Uplifting Excursion (vault)

BY JAKUB RADOSZEWSKI (POLAND)

Let $A = \sum_{\ell=-M}^{M} A_\ell$ and denote by $\mathcal{A}$ the collection of all art pieces. We assume that $A_0 = 0$ since we can otherwise simply add all art pieces with uplift 0 to the solution.

Subtask 1. $M, A_\ell \leq 50$

For the first subtask, we keep a DP table of size $\mathcal{O}(M \cdot A)$ where we save for every possible total uplift the maximum number of art pieces we can steal with that total uplift. Then, we add every art piece in linear time to the DP. Since there are A art pieces in total, this leads to a total runtime of $\mathcal{O}(M \cdot A^2)$ which passes the first subtask.

Subtask 2. $M, A_\ell \leq 100$

To solve the second subtask, we optimize the solution of the previous subtask by adding all art pieces with the same uplift ℓ more efficiently to the DP. There are several ways to do this.

For example, if we have three art pieces with uplift ℓ, we can replace them by one art piece each with uplift ℓ and 2ℓ respectively since both of these sets of art pieces yield exactly the same set of achievable total uplift values. By applying this transformation repeatedly to the A_ℓ art pieces with uplift ℓ, this will lead to $\mathcal{O}(\log A_\ell)$ many art pieces with uplift values $\ell, 2 \cdot \ell, 4 \cdot \ell, 8 \cdot \ell, \ldots$. By adding each of these art pieces in linear time to the DP, we get a solution with a runtime of $\mathcal{O}(M^2 \cdot A \log A)$.

There are also solutions which add all art pieces with uplift ℓ at once in linear time to the DP, yielding a runtime of $\mathcal{O}(M^2 \cdot A)$.

Subtask 3. $M \leq 30$

From this subtask on, there are several observations we have to do.

First, assume that there are uplift values $0 < \ell_1 < \ell_2 \leq M$ such that an optimal solution with total uplift L contains at most $A_{\ell_1} - M$ art pieces with uplift ℓ_1 and at least M art pieces with uplift ℓ_2. Then, we could replace ℓ_1 art pieces with uplift ℓ_2 from this solution by ℓ_2 art pieces with uplift ℓ_1. Because this would not change the total uplift of the solution but increase the total number of art pieces that we steal, it would follow that the original solution was not optimal, giving a contradiction.

Thus, there exists some uplift ℓ such that for all uplift values $0 < \ell_1 < \ell$ we take more than $A_{\ell_1} - M$ art pieces of uplift ℓ_1, and for all uplift values $\ell < \ell_2 \leq M$ we take less than M art pieces with uplift ℓ_2. This leads to the following solution to Subtask 3 for all testcases in which there are only art pieces with positive uplift.

At the beginning, we remove $B_\ell = \min(M - 1, A_\ell)$ art pieces with uplift ℓ from $\mathcal{A}$ and we order the remaining art pieces of $\mathcal{A}$ by uplift in increasing order. Let $\mathcal{B}$ be the set of the removed art pieces. Then, we know by the observation from above that every optimal solution can be constructed by taking a prefix $\mathcal{P}$ of the remaining art pieces of $\mathcal{A}$ (ordered by uplift) and adding a subset of the art pieces from $\mathcal{B}$ to it.

Since the sum of the uplift values of all art pieces from $\mathcal{B}$ is $\sum_{\ell=1}^{M} B_\ell \cdot \ell \leq M^3/2$, this can only yield a solution with total uplift L if the total uplift of $\mathcal{P}$ is already in the range $[L - M^3/2, L]$. If we fix the smallest prefix $\mathcal{P}$ of $\mathcal{A}$ whose total uplift is at least $L - M^3/2$ and the largest prefix $\mathcal{Q}$ of $\mathcal{A}$ whose total uplift is at most L, this means that every optimal solution can be constructed by taking all art pieces from $\mathcal{P}$, a subset of the art pieces from $\mathcal{Q} \setminus \mathcal{P}$, and a subset of the art pieces from $\mathcal{B}$. Thus, if we compute $\mathcal{P}$ and $\mathcal{Q}$, which can be done in $\mathcal{O}(M)$ time, and put all art pieces from $\mathcal{Q} \setminus \mathcal{P}$ and $\mathcal{B}$ into the DP from Subtask 1, we can obtain the maximal number of art pieces we can steal such that their total uplift is L. Because the sum of the uplift values of all art pieces from $\mathcal{Q} \setminus \mathcal{P}$ and $\mathcal{B}$ is at most $\mathcal{O}(M^3)$ and these sets contain at most $\mathcal{O}(M^3)$ art pieces in total, this results in a solution with runtime $\mathcal{O}(M^6)$.

