APPLICATION OF CONVEX ANALYSIS TO THE TREATMENT OF ELASTOPLASTIC SYSTEMS

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0. INTRODUCTION

Convex analysis, i.e. the study of convex subsets or of convex numerical functions in topological linear spaces, has progressed greatly during the recent years. This was the work of people with quite diverse backgrounds : potential theory, the general theory of topological linear spaces, partial differential equations and the calculus of variations, approximation theory, optimization and optimal control, economics ... Such very alive topics as variational inequalities, monotone operators and nonlinear semigroups are also closely intermingled with convex analysis.

The necessary facts for understanding the sequel of this lecture are sketched in Sect. 1 below ; for more details, the reader could refer to [7], [12], [19], [33].

It was with definite mechanical motivations that the author took part in the general development of convex analysis (see the reference lists in [19] or [25]). The very concept of subgradient, as formalized in [16], was devised on mechanical purposes : it allowed to treat frictionless unilateral constraints in mechanical systems as a special case of the force-configuration relations which admit a "super-potential" (Cf. [20] [24] [25] ; as an example of dynamical problem with unilateral constraints, see [17], [18], devoted to the inception of cavitation in a liquid). The "subdifferential calculus" proved also perfectly adapted to the formulation and handling of resistance laws such as the Coulomb law of friction (when the normal component of the contact force is treated as a state variable) or the Prandtl - Reuss law of perfect plasticity (cf. [21] [25]) ; strain hardening can also be tackled in the same way (cf. [28] [29]).

A short note [22] outlined how the quasi-static evolution of an elastoplastic system could be studied by these methods. An essential step in solving the problem consists in what is called the sweeping process (cf. Sect. 6 below) associated with a moving convex set in some normed space. The sweeping process has been extensively investigated in numerous reports of the series "Travaux du Séminaire d'Analyse Convexe, Montpellier" : existence of solutions under various assumptions, constructive algo-

rithms, asymptotic properties (for a basic exposition, see also [24] ; concerning the stochastic version of this process see [3]).

The method was first applied to an elastoplastic system whose configuration manifold is a Hilbert space (see [24]);the Hilbert structure is naturally associated with the elastic potential. However, the main assumption made in this case, involving a nonempty interior for the rigidity set, relatively to the Hilbert topology, is not satisfied by usual continuous elastoplastic systems, so that the practical interest of the theory at this stage is restricted to systems with only a finite number of degrees of freedom.

The purpose of the present lecture is to explain the general method by developing the yet impublished study of an elastoplastic rectilinear rod with small longitudinal displacements.

The occurence in that case of only one space variable brings much simplification . In further publications the author shall adapt the same ideas to two- or three - dimensional continuous systems. But in such more complicated situations, the elements of which the existence is obtained, with some approximation algorithms, can only be considered as "weak solutions" of the evolution problem. Determining some cases of "smoothness" still remains an open task.

The Hencky - Nadai model will not be considered here ; the phenomenon it describes appears more as an extreme case of nonlinear elasticity than proper plasticity. The reader could refer to [4] [5] [6] [11] [13].

1. SUBDIFFERENTIALS

1.a. DEFINITIONS

Let X and Y be a pair of real linear spaces placed in duality by a bilinear form $< .,. >$. For sake of simplicity it will be supposed that this duality is separating , i.e. the linear form defined on X by $x \mapsto < x,y >$ is identically zero only if y is the origin of Y and the symmetrical assumption is made regarding the linear form defined on Y by $y \mapsto < x,y >$.

Recall that a locally convex topology on X (resp. Y) is said compatible with the duality if the continuous linear forms relative to this topology are exactly those which can be expressed as above. The closed convex subsets of X or also the lower semicontinuous convex numerical functions on this space are the same for all these topologies ; therefore as soon as a dual pair of linear spaces is given, we shall refer to closed convex sets or to l.s.c. convex functions without specifying the topology.

In what follows, X and Y play symmetric roles. Let f be a function defined, for instance, on X with values in $] - \infty , + \infty]$. An element $y \in Y$ is called *a subgradient of* f *at the point* $x \in X$ *if the value* $f(x)$ *is finite and if the affine function, taking the same value as* f *at the point* x ,

$$u \mapsto < u - x, y > + f(x)$$

is a minorant of f *all over* X . The (possibly empty) set of these subgradients is called the *subdifferential* of f at the point x ; this is a subset of Y denoted by $\partial f(x)$.

For instance in the special case where f is convex on X and weakly differentiable at the point x with $y \in Y$ as gradient (or "Gâteaux differential") at this point, the subdifferential reduces to the singleton $\{y\}$.

The following concept is immediately connected with subdifferentials : The numerical function f^* defined on Y by

$$f^*(y) = \sup_{x \in X} [< x,y > - f(x)] \qquad (1.1)$$

is called the polar or conjugate function of f , relative to the duality $< X,Y >$. Then

(1.2) $\partial f(x) = \{ y \in Y \ : \ f(x) + f^*(y) - < x,y > \leq 0 \}$

(where the $\leq$ sign may equivalently be replaced by $=$) . As f^* is the supremum of a collection of affine functions on Y which are continuous for the l.c. topologies compatible with the duality, this function is convex and l.s.c. . Then (2.2) proves that $\partial f(x)$ is closed and convex.

Iterating the process one may consider f^{**}, the polar function of f^* . Standard separation arguments show that if f is convex l.s.c., with values in $]-\infty, +\infty]$, then $f^{**} = f$. If in addition it is specified that f is not the constant $+\infty$, this function is said proper closed convex and the same properties hold for $g = f^*$.

Suppose that f and g are, in that way, a pair of mutually polar proper closed function ; then for $x \in X$ and $y \in Y$ the four following properties are equivalent

(1.3)
$$\begin{cases} y \in \partial f(x) \\ x \in \partial g(y) \\ f(x) + g(y) - < x,y > \leq 0 \\ f(x) + g(y) - < x,y > = 0 \ ; \end{cases}$$

then x and y are said conjugate relatively to f,g.

1.b. EXAMPLE.

Let C be a subset of X ; its indicator function ψ_C (i.e. $\psi_C(x) = 0$ if $x \in C$ and $+\infty$ if not) is proper closed convex if and only if C is nonempty closed and convex. The polar function

$$y \mapsto \psi_C^*(y) = \sup_{x \in X} [< x,y > - \psi_C(x)] = \sup_{x \in C} < x,y >$$

is classically called the support function of C (relative to the duality $< X,Y >$). The nonzero subgradients of ψ_C are obviously related to the supporting hyperplanes of the set C . Precisely, for any $x \in X$, the set $\partial \psi_C(x)$ is a closed conic convex subset of Y (with vertex at the origin) ; it is empty if and only if $x \notin C$; as soon as $x \in C$ this set contains at least the origin of Y . Generally speaking, $\partial \psi_C(x)$ is called the normal outward cone to C associated with x .

More specially let U be a closed linear subspace of X ; then $\psi_U^* = \psi_V$, where

V denotes the subspace of Y orthogonal to U . And

$$(1.4) \qquad \partial\psi_U(x) = \begin{cases} V & \text{if } x \in U \\ \emptyset & \text{if } x \notin U \end{cases}.$$

1.c. ADDITION RULE

Here is the most usual problem of the subdifferential calculus :

Let f_1 and f_2 be two numerical functions on X ; for every $x \in X$ one trivially has

$$(1.5) \qquad \partial f_1(x) + \partial f_2 \subset \partial (f_1 + f_2)(x) \cdot$$

Various sufficient conditions have been established ensuring that this inclusion is actually an equality of sets. We shall only need the following one : <u>If f_1 and f_2 are convex and if there exists a point $x_o \in X$ at which both functions take finite values, one of them being continuous at this point</u> (for some topology compatible with the duality $< X,Y >$), <u>then</u> (1.5) <u>holds as an equality for every</u> $x \in X$.

