Analog Filter Design (EET 3132)

Lab 2: Design of Low Pass Filters with Maximally Flat Magnitude Response

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1 Introduction

An ideal filter gives us specifications of maximum pass band gain and flatness, minimum stop band attenuation and also a very steep pass band to stop band roll-off (the transition band) and it is therefore apparent that a large number of network responses would satisfy these requirements.

Not surprisingly then that there are a number of "approximation functions" in linear analogue filter design that use a mathematical approach to best approximate the transfer function we require for the filters design.

Such designs are known as Elliptical, Butterworth, Chebyshev, Bessel, Cauer as well as many others. Of these five "classic" linear analogue filter approximation functions only the Butterworth Filter and especially the low pass Butterworth filter design will be considered here as its the most commonly used function.

There are mainly three considerations in designing a filter circuits. They are

- The response of the pass band must be maximum flatness.
- There must be a slow transition from pass band to the stop band.
- The ability of the filter to pass signals without any distortions within the pass band.

These distortions are generally caused by the phase shifts of the waveforms. In addition to these three the rising and falling time parameters also play an important role. By taking these considerations for each consideration one type of filter is designed. For maximum flat response the Butterworth filter is designed. For slow transition from pass band to stop band the Chebyshev filter is designed and for maximum flat time delay Bessel filter is designed.

1.1 Low Pass Butterworth Filter Design

The frequency response of the Butterworth Filter approximation function is also often referred to as "maximally flat" (no ripples) response because the pass band is designed to have a frequency response which is as flat as mathematically possible from 0Hz (DC) until the cut-off frequency at -3dB with no ripples. Higher frequencies beyond the cut-off point rolls-off down to zero in the stop band at 20dB/decade or 6dB/octave. This is because it has a "quality factor", "Q" of just 0.707.

However, one main disadvantage of the Butterworth filter is that it achieves this pass band flatness at the expense of a wide transition band as the filter changes from the pass band to the stop band. It also has poor phase characteristics as well. The ideal frequency response, referred to as a "brick wall" filter, and the standard Butterworth approximations, for different filter orders are given below.

1.2 Ideal Frequency Response for a Butterworth Filter

The Figure. 1 shows the ideal frequency response of Butterworth filters. Note that the higher the Butterworth filter order, the higher the number of cascaded stages there are within the filter design, and the closer the filter becomes to the ideal "brick wall" response.

In practice however, Butterworth's ideal frequency response is unattainable as it produces excessive passband ripple.

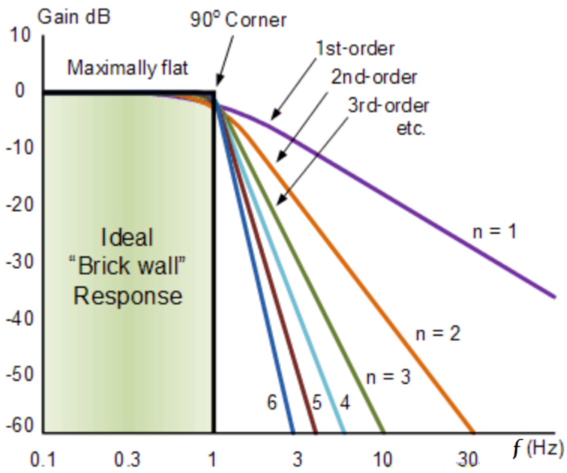

Figure 1: Butterworth Filter Response with varying order

The generalized equation representing a "n^{th}" Order Butterworth filter, the frequency response is given as:

$$H(j\omega) = \frac{1}{\sqrt{1+\epsilon^2(\frac{\omega}{\omega_p})^{2n}}} \tag{1}$$

Where: n represents the filter order, ω is equal to $2\pi f$ and ϵ is the maximum pass band gain, A_{max}. If A_{max} is defined at a frequency equal to the cut-off -3dB corner point f_c, ϵ will then be equal to one and therefore ϵ^2 will also be one.

