# Constant Elasticity of Substitution Production Functions[1]

1. The most common quantitative indices of production factor substitutability are forms of the elasticity of substitution (**E.S.**).[2] The defining formulae for these indices have the disadvantage of not allowing direct empirical evaluation. However, the assumption of *constant* **E.S.** leads to simple estimation methods, and has been widely used.[3]

The constant **E.S.** assumption is a restriction on the form of production possibilities, and one can characterize the class of production functions which have this property. This has been done by Arrow-Chenery-Minhas-Solow [3] for the two-factor production case.

2. For some empirical purposes, it would be desirable to have constant **E.S.** production functions with a larger number of factors. In this case, there is no "traditional" definition of the **E.S.**, but three forms have been suggested in the literature:[4]

*a.* the *Allen partial elasticity of substitution*,

*b.* the *Direct partial elasticity of substitution* (**D.E.S.**), defined by applying the two-factor **E.S.** formula to each pair of factors, holding fixed the remaining factor input levels, and

*c.* the *Shadow partial elasticity of substitution* (**S.E.S.**), defined by applying the two-factor **E.S.** formula to each pair of factors, holding fixed the *imputed prices* of the remaining factors and the imputed total cost.

Uzawa [9] has reformulated the definition of the Allen **E.S.**, and has characterized its class of constant **E.S.** production functions. This note considers the remaining two definitions. A detailed treatment of the **D.E.S.** case is given in Sections 3-6. The results for the **S.E.S.** case are similar, and are outlined in Section 7.

3. We assume the production process can be represented by a production function $y = f(x) = f(x_1, \ldots, x_n)$ specifying the output $y$ obtained by employing the $n$ possible inputs, numbered $1, \ldots, n$, in quantities $x_1, \ldots, x_n$.

In our analysis, we shall use the following conditions on $f$:

[1] The author is indebted to Hirofumi Uzawa, upon whose comments this note is based; and also to Leonid Hurwicz, Marc Nerlove, and Donald Katzner. All errors are the responsibility of the author This note was written at Stanford University and the University of Minnesota.

[2] For two factors of production, the **E.S.** is defined along an equal-product curve as the elasticity of the factor input ratio with respect to the marginal rate of substitution. See [1], pp. 340-3.

[3] The references in [3], and [7] include many of the empirical studies of the **E.S.** which make this assumption.

[4] The Allen **E.S.** and the **D.E.S.** were introduced in [2], pp. 202-6, 211-14, in the terminology "elasticity of complimentarily" and "elasticity of substitution between $Y$ and $Z$ in the $YZ$ indifference direction," respectively. The Allen **E.S.** is developed further in [1], p. 503. The **D.E.S.** has been used in [7], pp. 42, 49-52, and [6], pp. 77-82.

(**C1**) $f$ is a continuous function from the set of all non-negative input bundles onto the set of non-negative output levels, with $f(0) = 0$, and has continuous second order partial derivatives for positive input bundles.

(**C2**) The first order partial derivatives of $f$ are positive (and finite) for all positive input bundles.

(**W**) $f$ is quasi-concave.[1]

(**S**) $f$ is strictly quasi-concave.

(**L**) $f$ is linear homogeneous.

We term a production function *weakly classical* [Notation: (**CW**)] if it satisfies conditions (**C1**), (**C2**), and (**W**); *classical* [Notation: (**CS**)] if it satisfies conditions (**C1**), (**C2**), and (**S**); and *classical linear* [Notation: (**CSL**)] if it satisfies conditions (**C1**), (**C2**), (**S**) and (**L**).

For a weakly classical production function $f$, define the **D.E.S.** between two factors, say $i$ and $j$, as[2,3]

$$(1) \qquad \sigma_{ij}(x) = \frac{\dfrac{1}{x_i f_i(x)} + \dfrac{1}{x_j f_j(x)}}{-\dfrac{f_{ii}(x)}{f_i(x)^2} + 2\dfrac{f_{ij}(x)}{f_i(x)f_j(x)} - \dfrac{f_{jj}(x)}{f_j(x)^2}} \qquad (i \neq j).$$

$\sigma_{ij}(x)$ is a real-valued function from the set of positive input bundles into the extended interval $0 < \sigma_{ij} \leqq +\infty$. Furthermore, if $f$ satisfies (**CS**), then $\sigma_{ij}$ must be finite for some input bundle arbitrarily close to any given input bundle.[4]

$\sigma_{ij}$ is the elasticity of the input ratio of factors $i$ and $j$ with respect to the marginal rate of substitution of these factors, taken along a fixed isoproduct curve, with all remaining factor quantities fixed.

4. We now characterize the class of production functions that have all **D.E.S.** constant for all positive input bundles. First, we define a property (**K**) of the production function and the **D.E.S.**, and show that this property is necessary and sufficient for a production function to have all **D.E.S.** constant.

DEFINITION: *A production function $f$ has property* (**K**) *if*

(*i*) *there exists a partition $\{N_1, \ldots, N_S\}$ of the set $\{1, \ldots, n\}$ of factors*[5] *such that*

$$\sigma_{ij}(x) \equiv \begin{cases} 1 & \text{for } i, j \in N_s, \quad i \neq j, \\ \sigma & \text{for } i \in N_r, j \in N_s, \quad r \neq s, \end{cases}$$

*where $\sigma$ is a positive finite constant, and*

(*ii*) *$i$ and $j$ in the same set $N_s$ imply $x_i f_i(x) = x_j f_j(x)$ for all positive bundles $x$.*

The property (**K**) states that the factors can be divided into classes such that the **D.E.S.** in *one* between *any* pair of factors *within* a class, and is some *single value* $\sigma$ between any two factors drawn from *any* two *different* classes. Furthermore, the imputed distributive shares of two factors in the same class are identical (in competitive factor markets.) We have:

[1] A function $y = f(x)$ is (strictly) quasi-concave if the contour set $C(y) = \{x : f(x) \geqslant y\}$ is (strictly) convex for each $y$.

[2] We write $f_i(x)$ and $f_{ij}(x)$ for $\dfrac{\partial f}{\partial x_i}$ and $\dfrac{\partial^2 f}{\partial x_i \, \partial x_j}$, evaluated at $x$.

[3] A more conventional, but somewhat ambiguous, definition of the **D.E.S.** is $\sigma_{ij}(x) = \partial ln(x_i/x_j)/\partial ln(-dx_i/dx_j)$, taken with $y, x_k =$ const, $(k \neq i, j)$. Compare with Morrisset [7], p. 43. Under conditions (**CW**), form (1) is directly derivable from this expression (see Meade [6], p. 77.)

[4] A necessary condition for a function with continuous second partials to be strictly quasi-concave is that the associated Hessian matrix be negative definite under constraint on an open dense set in $x$. The denominator of (1) is the negative of a quadratic form of the Hessian satisfying the constraint, and hence is positive on such a set. (See Bernstein [4]).

[5] $N_1 \cup \ldots \cup N_S = \{1, \ldots, n\}$ and $N_r \cap N_s = \phi$ for $r \neq s$.

THEOREM 1:[1] *Suppose a production function $f$ is classical* (i.e., satisfies **(CS)**). *Then, $f$ can have all* **D.E.S.** *constant for all positive input bundles if and only if property* **(K)** *holds.*

We call a production function that satisfies **(K)** a *constant* **D.E.S.** production function.

5. It is possible to characterize classical constant **D.E.S.** production functions by their isoproduct curves. Initially, we define a form **(BAL)** of isoproduct curves, and show that a necessary and sufficient condition for a production function to be classical linear constant **D.E.S.** is that its isoproduct curves have the **(BAL)** form.

DEFINITION: *A production function $f$ is of the block additive linear homogeneous* **(BAL)** *form if it has the characterization:*

THERE EXIST

(i) *a partition $\{N_1, \ldots, N_S\}$ of the set $\{1, \ldots, n\}$ of factors*[2]

(ii) *a constant* $\rho \geqq -1/m$ (with the inequality strict if there is more than one set $N_s$ of size $m$), *and*

(iii) *a set of positive constants $\alpha_0, \alpha_1, \ldots, \alpha_s$ with $\alpha_1 + \ldots + \alpha_s = 1$*

SUCH THAT

*the isoproduct surface of output level $y$ is given by the set of positive input bundles $x$ satisfying*

$$(2) \qquad 1 = \alpha_0 \sum_{s=1}^{S} \alpha_s \left[ \prod_{k \in N_s} \left( \frac{x_k}{y} \right) \right]^{-\rho} \quad \textit{for } \rho \neq 0 \textit{ or,}$$

$$(3) \qquad y = \alpha_0 \prod_{s=1}^{S} [\prod_{k \in N_s} x_k]^{\alpha_s/m_s} \quad \textit{for } \rho = 0.$$

The form (3) is Cobb-Douglas. In several cases, form (2) can be written more conventionally: If each set $N_s$ contains the *same* number of elements $m$, then (2) reduces to

$$(4) \qquad y = a \, [\sum_{s=1}^{S} \alpha_s \prod_{k \in N_s} (x_k)^{-\rho}]^{-1/\rho m} \quad \left( \rho < \frac{-1}{m} \right),$$

where $a$ is a positive constant. When $m = 1$, this is the Arrow-Chenery-Minhas-Solow production function [3].

In the case of three factors and unequal class sizes, form (2) becomes

$$y = a \frac{x_1 x_2}{x_3} \{ [1 + b \, (x_3^2/x_1 x_2)^{\rho}]^{\frac{1}{2}} - 1 \}^{1/\rho} \qquad (\rho \geqq -\tfrac{1}{2}),$$

where factors 1 and 2 are in the same class, and $a$ and $b$ are positive constants.

We can now establish

THEOREM 2: *A production function is classical constant* **D.E.S.** *and linear homogeneous* (i.e., has properties **(CSL)** and **(K)**) *if and only if it is of the block additive linear homogeneous* **(BAL)** *form. Furthermore, the partitions of the factors introduced in* **(K)** *and* **(BAL)** *are identical, and* $\sigma = \dfrac{1}{1 + \rho}$.

We note in particular that $\sigma$ must lie in the range $0 < \sigma \leqq \dfrac{m}{m - 1}$, with the second inequality strict when there is more than one partition set of size $m$.

[1] The proofs of Theorems 1 and 2 are given in Appendix B.

[2] *Notation*: Let $m_s$ denote the number of elements in the set $N_s$, and let $m$ denote the maximum of the values $\{m_1, \ldots, m_S\}$.

6. It is useful to establish the forms of classical constant **D.E.S.** production functions without the homogeneity assumption **(L)**. This is done in Appendix A, Theorem 3. Another generalization of the result can be made by weakening the **(CS)** condition to the **(CW)** condition in Theorems 1, 2, and 3, and modifying the conditions on $\sigma$ and $\rho$ to allow $0 < \sigma \leqq \infty$ in **(K)** and $\rho \geqq -\frac{1}{m}$ without exception in **(BAL)** and **(BA)**. The proofs of Appendix B hold with the corresponding modifications.

7. Consider the Shadow elasticity of substitution. The **S.E.S.** can be defined in terms of the cost function $w = \lambda(y, p)$ of the producer, which specifies the minimum imputed cost $w$ of producing the output $y$ with the accounting price vector $p = (p_1, \ldots, p_n)$.[1] If the cost function is classical,[2] then the **S.E.S.** between a factor pair $i$ and $j$ has the form

$$\sigma_{ij}^*(y, p) = \frac{-\dfrac{\lambda_{ii}}{\lambda_i^2} + 2\,\dfrac{\lambda_{ij}}{\lambda_i\,\lambda_j} - \dfrac{\lambda_{jj}}{\lambda_j^2}}{\dfrac{1}{p_i\quad \lambda_i} + \dfrac{1}{p_j\quad \lambda_j}} \tag{5}$$

where $\lambda_i = \partial\lambda/\partial p_i$ and $\lambda_{ij} = \partial^2\lambda/\partial p_i \partial p_j$ are evaluated at $(y,p)$.

Using the obvious duality of (1) and (5), one can show that a classical cost function can have all **S.E.S.** constant if and only if it satisfies the property **(K)** of Section 4.[3] Further, the cost function is of the **(BAL)** form when $\lambda$ is linear in $y$.[4] The corresponding production function is determined by the condition:

*The set of positive input bundles $x$ in the equal-product surface of level $y$ satisfy*[5]

$$y = \beta_0 \prod_{s=1}^{S} \prod_{k \in N_s} x_k^{\alpha_s/m_s} \qquad \textit{for } \rho = 0, \textit{ or} \tag{6}$$

$$\left\{\begin{aligned} 1 &= \sum_{s=1}^{S} \beta_s \prod_{k\in N_s} (Qx_k/y)^{\rho/(\rho+1)} \\ Q &= \sum_{s=1}^{S} m_s\,\beta_s \prod_{k\in N_s} (Qx_k/y)^{\rho/(\rho+1)} \end{aligned}\right. \qquad \textit{for } \rho \neq 0, \tag{7}$$

*where the equations* (7) *are solved to eliminate the variable $Q$, and where $\beta_o = (1/\alpha_o) \prod_{s=1}^{S} (\alpha_s/m_s)^{-\alpha_s}$ and $\beta_s = (\alpha_0\alpha_s)^{1/(\rho+1)}$.*

[1] A function of this type has been used by Uzawa [9], who describes some of its properties. In particular, $\partial\lambda/\partial p_i$ gives the $i$th component of the minimum cost input bundle.

[2] The cost function $\lambda$ is classical if it has a continuous second differential, positive first partials in $p$ and is positive linear homogeneous and strictly quasi-concave in $p$ for fixed $y$. A detailed discussion of cost functions, as well as derivation of the **S.E.S.**, are in [5], pp. 41-8, 106-31.

[3] Technically, the cost function must satisfy the analogue of **(K)** determined by replacing "$f$" by "$\lambda$", "$\sigma_{ij}(x)$" by "$\sigma_{ij}^*(y,p)$", "$x_i f_i(x)$" by "$p_i\lambda_i(y,p)$", and "$x$" by "$p$" in the definition of **(K)**. The economic implications of the two conditions are the same.

[4] Replacing "$y$" by "$w/y$" and "$x$" by "$p$" in the **(BAL)** definition gives this class.

[5] By the Shephard duality theorem, the production function is uniquely determined by the cost function, and has classical properties. See [8], pp. 17-22, and [5], pp. 80, 106-31.

If $m_s = m$ for all $s$, then (7) reduces to (4). Computation shows that the " between-group " **S.E.S.** value for classical constant **S.E.S.** production functions is $\sigma^* = 1 + \rho \geqq 1 - 1/m$, with the inequality strict if there is more than one group of size $m$.

8. In the case of two factors of production, the assumption of a constant elasticity of substitution led to the Arrow-Solow production function, which has proved a fruitful tool in empirical analyses of production. However, generalization of this class to $n$ factors requires highly restrictive conditions on the elasticity values [Condition **(K)** in this note for the **D.E.S.** and **S.E.S.** cases, and equation (22) in [9] for the Allen **E.S.** case]. For many empirical purposes, these restrictions make the constant **E.S.** assumption untenable, even as a rough working hypothesis.[1] The need remains for an empirically tractable measure of substitutability under which a more general class of production functions is admissible.

*Pittsburgh.* DANIEL MCFADDEN.

## APPENDIX A: GENERAL BLOCK ADDITIVE PRODUCTION FUNCTIONS

Section 5 of the note established the form of classical linear constant **D.E.S.** production functions. We now characterize classical constant **D.E.S.** functions without the restriction of linear homogeneity.

DEFINITION: *A production function is of the block additive* **(BA)** *form if it has the characterization*:

THERE EXIST

(*a*) *parameters* $\{N_1, \ldots, N_S\}$, $\rho$, *and* $\alpha_0, \alpha_1, \ldots, \alpha_S$ *satisfying conditions* (i), (*ii*) *and* (iii) *of the* **(BAL)** *form, and*

(*b*) *a set of functions* $\eta_0(y), \eta_1(y), \ldots, \eta_S(y)$ *of the output level y with the property* **(Q)** *below*

SUCH THAT

*the isoproduct surface of the output level y is given by the set of positive input bundles x satisfying*

$$1 = \sum_{s=1}^{S} \left[ \prod_{k \in N_s} \left( \frac{x_k}{\eta_s(y)} \right) \right]^{-\rho} \qquad \text{for } \rho \neq 0, \text{ or} \tag{8}$$

$$\eta_0(y) = \prod_{s=1}^{S} \prod_{k \in N_s} x_k^{\alpha_s / m_s} \qquad \text{for } \rho = 0. \tag{9}$$

PROPERTY **(Q)**:

$(Q1)$ $\eta_t(y)$ *is a non-negative function of non-negative y with a continuous second derivative, and is positive for y positive, for* $t = 0, 1, \ldots, S$.

$(Q2)$ $\eta_t(y)$ *is monotone non-decreasing for* $t = 0, 1, \ldots, S$ *and has a positive (finite) first derivative for* $t = 0$ *and at least one* $t > 0$, *for each y*.

$(Q3)$ $\lim_{y \to 0} \eta_t(y) = 0$ *for* $t = 0$ *and at least one* $t > 0$. If $\rho > 0$, *this condition holds for all t*.

$(Q4)$ $\lim_{y \to \infty} \eta_t(y) = \infty$ for $t = 0$ *and at least one* $t > 0$. *If* $\rho < 0$, *this condition holds for all t*.

[1] Despite their apparent complexity, both the constant **D.E.S.** and **S.E.S.** production functions allow relatively simple parameter estimation. In the constant **D.E.S.** case, a system of log linear equations of the form (22), with the left-hand-side interpreted as an " observed " relative share, can be used to obtain linear estimators for $\alpha_1, \ldots, \alpha_s$ and $\rho$, provided that the partition of the factors is known. A similar formula holds for the S.E.S. case. The Allen **E.S.** does not yield a linear form for relative shares.

Noting that Lemmas 2.1 and 2.2 in Appendix B do not require homogeneity, one can use equations (23) and (24) to establish

THEOREM 3: *A production function f is classical constant* **D.E.S.** *if and only if it is of the block additive* **(BA)** *form.*

## APPENDIX B: THEOREM PROOFS

I. Proof of Theorem 1:

1. **(K)** is sufficient by definition. It will be proved necessary by obtaining a general solution of the system of partial differential equations determined by the **D.E.S.** definition (1), under the assumptions that the production function $f$ satisfies **(CS)**, and that all the **D.E.S.** are constant. Condition **(K)** is derived from this solution.

2. Since **(CS)** is satisfied, it follows from (1) that the constant **D.E.S.** values must be positive and finite. Denote these constants by $1/\omega_{ij} = \sigma_{ij}(x)$ for each factor pair $i$ and $j$. Equation (1) becomes

$$(10) \qquad \frac{\partial}{\partial x_i}\,[\ln g^{ij}(x)] + \omega_{ij}/x_i = g^{ij}(x)\left\{\frac{\partial}{\partial x_j}\,[\ln g^{ij}(x)] - \frac{\omega_{ij}}{x_j}\right\} \qquad (i \neq j),$$

where $g^{ij}(x) = f_i(x)/f_j(x)$. The first part of the proof is completed by showing

LEMMA 1.1: *For f satisfying* **(CS)**, *all solutions of the system* (10) *have the functional form*

$$(11) \qquad g^{ij}(x) = f_i(x)/f_j(x) = \frac{\theta^i(y)}{\theta^j(y)} \prod_{k=1}^{n} x_k^{\,\omega_{ki}-\omega_{kj}},$$

*where we define* $\omega_{kk} = 0$, *and where the* $\theta^k(y)$ *are arbitrary positive functions of* $y = f(x) > 0$ *with continuous first derivatives, for* $k = 1, \ldots, n$. *Furthermore, all functions of the form* (11) *are solutions of* (10).

Proof of Lemma 1.1: One can verify by substitution that (11) is a solution of (10) for each factor pair. To show (11) necessary, consider a general solution $G^{ij}(x) = f_i(x)/f_j(x)$ of (10), and define

$$(12) \qquad \varphi^{ij}(x) = G^{ij}(x)\left\{\prod_{k=1}^{n} x_k^{\,\omega_{kj}-\omega_{ki}}\right\}.$$

We show that $\varphi^{ij}(x)$ can be written as a function of $y = f(x)$ alone. Let $T_i$ be the transformation

$$(13) \qquad \begin{cases} v_k = v_k(x) = x_k & \text{for } k \neq i, \text{ and} \\ y = y(x) = f(x). \end{cases}$$

The Jacobian of $T_i$ is $f_i(x) > 0$ by **(C2)**, and the transformation is one-to-one. Hence, for each factor pair $i$ and $j$, define

$$(14) \qquad \theta^{ij}\,(y(x),\ v(x)) = \varphi^{ij}(x),$$

where $v(x)$ is the vector or elements $v_k(x)$, $k \neq i$.

Substitute (14) into (12), solve for $G^{ij}(x)$, and substitute this form in (10). We have for $i \neq j$

$$\begin{aligned}(15)\qquad \frac{\partial \theta^{ij}}{\partial v_j} &= \frac{\partial \varphi^{ij}}{\partial x_j} - f_i \frac{\partial \theta^{ij}}{\partial y} && \text{by (14),}\\ &= \frac{\partial \varphi^{ij}}{\partial x_j} - (f_j/f_i) f_i \frac{\partial \theta^{ij}}{\partial y},\\ &= \frac{\partial \varphi^{ij}}{\partial x_j} - (f_j/f_i) \frac{\partial \varphi^{ij}}{\partial x_i} && \text{by (14),}\\ &= 0 && \text{from (10).}\end{aligned}$$

Then also $\partial\theta^{ij}/\partial v_k = \theta^{ik}\,\partial\theta^{kj}/\partial v_k + \theta^{kj}\,\partial\theta^{ik}/\partial v_k = 0$ by (15) since $\theta^{jk}$ is independent of $v_k$ for $k \neq i, j$. Hence $\theta^{ij}$ is a function of $y$ alone.

The proof of the lemma is completed by defining $\theta^1(y) = 1$ and $\theta^j(y) = \theta^{j1}(y)$ for $j = 2, \ldots n$. Then, $\theta^{ij}(y) = \theta^{i1}(y)\,\theta^{1j}(y) = \theta^i(y)/\theta^j(y)$. Solution of (12) for $G^{ij}(x)$ then gives the form (11). Q.E.D. lemma 1.1.

Differentiating (11) with respect to $x_k$ ($k \neq i, j$) gives

$$\frac{f_{ik}}{f_i f_k} - \frac{f_{jk}}{f_j f_k} = \frac{\omega_{ki} - \omega_{kj}}{x_k f_k(x)} + \frac{\partial}{\partial y} \ln \theta^i - \frac{\partial}{\partial y} \ln \theta^j.$$

Permuting the indices $i \longrightarrow j \longrightarrow k$ in this expression and summing gives the identity in $x$

$$(16)\qquad \frac{\omega_{ij} - \omega_{ik}}{x_i f_i(x)} + \frac{\omega_{jk} - \omega_{ji}}{x_j f_j(x)} + \frac{\omega_{ki} - \omega_{kj}}{x_k f_k(x)} = 0 \qquad (i, j, k \neq \quad)$$

The proof of the theorem follows by establishing the following lemma, recalling that $\sigma_{ij} = 1/\omega_{ij}$:

LEMMA 1.2: *If a production function f satisfies* (CS), *and equations* (11) *and* (16), *then there exists a partition* $\{N_1, \ldots, N_S\}$ *of* $\{1, \ldots, n\}$ *and a positive constant* $\omega$ such that

$$(17)\qquad \begin{cases}\omega_{ij} = 1 \quad \text{and } x_i f_i \equiv x_j f_j & \text{for } i, j \in N_s,\ i \neq j,\\ \omega_{ij} = \omega \quad \text{and } x_i f_i \not\equiv x_j f_j & \text{for } i \in N_r,\ j \in N_s,\ r \neq s.\end{cases}$$

proof of Lemma 1.2: The identity relation "$\equiv$" partitions the imputed factor shares $x_k f_k$ — and hence the factors $\{1, \ldots, n\}$ — into equivalence classes $\{N_1, \ldots, N_S\}$ such that $x_i f_i(x) \equiv x_j f_j(x)$ if and only if $i, j \in N_s$. Using the following two propositions, one can verify that the $\omega_{ij}$ are one for $i$ and $j$ from the same class, and are equal for all $i$ and $j$ from different classes, proving the lemma.

(18) If the imputed relative share $x_i f_i(x)/x_j f_j(x)$ is constant for $x$ in an open sphere, then $\omega_{ij} = 1$ and $\omega_{ki} = \omega_{kj}$ for $k \neq i, j$. If, in addition, $x_i f_i(x) \neq x_j f_j(x)$, then $\omega_{ij} = \omega_{ik} = \omega_{jk} = 1$ for $k \neq i, j$.

(19) If none of the imputed relative shares $x_i f_i(x)/x_j f_j(x)$, $x_j f_j(x)/x_k f_k(x)$, $x_k f_k(x)/x_i f_i(x)$ are constant, then $\omega_{ij} = \omega_{jk} = \omega_{ki}$.

The proof is completed by establishing (18) and (19). To show (18), write (11) in the form

$$(20)\qquad \frac{x_i f_i(x)}{x_j f_j(x)} = \frac{\theta^i(y)}{\theta^j(y)} \left(\frac{x_i}{x_j}\right)^{1-\omega_{ij}} \prod_{k \neq i, j} x_k^{\omega_{ki} - \omega_{kj}}$$

When the left-hand-side of (20) is constant for $x$ in a sphere R, the supposition that $\theta^i(y)/\theta^j(y)$ is not constant for $y$ in the open set $\{y = f(x): x \in \mathrm{R}\}$ can be shown, by differentiation with respect to $x_i$ and $x_j$, to violate condition **(C2)** on $f$. Hence, this term is a constant for $x$ in R, and the resulting identity in $x$ holds only if $\omega_{ij} = 1$ and $\omega_{ik} = \omega_{jk}$ for $k \neq i, j$. Substituting this result in (16) gives $(1 - \omega_{ki})(1 - x_i f_i / x_j f_j) = 0$, and (18) follows.