As an illustration let C be a convex subset of X and f a convex function. If there exists a point x_o at which f takes a finite value and which is interior to C relatively to some topology compatible with the duality this is a point of continuity for ψ_C so that one has

$$\partial(\psi_C + f)(x) = \partial \psi_C(x) + \partial f(x)$$

for every $x \in X$.
Consequently x is a minimizing point of the restriction of f to C if and only if $\partial\psi_C(x) + \partial f(x)$ contains the origin of Y , or equivalently

$$\partial f(x) \cap - \partial\psi_C(x) \neq \emptyset .$$

1.d. CONVEX INTEGRAL FUNCTIONALS

Many rules of the subdifferential calculus involve several pairs of dual spaces at the same time. The following situation is of primary importance in continuum mechanics when, starting from the local behavior of the medium, one generates functional formulations.

Let (T,μ) be a measure space and let X and Y be two spaces of measurable mappings of (T,μ) into $\mathbb{R}^n$, such that the bilinear pairing

$$(1.6) \qquad \langle x,y \rangle = \int_T x(t) \cdot y(t)\, \mu(dt)$$

is meaningful for every $x \in X$ and $y \in Y$ (the dot represents the natural scalar product in $\mathbb{R}^n$). Let $u \mapsto f(t,u)$ and $v \mapsto g(t,v)$ be a pair of convex numerical functions on $\mathbb{R}^n$, depending on $t \in T$ and which, for each t, are the polar of each other in the sense of the scalar product of $\mathbb{R}^n$. Under mild assumptions concerning the function spaces X and Y (they are satisfied in particular by the Lebesgue spaces $L^p(T, \mathbb{R}^n)$ and $L^q(T, \mathbb{R}^n)$, $1/p + 1/q = 1$) and simple measurability assumptions concerning the numerical functions $f,g : T \times \mathbb{R}^n \to]-\infty, +\infty]$, R.T. Rockafellar [30] [33] has established that the functionals F and G respectively defined on the spaces X and Y by the integrals (possibly taking the value $+\infty$)

$$F : x \mapsto \int_T f(t,x(t))\, \mu(dt)$$

$$G : y \mapsto \int_T g(t,y(t))\, \mu(dt)$$

are mutually polar convex functions relatively to the pairing (1.6); furthermore x and y are conjugate relatively to F and G if and only if $x(t)$ and $y(t)$ are conjugate relative to $f(t,.)$ and $g(t,.)$, for almost every t in the sense of μ.

A very convenient account of this question may be found in [34]. Of course the case of a pair of functions $f(t,.)$ and $g(t,.)$ defined, instead of $\mathbb{R}^n$, on a pair of infinite dimensional dual spaces has also been investigated (see e.g. [7], [34] or various reports in "Travaux du Séminaire d'Analyse Convexe, Montpellier" by C. Castaing, P. Clauzure, M.F. Sainte - Beuve, M. Valadier).

1.e. THE $\mathcal{C}, \mathcal{M}$ DUALITY

In the same connection, we shall use in this lecture another result of R.T. Rockafellar [33]: Suppose now that K is a compact topological space, with no measure a priori given. Let $x \mapsto \gamma(x)$ be a <u>multifunction</u> of K into $\mathbb{R}^n$, with nonempty closed convex values (in other words, γ is a nonempty closed convex subset of $\mathbb{R}^n$, depending on $x \in K$). This multifunction is supposed <u>lower semicontinuous</u> in the classical sense that, for every open subset Ω of $\mathbb{R}^n$ the set $\{x \in K : \gamma(x) \cap \Omega \neq \emptyset\}$ is open. Let us consider the (Banach) space $\mathcal{C}$ of the continuous mapping of K into $\mathbb{R}^n$ and its dual, i.e. the space $\mathcal{M}$ of the n-dimensional Radon measures on K, with the

natural pairing. The set of the <u>continuous selectors</u> of γ , i.e.

$$C = \{ s \in \mathcal{C} : \forall\, x \in K,\ s(x) \in \gamma(x) \}$$

is a closed convex subset of $\mathcal{C}$ and it is nonempty by virtue of a theorem of E. Michael [15] . The statement is that the support function of C in the sense of the duality $\mathcal{C}, \mathcal{M}$ may be constructed as follows : For every $m \in \mathcal{M}$, there exists, non uniquely, a nonnegative (bounded) scalar Radon measure μ on K relatively to which m possesses a density $\frac{dm}{d\mu} \in L^1(K, \mu ; \mathbb{R}^n)$ and one has

$$\psi_C^*(m) = \int_K \psi_{\gamma(x)}^* \left(\frac{dm}{d\mu}(x) \right) \mu(dx) ;$$

changing the measure μ is clearly immaterial, because $\psi_{\gamma(x)}^*$, the support function of $\gamma(x)$ in the sense of the natural duality of $\mathbb{R}^n$, is positively homogeneous ; in particular, μ may be the "absolute value" of the n-dimensional measure m . In addition, for $s \in \mathcal{C}$ and $m \in \mathcal{M}$, the relation $m \in \partial\psi_C(s)$ is equivalent to : for every μ as above, the density function satisfies

$$\frac{dm}{d\mu}(x) \in \partial\psi_{\gamma(x)}\left(s(x)\right)$$

for every $x \in K$, except possibly on a set whose μ-measure is zero ; here again the choice of μ is immaterial because $\partial\psi_{\gamma(x)}$ is a cone in $\mathbb{R}^n$.

2. FORCES AND VELOCITIES

2.a. THE $\mathcal{V}$, $\mathcal{F}$ FORMALISM

One is used, in classical mechanics, to associate with each possible configuration of a material system a pair of real linear spaces, infinite dimensional if the system has an infinite number of degrees of freedom, which will be denoted in the sequel by $\mathcal{V}$ and $\mathcal{F}$. The elements of $\mathcal{V}$ constitute, in a general sense, the possible values of the velocity of the system if it comes to pass through the considered configuration. The elements of $\mathcal{F}$ are the possible values of various forces which may be exerted on the system in that event. Forces, in such an abstract sense, are merely items of the code under which the available physical information about the considered material system is fed into the calculating machinery of Mechanics.

Denoting by $< v,f >$ the power of the force $f \in \mathcal{F}$ if the systems happens to have the velocity $v \in \mathcal{V}$, one places the spaces $\mathcal{V}$ and $\mathcal{F}$ in duality. The traditional method of virtual power (or of virtual work) precisely consists in exploiting this duality.

Observe that, in a given mechanical situation, there are usually several ways of applying this $\mathcal{V}$, $\mathcal{F}$ formalism (comparative examples are developed in [24] or [25]). For instance, if the mechanical system is a continuous medium occupying in the considered configuration a region Ω of the physical space, it may be convenient to take as $\mathcal{V}$ and $\mathcal{F}$ two spaces of tensor fields defined in Ω. The element $v \in \mathcal{V}$ will be the field $\omega \mapsto \dot{\varepsilon}^{ik}(\omega)$, the time-rate of strain of the medium, while every "force" $f \in \mathcal{F}$ must take the form of a field of strain tensors $\omega \mapsto \sigma_{ik}(\omega)$. The latter may be the proper internal strain of the medium, but such an f may also depict some external mechanical action according to the following rule : for every $v \in \mathcal{V}$, $< v,f >$ must equal the corresponding power.

Recall that, under integrability assumptions relative to the Lebesgue measure $d\omega$ of Ω, one has classically

$$< v,f > \; = - \int_{\Omega} \dot{\varepsilon}^{ik}(\omega)\, \sigma_{ik}(\omega)\, d\omega \; . \tag{2.1}$$

The minus sign in this expression is a pure accident due to the sign conventions made when defining the components of stress in solid mechanics (while pressure, in fluid

mechanics, is counted with the opposite convention). Sometimes it will be found simpler to place such spaces in duality by using the natural functional analytic scalar product : then one shall remember that it represents the negative of the power.

2.b. RESISTANCE LAWS.

