Partial Ordering

* Partial order Relations

A relation R on a set A is called a Partial order relation iff R is

i) Reflexive relation

i.e $aRa \ \forall a \in A$

i.e $(a,a) \in R, \ \forall a \in A$

ii) Antisymmetric relation

if aRb and bRa then $a=b$

i.e $(a,b) \in R$, $(b,a) \in R \Rightarrow a=b$

Where $a,b \in A$

iii) Transitive relation

$aRb, bRc \Rightarrow aRc$

$(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$

Where $a,b,c \in A$

Ex:1) N is a set of natural numbers.

R is a relation defined on set N.

$R = \{(a,b) ; a$ divides b or $a|b\}$

aRb iff $a|b$. a is related with b iff a divides b.

Solution: i) $\forall a \in N \ a|a \Rightarrow (a,a) \in R$

hence R is Reflexive relation

ii) $a,b \in N$ and if $a|b$ and $b|a$ then $a=b$ hence R is antisymmetric relation.

iii) $a, b, c \in N$ if $a|b$ and $b|c$ then $a|c$ i.e $(a,b) \in R$, $(b,c) \in R \Rightarrow$

$(a,c) \in R$

hence R is transitive relation.

R is a Partial Order relation.

Ex: 2 $S = \{1, 2, 3\}$

$P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

R is a relation defined on set P(S).

ARB iff $A \subseteq B$

Then R is a Partial ordering relation on P(S).

Solution:

i) R is Reflexive

Every set is subset of itself hence

$A \subseteq A$

$\Rightarrow ARA, \forall A \in P(S)$

hence R is reflexive relation.

ii) R is antisymmetric

Let ARB and BRA

$\Rightarrow A \subseteq B$ and $B \subseteq A$

$A = B$

hence R is antisymmetric relation.

iii) R is transitive

Let ARB and BRC

$\Rightarrow A \subseteq B$ and $B \subseteq C$

$\Rightarrow A \subseteq C$

$\Rightarrow ARC$

R is transitive.

Hence R is a partial ordering relation.

Ex: Draw the digraph for the following relation and determine whether the relation is reflexive, symmetric, transitive relation and antisymmetric. (320)

$A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let xRy whenever y is divisible by x.

Solution: $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

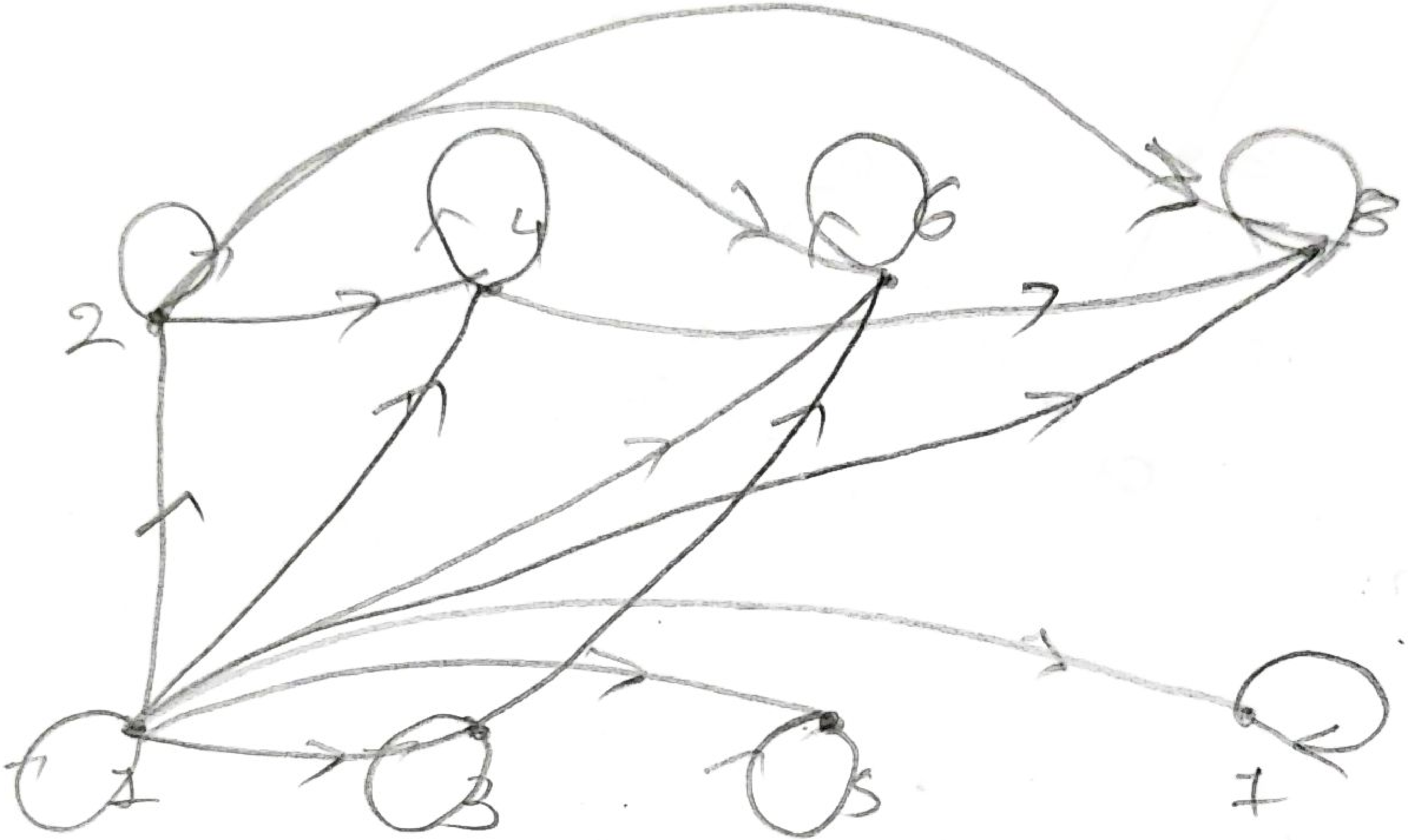
$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,2), (2,4), (2,6), (2,8), (3,3), (3,6), (4,4), (4,8), (5,5), (6,6), (7,7), (8,8)\}$

i) R is reflexive as $\forall a \in A$, $(a,a) \in R$

ii) R is not symmetric as $(1,2) \in R$ but $(2,1) \notin R$

iii) R is transitive

iv) R is antisymmetric.


2
4
6
8
1
3
5
7

* Hasse Diagrams

Poset can be represented by digraphs. A simpler way of representing poset is Hasse diagram.

→ Method to find Hasse diagram

1. Omitt loops as relation is reflexive on poset.
2. All arrows that appear on the edges are omitted.
3. Eliminate all edges that are implied by transitive relation.
   e.g if aRb, bRc, then aRc so eliminate (a,c) edge.
4. An arc pointing upward is drawn from a to b if $a \neq b$ and aRb.

(322)

Ex: Let R be the relation on the set A.

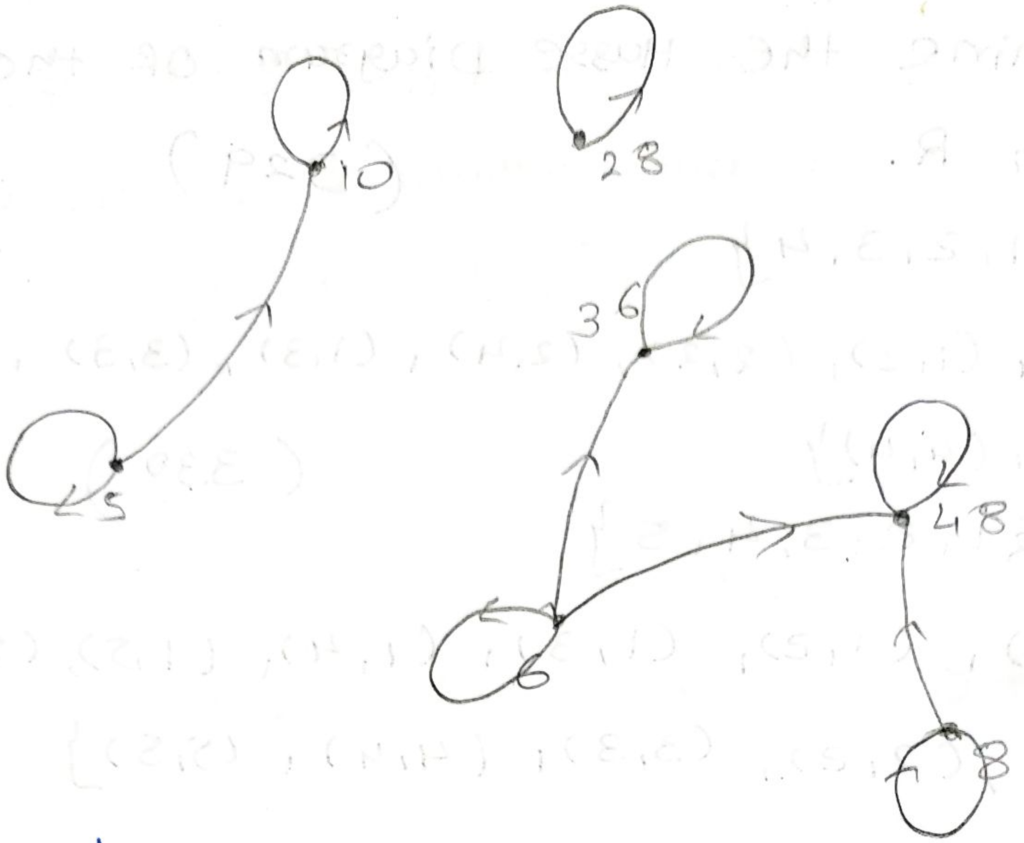
$A = \{5, 6, 8, 10, 28, 36, 48\}$

Let $R = \{(a,b) \mid a$ is divisior of $b\}$.

