,8), (4,4), (6,6), (6,8)\}$

Solution:

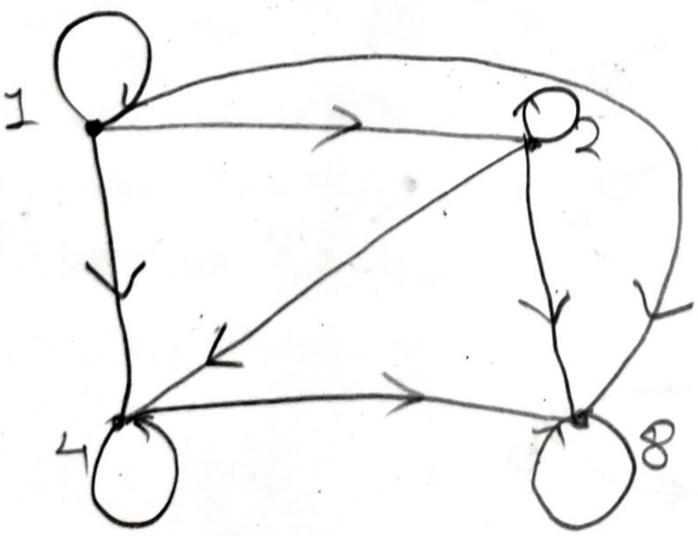


Ex: $A = \{1,2,4,8\} = B$

aRb iff $a|b$ (a divides b) then find the digraph of relation.

Solution:

$R = \{(1,1), (1,2), (1,4), (1,8), (2,2), (2,4), (2,8),$
$(4,4), (4,8), (8,8)\}$

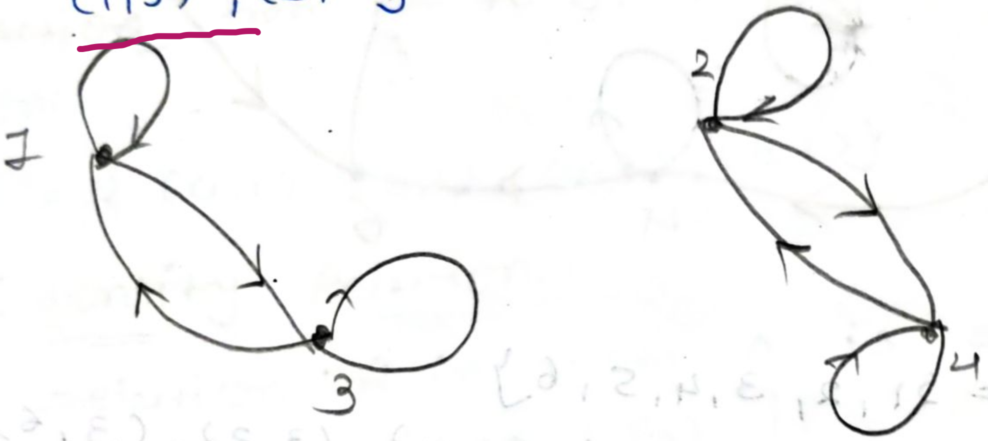


Ex: $A = \{1, 2, 3, 4\}$ (280)

If $R = \{(a, b) \mid (a-b)$ is an integral multiple of $2\}$ then find the digraph of relation.

Solution:

$R = \{(1,1), (2,2), (3,3), (4,4), (2,4), (4,2), (1,3), (3,1)\}$

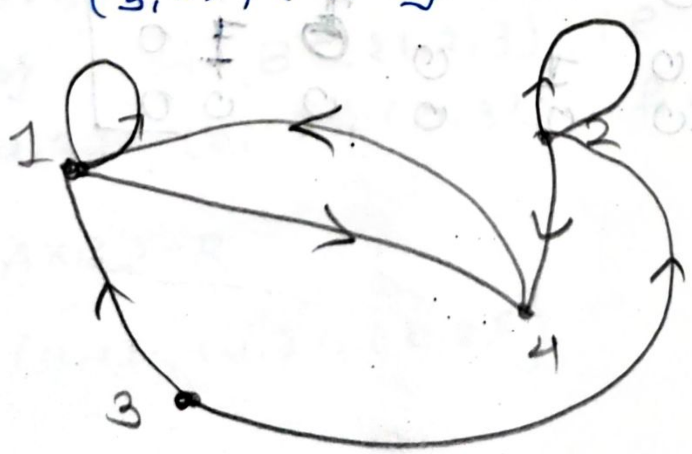


Ex: $A = \{1, 2, 3, 4\}$

and $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

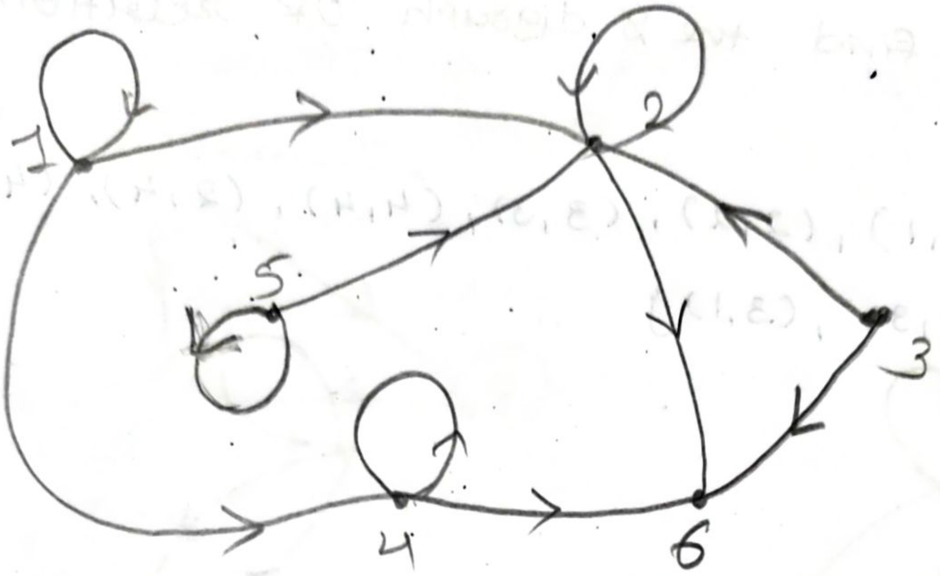
Draw the digraph of R.

Solution: $R = \{(1,1), (1,4), (2,2), (2,4), (3,1), (3,2), (4,1)\}$



Ex: Find the relation determined by the diagraph and give its matrix.

Solution:



Solution:

$A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(1,1), (1,2), (2,2), (1,4), (3,2), (3,6), (4,6), (4,4), (5,5), (5,2), (2,6)\}$

Relation Matrix

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

* Inverse Relation:-

Let R be a relation from set A to set B then inverse relation R^{-1} from set B to set A is ~~called~~ defined by $\{(b,a) : (a,b) \in R\}$

Ex: Let $A = \{1,2,3\}$ and $B = \{a,b\}$

$R = \{(1,a), (1,b), (3,a), (2,b)\}$ be a relation from A to B.

Solution:

$R^{-1} = \{(a,1), (b,1), (a,3), (b,2)\}$

* Identity Relation:-

A relation R in a set A is said to be identity relation (I_A). if

$I_A = \{(x,x) : x \in A\}$

Ex: $A = \{2,4,6\}$

$I_A = \{(2,2), (4,4), (6,6)\}$

* Complement of a Relation:-

$\bar{R} = \{(a,b) \mid (a,b) \in A \times B \text{ and } (a,b) \notin R\}$

$\bar{R} = (A \times B) - R$

$A = \{a,b\}$ and $B = \{1,2,3\}$, if

$R = \{(a,2), (b,1), (b,3)\}$ find $\bar{R}$.

$\bar{R} = (A \times B) - R$

$\bar{R} = \{(a,1), (a,3), (b,2)\}$

* Composition of Relations:-

Let A, B and C be sets.

Let R is the relation from A to B

i.e $R \subseteq A \times B$

S is the relation from B to C

i.e $S \subseteq B \times C$

The composition of R and S, denoted by R.S, Where

(RoS)

$R.S = \{(a,c) \in A \times C : \text{for some } b \in B, (a,b) \in R \text{ and } (b,c) \in S\}$

Ex: let $A = \{1,2,3\}$, $B = \{p,q,r\}$,

$C = \{x,y,z\}$

$R = \{(1,p), (1,r), (2,p), (2,q)\}$

$S = \{(p,y), (q,x), (q,y), (r,z)\}$

compute R.S

Solution: Given $R = \{(1,p), (1,r), (2,p), (2,q)\}$

$S = \{(p,y), (q,x), (q,y), (r,z)\}$

$R.S = \{(1,y), (1,z), (2,y), (2,x)\}$

* Universal Relation in a set

A relation R in a set A is said to be universal relation if $R = A \times A$

* **Void Relation (Empty Relation)**

A relation R in a set is said to be a void relation if R is a null set.
i.e if $R = \phi$

Ex: The relation R on the set $A = \{1,2,3\}$ by $R = \{(a,b) : a+b = 10\}$

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$R = \{\ \}$ or $R = \phi$

* **Types of Relation**

→ **Reflexive Relation**

Let R be a relation in set A, i.e $R \subset A \times A$ then R is known as reflexive relation if $\forall\ a \in A$. $(a,a) \in R$.
i.e Every element is related with itself.

Ex: $A = \{1,2,3\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$ - yes

$R_2 = \{(1,1), (2,2), (3,3), (1,2)\}$ - yes

$R_3 = \{(1,2), (2,2), (3,3), (2,1)\}$ NO

$R_4 = A \times A$ - yes

Ex: If A is a set of all straight lines in 2-D Plane and R is a relation.

$R = \{(a,b) : a \text{ is Parallel to } b\}$

then R is reflexive relation as every straight line is parallel to itself.

Ex: If I is a set of integers and R is a relation

$R = \{(a,b) : a|b \text{ (a divides b)}\}$

then R is reflexive relation as every element divide itself.

Ex: If I is a set of integers and R is a relation

$R = \{(a,b) ; 5|a-b \text{ (5 divides (a-b))}\}$

then R is reflexive relation as

$[5|a-a$ or $5|0]$ zero is divisible by every non-zero number.

→ If R is not reflexive, then R is known as irreflexive if $\forall a \in A$

$(a,a) \notin R$ Or $a \not R a$.

hence all the diagonal entries in M_R will be zero and digraph of irreflexive relation will have no loop for any element of A.

→ Symmetric Relation

Let R be a relation in a set A i.e $R \subseteq A \times A$. Then the Relation R is said to be Symmetric relation.

if $(a,b) \in R \Rightarrow (b,a) \in R$

i.e if a is related to with b then relation R then b should be related with a with the same relation R.

$$aRb \Rightarrow bRa$$

in other way $R = R^{-1}$ then relation is symmetric.

→ ϕ is a symmetric relation.

Ex: $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,2), (2,1), (3,4), (4,3)\}$ - yes

$R_2 = \{(1,3), (3,1), (3,4), (4,4)\}$ - NO

$R_3 = \{(1,1), (2,2), (3,3), (4,4)\}$ - yes

→ $(A \times A)$ is a symmetric relation.

Ex: A is a set of all straight line in 2-D plane and R is a relation.

$R = \{(a,b) ; a \text{ is parallel to } b\}$

then R is symmetric relation as if a is parallel to b then b is also parallel to a.

i.e $(a,b) \in R \Rightarrow (b,a) \in R$

→ **Compatible Relation**

A Relation R on a set S is said to be compatible if it is reflexive and symmetric.

→ **Antisymmetric Relation**

A relation R on a set A is said to be antisymmetric if $(a,b) \in R$ and $(b,a) \in R$, then $\boxed{a = b}$

i.e if a is related with b with relation R and also b is related with a with the same relation R then a and b are same.

Ex: $A \subseteq B, B \subseteq A \Rightarrow A = B$

So 'subset' relation is antisymmetric relation.

Ex: $a \leq b, b \leq a \Rightarrow a = b$

So 'less than or equal to' relation is antisymmetric relation.

→ Transitive Relation:

Let R be a relation on set A i.e $R \subset A \times A$, then R is called transitive relation on set A if

$(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$.

i.e if a is related with b and b is related with c and this shows a is related with c with the same relation then R is called transitive relation.

$aRb, bRc \Rightarrow aRc$

Ex:1 $A \subset B, B \subset C \Rightarrow A \subset C$

So subset relation is transitive relation.

Ex:2 R is a relation on the set of Integers I.

$R = \{(a,b) \text{ iff } 5 | a-b\}$

$aRb \Leftrightarrow 5 | a-b$

then R is transitive relation as

aRb, bRc

$\Rightarrow 5 | a-b$ and $5 | b-c$

$\Rightarrow 5 | (a-b+b-c)$

$\Rightarrow 5 | a-c$

$\Rightarrow aRc$

hence, $aRb, bRc \Rightarrow aRc$

So, R is transitive relation.

Ex: Let $A = \{1, 2, 3, 4\}$

(i) $R_1 = \{(1,2), (2,3), (1,3)\}$ - yes

(ii) $R_2 = \{(1,3), (3,2), (1,4)\}$ - NO

(iii) $R_3 = \{(1,2), (3,4)\}$ - yes

(iv) $R_4 = \phi$ - yes

(v) $R_5 = A \times A$ - yes

(vi) $R_6 = \{(1,2), (2,1), (1,1)\}$ - NO

(1,1) (2,1)

(1,2)

* Equivalence Relation

A relation on a set A is called an Equivalence relation, if it is

i) Reflexive relation

i.e. $(a,a) \in R, \forall a \in A$

ii) Symmetric relation

i.e $(a,b) \in R \Rightarrow (b,a) \in R, \forall a,b \in A$

iii) Transitive

i.e $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$
$\forall a,b,c \in A$

Ex: Let $A = \{1,2,3,4\}$

i) $R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$ - yes

ii) $R_2 = \{(1,1),(1,2),(2,2),(3,3),(4,4)\}$ - No

iii) $R_3 = \phi$ - No

iv) $R_4 = A \times A$ yes

Ex: Let $A = \{1,2,3\}$

i) $R_1 = \{(1,1),(2,2),(3,3)\}$ - yes

ii) $R_2 = \{(1,1),(2,2),(3,3),(2,1),(1,2)\}$ yes

iii) $R_3 = \{(1,1),(2,2),(3,2),(1,3)\}$ - No

* Transitive closure

The transitive closure of a relation R is the smallest relation containing R.

Transitive closure of R is denoted by R^*

→ To find the transitive closure of R.

Let A be a set and cardinality of A is n,

i.e $|A| = n$

and let R be a relation on A. Then,

$R^* = R \cup R^2 \cup R^3 \cup \dots \cup R^n$

where $R^2 = R.R$

$R^3 = R.R.R$ etc.

Ex:- If $A = \{1, 2, 3, 4, 5\}$ (298)

and $R = \{(1,2), (3,4), (4,5), (4,1), (1,1)\}$

find its transitive closure

solution: let R^* the transitive closure of R.

$R = \{(1,2), (3,4), (4,5), (4,1), (1,1)\}$

$|A| = 5$

$R^2 = R.R$ (ROR)

$= \{(1,2), (3,4), (4,5), (4,1), (1,1)\}$.

$\{(1,2), (3,4), (4,5), (4,1), (1,1)\}$

$= \{(3,5), (3,1), (4,2), (4,1), (1,2), (1,1)\}$

$R^3 = R^2 . R$

$= \{(3,5), (3,1), (4,2), (4,1), (1,1), (1,2)\} .$

$\{(1,2), (3,4), (4,5), (4,1), (1,1)\}$

$= \{(3,2), (3,1), (4,2), (4,1), (1,2), (1,1)\}$

$R^4 = R^3 . R$

$= \{(3,2), (3,1), (4,2), (4,1), (1,2), (1,1)\} .$

$\{(1,2), (3,4), (4,5), (4,1), (1,1)\}$

$= \{(3,2), (3,1), (4,2), (4,1), (1,2), (1,1)\}$

$R^5 = R^4 . R$

$= \{(3,2), (3,1), (4,2), (4,1), (1,2), (1,1)\} .$

$\{(1,2), (3,4), (4,5), (4,1), (1,1)\}$

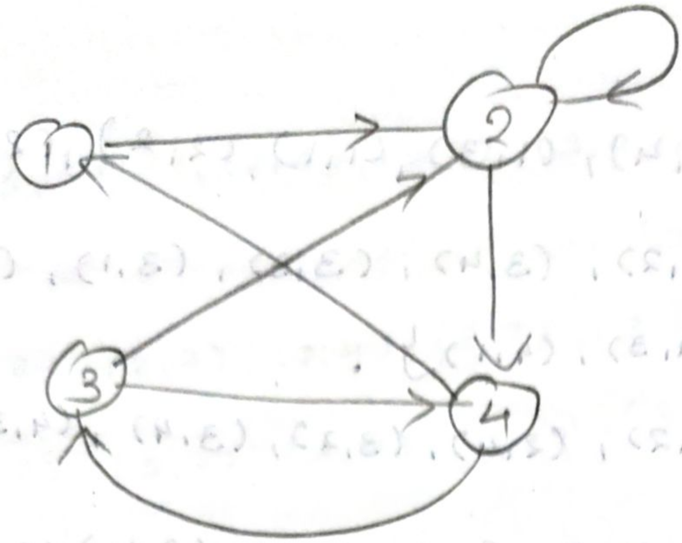
$= \{(3,1), (3,2), (4,2), (4,1), (1,2), (1,1)\}$

Hence $R^3 = R^4 = R^5$

$R^* = R \cup R^2 \cup R^3 \cup R^4 \cup R^5$

$= \{(1,2), (3,4), (4,5), (4,1), (1,1), (3,5), (3,1), (4,2), (3,2)\}$

Ex: 2 Find the transitive closure without using Warshall's Algorithm of a Relation R on $A = \{1,2,3,4\}$ defined by the directed graph as shown in the figure (299)



Solution:

$R = \{(1,2), (2,2), (2,4), (3,2), (3,4), (4,3), (4,1)\}$

$R^2 = R \cdot R$

$= \{(1,2), (2,2), (2,4), (3,2), (3,4), (4,3), (4,1)\}.$

$\{(1,2), (2,2), (2,4), (3,2), (3,4), (4,3), (4,1)\}$

$R^2 = \{(1,2), (1,4), (2,2), (2,4), (2,3), (3,2), (3,4), (3,3), (3,1), (4,2), (4,4)\}$

$R^3 = R^2 \cdot R$

$= \{(1,2), (1,4), (2,2), (2,4), (2,3), (3,2), (3,4), (3,3), (3,1), (4,2), (4,4)\}.$

$\{(1,2), (2,2), (2,4), (3,2), (3,4), (4,3), (4,1)\}$

$= \{(1,2), (1,4), (1,3), (1,1), (2,2), (2,4), (2,3), (2,1), (3,2), (3,4), (3,3), (3,1), \cancel{(3,4)}, (4,2), (4,4), (4,3), (4,1)\}$

$R^4 = R^3 . R$

$= \{(1,2), (1,4), (1,3), (1,1), (2,2), (2,4), (2,3),$

$(2,1), (3,2), (3,4), (3,3), (3,1), (4,2),$

$(4,4), (4,3), (4,1)\}.$

$\{(1,2), (2,2), (2,4), (3,2), (3,4), (4,3), (4,1)\}$

$= \{(1,2), (1,4), (1,3), (1,1), (2,2), (2,4), (2,3), (2,1),$

$(3,2), (3,4), (3,3), (3,1), (4,2), (4,4), (4,3),$

$(4,1)\}$

$R^* = R \cup R^2 \cup R^3 \cup R^4$

$= \{(1,2), (1,4), (1,3), (1,1), (2,2), (2,4), (2,3),$

$(2,1), (3,2), (3,4), (3,3), (3,1), (4,2), (4,4),$

$(4,3), (4,1)\}$

* Warshall's Algorithm

If relation R for a set is not transitive, we need to apply closure to make it transitive.

This closure is known as transitive closure.

We use Warshall's method to find the transitive closure.

Ex: Let $A = \{1,2,3,4\}$ and let

$R = \{(1,1), (1,2), (1,4), (2,4), (3,1), (3,2), (4,2), (4,3), (4,4)\}$. Find transitive closure by using Warshall's algorithm.

Solution:

$R = \{(1,1), (1,2), (1,4), (2,4), (3,1), (3,2), (4,2), (4,3), (4,4)\}$

$$M_R = W_0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 0 & 1 & 1 & 1 \end{array}$$

Step-1 In 1st column 1's are = 1, 3

In 1st Row 1's are = 1, 2, 4

Now add (1,1), (1,2), (1,4), (3,1), (3,2), (3,4) in W_1

$$W_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 0 & 1 & 1 & 1 \end{array}$$

? 1

Step-2 In 2^{nd} column 1's are:- 2, 3, 4

In 2^{nd} Row 1's are: 4

Now add. (2,4), (3,4), (4,4) in W_2

$$W_2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 0 & 1 & 1 & 1 \end{array}$$

Step-3 In 3^{rd} column 1's are:- 4

In 3^{rd} Row 1's are:- 1, 2, 4

Now add. (4,1), (4,2), (4,4) in W_3

$$W_3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 1 & 1 \end{array}$$

Step-4 In 4^{th} column 1's are:- 1, 2, 3, 4

In 4^{th} Row 1's are:- 1, 2, 3, 4

Now add; (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4) in W_4

$$W_4 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 & 1 \end{array}$$

$R^* = \{ (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4) \}$

Ex:1) Let I be set of all integers and if the relation R be defined as aRb iff $(a-b)$ is an even integers. Show that R is an equivalence relation.

Solution: $A = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ (A set of all integers)

aRb iff $(a-b)$ is an even integer.

→ Reflexive

$\forall a \in A \; (a,a) \in R$

Since $a-a=0$ and 0 is an even integer

aRa

∴ Therefore, R is Reflexive.

→ Symmetric

Let aRb → bRa

⇒ $a-b$ is an even integer

⇒ $b-a$ is an even integer.

⇒ aRb → bRa

∴ Therefore, R is Symmetric.

→ Transitive

Let aRb, bRc

⇒ $a-b$ is an even integer and also $b-c$ is an even integer.

⇒ $a-c$ is an even integer.

⇒ aRc.

Hence R is transitive Relation.

∴ Therefore, R is an equivalence relation.

Ex: If R be a relation in the set of Integers z defined by $R = \{(x,y) \mid x-y$ is divisible by $3\}$

Show that R is equivalence relation.

Solution:

$R = \{(x,y) \mid x-y = 3K, K \in I\}$

→ Reflexive

$xRx, \; \forall x \in X$

then $x - x = 0$ and 0 is an divisible by any non-zero number and hence divisible by 3.

Therefore xRx.

Hence R is Reflexive.

→ Symmetric

If xRy

⇒ $(x-y)$ is divisible by 3.

⇒ $-(x-y)$ is divisible by 3.

⇒ $y-x$ is divisible by 3.

⇒ yRx

$\Rightarrow x-y = 3K$

$\Rightarrow \cancel{x-y = 3K}$

$\Rightarrow \cancel{y-x = 6(-K)}$

$\Rightarrow y-x = 3(-K)$

Hence R is Symmetric

→ Transitive

If xRy and yRz

$\Rightarrow x-y = 3K_1$ and $y-z = 3K_2$

$\Rightarrow (x-y)+(y-z) = 3(K_1+K_2)$

$\Rightarrow x-z = 3(K_1+K_2)$

$\Rightarrow xRz$ [as if $K_1 \in I$, $K_2 \in I \Rightarrow K_1+K_2 \in I$]

So R is an equivalence Relation.

Ex: If R and S are equivalence relation on the set A, prove that $R \cap S$ is an equivalence Relation.

Solution:

→ Reflexive

→ $\forall\, a \in A,\ (a,a) \in R$ and

→ $\forall\, a \in A,\ (a,a) \in S$

⇒ $(a,a) \in R \cap S,\ \forall\, a \in A$.

Hence $R \cap S$ is reflexive.

→ Symmetric

→ $(a,b) \in R \Rightarrow (b,a) \in R$

→ $(a,b) \in S \Rightarrow (b,a) \in S$.

$\therefore (a,b) \in R \cap S \Rightarrow (b,a) \in R \cap S$.

Therefore $R \cap S$ is Symmetric.

→ Transitive

→ $(a,b) \in R,\ (b,c) \in R \Rightarrow (a,c) \in R$ and.

→ $(a,b) \in S,\ (b,c) \in S \Rightarrow (a,c) \in S$

Therefore $(a,b) \in R \cap S,\ (b,c) \in R \cap S$

$\Rightarrow (a,c) \in R \cap S$

Hence $R \cap S$ is transitive.

Thus $R \cap S$ is Equivalence Relation.

Ex:- Show that the relation $a \equiv b \pmod{n}$ is an equivalence Relation.

Solution:

$\therefore$ $a-b$ is divisible by n.

$\therefore n \mid a-b$

→ Reflexive

aRa, $\forall a \in A$.

then $a-a=0$ and 0 is divisible by any non-zero number

Therefor aRa

Hence $a \equiv b \pmod{n}$ is Reflexive.

$a \equiv a \pmod{n}$ is Reflexive.

→ Symmetric

If aRb

$\Rightarrow a-b=n.k$

$\Rightarrow -(a-b)=n(-k)$

$\Rightarrow b-a=n(-k)$

$\Rightarrow bRa$

$\therefore b \equiv a \pmod{n}$

→ Transitive

$a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$

$\Rightarrow (a-b)$ is divisible by n

$\Rightarrow (a-b) = nk_1$ — (1)

$\Rightarrow (b-c)$ is divisible by n

$\Rightarrow (b-c) = nk_2$ — (2)

Now from (1) & (2)

$a - b + b - c = mk_1 + mk_2$

$\therefore a - c = m(k_1 + k_2)$

$\therefore a - c = mk_3$

$\Rightarrow a \equiv c \pmod{n}$

Hence $a \equiv b \pmod{n}$ is Transitive.

Therefore the relation is equivalence relation.