Association of Teachers of Mathematics

The Proving Ground – an introduction to mathematical proof

Forty easy-to-understand mathematics problems - twenty have been solved and twenty haven't

The Proving Ground #progroebook

Written and designed by Jonny Griffiths

Author of the Risps website *@maxhikorski* *www.jonny-griffiths.net*

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2A Vernon Street, Vernon House, Derby DE1 1FR

Please notify any errors to the author at hello@jonny-griffiths.net

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Proving Ground (noun):

A venue or project in which new technologies, methods or techniques are tested.

Wiktionary

Prove all things; hold fast that which is good.

Paul of Tarsus, 1 Thessalonians 5: 21

Some mathematics problems look simple, and you try them for a year or so, and then you try them for a hundred years, and it turns out that they're extremely hard to solve. There's no reason why these problems shouldn't be easy, and yet they turn out to be extremely intricate.

Andrew Wiles

***Mathematics is Open Source*.**

Vishal Salgotra

For Steve Russ and David Loyn

How to use this ebook

You will find here forty problems, each set at three different levels. **Level 1** will hopefully be possible for most people who try the problem. **Level 2** will be harder, and **Level 3** could be extraordinarily hard, so hard, in fact, that no one in the history of mathematics has yet been able to resolve it. There are clickable links that take you from one level to the next.

Each problem has a **Notes** page where the current progress on that question is discussed. You might come to these once you have tried the problem for yourself. There are also Learning Pages, that introduce different proof techniques, and other more whimsical sections that carry mathematical stories and anecdotes. The Home page is also the Contents page, and this has links to every part of the ebook.

You can start now by diving straight in with a problem, or by reading the Introduction, or by picking up on some advice from the Learning Pages. If you get stuck on a mathematical word, then there is a glossary at the back that will hopefully help.

Good luck!

Jonny

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Introduction

Hello, and welcome to this book, which offers you forty tricky, sometimes extremely tricky, mathematical problems to think about. Twenty of these have been resolved by mathematicians, sometimes after many years of work, but the other twenty at the moment remain unsolved, despite the attention of some seriously clever people down the years. Will you be able to spot the difference?

First questions first. ‘Who’s this book aimed at?’ The answer is, ‘People old and young who enjoy a good puzzle, who are curious about what a pure mathematician gets up to, and who wonder if they could join in themselves.’ This might well include students of pure mathematics who are working towards exams and maybe a career in the subject.

Pure mathematics is the world's best game. It is more absorbing than chess, more of a gamble than poker, and lasts longer than Monopoly. It's free. It can be played anywhere - Archimedes did it in a bathtub. **Richard J. Trudeau**

Pure mathematics is, in its way, the poetry of logical ideas. **Albert Einstein**

Maths tends to fall into either the pure camp or the applied camp, although the boundary can be fuzzy. Applied mathematics is understood by all of us, if only vicariously, as we drive across bridges and walk beneath skyscrapers and watch probes land on distant bits of rock. But what does a pure maths professor really do to earn his or her salt? A two-word partial answer might be, ‘Prove things.’ We can date when mankind started to do this at around 500-300 BC.

Mathematics effectively began when a few Greek friends got together to talk about numbers and lines and angles. **C. S. Lewis**

It’s what makes maths different from other subjects. Elsewhere, evidence can be amassed to support one theory or another, but some other, more compelling, argument might come along in time. In pure maths, the idea is to establish something forever. If you’re looking to make a bid for immortality, you’ve picked up the right book.

Mathematics is the supreme judge; from its decisions there is no appeal. **Tobias Dantzig**

‘What exactly do you mean,’ you ask, ‘by “proof”?’ It’s a good question.

How does one go about proving? We are trying to craft a ‘poem of reason’ that explains fully and clearly and satisfies the pickiest demands of logic, while at the same time giving us goosebumps. **Paul Lockhart**

‘But I don’t have a mathematical PhD,’ you might add, ‘or a degree in maths, or an A level either, for that matter.’

‘Good,’ I reply, ‘so you’re unlikely to have met these problems before, and you’ll come to them fresh.’

‘But if the top people can’t sort them out, how am I going to understand them?’

Well, I’ve had to pick my examples carefully, because you’re right, some of the questions left for future maths students to take on are really hard to understand using general knowledge. Like every subject, mathematics has its jargon, and it can be off-putting. The unsolved André–Oort conjecture, for example, runs like this:

Let S be a Shimura variety and let V be a set of special points in S. Then the irreducible components of the Zariski closure of V are special subvarieties.

What’s a variety, let alone a ‘Shimura variety’? What does ‘special’ mean here? Irreducible? Component? Closure? Presumably a subvariety is something within a variety, but that doesn’t help us much.

There’s no reason why we couldn’t all comprehend these new concepts if we were prepared to invest substantial time in reading, talking and exploring. But that’s not quite what this book’s about. I promise you that all of the problems here, however hard to solve they are, will be understandable to anyone who studied maths up to the age of 16 (if you need some gentle reminders along the way, I hope to supply these as well). Everyone can understand what’s required when they’re handed a mixed-up Rubik’s cube, but to work out how to unmix it can be hard. I hope the same will be true here.

Some of these questions are new, while others have occupied mathematicians for centuries.

The definition of a good mathematical problem is the mathematics it generates rather than the problem itself. **Andrew Wiles**

With that in mind, I hope that these problems, both the solved and the unsolved, are good ones.

A mathematician is a device for turning coffee into theorems. **Paul Erdős**

In that case the unsolved problems here have seen more coffee drunk in their pursuit than most, and no one has yet got as far as the washing up.

There's a passion for puzzles in the world at the moment; I'm here offering you the best puzzles going. A crossword is a relaxation for half an hour, and there's a sense of satisfaction when it's finished that's probably soon forgotten. The problems in this book, however, are all cultural milestones. Study them, and you join a worldwide community of people who've done the same, and maybe your sense of satisfaction will last a lifetime.

Without enlightenment, one is merely reduced to memorizing proofs. With enlightenment about a proof, its flow becomes clear and it can become an item of astonishing beauty. The need to memorize disappears because the proof has become part of your soul. **Herbert S. Gaskill**

That brain of mine is something more than merely mortal, as time will show. **Ada Lovelace**

One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers.
Heinrich Hertz

Each of the forty problems comes in three levels. **Level 1** I'll endeavour to make possible for everyone. **Level 2** will be within the grasp of maybe more than half of you. **Level 3**, however, carries no guarantees: it may be that no one has ever solved **Level 3** on many of these problems. When you try a **Level 3** task, try to use your intuition; is this one of the unsolved ones? You'll be able to check afterwards.

I should add that solving problems is only part of what mathematics is about.

In mathematics the art of proposing a question must be held of higher value than solving it.
Georg Cantor

The best outcome of this book might be that you come up with a brand-new conjecture of your own for future mathematicians to take on.

I invite you to try all of these problems. Just because a problem can't be solved completely doesn't mean that we can't at least make a start. When learning maths it's really important to 'have a go' rather than simply 'read about other people having a go'.

It is active experience in mathematics itself that can alone answer the question: what is mathematics?
Richard Courant

Mental acuity of any kind comes from solving problems yourself, not from being told how to solve them.
Paul Lockhart

For a lot of people, mathematics is associated with fear. There's no other subject that can turn into phobia quicker than mathematics, and that's a tragedy. If this book does something to turn that around, I'd say, 'About time.'

The important thing to remember about mathematics is not to be frightened. **Richard Dawkins**

It's worth remembering that teachers can be afraid of their students too.

There was a seminar that I was teaching and von Neumann was in the class. I came to a certain theorem, and I said it's not proved and it may be difficult. Von Neumann didn't say anything but after five minutes

he raised his hand. When I called on him he went to the blackboard and proceeded to write down the proof. After that I was afraid of von Neumann. **George Pólya**

So for the maths techniques shown here, it's okay to skip passages you find hard until the flow becomes manageable again.

This skipping is another important point. It should be done whenever a proof seems too hard or whenever a theorem or a whole paragraph does not appeal to the reader. In most cases he will be able to go on and later he may return to the parts which he skipped. **Emil Artin**

I can't promise you will understand every line of every proof, but I'm hoping you'll understand more than you expect. The challenge for me is to make any proofs crystal clear to you.

I'm someone who earned their daily crust by teaching maths in a sixth form college for many years. I guess it's possible that I myself will solve one of these problems; I'm still as passionate about maths as ever I was. But if I were a betting man, I would back younger mathematicians to make the breakthroughs required. My thinking may well have settled into ruts that sadly it would be too traumatic for me to jump out of now. But if you have an eager young mind, then it could be you; why not? It's my hope that this book will put ideas into your head.

‘And, most important of all,’ added the Mathemagician, ‘here is your own magic staff. Use it well and there is nothing it cannot do for you.’ He placed in Milo's breast pocket a small gleaming pencil which, except for the size, was much like his own. **Norton Juster**

Home

What Maths Will You Need?

For **Levels** **1** and **2**

Included:

Whole numbers, powers, factors, primes, rational/algebraic/ transcendental numbers, simple Cartesian graphs, factorials, binomial coefficients, simple graph theory, Pythagoras's Theorem, factorising, expanding, intuitive limits, simple programming, algebra, logic.

These things are explained in the text and in the glossary.

Excluded:

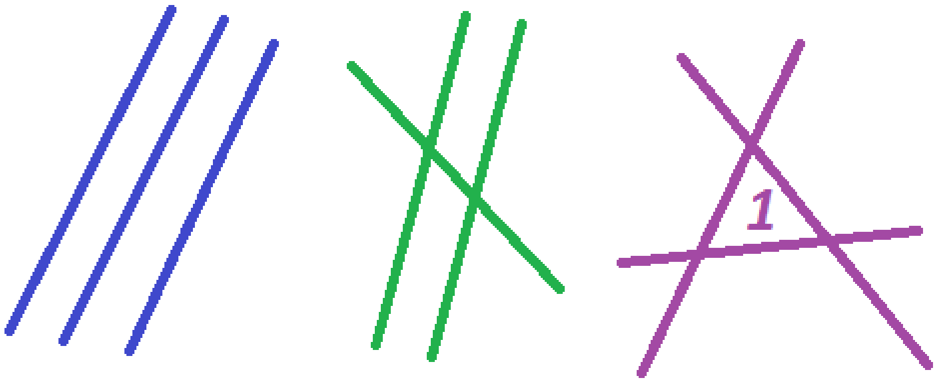
Trigonometry, logarithms, calculus, formal limits, advanced geometry, complex numbers, recurrence relations.

For **Level 3**: you could need anything!

Home

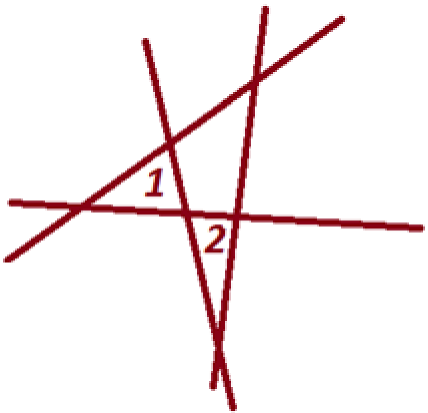
1: The Lines-into-Triangles Problem

Level 1: Start with a blank piece of paper. You aim to make triangles, as many as you can, and these triangles have to be non-overlapping. Suppose you are only allowed to draw three straight lines.



If you pick your straight lines to be ***parallel*** (left), you can't make a triangle, and the same happens if a pair is parallel (middle). If no pair is parallel, you get one triangle (right), and that's the best you can do.

What if you can use four lines?



You can add the extra line to get two triangles. Remember, they must be non-overlapping. Try as you might, two is the best you can do.

What's the best you can do with five lines?

Level 2 Notes Home

2: The Partition Problem

You can write **6 = 1 + 1 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 3 = 3 + 3 = 1 + 5.** So there are four ways to write 6 as the sum of positive odd numbers (you ARE allowing repeats here!)

You can also write **6 = 1 + 5 = 2 + 4 = 1 + 2 + 3 = 6.** So there are four ways to write 6 as the sum of any positive whole numbers (but this time you are NOT allowing repeats!)

Level 1: Pick a positive whole number between 10 and 20, call it k, and work out

1. how many ways you can write k as the sum of positive odd numbers (repeats allowed) and
2. how many ways you can write k as the sum of any positive whole numbers (repeats not allowed).

What do you discover?

Level 2 Notes Home

3: The Twin Prime Problem

What is a prime number?

Level 1: Write down the prime numbers between 1 and 50. What are the possible gaps between prime numbers here?

So between 13 and 17 there is a gap of 4.

There's a gap of 2 between...which primes?

We call pairs where there is a gap of 2 between primes, like 3 and 5, ***twin primes.***

Level 2 Notes Home

4: The Arithmetic Simultaneous Equations Problem

...1, 3, 5, 7, 9, 11...

...-16, -5, 6, 17, 28, 39....

...78, 76, 74, 72, 70, 68...

Each of the above sequences is called ***arithmetic*** - they change by a constant amount (which can be negative) each time you go one term to the right.

Level 1: Can you find an arithmetic sequence that has $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ as three of its terms? *(The terms will not be consecutive!)*

Level 2 Notes Home

5: The Sum of single powers, Sum of cubes Problem

Level 1: Pick a whole number between 5 and 20, let's call it a.

Work out $1 + 2 + 3 + \ldots + a$, call this **S.**

Now work out $1^3 + 2^3 + 3^3 + \ldots + a^3$**,** call this **C**.

Do you notice anything?

Level 2 Notes Home

6: The Triangle of 1s Problem

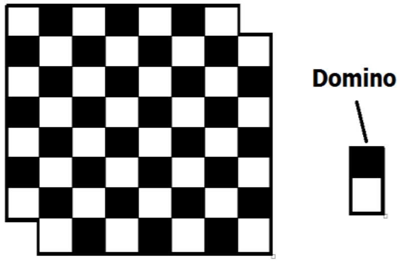
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
. . .
. . .
. . .

Level 1: How is this triangle of numbers constructed? Can you find the next three rows?

Level 2 Notes Home

7: First Triomino Tiling Problem

Level 1: There's a famous tiling problem that goes like this: suppose you have the standard 8 by 8 chessboard, and you cut off opposite corners.

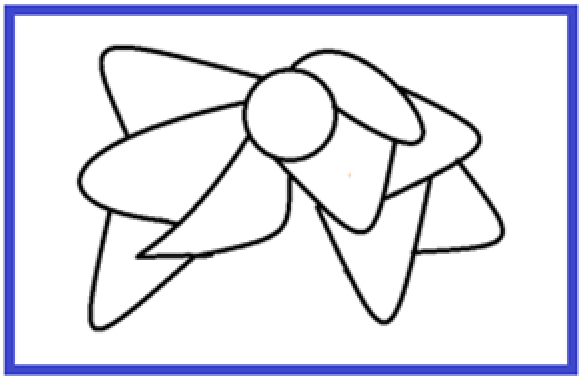


Can you cover what's left of the board with dominoes to leave no gaps?