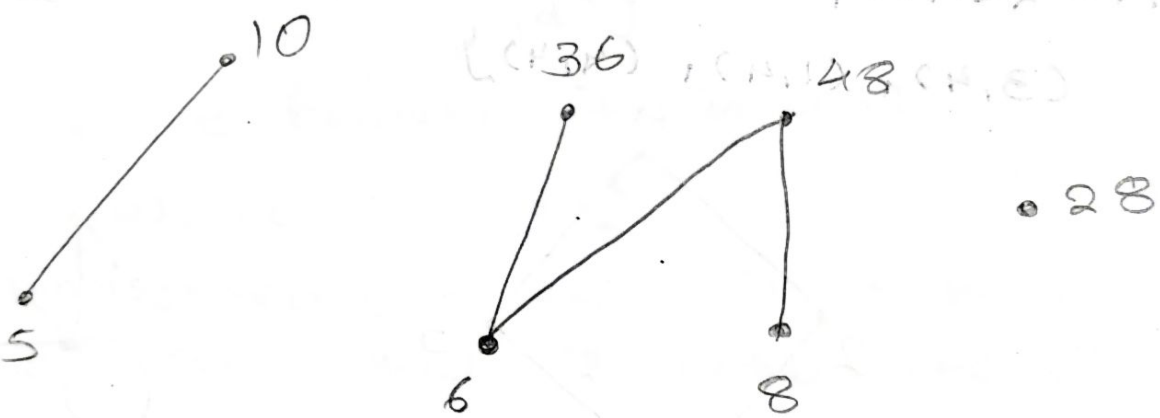
Draw the Hasse diagram.
Compare with digraph. Determine whether R is equivalence relation.

Solution: $A = \{5, 6, 8, 10, 28, 36, 48\}$.

$R = \{(5,5), (6,6), (8,8), (10,10), (28,28), (36,36), (48,48), (5,10), (6,36), (6,48), (8,48)\}$



Hasse diagram



R is reflexive relation, but R is not Symmetric relation. Hence R is not equivalence relation.

Ex: 2 Determine the Hasse Diagram of the relation R. (329)

(i) $A = \{1, 2, 3, 4\}$

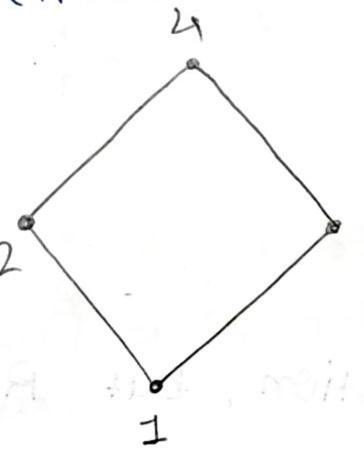
$R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$ (330)

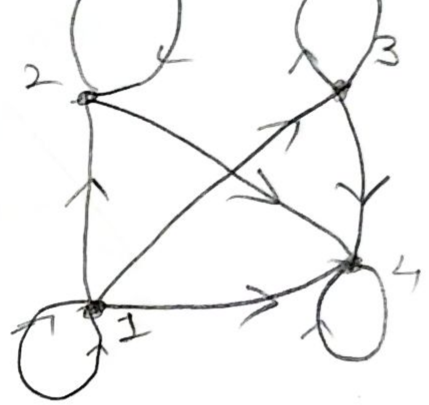
(ii) $A = \{1, 2, 3, 4, 5\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,4), (3,5), (2,2), (3,3), (4,4), (5,5)\}$

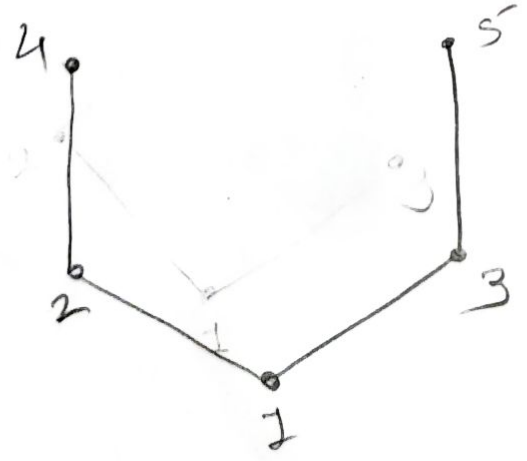
Solution:

(i) $R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$





(ii) $R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,4), (3,5), (2,2), (3,3), (4,4), (5,5)\}$



Ex:3 Let $A = \{a, b, c, d\}$ and x be a relation on A whose matrix is

(328)

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) Prove that R is Partial order.

(ii) Draw the Hasse diagram of R.

Solution:

(i) $R = \{(a,a), (a,c), (a,d), (b,b), (b,c), (c,c), (b,d), (c,d), (d,d)\}$

-> R is Reflexive because it contain $(a,a), (b,b), (c,c), (d,d)$

-> R is antisymmetric because it contain a and b such that if $a \neq b$ then $a \not R b$ or $b \not R a$.

R Satisfies this condition, hence R is anti-Symmetric.

-> R is transitive Since it contain (a,d) and (b,d).

$\therefore$ R is Partial order.

Ex:3 Let $A = \{a, b, c, d\}$ and x be a relation on A whose matrix is

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$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) Prove that R is Partial order.

(ii) Draw the Hasse diagram of R.

Solution:

(i) $R = \{(a,a), (a,c), (a,d), (b,b), (b,c), (c,c), (b,d), (c,d), (d,d)\}$

-> R is Reflexive because it contain $(a,a), (b,b), (c,c), (d,d)$

-> R is antisymmetric because it contain a and b such that if $a \neq b$ then $a \not R b$ or $b \not R a$.

R satisfies this condition, hence R is anti-symmetric.

-> R is transitive since it contain (a,d) and (b,d).

$\therefore$ R is Partial order.

(ii)

<u>Ex: 4</u> Determine the Hasse diagram of the relation on

$A = \{1, 2, 3, 4, 5\}$, whose matrix is shown.

(i)

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(331)

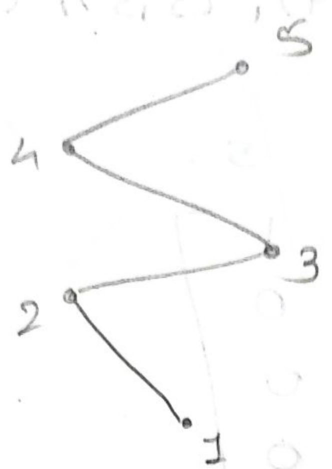
(ii)

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

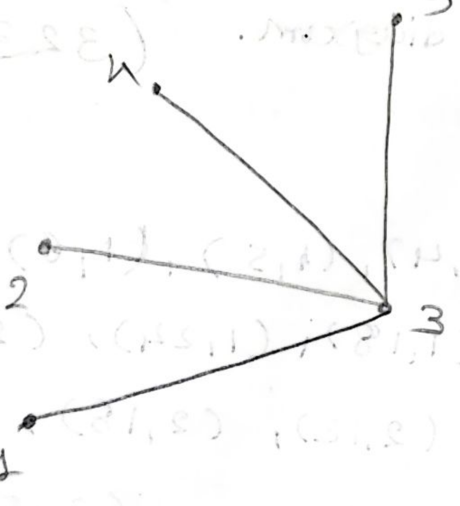
<u>Solution:</u> ~~[illegible]~~

(i) Hasse diagram:

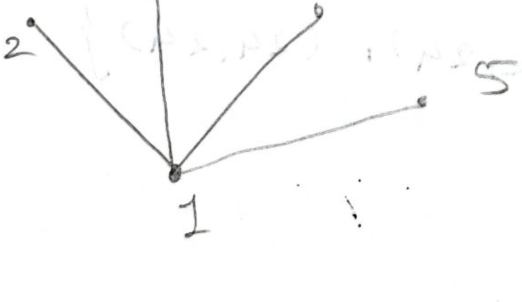
$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,2),$
$(2,3), (2,4), (2,5), (3,3), (3,4), (3,5),$
$(4,4), (4,5), (5,5)\}$



(ii) $R = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (5,5)\}$



<u>Ex</u>:5 Determine the matrix of the Partial Order whose Hasse diagram is given in the figure. (332)



Solution:

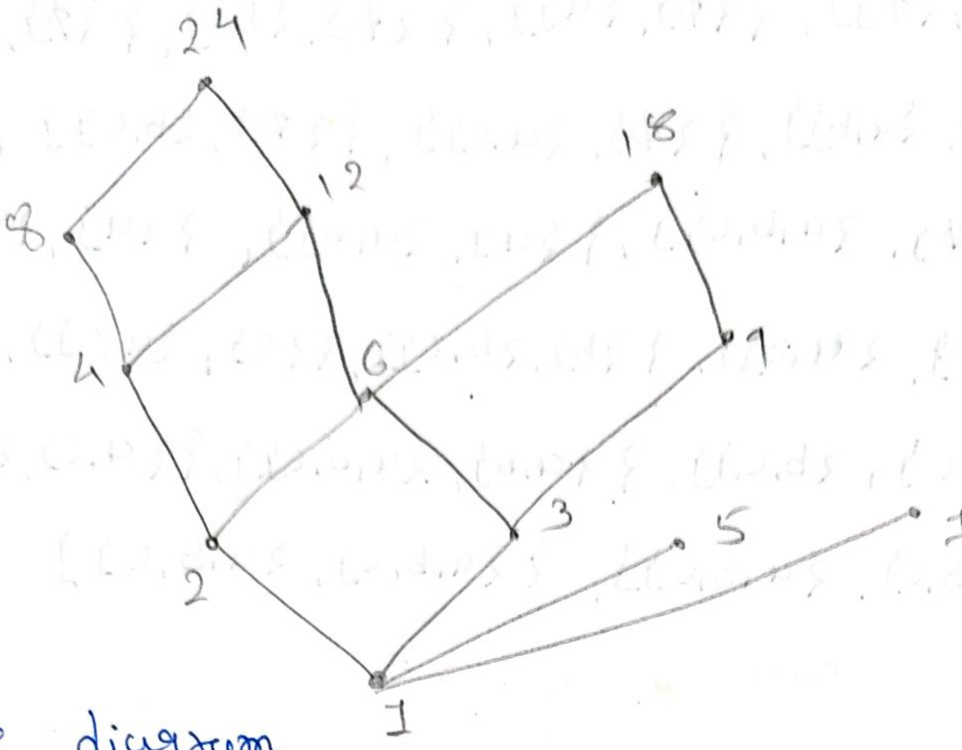
$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (1,3), (1,4), (1,5)\}$

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Ex:6 Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24\}$ be ordered by the relation x divides y. Show that the relations is partial ordering and draw the Hasse diagram. (323)

Solution:

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,12), (1,18), (1,24), (2,2), (2,4), (2,6), (2,8), (2,12), (2,18), (2,24), (3,3), (3,6), (3,9), (3,12), (3,18), (3,24), (4,4), (4,8), (4,12), (4,24), (5,5), (6,6), (6,12), (6,18), (6,24), (7,7), (8,8), (8,24), (9,9), (9,18), (12,12), (12,24), (24,24)\}$



Hasse diagram.

