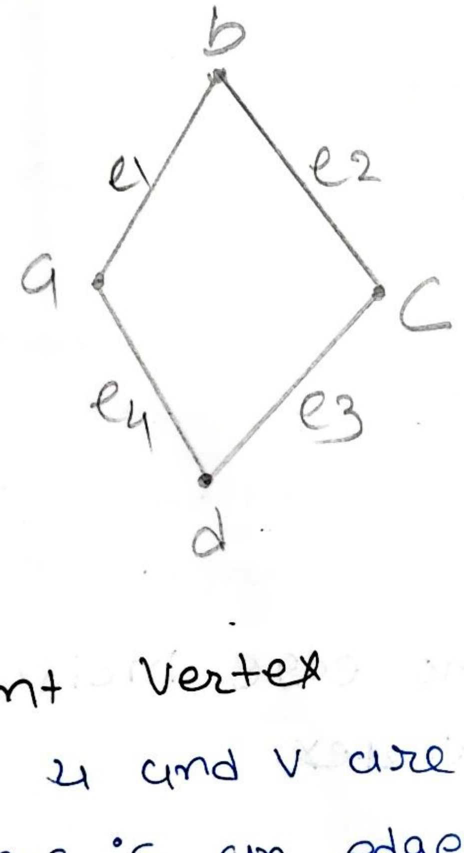
Graph Theory

A Graph is defined as an ordered Pair of a set of vertices and a set of edges.

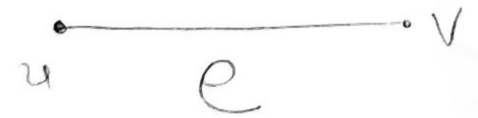
$G=(V,E)$ or $G(V,E)$

Here, V is the set of vertices and E is the set of edges connecting the vertices.



→ Adjacent Vertex

Vertex u and v are said to be adjicent if there is an edge $= \{u,v\}$



$\{u,v\}$ is adjacent with $e=\{u,v\}$

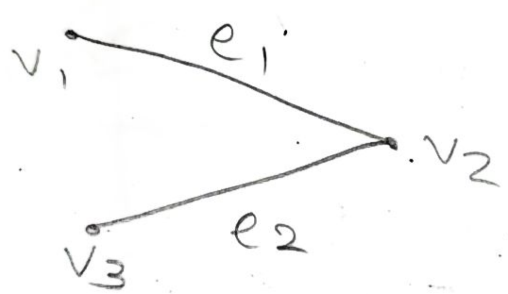
→ Incident edges

The edge 'e' that joins the nodes u and v is said to be incident on each of its end points u and v.

incident edges

→ Adjacent edges

Two edges are said to be adjacent if they are incident on a common vertex.



→ Self loop

→ Parallel edges.

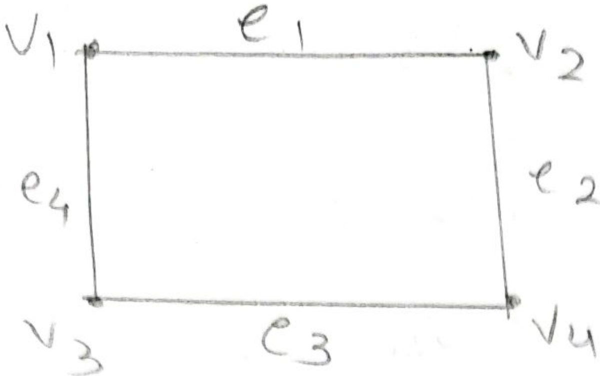
→ Isolated vertex

A vertex having no edge incident on it is an isolated vertex.

$\cdot v_3$ → isolated vertex

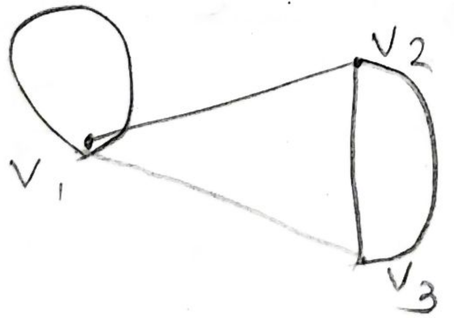
→ Simple graph - (No loops, No parallel edges)

A graph which is neither self loops nor parallel edge is simple graph.

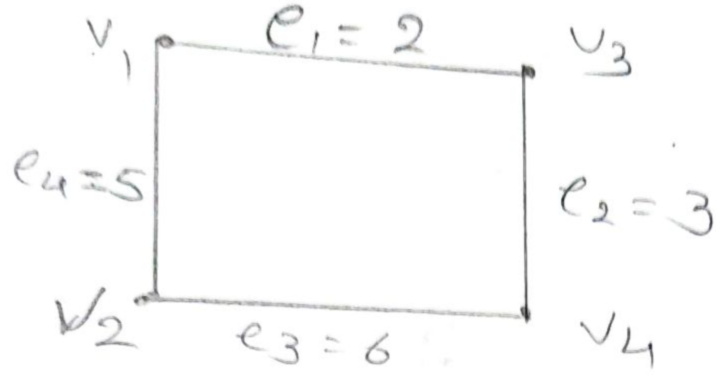


-> Multigraph- (Parallel edges but no self loop)
A graph which has more than one edge between a pair of vertices is multigraph.

-> Pseudograph- (self loop + Parallel edges)
A graph in which loops and Parallel edges | multiple edges are allowed is called pseudograph.

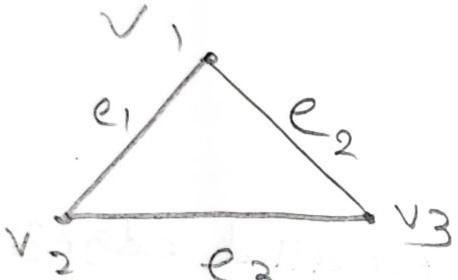


-> Weighted graph-
A graph in which weights are assigned to every edge is called weighted graph.



→ Finite graph -

A graph $G = \{V, E\}$ in which both V and E are finite set is called a finite graph.

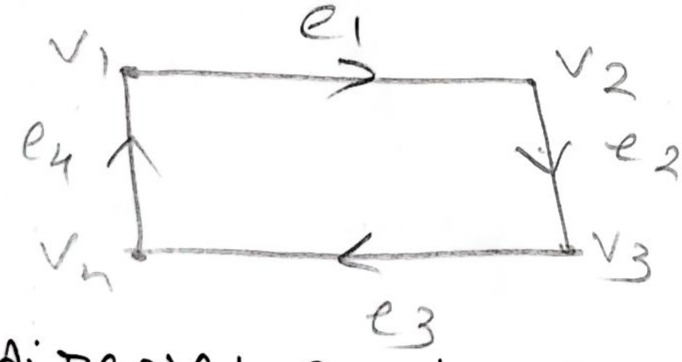


$V = \{V_1, V_2, V_3\}$

$E = \{e_1, e_2, e_3\}$

→ Directed graph / Diagraph

A graph in which every edge is directed is diagraph.

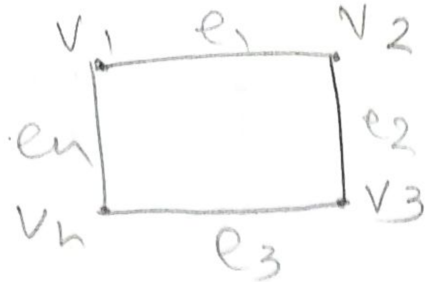


$e_1 = (V_1, V_2)$

$\cancel{(V_2, V_1)}$

→ undirected graph

A graph in which an edge is associated with an unordered pair of vertices is undirected graph.



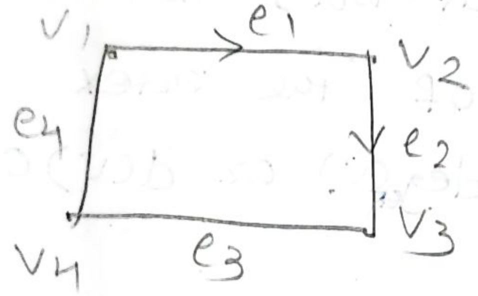
$e_1 = (v_1, v_2)$

or

(v_2, v_1)

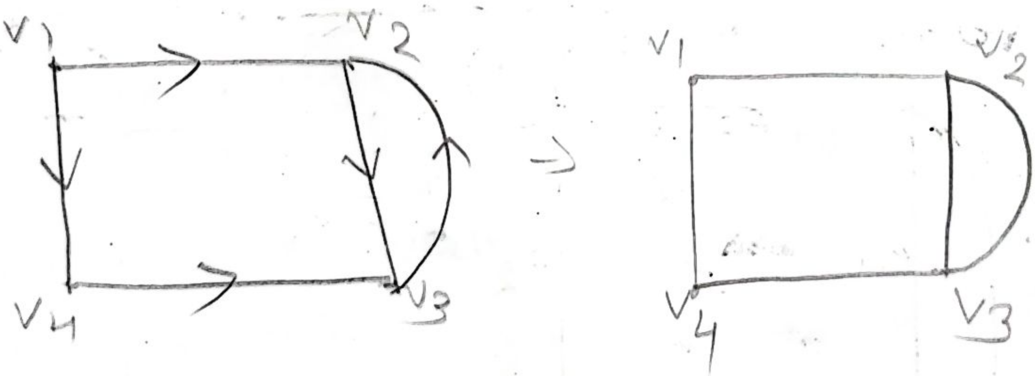
→ Mixed graph-

If some edges are directed and some are undirected in a graph, the graph is called mixed graph.



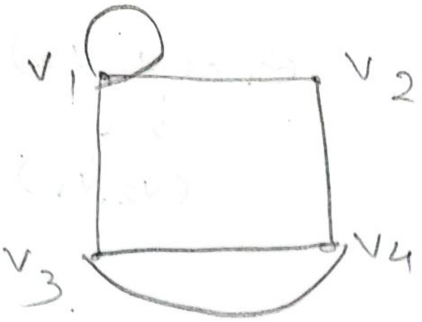
→ underlying undirected graph-

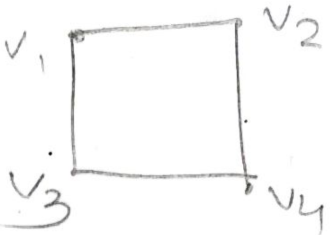
A graph obtained by ignoring the direction of edges in a directed graph is called underlying undirected graph.



→ Underlying simple graph-

A graph obtained by deleting all loops and parallel edges from a graph is called a underlying simple graph.