Level 2 Notes Home

8: The Map-colouring Problem

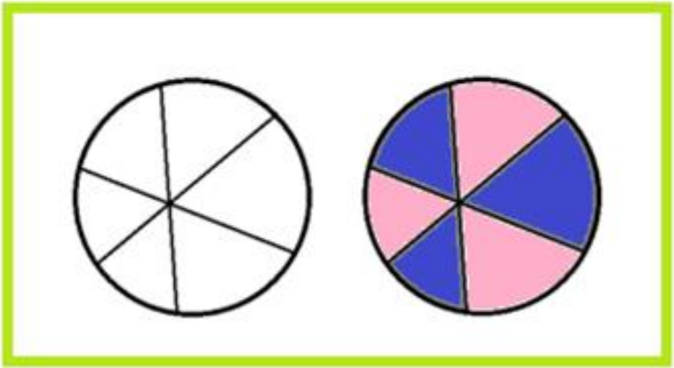
Level 1: On a piece of paper, draw out a map of ten imaginary countries that border each other in some way.
For example;



Try to colour the map so that no two countries that share a border have the same colour.
(Don't worry about the outside region.)

Put numbers into countries to stand for colours if you wish.

What's the smallest number of colours needed?



Warning!

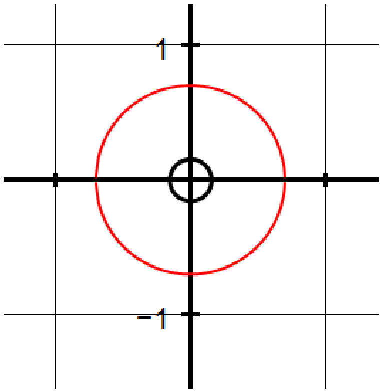
In the map above, the six countries are not said to all border each other; only two colours are necessary.

A border must be a line, however short, and not a point.

Level 2 Notes Home

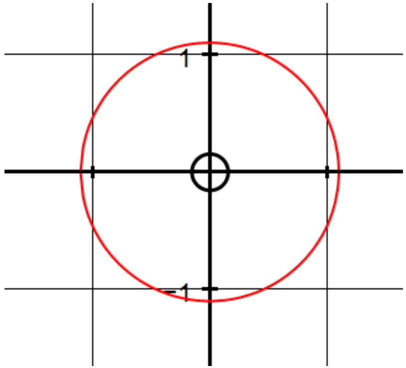
9: The Points-in-a-Circle Problem

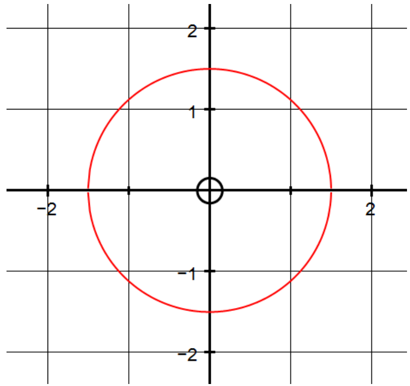
Level 1: Imagine you have a grid (or lattice) made up of squares of side 1 unit.
Lattice points occur where the grid lines meet.
Create an origin, and centre a small circle (in red below) there, radius r.



How many grid points does the circle contain? (Points that are ON the circle don't count).

Now you are going to let r increase, and we'll monitor the number of contained lattice points as you go. What happens to the count of lattice points as r increases just beyond 1?





r just bigger than 1

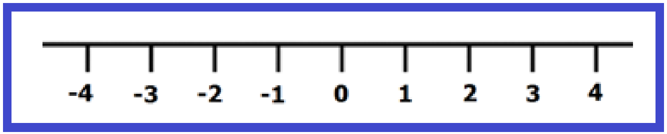
r just bigger than $\sqrt{2}$

What does the lattice point count jump to next? For which value of r?

Level 2 **Notes** **Home**

10: The Is-Every-Number-a-Fraction Problem

Level 1: You know that a fraction is a number that you can write as a/b, where a and b are whole numbers and b is not zero (the technical maths word for a number that can be written as a fraction is a ***rational*** number).



Contemplate this number line for a minute, a line that might be the x-axis if you were dealing with coordinate geometry.

Given two whole numbers, can you always find a whole number that lies between them?

Level 2 Notes Home

11: The Handshake Problem

Level 1: Suppose Person A throws a dinner party for himself and five guests, let's call them B, C, D, E and F.

Some of the guests are shyer than others,

so you can't assume that everyone shakes hands with everyone else;

indeed, it could be there are no handshakes at all.

No pair shakes hands more than once.

What's the maximum number of handshakes there could be?

Level 2 Notes Home

12: The How-Many-Tilings Problem

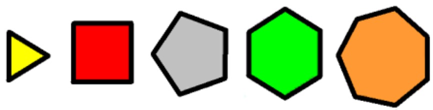
A *polygon* is a shape with straight line sides, and a *regular polygon* is one with all angles equal, and all sides equal.

Level 1: Which regular polygons tile the plane?

In other words, if you're given an endless supply of tiles

shaped like the regular n-sided polygon,

for which values of n can you tile the plane without gaps?



Level 2 Notes Home

13: The Lonely Runner Problem

Level 1: You have 8 runners, who all start running together on a track.

The track is circular and of length 1 unit.
All the runners run at different constant speeds, measured in units/hour,
and they run theoretically forever(!)
If someone has run 17.135 units, their position on the track is said to be 0.135.

Let's pick a time; say 0.264 hours.
Pick eight permitted integer speeds for your eight runners,
and find out where they are on the track after 0.264 hours.

Level 2 Notes Home

14: The Odd Perfect Number Problem

Level 1: Let's define s(n) as the sum of all the factors of n (except for n itself) added together.

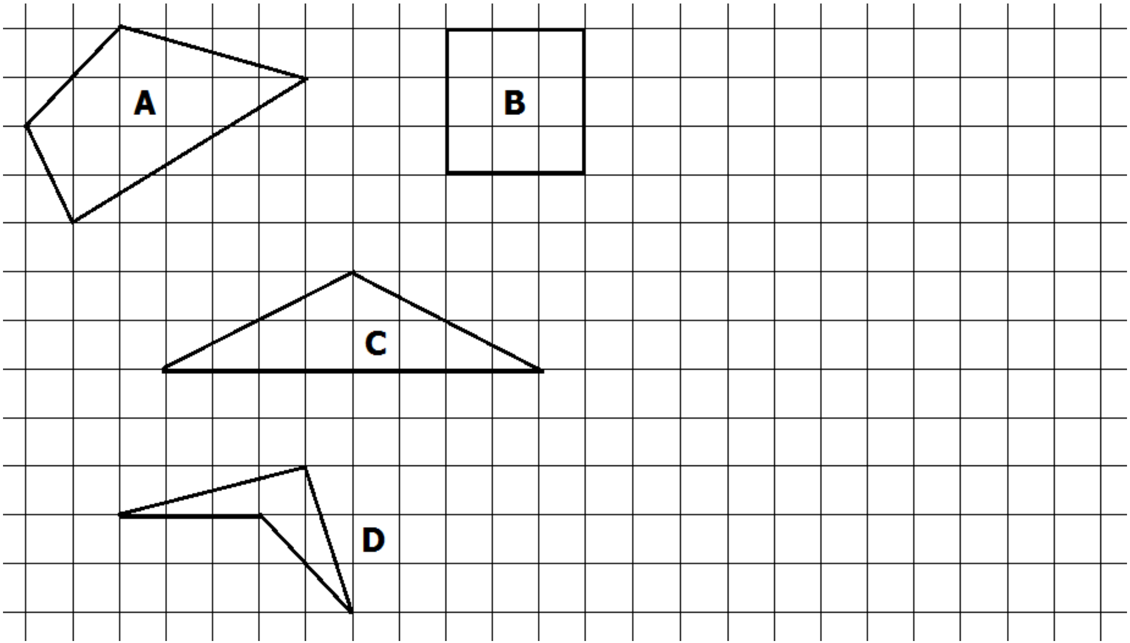
So 12 has the factors 1, 2, 3, 4, 6 and 12,

and **s(12) = 1 + 2 + 3 + 4 + 6 = 16.**

Can you work out s(n) for n from 1 to 30, putting your results into a table?

Level 2 Notes Home

15: The Dots and Area Problem



Level 1: You are given four shapes on a grid.

You might like to copy this, and add some more shapes to the grid if you wish,

as long as they are shapes with straight sides (polygons)

with corners that lie on the square grid intersections (lattice points).

What are the areas of your shapes?

Level 2 Notes Home

16: The How-Many-Primes-Are-There Problem

Level 1: List the primes from 1 to 100. How many are there?

It is tempting to ask the question, when does this list stop?

Some extremely big primes have been discovered, some with over ten million digits. But then, ten million digits is nothing compared to infinity.

It’s true that primes get more spread out as your numbers get larger.

What does your intuition tell you; is there a largest prime?

Level 2 Notes Home

17: The HOTPO problem

HOTPO = Halve Or Treble Plus One

Level 1: Pick a whole number between 10 and 20.

If it's even, halve it. If it's odd, multiply by 3 and then add 1.

Now repeat the process, again and again.

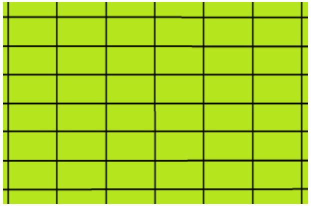
For example, 5 goes to 16, goes to 8, goes to 4, goes to 2,

goes to 1, goes to 4, and then you are in a loop.

What do you find for your choice of number?

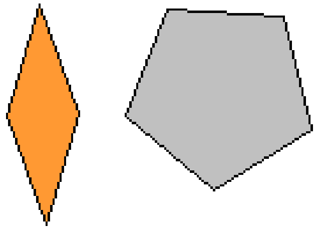
Level 2 Notes Home

18: The One Tile Problem



A ***tessellation*** is called ***periodic*** if we can lift it up, shift it and put it down again (without rotating it) so that it sits exactly on top of itself (see left).

Level 1: Can you make a periodic tiling using these two tiles? A rhombus with angles 144° and 36°, and a regular pentagon (with 108° angles).



Level 2 Notes Home

19: The Are-The-Infinities-Equal problem

The set of the natural numbers **0, 1, 2, 3, 4...** (we call this $\mathbb{N}$), is clearly infinite. Any set of numbers that can be put into an infinite list (one that pairs off with the natural numbers) is called ***countably infinite.*** For example, the non-negative even numbers are countably infinite.

0	**1**	**2**	**3**	**4**	**5**	**6...**
0	**2**	**4**	**6**	**8**	**10**	**12...**

Level 1: Is the set of numbers **...-3, -2, -1, 0, 1, 2, 3...** of positive and negative whole numbers (that is, the set of integers, $\mathbb{Z}$) countably infinite?

Level 2 ***Notes*** ***Home***

20: The Even = Prime + Prime Problem

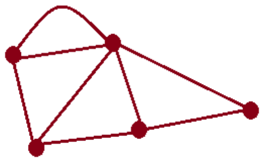
Level 1: Write down the even numbers from 20 to 30;

can you write each of them as the sum of two primes?

Level 2 Notes Home

21: The Dot-Line-Region Problem

Level 1: Take a piece of paper and draw some dots (vertices). Now join them with lines (edges) to form a graph, so that no part of the graph is isolated (it's possible to travel from any dot to any other along edges - such graphs are called ***connected***). For example;



Call your graph Graph 1, and draw a table like the one below. Count the number of vertices V, the number of edges E and the number of regions R (count the infinite region on the outside as one region), and fill in your values. Repeat this for a second graph Graph 2.

Graph	V	E	R
Graph 1			
Graph 2			

Are V, E and R connected in similar ways for Graphs 1 and 2?

Level 2 Notes Home

22: The Three Averages Problem

Given two positive numbers, there are several methods to find what you might call their ‘average’.

Method 1: $\dfrac{a+b}{2}$ (the Arithmetic Mean, or AM)

Method 2: $\sqrt{ab}$ (the Geometric Mean, or GM)

Method 3: $\dfrac{1}{\left(\dfrac{\frac{1}{a}+\frac{1}{b}}{2}\right)}$ (the Harmonic Mean, or HM)

Suppose you pick a = 9 and b = 25.

Method 1: $(9 + 25)/2 = 17 =$ AM, **Method 2**: $\sqrt{(9\times25)} = 15 =$ GM,

Method 3: $1/9 + 1/25 = 34/225$, divided by 2 is $17/225$,
and $1/(17/225)= 225/17 = 13.235... =$ HM.

Level 1: Show that the HM of a and b can be written as

$$\frac{2ab}{a+b}.$$

Find the AM, the GM and the HM for the numbers 3 and 6.

If you order these three averages by size,

in which order do the AM, GM and HM come?

Level 2 Notes Home

23: The How-Many-Powers Problem

Take the square numbers, **0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100...**

20 = 1 + 1 + 9 + 9, so you can write 20 as the sum of four squares.

Can you reduce this number?

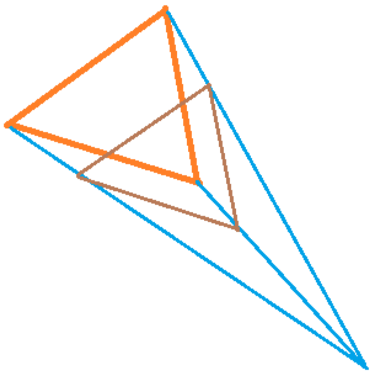
In fact, **20 = 4 + 16**, so you can reduce it to two.

Level 1: Express the whole numbers from 0 to 20 as sums of squares, finding the minimum number possible each time.

Level 2 Notes Home

24: The Shape-Covering Problem

Level 1: Draw a triangle, let's call it Triangle A, then make a smaller copy of it, Triangle B, like so:

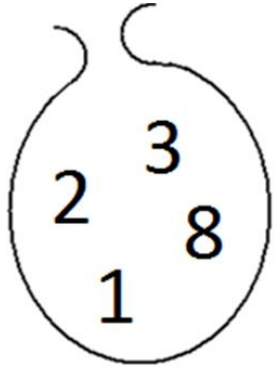


Triangle B can be any size, as long as it's smaller than your starting one.
What's the minimum number of copies of B you need to cover A?
(You're not allowed to rotate your copies of Triangle B!)

Level 2 Notes Home

25: The Four-in-a-Bag Triple Problem

$$\bigcirc\, x + \bigcirc = \bigcirc\, x + \bigcirc$$

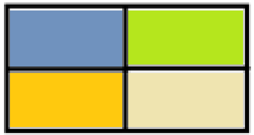


Level 1: Put the numbers in the bag into the circles in some order (no repeats!) and solve the resulting equation for x. How many different solutions can you get as you vary the order of your four numbers?

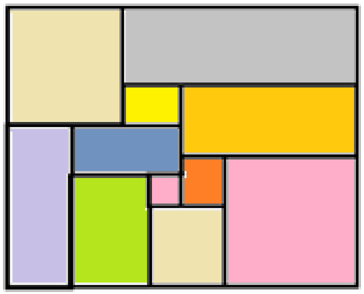
Level 2 Notes Home

26: The Squaring the Square Problem

You can always cut a rectangle with integer sides
into smaller rectangles that also have integer sides.