In this general $\mathcal{V}$, $\mathcal{F}$ formalism, let us call a <u>resistance law</u> some relation between the possible velocity $v \in \mathcal{V}$ of the system and the value $f \in \mathcal{F}$ of some of the forces it undergoes.

The most elementary case is that of a linear of <u>viscous</u> resistance. Then the relation has the form $-f = L\,v$, where $L : \mathcal{V} \to \mathcal{F}$ is a linear mapping, self-adjoint with regard to the power pairing $< .,. >$ and monotone, i.e.

$$(2.2) \qquad \forall\, v \in \mathcal{V} : \qquad < v,\, L\,v > \geq 0 \ .$$

Trivially, the numerical function $Q : v \mapsto < v,\, L\,v > / 2$ is a quadratic form on the space $\mathcal{V}$ (the Rayleigh function) and $L\,v$ is its weak gradient at the point v . By (2.2) the quadratic form Q is nonnegative, thus convex and the preceding relation may equivalently be written as

$$(2.3) \qquad -f \in \partial\, Q\,(v) \ .$$

The advantage of the subdifferential notation manifests itself if one is dealing with resistance laws of the <u>dry friction</u> type, for in that case, the relation cannot be "solved" to express one of the elements v or f as a function of the other.

Generally speaking, a dry friction law is defined by giving a nonempty closed convex subset C of $\mathcal{F}$, containing the origin, and by stating the <u>maximal dissipation principle</u>, i.e. the values of $f \in \mathcal{F}$ which correspond to some given $v \in \mathcal{V}$ are the elements of C which minimize the numerical function $f \mapsto < v,f >$ (usually $-< v,f >$ is called the dissipated power). Such a relation between v and f is immediately found equivalent to

$$(2.4) \qquad -v \in \partial \psi_C(f) \ .$$

In view of (1.3) this is also equivalent to

$$(2.5) \qquad f \in \partial \psi_C^{*}\,(-v)$$

equivalent to

$$(2.6) \qquad \psi_C^*(-v) + \psi_C(f) + \langle v,f \rangle = 0$$

(where the = sign may be replaced by $\leq$).

Denoting by φ the function $v \mapsto \psi_C^*(-v)$ (it is the support function of the set $-C$) one gives to (2.5) the form

$$-f \in \partial\varphi(v)$$

similar to (2.3) .

Observe that (2.6) is equivalent to

$$(2.7) \qquad \begin{cases} f \in C \\ -\langle v,f \rangle = \varphi(v) \ . \end{cases}$$

In other words the values of f that the considered relation associates with a given $v \in \mathcal{V}$ are the elements of C for which the dissipated power $-\langle v,f \rangle$ exactly equals $\varphi(v)$; hence φ may be called in the present case the dissipation function of the considered resistance law (for a general discussion of this concept see [21], [24]).

2.c. SYSTEMS WITH LINEAR CONFIGURATION MANIFOLD

Many problems of applied mechanics are treated under the small deviation approximation, i.e. the considered system is assumed to remain "infinitely close" to a given reference configuration. Then all the geometrical and kinematical relations concerning the possible motions are linearized ; thereby the set of the possible configurations of the system is treated as a linear space $\mathcal{U}$, whose the considered reference configuration constitutes the origin. A motion of the system being defined as a mapping $t \mapsto u(t) \in \mathcal{U}$, the velocity at the instant t is the derivative $\dot{u}(t) \in \mathcal{U}$, supposed to exist relatively to some topology on $\mathcal{U}$. Thus, in the present situation one has $\mathcal{V} = \mathcal{U}$, the same velocity space for all configurations, and a single force space $\mathcal{F}$, in duality with $\mathcal{U}$, will be considered.

2.d. THE PRANDTL - REUSS LAW

The classical treatment of plasticity consists in introducing, beside the geometric or "visible" elements depicting the configuration of the considered system, some internal variables or hidden parameters. If the framework of the preceding

paragraph, where the configuration manifold is a linear space $\mathcal{U}$, one is naturally induced to interpret also the hidden parameters as defining an element of some linear space. Each possible state of the system is thus described by two components : the visible or exposed component $q \in \mathcal{U}$ and the hidden or plastic component p . The phenomenological representation of strain hardening requires of p to range over a larger space than q (see [28], [29]) ; but as far as perfect plasticity is concerned it suffices to take as p an element of the same space $\mathcal{U}$. The elastic potential is then assumed to depend only on the difference $q - p$, the elastic deviation : this implies that the elastic forces "acting" respectively on the components q and p are two elements of $\mathcal{F}$ with zero sum.

On the other hand, the plastic component p is submitted to a resistance to yielding, which is a resistance law of the dry friction type.

Such is in particular the underlying pattern of the Prandtl - Reuss law of perfect platicity for a continuous medium occupying a domain Ω of the physical spaces. The linear space $\mathcal{U}$ consists of strain tensor fields such as $q : \omega \mapsto \varepsilon(\omega)$, the visible strain. The plastic strain is an element $p : \omega \mapsto \varepsilon_p(\omega)$ of the same space. This latter element is assumed to present only a quasistatic evolution, i.e. at every instant, the elastic force on it equilibrates the resistance to yielding. The linear space $\mathcal{F}$ consists of stress tensor fields such as $s : \omega \mapsto \sigma(\omega)$, the stress in the medium. At every point ω of the medium is given the rigidity set $\gamma(\omega)$ a closed convex subset of E_6 , the six-dimensional linear space of the second order symmetric tensors. The resistance to yielding is an element $\omega \mapsto r(\omega)$ of $\mathcal{F}$, locally related to the "plastic strain velocity" $\dot{\varepsilon}_p(\omega)$ by

$$\dot{\varepsilon}_p(\omega) \in \partial \psi_{\gamma(\omega)}(r(\omega)) \tag{2.8}$$

We choose to understand this subdifferential in the sense of the self-duality of E_6 , defined by the natural euclidean scalar product of second order tensors ; hence the sign discrepancy with (2.4) . As the stress s is conceived as "acting" on the visible component, the elastic force acting on the component p is $-s$ and the quasi-equilibrium of this component is finally expressed by

$$\dot{\varepsilon}_p(\omega) = \partial \psi_{\gamma(\omega)}(s(\omega)) \tag{2.9}$$

for every ω in Ω .

3. ELASTOPLASTIC ROD : THE PRIMARY FORMULATION

3.a. THE SYSTEM

One considers a rectilinear thin rod occupying the interval $0 \leq x \leq 1$ of the x axis. The elements of this rod are supposed to perform only displacements along the x axis ; let $u(t,x)$ denote the displacement, at the time $t \geq 0$, of the element whose position in some reference state of the rod is $x \in [0,1]$. The values of u are treated as "infinitely small" ; thus the derivative

$$\partial u/\partial x = \varepsilon\ (t,x) \tag{3.1}$$

constitutes the <u>strain</u> at the time t and at the point x of the rod.

3.b. BOUNDARY CONDITIONS.

The extremity $x = 0$ is maintained fixed, i.e.

$$\forall\ t \geq 0\ :\qquad u(t,0) = 0\quad . \tag{3.2}$$

A given motion is imposed to the extremity $x = 1$, i.e.

$$\forall\ t \geq 0\ :\qquad u(t,1) = h(t) \tag{3.3}$$

where $t \mapsto h(t)$ is a given function.

3.c. LOAD

Distributed external forces parallel to the x axis, depending on time, are exerted on the various elements of the rod. They are at the present stage described by a function $(t,x) \mapsto f(t,x)$, the density of this distribution of forces relatively to the Lebesgue measure on $[0,1]$.