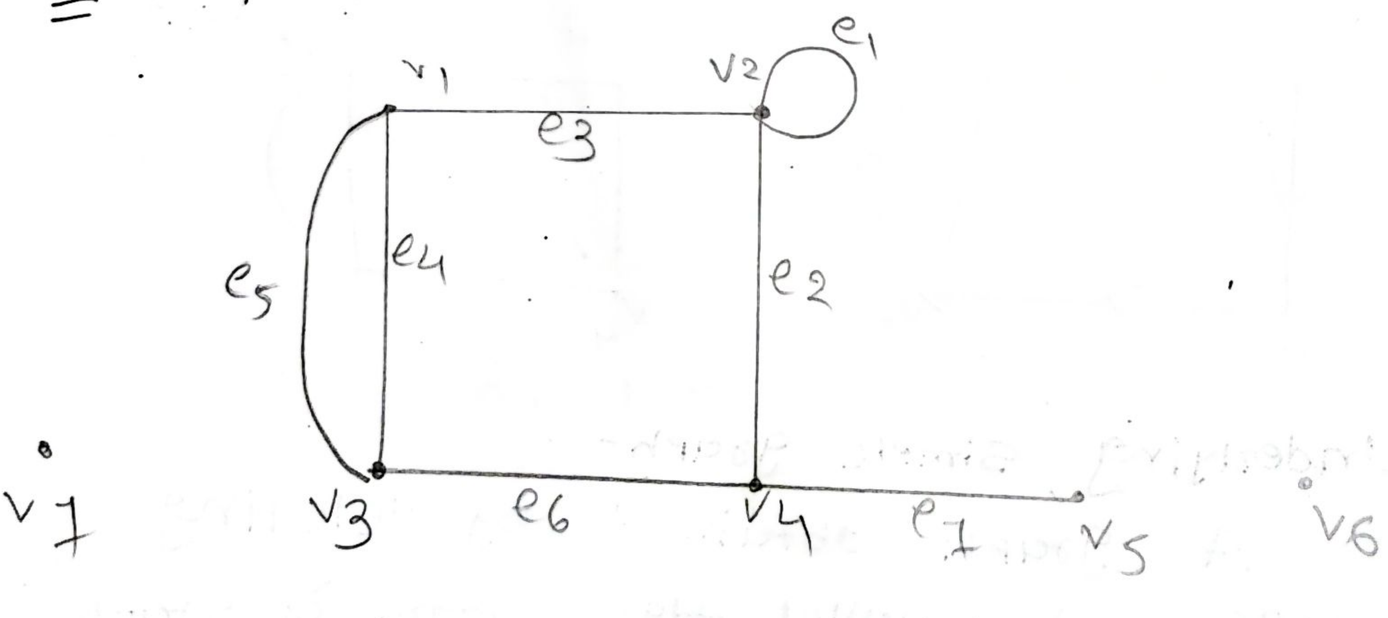
→ Degree of vertex

The number of edges incident on a vertex v, with self-loops counted twice, is called the degree of the vertex v and is denoted by $\deg_G(v)$ or $d(v)$ or $\deg(v)$.

→ Pendent vertex

A vertex of degree 1, is called a pendent vertex or an end vertex.

Ex: Find the degree of all vertex.



Graph

- undirected
  - degree of a vertex
- directed
  - Indegree
  - outdegree

Solution: $\deg(v_1) = 3$, $\deg(v_2) = 4$, $\deg(v_3) = 3$, $\deg(v_4) = 3$ — internal vertex.

$\deg(v_5) = 1$ — Pendant vertex

$\deg(v_6) = 0$, $\deg(v_7) = 0$ — isolated vertex

~~Directed graph~~

Degrees in Digraph

Indegree of a vertex -

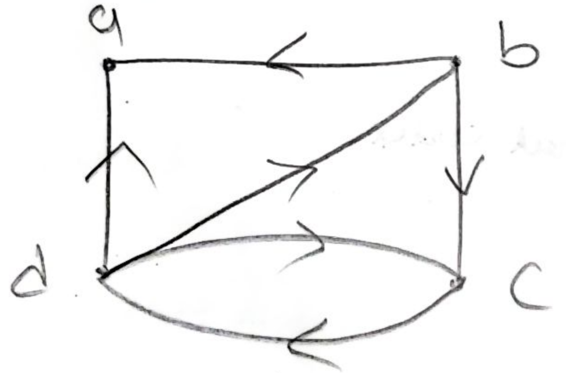
The indegree of vertex v in a directed graph G is the number of edges which are coming into the vertex v (that is number of incoming edges) and is denoted by $\bar{d}(v)$, $\deg^{-}(v)$ or $\mathrm{Indeg}(v)$.

Outdegree of a vertex:-

The out degree of vertex v in a directed graph G is the number of edges which are going out from the vertex v (that is, the number of outgoing edges) and is denoted by

$d^+(v)$, $\deg^+(v)$ or outdeg(v)

Ex: Find the degrees of vertex of the directed graph.



Solution:

Vertex	$d^-(v)$ Indegree	$d^+(v)$ outdegree	Total degree
a	3	1	4
b	1	2	3
c	2	1	3
d	1	3	4

Types of Graph

→ **Null graph-**

A graph without edges is called an empty graph or a null graph

Or

A graph which contains only isolated vertex is called a null graph i.e the set of edges in a null graph is empty. Null graph is denoted on n vertices by N_n.

E.g

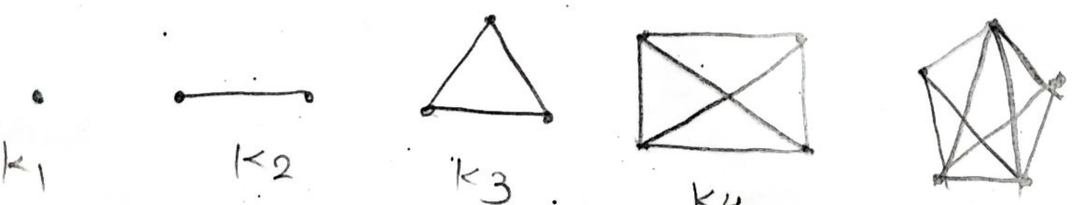
N_2 N_3 N_4

→ **Complete graph-**

A Simple graph G is said to be Complete if every vertex in G is connected with every other vertex i,e if G contains exactly one edge between each pair of distinct vertices.

A complete graph on n vertices is denoted by K_n.

E.g



K_1 K_2 K_3 K_4 K_5

Note: ① A complete graph K_n has $\frac{n(n-1)}{2}$ edges.

E.g. $K_4 = \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6$ edges.

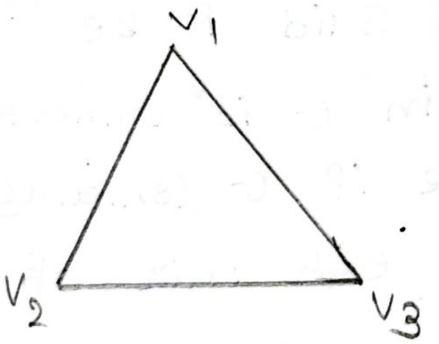
② The size of every graph of order n is at most $\frac{n(n-1)}{2}$

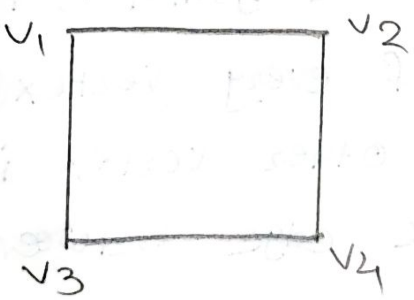
K_n has exactly $\frac{n(n-1)}{2}$ edges.

→ **Regular graph-**

A graph in which all vertices are of equal degree is called a regular graph. If the degree of each vertex is r, then the graph is called a regular graph of degree r or r-regular graph.

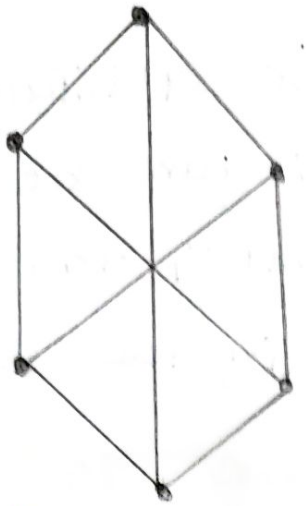
E.g:-

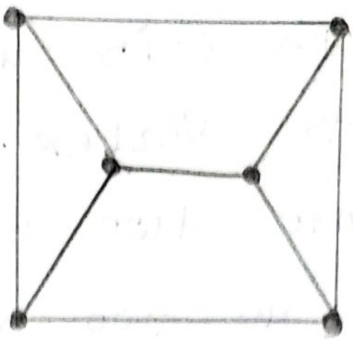




$d(v_1) = d(v_2) = d(v_3) = 3$ $\qquad$ $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2$

← 2-Regular graph →



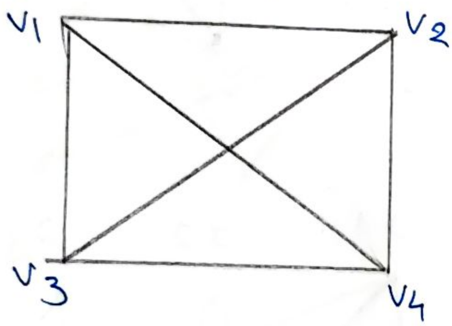


← 3-Regular graph →

Note:

① The complete graph K_n is a regular of degree $n-1$

K_4



$d(v_1) = d(v_2) = d(v_3) = d(v_4)$
$= 3$

3-Regular graph.

② If G has n vertices and is regular of degree r, then G has $\frac{rn}{2}$ edges.

E.g. $n=4$, $r=3$

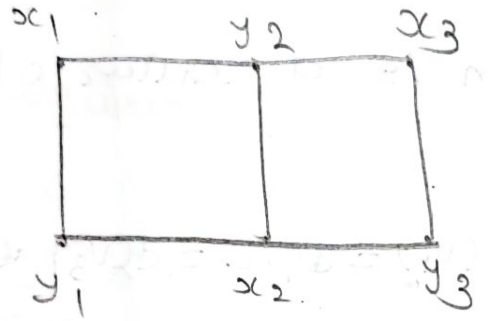
No. of edges $= \frac{rn}{2} = \frac{3 \times 4}{2} = 6$ edges.

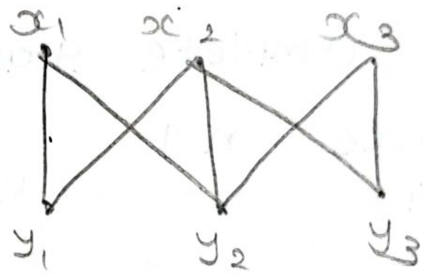
→ Bipartite graph.

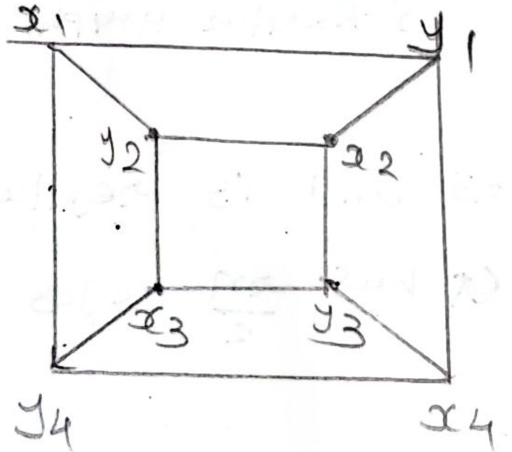
A graph G is said to be a bipartite graph if it's vertex set V can be partitioned into two sets, say V_1 and V_2, such that no two vertices in the same partition can be adjacent.

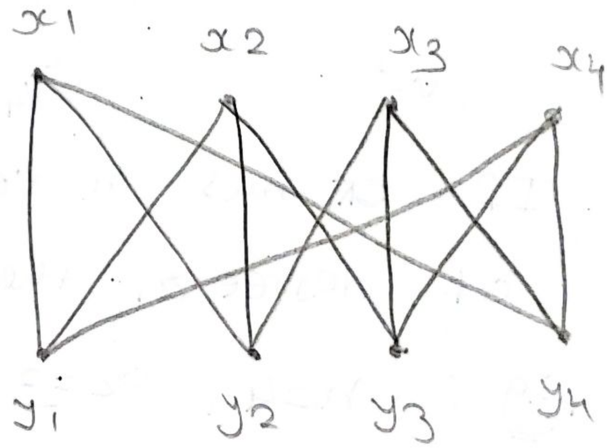
Here, the pair (V_1, V_2) is called the bipartition of G.

Ex:

① 



② 

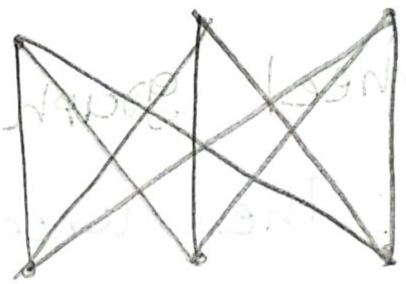


→ Complete Bipartite graph.

