Ex 14 QB 262 Let the relation R in the set of all triangles in a plane be defined by "is congruent to". Show that this relation is an equivalence relation.

Soln Let A be the set of all triangles in a plane.

① R is Reflexive:
we know that every triangle $x \in A$ is congruent to itself x.
Hence, $\forall x \in A$, xRx is true.
$\therefore$ R is reflexive.

② R is Symmetric:
Let $x, y \in A$,
If triangle x is congruent to triangle y,
then triangle y is congruent to triangle x.
ie $xRy \Rightarrow yRx$
$\therefore$ R is Symmetric.

③ R is Transitive:
Let $x, y, z \in A$.
If 'triangle x' is congruent to triangle y and triangle y is congruent to triangle z then triangle x is congruent to triangle z.
xRy and $yRz \Rightarrow xRz$
$\therefore$ R is Transitive.

Ex 15 QB 277 Show that the relation 'is equal to' in the sets, is an equivalence relation.

Soln Let the relation 'is equal to' in the set A.
R = 'is equal to'

① R is Reflexive: Let $a \in A$, then $a = a$
$\therefore (a, a) \in R$, $\forall a \in A$

Hence, R is Reflexive relation.

(2) R is Symmetric: Let $a, b \in A$, then

$$a = b \Rightarrow b = a$$

$\therefore (a,b) \in R \Rightarrow (b,a) \in R$ is true.

Hence, R is Symmetric relation

(3) R is Transitive: Let $a, b, c \in A$, then

$$a = b \text{ and } b = c \Rightarrow a = c.$$

$\therefore (a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

Hence, R is transitive relation

Thus, R is an equivalence relation.

Ex 16 Let I be the set of all integers excluding
QB zero and R be a relation defined by
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$$xRy \text{ if } x^y = y^x \text{ where } x, y \in I.$$

Is this relation R an equivalence relation?

Sol^n (1) R is Reflexive: We have

$$x^x = x^x, \ \forall x \in I.$$

$\therefore xRx, \ \forall x \in I$ is true

$\therefore$ R is Reflexive

(2) R is Symmetric: Let $x, y \in I$ such that

$$x^y = y^x.$$

If we replace x by y and y by x in relation then, it remains unchanged.

$$y^x = x^y$$

$\therefore xRy \Rightarrow yRx$

$\therefore$ R is Symmetric.

③ R is transitive: Let $x, y, z \in I$ such that $x^y = y^x$ and $y^z = z^y$ then $x^z \neq z^x$

xRy and $yRz \Rightarrow x \not R z$

$\therefore$ R is not transitive

Thus the relation R is reflexive and symmetric but not transitive.

Hence, R is not an equivalence relation.

Ex 17. If R and R' are equivalence relations in a set A, show that $R \cap R'$ is an equivalence relation in A.

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Solⁿ Since R and R' are relations in A, so

$R \subseteq A \times A$ and $R' \subseteq A \times A$.

Hence, $R \cap R' \subseteq A \times A$ and $R \cap R'$ is also a relation in A.

① $R \cap R'$ is reflexive:

Since R is reflexive $\therefore (a,a) \in R, \forall a \in A$

Also R' is reflexive $\therefore (a,a) \in R', \forall a \in A$.

$\therefore \forall a \in A$ we have $(a,a) \in R \cap R'$

$\therefore R \cap R'$ is reflexive.

② $R \cap R'$ is symmetric: Let $(a,b) \in R \cap R'$

Now, $(a,b) \in R \cap R' \Rightarrow (a,b) \in R$ and $(a,b) \in R'$

$\Rightarrow (b,a) \in R$ and $(b,a) \in R'$

$\Rightarrow (b,a) \in R \cap R'$

$\therefore R \cap R'$ is symmetric

③ $R \cap R'$ is transitive: we have

$(a,b) \in R \cap R'$ and $(b,c) \in R \cap R'$

$\Rightarrow [(a,b) \in R$ and $(a,b) \in R']$ and $[(b,c) \in R$ and $(b,c) \in R']$

$\Rightarrow [(a,b)\in R$ and $(b,c)\in R]$ and $[(a,b)\in R'$ and $(b,c)\in R']$

$\Rightarrow (a,c)\in R$ and $(a,c)\in R'$

$\Rightarrow (a,c)\in R\cap R'$

$\therefore R\cap R'$ is transitive.