If we have also art pieces with negative uplift, we can perform similar observations. If there were uplift values $\ell_2 < \ell_1 < 0$ such that an optimal solution contains at most $A_{\ell_1} - M$ art pieces with uplift ℓ_1 and at least M art pieces with uplift ℓ_2, the same argument as above applies. Moreover, if there were uplift values $\ell_2 < 0 < \ell_1$ such that an optimal solution contains at most $A_{\ell_1} - M$ and $A_{\ell_2} - M$ art pieces with uplift ℓ_1 and ℓ_2 respectively, we could again increase the number of stolen art pieces without changing the total uplift by adding ℓ_1 art pieces with uplift ℓ_2 and $-\ell_2$ art pieces with uplift ℓ_1 to our solution.

This implies that if we also remove $B_\ell = \min(M-1, A_\ell)$ art pieces from $\mathcal{A}$ for every negative uplift ℓ, then every optimal solution will either use all remaining art pieces with negative uplift from $\mathcal{A}$ and a prefix $\mathcal{P}$ of the art pieces with positive uplift, or all remaining art pieces with positive uplift and a prefix $\mathcal{P}$ of the art pieces with negative uplift. For both cases, we can use the same DP as above to obtain a solution with runtime $\mathcal{O}(M^6)$.

Subtask 4. $M \leq 50$

In this subtask, we use the same solution as in Subtask 3, but we compute it slightly differently. We first create a DP with the art pieces from $\mathcal{B}$ only. Then, we simply iterate through every prefix $\mathcal{P}$ as above whose total uplift T falls in the range $[L - M^3/2, L + M^3/2]$ and check whether this prefix can be extended to a solution with total uplift L by checking in our DP whether the difference $L - T$ can be achieved with the art pieces from $\mathcal{B}$. Since each of these checks takes constant time and the computation of the DP with the art pieces from $\mathcal{B}$ can be done in time $\mathcal{O}(M^5)$ as in Subtask 1, this yields a solution with a runtime of $\mathcal{O}(M^5)$.

Subtask 5. $M \leq 100$

To solve Subtask 5, we can use the same solution as in Subtask 4 and apply the optimizations from Subtask 2. This yields solutions with a runtime of $\mathcal{O}(M^4 \log M)$ or $\mathcal{O}(M^4)$.

Subtask 6. No further constraints.

For the full solution, we use a different approach. First, we greedily take as many art pieces as we can without "exceeding" the total uplift L. This means that if total uplift of all our art pieces is at least L, we take all art pieces with negative uplift and the largest prefix of the art pieces with positive uplift such that the total uplift stays below L, and if this total uplift is less than L, we take all art pieces with positive uplift and the largest prefix of the art pieces with negative uplift such that the total uplift stays above L. Since both cases are symmetric, let us focus only the first case.

Consider any optimal solution and let $\mathcal{B}$ be a minimal set of art pieces we would have to add or remove to get from our greedy solution to the optimal solution. First, note that our greedy solution has a total uplift in the interval $[L - M, L]$. If the total uplift were smaller, we would have extended the prefix of the art pieces with positive uplift that we have taken. Moreover, when changing our greedy solution to the optimal solution, we can ensure that the total uplift always stays in the interval $[L - M, L + M]$. Indeed, if the current total uplift is smaller than L, there has to exist some change in $\mathcal{B}$ that increases the total uplift because all these changes together result in the total uplift L of the optimal solution, and performing this change will increase the current total uplift by at most M since every art piece has an uplift of at most M. Thus, after the change, the total uplift will still be in the interval $[L - M, L + M]$. A similar argument holds if the current total uplift is larger than L.

Moreover, assume that we hit the same total uplift ℓ twice while performing all changes from $\mathcal{B}$ according to the above algorithm. Then, we could simply delete all changes in-between. Clearly, this results in the same total uplift in the final solution. Even better, this can only increase the number of art pieces that we steal. Indeed, if we have removed some art pieces and added others to get back to ℓ, the fact that our original solution was constructed greedily ensures that the uplift values of the removed art pieces must have been smaller than the uplift values of the added art pieces, and since their total uplift must have been the same, we must have removed at least as many art pieces as we added. Thus, deleting all these changes cannot decrease the number of art pieces that we steal.

We may therefore assume that we hit no total uplift $\ell \in [L - M, L + M]$ twice, implying that we have to add or remove at most $2M$ art pieces to get from our greedy solution to the optimal solution. This means that we can keep a DP of size $\mathcal{O}(M^2)$ where we have to add or remove at most $\mathcal{O}(M)$ art pieces of every uplift ℓ to compute the maximal number of art pieces that we can steal such that their total uplift is exactly L. By adding each art piece in linear time, this would only yield a solution with runtime $\mathcal{O}(M^4)$, but using the optimizations from Subtask 2 gives a runtime of $\mathcal{O}(M^3 \log M)$ or $\mathcal{O}(M^3)$, solving the last subtask.

Note: Due to the high number of subtasks with similar constraints, the time limit was relatively tight in this problem. Thus, it might have been difficult to get inefficient implementations to run within the time limit. A careful choice of the DP size might therefore have been necessary to pass the given subtasks as described.