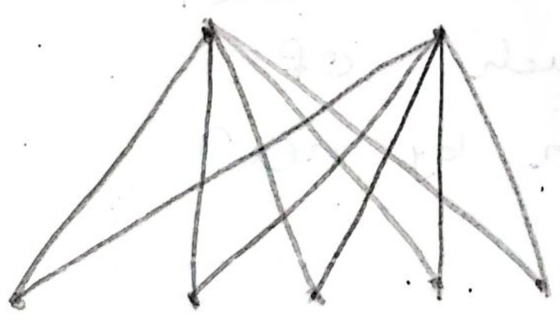
A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between every pair of vertices if and only if one vertex in the pair is in the first subset and the other vertex is in the second subset.

E.g

$K_{2,3}$



$K_{3,3}$

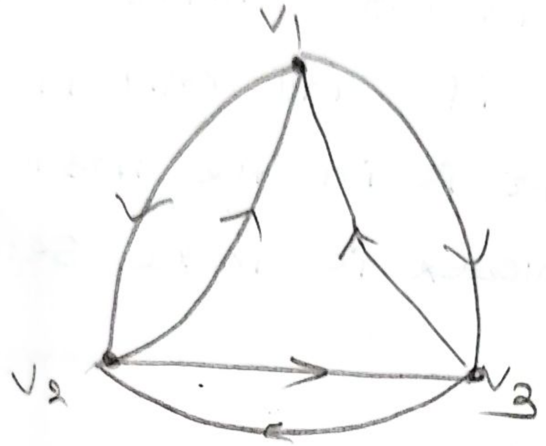


$K_{2,5}$

Note: A complete bipartite graph $K_{m,n}$ is not a regular graph if $m \neq n$.

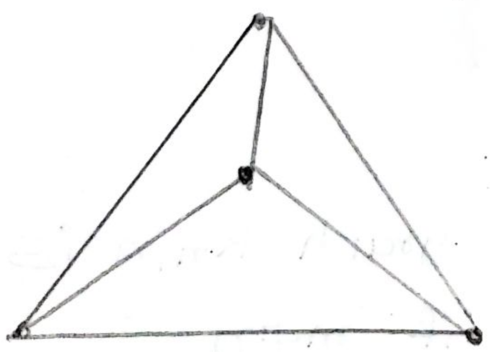
→ Directed complete graph -

A directed complete graph of n vertices is a graph in which there is exactly one arrow between each pair of distinct vertices.

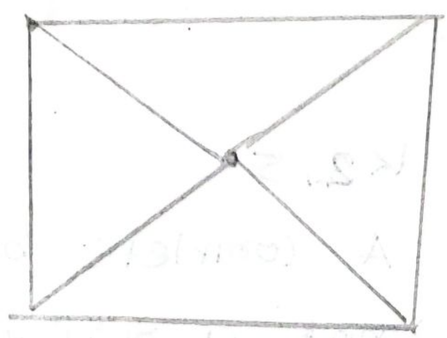


→ Wheel graph -

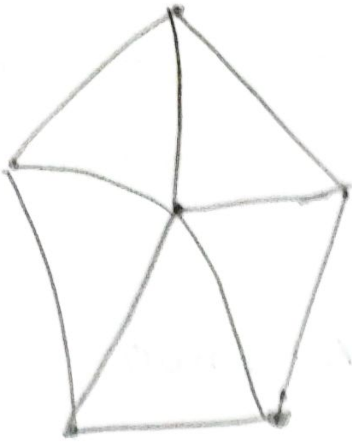
The wheel W_n is obtained when an additional vertex to the cycle C_n for $n \geq 3$ and connect this new vertex to each of the n vertices in C_n by new edges.



W_3



W_4



W_5

<u>Note</u>: ① $n+1$ vertices

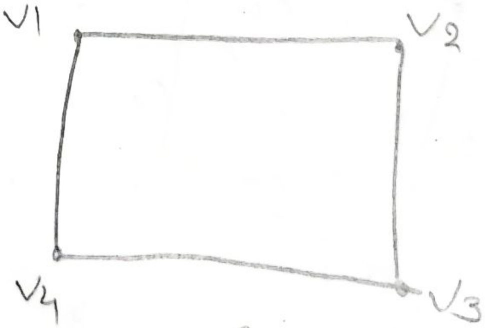
② $2n$ edges

→ Cycles – cyclic graph –

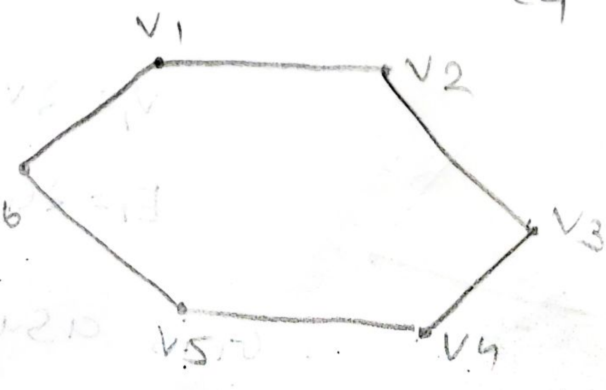
The cycle C_n, $n \geq 3$ consists of n vertices $V_1, V_2, ---V_n$ and edges $\{V_1, V_2\}$, $\{V_2, V_3\}$--- $\{V_{n-1}; V_n\}$ and $\{V_n, V_1\}$

A graph containing atleast one cycle in it is known as cyclic graph.

C_3



C_4



C_6

Note:

(1) n vertices, n edges.

(2) Degree of each vertex is two.

-> Subgraph -

$G_1 \subseteq G$

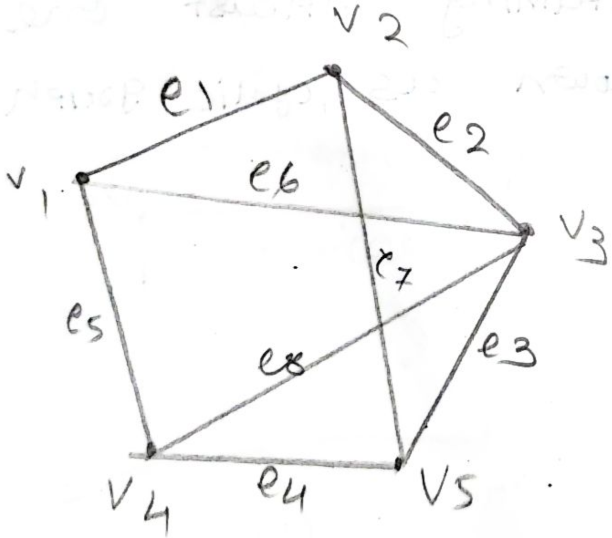
Given two graphs G and G_1, we say that G_1 is a subgraph of G if the following conditions hold:

(1) All the vertices and all the edges of G_1 are in G.

(2) Each edge of G_1 has the same end vertices in G as in G_1.

Ex:

G

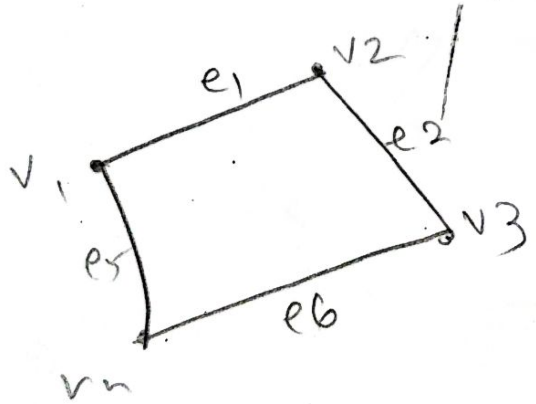


G(V, E)

$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

G_1.



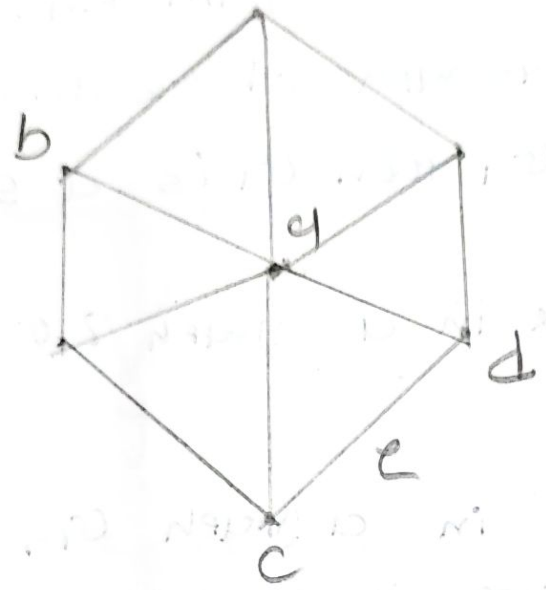
$V_1 = \{v_1, v_2, v_3, v_4\}$

$E_1 = \{e_1, e_2, e_5, e_8\}$

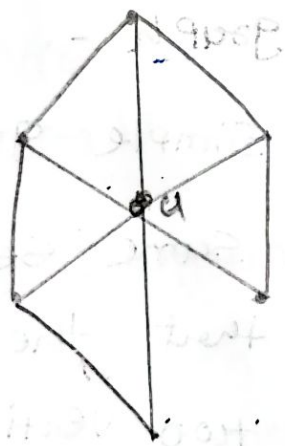
G_1 is a subgraph of G.

Ex: for the graph G shown below, draw the subgraphs.

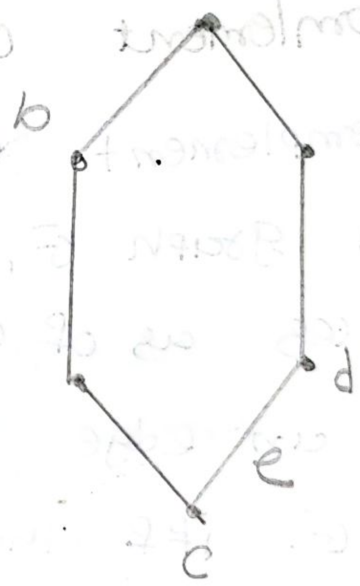
(a) G − e (b) G − a



Solution: (a) G − e



(b) G − a



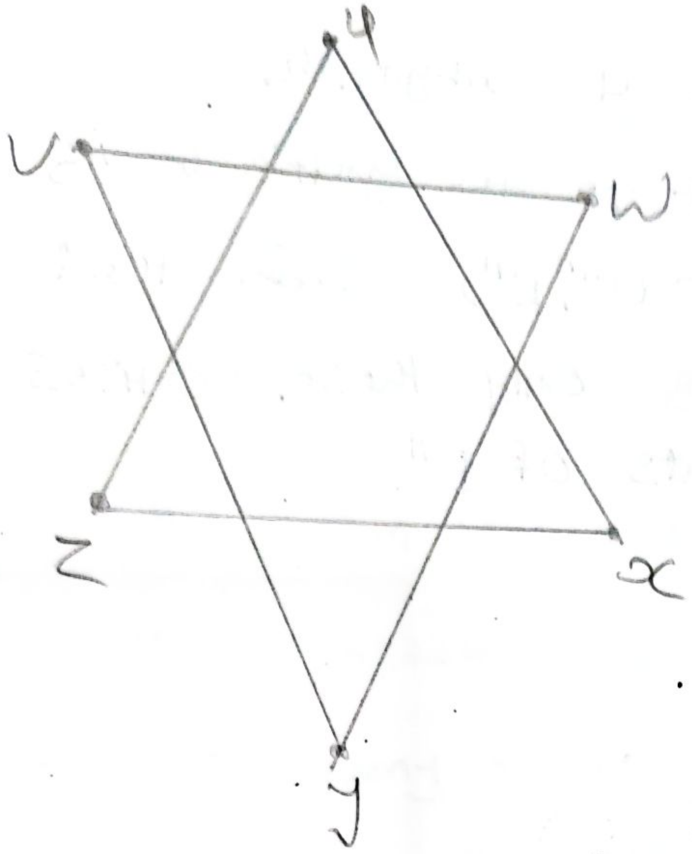
Result on Subgraph

1. Every graph is a subgraph of itself.
2. Every simple graph of n vertices is a subgraph of complete graph K_n.
3. If G_1 is a subgraph of G_2 and G_2 is a subgraph of G, then G_1 is a subgraph of G.
4. A single vertex in a graph G is a subgraph of G.
5. A single edge in a graph G, together with its vertices is a subgraph of G.