Hence, $R\cap R'$ is an equivalence relation.

Ex 18 If R is an equivalence relation in a set A
OB then prove that R^{-1} is an equivalence
&90 relation in the set A.

Soln Given R is an equivalence relation in the set A.

To prove: R^{-1} is an equivalence relation in A

① R^{-1} is Reflexive:-

$\because R$ is reflexive.

$$\forall a\in A \Rightarrow (a,a)\in R$$

Now, $(a,a)\in R \Rightarrow (a,a)\in R^{-1}$

$\therefore \forall a\in A,\ (a,a)\in R^{-1}$

$\therefore R^{-1}$ is reflexive.

② R^{-1} is Symmetric:-

Let $a,b\in A$ and $(a,b)\in R$

then $(a,b)\in R \Rightarrow (b,a)\in R^{-1}$

Now, $(a,b)\in R \Rightarrow (b,a)\in R$

$\Rightarrow (a,b)\in R^{-1}$

$(b,a)\in R^{-1} \Rightarrow (a,b)\in R^{-1}$

$\therefore R^{-1}$ is Symmetric.

③ R^{-1} is Transitive:

Let $(a,b)\in R$ and $(b,c)\in R$, then

$(b,a) \in R^{-1}$ and $(c,b) \in R^{-1}$

Now, $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$

$\Rightarrow (c,a) \in R^{-1}$

$\therefore (c,b) \in R^{-1}$ and $(b,a) \in R^{-1} \Rightarrow (c,a) \in R^{-1}$

$\therefore R^{-1}$ is transitive

$\therefore R^{-1}$ is an equivalence relation.

Ex 19 give an example of a relation which is

(a) reflexive and transitive but not symmetric

(b) symmetric and transitive but not reflexive

(c) reflexive and symmetric but not transitive

(d) reflexive and transitive but neither symmetric nor antisymmetric.

Sol^n (a) Define a relation R in $\mathbb{R}$ as.

$R = \{(a,b) : a^3 \geq b^3\}$

(1) Reflexive : $(a,a) \in R$

$\therefore a^3 = a^3, \forall a \in \mathbb{R}$

$\therefore R$ is reflexive

(2) Symmetric : $(a,b) \in R$ then $(b,a) \in R$

$\therefore a^3 \geq b^3$ then $b^3 \ngeq a^3$ not possible

Now, $(2,1) \in R \Rightarrow 2^3 \geq 1^3$

$(1,2) \notin R \Rightarrow 1^3 \geq 2^3$

$\therefore R$ is not symmetric.

(3) Transitive : $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$

$\Rightarrow a^3 \geq b^3$ and $b^3 \geq c^3$

$\Rightarrow a^3 \geq c^3$

$\Rightarrow (a,c) \in R$

$\therefore R$ is Transitive

Hence, R is reflexive and transitive but not symmetric.

(b) Let $A = \{-5, -6\}$

Define a relation R on A as.

$R = \{(-5,-6), (-6,-5), (-5,-5)\}$

① R is reflexive :- $(a,a) \in R$

$(-6,-6) \notin R$

$\therefore$ R is not reflexive

② R is Symmetric : $(a,b) \in R$ then $(b,a) \in R$

$(-5,-6) \in R$ $(-6,-5) \in R$

So, $(-5,-5) \in R$.

$\therefore$ R is symmetric

③ R is Transitive : $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$

$(-5,-6) \in R$ and $(-6,-5) \in R$ then $(-5,-5) \in R$

$\therefore$ R is transitive

Hence, R is symmetric and transitive but not reflexive.

(c) Let $A = \{4, 6, 8\}$

Defined a relation R on A as:

$A = \{(4,4), (6,6), (8,8), (4,6), (6,4), (6,8), (8,6)\}$

(1) Reflexive :- $(a,a) \in R$

$(4,4), (6,6), (8,8) \in R$

$\therefore$ R is Reflexive

(2) Symmetric : $(a,b) \in R$ then $(b,a) \in R$

$(4,6) \in R$ then $(6,4) \in R$

$(6,8) \in R$ then $(8,6) \in R$

$\therefore$ R is Symmetric

(3) Transitive : $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$

$(4,6), (6,8) \in R$ but $(4,8) \notin R$

$\therefore$ R is not transitive

Hence, R is reflexive and symmetric but not transitive

* Transitive Closure :-
Transitive closure of a relation R is the smallest transitive relation containing R.

$R^{\infty} = R^1 \cup R^2 \cup R^3 \dots \cup R^n$

Ex 1 Let $A = \{1,2,3\}$, and $M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$

Find transitive closure ?

Sol^n $R = \{(1,1), (1,3), (3,1), (2,2), (3,2)\}$

$R^* = R \cup R^2 \cup R^3$

Here $R = \{(1,1), (1,3), (2,2), (3,1), (3,2)\}$

$R^2 = R \cdot R$

$= \{(1,1), (1,3), (2,2), (3,1), (3,2)\} \cdot \{(1,1), (1,3), (2,2), (3,1), (3,2)\}$

$R^2 = \{(1,1), (1,3), (1,2), (2,2), (3,1), (3,2), (3,3)\}$

$R^3 = R \circ R^2$

$= \{(1,1), (1,3), (2,2), (3,1), (3,2)\}$
$\cdot \{(1,1), (1,3), (1,2), (2,2), (3,1), (3,2), (3,3)\}$

$= \{(1,1), (1,3), (1,2), (2,2), (3,1), (3,2), (3,3)\}$

Hence, $R^* = \{(1,1), (2,2), (3,3), (1,2), (1,3), (3,2), (3,1)\}$.
This is transitive closure of given set.

Ex 2 If $A = \{1,2,3,4,5\}$. and
QB 298 $R = \{(1,2), (3,4), (4,5), (4,1), (1,1)\}$.
Find transitive closure ?

Sol^n
$R^* = R^1 \cup R^2 \cup R^3 \cup R^4 \cup R^5$

$R^2 = R \circ R$

$= \{(1,2), (3,4), (4,5), (4,1), (1,1)\} \cdot \{(1,2), (3,4), (4,5), (4,1), (1,1)\}$

$= \{(3,5), (3,1), (4,2), (4,1), (1,2), (1,1)\}$

$R^3 = R \cdot R^2$

$= \{(1,2),(3,4),(4,5),(4,1),(1,1)\}$
$\cdot \{(3,5),(3,1),(4,2),(4,1),(1,2),(1,1)\}$

$= \{(3,2),(3,1),(4,2),(4,1),(1,2),(1,1)\}$

$R^4 = R \cdot R^3$

$= \{(1,2),(3,4),(4,5),(4,1),(1,1)\}$
$\cdot \{(3,2),(3,1),(4,2),(4,1),(1,2),(1,1)\}$

$= \{(3,2),(3,1),(4,2),(4,1),(1,2),(1,1)\}$

$R^5 = R \cdot R^4$

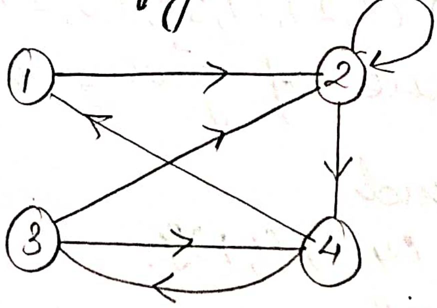
$= \{(1,2),(3,4),(4,5),(4,1),(1,1)\}$
$\cdot \{(3,2),(3,1),(4,2),(4,1),(1,2),(1,1)\}$

$= \{(3,2),(3,1),(4,2),(4,1),(1,2),(1,1)\}.$

$R^* = \{(1,2),(1,1),(3,2),(3,1),(4,1),(4,2),(3,4),$
$(4,5),(3,5)\}.$

Ex 3 Find the transitive closure without using
QB Warshall's Algorithm of the relation R on
299 $A = \{1,2,3,4\}$ defined by the directed graph as shown in the figure.

Sol^n



$R = \{(1,2),(2,2),(2,4),(3,2),(3,4),(4,1),(4,3)\}$

$R^* = R^1 \cup R^2 \cup R^3 \cup R^4$

$R^2 = R \circ R$

$= \{(1,2),(2,2),(2,4),(3,2),(3,4),(4,1),(4,3)\}$
$\cdot \{(1,2),(2,2),(2,4),(3,2),(3,4),(4,1),(4,3)\}$

$= \{(1,2),(1,4),(2,2),(2,4),(2,1),(2,3),(3,2),(3,4),(3,1),(3,3),$
$(4,2),(4,4)\}.$

$R^3 = R \cdot R^2$

$= \{(1,2),(2,2),(2,4),(3,2),(3,4),(4,1),(4,3)\}$
$\cdot \{(1,2),(1,4),(2,2),(2,4),(2,1),(2,3),(3,2),(3,4),(3,1),$
$(3,3),(4,2),(4,4)\}.$

$= \{(1,2),(1,4),(1,1),(1,3),(2,2),(2,4),(2,1),(2,3),(3,4),(3,2),$
$(3,1),(3,3),(4,2),(4,4),(4,1),(4,3)\}.$

$R^4 = R \cdot R^3$

$= \{(1,2),(2,2),(2,4),(3,2),(3,4),(4,1),(4,3)\}$
$\cdot \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),$
$(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}.$

$= \{(1,2),(1,4),(1,1),(1,3),(2,2),(2,4),(2,1),(2,3),(3,2),$
$(3,1),(3,4),(4,2),(4,4),(4,1),(4,3)\}.$

$R^* = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),$
$(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$

* Warshall's Algorithm

Ex 1. Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1,2), (2,3), (3,4), (2,1)\}$
Find transitive closure of R using Warshall's algorithm.

Soln

$$W_0 = M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

Step-1 In 1^{st} column, 1's are :- 2.
In 1^{st} row, 1's are : 2.

Now add (2,2)

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-2 In 2^{nd} column : 1's are : 1, 2
In 2^{nd} row : 1's are : 1, 2, 3.

Now add (1,1), (1,2), (1,3), (2,1), (2,2), (2,3)

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-3 In 3^{rd} column : 1's are : 1, 2
In 3^{rd} row : 1's are : 4

Now add (1,4) and (2,4).

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, W_3 has 1's in 1, 2, 3 of column 4 and no 1's in row 4, so no new 1's are added and $M_{R^\infty} = W_4 = W_3$.

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R^* = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)\}$

Ex 2 Let $A = \{1, 2, 3, 4, 5\}$ and $R = \{(1,1), (1,2), (3,4), (4,1), (4,5)\}$. Find transitive closure by using Warshall's algorithm.

Sol^n Here

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have

$$W_0 = M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-1 In 1st column : 1's are : 1, 4
In 1st row : 1's are : 1, 2

Now add $(1,1), (1,2), (4,1), (4,2)$

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-2 In 2nd column : 1's are : 1, 4

In 2nd row : 1's are : –

$$W_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step - 3 In 3rd column : 1's are : –

In 3rd row : 1's are : 3

$$W_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step - 4 In 4th column : 1's are : 3

In 4th row : 1's are : 1, 2, 5

Now, add (3,1), (3,2), (3,5)

$$W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-5 In 5^{th} column 1's are: 3, 4
In 5^{th} row 1's are: –

$$W_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R^* = \{(1,1),(1,2),(3,1),(3,2),(3,4),(3,5),(4,1),(4,2),(4,5)\}$

Ex 3 Let $A = \{1,2,3,4\}$ and let $R = \{(1,1),(1,2),(1,4),(2,4),$
QB $(3,1),(3,2),(4,2),(4,3),(4,4)\}$. Find transitive closure by using Warshall's algorithm.

Soln

$$W_0 = M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Step-1 In 1^{st} column : 1's are : 1, 3
In 1^{st} row : 1's are : 1, 2, 4
Now add $(1,1),(1,2),(1,4),(3,1),(3,2),(3,4)$.

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Step-2 In 2^{nd} column : 1's are : 1, 3, 4
In 2^{nd} row : 1's are : 4
Now add $(1,4),(1,3),(4,4)$

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Step-3 In 3rd column : 1's are : 1, 4
In 3rd row : 1's are : 1, 2, 4.
Now add (1,1), (1,2), (1,4), (4,1), (4,2), (4,4)

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Step-4 In 4th column : 1's are : 1, 2, 3, 4
In 4th row : 1's are : 1, 2, 3, 4
Now add {(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3)
(2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)}

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$R^* = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4),$
$(3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$.

Ex 4 Let $A = \{11, 12, 13, 14\}$ and let
OB 301 $R = \{(11,12), (12,13), (13,14), (12,11)\}$. Find transitive closure of R using Warshall's algorithm

Sol^n

$$W_0 = M_R = \begin{array}{c|cccc} & 11 & 12 & 13 & 14 \\ \hline 11 & 0 & 1 & 0 & 0 \\ 12 & 1 & 0 & 1 & 0 \\ 13 & 0 & 0 & 0 & 1 \\ 14 & 0 & 0 & 0 & 0 \end{array}$$

Step-1 In 1st column : 1's are = 12
In 1st row : 1's are = 12
Now add (12,12).

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-2 In 2nd column : 1's are = 11, 12
In 2nd row : 1's are = 11, 12, 13
Now add (11, 11), (11, 12), (11, 13), (12, 11), (12, 12), (12, 13)

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-3 In 3rd column : 1's are : 11, 12
In 3rd row : 1's are : 14
Now add (11, 14), (12, 14).

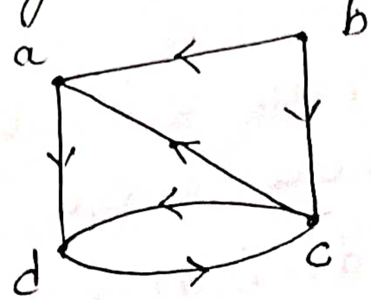
$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-4 In 4th column : 1's are : 11, 12, 13
In 4th row : 1's are : –

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R^* = \{(11, 11), (11, 12), (11, 13), (11, 14), (12, 11), (12, 12), (12, 13), (12, 14), (13, 14)\}$

Ex 5 Let R be a relation with given directed graph. Find the matrix of transitive closure of R using Warshall's algorithm.



Sol^n $R = \{(a,d), (b,a), (b,c), (c,a), (c,d), (d,c)\}$.

$$W_0 = M_R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array}$$

Step-1 In 1st column : 1's are : b, c
In 1st row : 1's are : d

Now add $(b,d), (c,d)$

$$W_1 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array}$$

Step-2 In 2nd column : 1's are : –
In 2nd row : 1's are : a, c, d

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Step-3 In 3rd column : 1's are : b, d
In 3rd row : 1's are : a, d

Now add $(b,a), (b,d), (d,a) (d,d)$

$$W_3 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Step-4 In 4th column: 1's are : a, b, c, d
In 4th row : 1's are : a, c, d

Now add (a,a), (a,c), (a,d) (b,a), (b,c), (b,d), (c,a), (c,c), (c,d), (d,a), (d,c), (d,d)

$$W_4 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$R^* = \{(a,a), (a,c), (a,d), (b,a), (b,c), (b,d), (c,a), (c,c), (c,d), (d,a), (d,c), (d,d)\}$

Ex6 Let R be a relation on set $A = \{1, 2, 3, 4, 5\}$ and
OB $R = \{(1,1), (1,2), (1,3), (1,4), (3,1), (3,2), (5,1), (5,2), (5,3), (5,4), (5,5)\}$
303 find transitive closure for R using Warshall's algorithm

Soln

$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Step-1 : In 1st column : 1's are = 1, 3, 5
In 1st row : 1's are = 1, 2, 3, 4

Now add (1,1), (1,2), (1,3), (1,4), (3,1), (3,2), (3,3), (3,4), (5,1), (5,2), (5,3), (5,4)

$$W_1 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Step-2 : In 2nd column : 1's are = 1, 3, 5
In 2nd row : 1's are : –

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Step-3 : In 3rd column : 1's are : 1, 3, 5
In 3rd row : 1's are : 1, 2, 3, 4
Now add (1,1), (1,2), (1,3), (1,4), (3,1); (3,2), (3,3), (3,4), (5,1), (5,2), (5,3), (5,4).

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Step-4 : In 4th column, 1's are = 1, 3, 5
In 4th row : 1's are = –

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Step-5 In 5th column : 1's are = 5
In 5th row , 1's are = 1, 2, 3, 4, 5
Now add (5,1), (5,2), (5,3), (5,4), (5,5).

$$W_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$R^* = \{(1,1),(1,2),(1,3),(1,4),(3,1),(3,2),(3,3),(3,4),$
$(5,1),(5,2),(5,3),(5,4),(5,5)\}.$

Ex7 Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and let R be a relation
OB on A whose matrix is
S04

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find transitive closure of R using Warshall's alogrithm

Sol^n

$$W_0 = M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Step-1 : In 1st column : 1's are $= a_1, a_4$
In 1st row : 1's are $= a_1, a_4$
Now add $(a_1,a_1)(a_1,a_4),(a_4,a_1),(a_4,a_4)$

$$W_1 = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Step-2 In 2nd column : 1's are : a_2, a_5
In 2nd row : 1's are : a_2
Now add $(a_2,a_2),(a_5,a_2)$

$$W_2 = \begin{array}{c|ccccc} & a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline a_1 & 1 & 0 & 0 & 1 & 0 \\ a_2 & 0 & 1 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 & 1 & 1 \\ a_4 & 1 & 0 & 0 & 1 & 0 \\ a_5 & 0 & 1 & 0 & 0 & 1 \end{array}$$

Step-3 In 3rd column : 1's are : –
In 3rd row : 1's are : a_4, a_5

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Step-4 In 4th column : 1's are – a_1, a_3, a_4
In 4th row : 1's are – a_1, a_4

Now add $(a_1,a_1), (a_1,a_4), (a_3,a_1), (a_3,a_4), (a_4,a_1), (a_4,a_4)$

$$W_4 = \begin{array}{c|ccccc} & a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline a_1 & 1 & 0 & 0 & 1 & 0 \\ a_2 & 0 & 1 & 0 & 0 & 0 \\ a_3 & 1 & 0 & 0 & 1 & 1 \\ a_4 & 1 & 0 & 0 & 1 & 0 \\ a_5 & 0 & 1 & 0 & 0 & 1 \end{array}$$

Step-5 In 5th column : 1's are = a_3, a_5
In 5th row : 1's are = a_2, a_5

Now add $(a_3,a_2), (a_3,a_5), (a_5,a_2), (a_5,a_5)$

$$W_5 = \begin{array}{c|ccccc} & a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline a_1 & 1 & 0 & 0 & 1 & 0 \\ a_2 & 0 & 1 & 0 & 0 & 0 \\ a_3 & 1 & 1 & 0 & 1 & 1 \\ a_4 & 1 & 0 & 0 & 1 & 0 \\ a_5 & 0 & 1 & 0 & 0 & 1 \end{array}$$

$R^* = \{(a_1,a_1), (a_1,a_4), (a_2,a_2), (a_3,a_1)\ (a_3,a_2), (a_3,a_4), (a_3,a_5)$
$(a_4,a_1), (a_4,a_4), (a_5,a_2), (a_5,a_5)\}$