Bachelor's-Thesis Project

Topic: On energy-based vector hysteresis models

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Scope. We review the two hysteresis models of [**Lavet**] and [**Prigozhin**], establish their equivalence, and investigate their efficient numerical realization. Both models are based on a magneto-quasistatic consideration with magnetic energy density given by

$$E(b, m) = \frac{\mu_0}{2}|h|^2 + U(m), \tag{1}$$

with $U(m)$ the energy stored in the magnetization m, and magnetic field strength

$$h = \frac{1}{\mu_0}b - m. \tag{2}$$

Here b denotes the magnetic flux density, and μ_0 the permeability of vacuum.

Note. All the following considerations concern a single material point. Thus m, h, b are vectors in $\mathbb{R}^d$ in dimension $d = 2$ or $d = 3$.

1 Models

1.1 First Model

The constitutive model in [**Lavet**] is based on the variational principle

$$m = \arg\min_m U(m) - \langle h, m\rangle + \chi|m - m_p|, \tag{3}$$

with $\langle\cdot,\cdot\rangle$ denoting the Euclidean inner product, $\chi > 0$ a given parameter, and $m_p \in \mathbb{R}^d$ the given magnetization of the "previous time step". If $U(\cdot)$ is assumed strictly convex, this uniquely determines m as a function of h and m_p consequently allows to express $m = m(h, m_p)$ and $b = b(h, m_p)$ using (2).

1.2 Second Model

In [**Prigozhin**], the following alternative problem is proposed:

$$h_r = \arg\min_{u \in K(h)} S(u) - \langle m_p, u \rangle, \tag{4}$$

with $S(u)$ defined as the Legendre-Fenchel conjugate function of $\frac{1}{\mu_0}\frac{\partial U}{\partial m}(m)$, which simply means that

$$\frac{\partial S}{\partial u}(h_r) = m \quad \Leftrightarrow \quad h_r = \frac{\partial U}{\partial m}(m). \tag{5}$$

The set $K(h)$ over which is minimized is given by

$$K(h) = \{u : |u - h| \leq \chi\}. \tag{6}$$

Once h_r is found, we can determine m as a function of h_r (and of h and m_p) using (5), and then find b as a function of h and m_p using (2). Both models are motivated by an analogy with mechanics, i.e., a model for dry friction; see [**Moreau**].

1.3 Equivalence

Theorem 1. *The two proposed models are equivalent in the sense that if $\bar{m}$ minimizes the unrestrained model, then $h_r = \frac{\partial U}{\partial m}(\bar{m})$ minimizes the restrained problem and inversely if h_r minimizes the restrained problem, then $\bar{m} = \frac{\partial S}{\partial u}(h_r)$ minimizes the unrestrained problem.*

Proof. Let $\mathcal{L}_\lambda(u) = S(u) - \langle u, m_p \rangle + \lambda(\|u - h\| - \chi)$ be the Lagragian of the restrained problem. Let h_r be a minimizer of the restrained problem. Then there exists $\bar{\lambda} \geq 0$, so that the following conditions holds:

$$\bar{\lambda}(\|h_r - h\| - \chi) = 0 \tag{7}$$

$$0 = \frac{\partial \mathcal{L}_{\bar{\lambda}}}{\partial u}(h_r) = \frac{\partial S}{\partial u}(h_r) - m_p + \bar{\lambda}\frac{h_r - g}{\|h_r - h\|} = m - m_p + \bar{\lambda}\frac{h_r - h}{\|h_r - h\|} \tag{8}$$

The last equality holds, because of the Legendre-Fenchel property. First assume $\lambda = 0$, then from (8) it follows, that $m - m_p = 0$. Also $\|\frac{\partial U}{\partial m}(\bar{m}) - h\| = \|h_r - h\| \leq \chi$ follows directly from the Legendre Fenchel property and the constraints. By combining these two properties we get that the optimality condition $\frac{\partial U}{\partial m}(\bar{m}) - h \in \partial\|\bar{m} - m_p\|$ is indeed fulfilled, because $\partial\|0\| = \{x\colon \|x\| \leq 1\}$ Now let $\|h_r - h\| = \chi$. Inserting into (8) yields .

$$m - m_p = -\frac{\lambda}{\chi}(h_r - h)$$

Inserting into the optimality condition yields $\frac{\partial U}{\partial m} - h - \chi(\frac{\lambda}{\chi}\frac{h_r - h}{\|\frac{\lambda}{\chi}(h_r-h)\|}) = h_r - h - (h_r - h) = 0$. So the optimality condition of the unrestrained problem is indeed fulfilled.
Conversely, let $\bar{m}$ be a minimizer of the unrestrained problem and $h_r = \frac{\partial U}{\partial m}(\bar{m})$. Then the optimality condition holds. So

$$\frac{\partial U}{\partial m}(\bar{m}) - h = -\chi\partial\|\bar{m} - m_p\| \tag{9}$$

Assume $\bar{m} = m_p$, then for $\lambda = 0$, we get that $\mathcal{L}_0(h_r) = \frac{\partial S}{\partial u}(h_r) - m_p = m - m_p = 0$
Now let $m \neq m_p$, then the subdifferential only contains the derivative, and out of (9) with the use of the Legendre-Fenchel conjugate one get's

$$\frac{\partial U}{\partial m}(\bar{m}) - h = h_r - h = -\chi\frac{\bar{m} - m_p}{\|\bar{m} - m_p\|} \tag{10}$$

So $\|h_r - h\| = \chi$, i.e (7) is fulfilled for any $\lambda > 0$. Also

$$\mathcal{L}_{\|m-m_p\|}(h_r) = \bar{m} - m_p - \chi\|\bar{m} - m_p\|\frac{\bar{m} - m_p}{\|\bar{m} - m_p\|}(\chi\|\frac{\bar{m} - m_p}{\|\bar{m} - m_p\|}\|)^{-1} = 0$$

$\square$