→ Comlement of a graph -

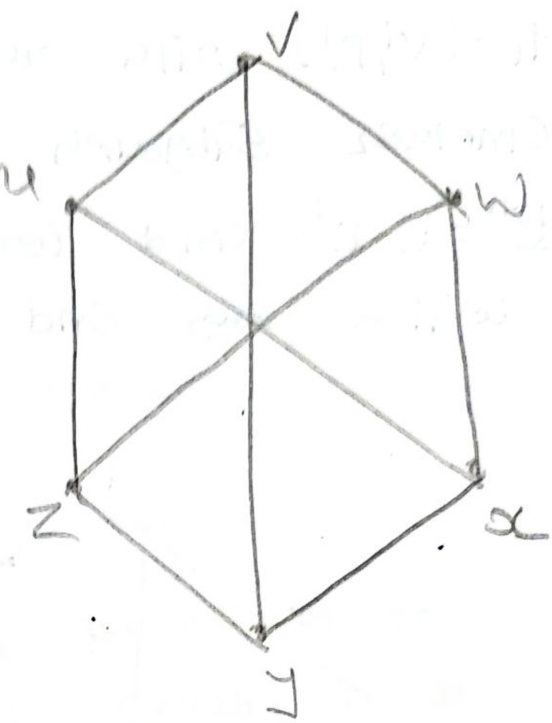
Complement of a simple graph G is a graph $\bar{G}$, on the same set of vertices as of G such that there will be an edge between two vertices (u,v) in $\bar{G}$ iff there is no edge between (u,v) in G.

→ The complement G' (or $\bar{G}$) is defined as a simple graph with the same vertex set as G and where two vertices u and v are adjacent only when they are not adjacent in G.



Complement of G.

(G')



Note: for a graph G and its complement G'

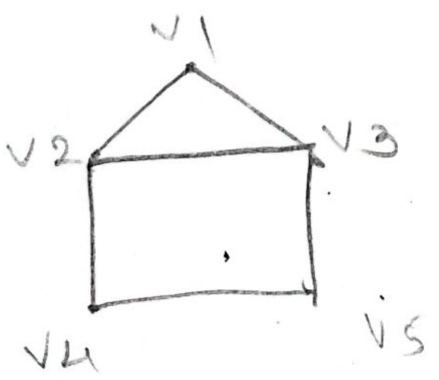
(i) $G \cup G' = K_n$

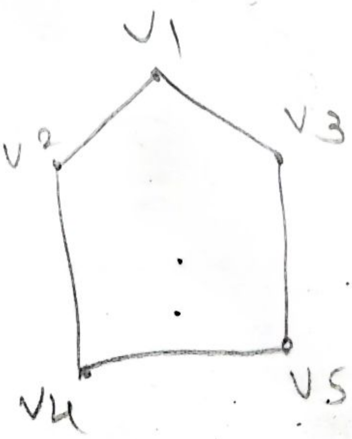
(ii) $V(G) = V(G')$

(iii) $E(G) \cup E(G') = E(K_n)$

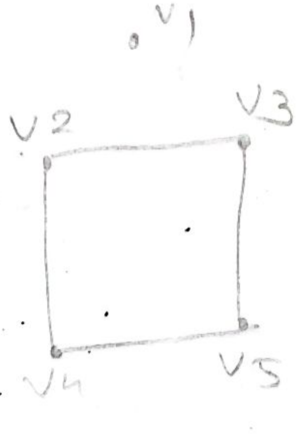
→ Spanning Subgraph.

A Subgraph of G is said to be a spanning subgraph if it contain all the vertices of G.





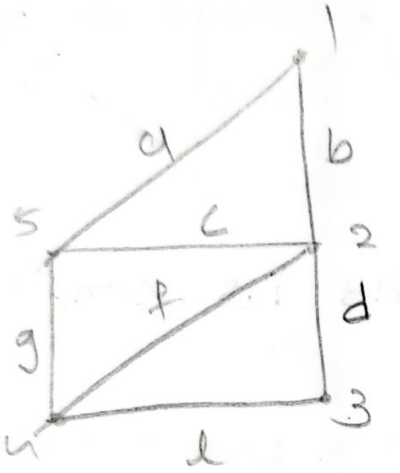
H_1



H_2

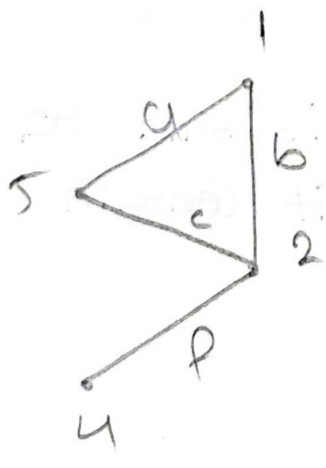
→ Complement of Subgraph -

The complement of a subgraph $H_1 = (V', E')$ with respect to the graph G is another subgraph $H_2 = (V'', E'')$ such that $E'' = E - E'$ and containg only those vertices which has end points of E''.



$G = (V, E)$

$H_1 = (V', E')$



$V' = \{1, 2, 4, 5\}$

$E' = \{a, b, c, f\}$

$H_2 = (V'', E'')$

$V'' = \{1, 2, 3, 4, 5\}$

$E - E' = \{d, l, g\}$

→ Handshaking Theorem-

If $G=(V,E)$ be an undireted graph With e edges. Then

$$\sum_{v \in V} deg(v) = 2e$$

i.e the sum of degrees of the vertices is an undireeted graph is even.

Ex:1 How many nodes are necessary to construct a graph with exactly 8 edges in which each nodes is of degree 2.

Solution:

Suppose there are n nodes (vertices) in the graph with 8 edges.
Each vertex is of degree 2.

$$\sum_{i=1}^{n} deg(v_i) = 2e$$

$\therefore 2 \times n = 2e$

$\therefore 2 \times n = 2(8)$

$\therefore 2 \times n = 16$

$\therefore n = 16/2$

$\therefore n = 8$

Ex: 2 Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw two such graphs.

Solution:

By Hand shaking lemma

$$\sum \deg(v_i) = 2e$$

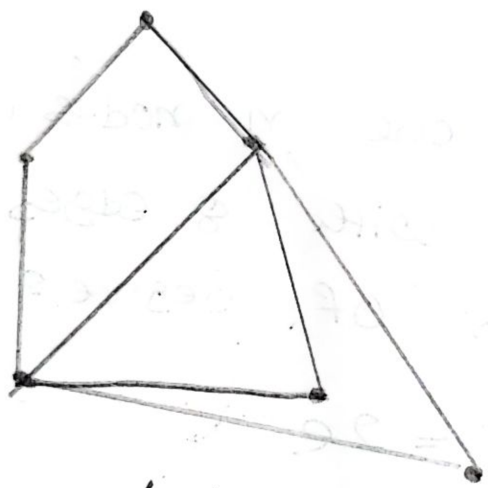
$$\therefore (2 \times 4) + (4 \times 2) = 2e$$

$$\therefore 8 + 8 = 2e$$

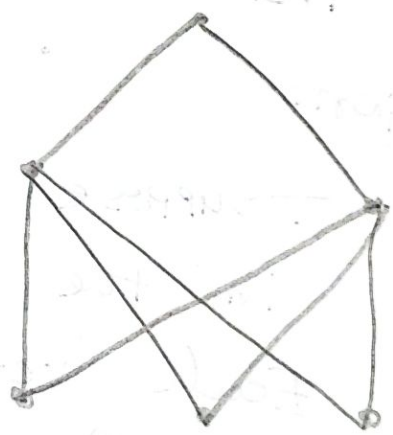
$$\therefore 16 = 2e$$

$$\therefore 16/2 = e$$

$$\therefore e = 8$$



(a)



(b)

Ex:3 A Simple graph G has 24 edges and degree of each vertex is 4 find the number of vertices.

Solution:

$$|E| = 24$$

$$\sum deg(v_i) = 2e$$

$$\therefore n \times 4 = 2(24)$$

$$\therefore n \times 4 = 48$$

$$\therefore n = \frac{48}{4}$$

$$\therefore n = 12$$

Ex:4 Simple graph with 35 edges four vertices of degree 5, five vertices of degree 4. four vertices of degree 3, find the number of vertex with degree 2?

Solution:

4 vertices of degree 5.

5 vertices of degree 4.

4 vertices of degree 3.

n vertices of degree 2.

$$e = 35$$

$$\therefore \sum deg(v_i) = 2e$$

$$\therefore (4 \times 5) + (5 \times 4) + (4 \times 3) + (n \times 2) = 2(35)$$

$$\therefore 20 + 20 + 12 + 2n = 70$$

$$\therefore 52 + 2n = 70$$

$$\therefore 2n = 70 - 52$$

$\therefore 2n = 18$

$\therefore n = 18/2$

$\therefore n = 9$

→ Isomorphic Graphs

Two graphs G_1 and G_2 are said to be isomorphic if there is one to one corrspondance between their vertices and between their edges. Such that incidences are preceVered In Other words. Two graphs are isomorphic if

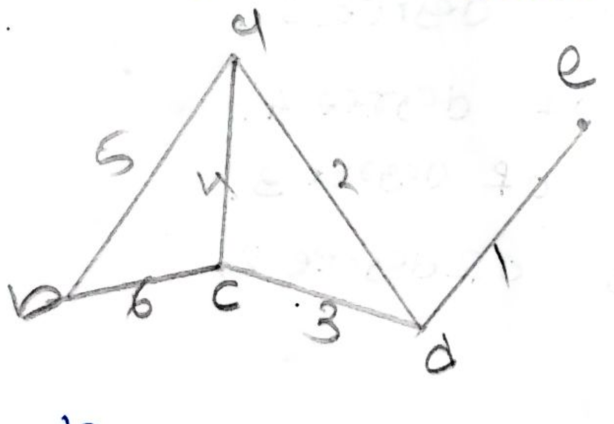
i) Number of vertices are same.

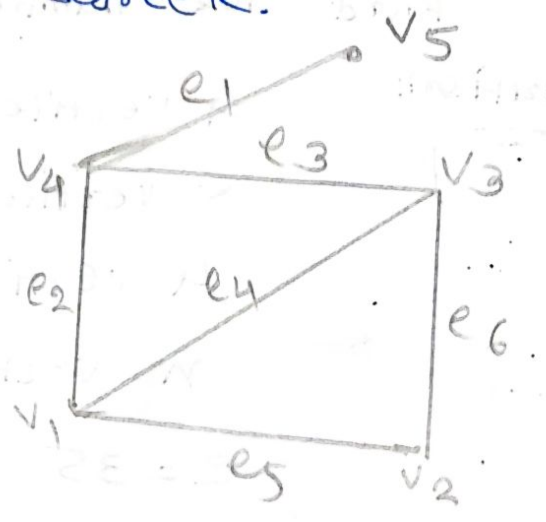
ii) Number of edges are same.

iii) In G_1 has n vertices of degree k then G_2 must have n vertices of degree k.

iv) Adjacency is preserved.