(19) is proved by contradiction. Suppose for some triplet $(i, j, k)$ we have $\omega_{ij} \neq \omega_{ik}$. To avoid a contradiction of (16) and the hypothesis, we must have $\omega_{ij} \neq \omega_{jk} \neq \omega_{ik}$. Differentiating (16) with respect to $x_i$ and $x_j$, after substituting (20) for the imputed relative shares, yields

(21) $$A\, x_i f_i = (\omega_{ik} - 1)\, q^1 + (\omega_{ik} - \omega_{ij})\, q^2, \text{ and}$$
$$A\, x_j f_j = (\omega_{jk} - \omega_{ji})\, q^1 + (\omega_{kj} - 1)\, q^2,$$

where $q^1 = q^1(y,x) = (\omega_{ij} - \omega_{ik})\, x_k f_k / x_i f_i$;

$q^2 = q^2(y,x) = (\omega_{jk} - \omega_{ji})\, x_k f_k / x_j f_j$; and $A = \dfrac{-\partial}{\partial y}(q^1 + q^2)$.

If $A \equiv 0$, then, from (21), $q^3 = q^1/q^2 = \dfrac{\omega_{ij} - \omega_{ik}}{\omega_{jk} - \omega_{ji}} \dfrac{x_j f_j(x)}{x_i f_i(x)}$

is constant, contradicting the hypothesis. If $A \neq 0$, then it is non-zero for $x$ in an open sphere R. Dividing the equations (21) gives

$$q^3 \frac{\omega_{jk} - \omega_{ji}}{\omega_{ij} - \omega_{ik}} = \frac{(\omega_{jk} - \omega_{ji})\, q^3 + (\omega_{kj} - 1)}{(\omega_{ik} - 1)\, q^3 + (\omega_{ik} - \omega_{ij})}$$

for $x$ in R. But this cannot be true as $q^3$ varies, giving a contradiction. Hence, (19) follows.

Q.E.D. Lemma 1.2 and Theorem 1.

II. Proof of Theorem 2:

1. To show the **(BAL)** form necessary, we first prove a lemma giving a general solution of the partial differential equation system (11). Under **(K)** and **(CS)**, this system is

(22) $$\frac{x_i f_i}{x_j f_j} = \frac{\theta_s(y)}{\theta_t(y)} \prod_{k \in N_s} x_k^{1-\omega} \prod_{k \in N_t} x_k^{\omega - 1}, \quad i \in N_s, \quad j \in N_t,$$

where the $\theta_r(y)$ are arbitrary positive functions with continuous first derivatives, $r = 1, \ldots, S$, and $\omega = 1/\sigma$.[1] We have

LEMMA 2.1: *If a function $f$ satisfies* **(CS)** *and* (22), *then it must necessarily have the functional form*

(23) $$1 = \sum_{s=1}^{S} \left[ \prod_{k \in N_s} \left( \frac{x_k}{e_s(y)} \right) \right]^{1-\omega} \qquad \textit{for } \omega \neq 1, \textit{ or}$$

(24) $$\ln e_0(y) = \sum_{s=1}^{S} \left( \frac{\alpha_s}{m_s} \right) \sum_{k \in N_s} \ln x_k \qquad \textit{for } \omega = 1,$$

*where $e_0(y)$ and the $e_s(y)$ are positive functions with continuous first derivatives for $y = f(x) > 0$, and the $\alpha_s$ are positive constants with $\alpha_1 + \ldots + \alpha_s = 1$.*

Proof of Lemma 2.1: Suppose $y = f(x)$ is a general solution of (22). In the case $\omega \neq 1$, define

(25) $$F(x) = \sum_{s=1}^{S} \left[ \prod_{k \in N_s} \left( \frac{x_k}{e_s^*(f(x))} \right) \right]^{1-\omega}$$

where $e_s^*(y) = [\theta_s(y)]^{\frac{-1}{(1-\omega)m_s}}$.

[1] The econometrician will note that when $f$ is homothetic, $\theta_s/\theta_t$ is proportional to $y^{(1-\omega)(m_t - m_s)}$. The system (22) is then linear in logs.

Applying the transformation $T_i$ of (13) to (25), define $G(y(x), v(x)) = F(x)$. Substituting this expression in (25) and differentiating with respect to $x_k$,

$$A f_k = \begin{cases} \dfrac{1-\omega}{x_i} R_s & \text{for } k = i, \quad i \in N_s \\[2ex] \dfrac{1-\omega}{x_k} R_t - \partial \ln G/\partial v_k & \text{for } k \neq i, \quad k \in N_t, \end{cases} \tag{26}$$

where $R_s(x) = \dfrac{1}{G} \left[ \prod_{k \in N_s} \left( \dfrac{x_k}{e_s^*(f(x))} \right) \right]^{1-\omega}$ and

$$A(x) = \left[ \frac{\partial \ln G}{\partial y} + (1-\omega) \sum_{s=1}^{S} m_s R_s(x) \frac{\partial}{\partial y} \ln e_s^* (y(x)) \right].$$

From (26) for $k = i$, $A(x) \neq 0$. From (22) and (25)

$$\frac{x_k f_k(x)}{x_i f_i(x)} = \frac{R_t}{R_s} - \frac{x_k}{(1-\omega) R_s} \frac{\partial \ln G}{\partial v_k} = \frac{R_t}{R_s} \tag{27}$$

for $k \in N_t$, $i \in N_s$, $i \neq k$,

giving $\partial \ln G/\partial v_k = 0$. Hence, $G$ is a positive function of $y = f(x)$ alone, and (25) reduces to (23) after division by $G$.

In the case $\omega = 1$, consider the function

$$G(x) = \sum_{s=1}^{S} (\alpha_s(y)/m_s) \sum_{k \in N_s} \ln x_k,$$

where $\alpha_t(y) = m_t\, \theta_t(y)/\sum_{s=1}^{S} m_s\, \theta_s(y)$. An argument similar to that above shows that $G(x)$ can be written as a function $\ln e_0(y)$ of $y = f(x)$ alone. The solution (24) is then obtained by showing the $\alpha_t(y)$ constant. If the $\alpha_t(y)$ are not constant, then $d\alpha_s/dy < 0$ and $d\alpha_r/dy > 0$ for some $r,s,y$. Then, the input bundle

$$\ln x_k = \begin{cases} (M/\alpha_r)\ln 2 + (1/2\alpha_r)\ln e_0(y) & k \in N_r, \\ -(M/\alpha_s)\ln 2 + (1/2\alpha_s)\ln e_0(y) & k \in N_s, \\ 0 & k \in N_t,\ t \neq r,s. \end{cases}$$

gives $y = f(x)$, and yields a contradiction of **(C2)** for sufficiently large $M$.

Q.E.D. Lemma 2.1

If the function $f$ satisfies **(CSL)**, then equation (3) of the **(BAL)** form follows by application of **(L)** to (24). In the case of (23), using the notation of (26) and noting that (27) is homogeneous of degree zero, we have

$$1 = \sum_{s=1}^{S} R_s(x) = R_t(x) \sum_{s=1}^{S} \frac{R_s(x)}{R_t(x)}.$$

Hence, $R_t(x)$ is homogeneous of degree zero, and $e_t$ must be linear homogeneous, giving equation (2) of the **(BAL)** form with $1 - \omega = \rho$.

The proof of necessity is completed by showing that condition **(S)** on $f$ requires the restriction (ii) of the **(BAL)** form. Setting $\beta_s = \alpha_0 \alpha_s / y^{m_s}$, for $y$ fixed, we have

LEMMA 2.2 : *The surface defined by the positive bundles x satisfying*

$$1 = \sum_{s=1}^{S} \beta_s \Big(\prod_{k \in N_s} x_k\Big)^{-\rho}, \tag{28}$$

*for* $\{N_1, \ldots, N_S\}$ *a partition of* $\{1, \ldots, n\}$, *the* $\beta_s$ *positive constants,* $\rho \neq 0$, *and m the size of the largest set* $N_S$, *is strictly convex if and only if* $\rho \geqq -1/m$ *(with the inequality strict if more than one partition set has size m.)*

Proof of Lemma 2.2: Let $u_s(x) = (\prod_{k \in N_s} x_k)^{1/m_s}$. The following two propositions follow from elementary properties of convex functions:

(29) For a variable $u_s > 0$, the function $u_s^{-c_s}$ is strictly convex (in $u_s$) for $c_s > 0$ or $c_s < -1$, and is strictly concave for $0 > c_s > -1$.

(30) If $x$ and $x'$ are two distinct bundles in the surface (28), and $\varepsilon$ and $\varepsilon'$ are positive constants with $\varepsilon + \varepsilon' = 1$, then $u_s(\varepsilon x + \varepsilon' x') \geqq \varepsilon u_s(x) + \varepsilon' u_s(x')$, with equality holding only if $x'_k = \lambda_s x_k$ for a constant $\lambda_s$, all $k \in N_S$.

For $\rho > 0$,

$$
\begin{aligned}
(31) \qquad 1 &= \sum_{s=1}^{S} \beta_s \,[\varepsilon u_s(x)^{-\rho m_s} + \varepsilon' u_s(x')^{-\rho m_s}] \text{ from (28),} \\
&\geqq \sum_{s=1}^{S} \beta_s \,[\varepsilon u_s(x) + \varepsilon' u_s(x')]^{-\rho m_s} \text{ from (29)} \\
&\geqq \sum_{s=1}^{S} \beta_s \, u_s(\varepsilon x + \varepsilon' x')^{-\rho m_s} \text{ from (30)}
\end{aligned}
$$

For $0 > \rho \geqq -1/m$ (with the second inequality strict if there is more than one partition set of size $m$), (31) holds with the inequalities reversed. Denote this modified equation as (32).

If $u_s(x) = u_s(x')$ for each $s$, then for some $t$, $x'_k \neq \lambda_t x_k$ for some $k \in N_t$ for any constant $\lambda_t$, and the second inequality in (31) or (32) is strict. Otherwise, since the right-hand-side of (28) is monotone in $(u_1, \ldots, u_S)$, $u_s(x) \neq u_s(x')$ for at least two values of $s$, and the first inequality in (31) or (32) is strict. It follows from the monotonicity property of (28) that $\varepsilon x + \varepsilon' x'$ lies above the surface in these two cases, and the lemma follows.

Now consider $\rho < -1/m$, with equality allowed if there is more than one partition set of size $m$. If $\rho m_s \leqq -1$ for $s = r, t$, then define a set of positive constants $\{\mu_s\}$ with $\mu_1 + \ldots + \mu_s = 1$ and $\mu_r \neq \mu_t$, and consider the bundles $x$ and $x'$ defined by

$$
\begin{aligned}
x_k &= (\mu_s/\beta_s)^{-1/\rho m_s} \quad \text{for } k \in N_t, \quad s = 1, \ldots, S \\
x'_k &= \begin{cases} (\mu_r/\beta_t)^{-1/\rho m_t} & \text{for } k \in N_t \\ (\mu_t/\beta_r)^{-1/\rho m_r} & \text{for } k \in N_r \\ (\mu_s/\beta_s)^{-1/\rho m_s} & \text{for } k \in N_s, \quad s \neq r, t \end{cases}
\end{aligned}
$$

These bundles satisfy (28) and any linear combination of them violates the condition of strict convexity.

If $\rho m < -1$ and $\rho m_s > -1$ for $s \neq t$, $m_t = m$ $(S > 1)$, consider any two bundles $x$ and $x'$ satisfying (28), and define

$$
q_s = \frac{u_s(x')}{u_s(x)} \text{ and } \delta_s = \varepsilon + \varepsilon' q_s^{-\rho m_s} - [\varepsilon + \varepsilon' q_s]^{-\rho m_s}.
$$

Then $\delta_t > 0$ and $\delta_s < 0$ for $s \neq t$, and from (28),

$$
\begin{aligned}
(33) \qquad 1 &= \sum_{s=1}^{S} \beta_s \,[\varepsilon u_s(x)^{-\rho m_s} + \varepsilon' u_s(x')^{-\rho m_s}] \\
&= \sum_{s=1}^{S} \beta_s \,[\varepsilon u_s(x) + \varepsilon' u_s(x')]^{-\rho m_s} + \sum_{s=1}^{S} \delta_s \beta_s u_s(x)^{-\rho m_s}.
\end{aligned}
$$

Consider bundles $x$ and $x'$ defined for a constant $\mu$ with $0 < \mu < 1$ by

$$
x_k = \begin{cases} [\mu/\beta_t]^{-1/\rho m_t} & \text{for } k \in N_t \\ [(1-\mu)/\beta_s(S-1)]^{-1/\rho m_s} & \text{for } k \in N_s, \quad s \neq t. \end{cases}
$$

$$
x'_k = \begin{cases} [2\mu/\beta_t]^{-1/\rho m_t} & \text{for } k \in N_t, \\ [(1-2\mu)/\beta_s(S-1)]^{-1/\rho m_s} & \text{for } k \in N_s, \quad s \neq t. \end{cases}
$$

These bundles satisfy (28), and for sufficiently small $\mu$, the last term in (33) is positive, so that

$$1 \geqq \sum_{s=1}^{S} \beta_s [\varepsilon u_s(x) + \varepsilon' u_s(x')]^{-\rho m_s} = \sum_{s=1}^{S} \beta_s u_s(\varepsilon x + \varepsilon' x')^{-\rho m_s},$$

yielding a contradiction of the hypothesis. Q.E.D. Lemma 2.2.

2. We now establish that the **(BAL)** form is sufficient to define a function $f$ satisfying **(K)** and **(CSL)**.

Consider equation (2) of the **(BAL)** form. For fixed $x$, the right-hand-side of (2) is a continuous, strictly monotone function from positive $y$ onto the positive real numbers, and hence there exists a unique positive function $y = f(x)$ satisfying the identity (2).

The implicit function theorem ensures that $f$ has continuous derivatives of all orders, and **(C1)** is satisfied. Computation gives

$$\frac{f_i(x)}{f(x)} = \frac{1}{x_i} \frac{R_t(x)}{R(x)} \quad \text{for } i \in N_t,$$

and

$$\frac{f_{ij}(x)}{f_i(x) f_j(x)} = \frac{1}{f(x)} \left\{ 1 + \rho(m_r + m_t) - \rho \frac{R^*(x)}{R(x)} \right\} - \frac{1}{x_i f_i(x)} \; \{ \delta_{ij} + \rho \delta_{rt} \}$$

for $i \in N_t$, $j \in N_r$, where $R_s(x) = \alpha_s \left[ \prod_{k \in N_s} \left( \frac{x_k}{f(x)} \right) \right]^{-\rho}$;

$R(x) = \sum_{s=1}^{S} m_s R_s(x)$; and $R^*(x) = \sum_{s=1}^{S} m_s^2 R_s(x)$.

**(C2)** and **(K)** can be verified directly using these formulae. To show **(L)**, we note that

$$1 = \sum_{s=1}^{S} \alpha_s \left[ \prod_{k \in N_s} \left( \frac{x_k}{f(x)} \right) \right]^{-\rho} = \sum_{s=1}^{S} \alpha_s \left[ \prod_{k \in N_s} \left( \frac{x_k}{\frac{1}{\lambda} f(\lambda x)} \right) \right]^{-\rho}$$

for $\lambda > 0$. But for fixed $x$, equation (2) is satisfied for a unique $y$, and hence $y = f(x)$ $= \frac{1}{\lambda} f(\lambda x)$.

Finally, application of Lemma 2.2 to equation (2) for $y$ fixed establishes **(S)**.

The proof that the Cobb-Douglas form (3) satisfies **(CSL)** and **(K)** is left to the reader. Q.E.D. Theorem 2.

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