1.3 Normalised Low Pass Butterworth Filter Polynomials

To help in the design of his low pass filters, Butterworth produced standard tables of normalised second-order low pass polynomials given the values of coefficient that correspond to a cut-off corner frequency of 1 radian/sec. This has been listed in Table 1.

n	**Normalised Denominator Polynomials in Factored Form**
1	$(1+s)$
2	$(1+1.414s+s^2)$
3	$(1+s)(1+s+s^2)$
4	$(1+0.765s+s^2)(1+1.848s+s^2)$
5	$(1+s)(1+0.618s+s^2)(1+1.618s+s^2)$
6	$(1+0.518s+s^2)(1+1.414s+s^2)(1+1.932s+s^2)$
7	$(1+s)(1+0.445s+s^2)(1+1.247s+s^2)(1+1.802s+s^2)$
8	$(1+0.390s+s^2)(1+1.111s+s^2)(1+1.663s+s^2)(1+1.962s+s^2)$
9	$(1+s)(1+0.347s+s^2)(1+s+s^2)(1+1.532s+s^2)(1+1.879s+s^2)$
10	$(1+0.313s+s^2)(1+0.908s+s^2)(1+1.414s+s^2)(1+1.782s+s^2)(1+1.975s+s^2)$

Table 1: Normalised Low Pass Butterworth Filter Polynomials

$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
−0.7071068	−0.5000000	−0.3826834	−0.8090170	−0.2588190	−0.9009689	−0.1950903	−0.9396926	−0.1564345
±j0.7071068	±j0.8660254	±j0.9238795	±j0.5877852	±j0.9659258	±j0.4338837	±j0.9807853	±j0.3420201	±j0.9876883
	−1.0000000	−0.9238795	−0.3090170	−0.7071068	−0.2225209	0.5555702	−0.1736482	−0.4539905
		±j0.3826834	±j0.9510565	±j0.7071068	±j0.9749279	±j0.8314696	±j0.9848078	±j0.8910065
			−1.0000000	−0.9659258	0.6234898	−0.8314696	−0.5000000	−0.7071068
				±j0.2588190	±j0.7818315	±j0.5555702	±j0.8660254	±j0.7071068
					−1.0000000	−0.9807853	−0.7660444	−0.8910065
						±j0.1950903	±j0.6427876	±j0.4539905
							−1.0000000	−0.9876883
								±j0.1564345

Table 2: Pole Locations Butterworth Polynomials of different orders

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	1.0000000	1.4142136								
3	1.0000000	2.0000000	2.0000000							
4	1.0000000	2.6131259	3.4142136	2.6131259						
5	1.0000000	3.2360680	5.2360680	5.2360680	3.2360680					
6	1.0000000	3.8637033	7.4641016	9.1416202	7.4641016	3.8637033				
7	1.0000000	4.4939592	10.0978347	14.5917939	14.5917939	10.0978347	4.4939592			
8	1.0000000	5.1258309	13.1370712	21.8461510	25.6883559	21.8461510	13.1370712	5.1258309		
9	1.0000000	5.7587705	16.5817187	31.1634375	41.9863857	41.9863857	31.1634375	16.5817187	5.7587705	
10	1.0000000	6.3924532	20.4317291	42.8020611	64.8823963	74.2334292	64.8823963	42.8020611	20.4317291	6.3924532

Table 3: Coefficients of Butterworth Polynomial $B_n(s) = s^n + \sum_{k=1}^{n-1} a_k s^k$

n even								*n odd**						
2	4	6	8	10	12	14	16	3	5	7	9	11	13	15
0.71	0.54	0.52	0.51	0.51	0.50	0.50	0.50	1.00	0.62	0.55	0.53	0.52	0.51	0.51
	1.31	0.71	0.60	0.56	0.54	0.53	0.52		1.62	0.80	0.65	0.59	0.56	0.55
		1.93	0.90	0.71	0.63	0.59	0.57			2.24	1.00	0.76	0.67	0.62
			2.56	1.10	0.82	0.71	0.65				2.88	1.20	0.88	0.75
				3.20	1.31	0.94	0.79					3.51	1.41	1.00
					3.83	1.51	1.06						4.15	1.62
						4.47	1.72							4.78
							5.10							

* For n odd there is also a real pole for which $Q = 0.5$.

Table 4: Q of Butterworth poles

2 Steps for Designing Butterworth filters

Requirement

$$\alpha_{max}, \alpha_{min}, \omega_p, \omega_s$$

Steps

I. Find normalized stopband frequency ω_{ns}

$$\omega_{ns} = \frac{\omega_s}{\omega_p} \tag{2}$$

II. Find ϵ

$$\epsilon = \sqrt{10^{0.1\alpha_{max}} - 1} \tag{3}$$

III. Find n

$$n \geq \frac{10log[(10^{0.1\alpha_{min}} - 1)/(10^{0.1\alpha_{max}} - 1)]}{2log[\omega_{ns}]} \tag{4}$$

IV. Find ω_B

$$\omega_B = \epsilon^{-1/n}\omega_p \tag{5}$$

V. List down the normalized pole and Q-factors from the Butterworth Design Tables (Table 2, 3 and 4). Find actual poles by multiplying ω_B with them.