Ex:





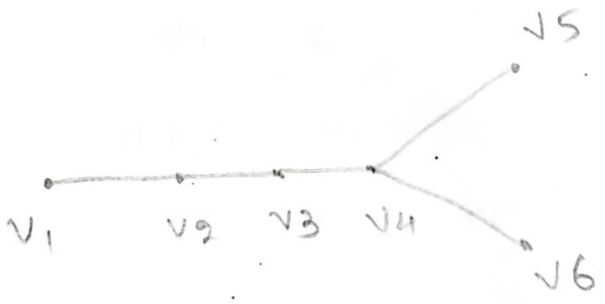
deg seq (G) = 3,3,3,2,1

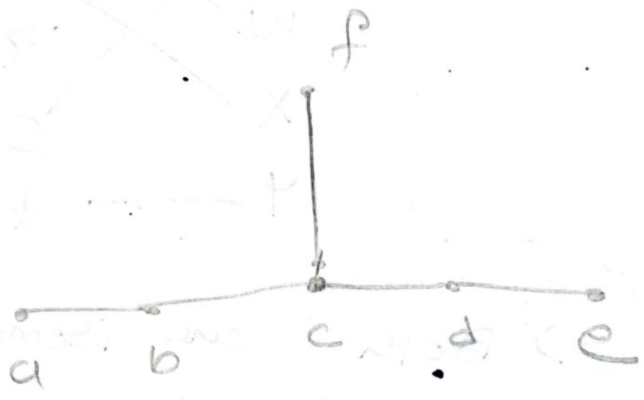
deg(G') = 3,3,3,2,1

vertex	edges
$f(a) = v_1$	$f(1) = e_1$
$f(b) = v_2$	$f(2) = e_2$
$f(c) = v_3$	$f(3) = e_3$
$f(d) = v_4$	$f(4) = e_4$
$f(e) = v_5$	$f(5) = e_5$
	$f(6) = e_6$

Adjacency also preserved.
Therefore G and G' said to to be isomorphic.

Ex:



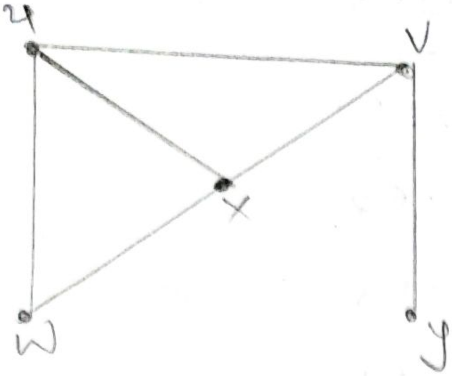


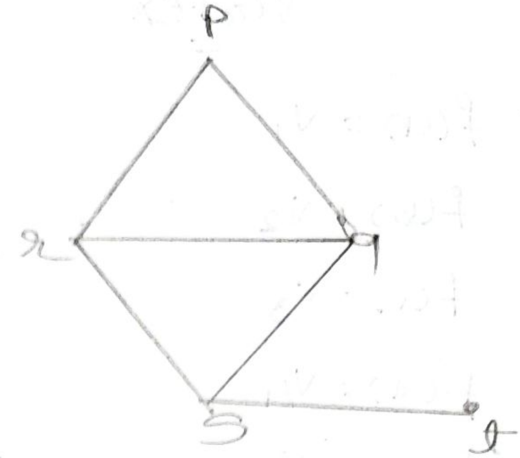
deg(seq): 3, 2, 2, 1, 1, 1

deg(seq): 3, 2, 2, 1, 1, 1

Same no. of vertices
Same no. of edges.
These graphs are not isomorphic because v_4 is adjacent to two pendent vertex is not preserved.

Ex:



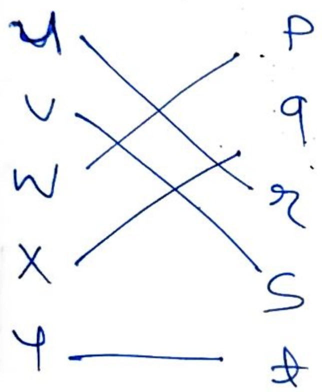


Solution: (i) Number of vertices are same.

(ii) Number of edges are same.

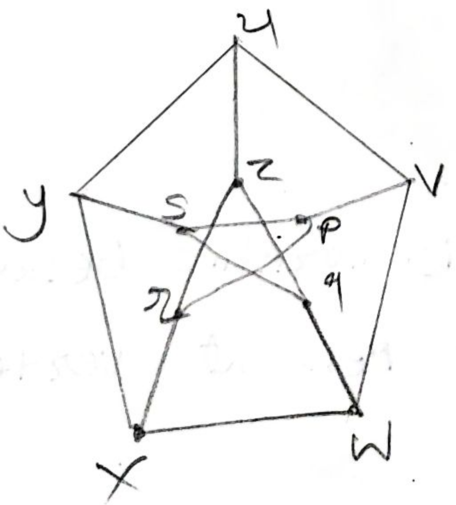
(iii) An equal number of vertices with given degree.

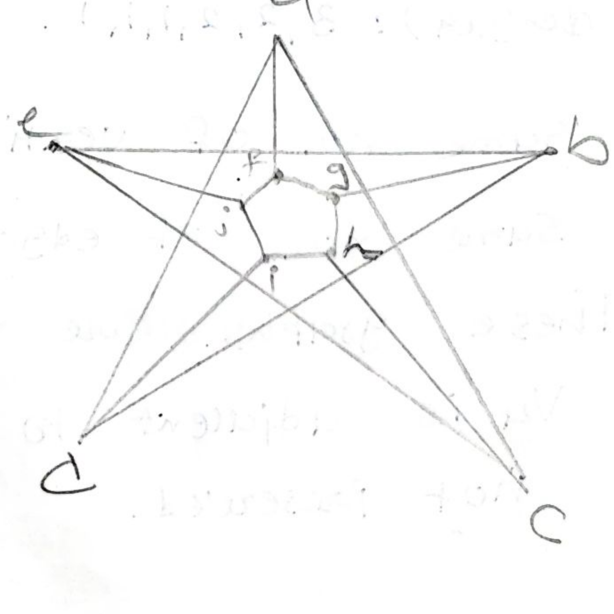
(iv)



=> Both are isomorphic

Ex: Show that the following graphs are isomorphic.





(i) Number of vertices are same

(ii) Number of edges are same.

(iii) An equal number of vertices with given degree.

$y \to e$ $r - h$

$u \to b$ $p - i$

$v \to d$ $s - j$

$w \to a$ $a - f$

$x \to c$ $z - g$

* Operations on Graph

→ Union of two graph.

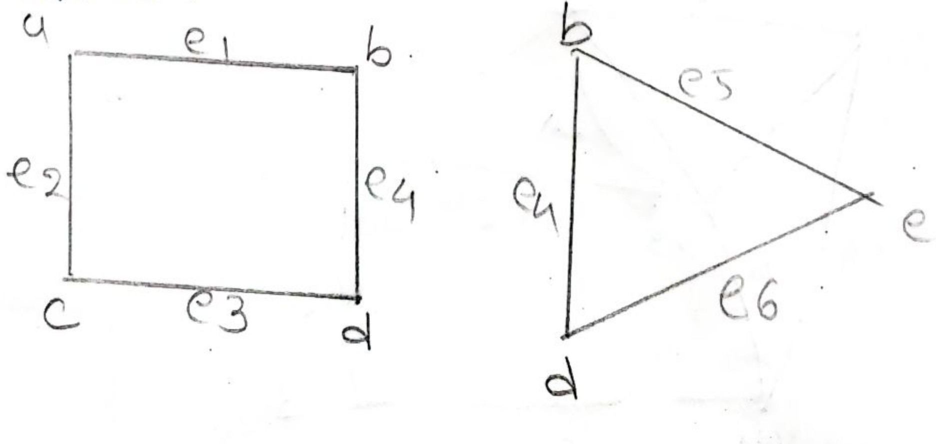
Given two graphs G_1 and G_2 their union will be a graph such that

$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$

and $E_1(G_1 \cup G_2) = E(G_1) \cup E(G_2)$

The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

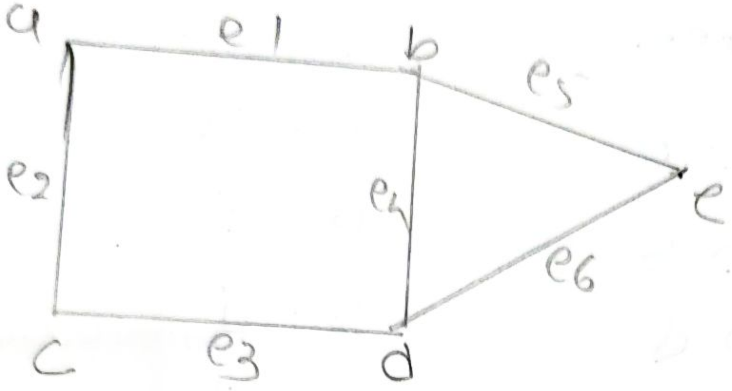
Ex:



$V(G_1) = \{a, b, c, d\}$ $E(G_1) = \{e_1, e_2, e_3, e_4\}$

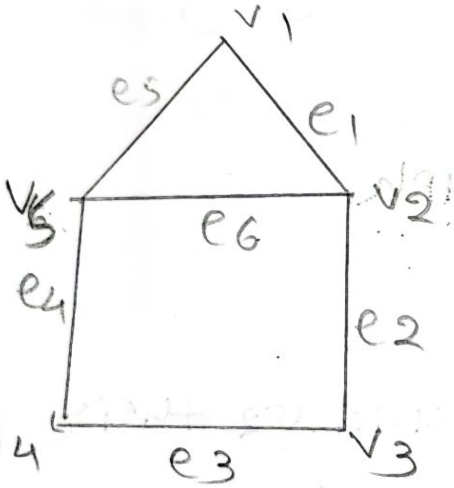
$V(G_2) = \{b, d, e\}$ $E(G_2) = \{e_4, e_5, e_6\}$

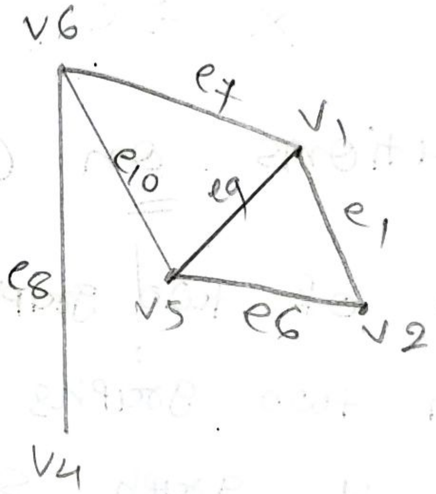
$V(G_1 \cup G_2) = \{a, b, c, d, e\}$ $E(G_1 \cup G_2) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$



$G_1 \cup G_2$

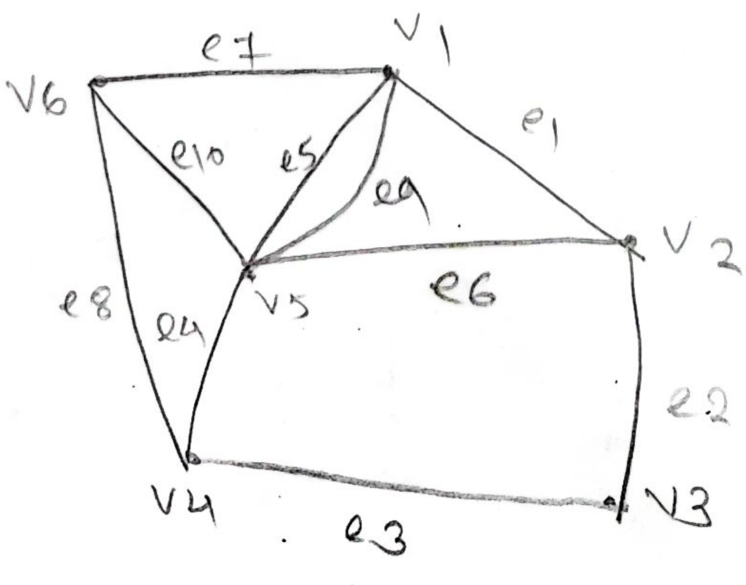
Ex:





$V(G_1 \cup G_2) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$E(G_1 \cup G_2) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$



→ Intersection of graph

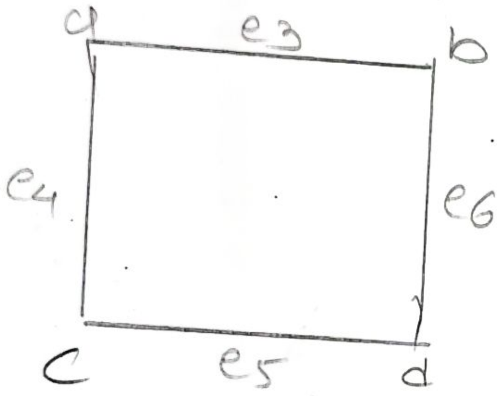
Given two graphs G_1 and G_2 with at least one vertex in common then their intersection will be a graph such that

$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$

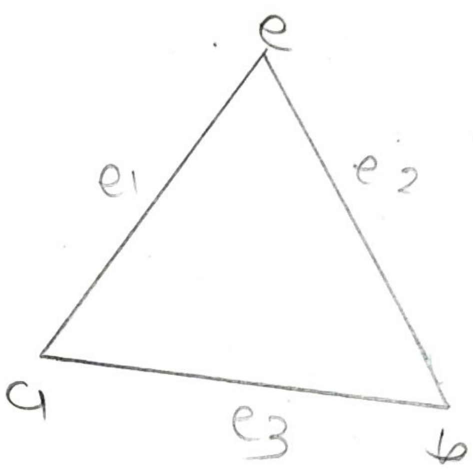
and $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$

The intersection of G_1 and G_2 is denoted by $G_1 \cap G_2$.

<u>Ex</u>:



G_1



G_2

$V(G_1) = \{a, b, c, d\}$

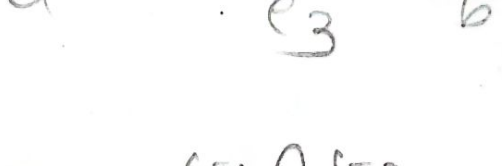
$V(G_2) = \{a, b, e\}$

$V(G_1 \cap G_2) = \{a, b\}$

$E(G_1) = \{e_3, e_4, e_5, e_6\}$

$E(G_2) = \{e_1, e_2, e_3\}$

$E(G_1 \cap G_2) = \{e_3\}$



$G_1 \cap G_2$

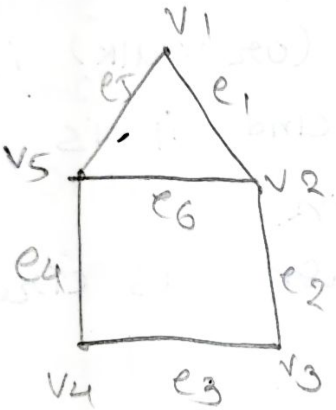
→ Ring sum of graph

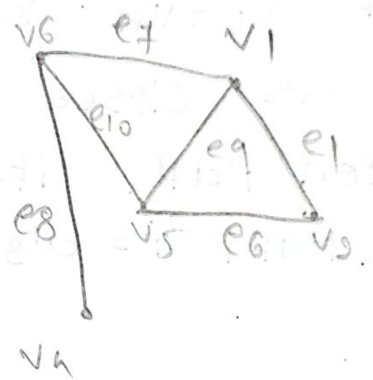
Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graph. Then the ring sum of G_1 and G_2, denoted by $G_1 \oplus G_2$ is defined as the graph G such that

(i) $V(G) = V(G_1) \cup V(G_2)$

(ii) $V(E) = E(G_1) \cup E(G_2) - E(G_1) \cap E(G_2)$

Ex:





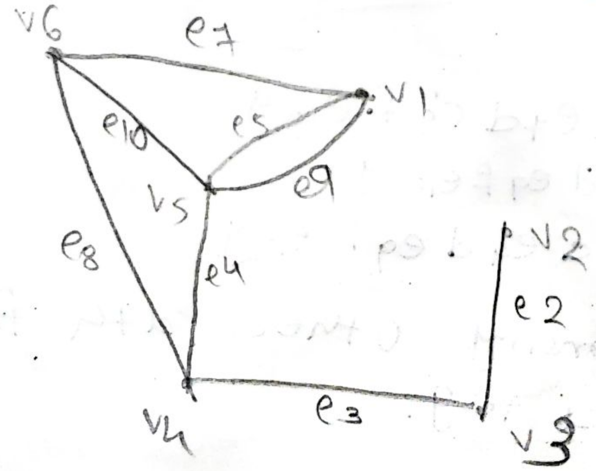
$V(G) = V(G_1) \cup V(G_2)$

$= \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$E(G) = E(G_1) \cup E(G_2) - E(G_1) \cap E(G_2)$

$= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\} - \{e_1, e_6\}$

$= \{e_2, e_3, e_4, e_5, e_7, e_8, e_9, e_{10}\}$



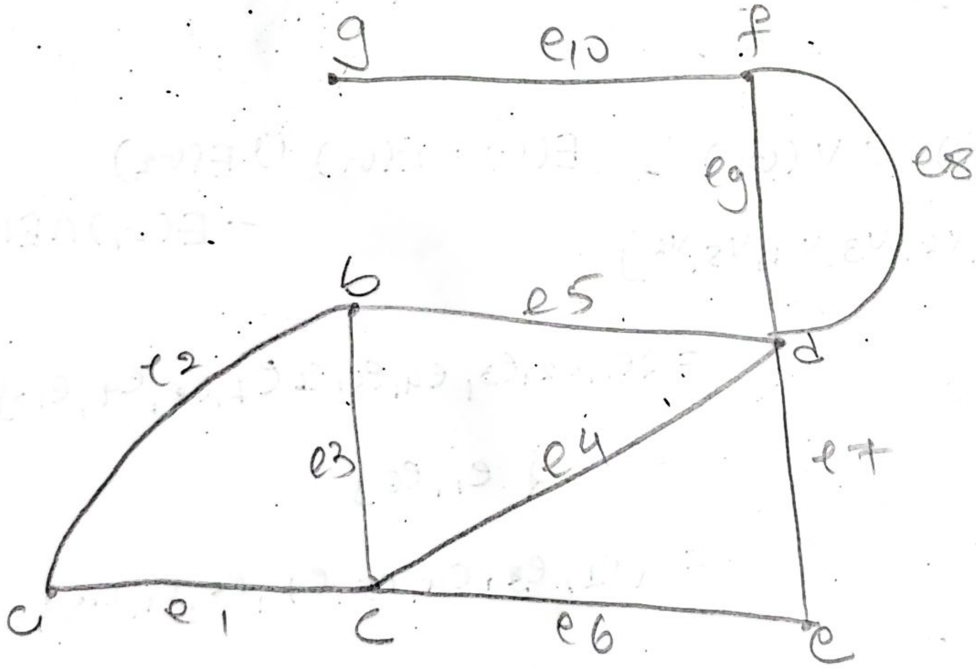
$G_1 \oplus G_2$

* Path (or Walk) and circuits

A walk of a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \ldots v_{n-1}, e_n, v_n$ begining and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it.

If $v_0 = v_n$ the path (or walk) is known as closed path and it is known as open path if $v_0 \neq v_n$.

Closed path is also known as circuit.



Path-I :- $a\, e_1\, c\, e_6\, e\, e_7\, d\, e_8\, f\, e_{10}\, g$.

Path-II :- $a\, e_2\, b\, e_5\, d\, e_9\, f\, e_{10}\, g$

Path III :- $a\, e_2\, b\, e_3\, c\, e_4\, d\, e_9\, f\, e_{10}\, g$

There are so many other path from a to g.

* Simple Path:- It is a Path in which edges do not repeat.

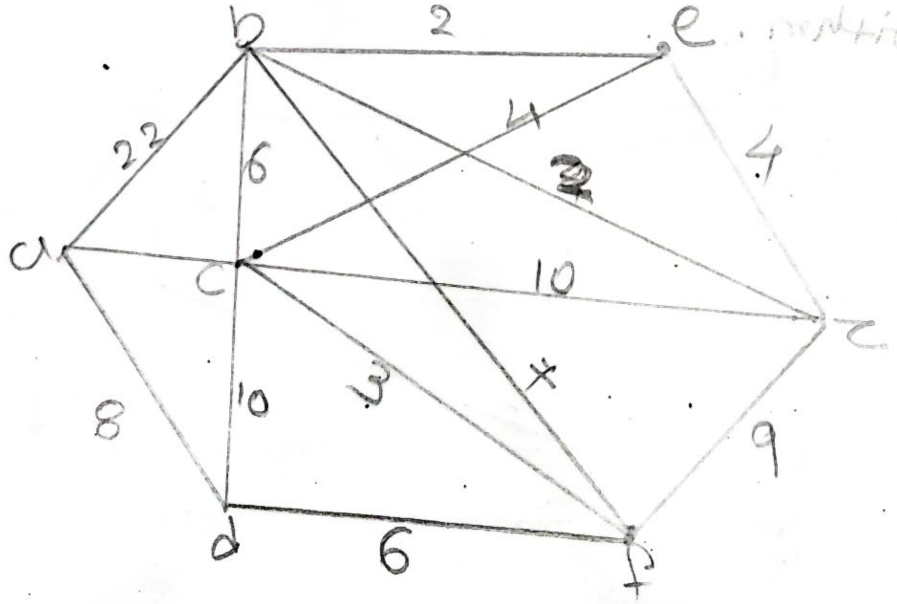
* Elementery Path-
It is a Path in which vertices do not repeat

* Simple circuits
A circuit in a graph G is called as simple circuit if it does not include the same edge twice.

* Elementery circuits
It is a circuit in which vertex do not repeated except the end point

* Dijkstra's Shortest Path Algorithm

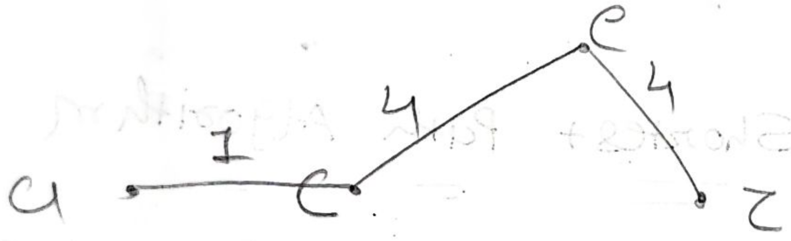
Ex:- Using Dijkstra's algorithm, find the Shortest Path between vertices a and z.