3.d. BEHAVIOR OF THE MATERIAL

According to the Prandtl - Reuss model recalled in Sect. 2, the strain $\varepsilon(t,x)$ is decomposed into the sum of the <u>elastic strain</u> ε_e and of the plastic strain ε_p

$$\varepsilon\ (t,x) = \varepsilon_e(t,x) + \varepsilon_p(t,x) \tag{3.4}$$

For each (t,x) the value of ε_e is related to the stress or tension s (here reduced to a single scalar component) by a linear law

$$\varepsilon_e(t,x) = a(x)\, s(t,x) \quad . \tag{3.5}$$

The given scalar $a(x) \geq 0$, independent of time, is the elastic compliance of the rod at the point x .

For every $x \in [0,1]$ the local rigidity set is a given interval of $\mathbb{R}$

$$\gamma(x) = [\alpha(x),\ \beta(x)]$$

whose extremities are the local yield limits. Putting $\partial\varepsilon_p/\partial t = \dot{\varepsilon}_p$, we require as in (2.9) that

$$\forall\, t \geq 0 \ , \ \forall\, x \in [0,1] \ : \ \dot{\varepsilon}_p\,(t,x) \in \partial\psi_{\gamma(x)}\,(s(t,x)) \tag{3.6}$$

where the subdifferential is relative to the natural self-duality of $\mathbb{R}$.

3.e. INITIAL CONDITIONS

As usual in elastoplastic problems, the initial stress must be given (not arbitrarily, see (3.10) below) :

$$\forall\, x \in [0,1] \ : \quad s(0,x) = s_o(x) \tag{3.7}$$

as well as the initial deviation :

$$\forall\, x \in [0,1] \ : \quad u(0,x) = u_o(x). \tag{3.8}$$

The latter implies

$$\forall\, x \in [0,1] \ : \quad \varepsilon(0,x) = \frac{\partial u_o}{\partial x}\,(x). \tag{3.9}$$

3.f. QUASI - EQUILIBRIUM

The problem is that of determining the evolution of the rod under conditions (3.1) to (3.9), supposing that this evolution is quasistatic, i.e. the data are subject to such limitations that inertia is negligible in the motion which actually takes place. Thus the dynamical equation reduces to that of pure statics

(3.10) $\qquad \forall\, t \geq 0\,, \quad \forall\, x \in [0,1] \;:\; \dfrac{\partial s}{\partial x} + f = 0\,.$

3.g NECESSARY WEAKENING OF THE REQUIREMENTS.

As usual in the primary formulation of mechanical problems, such as they arise from engineering situations, the requirements listed above implicitely involve the smoothness of the considered functions. But in the present example, one easily observes that, even under very strong regularity assumptions regarding the data, the existence of a solution in terms of smooth functions cannot be expected in general.

Suppose for instance that the given load density f is time-independent, continuous with regard to x and vanishing only at a finite number of points. Then the function

$$x \mapsto F(x) = -\int_0^x f(\xi)\, d\xi$$

attains its extrema at some of these points $x_1, x_2, \ldots, x_n$. The quasi-equilibrium condition (3.10) is equivalent to $s(x,t) = y(t) + F(x)$ where $t \mapsto y(t)$ denotes an unknown function. Suppose that the yield limits α and β are independent of x and that the initial tension $s_o(x) = y(0) + F(x)$ verifies

$$\forall\, x \in [0,1] \quad : \quad \alpha < s_o(x) < \beta \quad .$$

This evidently implies the existence of a time interval $[0,t_1[$ during which the evolution, caused by the given continuous motion imposed by (3.3) to the extremity 1 of the rod, takes place in a purely elastic way, i.e. $\dot{\varepsilon}_p = 0$ everywhere. Thus $x \mapsto \varepsilon_p(x)$ is independent of t and, by easy calculation,

$$y(t) = y(0) + (h(t) - h(0)) \,/ \int_0^1 a(x)\, dx \quad .$$

The function $x \mapsto s(t,x) = y(t) + F(x)$ attains its extrema at some of the points $x_1, x_2, \ldots, x_n$. Supposing for instance that $t \mapsto h(t)$ continuously increases, one finds that this phase of motion ends at the instant where the maximum of $x \mapsto y(t) + F(x)$ attains the yield limit β. Henceforward the condition $\alpha \leq s(x) \leq \beta$ makes that $t \mapsto y(t)$ cannot increase anymore, thus

$$\int_0^1 \varepsilon_e(t,x)\, dx = \int_0^1 a(x)\, (y(t) + F(x))\, dx$$

remains constant. The requirement $u(t,1) = h(t)$ can only be met as the result of yielding, localized to some of the points $x_1, x_2, \ldots, x_n$. Consequently the plastic strain cannot be depicted as a function $x \mapsto \varepsilon_p(t,x)$ nor its time-rate as a function $x \mapsto \dot{\varepsilon}_p(t,x)$. One is definitely induced to consider this elements as measures, thus to turn to a weaker formulation of the problem.

4. THE CHOICE OF A PAIR OF SPACES

4.a THE STRAIN AND STRAIN-RATE MEASURES

The elementary treatment sketched in the preceding section suggests that, under reasonable assumptions concerning the data, one may expect a solution of the problem in which the stress in all part of the rod is depicted, for every t, by a continuous numerical function $x \mapsto s(t,x)$. But to describe the strain or its decomposition into the sum of the elastic and plastic terms and also in what concerns the time-rates of these elements, it turned out that the suitable mathematical objects should not be functions but measures.

Therefore we choose as the mathematical framework for all the sequel the following pair of linear spaces :

1° The space $\mathcal{C}([0,1], \mathbb{R})$, abreviatively denoted by $\mathcal{C}$, of the continuous numerical functions defined on the compact interval $[0,1]$.

2° The space $\mathcal{M}([0,1], \mathbb{R})$, abreviatively denoted by $\mathcal{M}$, of the (bounded) scalar measures on the same interval.

$\mathcal{C}$ is a separable nonreflexive Banach space and $\mathcal{M}$ is its dual ; the corresponding bilinear pairing will be denoted by $\ll . , . \gg$

In the elementary setting of Sect.3, the time-rate of strain at the instant t was a numerical function $x \mapsto \dot{\varepsilon}(t,x)$. Similarly to the three-dimensional expression recalled in (2.1), the corresponding power of some stress $x \mapsto \sigma(t,x)$ should be expressed by an integral relative to the Lebesgue measure of $[0,1]$

$$\mathcal{P} = - \int_{[0,1]} \sigma(t,x)\, \dot{\varepsilon}(t,x)\, dx \qquad . \tag{4.1}$$

This may equivalently by read as the integral of the function $x \mapsto \sigma(t,x)$ relatively to the measure $\dot{q}$ whose density with regard to the Lebesgue measure is $\dot{\varepsilon}$. The step we take now consists in depicting the time-rate of strain of the rod at some instant t by an element $\dot{q}(t)$ of $\mathcal{M}$ which does not necessarily possess a density with regard to the Lebesgue measure. Then if the stress $x \mapsto \sigma(t,x)$ is a continuous function on $[0,1]$, i.e. an element $\sigma(t)$ of $\mathcal{C}$, (4.1) is to be replaced by

$$\mathcal{P} = - \ll \sigma, \dot{q} \gg \quad . \tag{4.2}$$

By this minus sign the functional analytic pairing $\ll . , . \gg$ we shall use in the sequel differs from the mechanical pairing which would be defined, according to Sect.2, between $\mathcal{M}$ considered as the velocity space and $\mathcal{C}$ considered as the force space of our mechanical system.

Similarly, the strain of the rod at each instant t will depicted by an element $q(t)$ of $\mathcal{M}$, without reference to the Lebesgue measure. The connection between q and the displacement u is stated as follows : it is required of the numerical function $x \mapsto u(t,x)$ to have for every t a bounded variation and to admit $q(t)$ as its differential measure.

This classically means that the function $x \mapsto u(t,x)$ possesses at every point $x \in]0,1[$ a left limit and a right limit respectively denoted by $u^-(t,x)$ and $u^+(t,x)$ and that

$$\int_{[0,x[} q(t) = u^-(t,x) - u(t,0) \tag{4.3}$$

$$\int_{[0,x]} q(t) = u^+(t,x) - u(t,0) \quad . \tag{4.4}$$

This will still be true for $x = 0$ or $x = 1$ if we agree to write $u^-(t,0) = u(t,0)$ and $u^+(t,1) = u(t,1)$.

Of course we shall require of q to be, in a certain sense, the time integral of $\dot{q}$: see the formulation in Sect.5 below. In addition, some connection will be found between the measure $\dot{p}(t)$ and the derivatives of the functions $t \mapsto u^-(t,x)$ and $t \mapsto u^+(t,x)$.

4.b ELASTICITY LAW

The general pattern of elastoplasticity (cf Parag. 2.d) is now applied by taking $\mathcal{M}$ as the configuration space and $\mathcal{C}$ as the force space. Thus, for every t, the "visible" strain $q(t)$ is decomposed into

$$q = e + p \quad , \tag{4.5}$$

where $e \in \mathcal{M}$ and $p \in \mathcal{M}$ are respectively called the elastic strain and the plastic strain.

The local linear elasticity relation (3.5) will be replaced by

$$e = As \tag{4.6}$$

where $s \in \mathcal{C}$ is the stress of the rod at the considered instant and A a nonnegative element of $\mathcal{M}$, independent of time, called the compliance measure of the rod ; the right member of (4.6) is to be read as the product of this measure by the continuous function s .

In the elementary setting, the measure A possessed, with regard to the Lebesgue measure of $[0,1]$, a density which was precisely the compliance function $x \mapsto a(x)$. But the Lebesgue measure has no mechanical relevance to the present situation and defining directly the compliance of the rod as a measure appears definitely more convenient. As an illustration, suppose that the measure A presents an atom at some point $x \in]0,1[$, i.e. this measure is the sum of a diffuse measure and of the punctual mass $\alpha > 0$ at the point x . Then (4.6) implies that e is the sum of a diffuse measure and of the punctual mass $\alpha\, s(x)$ at the point x . If on the other hand, the plastic strain p is zero at the considered instant, (4.3) and (4.4) imply

$$u^+(t,x) - u^-(t,x) = \alpha\, s(x) \quad .$$

Such a jump of u means that a gap occurs at the point x of the rod, proportional to the local value of the stress. This may represent some loose elastic connection between the parts $[0,x[$ and $]x,1]$ of the rod ; something like a crack.

4.c PLASTICITY LAW

In the same way as for q , it will be required in Sect.5 of the function $t \mapsto p(t) \in \mathcal{M}$ to be the (weak) integral of some function $t \mapsto \dot{p}(t) \in \mathcal{M}$. The measure $\dot{p}(t)$ is called the time-rate of plastic strain of the rod at the instant t .

Let us denote by C the set of the continuous selections of the multifunction $x \mapsto \gamma(x)$, i.e.

$$C = \{ s \in \mathcal{C} \ : \ \forall\, x \in [0,1], \ \ \alpha(x) \leq s(x) \leq \beta(x) \ \}$$

This is a closed convex subset of $\mathcal{C}$; Assumptions 2 and 3 to be formulated in Parag.5.c below, will ensure that this set is nonempty and that Rockafellar's result of Parag. 1.e apply. Therefore the writing

$$\dot{p}(t) \in \partial \psi_C \ (s(t) \ ) \tag{4.7}$$

in the sense of the duality $\mathcal{C}$, $\mathcal{M}$ will have a local meaning quite similar to the Prandtl - Reuss law (3.6) of the elementary case. The only difference is that the function $x \mapsto \dot{\varepsilon}_p \ (t,x)$ will then be the density of the measure $\dot{p}(t)$ with regard to some nonnegative measure on $[0,1]$ possibly other than the Lebesgue measure.

5. THE MATHEMATICAL PROBLEM

5.a FORMULATION

Let us choose an interval of time $[0,T]$.

We are to determine a mapping $u : [0,T] \times [0,1] \to \mathbb{R}$, three mappings $q,e,p : [0,T] \to \mathcal{M}$, a mapping $s : [0,T] \to \mathcal{C}$ such that :

1° For every $t \in [0,T]$, the function $x \mapsto u(t,x)$ has a bounded variation on $[0,1]$, it agrees with the boundary conditions

(5.1) $$u(t,0) = 0$$

(5.2) $$u(t,1) = h(t)$$

and the measure $q(t)$ is its differential measure according to (4.3) and (4.4) .

2° For every $x \in [0,1]$, the functions $t \mapsto u^-(t,x)$ and $t \mapsto u^+(t,x)$ (i.e. the left and right limits at the point x ; cf. Parag. 4.a : recall that, by convention, $u^- = u$ for $x = 0$ and $u^+ = u$ for $x = 1$) are Lipschitz and they agree with the initial conditions

(5.3) $$u^-(0,x) = u_o^-(x) \quad , \qquad u^+(0,x) = u_o^+(x) \quad .$$

3° The function $t \mapsto s(t)$ is Lipschitz in the norm topology of $\mathcal{C}$; it agrees with the initial condition

(5.4) $$s(0) = s_o \in \mathcal{C}$$

and, for every $t \in [0,T]$ the quasi-equilibrium condition

(5.5) $$\frac{\partial s}{\partial x} + f = 0$$

holds in the elementary sense ; here the given load density $x \mapsto f(t,x)$ is a continuous function on $[0,1]$. More generally the given load could be depictedas a <u>diffuse measure</u> on $[0,1]$; then (5.5) would be replaced by the equality of this measure to the negative of the differential measure of s (in that case s should be a continuous function with bounded variation) : the essential fact is that this quasi-equilibrium condition could be translated into the form (5.15) below.

4° For every $t \in [0,T]$

(5.6) $$q = e + p$$

(5.7) $$e = As$$

with $A \in \mathcal{M}$, a nonnegative given measure.

5° There exists $\dot{p} : [0,T] \to \mathcal{M}$, <u>weakly integrable</u> with values in a ball of $\mathcal{M}$, such that for every $t \in [0,T]$

(5.8) $$p(t) = p(0) + \int_0^t \dot{p}(\tau)\,d\tau \quad ,$$

with $p(0)$ related to the initial data (see (5.17) below) and that, for almost every t

$$\dot{p} \in \partial\psi_C(s) \quad , \tag{5.9}$$

with C defined as in Parag. 4.c .

6° There exists $\dot{q} : [0,T] \to \mathcal{M}$, weakly integrable with values in a ball of $\mathcal{M}$ such that for every t

$$q(t) = q(0) + \int_0^t \dot{q}(\tau)\, d\tau \quad . \tag{5.10}$$

For almost every t , the measure $\dot{q}(t)$ is the differential measure of a function $x \mapsto \dot{u}(t,x)$ with bounded variation on $[0,1]$, vanishing at $x = 0$ and such that

$$\frac{d}{dt}\, u^-(t,x) = \dot{u}^-(t,x) \tag{5.11}$$

$$\frac{d}{dt}\, u^+(t,x) = \dot{u}^+(t,x). \tag{5.12}$$

Remark. As $\mathcal{C}$ is separable, (5.8) and (5.10) imply that for almost every t the elements $\dot{p}(t)$ and $\dot{q}(t)$ of $\mathcal{M}$ are the derivatives of the functions $t \mapsto p(t)$ and $t \mapsto q(t)$ in the sense of the weak-star topology $\sigma(\mathcal{M}, \mathcal{C})$.

5.b EQUIVALENT DATA

As we are to focus on the construction of mappings of $[0,T]$ into $\mathcal{C}$ or $\mathcal{M}$, we first represent all the data by elements of these spaces. So to speak, the dual pair of linear spaces $\mathcal{C}$, $\mathcal{M}$ constitute a calculating device and the data must be fed into it under a suitably adapted form.

Let us denote by U the subspace of $\mathcal{M}$ consisting of the measures whose sum is zero ; let us denote by V the subspace of $\mathcal{C}$, isomorphic to $\mathbb{R}$, of the constant numerical functions on $[0,1]$. Clearly V and U are the orthogonal of each other relatively to the duality $\mathcal{C}$, $\mathcal{M}$.

To the given $h : [0,1] \mapsto \mathbb{R}$ depicting the motion of the extremity 1 of the rod, we associate $g : [0,T] \to V$ by

$$g(t) = h(t) / \int A \tag{5.13}$$

Thus, the existence of u meeting the requirements listed in 1° above is equivalent to

(5.14) $\forall\, t \in [0,T] \;:\quad q(t) \in U + A\, g(t)$.

On the other hand, the quasi-equilibrium condition (5.5) is equivalent to

(5.15) $\forall\, t \in [0,T] \;:\quad s(t) \in V + c(t)$

where $c(t) \in \mathcal{C}$ denotes, for each t, a primitive of the function $x \mapsto -f(t,x)$. As an arbitrary constant may be added to this primitive, we suppose it chosen in such a way that

(5.16) $\forall\, t \in [0,T] \;:\quad A\, c(t) \in U$.

Concerning the initial data, let us denote by $q_o \in \mathcal{M}$ the differential measure of the given function $x \mapsto u_o(x)$ (function with bounded variation on $[0,1]$). Thus, in view of (5.4), (5.6), (5.7), condition (5.8) shall be understood with

(5.17) $p(0) = q_o - A\, s_o$

and (5.10) with $q(0) = q_o$.

5.c HYPOTHESES CONCERNING THE DATA

Assumption 1 : The nonnegative measure A does not vanish, i.e. $\int A > 0$.

Assumption 2 : The multifunction $x \mapsto \gamma(x)$ of $[0,1]$ into $\mathbb{R}$ is lower semi continuous (cf Parag. 1.e) ; equivalently the yield limits $x \mapsto \alpha(x)$ and $x \mapsto \beta(x)$ are respectively u.s.c. and l.s.c. numerical functions.

Assumption 3 : For every $t \in [0,T]$, the space of constants V has a nonempty intersection with the interior of the convex subset $C - c(t)$ of $\mathcal{C}$; equivalently

$$\forall\, t \in [0,T] \;:\quad \sup_x \,(\alpha(x) - c(t,x)\,) < \inf_x \,(\beta(x) - c(t,x)\,) \;.$$

One may call this Assumption the safe load hypothesis because of the following remark : For (5.9) and (5.15) to be satisfied it is obviously necessary that $V + c$ meets C, i.e. $V \cap (C - c) \neq \emptyset$; otherwise the system could not present a quasistatic evolution. The above assumption means that the latter requirement is fullfilled with a certain safety margin. In view of the unavoidable uncertainty in the physical measu-

rement of data, such a margin may be considered as necessary for the problem to be physically well set.

Assumption 4 : The numerical function $t \mapsto h(t)$ is Lipschitz and the function $t \mapsto c(t)$ is Lipschitz in the sense of the norm of $\mathcal{C}$.

This last Assumption is less restrictive than it looks. Intuitively, the evolution of our system, being regulated by some resistance phenomenon of the dry friction type (i.e. the resistance force depends on the oriented direction of the velocity but not on its magnitude), associates the successive configurations of the system to the successive values of the data in a way which does not depend on the timing. This means that the mathematical conditions of the problem are invariant under any absolutely continuous non decreasing change of variable $t \mapsto t'$. Starting from some data $t \mapsto h$ and $t \mapsto c$ which would be only absolutely continuous, one could use such a change of variables to reduce to the Lipschitz case.

5.d AUXILIARY PROBLEM

Let us introduce two new unknowns functions $y : [0,T] \to \mathcal{C}$ and $z : [0,T] \to \mathcal{U}$ by

$$y(t) = s(t) - c(t) - g(t) \tag{5.18}$$

$$z(t) = q(t) - A\,c(t) - A\,g(t) \quad . \tag{5.19}$$

Then, in view of (5.6) and (5.7)

$$p(t) = z(t) - A\,y(t) \tag{5.20}$$

and the requirements of Parag. 5.a take the following equivalent form :

1° For every $t \in [0,T]$:

$$y(t) \in V \quad , \qquad z(t) \in U \tag{5.21}$$

(in other words y is a numerical function) .

2° For $t = 0$:

$$y(0) = s_0 - c(0) - g(0) \quad \text{denoted by} \quad y_0 \in V \tag{5.22}$$

$$z(0) = q_0 - A\,c(0) - A\,g(0) \quad \text{denoted by} \quad z_0 \in U \tag{5.23}$$

3° There exists $\dot{y} \in L^{\infty}(0,T\ ;\ \mathbb{R})$ such that

$$y(t) = y_o + \int_0^t \dot{y}(\tau)\ d\tau \ . \tag{5.24}$$

4° There exists $\dot{z} : [0,T] \to U$, weakly integrable with values in a ball of $\mathcal{M}$ such that

$$z(t) = z_o + \int_0^t \dot{z}(\tau)\ d\tau \quad . \tag{5.25}$$

5° For almost every t , in view of (5.9) and(5.20),

$$\dot{z} - A\,\dot{y} \in \partial\psi_C\ (y + c + g) \tag{5.26}$$

or equivalently

$$\dot{z} - A\,\dot{y} \in \partial\psi_C - c(t) - g(t)\quad (y) \ . \tag{5.27}$$

6. DETERMINATION OF s OR y .

6.a. THE PROBLEM FOR y

As usual in the study of elastoplastic systems, determining the stress is the easiest part and it will be found that this preliminary problem possesses a unique solution.

In view of (5.18) it is equivalent to determine the function $t \mapsto y$; this will be done by drawing some consequences of the various conditions formulated in Parag. 5.d.

From $y \in V$ it results (see Parag. 1.b)

$$\partial\psi_V(y) = U \ .$$

As (5.21) implies $\dot{z} \in U$, (5.27) implies

$$(6.1) \qquad -A\,\dot{y} \in \partial\psi_{C-c-g}(y) + \partial\psi_{V}(y) \quad .$$

Considering the general inclusion (1.5), this in turn implies that for almost every t,

$$(6.2) \qquad -A\,\dot{y} \in \partial\psi_{(C-c-g)\cap V}(y).$$

For every $t \in [0,T]$, the intersection

$$(C - c(t) - g(t)) \cap V = I(t)$$

is a convex subset, i.e. an interval, of the one-dimensional space V. Let us identify the elements of V (the constant numerical functions on $[0,1]$) with the real numbers. An element of $I(t)$ is then a real number η such that

$$\forall\, x \in [0,1] : \alpha(x) - c(t,x) - g(t) \leq \eta \leq \beta(x) - c(t,x) - g(t).$$

In other words

$$I(t) = \Big[\ \sup_x\,(\alpha(x) - c(t,x) - g(t)),\ \inf_x\,(\beta(x) - c(t,x) - g(t))\ \Big]$$

a non empty interval, by virtue of Assumption 3.

The subdifferential in (6.2) must be read in the sense of the duality $\mathcal{C}$, $\mathcal{M}$. But, as (6.2) requires that the function $t \mapsto y$ takes its values in V (otherwise the right member would be empty), we shall be able now to convert (6.2) into an equivalent form, without reference to the imbedding of V in $\mathcal{C}$. In fact, when η, y, $\dot{y}$ are interpreted as real functions on $[0,1]$, (6.2) means

$$\begin{cases} y \in I(t) \\ \forall\, \eta \in I(t) : \quad \ll \eta - y,\ -A\,\dot{y} \gg \leq 0 \end{cases}$$

and the latter inequality is merely

$$-(\eta - y)\,\dot{y} \int A \leq 0 \quad .$$

As the strictly positive factor $\int A$ may be omitted this comes to be equivalent to

$$(6.3) \qquad -\dot{y} \in \partial\psi_{I(t)}(y)$$

in the sense of the conventional self-duality of $\mathbb{R}$

6.b. THE SWEEPING PROCESS.

Condition (6.3) is a very special case of the following : Let H be a real Hilbert space and let $\Gamma : [0,T] \to H$ be a given moving closed convex subset of H, i.e. a multifunction of $[0,T]$ into H with nonempty closed convex values. One looks for a moving point, i.e. an absolutely continuous mapping $u : [0,T] \to H$, agreeing with some initial condition $u(0) = u_0 \in \Gamma(0)$ and such that for almost every t in $[0,T]$

$$(6.4) \qquad -\dot{u} \in \partial\psi_{\Gamma(t)}(u)$$

(because H is a reflexive Banach space, the absolutely continuous function u is known to possess a strong derivative $\dot{u}$ for almost every t).

Recall (cf. Parag. 1.b) that $\partial\psi_{\Gamma(t)}(u)$ is the normal outward cone to the set $\Gamma(t)$ at the point u .

In the case where Γ possesses a nonempty interior the meaning of (6.4) may be illustrated as follows : as long as the point u happens to lie in this interior, it remains at rest, for the normal cone reduces at this time to the origin of H . When u is caught up with by the boundary of the moving set, this point can only take a motion in an inward normal direction as if pushed by this boundary, so as to go on belonging to $\Gamma(t)$.

We call this a sweeping process.

Condition (6.4) is a special case of

$$(6.5) \qquad -\dot{u} \in M(t,u)$$

where $x \mapsto M(t,x)$ denotes, for each t, a multifunction of the Hilbert space H into itself which is monotone in the sense of Minty ; monotony trivially implies that, for each initial condition $u(0) = u_0$, (6.5) possesses at most one solution.

The simplest sufficient condition we found for the existence of solutions of (6.4) is the absolute continuity of the multifunction $t \mapsto \Gamma(t)$ in the sense of the Hausdorff distance (see [24]). Such is in particular the case when the multifunction is Lipschitz in the sense of the Hausdorff distance ; then every solution $t \mapsto u(t)$ is also Lipschitz, in the sense of the norm of H , with the same Lipschitz ratio as Γ.

In the present situation $H = \mathbb{R}$ and the Lipschitz property for the multifunction $t \mapsto I(t)$ is an immediate consequence of Assumption 4. We conclude the existence of a unique solution $t \mapsto y(t)$ of (6.3), agreeing with the initial condition (5.22)

This solution is Lipschitz, so that $\dot{y} \in L^{\infty}(0,T ; \mathbb{R})$.

6.c. THE CATCHING-UP ALGORITHM.

For the numerical solution of (6.4) we proposed (see e.g. [24]) the following algorithm : An increasing sequence is chosen in the interval $[0,T]$:

$$0 = t_o < t_1 < \ldots < t_n = T$$

and a sequence of points of H is constructed by successive projections, i.e.

$$u^o = u_o$$
$$u^{i+1} = \text{proj}\,(u^i , \Gamma(t_{i+1}))$$

(this denotes the nearest point to u^i in the closed convex set $\Gamma(t_{i+1})$). It is proved that the step function based on this sequence (or also the continuous piecewise linear function of t which interpolates the sequence) converges uniformly to the desired solution when the division of $[0,T]$ is refined in such a way that. $\max_i (t_{i+1} - t_i)$ tends to zero.

In the present one-dimensional case the projection operation is especially simple : u^{i+1} either equals u^i or is one of the two extremities of the interval $I(t_{i+1})$.

A graphical solution would also easily be devised.

7. EXISTENCE PROOF

7.a. THE CONSTRUCTION OF $\dot{p}$ AND p .

By combining the various conditions imposed in section 5 to the unknowns, we obtained condition (6.3), which involves the single unknown function $t \mapsto y$. It was found that, with attention to the initial data, (6.3) possessed a unique solution. We are now to establish that such a procedure properly constitutes an <u>elimination</u>, i.e. (6.3) is not only necessary but also sufficient for the existence of functions $t \mapsto p$, $t \mapsto q$, $t \mapsto s$, $(t,x) \mapsto u$ fullfilling with the considered $t \mapsto y$ all the requirements formulated in Sect. 5.

Henceforth it is supposed that $t \mapsto y$ satisfies (6.3), or equivalently (6.2), for almost every t ; then it satisfies also (6.1) by virtue of the <u>addition rule for subdifferentials</u> (cf. Parag. 1.c) and of Assumption 3 which implies that V intersects the interior of $C - c(t) - g(t)$. Now (6.1) means that, for the considered values of t , there exists a non empty set of elements $\dot{z}$ of $\partial\psi_V(y) = U$ such that (5.26) holds. Equivalently there exists a nonempty set $\Gamma(t)$ of elements $\dot{p} = \dot{z} - A\,\dot{y}$ of $U - A\,\dot{y}(t)$ such that

$$\dot{p} \in \partial\psi_C\,(y + c + g). \tag{7.1}$$

We are to prove that, from the multifunction $t \mapsto \Gamma(t)$, a single-valued function $t \mapsto \dot{p}(t)$ can be <u>selected</u>, which is weakly integrable, so as to permit the construction of $t \mapsto p(t)$ according to (5.8). Then, if we construct $t \mapsto s(t)$ by (5.18), condition (5.9) will be satisfied for almost every t :

Owing to the initial conditions, the fact that $\dot{z}$ takes its values in U implies $z(t) \in U$ for every t if z is the weak integral of $\dot{z}$.

Since by construction $y \in C - c - g$, one has $\psi_C(y + c + g) = 0$; the equivalence of the various relations (1.3) makes that (7.1) may as well be written under the form

$$\psi_C^*(\dot{p}) - \ll y + c + g\,,\ \dot{p} \gg\ \leq\ 0\ .$$

Therefore $\Gamma(t)$ is the intersection of $U - A\,\dot{y}(t)$ with the set

$$\Phi(t) = \{w \in \mathcal{W} : \psi_C^*(w) - \ll y(t) + c(t) + g(t),\ w \gg \leq 0\ \} \tag{7.2}$$

7.b. BOUNDEDNESS OF $\Gamma(t)$

Assumptions 2 and 3 imply the existence of some $\rho > 0$ and of some continuous mapping $k : [0,T] \to \mathcal{C}$ such that, for every $t \in [0,T]$, one has $k(t) \in V + c(t)$ and the closed ball in $\mathcal{C}$ with radius ρ, with center $k(t)$, is contained in C. In fact the interval

$$\left[\sup_x (\alpha(x) - c(t,x)) , \inf_x (\beta(x) - c(t,x)) \right]$$

has a non zero length, depending continuously on $t \in [0,T]$. Let us choose $\rho > 0$ such that 2ρ minorizes this length for every t. The interval

$$\left[\sup_x (\alpha(x) - c(t,x) + \rho) , \inf_x (\beta(x) - c(t,x) - \rho) \right]$$

is nonempty and depends on t in a Lipschitz way (see Sect. 6). Let $t \mapsto u(t) \in \mathbb{R}$ be a solution of the sweeping process for this moving interval ; $u(t)$ may be interpreted as an element of V and $t \mapsto k(t) = u(t) + c(t)$ meets the above requirements.

The inclusion of the ball in C is equivalent to the following inequality between the respective support functions of these closed convex sets

$$(7.3) \qquad \forall w \in \mathcal{M}_0 : \quad \rho \|w\|_{\mathcal{M}_0} + \ll k(t), w \gg \leq \psi_C^*(w) .$$

For any w in $\mathcal{M}$ let us write

$$\ll y + c + g, w \gg = \ll k, w \gg + \ll y + c + g - k, w + A\dot{y} \gg - \ll y + c + g - k, A\dot{y} \gg .$$

If $w \in \Gamma(t)$, one has $w + A\dot{y} \in U$, thus the second bracket in the right member vanishes, since $y \in V$, $k - c \in V$, $g \in V$. On the other hand, in view of (5.16),

$$\ll c, A\dot{y} \gg = \ll \dot{y}, Ac \gg = 0 .$$

Then, by comparing (7.2) with (7.3) one obtains that, for every $w \in \Gamma(t)$,

$$\rho \|w\|_{\mathcal{M}_0} + \ll y + g - k, A\dot{y} \gg \leq 0$$

thus

$$(7.4) \qquad \|w\|_{\mathcal{M}_0} \leq \frac{1}{\rho} M \int_{\mathbb{R}} |\dot{y}| \int A ,$$

where M denotes a majorant of the continuous function $t \mapsto \|y(t) + g(t) - k(t)\|_{\mathcal{C}}$

on $[0,T]$.

Recall that $y : [0,T] \to \mathbb{R}$ is Lipschitz i.e. $\dot{y}$ is bounded. Then (7.4) implies the existence of a closed ball B in $\mathcal{M}$ containing the set $\Gamma(t)$ for almost every t .

7.c. MEASURABILITY OF THE MULTIFUNCTION Γ .

As $\mathcal{C}$ is separable, the topology induced by $\sigma(\mathcal{M}, \mathcal{C})$ (the weak-star topology) makes of the preceding ball a compact metrizable topological space B_s . For almost every t the set $\Gamma(t)$ is nonempty and

$$\Gamma(t) = B_s \cap \Phi(t) \cap (U - A \dot{y}(t))$$

Let us make use now of the theory of measurable selectors, initiated in [2] and [10] . We are here in the simple case of a multifunction whose values are closed subsets of the metrizable separable complete space B_s (a convenient account of this case may be found in [34]) . A necessary and sufficient condition for the multifunction to possess a dense collection of measurable selectors is that its graph, i.e.

$$G = \{ (t,m) \in [0,T] \times B_s \ : \ m \in \Gamma(t) \}$$

belong to the σ-algebra $\mathcal{L} \otimes \mathcal{B}$ generated by the products of Lebesgue-measurable subsets of $[0,T]$ by Borel subsets of B_s.

Let us establish this property separately for the two multifunctions $t \mapsto B_s \cap \Phi(t)$ and $t \mapsto B_s \cap (U - A \dot{y}(t))$, as G is the intersection of their respective graphs.

Concerning the first one, we observe that in the definition (7.2) of $\Phi(t)$, the function $w \mapsto \psi_C^*(w)$ is l.s.c. for $\sigma(\mathcal{M}, \mathcal{C})$; besides, $t \mapsto y + c + g$ is continuous on $[0,T]$ for the norm topology of $\mathcal{C}$; therefore $(t,w) \mapsto \ll y + c + g, w \gg$ is continuous from $[0,T] \times B_s$ into $\mathbb{R}$. Consequently the graph of $t \mapsto B_s \cap \Phi(t)$ is a closed subset of $[0,T] \times B_s$, thus a member of $\mathcal{L} \otimes \mathcal{B}$.

On the other hand, by the definition of U , a couple (t,m) belongs to the graph of $t \mapsto U - A \dot{y}(t)$ if and only if

$$\int m + \dot{y}(t) \Big\| A = 0 \ .$$

As $m \mapsto \int m$ is continuous from B_s into $\mathbb{R}$ and as $t \mapsto \dot{y}(t)$ is Lebesgue-measurable from $[0,T]$ into $\mathbb{R}$, one easily sees that this graph belongs to $\mathcal{L} \otimes \mathcal{B}$.

We observe at this stage that our problem will possess in general an <u>infinity of solutions</u> ; <u>let us take as</u> $t \mapsto \dot{p}(t)$ <u>any one of the</u> $\sigma(\mathcal{M}, \mathcal{C})$ - <u>measurable selectors of</u> Γ. In view of $\Gamma(t)$ being contained in B_s for every t, this $\dot{p}$ is an element of the space $L^{\infty}_{\mathcal{M}_s}$ (cf. [1], Chap.6, § 2 n° 6), the dual of $L^1_{\mathcal{C}}$. Then p is constructed in accordance with (5.8)

$$p(t) = p(0) + \int_0^t \dot{p}(\tau)\, d\tau \tag{7.5}$$

(the weak integral is in fact an element of $\mathcal{M}$, see e.g. [1], Chap. 6, § 1 n° 4). The function $t \mapsto p(t) \in \mathcal{M}$ is Lipschitz ; it admits for almost every t a $\sigma(\mathcal{M}, \mathcal{C})$ - derivative equal to $\dot{p}(t)$.

7.d. CONSTRUCTION OF e AND $\dot{e}$

Recall that we constructed s according to (5.18), i.e.

$$s(t) = y(t) + c(t) + g(t) .$$

This defines a mapping $s : [0,T] \to \mathcal{C}$ which is Lipschitz relatively to the norm of $\mathcal{C}$, by virtue of Assumption 4.

In order to comply with (5.7) we take now $e(t) = A\, s(t)$. This defines a mapping $e : [0,T] \to \mathcal{M}$ which is Lipschitz relatively to the norm of $\mathcal{M}$. One concludes the existence of $\dot{e} \in L^{\infty}_{\mathcal{M}_s}$ such that

$$e(t) = e(0) + \int_0^t \dot{e}(\tau)\, d\tau . \tag{7.6}$$

This existence may be established as follows : The product of some time-independent element a of $\mathcal{C}$ by the characteristic function $\chi_{[t_1,t_2]}$ of some subinterval of $[0,T]$ yields an element u of $L^1_{\mathcal{C}}$. Denoting as before by $\ll . , . \gg$ the pairing between $\mathcal{C}$ and $\mathcal{M}$, put

$$\ell(u) = \ll a,\ e(t_2) - e(t_1) \gg \tag{7.7}$$

If λ is the Lipschitz constant of e, one has

$$|\ell(u)|_{\mathbb{R}} \leq \lambda \|a\|_{\mathcal{C}} (t_2 - t_1) \leq \lambda \|u\|_{L^1_{\mathcal{C}}} .$$

This inequality holds more generally when $u \mapsto \ell(u)$ is extended by linearity to the space of the <u>step-functions</u> from $[0,T]$ into $\mathcal{C}$. As this space is dense in $L^1_{\mathcal{C}}$, one can extend ℓ as an element of the dual of $L^1_{\mathcal{C}}$: let us define $\dot{e}$ as this element of $L^\infty_{\mathcal{M}_s}$. Then (7.6) immediately follows from (7.7), since the weak integral may be characterized by

$$\ll a, \int_0^t \dot{e}(\tau)\, d\tau \gg = \lll a\, \chi_{[0,t]}, \dot{e} \ggg$$

for every $a \in \mathcal{C}$, where $\lll . , . \ggg$ denotes the pairing between $L^1_{\mathcal{C}}$ and $L^s_{\mathcal{M}_s}$.

7.e. CONSTRUCTION OF u^- and u^+ .

Let us take q in accordance with (5.6) ; by addition it comes that

$$\dot{q} = \dot{e} + \dot{p}$$

is an element of $L^\infty_{\mathcal{M}_s}$ and that

$$q(t) = q(0) + \int_0^t \dot{q}(\tau)\, d\tau .$$

For every $t \in [0,T]$ we construct now u according to (4.3), (4.4) and (5.1) :

$$u^-(t,x) = \int_{[0,x[} q(t)$$

$$u^+(t,x) = \int_{[0,x]} q(t)$$

(the very value of u at some discontinuity point is immaterial : one may choose any value between u^- and u^+).

As (5.21) holds, thus also (5.14), one concludes that the boundary condition (5.2) is satisfied.

To verify that the final requirements of Parag. 5.a , are fullfilled is simply a matter of commuting integrations. Some results about weak vector integration (see [1], Chap. 6, § 1) might be used. More elementarily a sequence (φ_n) of elements of

$\mathcal{C}$ may be chosen, with $|\varphi_n| \leq 1$, converging pointwise to the characteristic function of the interval $[0,x[$ (resp. the interval $[0,x]$) ; then one applies the dominated convergence theorem repeatedly .

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