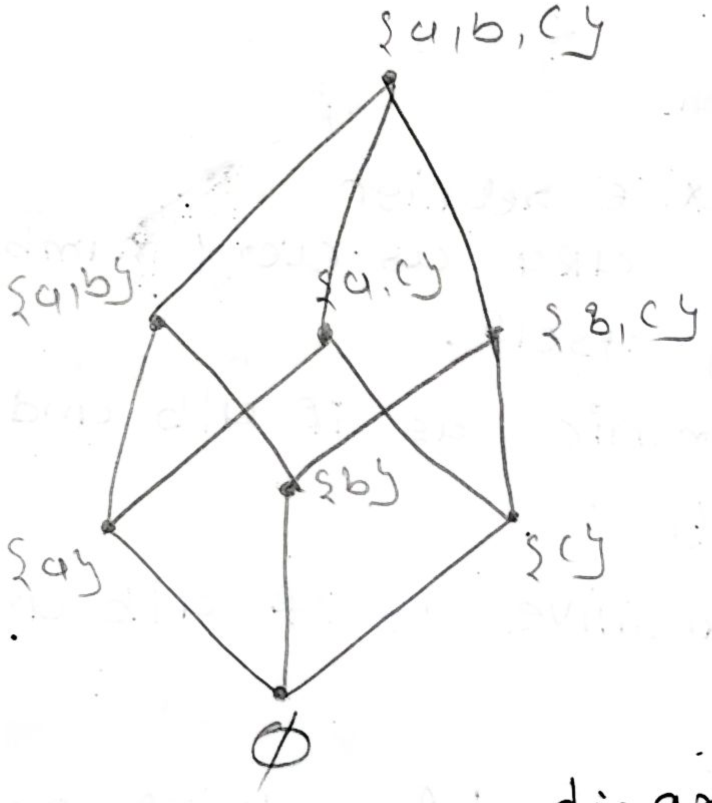
1. R is reflexive relation
As $\forall$ $a \in A$ aRa as every number is divisible by itself.
2. R is antisymmetric as if $a|b$ and $b|a$
then $a=b$
3. R is transitive as if $a|b$ and $b|c$
$\Rightarrow a|c$

Hence R is a Partial ordering relation.

Ex: 7 Draw the Hasse diagram for the Partial ordering $\{(A,B) / A \subseteq B\}$ on the Power set $P(S)$ where $S = \{a,b,c\}$.
(325)

Solution:
$S = \{a,b,c\}$
$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\},$
$\{b,c\}, \{a,b,c\}\}$

$R = \{\{\{\phi\}, \{\phi\}\}, \{\{\phi\}, \{a\}\}, \{\{\phi\}, \{b\}\}, \{\{\phi\}, \{c\}\},$
$\{\{\phi\}, \{a,b\}\}, \{\{\phi\}, \{a,c\}\}, \{\{\phi\}, \{b,c\}\},$
$\{\{\phi\}, \{a,b,c\}\}, \{\{a\}, \{a,b\}\}, \{\{a\}, \{a,c\}\},$
$\{\{b\}, \{a,b\}\}, \{\{b\}, \{b,c\}\}, \{\{c\}, \{a,c\}\},$
$\{\{c\}, \{b,c\}\}, \{\{a,b\}, \{a,b,c\}\}, \{\{a,c\}, \{a,b,c\}\},$
$\{\{b,c\}, \{a,b,c\}\}, \{\{a,b,c\}, \{a,b,c\}\}\}$

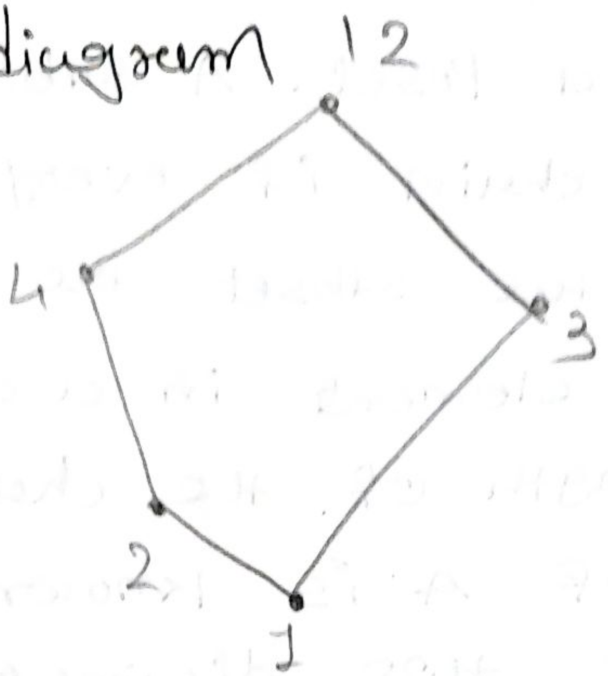


Ex: 8 Draw the Hasse diagram for the following relation on Set $A = \{1,2,3,4,12\}$. (327)

$R = \{(1,1), (2,2), (3,3), (4,4), (12,12),$
$(1,2), (4,12), (1,3), (1,4), (1,12),$
$(2,4), (2,12), (3,12)\}$

Solution:

Hasse diagram

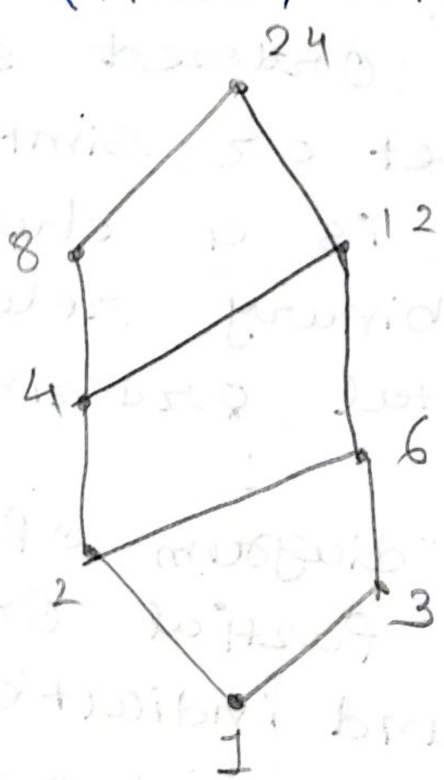


Ex: 9 Draw the Hasse diagram of D_{24}.

(324)

Solution:

$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$



* Chains and Antichains

Let $(A, \le)$ be a Poset. A subset of A is known as chain if every pair of elements in the subset are related. The number of element in a chain is called the length of the chain.

→ A subset of A is known as antichain if no two distinct element in a subset are related.

* Totally Ordered set

A Partially ordered set $(A, \le)$ is called a "totally ordered set" or linear ordered set or simple ordered set. if A is a chain. In this case, the binary relation '$\le$' is called a total ordering relation.

Ex: 1 Draw the Hasse diagram of the following sets under partial ordering relation "divides" and indicate those which are chains. (333)

(a) $\{1, 3, 9, 18\}$

(b) $\{1, 2, 5, 10, 20\}$

(c) $\{1, 2, 3, 5, 6, 10, 15, 30\}$

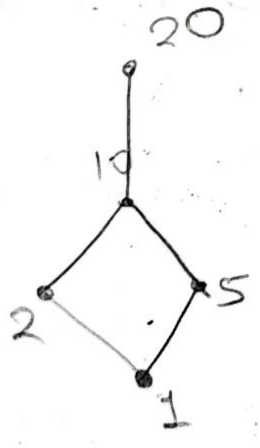
Solution:

(a) $R = \{(1,1), (1,3), (1,9), (1,18), (3,3), (3,9), (3,18), (9,9), (9,18), (18,18)\}$

This Poset is a chain because every two elements are related.

(b) $\{1, 2, 5, 10, 20\}$

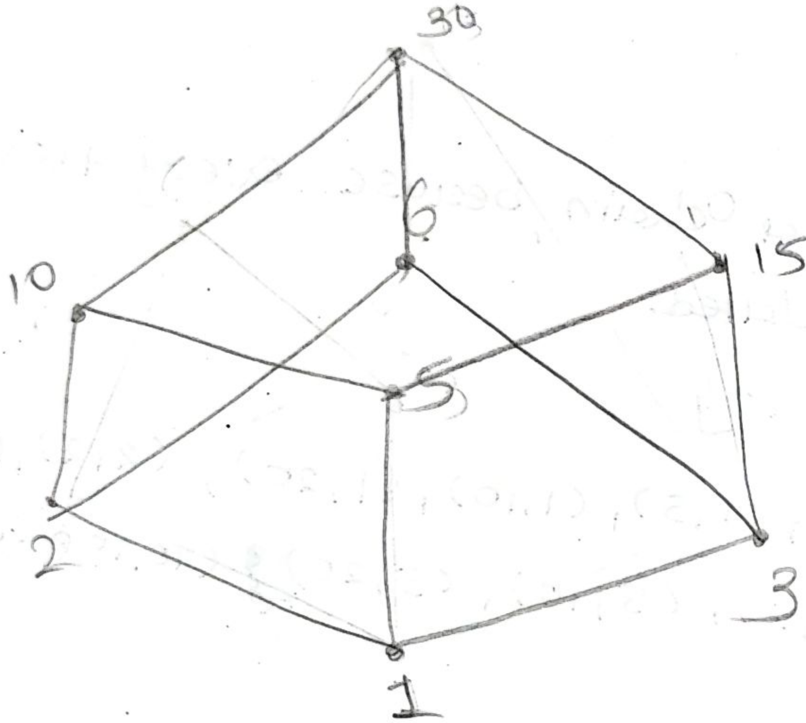
$R = \{(1,1), (1,2), (1,5), (1,10), (1,20), (2,2), (2,10), (2,20), (5,5), (5,10), (5,20), (10,10), (10,20), (20,20)\}$



Above Poset is not a chain because $2 \not R 5$ or $5 \not R 2$.