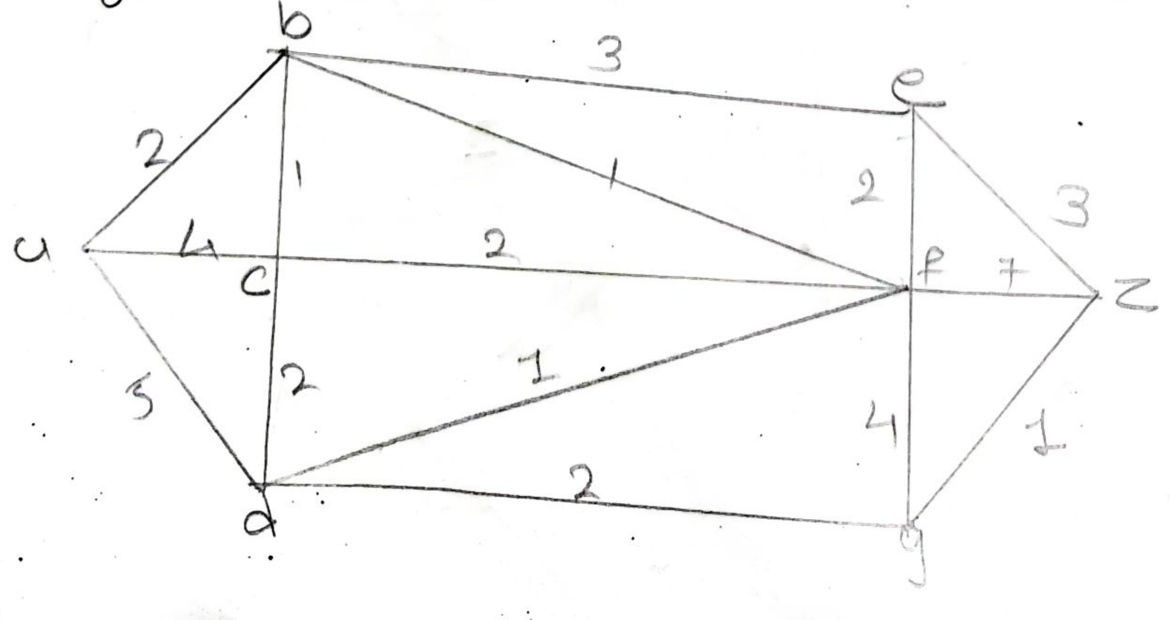
	a	b	c	d	e	f	z	Z
a	-	22a	$\boxed{1a}$	8a	∞	∞	∞	
c	-	7c	-	8a	5c	$\boxed{4c}$	11c	
f	-	7c	-	8a	$\boxed{5c}$	-	11c	
e	-	$\boxed{7c}$	-	8a	—	-	9e	
b	-	-	-	$\boxed{8a}$	-	-	9e	
d	-	-	-	-	-	-	$\boxed{9e}$	

Hence the length of Shortest Path from a to z is 9.

The Shortest Path is a c e z.



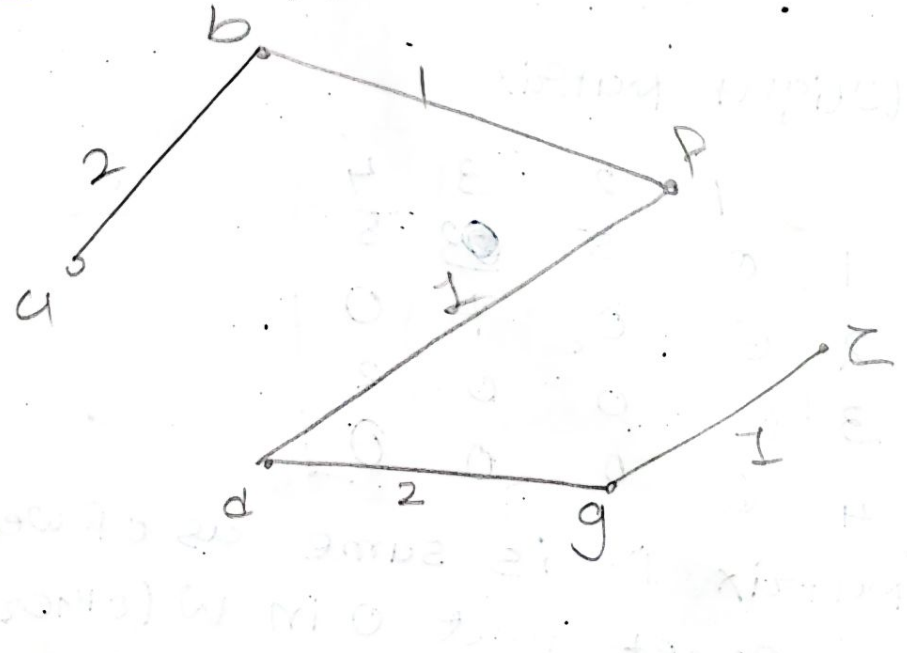
Ex:2 Find the Shortest Path between a - z for the given graph: using Dijkstra's algorithm.



	a	b	c	d	e	f	g	z
a	-	$\boxed{2a}$	4a	5a	∞	∞	∞	∞
b	-	-	$\boxed{3b}$	5a	5b	3b	∞	∞
c	-	-	-	5a	5b	$\boxed{3b}$	∞	∞
f	-	-	-	$\boxed{4f}$	5b	-	7f	10f
d	-	-	-	-	$\boxed{5b}$	-	6d	10f
e	-	-	-	-	-	-	$\boxed{6d}$	8e
g	-	-	-	-	-	-	-	$\boxed{7g}$

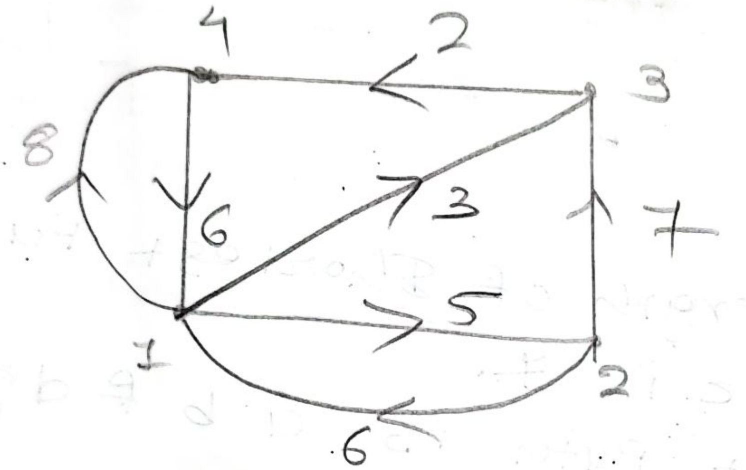
Hence the length of Shortest Path from a to z is 7.

The shortest path is a b f d g z.



* Floyd Warshall's Algorithm to produce shortest path.

Ex:1 Find the shortest path between each pair of vertices for a simple digraph using Warshall's algorithm.



Solution: Weight matrix

$$W = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 5 & 3 & 8 \\ 2 & 6 & 0 & 7 & 0 \\ 3 & 0 & 0 & 0 & 2 \\ 4 & 6 & 0 & 0 & 0 \end{array}$$

Distance matrix D_0 is same as of weight matrix W except that 0 in W (other than digonal) are replaced by ∞ (a very large Number)

$$D_0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 5 & 3 & 8 \\ 2 & 6 & 0 & 7 & \infty \\ 3 & \infty & \infty & 0 & 2 \\ 4 & 6 & \infty & \infty & 0 \end{array}$$

In distance matrix D_1, keep first row and first column as D_0 and diagonal entries as zero. To fill other entries of D_1.

$D_1(2,3) = \text{Min}[D_0(2,3),\ D_0(2,1)+D_0(1,3)]$
$= \text{Min}[7,\ 6+3]$
$= \text{Min}[7,9]$
$= 7$

$D_1(2,4) = \text{Min}[D_0(2,4),\ D_0(2,1)+D_0(1,4)]$
$= \text{Min}[\infty,\ 6+8]$
$= \text{Min}[\infty, 14]$
$= 14$

$D_1(3,2) = \text{Min}[D_0(3,2),\ D_0(3,1)+D_0(1,2)]$
$= \text{Min}[\infty,\ \infty+5]$
$= \infty$

$D_1(3,4) = \text{Min}[D_0(3,4),\ D_0(3,1)+D_0(1,4)]$
$= \text{Min}[2,\ \infty+8]$
$= 2$

$D_1(4,2) = \text{Min}[D_0(4,2),\ D_0(4,1)+D_0(1,2)]$
$= \text{Min}[\infty,\ 6+5]$
$= 11$

$D_1(4,3) = \text{Min}[D_0(4,3),\ D_0(4,1)+D_0(1,3)]$
$= \text{Min}[\infty,\ 6+3]$
$= 9$

$$D_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 5 & 3 & 8 \\ 2 & 6 & 0 & 7 & 14 \\ 3 & \infty & \infty & 0 & 2 \\ 4 & 6 & 11 & 9 & 0 \end{array}$$

Similarly, to find D_2 from D_1, keep second row and second column of D_1 as it is and diagonal entries as zero.

$D_2(1,3) = \text{Min}[D_1(1,3), D_1(1,2) + D_1(2,3)]$

$= \text{Min}[3, 5+7] = 3$

$D_2(1,4) = \text{Min}[D_1(1,4), D_1(1,2) + D_1(2,4)]$

$= \text{Min}[8, 5+14] = 8$

$D_2(3,1) = \text{Min}[D_1(3,1), D_1(3,2) + D_1(2,1)]$

$= \text{Min}[\infty, \infty+6] = \infty$

$D_2(3,4) = \text{Min}(D_1(3,4), D_1(3,2) + D_1(2,4)]$

$= \text{Min}[2, \infty+14] = 2$

$D_2(4,1) = \text{Min}[D_1(4,1), D_1(4,2) + D_1(2,1)]$

$= \text{Min}[6, 11+6] = 6$

$D_2(4,3) = \text{Min}[D_1(4,3), D_1(4,2) + D_1(2,3)]$

$= \text{Min}[9, 11+7] = 9$

$$D_2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 5 & 3 & 8 \\ 2 & 6 & 0 & 7 & 14 \\ 3 & \infty & \infty & 0 & 2 \\ 4 & 6 & 11 & 9 & 0 \end{array}$$

Similarly, to find D_3 from D_2, keep third row and third column of D_2 unchanged and diagonal entries as zero.

$$D_3(1,2) = \text{Min}\,[D_2(1,2),\ D_2(1,3) + D_2(3,2)]$$
$$= \text{Min}\,[5,\ 3+\infty] = 5$$

$$D_3(1,4) = \text{Min}\,[D_2(1,4),\ D_2(1,3) + D_2(3,4)]$$
$$= \text{Min}\,[8,\ 3+2] = 5$$

$$D_3(2,1) = \text{Min}\,[D_2(2,1),\ D_2(2,3) + D_2(3,1)]$$
$$= \text{Min}\,[6,\ 7+\infty] = 6$$

$$D_3(2,4) = \text{Min}\,[D_2(2,4),\ D_2(2,3) + D_2(3,4)]$$
$$= \text{Min}\,[14,\ 7+2] = 9$$

$$D_3(4,1) = \text{Min}\,[D_2(4,1),\ D_2(4,3) + D_2(3,1)]$$
$$= \text{Min}\,[6,\ 9+\infty] = 6$$

$$D_3(4,2) = \text{Min}\,[D_2(4,2),\ D_2(4,3) + D_2(3,2)]$$
$$= \text{Min}\,[11,\ 9+\infty] = 11$$

$$D_3 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 5 & 3 & 5 \\ 6 & 0 & 7 & 9 \\ \infty & \infty & 0 & 2 \\ 6 & 11 & 9 & 0 \end{bmatrix} \end{array}$$

To find D_4 from D_3, keep fourth row and fourth column of D_3 unchanged and diagonal entries as zero.