It gets harder if you insist that all the smaller rectangles are different!



Level 1: Can you find a rectangle with integer sides
that you can cut up
into smaller rectangles with integer sides that are all different?

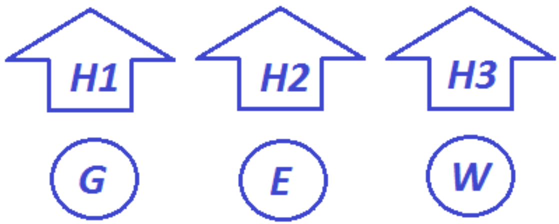
Level 2 Notes Home

27: The Factor Graph Problem

Level 1: You’re given three houses.

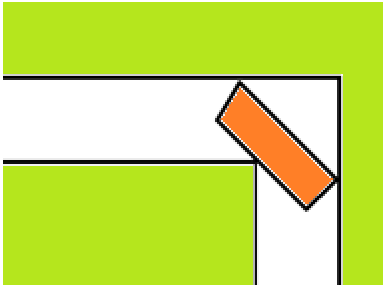
Can you connect them to the gas, the electricity and the water supplies

so that no pipe or cable crosses any other pipe or cable?



Level 2 Notes Home

28: The Moving Sofa Problem



You have the task of manoeuvring a sofa, which does not have to be rectangular, around an L-shaped band in a corridor. Let's say for simplicity that the corridor is 1 unit wide on both sides. You are restricting yourself to 2 dimensions here, so you're not allowed to put the sofa up on a slant.

Level 1: If the sofa is shaped like a circle, what's the largest area of sofa that you can successfully get around the bend?

Level 2 Notes Home

29: The Infinite Sum of Fractions Problem

Level 1: Have a think about the sum

$$S_n = 1 + 1/2 + 1/4 + 1/8 + \ldots + 1/2^n.$$

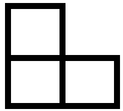
What happens to S_n as n gets bigger and bigger and bigger?

We're adding together infinitely many things,
so surely S_n must get infinitely big?
But you must remember, the things you are adding together
are getting infinitely small...

Level 2 Notes Home

30: Second Triomino Tiling Problem

Level 1: Take a standard chessboard, together with lots of tiles like this:



Can you tile the board with these in a way that leaves no gaps?

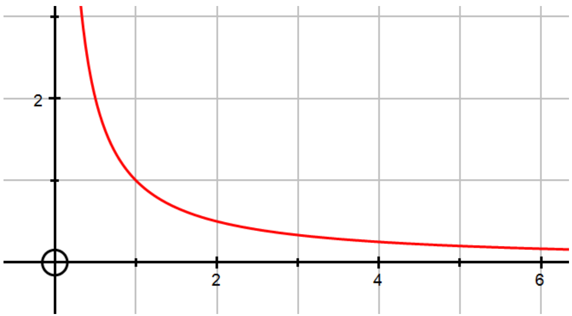
Level 2 Notes Home

31: The y = 1/x Number Problem

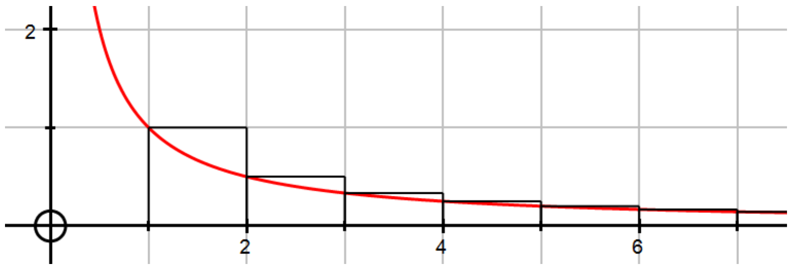
Level 1: Start with the curve y = 1/x. You draw up this table:

x	1/2	1	2	3	4	5	6
y	2	1	1/2	1/3	1/4	1/5	1/6

and then plot the points to get this:

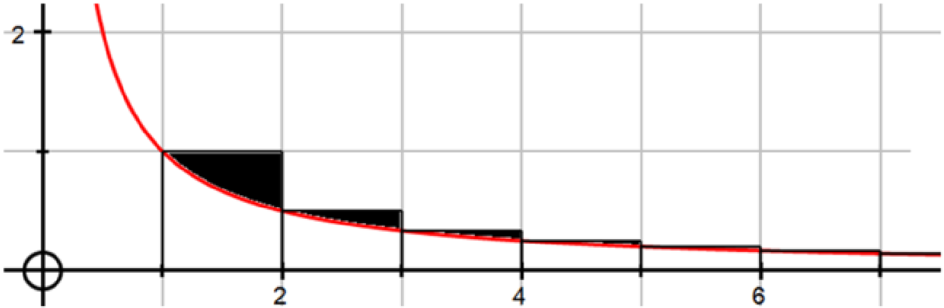


Now you add some rectangles, like this:



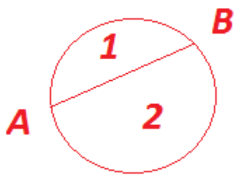
Think about the black area shaded below (let's call it γ). Show

$$\frac{1}{2} < \gamma < 1.$$



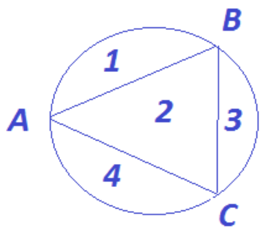
Level 2 Notes Home

32: The Circle-Region Problem



Draw a circle, and pick two points, call them A and B on its edge. Draw the chord AB. We've made two regions (don't count the outside region).

Now add a point C, and connect to A and B.



You now have four regions. Keep adding points and drawing in all possible chords in such a way that you make the maximum number of regions possible each time (you can't have three lines meeting at a point within the circle, for example).

Level 1: How many regions do you get with four dots?

Level 2 Notes Home

33: The Transversals in a Latin Square Problem

This conjecture is about Latin Squares. You've encountered these if you've ever done a sudoku - the finished grid is an example of a 9 by 9 Latin square. The rule is simple; if you have

1. **n values, and**
2. **an n by n square, so that**
3. **every row contains each of the n possible values, and**
4. **every column contains each of the n possible values,**

then you have a Latin square of order n (the diagonals are not involved here).

So what does the word ***transversal*** mean? Take this small example of a latin square.

C	*W*	*B*	*T*
W	*C*	*T*	*B*
B	*T*	*C*	*W*
T	*B*	*W*	*C*

Can you pick four letters out of the sixteen so that you have C, B, T and W once each, with one letter in each row, and one in each column?

C	W	B	T
W	C	T	B
B	T	C	W
T	B	W	C

You can. This is what is meant by a transversal, this set of four entries containing the four different values.

Level 1: How many different latin squares are there of order 3?

Can you find them all, and give the number of transversals for each of them?

Level 2 Notes Home

34: The Inscribed Square Problem

This problem asks you to draw squares with corners that lie on a given curve.

Level 1: Given a circle, how many different squares can you draw inside it, so that all four corners of the square are on the circle?

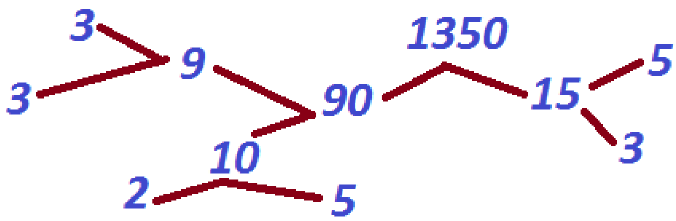
(Differently-positioned squares are counted as different here).

Suppose you now change ‘circle’ to ‘rectangle’?

How many different squares are possible now?

Level 2 Notes Home

35: The Same-Prime Trees Problem



Level 1: Pick a number between 1000 and 5000 that has lots of factors (you could do this by multiplying together small numbers until you have one between 1000 and 5000).

Then create a tree like the one above by splitting the starting number into two factors greater than 1 (if you can) time and time again, until eventually you're left with numbers at the end of the branches that can't be split any further (splitting a number into itself and 1 is not allowed).

What kind of numbers are you left with at the end of the branches?

Level 2 Notes Home

36: The Perfect Box Problem

A *cuboid* (or *brick*) is a name for a box with six rectangular faces.

A *cube* is a special case of a cuboid, when all six faces are identical.

Level 1: Given a cuboid with sides 3, 4 and 5 units,

can you find the lengths of the three diagonals for the three faces?

Level 2 Notes Home

37: The Friends and Strangers Problem

A number of people go to a party.

Every pair of people at the party either know each other, or they don't (they are either friends or strangers).

Let's now look at trios of people. It's possible for a chosen trio of guests to be all friends with each other, or all strangers to each other.

Level 1: Show that if there are four people in the room, there is no guarantee that there will be either a trio of friends, or a trio of strangers.

Level 2 Notes Home

38: The Minimum-Number-of-Points Problem

Level 1: Suppose you put four points onto a page, with no three in a straight line.

Is it guaranteed that these will make a convex quadrilateral when you join them up?

Level 2 Notes Home

39: The Cubes + Cubes = Cubes Problem

You know from Fermat's Last Theorem that $x^3 + y^3 = z^3$ has no natural number solutions beyond (x, y, z) = (0, 0, 0).

But could the equation $ax^3 + by^3 = cz^3$, where a, b and c are natural numbers, have natural number solutions for x, y and z?

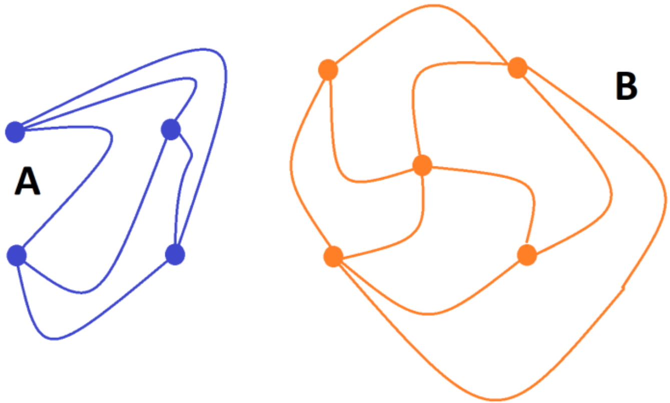
You could write any such solution as (a, b, c, x, y, z).

Level 1: Show (1, 1, 35, 2, 3, 1) is one solution,

and that (0, 1, 1, j, k, k), where j and k are any natural numbers, is another.

Level 2 Notes Home

40: The Integer Edge-Length Problem



A *graph* (a collection of dots on a page joined by edges) is ***planar*** if no edges cross.

Two examples A and B are above.

Level 1: Can you move the vertices for these graphs around, while preserving which dot is connected to which, keeping the graph planar, until all your edges could be straight lines?

The degree of a dot is the number of lines coming from it, so the degrees of the dots in your straight line diagram have to be the same as the degrees for your curved line version.

Level 2 Notes Home

1: The Lines-into-Triangles Problem

Level 2: As the number of lines gets bigger, the problem gets harder. You may know you have a good solution, but is it the best possible? Let's say n is the number of lines.

Note: to revise the use of algebra please go to Proving and Algebra.

What happens if n = 6 or 7? You have the beginnings of a table here. Let's call the maximum number of triangles that you can make with n lines N(n).

n	3	4	5	6	7	8
N(n)	1	2				

Try to fill in the gaps in the table for n up to 8.

Level 1 Level 3 Notes Home

2: The Partition Problem

Level 2: Try this with some other whole numbers; does this ever fail, or does it seem to work every time?

Level 1 Level 3 Notes Home

3: The Twin Prime Problem

Level 2: Write down all the twin primes between 1 and 100.

Is there any reason why we should eventually run out of pairs of primes that have a gap of 2 between them?

Find a table of primes on the internet and investigate.

Level 1 Level 3 Notes Home

4: The Arithmetic Simultaneous Equations Problem

Level 2: Pick six consecutive terms from an arithmetic sequence

and put them in order into the squares below.

$$\square\,x + \square\,y = \square, \quad \square\,x + \square\,y = \square$$

Now solve the pair of ***simultaneous equations*** you have created.

Level 1 | Level 3 | Notes | Home

5: The Sum of single powers, Sum of cubes Problem

Level 2: Try this task out

for a few more positive whole numbers.

How do **S** and **C** appear to be related?

Can you make a conjecture?

Level 1 Level 3 Notes Home

6: The Triangle of 1s Problem

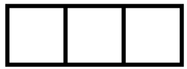
Level 2: The numbers in the triangle get bigger quickly as you move downwards.

Can you find a way of writing down the n^{th} row without having to write down all the rows that come before it?

Level 1 Level 3 Notes Home

7: First Triomino Tiling Problem

Level 2: Let’s refine this problem. Take the 8 by 8 chessboard again, with corners in place this time, and try to cover it with ***triomino*** tiles like this one:



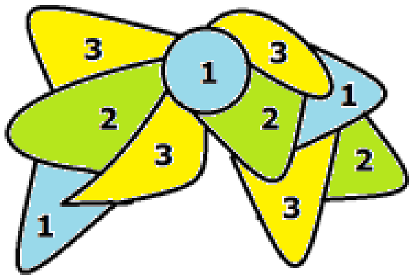
There are 64 squares on the chessboard, and 21 lots of 3 is 63, so the largest number of triominoes you can place onto the board is 21, with one square left over.

Can you find a tiling that leaves one square left over?

Level 1 Level 3 Notes Home

8: The Map-colouring Problem

Level 2: The example map in **Level 1** can be coloured with just three colours while obeying the rule that no two countries that share a border share a colour.



Are three colours going to be enough for any map?
Can you find a map that shows four are essential sometimes?

Level 1 *Level 3* *Notes* *Home*

9: The Points-in-a-Circle Problem

Level 2: Draw up a table for r and N(r), where N(r) is the number of lattice points inside the circle with radius r, as far as r just greater than 3.

Level 1 Level 3 Notes Home

10: The Is-Every-Number-a-Fraction Problem

Level 2: It should be clear that if you pick two consecutive whole numbers, you won't be able to find a whole number between them.

But what about if you pick two different fractions;
Is it always possible to find a fraction between the fractions you've picked?

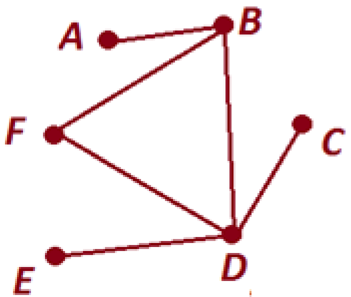
Given two rational numbers,
can you always find a rational number that lies between them?

If you marked in all the fractions on the number line,
would there be any 'gaps'?

Level 1 Level 3 Notes Home

11: The Handshake Problem

Level 2: You could draw a graph (a collection of dots and lines joining them) where you connect two dots if a handshake takes place between the two people the dots represent (of course, nobody shakes their own hand).