(c) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ (334)

$R = \{(1,1), (2,2), (3,3), (5,5), (6,6), (10,10),$
$(15,15), (30,30), (1,2), (1,3), (1,5), (1,6),$
$(1,10), (1,15), (1,30), (2,6), (2,10), (2,30),$
$(3,6), (3,15), (3,30), (5,10), (5,15), (5,30),$
$(6,30), (10,30), (15,30)\}$



Above Poset is not chain or linearly Ordered set, because 5 is not divisible by 3, 6 is not divisible by 5.

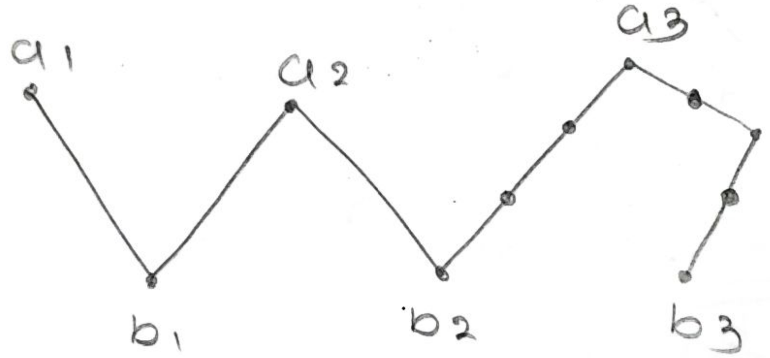
* Maximal and Minimal Elements

Let A be a non-empty set and '≤' is Partial Order relation on A. (A, ≤) is Poset.

-> An element $a \in A$ is known as maximal element of A if there is no element c in A such that $a \leq c$.

-> An element $b \in A$ is known as minimal element of A if there is no element c in A such that $c \leq b$.

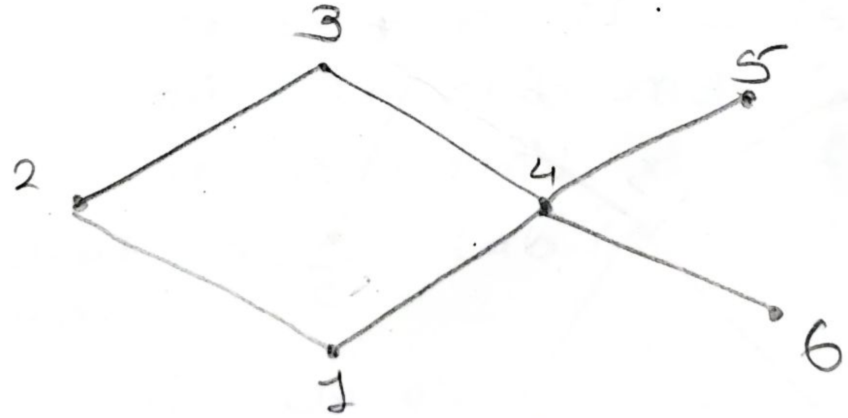
Ex: 1



Maximal elements: a_1, a_2, a_3

Minimal elements: b_1, b_2, b_3

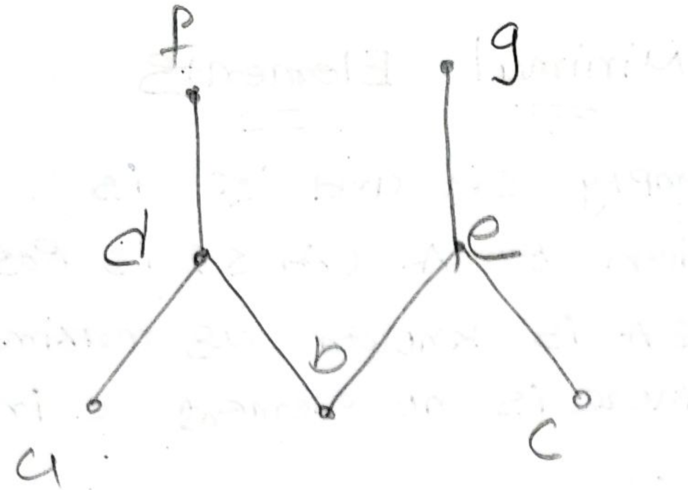
Ex: 2



Maximal elements: 3, 5

Minimal elements: 1, 6

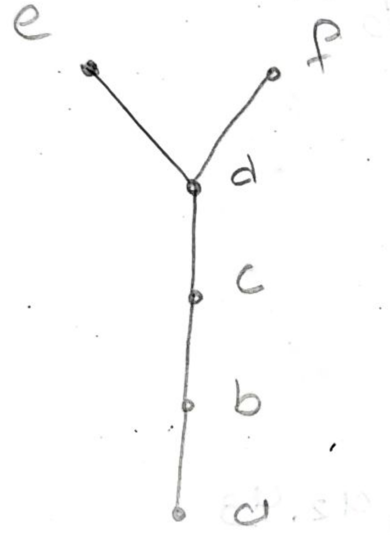
Ex: 3



Maximal elements : f, g

Minimal elements: a, b, c

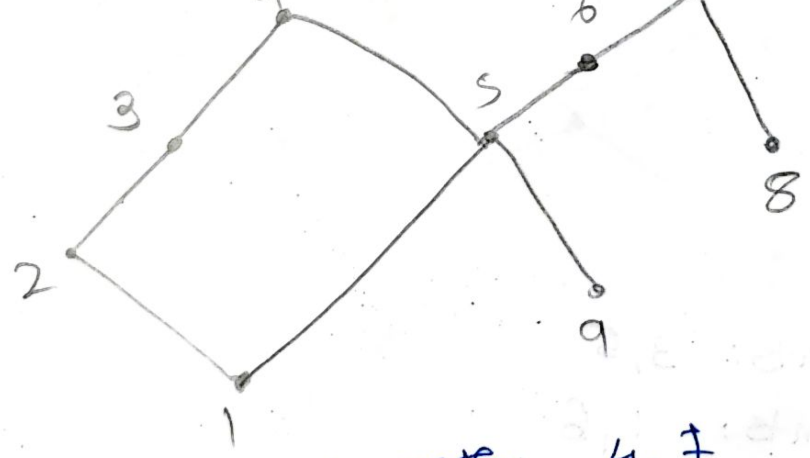
Ex: 4



Maximal elements : e, f

Minimal elements: a

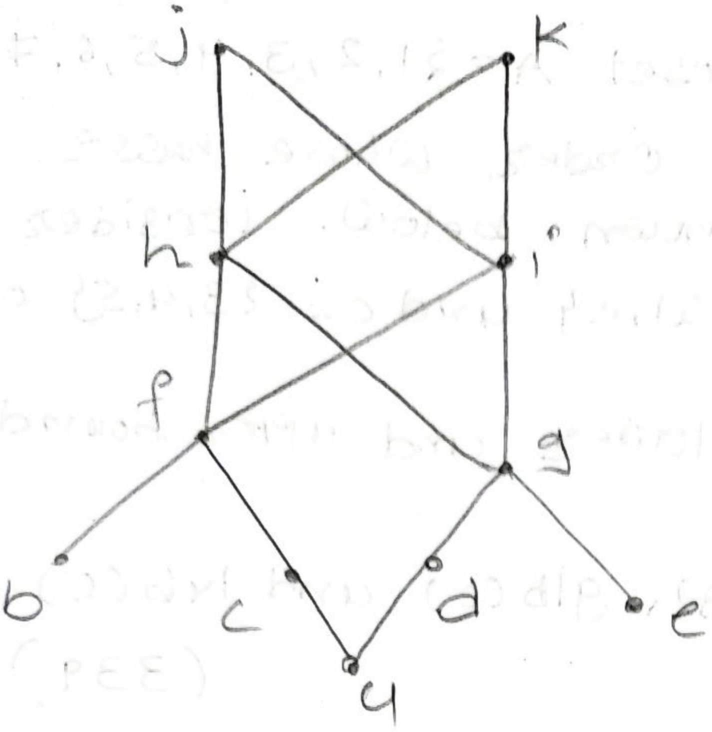
Ex: 5



Maximal elements: 4, 7

Minimal elements: 1, 9, 8

Ex: 6



Maximal elements: j, k

Minimal elements: a, b, e

* Upper Bounds and Lower Bounds

Let $(A, \leq)$ be a poset. For elements $a, b \in A$, an element $c \in A$ is called upper bound of a and b if $a \leq c$ and $b \leq c$

c is known as least upper bound (lub) of a and b if c is an upper bound of a, b and if there is no other upper bound d of a and b such that $d \leq c$. Lub is also known as supremum.

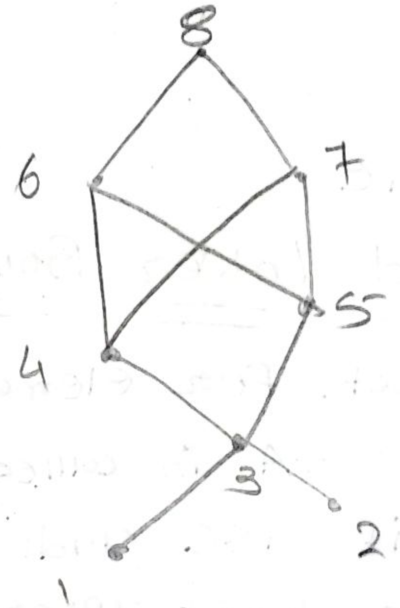
Similarly an element e is said to be a lower bound of a and b if $e \leq a$ and $e \leq b$; and e is known as greatest lower bound. (glb) of a and b if there is no other lower bound f of a and b such that $e \leq f$. glb is known as infimum.

Ex:1 Consider the Poset $A = \{1,2,3,4,5,6,7,8\}$ under the Partial order whose hasse diagram is as shown below. Consider the subsets $B = \{1,2\}$ and $C = \{3,4,5\}$ of A.

Find (i) All the lower and upper bound of B and C.