VI. Write down the expression for the transfer function $T(s)$ as per the following.

For $n =$ odd,

$$T(s) = \frac{(\omega_B)^n}{(s + \omega_B)\prod_{k=1}^{(n-1)/2}(s^2 + \frac{\omega_B}{Q_k}s + \omega_B^2)} \tag{6}$$

For $n =$ even,

$$T(s) = \frac{(\omega_B)^n}{\prod_{k=1}^{n/2}(s^2 + \frac{\omega_B}{Q_k}s + \omega_B^2)} \tag{7}$$

3 Assignments

3.1 Assignment 01

Design a low pass filter with a maximally flat magnitude Response with the given specifications.

$$\alpha_{max} = 0.5dB, \alpha_{min} = 20dB, \omega_p = 1000rad/s, \omega_s = 2000rad/s$$

(a) Derive the expression for the transfer function $T(s)$.

(b) Using MATLAB generate the pole-zero plot and the frequency response of the corresponding transfer function $T(s)$.

   i. MATLAB Program
   ii. Pole - Zero Plot
   iii. Frequency Response / Bode Plot

(c) From the results obtained from MATLAB answer the following.

   i. Does the filter satisfy the specifications cited in this question?
   ii. What is the slope of the response in the stopband?

Solution

3.2 Assignment 02

we wish to design an LPF with the following specifications.

$$\alpha_{max} = 0.5dB, \alpha_{min} = 30dB, \omega_p = 1000rad/s, \omega_s = 2330rad/s$$

(a) Determine the degree n of the required maximally flat filter.

(b) Determine the poles in the s-plane and quality factor of each pole.

(c) Determine the actual loss $\alpha_{min}(\omega_p)$ and $\alpha_{max}(\omega_s)$ at the passband and stopband corners.

(d) Using MATLAB generate the pole-zero plot and the frequency response of the corresponding transfer function $T(s)$.

   i. MATLAB Program
   ii. Pole - Zero Plot
   iii. Frequency Response / Bode Plot

(e) From the results obtained from MATLAB answer the following.

   i. Does the filter satisfy the specifications cited in this question?
   ii. What is the slope of the response in the stopband?

Solution

3.3 Assignment 03

we wish to design an LPF with the following specifications.

$$\alpha_{max} = 1dB, \alpha_{min} = 35dB, \omega_p = 1000rad/s, \omega_s = 3500rad/s$$

(a) Determine the degree n of the required maximally flat filter.

(b) Determine the poles in the s-plane and quality factor of each pole.

(c) Determine the actual loss $\alpha_{min}(\omega_p)$ and $\alpha_{max}(\omega_s)$ at the passband and stopband corners.

(d) Using MATLAB generate the pole-zero plot and the frequency response of the corresponding transfer function $T(s)$.

  i. MATLAB Program
  ii. Pole - Zero Plot
  iii. Frequency Response / Bode Plot

(e) From the results obtained from MATLAB answer the following.

  i. Does the filter satisfy the specifications cited in this question?
  ii. What is the slope of the response in the stopband?

Solution

3.4 Assignment 04

we wish to design an LPF with the following specifications.

$$\alpha_{max} = 1dB, \alpha_{min} = 35dB, \omega_p = 1000rad/s, \omega_s = 3500rad/s$$

(a) Determine the degree n of the required maximally flat filter.

(b) Determine the poles in the s-plane and quality factor of each pole.

(c) Determine the actual loss $\alpha_{min}(\omega_p)$ and $\alpha_{max}(\omega_s)$ at the passband and stopband corners.

(d) Using MATLAB generate the pole-zero plot and the frequency response of the corresponding transfer function $T(s)$.

  i. MATLAB Program
  ii. Pole - Zero Plot
  iii. Frequency Response / Bode Plot

(e) From the results obtained from MATLAB answer the following.

  i. Does the filter satisfy the specifications cited in this question?
  ii. What is the slope of the response in the stopband?

Solution

3.5 Assignment 05

Write a MATLAB program which takes the specifications as the inputs and provides the Filter Transfer Function as the output. The structure of the program should be as given below.

```
[a, b] = myButterworth(alpha_max, alpha_min, wp, ws)
```

where,

$$\begin{aligned} \texttt{a} &= \text{denominator co-efficient vector} \\ \texttt{b} &= \text{numerator co-efficient vector} \quad . \end{aligned}$$

Use the data of Q1, Q2, Q3, and Q4 to test this program.

4 Explain what you have learned in this assignment set.

5 References used

1.
2.