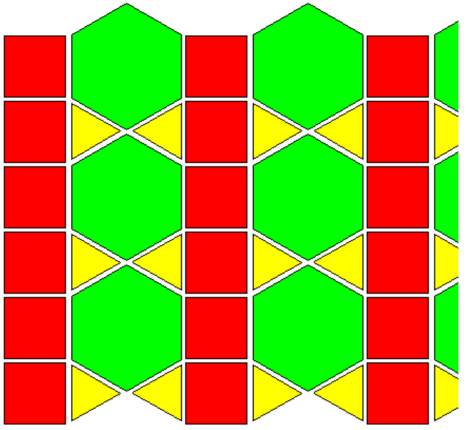
How many handshakes are there here?

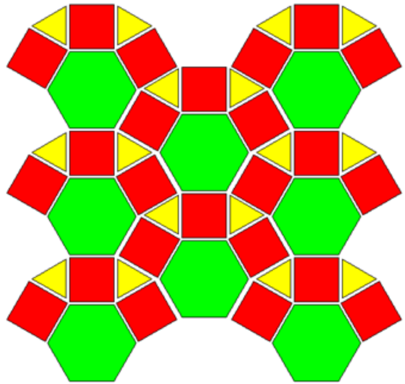
How is the total number of handshakes connected to the sum of the individual handshake counts?

Level 1 Level 3 Notes Home

12: The How-Many-Tilings Problem

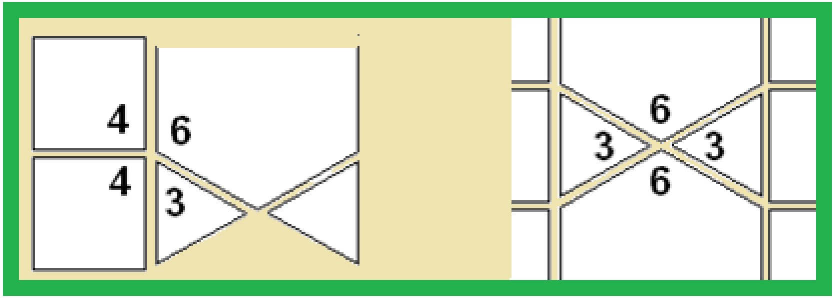
Level 2: Suppose you allow more than one regular polygon in a tiling? Here are two examples.





They're different tilings, but using the same regular tiles.

Consider the combinations of tiles around corners in the left-hand tiling.



There are two combinations, (3, 4, 4, 6) and (3, 6, 3, 6).

For the right-hand tiling, on the other hand, there's only one combination at every point, (3, 4, 6, 4).

Can you find another tiling that combines different regular polygons, but which has the same combination at every point, like the right-hand tiling above?

Level 1 Level 3 Notes Home

13: The Lonely Runner Problem

Level 2: You have eight runners at various points on your track when t = 0.264.

Concentrate on one particular runner from your eight
who is k units around the track at this time.

Is there at least one other runner in the interval $\left(k-\frac{1}{8}, k+\frac{1}{8}\right)$?

(With n runners, the fraction would be 1/n).

If your answer is 'No', then your runner is said to be ***lonely*****.**

Are any of your eight runners lonely when t = 0.264?

Level 1 Level 3 Notes Home

14: The Odd Perfect Number Problem

Level 2: Sometimes $s(n) < n$, sometimes $s(n) > n$, and sometimes $s(n) = n$.

It seems hard for $s(n)$ to exceed n if n is an odd number.
Can you find an odd number n so that $s(n) > n$?

Which values of n from 1 to 30 give $s(n) = n$?

Such numbers are called ***perfect*.** They don't come along often.
The next one in the list is 496 – can you check this is perfect?

Now for something at Level 3 that looks completely different...

Level 1 Level 3 Notes Home

15: The Dots and Area Problem

Level 2: Call the number of dots (lattice points) completely inside each shape I, and the number of dots on the perimeter (the edge) of each shape P.

Draw up a table with four columns for Name of shape, I, P and Area, and fill in the values for the shapes you have.

Level 1 Level 3 Notes Home

16: The How-Many-Primes-Are-There Problem

Level 2: Find the primes in the interval from 1 001 to 1 100 inclusive.

(You could find a list of primes on the net to help you).

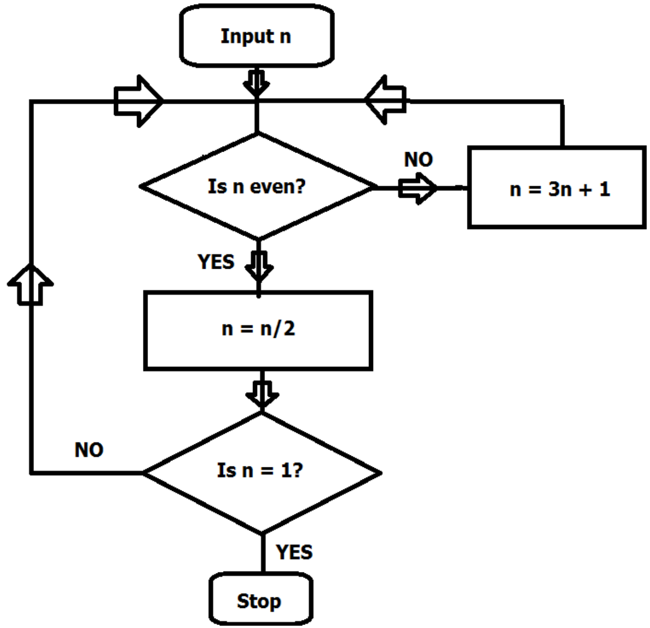
How many are there?

Compare the number of primes between 1 and 100 with the number of primes between 1 001 and 1 100, between 2 001 and 2 100, between 3 001 and 3 100, between 4 001 and 4 100, and between 5 001 and 5 100.

What are your conclusions?

Level 1 Level 3 Notes Home

17: The HOTPO problem

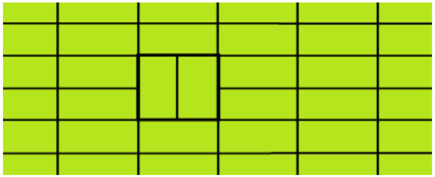


Level 2: Try this with several starting numbers.

Can you make a conjecture?

18: The One Tile Problem

Level 2: Sometimes a periodic tiling can be tweaked so that it becomes non-periodic.



So the 2 by 1 rectangle tile used above can tessellate both periodically and non-periodically. Can you make a non-periodic tiling with the rhombus/pentagon tiles from **Level 1**?

Level 1 Level 3 Notes Home

19: The Are-The-Infinities-Equal problem

Level 2: Are the rational numbers countably infinite?

Level 1 Level 3 Notes Home

20: The Even = Prime + Prime Problem

Level 2: How many different ways of writing 100 as the sum of two primes are there?

Level 1 Level 3 Notes Home

21: The Dot-Line-Region Problem

Level 2: Draw some more connected graphs, Graph 3, Graph 4 and so on, counting V, R and E each time, and extend your table.

Is there a connection between V, E and R that always seems to hold?

Can you make a conjecture?

Level 1 Level 3 Notes Home

22: The Three Averages Problem

Level 2: A car travels from A to B and back again.

From A to B it travels at x km/hr, while on the journey back it travels at y km/hr.

What is its average speed for the whole journey?

Which average for x and y, AM, GM or HM, have you used here?

Level 1 Level 3 Notes Home

23: The How-Many-Powers Problem

Level 2: If you write each of the numbers from 1000 to 1010 as a sum of squares,

what is the minimum number of squares you need to use each time?

You know some numbers need four squares; for example, **7 = 1 + 1 + 1 + 4**.

Will allowing four squares be enough for every positive whole number?

Or is there a point beyond which three squares will always do?

If you can write every positive whole number as the sum of at most n squares, what is n?

What if you ask the same question about cubes?

If every positive whole number can be written as the sum of at most m cubes, what is m?

Level 1 Level 3 Notes Home

24: The Shape-Covering Problem

Level 2: How would this work for a square?
How many smaller unrotated copies would you need to cover it?

What about rhombuses? Kites? Parallelograms?
Trapezia? Rectangles? Pentagons?
What about other two dimensional shapes?

Level 1 Level 3 Notes Home

25: The Four-in-a-Bag Triple Problem

Level 2: Pick any four numbers for the bag (as long as they are distinct).

Now say **S** is the set of possible solutions for the equation as you place your numbers from the bag into the circles in all possible orders without repeats.

What is the largest number of positive whole numbers **S** can contain?

If **S** contains the three whole number solutions

$$\mathbf{a, b, c \geq 1,}$$

then we'll say that **(a, b, c)** is a **Triple**.

(The Triple **(n, 1, 1)** is called the ***trivial Triple***.)

Is there an equation that links a, b and c if they form a Triple?

How common are Triples?

Could you answer this with a computer search?

Level 1 Level 3 Notes Home

26: The Squaring the Square Problem

Level 2: Now let's make it harder still.

Can you find a rectangle with integer sides
that you can cut up into SQUARES,
that are all different, and which all have integer sides?

Level 1 Level 3 Notes Home

27: The Factor Graph Problem

Level 2: Draw five dots on a page.

Can you connect every dot directly to every other dot so that lines don’t cross?

Level 1 Level 3 Notes Home

28: The Moving Sofa Problem

Level 2: If the sofa is shaped like a rectangle,
what's the largest area of sofa
that you can successfully get around the bend?

Level 1 Level 3 Notes Home

29: The Infinite Sum of Fractions Problem

Level 2: Now think about T_n =

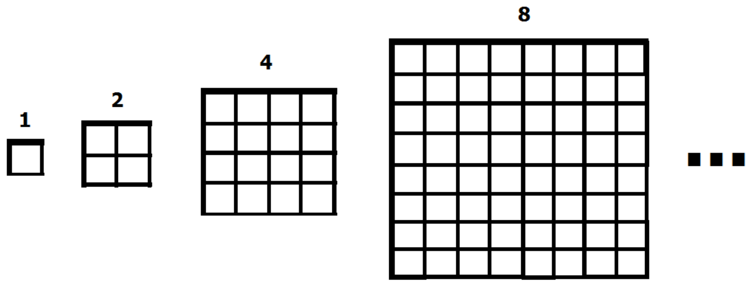
$$\frac{1}{1}+\frac{5}{8}+\frac{4}{8}+\frac{11}{27}+\frac{10}{27}+\frac{9}{27}+\frac{19}{64}+\frac{18}{64}+\frac{17}{64}+\frac{16}{64}+\frac{29}{125}+\ldots+\frac{n^2}{n^3}.$$

What happens to T_n as n gets bigger and bigger?

Level 1 Level 3 Notes Home

30: Second Triomino Tiling Problem

Level 2: Now consider all grids with side-lengths that are a power of 2.



Can you tile any of these grids using the L-shaped tile leaving no gaps?

Level 1 Level 3 Notes Home

31: The y = 1/x Number Problem

Level 2: Can you think of a way to improve on these upper and lower bounds for γ?

Level 1 Level 3 Notes Home

32: The Circle-Region Problem

Level 2: How many regions do you get with five dots?

Can you draw up a table?

n = # of dots	1	2	3	4	5	6
R = # of regions	1	2	4			

Level 1 Level 3 Notes Home

33: The Transversals in a Latin Square Problem

Level 2: Can you find a latin square of order 4

that has no transversals?

Level 1 | Level 3 | Notes | Home

34: The Inscribed Square Problem

Level 2: Suppose you now change ‘rectangle’ to ‘triangle’?

How many different squares are possible now?

Level 1 Level 3 Notes Home

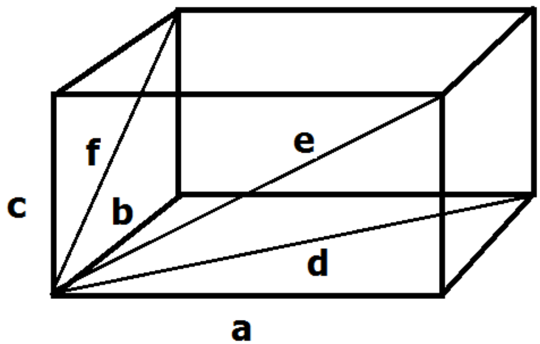
35: The Same-Prime Trees Problem

Level 2: Try creating two different trees starting with the same number, by splitting in different ways as you go.

Level 1 Level 3 Notes Home

36: The Perfect Box Problem

Level 2: An ***Euler Brick*** is a cuboid where all three lengths for the sides, and all three lengths for the diagonals of the faces, are whole numbers.



So in the diagram, a, b, c, d, e and f are all integers.

Can you find equations connecting the lengths here?

Can you find any solutions to these equations?

(A short computer program might help).

Do Euler bricks exist at all?

Level 1 Level 3 Notes Home

37: The Friends and Strangers Problem

Level 2: Show that if there are five people in the room, there is no guarantee that there will be either a trio of friends, or a trio of strangers.

What if there are six people in the room?

Will there be either a trio of friends, or a trio of strangers for sure now?

Can you prove it?

Level 1 Level 3 Notes Home

38: The Minimum-Number-of-Points Problem

Level 2: What if you add a fifth point; is it now guaranteed that some choice of four of your dots will make a convex quadrilateral?

Define g(n) as the minimum number of points that you need to draw on a page to guarantee you can choose n of these to form an n-sided convex polygon.

A table will help. What can you fill in here?

n	3	4	5	6
g(n)				

Level 1 Level 3 Notes Home

39: The Cubes + Cubes = Cubes Problem

Level 2: Can you find any more solutions (a, b, c, x, y, z),

where none of x, y and z are 1,

and where x, y and z have no common factor?

Level 1 Level 3 Notes Home

40: The Integer Edge-Length Problem

Level 2: Can you always find a straight line version for a planar graph?

Or are there planar graphs out there that are so complicated

that you cannot come up with a straight line version

by moving the vertices around sensibly?

Level 1 Level 3 Notes Home

1: The Lines-into-Triangles Problem

Level 3: Can you find a ***formula*** (in terms of n) for N(n), the maximum number of non-overlapping triangles you can make using n lines?

Level 1 Level 2 Notes Home

2: The Partition Problem

Level 3: Can you come up with a ***conjecture*** based on your findings?

Can you prove it?

Level 1 Level 2 Notes Home

3: The Twin Prime Problem

Level 3:

You can see that as primes get larger and more spread out, there will be fewer such twin prime pairs.

But will they ever stop altogether?

Are there infinitely many twin primes?

Can you show there are?

Level 1 Level 2 Notes Home

4: The Arithmetic Simultaneous Equations Problem

Level 3: Can you make a ***conjecture*** based on this?

Can you prove it?

Level 1 *Level 2* *Notes* *Home*

5: The Sum of single powers, Sum of cubes Problem

Level 3: Does this always work?

Can you come up with a ***conjecture***?

Can you prove this? Or find a ***counterexample***?

Can you find a ***formula*** for **S**, and so for **C**?

Level 1 Level 2 Notes Home

6: The Triangle of 1s Problem

Level 3: How many times can a number be repeated in this triangle?

Clearly the number 1 appears an infinite number of times,
but that's not especially interesting.

There are two 3s in the third row; can we find any more 3s lower down?

Surely two 3s is as many as we get,
since the numbers below must all be bigger than 3.

We can counts three 6s,
and then four 10s if we go down as far as the tenth row.

Is there a limit to the number of times a number can appear?

If I give you a number k, can you always find some number m so that m appears more than k times in the triangle?

Or is there some number k where you can be sure that no number appears more than k times in this triangle?

Level 1 Level 2 Notes Home

7: First Triomino Tiling Problem

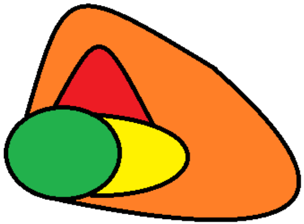
Level 3: Can this left-over square be anywhere?

If you pick any square on the board,
can you find a tiling of triominoes
that leaves this square and this square alone blank?

Level 1 Level 2 Notes Home

8: The Map-colouring Problem

Level 3: It’s not too hard to find a map where four colours are essential.



Can you find a map where you must use five colours?

Where you must use six colours?
What is the smallest number of colours that will be enough for *any* map?

Can you prove it?

Level 1 Level 2 Notes Home

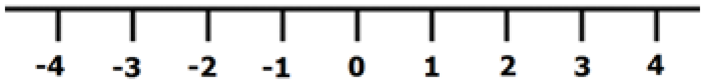
9: The Points-in-a-Circle Problem

Level 3: Can you come up with a ***formula*** in terms of r that gives you the number of lattice points inside the circle with radius r?

Can you prove the formula always works?

Level 1 **Level 2** **Notes** **Home**

10: The Is-Every-Number-a-Fraction Problem

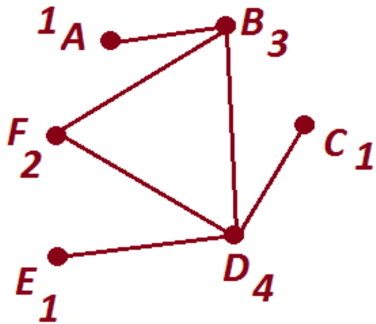


Level 3: Are there any numbers on this line that are not fractions?

How about the number ***'the square root of 2'***?
Can you write **√2** as a fraction, or is that impossible?

Level 1 Level 2 Notes Home

11: The Handshake Problem



Level 3: You can add 'the handshake count' to each dot (left). You notice here that A's handshake count is the same as E's (and the same as C's too, in this case).

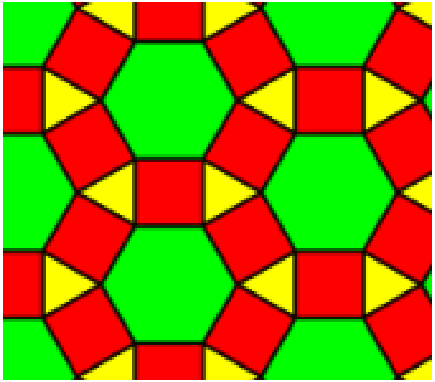
So suppose you know n people enter a room, and some handshakes take place. No one shakes their own hand, and no one shakes anybody's hand more than once. Is there always some pair of people who have shaken hands the same number of times?

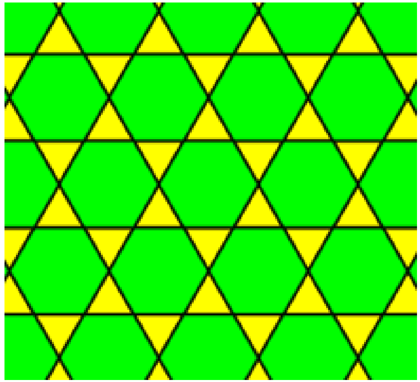
Can you prove your conjecture? Or find a counterexample?

Level 1 Level 2 Notes Home

12: The How-Many-Tilings Problem

Level 3: It's possible to find tilings of the plane that employ a mix of regular polygons, but which have the same combination of tiles around each vertex. Here are examples;





Can you find all possible tilings that fit this description?

Level 1 Level 2 Notes Home

13: The Lonely Runner Problem

Level 3: You can generalise the situation to n runners.

So if one of your n runners is at position k units on the track,
then she is ***lonely*** at some time t if
the interval $\left(k-\dfrac{1}{n}, k+\dfrac{1}{n}\right)$ is empty of other runners.

Will there always be some value of t for each runner for which they are lonely?

Or is it possible for one runner to never be lonely?

Can you prove this one way or the other?

Level 1 Level 2 Notes Home

14: The Odd Perfect Number Problem

Level 3: A number of the form $2^n - 1$, for some whole number n, is called a ***Mersenne* number**.

Which of the first ten Mersenne numbers are prime?
Can you come up with a conjecture? Can you prove it?

Suppose 2^n-1 is prime, and $m = 2^{n-1}(2^n-1)$. What will s(m) be?

Can an odd perfect number exist? What would it look like?

Level 1 Level 2 Notes Home

15: The Dots and Area Problem

Level 3: Working with the values of I and P,

can you find a general formula for the area of any such polygon?

Can you prove your formula to be true?

Level 1 Level 2 Notes Home

16: The How-Many-Primes-Are-There Problem

Level 3: Can you prove that the list of primes goes on forever?

Can you prove that the list is infinite?

Or might the list of primes get so thin that it eventually just stops?

In which case, how many prime numbers are there?

Level 1 Level 2 Notes Home

17: The HOTPO Problem

Level 3: Does every starting number end up in the same loop?

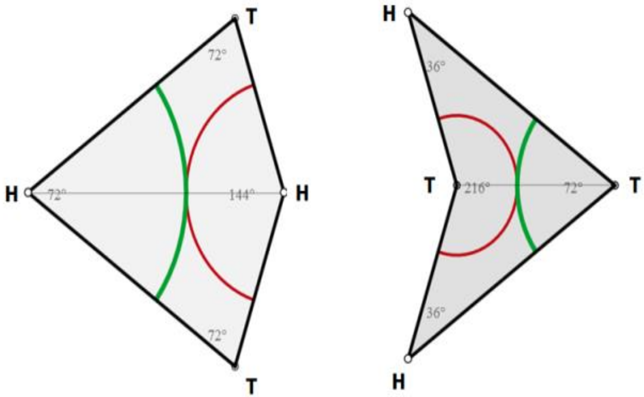
Or is there more than one loop?

Or do some numbers never end up in a loop?

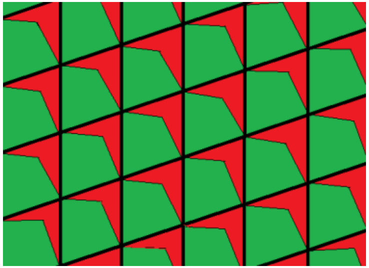
Level 1 Level 2 Notes Home

18: The One Tile Problem

Level 3: The question arises, is there a tile or a set of tiles so that EVERY infinite tiling of the plane they make is non-periodic? Roger Penrose came up with a pair of tiles that accomplish this in 1974. He called them *the kite and the dart*.

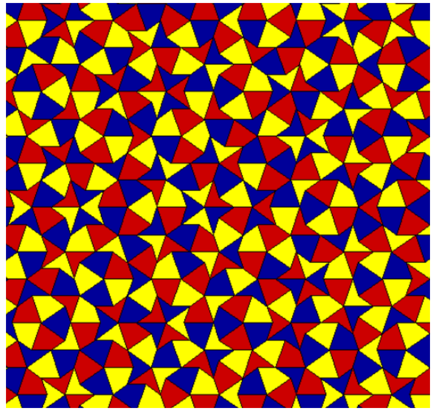


The kite and dart shapes can be used on their own and together to generate periodic tilings, like the green and red tiling below…



But we can add a matching rule that stop tilings like this from being legal. We'll say the kites and darts can only be placed with the Hs together and the Ts together (see above). With this matching rule, every infinite tiling that they make is non-periodic (see below right).

There are other pairs of shapes that always give non-periodic tilings too. Is there a SINGLE tile that tiles only non-periodically? Is there a tile that covers the plane on its own, where every such tiling is non-periodic?



19: The Are-The-Infinities-Equal problem

Think about the set of all the numbers on the part of the number line from 0 to 1. This includes all of the rational numbers, like 2/3, and all of the irrational numbers, like $\sqrt{2} - 1$, $\pi - 3$ and $e - 2$.

Level 3: Is the set of numbers on the number line from 0 to 1 countably infinite? Can you put them into an infinite list? Or is the infinity of numbers on the number line between 0 and 1 a different size of infinity altogether?

Level 1 Level 2 Notes Home

20: The Even = Prime + Prime Problem

Level 3: If you run the primes down one side of a spreadsheet, and then across the top row, you can get Excel to do some adding.

+	2	3	5	7	11	13	17	19	23	29	31	37
2	4	5	7	9	13	15	19	21	25	31	33	39
3	5	6	8	10	14	16	20	22	26	32	34	40
5	7	8	10	12	16	18	22	24	28	34	36	42
7	9	10	12	14	18	20	24	26	30	36	38	44
11	13	14	16	18	22	24	28	30	34	40	42	48
13	15	16	18	20	24	26	30	32	36	42	44	50
17	19	20	22	24	28	30	34	36	40	46	48	54
19	21	22	24	26	30	32	36	38	42	48	50	56
23	25	26	28	30	34	36	40	42	46	52	54	60
29	31	32	34	36	40	42	46	48	52	58	60	66
31	33	34	36	38	42	44	48	50	54	60	62	68
37	39	40	42	44	48	50	54	56	60	66	68	74

Does every even number appear somewhere in the white part of the grid?

Starting with 4, 6, 8, 10..., you can see them all, as far as 62,

somewhere in there, sometimes in more than one place (10 = 5 + 5 = 3 + 7).

Can you write every even number bigger than 2

as the sum of two prime numbers?

Or can you find an even number that can't be split in this way?

Can you prove your conjecture? Or find a counterexample?

Level 1 Level 2 Notes Home

21: The Dot-Line-Region Problem

Level 3:

Take your conjecture as to how V, R and E are linked;

can you prove this to be true?

Or can you find a counterexample?

Level 1 Level 2 Notes Home

22: The Three Averages Problem

Level 3: In your calculations for Level 1, AM $\geq$ GM $\geq$ HM.

Is this always true, for all pairs of positive numbers?

Can you find a case when it is not true?

Can you prove any conjectures you have made?

Level 1 Level 2 Notes Home

23: The How-Many-Powers Problem

Level 3: Suppose we're dealing with k^{th} powers, where k can be 4, 5, 6...

Let g(k) be the minimum number of k^{th} powers you need

to construct any positive whole number.

Can you find a formula for g(k)?

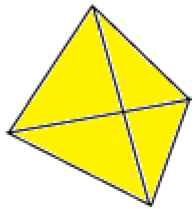
Or is it possible that for some value of k,

there is no minimum number of k^{th} powers that will do?

Level 1 Level 2 Notes Home

24: The Shape-Covering Problem

Level 3: What happens if you work in three dimensions? How many smaller copies are needed to fill the tetrahedron?



How about a cube? A dodecahedron? Can you put together a conjecture for what might happen in general? In two dimensions? In three dimensions? In n dimensions? Can you prove it for n = 2 or 3?

Level 1 Level 2 Notes Home

25: The Four-in-a-Bag Triple Problem

Level 3: You can show that (7, 5, 3) and (11, 4, 3) are both Triples.

Now **$7 \times 5 \times 3 = 105$, $11 \times 4 \times 3 = 132$,** and **$105 \neq 132$.**

In fact, computer checking suggests that whenever you multiply the three elements of a non-trivial Triple together, you get a number given by no other triple.

So you might pose the Triple Uniqueness Conjecture:

if (a, b, c) and (p, q, r) are non-trivial Triples
so that abc = pqr, then (a, b, c) = (p, q, r).

Is this true? Can you prove it?

Level 1 *Level 2* *Notes* *Home*

26: The Squaring the Square Problem

Level 3: One last increase in difficulty:

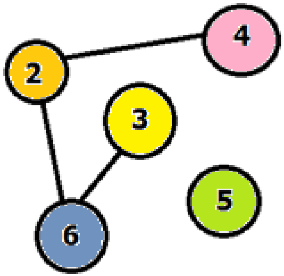
can you cut up a SQUARE with an integer side into squares,

that are all different, and which all have whole number sides?

Level 1 Level 2 Notes Home

27: The Factor Graph Problem

Level 3: Take a fresh bit of paper. Start by writing down ‘2’. Add consecutive whole numbers to the diagram in order, joining up numbers where one is a factor of another. So 2 and 3 both connect to 6, and 2 connects to 4.



Keep adding successive whole numbers to the diagram, obeying the rule that lines aren’t allowed to cross (the graph must be planar). How far can we get obeying these rules?

Can you keep on adding whole numbers, or will there come a point where you have to stop?

Can you formulate a conjecture? Can you prove it?

Level 1 Level 2 Notes Home

28: The Moving Sofa Problem

Level 3: If the sofa can take any 2D shape at all,
what's the largest area of sofa
that you can successfully get around the bend?

Level 1 Level 2 Notes Home

29: The Infinite Sum of Fractions Problem

Level 3: Now think about

$U_n = 1 + 1/2 + 1/3 + 1/4 + 1/5 \ldots + 1/n$, and

$V_n = 1/2 + 1/3 + 1/5 + 1/7 \ldots + 1/p_n$,

where p_n is the n^{th} prime number.

What happens to U_n and V_n as n gets larger and larger?

Do they grow without limit? Or do they home in on some finite number?

Level 1 Level 2 Notes Home

30: Second Triomino Tiling Problem

Level 3: What will be the smallest number of squares you can leave untiled with your best possible tiling for one of these $2^n \times 2^n$ boards?

Is it possible to leave just one square blank each time?

If so, can you leave this blank square anywhere on the grid,
or is it limited to a certain number of positions?

Level 1 Level 2 Notes Home

31: The y = 1/x Number Problem

Level 3: How big is this area γ exactly?

Is this number γ rational or irrational?

Level 1 | Level 2 | Notes | Home

32: The Circle-Region Problem

Level 3: What happens for larger n?

Can you find a formula for R in terms of n?

(Beware of saying, 'This is obvious!')

Level 1 Level 2 Notes Home

33: The Transversals in a Latin Square Problem

Level 3: Pick a value for n, and imagine all the possible latin squares for that value of n (as n gets larger, the number of possible latin squares gets big quickly).

Now imagine working out **T(n) = number of transversals in a Latin Square** for each of these.

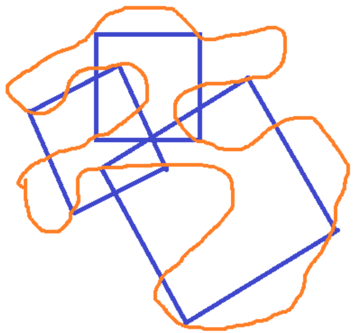
What is the maximum value that **T(n)** can be?

Can you come up with a formula for **T(n)** that works for all n?

Level 1 Level 2 Notes Home

34: The Inscribed Square Problem

Level 3:



Now draw a closed curve on a piece of paper, any closed curve so that the curve doesn't cut itself. How many squares can we draw so that all four corners of the square are on the curve?

The diagram shows one possible closed curve with three inscribed squares.

Is it always possible to draw at least one square on every such curve?

Level 1 Level 2 Notes Home

35: The Same-Prime Trees Problem

Level 3: When you create different trees starting with the same number,

what do you notice about the numbers

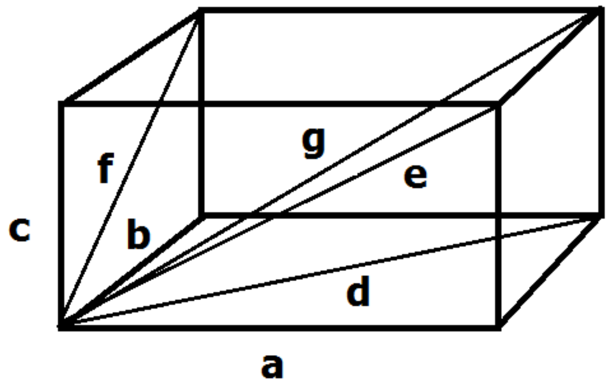
at the end of the branches of the different trees?

Is this always going to work?

Can you come up with a conjecture? Can you prove it?

Level 1 Level 2 Notes Home

36: The Perfect Box Problem



Level 3: *A **Perfect Box*** is defined to be an Euler brick whose diagonal from front left corner to back right corner is also an integer length (labelled g in the diagram). This is known as the ***space diagonal**,* as opposed to the ***face diagonals***.

Can you find another equation to add to your system of equations that have to hold, involving g this time?

Can you find an integer solution to your full system of equations?

How many perfect boxes are there?

Level 1 Level 2 Notes Home

37: The Friends and Strangers Problem

Level 3: Suppose you want to guarantee that at the party there is a quartet who all know each other, or a quartet that are all strangers to each other.

How big a party do you need now?

Suppose that you replace ‘quartet’ with ‘quintet’ in the above.
How big does the party need to be now?

Suppose you look at n-tets of people;

can you find a general formula for the size of party needed?

Level 1 Level 2 Notes Home

38: The Minimum-Number-of-Points Problem

Level 3: Can you find a formula for g(n)?

Does such a formula exist?

If so, can you prove it to be true?

Level 1 Level 2 Notes Home

39: The Cubes + Cubes = Cubes Problem

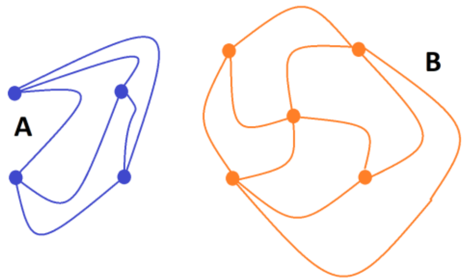
Level 3: The equation $x^3 + 2y^3 = 4z^3$

clearly has the solution **(1, 2, 4, 0, 0, 0).**

If x, y and z are natural numbers, are there any other solutions?

Level 1 Level 2 Notes Home

40: The Integer Edge-Length Problem



Level 3: Can you find straight-line versions of the two graphs above that have integer lengths?

Can you always come up with a straight-line version of a planar graph

so that every edge in the graph is of integer length?

Level 1 Level 2 Notes Home

Proving and Algebra

So how do pure mathematicians go about proving things? Let me dive in with an example. Suppose you are faced with this problem:

What happens if you add the first however many odd numbers, starting with 1?

We'd probably have to try out a few examples, to get a feel for this.

$$1 + 3 = 4,\ 1 + 3 + 5 = 9,\ 1 + 3 + 5 + 7 = 16.$$

Now **4, 9, 16**; we recognise these, they're the square numbers. So straight away we have a conjecture; that adding the first however many positive odd integers gives a square number.

That's fine as far as it goes, but any intention of checking all possibilities is going to come to grief. We can check (or get a computer to check) the sum of the first however many odd numbers, but we are still stuck with an infinite list to check beyond that. Something more imaginative is going to be needed, and that is algebra.

The human mind has never invented a labour-saving machine equal to algebra. **Anon**

The word ‘algebra’ can summon up profoundly different responses in different people. For some, it is a breakthough idea that opens the door to generalising in mathematics. For others, I can almost hear a sigh. I know of one student who would regularly complain, ‘Here we go with the ns again!’ Why do we have to mix up letters and numbers?

When I was introduced to algebra many years ago, I recall being presented with arcane questions about apples and bananas that no one in their right mind would ask, alongside equations that came out of nowhere, and to which no one really wanted an answer. But things have changed; if you are fortunate, you will have met algebra in ways that are fresh and meaningful, maybe beginning that journey by using Cuisenaire rods at primary school before encountering even arithmetic.

In this ebook, algebra has a precise purpose throughout; it allows us to deal with that phrase ‘however many’. It lets us get beyond mere examples to generality. We need to prove something for all whole numbers, say, and offering examples, however many, won’t do the job. It’s by letting a letter stand for a number, any number, that this surprising magic happens; we can deal with all numbers at the same time.

So how would this work for the problem above? We need to clarify in our mind just what it is that makes a number odd or even in the first place.

An even number leaves no remainder when divided by 2.
An odd number leaves a remainder of 1 when divided by 2.

Now here's the leap with algebra; let n be a whole number.

2n will be even, while 2n - 1 will be odd.

n	1	2	3	4	5	6
2n	2	4	6	8	10	12
2n-1	1	3	5	7	9	11

You can see that as n runs through all the positive whole numbers, 2n runs through the positive even numbers, and 2n - 1 runs through the positive odd numbers. Moreover, you give me any even number, and I can give you an n so that 2n is that even number, and the same goes for odd numbers. Algebra is a powerful thing.

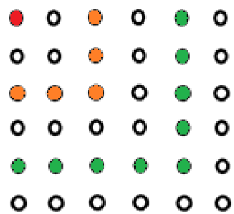
So we have **1 + 3 + 5 + ….. + (2n – 1) = S,** and we have to find S. Write S down backwards:

S = (2n-1) + (2n-3) + … + 1

Now we can add our two equations:

$$
\begin{array}{rl}
S = & 1 + 3 + \dots + (2n-1) \\
S = & (2n-1) + (2n-3) + \dots + 1 \\
\hline
2S = & \underbrace{2n + 2n + \dots + 2n}_{n \text{ lots}}
\end{array}
$$

So $2S = n \times 2n = 2n^2$, and so $S = n^2$, as we hoped. We've proved our conjecture for all positive whole numbers. There is also what we might call a Show-Me proof for our result.



Immediately from this diagram we can see that adding the first n odd numbers gives us the square number n^2. Such a visual demonstration of a result is always a good thing!

Trying out a few examples is a sound way into proving a problem. It gives you a feel for what it is going on, and it may suggest a strategy that you can use later. In fact, there's something bogus about the way you may see theorems presented in maths textbooks. You'll find a precise statement of the theorem, followed by a clean and tidy proof, followed by examples of the theorem in action.

In real life, the creation of the theorem is likely to have followed precisely the reverse route. A mathematician plays with a few examples; this is followed by a tentative guess at an argument coming out of these, with the tidying up of how precisely to word their theorem coming last. A maths book gives us **Theorem-Proof-Examples,** while in real life, where the maths is created, we have **Examples-Proof-Theorem.**

The difference is the difference between life and death. All the excitement of the chase is leached out of the textbook presentation. The refined version protects your reputation as a mathematician, with all your mistakes and failed intuition airbrushed out, but by doing that, your motivation for proceeding as you did is often masked.

But then, to speak for the other side, most mathematicians don't have the time to pursue all the possible highways and byways, the dead-ends and the red herrings that the whole story of your proof would involve. Maybe there's sadly something inevitable about wanting to dress your theorem in smart clothes before presenting it to the world**.**

Home

1 Notes: The Lines-into-Triangles Problem

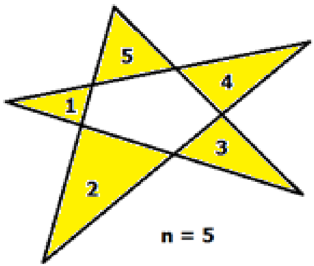
Status: unsolved **Name: The Kobon triangle problem, first stated by Kobon Fujimura**

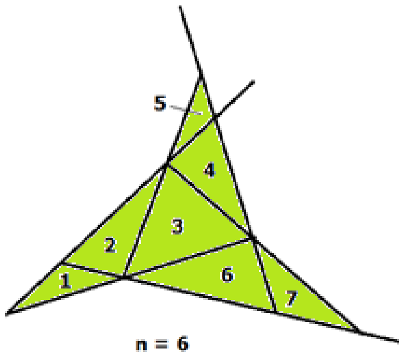
Mathematics may be defined as the economy of counting. There is no problem in the whole of mathematics which cannot be solved by direct counting. **Ernst Mach**

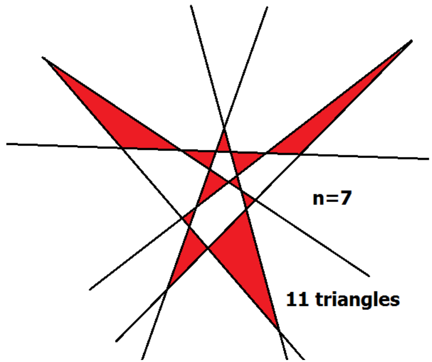
Level 1: 5 lines can give 5 triangles.

Level 2: The best triangle-scores for 6, 7 and 8 lines are given in the table below.

n	3	4	5	6	7	8
N(n)	1	2	5	7	11	15







Level 3: Finding a formula that works generally for N(n) has proved to be remarkably difficult. The mathematician Saburo Tamura made major progress with the problem. His result says

the greatest whole number not exceeding n(n-2)/3 is an upper bound for K(n).

In 2007, Bader and Clément managed to improve on this result. They showed that if we get the remainder 0 or 2 when dividing n by 6, then Tamura's upper limit could not be reached (reducing the bound for K(n) for these numbers by 1). These results are shown in the table below;

n	*Remainder on dividing n by 6*	*K(n)*	*Tamura's bound*	*B-C Bound*
3	3	1	1	1
4	4	2	2	2
5	5	5	5	5
6	0	7	8	7
7	1	11	11	11
8	2	15	16	15
9	3	21	21	21
10	4	25	26	26
11	5	32	33	33
12	0	38	40	39
13	1	47	47	47
14	2	53	56	55
15	3	65	65	65
16	4	72	74	74
17	5	85	85	85
18	0	93	96	95
19	1	104	107	107
20	2	115	120	119

K(n) is the number of non-overlapping triangles that the best known diagram at the time of writing attains.

Perfect solutions are diagrams that hit the maximum under Tamura's formula as modified by Bader and Clément. A list of n for which perfect solutions are known begins

3, 4, 5, 6, 7, 8, 9, 13, 15, 17...

For the numbers 10, 11 and 12, the best known solutions are one less than the bound. There's a nice way in which one perfect solution leads to another, since

given a perfect solution for n lines,

we can create from it a perfect solution for 2n-1 lines

(this was proved by Forge and Alfonsin). So a perfect solution for n = 3 leads to perfect solutions for 5, 9, 17, 33, 65... and so on.

Home Level 1 Level 2 Level 3

Prime Numbers

A ***factor*** of a number is a number that goes into it exactly. If we write down the numbers from 1 to 10 together with their factors, we get a table like this.

	Number	Factors
	1	1
Prime	2	1, 2
Prime	3	1, 3
	4	1, 2, 4
Prime	5	1, 5
	6	1, 2, 3, 6
Prime	7	1, 7
	8	1, 2, 4, 8
	9	1, 3, 9
	10	1, 2, 5, 10

The numbers that have exactly two factors, 1 and themselves, are called **prime**. If you're working on a problem about whole numbers, prime numbers are extremely likely to make an entrance.

The list of primes begins

2, 3, 5, 7, 11, 13, 17, 19, 23, 29…

1 is not considered these days to be a prime, for good reasons.

Just as all the compounds we know are made up of molecules that can be broken down into their constituent elements, so the prime numbers are the building blocks for all of our whole numbers.

Home

2 Notes: The Partition Problem

Status: solved

Let's call a partition of n into odd numbers (with repeats) a Type-1 partition of n, and one into any numbers (without repeats) a Type-2 partition of n.

Level 1: Type-1 partitions of 7 are **1 + 1 + 1 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 3 = 1 + 3 + 3 = 1 + 1 + 5 = 7.**

Type-2 partitions of 7 are **1 + 2 + 4 = 1 + 6 = 2 + 5 = 3 + 4 = 7,** so five again.

Level 2: This seems to work for every whole number we pick. So our conjecture becomes that for any number n there are the same number of Type-1 partitions as Type-2 Partitions.

Level 3: Can we prove this? Let's take an example. We can write

52 = 5 + 5 + 5 + 5 + 5 + 5 + 5 + 3 + 3 + 3 + 3 + 3 + 1 + 1,

so we have a Type-1 partition of 52 into odd numbers, with repeats.

We can manufacture a Type-2 partition of 52 from this, using any positive numbers, but without repeats.

52 = (7)5 + (5)3 + (2)1

Write the frequencies 7, 5 and 2 as sums of powers of 2

= (4 + 2 + 1)5 + (4 + 1)3 + (2)1

Now multiply this out

= 20 + 10 + 5 + 12 + 3 + 2

Put in order

= 20 + 12 + 10 + 5 + 3 + 2.

We know we can't have a repeat here, since the second red line above makes that impossible.

So we have a Type-2 partition of 52.

Can we reverse this? Given this Type-2 partition of 52;

52 = 20 + 12 + 10 + 5 + 3 + 2

Collect together the multiples of the various powers of 2

= (20 + 12) + (10 + 2) + (5 + 3)

That's multiples of 4 + multiples of 2 (but not 4) + multiples of 1 (but not 2)

= 4(5 + 3) + 2(5 + 1) + 1(5 + 3)

Multiply this out

= 4×5 + 4×3 + 2×5 + 2×1 + 1×5 + 1×3

Collect together the odd numbers to the right side of each product

= (4 + 2 + 1)5 + (4 + 1)3 + 2(1)

= 5 + 5 + 5 + 5 + 5 + 5 + 5 + 3 + 3 + 3 + 3 + 3 + 1 + 1.

Given the third purple line above, you can be sure that there are no even numbers here. We now have the Type-1 partition of 52 that we started out with.

We have to be careful, making sure that every Type-1 partition transfers by this process to a unique Type-2 partition and back again. That does work here, so we have our theorem;

There are the same number of Type-1 partitions as there are Type-2 partitions for any number n.

Home Level 1 Level 2 Level 3

Implication

Algebra is our friend in constructing proofs; clear-headed logic will be its bedfellow. We need to be happy with the idea of *implication*. Consider these statements about real numbers:

$x = 2$ implies that $x^2 = 4$, $x^2 = 4$ implies that $x = 2$.

The first statement is surely true, but the second isn't; if x^2 is 4, then x might be 2, but it could be -2. Compare these with the following;

$x = 2$ implies that $x^3 = 8$, $x^3 = 8$ implies that $x = 2$.

This time both statements are true, since a number has only one real cube root.

We have the symbols $\Rightarrow$ (for 'implies'), $\Leftarrow$ (for 'is implied by') and $\Leftrightarrow$ (for 'implies and is implied by'), and we can use them here:

$$x = 2 \Rightarrow x^2 = 4, \qquad x^2 = 4 \Leftarrow x = 2, \qquad x = 2 \Leftrightarrow x^3 = 8.$$

Suppose we’ve established the theorem **‘Statement A implies Statement B’**. We could come up with a conjecture, **‘Statement B implies Statement A’**. This is called the ***converse*** of our first theorem; as our examples above show, it might be true, but it might not.

$x = 2 \Rightarrow x^2 = 4$ **(which is true) has the converse** $x^2 = 4 \Rightarrow x = 2$ **(which is false).**

$x = 2 \Rightarrow x^3 = 8$ **(which is true) has the converse** $x^3 = 8 \Rightarrow x = 2$ **(which is true).**

If a statement and its converse are both true, we can use the ‘implies and is implied by’ symbol (this is often written as ‘if and only if’, or sometimes ‘iff’, or even ‘just if’).

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3 Notes: The Twin Prime Problem

Status: unsolved **Name: The Twin Prime conjecture**

If you're looking for a new conjecture that mathematicians would find hard to prove, the prime numbers are a fertile starting point. There are lots of prime number problems that mathematics just doesn't currently seem equipped to resolve. The Twin Prime problem is one of those.

Level 1: The primes from 1 to 30 are **2, 3, 5, 7, 11, 13, 17, 19, 23, 29,** so the gaps between these primes are **1, 2, 2, 4, 2, 4, 2, 4, 6.** We are interested in the gaps length 2.

Level 2: The twin primes up to 100 are (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61) and (71, 73).

Level 3: Do pairs of primes (p, p + 2) roll on forever? Nobody knows.

There are plenty more unsolved prime problems. A prime number p is a Sophie Germain prime if 2p + 1 is also prime. For example, 2 x 5 + 1 = 11, so 5 is a Sophie Germain Prime. The list of these starts

2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191, 233, 239, 251, 281, 293, 359,…

The conjecture here is that there are infinitely many; this is unresolved.

This list comes from the ***Online Encyclopedia of Integer Sequences*, or** OEIS, which is an indispensible tool for the up-and-coming prover. I was once investigating a problem I couldn't get a handle on, but certain numbers kept popping up repeatedly (for example, 18817 and 19601). I turned to the OEIS. Searching for these numbers led me towards the Chebyshev polynomials, and that was the connection I needed. Suddenly, all the theory related to these polynomials became available to me, and I was able to make a quantum leap forward with my problem.

OEIS is a sensational collaborative enterprise; it's part of what make the internet great. As you gain experience as a mathematician, you can donate the number sequences that you discover along the way to the project, although with zillions of sequences already in place, be warned, it's hard to find a gap!

Three more unsolved prime problems…

1. A Fibonacci number that's prime is called a *Fibonacci prime*. The first few are

2, 3, 5, 13, 89, 233, 1597, 28657, 514229, 433494437,....

Are there infinitely many Fibonacci primes? Nobody can prove there are.

2. Is there always a prime between n^2 and $(n+1)^2$? Posed by Legendre, this question remains unsolved.
3. Are there infinitely many primes of the form n^2+1? Nobody knows.

It seems only fair to show something that can be proved about primes: this is Dirichlet's Theorem (proved in 1837).

There are infinitely many integers for which a + bn is prime

if a and b are natural numbers with no common factor.

So if we pick a = 3, b = 7, then the sequence 10, 17, 24, 31,… is guaranteed to contain infinitely many primes.

Home Level 1 Level 2 Level 3

The Square-and-Add-the-Other Problem

Here's a task to get your proving teeth into.
This one is certainly doable; can you find a proof?

Pick two numbers, let's call them x and y, that add to 1.

They can be whole numbers, fractions or irrational numbers, positive or negative.

Now square x and add y. Then square y and add x.

What do you find?

Does this always work? Can you make a conjecture? Can you prove it?

Can you come up with a visual proof of this?

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4 Notes: The Arithmetic Simultaneous Equations Problem

Status: solved

Level 1: $\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}$ are certainly six consecutive terms from an arithmetic sequence that increases by the constant amount $\frac{1}{12}$ with every step to the right. You can write these terms as $\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}$, which gives you what you need.

Level 2: Suppose we choose the simplest possible set of six numbers from an arithmetic sequence, 1, 2, 3, 4, 5 and 6. These generate the equations

$$x + 2y = 3$$
$$4x + 5y = 6$$

If we multiply the top equation by 4, we get

$$4x + 8y = 12$$
$$4x + 5y = 6$$

Subtracting, we get 3y = 6, and y = 2. Substituting back in for y, we get x = -1.

Taking -2, -1, 0, 1, 2 and 3 as our terms generates the equations

$$-2x - y = 0$$
$$x + 2y = 3$$

Solving these once again gives us x = -1, y = 2 as our solution. Trying more examples gives us the same solution each time. Why should this be?

Level 3: Turning to algebra, let's us tackle the general case.

$$ax + (a + d)y = a + 2d$$
$$(a + 3d)x + (a + 4d)y = a + 5d$$

Subtracting straight away, we get 3dx + 3dy = 3d, so x + y = 1.

So $y = 1 - x$, and substituting back into the first equation gives

$ax + a + d - ax - dx = a + 2d$, which gives $x = -1$, $y = 2$, every time.

So we have that choosing six coefficients in this way gives a solution of (-1, 2).

Is the converse true? No, since

$$\begin{aligned} 4x + 5y &= 6 \\ 8x + 7y &= 6 \end{aligned}$$

give the solution (-1, 2) too.

Note that if the point (-1, 2) lies on $ax + by = c$, then $-a + 2b = c$, or $a + c = 2b$, which happens if and only if a, b, c are three consecutive terms from an arithmetic sequence.

$$\mathbf{(a, b, c) = (a, a + d, a + 2d) \text{ if and only if } a + c = 2b.}$$

So if we view each equation we create as a line in the x-y plane, then if we follow our rule, any such line is compelled to go through the point (-1, 2), which means (-1, 2) must be the solution if we solve two such equations simultaneously.

So our theorem becomes

The solution to a pair of simultaneous equations is (-1, 2)

if and only if

the first three coefficients come consecutively from an arithmetic sequence, and so do the second three.

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Counterexamples

Some of our conjectures will turn out to be false. Indeed, if we doubt our conjecture, it's tempting to start by trying to find an example that will disprove it. Such an example is called a ***counterexample***. Suppose we have the following conjecture;

$$x^2 + x + 41 \text{ is always prime if } x \text{ is a positive integer.}$$

How to get started? Why not try a few examples? We're attempting to ***verify*** the conjecture for some early numbers (notice that verifying is an easier activity than proving!)

The value x = 0 gives 41, which is prime, while x = 1 gives 43 and x = 2 gives 47, which are also prime. We can carry on for a while like this; putting in early values for x gives us prime numbers every time. But then it dawns, if we put x = 41, we're guaranteed to get a non-prime (composite) number, since we have

$41^2 + 41 + 41$, which 41 definitely goes into.

We have our counterexample, and in fact we had no need to check out all those earlier values; we could have gone straight there.

Now that you understand the notion of a counterexample, you're able to appreciate this maths joke:

At a large mathematical conference, a young mathematician demonstrates his first original result to a gaggle of respected mathematicians.

When he concludes its proof, someone in the audience interrupts him:

'That proof must be wrong - I have two counterexamples to your theorem.'

The young mathematician shrugs. 'That's okay - I've got three more proofs.'

It always seems unfair to me, that to prove a theorem, you have to show the theorem is true for infinitely many examples. To disprove the theorem with a counterexample, however, just one will suffice, which seems like a lot less work. But finding a counterexample can be extremely hard; sometime the first billion examples will check out correctly, but then the billion-plus-oneth will be the stumbling block. Computers are likely to be your friend in finding such a beast.

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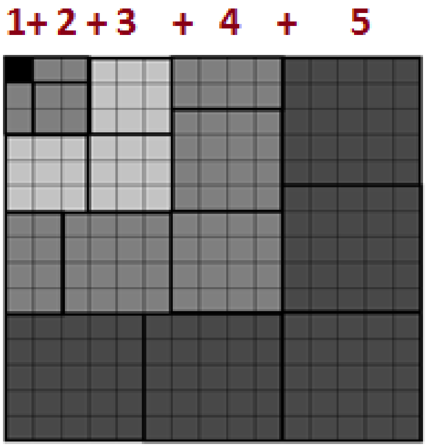
5 Notes: The Sum of single powers, Sum of cubes problem

Status: solved

Level 1 and **2:** You should find that C = S^2 every time. Can you prove this for all positive whole numbers?

Level 3: This is one of those beautiful theorems proved by a single picture.



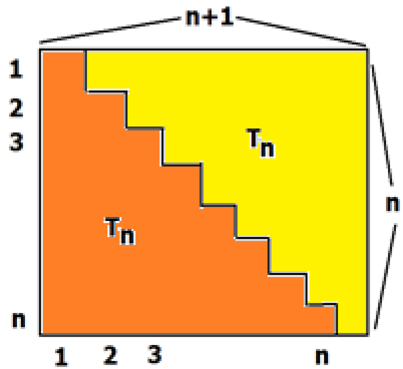
$$(1 + 2 + 3 + 4 + 5)^2$$
$$= 5.5^2 + 4.4^2 + 3.3^2 + 2.2^2 + 1.1^2$$
$$= 5^3 + 4^3 + 3^3 + 2^3 + 1^3.$$

We can see how this will generalise to the case for n.

We can in fact do better here, and find a formula for;

$$T_n = 1 + 2 + 3... + n$$

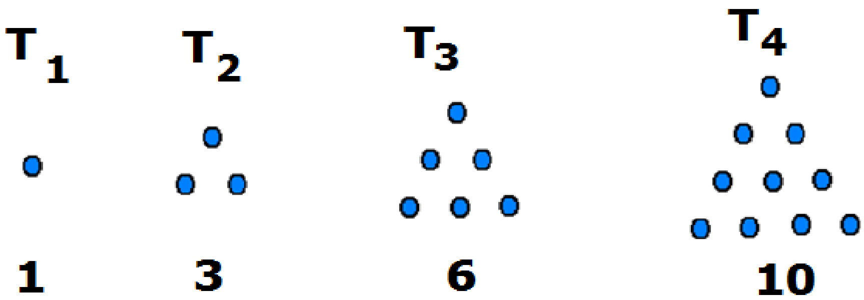
Once again a picture comes to our aid.



So we can see that $2T_n = n(n + 1)$, and so $T_n = n(n + 1)/2$, and

$$C_n = 1^3 + 2^3 +... n^3 = n^2(n + 1)^2/4.$$

The numbers T_n are called ***the Triangle numbers,***
and they appear in many different mathematics settings.



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Exhaustion

Time for another way of proving things, proof by exhaustion. I offer you the following problem:

Show that no perfect square can ever end in 14.

Try to prove this before reading the Notes page.

The idea with proof by exhaustion is to 'exhaust all the possibilities'.
So we narrow the list of possible counterexamples to be as small as we can,
and then we check out everything on the list.

There is an element of brute force about this,
but sometimes it is the simplest way to prove something.

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6 Notes: The Triangle of 1s Conjecture

Status: unsolved ***Name: Singmaster's Conjecture***

This 'Triangle of 1s' is called Pascal's Triangle, after the brilliant but sadly short-lived French mathematician Blaise Pascal (you may know that he has a computer language named in his honour). He's fortunate to have this triangle attributed to him, because it was well known to Indian and Chinese mathematicians long before his time.

Level 1: The triangle is built from rows of numbers, one more for each new row. The first and last number in each row is a 1, while the rule for finding the other numbers is to add the two numbers immediately above. So in the diagram, the 4 and 6 are next to each other in the fourth row, and the number beneath them is a 10.

It makes sense to call the first 1 the 0th row, so that the nth row begins '1 n …'.

So the next three rows beyond the one starting **1 6 …** are

1 7 21 35 35 21 7 1, 1 8 28 56 70 56 28 8 1, 1 9 36 84 126 126 84 36 9 1.

We could use a simple spreadsheet to tell us the values for larger n (how would you create this spreadsheet?). The numbers get big quickly...

1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	12	13
1	3	6	10	15	21	28	36	45	55	66	78	91
1	4	10	20	35	56	84	120	165	220	286	364	455
1	5	15	35	70	126	210	330	495	715	1001	1365	1820
1	6	21	56	126	252	462	792	1287	2002	3003	4368	6188
1	7	28	84	210	462	924	1716	3003	5005	8008	12376	18564
1	8	36	120	330	792	1716	3432	6435	11440	19448	31824	50388
1	9	45	165	495	1287	3003	6435	12870	24310	43758	75582	125970
1	10	55	220	715	2002	5005	11440	24310	48620	92378	167960	293930
1	11	66	286	1001	3003	8008	19448	43758	92378	184756	352716	646646
1	12	78	364	1365	4368	12376	31824	75582	167960	352716	705432	1352078
1	13	91	455	1820	6188	18564	50388	125970	293930	646646	1352078	2704156
1	14	105	560	2380	8568	27132	77520	203490	497420	1144066	2496144	5200300
1	15	120	680	3060	11628	38760	116280	319770	817190	1961256	4457400	9657700
1	16	136	816	3876	15504	54264	170544	490314	1307504	3268760	7726160	17383860
1	17	153	969	4845	20349	74613	245157	735471	2042975	5311735	13037895	30421755
1	18	171	1140	5985	26334	100947	346104	1081575	3124550	8436285	21474180	51895935
1	19	190	1330	7315	33649	134596	480700	1562275	4686825	13123110	34597290	86493225
1	20	210	1540	8855	42504	177100	657800	2220075	6906900	20030010	54627300	1.41E+08
1	21	231	1771	10626	53130	230230	888030	3108105	10015005	30045015	84672315	2.26E+08
1	22	253	2024	12650	65780	296010	1184040	4292145	14307150	44352165	1.29E+08	3.55E+08
1	23	276	2300	14950	80730	376740	1560780	5852925	20160075	64512240	1.94E+08	5.48E+08
1	24	300	2600	17550	98280	475020	2035800	7888725	28048800	92561040	2.86E+08	8.34E+08
1	25	325	2925	20475	118755	593775	2629575	10518300	38567100	1.31E+08	4.17E+08	1.25E+09

What's shown here in black are the 12th and part of the 24th rows.

It turns out that Pascal's Triangle is made up of the How Many Ways numbers, which gives us a formula for the n^{th} row.

Level 3: Can we have as many appearances of some value as we like, if we pick our value carefully? Singmaster's Conjecture says, 'No', there's some number k so that there are guaranteed to be less than k appearances of any number in Pascal's triangle.

The amazing bit of that, I think, is the word 'any'. However far you go down Pascal's Triangle, however long your rows get, you will never find a whole number, however large, that appears more than k times. Can it be true? The work done on Pascal's Triangle, which involves lots of computer help, suggests that Singmaster's Conjecture is likely.