(ii) glb (B), lub (B), glb (C) and lub (C).

(339)



Solution:

(i) upper bound of B = 3, 4, 5, 6, 7, 8

lower bound of B = ϕ (None)

upper bound of C = 6, 7, 8

lower bound of C = 3, 1, 2

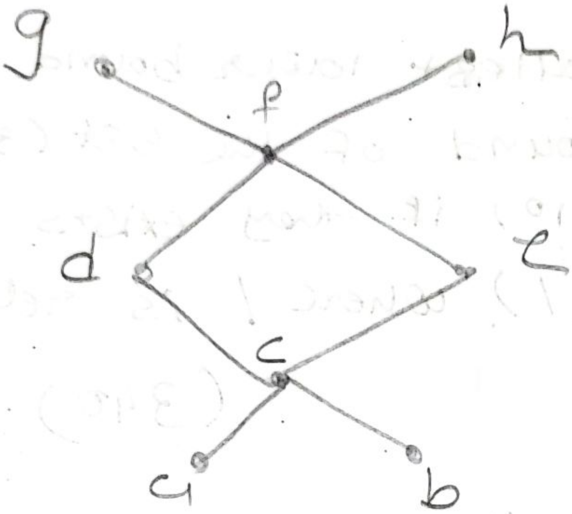
(ii) glb of B = None

lub of B = 3

glb of C = 3

lub of C = None

Ex: 2 let A = {a, b, c, d, e, f, g, h} be the Poset Whose Hasse diagram in shown in figure. Find GLB and LUB of B = {c, d, e} (335).



Solution:

upper Bound of B = f, g, h

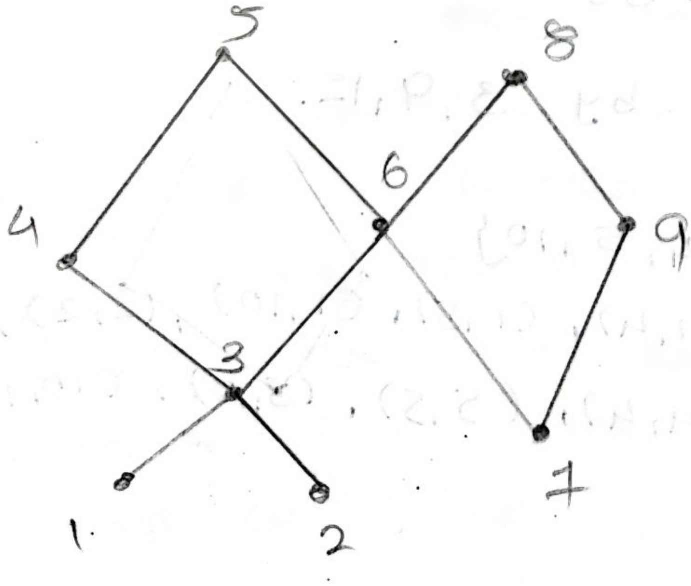
least upper bound of B = f.

lower Bound of B = c, a, b

Greatest lower bound of B = c

Ex: 3 Let A be Poset Whose Hasse diagram is Shown in figure A = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Find GLB, LUB of set B = {3, 4, 6}

(336)



Solution:

Lower bound of B = 3, 1, 2

glb of B = 3

upper bound of B = 5

lub of B = 5

Ex: 4 Find the greatest lower bound and least upper bound of the set (3, 9, 12) and (1, 2, 4, 5, 10) if they exists in the Poset (Z+, /) where / is relation of diversibility. (340)

Solution:

→ Set A = {3, 9, 12}

R = {(3, 3), (3, 9), (3, 12), (9, 9), (12, 12)}

Hasse diagram

GLB of {3, 9, 12} = 3

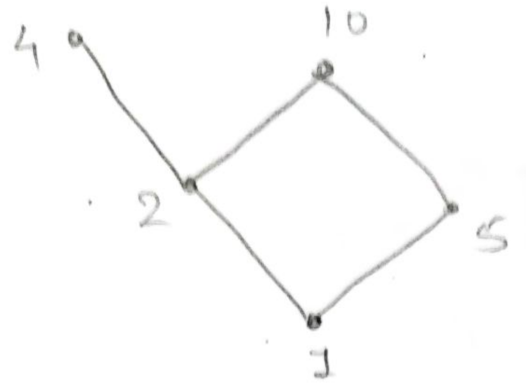
LUB of {3, 9, 12} = 36.

→ 36 is divisible by 3, 9, 12.

→ Set B = {1, 2, 4, 5, 10}

R = {(1, 1), (1, 2), (1, 4), (1, 5), (1, 10), (2, 2), (2, 4), (2, 10), (4, 4), (5, 5), (5, 10), (10, 10)}

Husse diagram



GLB of $\{1,2,4,5,10\} = 1$

LUB of $\{1,2,4,5,10\} = 20$

* Greatest Element, Least Element

→ An element $a \in A$ is called a greatest element of A if $x \leq a$ for all $x \in A$.

An element $a \in A$ is called a least element of A if $a \leq x$ for all $x \in A$.

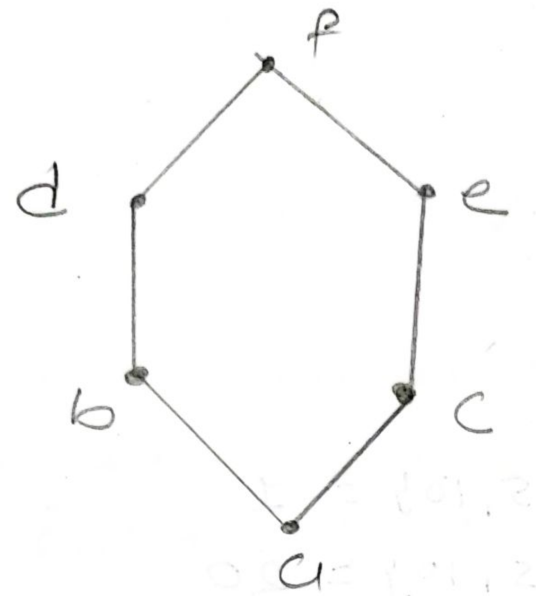
→ An element a of $(A, \leq)$ is a greatest (or least) element if and only if is a least (or greatest) element of $(A, \geq)$

A Poset has atmost one greatest element and atmost one least element.

The greatest elements of a poset if it exists, is denoted by I and is often called the unit element.

Similarly, the least element of a poset if it exists, is denoted by 'o' and is often called the zero element.

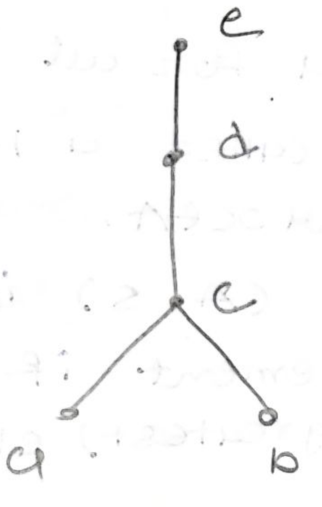
Ex: 1



Greatest element $I = f$

Least element $O = a$

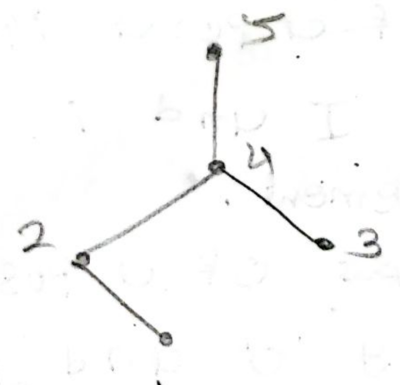
Ex: 2



Greatest element $I = e$

Least element $O =$ None

Ex: 3



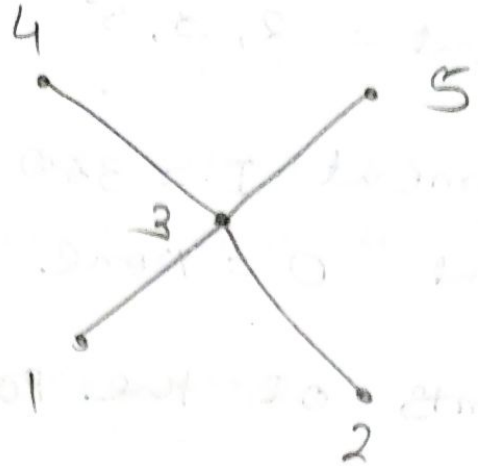
Greatest element

$I = 5$

Least element

$O =$ None

Ex: 4

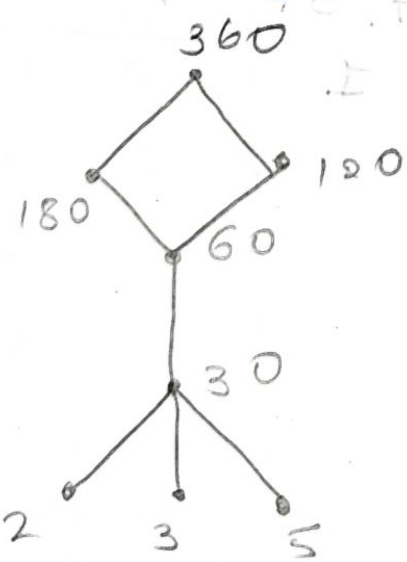


Greatest element I = None
least element O = None

Note: Greatest element is also called as universal upper bound, least element is also called as universal lower bound.

Ex: 1 Consider the divides relation on
$S = \{2, 3, 5, 30, 60, 120, 180, 360\}$.
Draw the Hasse diagram and find
(a) all minimal and maximal element.
(b) Greatest and least element. (338)

Solution: Hasse diagram.



(a) Maximal element = 360

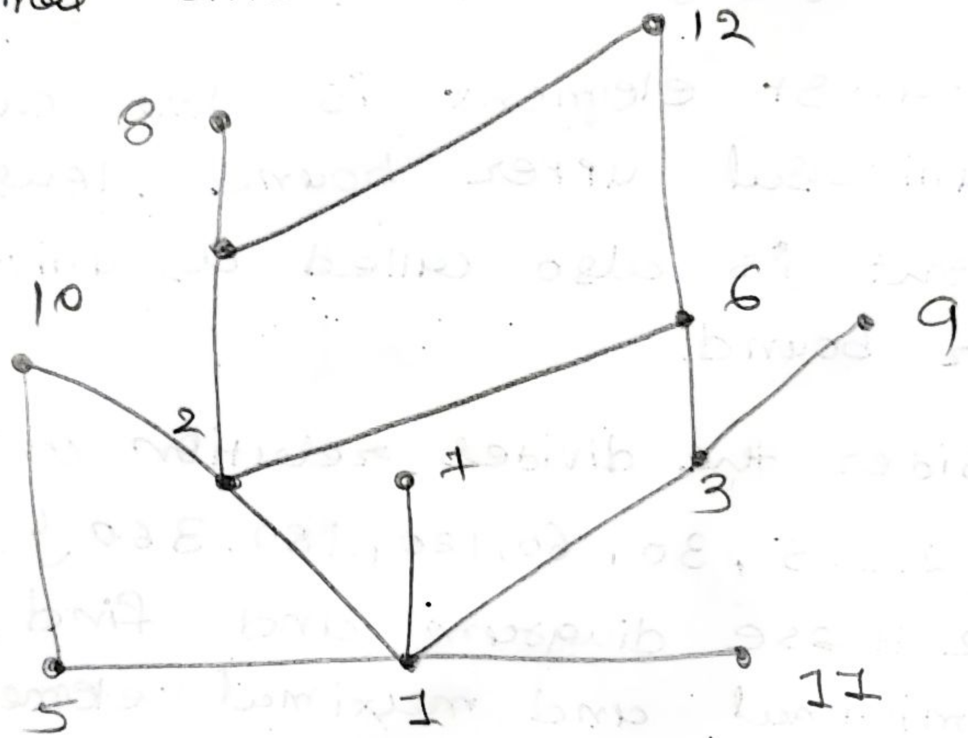
Minimal element = 2, 3, 5

(b) Greatest element I = 360

Least element O = None

Ex: 2 Which elements of the Poset

$\{\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\}$ are maximal and which are minimal? (337)



Solution:

Maximal element = 7, 8, 9, 10, 11, 12

Minimal element = 1

* Lattice

A lattice is a Poset $(A, \leq)$ in which every subset $\{a, b\}$ of A, has a least upper bound and a greatest lower bound.

* Lattice operators

lub	$a \vee b$	a join b
Glb	$a \wedge b$	a meet b

Ex:1 Which of the following diagram in the figure represents a lattice? Justify.

Solution: LUB:

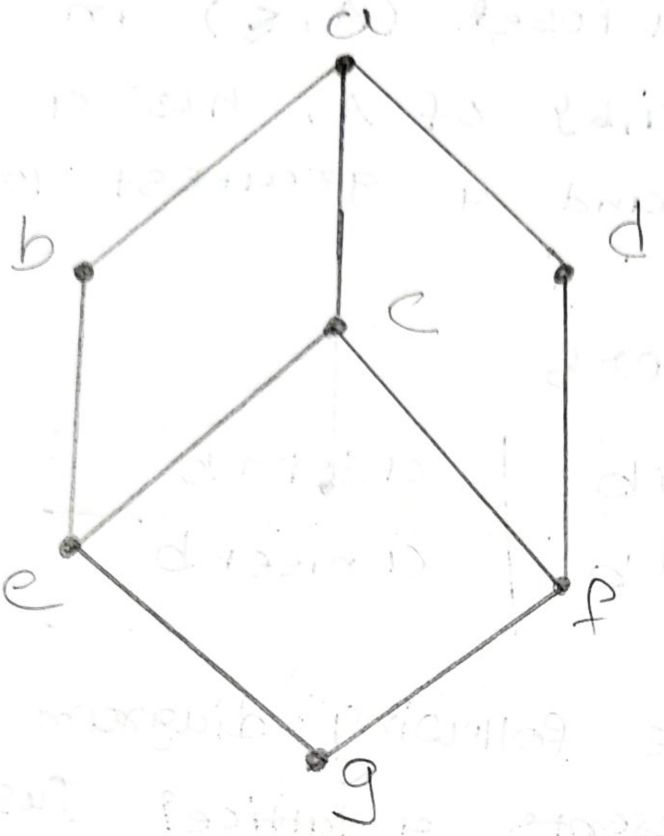
$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

GLB:

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

This is a lattice because every pair of elements has a least upper bound and a greatest lower bound.

Ex: 2 Which of the following diagram in the figure represents a lattice? Justify. (341)



Solution:

LUB:

∨	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	a	b	a	b
c	a	a	c	a	c	c	c
d	a	a	a	d	a	d	d
e	a	b	c	a	e	c	e
f	a	a	c	d	c	f	f
g	a	b	c	d	e	f	g

GLB:

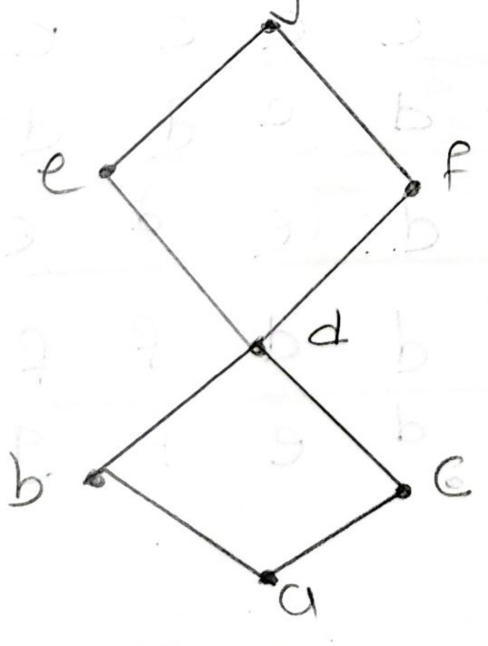
$\wedge$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	e	g	e	g	g
c	c	e	c	f	e	f	g
d	d	g	f	d	g	f	g
e	e	e	e	g	e	g	g
f	f	g	f	f	g	f	g
g	g	g	g	g	g	g	g

Each subset of two elements has a least upper bound and a greatest lower bound, so it is a lattice.

Ex:3 Which of the following diagram in the figure represents a lattice? Justify.

(342)

Solution.



Solution:

LUB:

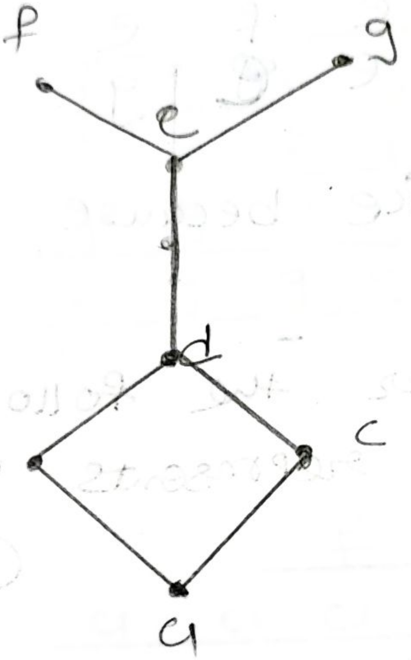
$\vee$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	g	g
f	f	f	f	f	g	f	g
g	g	g	g	g	g	g	g

GLB:

$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	d	e
f	a	b	c	d	d	f	f
g	a	b	c	d	e	f	g

Given Hasse diagram is a lattice because every Pair of element has a least upper bound and a greatest lower bound.

Ex:4 Which of the following diagram in the figure represent a lattice? Justify. (343)



Solution: LUB

$\vee$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	f	g
f	f	f	f	f	f	f	-
g	g	g	g	g	g	-	g

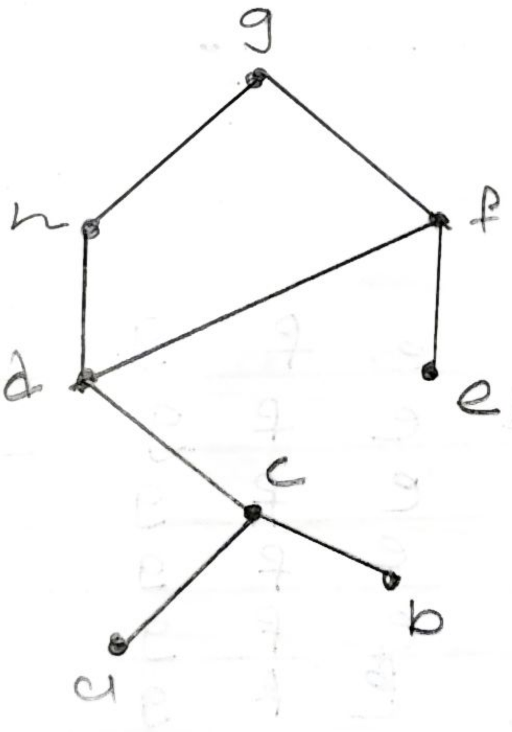
GLB:

$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	e	e
f	a	b	c	d	e	f	e
g	a	b	c	d	e	e	g

This is not a lattice because $f \vee g$ does not exist.

Ex: 5 Check whether the following Hasse diagrams represents a lattice?

(344)



Solution:

LUB:

∨	a	b	c	d	e	f	g	h
a	a	c	c	d	f	f	g	h
b	c	b	c	d	f	f	g	h
c	c	c	c	d	f	f	g	h
d	d	d	d	d	f	f	g	h
e	f	f	f	f	e	f	g	g
f	f	f	f	f	f	f	g	g
g	g	g	g	g	g	g	g	g
h	h	h	h	h	g	g	g	h

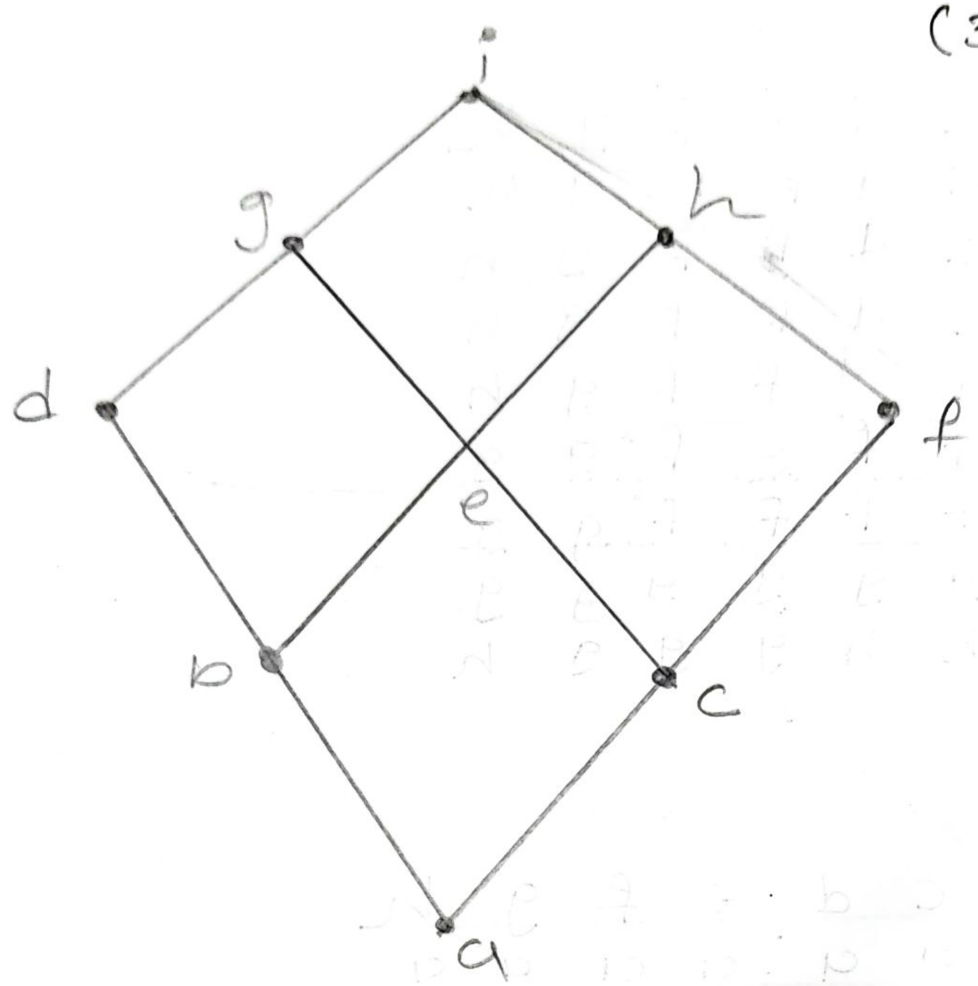
GLB:

∧	a	b	c	d	e	f	g	h
a	a	-	a	a	a	a	a	a
b	-	b	b	b	b	b	b	b
c	a	b	c	c	-	c	c	c
d	a	b	c	d	-	d	d	d
e	a	b	-	-	e	e	e	-
f	a	b	c	d	e	f	f	d
g	a	b	c	d	e	f	g	h
h	a	b	c	d	-	d	h	h

NO, it is not a lattice as pairs (a,b), (e,c), (e,d), (h,e) do not have GLB.

Ex: 6 Check whether the following Hasse diagrams represents a lattice?

(345)



Solution:

LUB:

$\vee$	a	b	c	d	e	f	g	h	i
a	a	b	c	d	e	f	g	h	i
b	b	b	e	d	e	h	g	h	i
c	c	e	c	g	e	f	g	h	i
d	d	d	g	d	g	i	g	i	i
e	e	e	e	g	e	h	g	h	i
f	f	h	f	i	h	f	i	h	i
g	g	g	g	g	g	i	g	i	i
h	h	h	h	i	h	h	i	h	i
i	i	i	i	i	i	i	i	i	i

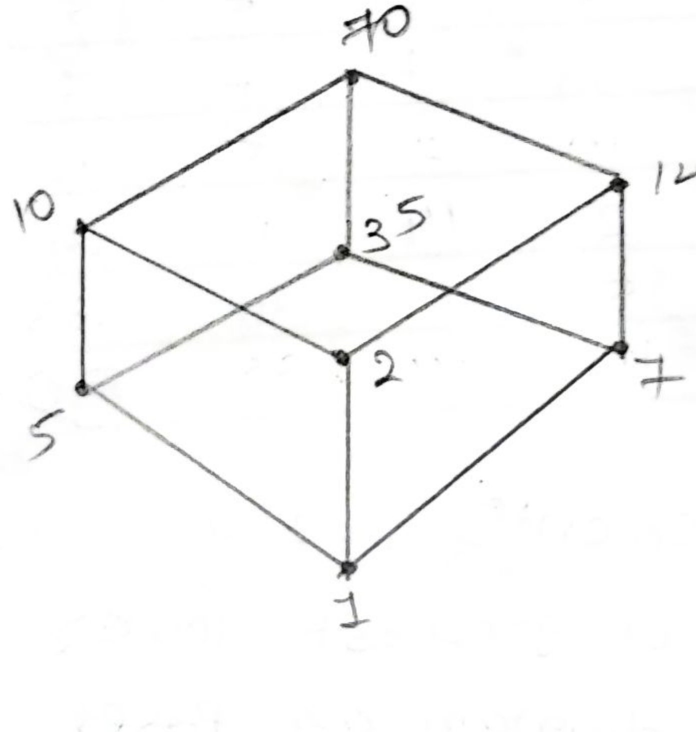
GLB:

$\wedge$	a	b	c	d	e	f	g	h	i
a	a	a	a	a	a	a	a	a	a
b	a	b	a	b	b	a	b	b	b
c	a	a	c	a	c	c	c	c	c
d	a	b	a	d	b	a	d	b	d
e	a	b	c	b	e	c	e	e	e
f	a	a	c	a	c	f	c	f	f
g	a	b	c	d	e	c	g	e	g
h	a	b	c	b	e	f	e	h	h
i	a	b	c	d	e	f	g	h	i

yes it is a lattice as every Pair of element has GLB and LUB.

Ex: 7 Show that the set of all divisiors of 70 forms a lattice. (346)

Solution:



LUB:

V	1	2	5	7	10	14	35	70
1	1	2	5	7	10	14	35	70
2	2	2	10	14	10	14	70	70
5	5	10	5	35	10	70	35	70
7	7	14	35	7	70	14	35	70
10	10	10	10	70	10	70	70	70
14	14	14	70	14	70	14	70	70
35	35	70	35	35	70	70	35	70
70	70	70	70	70	70	70	70	70

GLB:

$\wedge$	1	2	5	7	10	14	35	70
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
5	1	1	5	1	5	1	5	5
7	1	1	1	7	1	7	7	7
10	1	2	5	1	10	2	5	10
14	1	2	1	7	2	14	1	14
35	1	1	5	7	5	1	35	35
70	1	2	5	7	10	14	35	70

Every Pair of elements has a least upper bound and a greatest lower bound so this Husse diagram or Poset is lattice.

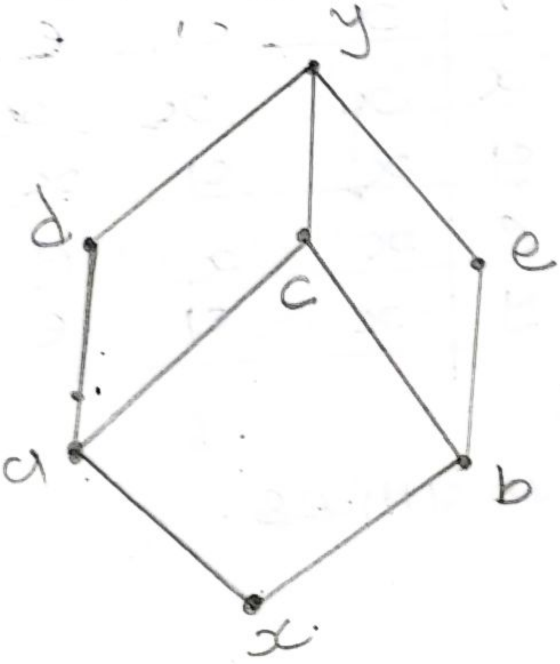
* Sublattice

Let $(A, \leq)$ be a lattice. A non-empty subset S of A is called a sublattice of A, if $a \vee b \in S$ and $a \wedge b \in S$ whenever $a \in S$ and $b \in S$.

Ex:1 Consider the lattice L in figure. Determine whether or not each of the following is a sublattice of L.

$L_1 = \{x, a, b, y\}$, $L_2 = \{x, a, e, y\}$

(347)



Solution: A subset is a sublattice if it is closed under $\wedge$ and $\vee$.

(i) $L_1 = \{x, a, b, y\}$

LUB:

$\vee$	x	a	b	y
x	x	a	b	y
a	a	a	$\boxed{c}$	y
b	b	$\boxed{c}$	b	y
y	y	y	y	y

GLB:

$\wedge$	x	a	b	y
x	x	x	x	x
a	x	a	x	a
b	x	x	b	x
y	x	a	b	y

Now L_1 is not a sublattice since $a \vee b = c$ which does not belong to L_1.

(ii) $L_2 = \{x, a, e, y\}$

LUB:

$\vee$	x	a	e	y
x	x	a	e	y
a	a	a	y	y
e	e	y	e	y
y	y	y	y	y

GLB:

$\wedge$	x	a	e	y
x	x	x	x	x
a	x	a	x	a
e	x	x	e	e
y	x	a	e	y

The set L_2 is sublattices.

* Types of Lattices

-> Bounded Lattice

A lattice L is said to be bounded if it has a greatest element I and least element O. If L is a bounded lattice, then for all $a \in A$.

$$O \le a \le I$$

$a \vee O = a$, $\quad a \wedge O = O$

$a \vee I = I$, $\quad a \wedge I = a$

Example:

1. The lattice Z^+ under the partial order of divisibility is not a bounded lattice since it has a least element, the number 1, but no greatest element.

2. The lattice Z under the partial order $\le$ is not bounded since it has neither a greatest nor a least element.

3. The lattice P(S) of all subsets of a set S, is bounded. It's greatest element is S and it's least element is ϕ.

→ Distributive lattice

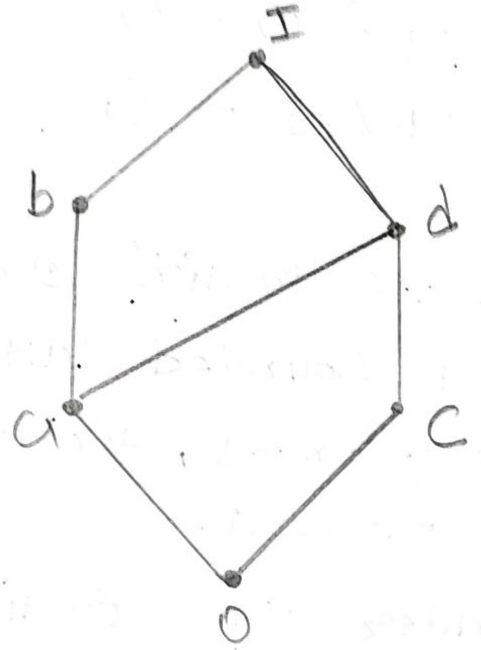
A lattice $(A, \leq)$ is called distributive lattice if for any elements $a, b, c \in A$.

i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Remark: If $(A, \leq)$ does not satisfy distributive property $(A, \leq)$ is known as non-distributive lattice.

Ex:



Given Hasse diagram is distributive or not.

Solution:

$a \wedge (d \vee c) = (a \wedge d) \vee (a \wedge c)$.

$d \vee c = d$ $\quad \therefore a \wedge (d \vee c) = a \wedge d = a$

$a \wedge d = a$ $\quad \because (a \wedge d) \vee (a \wedge c)$

$a \wedge d = a$ $\quad = a \vee 0$

$a \wedge c = 0$ $\quad = a$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$b \vee c = I$$

$$a \wedge I = a$$

$$a \wedge b = a.$$

$$a \wedge c = 0$$

$$a \wedge (b \vee c) = a \wedge I = a$$

$$(a \wedge b) \vee (a \wedge c) = a \vee 0 = a$$

The lattice shown in figure is distributive. as can be seen by verifying the distributive properties for all ordered triples chosen from the elements a, b, c and d.

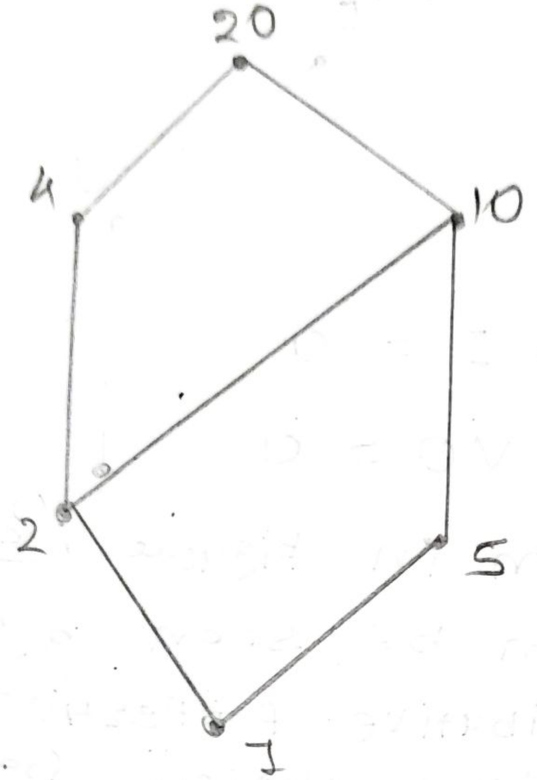
→ **Complemented lattice.**

Let L be a bounded lattice with greatest element I and least element 0, and let $a \in L$ An element $a' \in A$ is known as complement of $a \in A$.

A lattice L is called complemented if it is bounded and if every element in L has a complement.

Ex:- Find the complement of each element in D_{20}. (348)

Solution:- $D_{20} = \{1, 2, 4, 5, 10, 20\}$



→ In D_{20} universal upper bound or greatest element is 20 and universal lower bound or least element is 1.

$4 \vee 5 = 20$ $\qquad$ $4 \wedge 5 = 1$

∴ Complement of 4 is 5 and complement of 5 is 4

$2 \vee 10 = 10$ $\qquad$ $2 \wedge 10 = 2$

$2 \vee 5 = 10$ $\qquad$ $2 \wedge 5 = 1$

$2 \vee 4 = 4$ $\qquad$ $2 \wedge 4 = 2$

But the element 2 and 10 in D_{20} have no complements.

Similarly

$10 \vee 5 = 10$ $\qquad$ $10 \wedge 5 = 5$

$10 \vee 4 = 20$ $\qquad$ $10 \wedge 4 = 2$

$\therefore$ D_{20} is not complemented lattice.

* Boolean Algebra

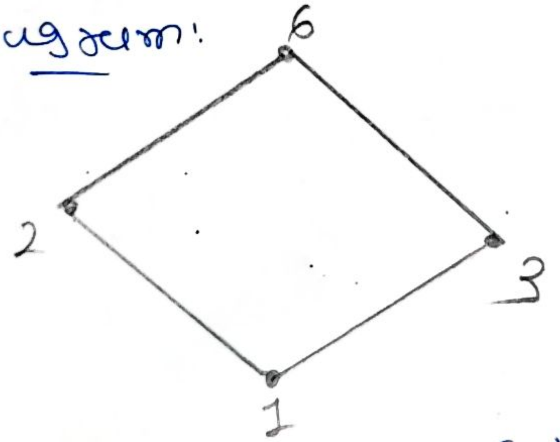
A boolean algebra is a lattice which contains

1. 2^n elements for any integer $n \geq 0$.
2. A greatest element and a least element.
3. Which is both complemented and distributive.

Ex:1, Determine whether the following posets is Boolean algebras. Justify your answer. $A = \{1, 2, 3, 6\}$ with divisibility. (352)

Solution: $A = \{1, 2, 3, 6\}$

Hasse diagram:



LUB

$\vee$	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

GLB:

$\wedge$	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

Every Pair of element in A has a GLB and a LUB.

Therefore A is lattice.

A has least element = 1

A has greatest element = 6

-> Complement

$2 \vee 3 = 6$ $\quad\quad$ $2 \wedge 3 = 1$

Complement of 2 is 3.

or Complement of 3 is 2.

∴ A is a complemented lattice.

Also we can show that the operations $\vee, \wedge$ are distributive

$$2 \vee (2 \wedge 3) = (2 \vee 2) \wedge (2 \vee 3)$$

$$2 \vee 1 = 2 \wedge 6$$

$$2 = 2$$

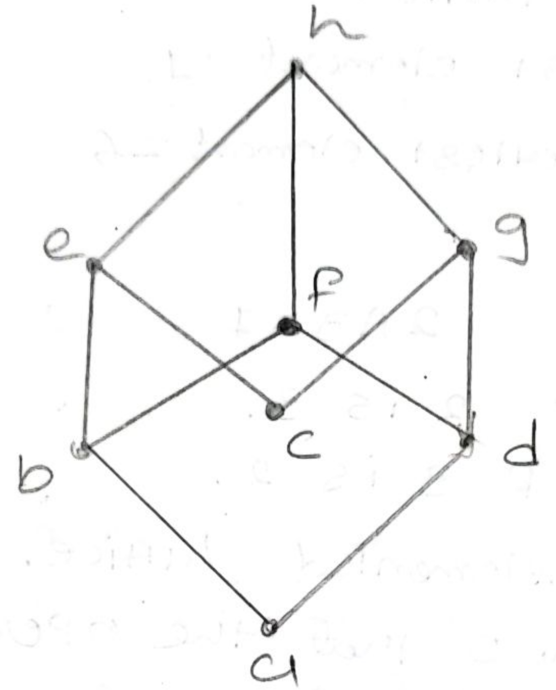
A is distributive lattice.

A satisfies all requirments of Boolean Algebra.

∴ A under divisibility is a Boolean Algebra.

Ex: 2 Determine whether the following Hasse diagram represent Booleam Algebra. (355)

Solution:



Solution:

GLB:

∧	a	b	c	d	e	f	g	h
a	a	a	–	a	a	a	a	a
b	a	b	–	a	b	b	a	b
c	–	–	c	–	c	–	c	c
d	a	a	–	d	a	d	d	d
e	a	b	c	a	e	b	a	e
f	a	b	–	d	b	f	d	f
g	a	a	c	d	a	d	g	g
h	a	b	c	d	e	f	g	h

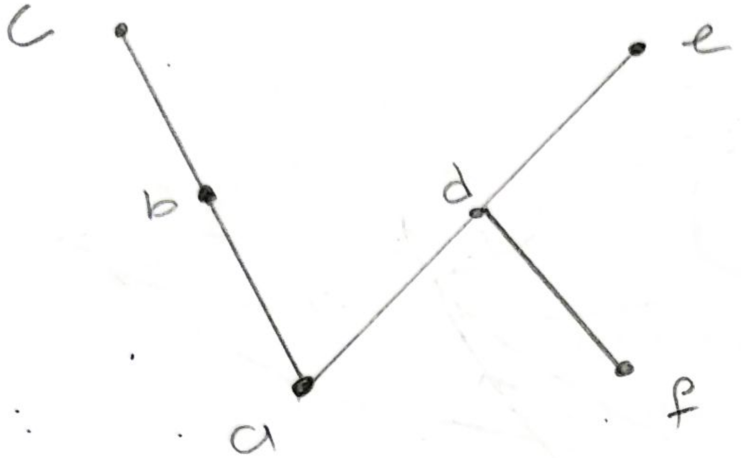
There is no GLB for Pairs (a, c), (b, c), (c, d), (c, f).

So the Hasse diagram does not represent a lattice.

∴ It does not represent a Booleam Algebra

Ex: 3 Determine whether the following Posets represents Boolean Algebra.

(356)



Solution:

LUB:

∨	a	b	c	d	e	f
a	a	b	c	d	e	d
b	b	b	b	-	-	-
c	c	c	c	-	-	-
d	d	-	-	d	e	d
e	e	-	-	e	e	e
f	d	-	-	d	e	f

There is no LUB for Pairs (b, d), (b, e), (b, f), (c, d), (c, e); (c, f).

So the Hasse diagram doesnot represent a lattice.

∴ It doesnot represent a Boolean algebra.