$D_4(1,2) = Min[D_3(1,2), D_3(1,4) + D_3(4,2)]$
$= Min[5, 5+11] = 5$

$D_4(1,3) = Min[D_3(1,3), D_3(1,4) + D_3(4,3)]$
$= Min[3, 5+9] = 3$

$D_4(2,1) = Min[D_3(2,1), D_3(2,4) + D_3(4,1)]$
$= Min[6, 9+6] = 6$

$D_4(2,3) = Min[D_3(2,3), D_3(2,4) + D_3(4,3)]$
$= Min[7, 9+9] = 7$

$D_4(3,1) = Min[D_3(3,1), D_3(3,4) + D_3(4,1)]$
$= Min[\infty, 2+6] = 8$

$D_4(3,2) = Min[D_3(3,2), D_3(3,4) + D_3(4,2)]$
$= Min[\infty, 2+11] = 13$

$$D_4 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 5 & 3 & 5 \\ 6 & 0 & 7 & 9 \\ 8 & 13 & 0 & 2 \\ 6 & 11 & 9 & 0 \end{bmatrix} \end{array}$$

Now D_4 is the matrix of shortest path between the vertices.

Eulerian Path and circuit.

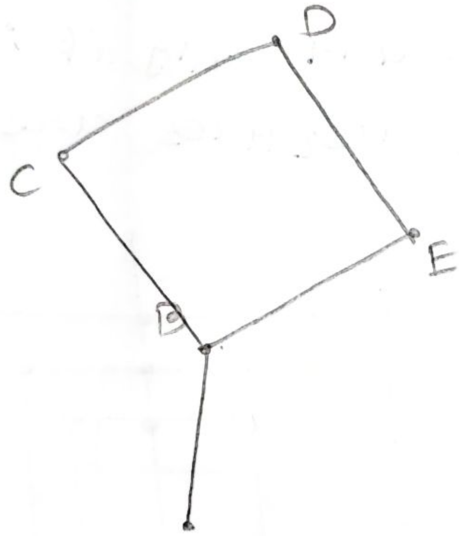
* Euler Path

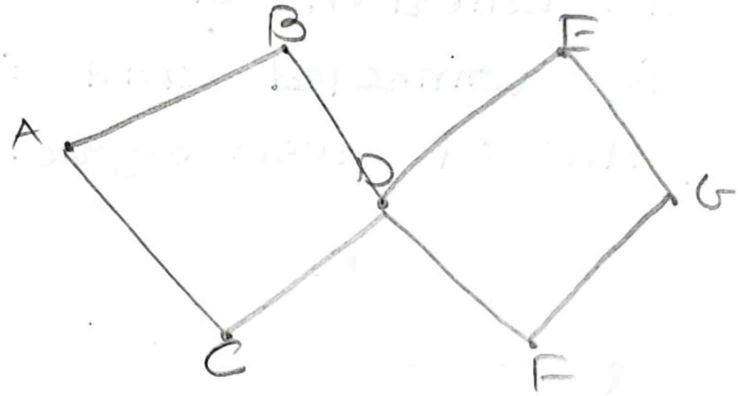
A Path is known as Eulerian Path if every edge of the graph G appears exactly once in the Path.

-> Eulerian Circuit.

The circuit which contains every edge of the graph exactly once is called eulerian circuit.

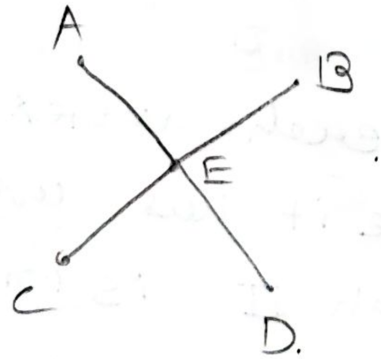
Ex:

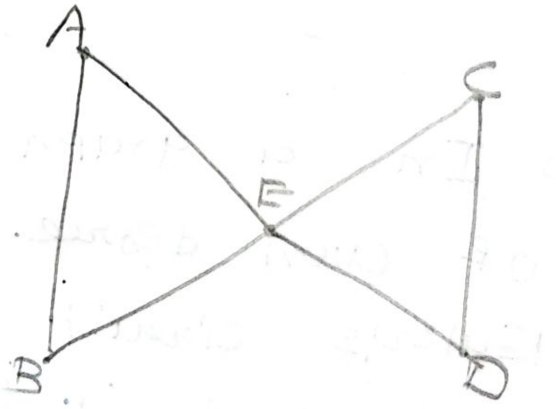




Euler Path
but Euler circuit

Ex:





Euler Path & circuit

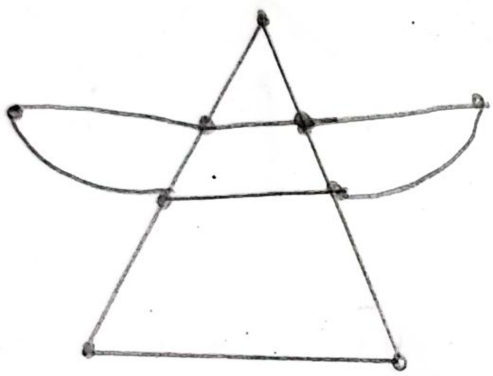
-> Eulerian graph

→ Eulerian Graph

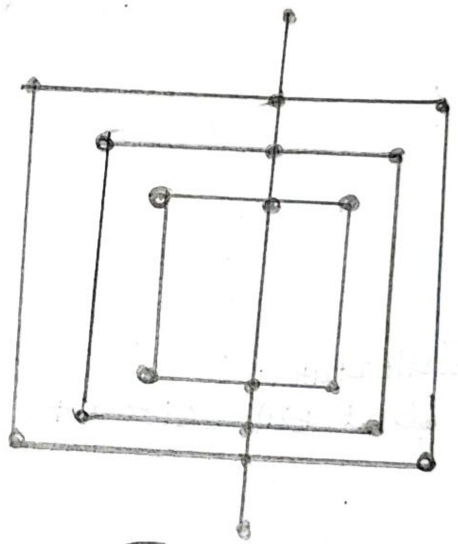
A Graph which has Eulerian circuit is known as Eulerian graph.

Theorem-1: An undirected graph has a Eulerian Path if and only if it is connected and has either zero or two vertices of odd degree.

Theorem-2. An undirected graph has an Eulerian circuit if and only if it is connected and its vertices are all of even degree.



I



II

→ In a graph I, each vertex is of even degree. Hence it has an Euler's circuit. Graph II is connected graph and it has exactly 2 vertices of odd degree. Hence this graph has an Euler's path.

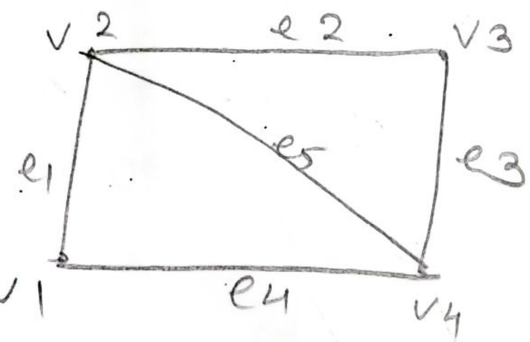
* Hamiltonian Path and circuit.

-> Hamiltonian Path.

A Path in a connected graph G is a Hamiltonian Path if it contains every vertex of G exactly once.

-> Hamiltonian circuit.

A circuit in a connected graph G is a Hamiltonian circuit if it contains every vertex of G exactly once except the first and the last vertex.



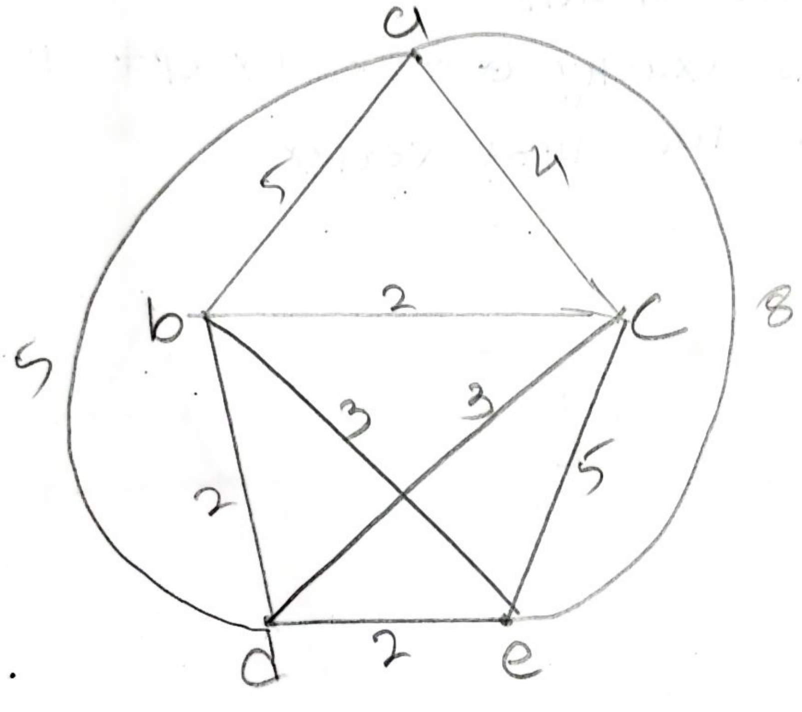
-> Maximum number of edge in a simple graph with n vertices is $\frac{n(n-1)}{2}$

-> Degree of any vertex of a simple graph with n vertices is at the most $(n-1)$.

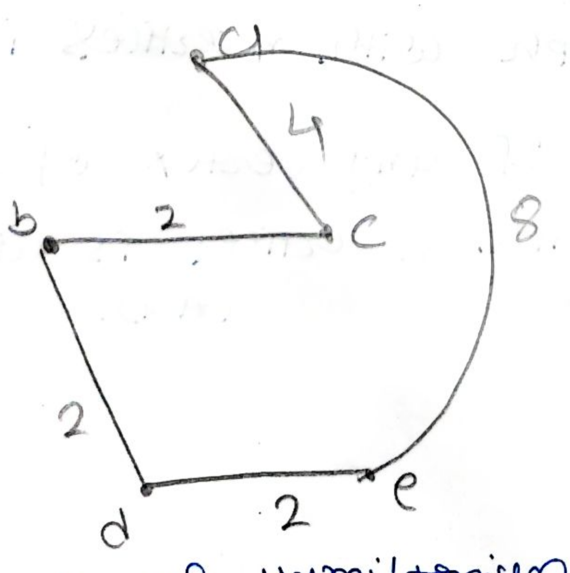
* Travelling Salesman Problem

→ Nearest - Neighbour Method

Ex: 1 Use nearest - neighbourhood method to find out Hamiltonian circuit for the graph in the following fig. Starting at vertex a.



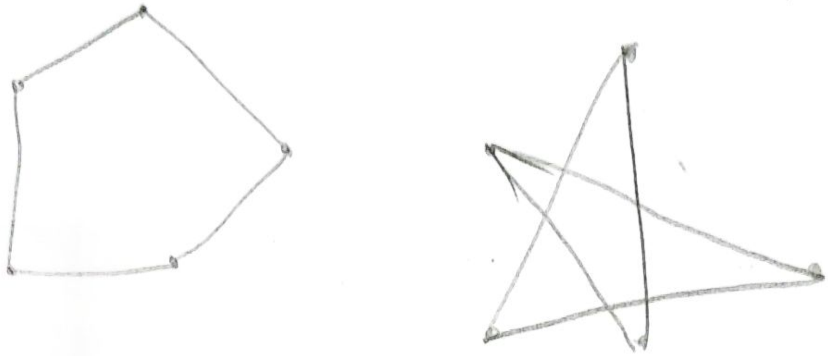
Solution:



The weight of Hamiltonian circuit = 18

-> Self-complementery

A graph G is self-complemetery if it is isomorphic to its complement.



-> planar Graphs-

A graph G is said to be planar if G can be drawn in a plane so that no edges cross.

Ex:

Ex: