Propositional Logic

Logic:- Logic is analysis of language, Which consists of signs. Logic is a set of rules (or axioms) which we can use to draw valid conclusions.

* **Statements (or propositions) :-**

A proposition or statement is a declarative sentence which is either true or false, but not both.

Ex:

i) There are seven days in a week.

ii) $2+2=5$

iii) Four is even

iv) $4 \in \{1,3,5\}$

v) When is your interview?

Example (i) & (iii) are true statements.
Example (ii) & (iv) are false statements
Example (v) is not a declarative sentence.

Open statement

In a example $y+8=12$ if we put $y=4$, it become a true statement, if we take value of $y \neq 4$ it become false. Such statements are open statements. Thus if a mathematical sentence is neither true nor false it is called open sentence.

* Truth value of a statement

Statement has a definite truth value which is either true or false. True values are denoted by (T) and false values are denoted by (F)

* Logical connectives

If ∃ two or more statements, they can be combined to produce a new statement. These statements are called compound statements.

-> To combine statements we use following symbols.

* Conjunction ('∧' or 'and')

When two or more statements are combined by the word 'and' the compound statement is known as 'conjunction'.

If p and q are statements, the compound statement 'p∧q' is 'p and q' and is called 'p conjunction q' or 'p meet q'.

Truth table for $p \wedge q$

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

* Disjunction ('$\vee$' or 'or')

When two or more statements are combined by the word 'or', the compound statement is known as 'disjunction'.

The symbol '$p \vee q$' is read as 'p or q' or 'p disjunction q' or 'p join q'.

Truth table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

→ Exclusive 'OR'

Truth table for $p \oplus q$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

* Negation ($\neg$ and $\sim$)

Let p be any statement then negation of p is denoted by '$\neg p$' (or $\bar{p}$) is read as 'not p'.

Truth table

p	$\neg p$
T	F
F	T

* Conditional connectives (if --- then ---) or Implication

Let p and q be two propositions, then the statement 'if p then q' is denoted by $p \to q$ or 'p is sufficient for q' or 'p only if q'. or $p \to q$ 'q if p', or $p \to q$ 'q is necessary for p'.

Truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

* Converse ($p \to q$)

→ Converse of $p \to q$ is $q \to p$

* Inverse ($p \to q$).

→ Inverse of $p \to q$ is $\neg p \to \neg q$

* Contrapositive ($p \to q$)

→ Contrapositive of $p \to q$ is $\neg q \to \neg p$

* Biconditional (or Double implication or Equivalence)

If p and q are two propositions, then '$p \to q$' and '$q \to p$' is called biconditional or double implication. It is written as, '$p \leftrightarrow q$' and it is known as 'p if and only if q'

'p iff q'

'~~p is sufficient~~ ~~f~~

'p is necessary and sufficient for q'.

Truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F.
F	T	F
F	F	T.

Example: Express following statements in propositional form:

i) There are many clouds in the sky but it did not rain.

ii) I will get first class if and only if I study well and score above 80 in mathematics.

iii) Computers are cheap but softwares are costly..

iv) ~~It~~ is very hot and humid or Ramesh is having heart problem.

v) In small restaurants the food is good and service is poor

vi) If I finish my submission before 5.00 in the evening and it is not a very hot I will go and play game of hockey.

Solution:

i) p: There are many clouds in the sky.

q: It rain.

$\therefore p \land \neg q$

ii) p: I will get first class.

q: I study well.

r: Score above 80 in mathematics.

$\therefore p \leftrightarrow (q \land r)$

iii) p : Computers are cheap.

q : Softwares are costly.

$\therefore$ $p \wedge q$

iv) p : It is very hot.

q : It is very humid.

r : ramesh is having heart problem.

$\therefore$ $(p \wedge q) \vee r$

v) p : In small ~~reast~~ restaurant food is good.

q : Service is poor.

$\therefore$ $p \wedge q$

vi) p : I finish my submission before 5.00 pm.

q : It is very hot.

r : I will go.

s : I will play a game of hockey.

$\therefore$ $(p \wedge \neg q) \rightarrow (r \wedge s)$

Ex: Express the contrapositive, converse, inverse and negation forms of the conditional statement given below:

" if x is rational, then x is real !"

Solution: Let p: x is rational

q: x is real.

Symbolic form $p \to q$

contrapositive : $(\neg q \to \neg p)$

If x is not real, then x is not rational.

Converse : $(q \to p)$

If x is real then x is rational.

Inverse : $(\neg p \to \neg q)$

If x is not rational. then x is not real.

Negation : $\neg(p \to q)$

$\Rightarrow \neg(\neg p \vee q)$

$\equiv \neg(\neg p) \wedge (\neg q)$

$\equiv p \wedge \neg q$

x is rational and x is not real.

* Tautology

A tautology is a proposition which is true for all truth values of its sub propositions.

In other words a propositions is a tautology if it is always true for all assignments of truth values.

* Contradiction

A proposition is a contradiction if it is always false for all assignments of truth values.

Remark:- A proposition ~~is a~~ which is neither a tautology nor a contradiction is called a contingency.

Ex:- Show that $(p \to q) \leftrightarrow (\neg p) \vee q$ is a tautology.

Solution:

p	q	$\neg p$	$p \to q$	$(\neg p) \vee q$	$(p \to q) \leftrightarrow (\neg p) \vee q$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Hence $(p \to q) \leftrightarrow (\neg p) \vee q$ is a tautology.

* Precedence Rule

The order of Preference in which the Connectives are applied in a formula of Propositions that has no brackets is

i) $\neg$

ii) $\wedge$

iii) $\vee$ and $\oplus$

iv) $\rightarrow$ and $\leftrightarrow$

* Logical Equivalence

Two propositions A and B are logically equivalent if and only if they have the same truth value for every choice of truth values of simple propositions involved in them.

$A \equiv B$

Example: Prove that $(p \vee q) \wedge \neg p \equiv \neg p \wedge q$.

p	q	$\neg p$	$p \vee q$	$(p \vee q) \wedge \neg p$	$\neg p \wedge q$
T	T	F	T	F	F
T	F	F	T	F	F
F	T	T	T	T	T
F	F	T	F	F	F

$\therefore (p \vee q) \wedge \neg p \equiv \neg p \wedge q$

* Duality law

Two formulas A and A^* are said to be the duals of each other if either one can be obtained from the other by replacing '$\wedge$' by '$\vee$' and '$\vee$' by '$\wedge$'. The connectives '$\wedge$' and '$\vee$' are also called dual of each other.

* Logical identities

1. Demorgan's laws

i) $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$

ii) $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$

2. Associative laws

i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

3. Commutative laws

i) $p \vee q \equiv q \vee p$

ii) $p \wedge q \equiv q \wedge p$

4. Idempotent laws

i) $p \vee p \equiv p$

ii) $p \wedge p \equiv p$

5. Double negation

$\neg(\neg p) \equiv p$

6. Distributive laws

i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

~~ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$~~

ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

7. Absorption laws

i) $p \vee (p \wedge q) \equiv p$

ii) $p \wedge (p \vee q) \equiv p$

Example: $p \to q$ and $\neg p \vee q$ are logically equivalent.

Solution:

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T