So do we have any idea what k might be? The number 120 appears 6 times. In fact, there are lots of numbers that appear 6 times (it's been possible to come up with a formula using the Fibonacci numbers that generates as many such numbers as we wish).

There's only one number that's known to appear 8 times, and that's 3003. It appears twice each in the 3003th row, the 78th, the 15th and the 14th.

Here’s a list of numbers that appear at least six times in Pascal’s Triangle.

1, 120, 210, 1540, 3003, 7140, 11628, 24310, 61218182743304701891431482520, ...

Notice the huge jump from **24310** to **61218182743304701891431482520**. Such numbers are rare, and that makes Singmaster’s conjecture seem more likely.

Mathematicians who work on a problem over a period of time develop a sixth sense over what might be true. Singmaster has devoted many hours to this task, and his intuition suggests that although k could be 8 (and certainly no counterexample to this idea has been found) it could be 10, and it could be 12. Even numbers seem to be the most likely possibilities for k; no one knows if a number can appear five or seven times in the triangle, since no such example has been found. The great mathematician Erdős said that he sensed that Singmaster’s conjecture was true, but he also sensed that any proof would be tough. So beware before making this your life’s work...

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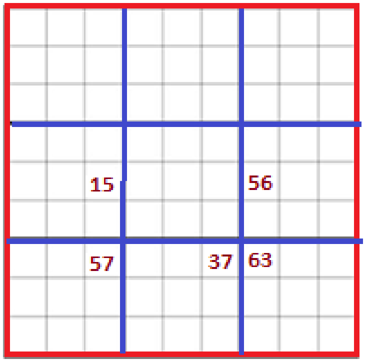
Contradiction

Here’s another proof method - by contradiction (this is also known as ***reductio ad absurdum***). I hope I can take it as read that you’ve tried a Sudoku at some time in your life. To recap, you’re given a 9 by 9 square, divided into nine 3 by 3 boxes. You are told each row, each column and each box contain the digits from 1 to 9 with no repeats. With the help of some digits chalked in for you to get you started, the challenge is to complete the whole grid.

This turns out to be a nice way to demonstrate proof by contradiction. Suppose we have deduced that that some of our cells look like this:



Notation; the cell showing pq is definitely either a p or a q. Suppose you think the 15 cell contains a 1. You can prove this as follows; suppose it contains a 5. That means the 56 cell is a 6, which means the 63 cell is a 3, which means the 37 cell is a 7, and the 57 cell is a 5. But then you have two 5s in the same column. This can’t happen, so you were right to say that the 15 cell contains a 1.

This is an unlikely example, I grant you, since not many Sudoku grids will ever look quite like this, but the principle is sound. With a tough Sudoku you may get stuck, and this kind of argument could get you out of that.

So proof by contradiction looks like this: you want to prove a statement S. You assume instead that S is false, and try to argue from this into a contradiction. Now mathematics is not allowed to contain contradictions, that's a golden rule (if you allow contradictions, then our logic implies that ANYTHING is true). The only way out of this impasse is to say that your initial assumption must have been wrong, and so S is not false but true.

This may seem like a convoluted way of going about things, but proof by contradiction is a vital tool at the highest level of pure mathematics.

Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess play: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game. **G.H. Hardy**

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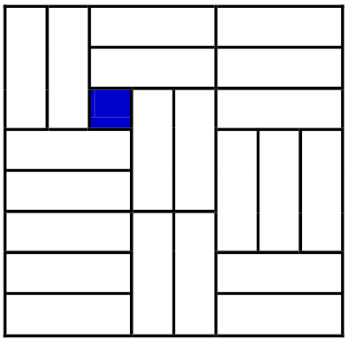
7 Notes: First Triomino Tiling Problem

Status: solved

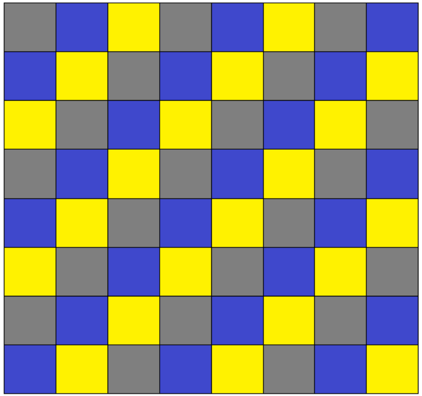
Level 1: Starting with the famous tiling problem; since every domino takes up one black and one white square, we are bound to end up with two white squares empty, however we tile. So no, we can’t use dominoes to cover a chess-board with opposite corners missing.

Level 2: If we try to tile with the triomino, we find that our best possible effort leaves our one empty square stubbornly in the same position every time (counting rotations as the same).

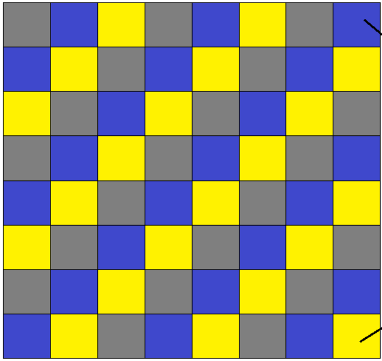


Is this the only possible blank? We feel that surely we can manage things so that that empty square appears anywhere on the board. But our experience of working on this suggests that this is the only possible spot we can achieve.

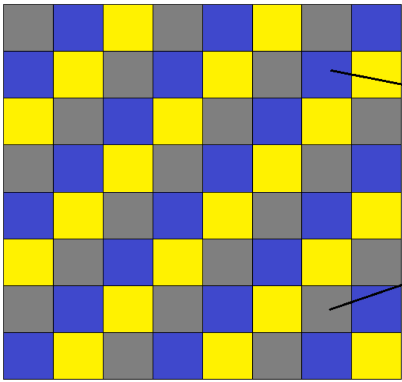
Level 3: How would we set about proving this? Let’s paint the board like this:



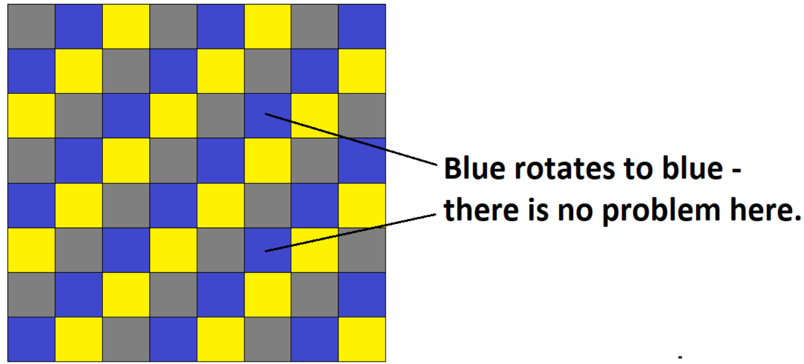
So every triomino we put down covers one blue, one grey and one yellow square.
The way we’ve coloured the board, we have 21 yellow, 21 grey and 22 blue squares, which means that the empty square must be blue. But...

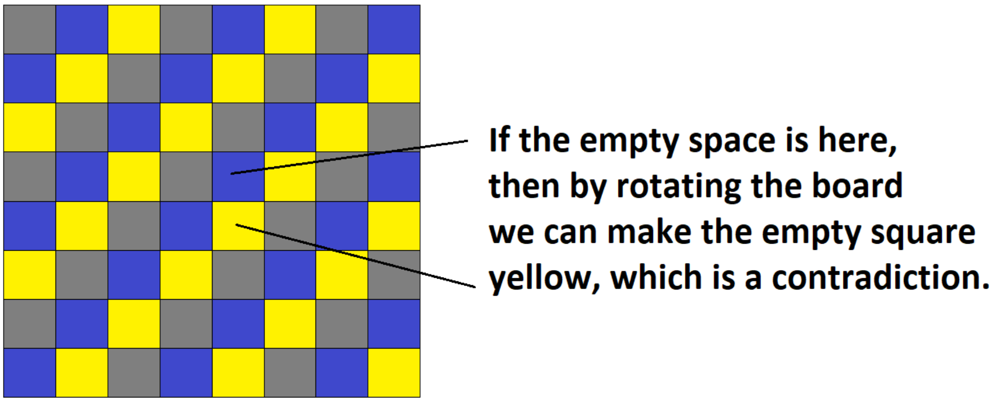


If this could be the empty square, then we could rotate the board so that the empty square was yellow - impossible!



If this could be the empty square, then we could rotate the board so that the empty square was grey - impossible!


Blue rotates to blue -
there is no problem here.


If the empty space is here,
then by rotating the board
we can make the empty square
yellow, which is a contradiction.

We can carry the argument on for the other blue squares, perhaps using reflections.

The only position for the blank square that does not lead to a contradiction like this is the one that we know can be achieved.

With this simple colouring argument, we've proved something that looked to be really difficult at the start.

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Induction

Imagine you can see a line of dominoes, all placed vertically on their ends in a rather unstable arrangement. We're about to administer a little push to the first domino. What do we need to know in order to say that the entire set falls over?
We need to know that:

1. **The first domino topples.**
2. **If the n^{th} domino topples, the $n+1^{th}$ domino will topple.**

Mathematics being as wonderful as it is, we can construct in our heads a line of dominoes that is infinite, which leads to the technique of proof by induction to establish theorems for all natural numbers.

Let me show you how this works. We must have a statement about whole numbers, let's call it S_n (proof by induction relies on you trying to prove a theorem featuring positive whole numbers.) Suppose that S_n says (using our earlier example), the sum of the first n odd numbers is n^2. Using algebra, this becomes:

$$\mathbf{1 + 3 + \ldots + (2n - 1) = n^2.}$$

Our first job is to check S_1. This says $1 = 1^2$, which is true. So we know that the first domino falls.

Now assume S_n, so $\mathbf{1 + 3 \ldots + (2n - 1) = n^2}$ (the n^{th} domino falls). We need to deduce that now S_{n+1} is true (the $n+1^{th}$ domino falls). Adding $2n + 1$ to both sides of our S_n equation gives us

$$\mathbf{1 + 3 \ldots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2}, \text{ which is } S_{n+1}.$$

So if the n^{th} statement S_n is true, then the $n + 1^{th}$ statement S_{n+1} is true also.

We know S_1 is true, so S_2 is true, so S_3 is true, so S_4 is true... off to infinity.

This example is, in fact, regarded as the first explicit example of proof by induction, offered by **Francesco Maurolico** in his book ***Arithmeticorum libri duo*** (1575).

Proof by induction is a mightily powerful technique in our proof arsenal, but we should never wheel it out routinely. We should always ask first, ‘Is there a nicer way to prove this?’ There may well be.

Home

8 Notes: The Map-Colouring Problem

Status: solved **Name: The Four Colour Theorem**

Levels 1 and **2** and **3:** The question is, are four colours enough? However big the map we draw, and how ever complicated the boundaries between countries become, four always seems to work. (If we get stuck somewhere, we can always start again and this time make it work.) This is a ridiculously simple conjecture that a five-year-old can understand, yet it turns out to be extremely hard to prove. The problem was resolved, however, by Kenneth Appel and Wolfgang Haken in 1976; four colours are always enough.

This proof has a fascinating history. Francis Guthrie was the first person to bring the conjecture to full attention in 1852. The mathematician Alfred Kempe claimed to have a proof in 1879, but a flaw in his argument emerged. In 1922, Philip Franklin proved four colours were sufficient for every map with 25 or fewer countries, and by 1976 this number had been improved to 95. But then in 1976, Appel and Haken managed to reduce the problem to around 2 000 maps, a collection that had to contain any possible counterexample. Checking was not feasible for humans, so they ran the maps through a computer (a proof by exhaustion). Finally the program came up with the answer; the four colour theorem was true.

This created something of a philosophical conundrum for the maths community; checking the logic of a proof on a page was one thing, but accepting a result from a machine whose calculations could not be verified? This caused some grief, but nowadays most people are happier about including the results from computer work in proofs (if a computer generates a counterexample, then that can be checked as an individual case, and there is no problem).

We’ve only been talking about maps on a page (which is effectively the same as for maps on a sphere). Imagine if we were drawing our map of countries on a torus (a doughnut with a hole in the middle). Would we need fewer colours, or more colours? The answer is that this time we need seven.

One last fact; if we work in three dimensions, colouring three-dimensional regions, there’s no limit to the number of colours we need. Given some number k, we can always find a set of bordering regions in 3D space that needs more than k colours.

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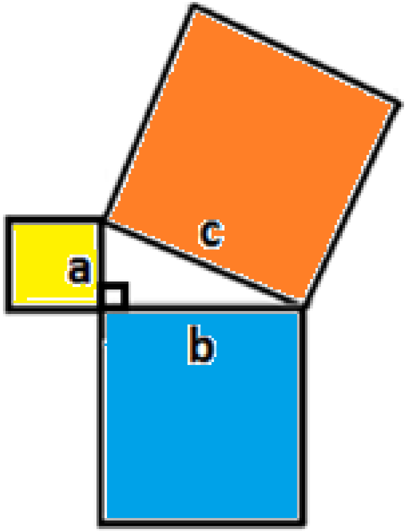
Pythagoras

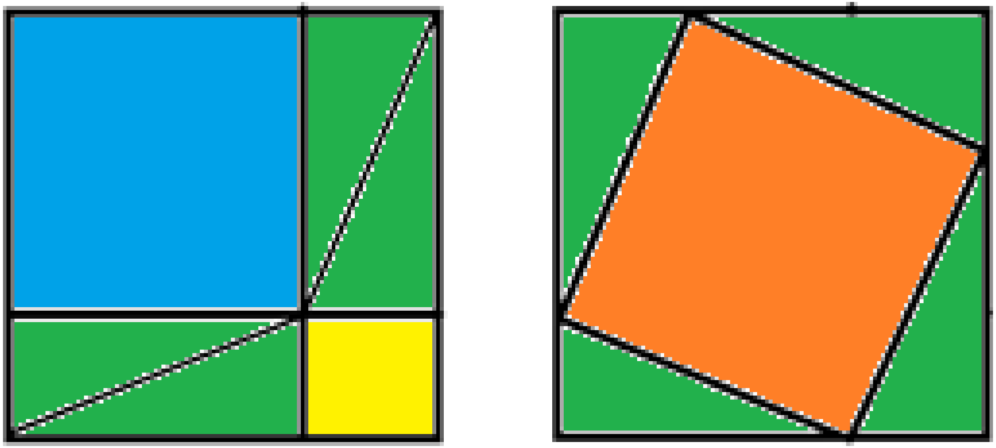
We've concentrated so far on conjectures about whole numbers - what about proofs in geometry? How could we prove the most famous theorem of all, that is, Pythagoras's?



Pythagoras's Theorem says that the area of the biggest square of the three adds to the area of the two smaller squares.

$$a^2 + b^2 = c^2$$

Note that the biggest square must be on the side opposite the right angle (the hypotenuse).



Here’s a compelling proof-without-words argument.
The two smaller squares + 4 copies of the triangle equal
the biggest square + 4 copies of the triangle. We’re done!

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9 Notes: The Points-in-a-Circle Problem

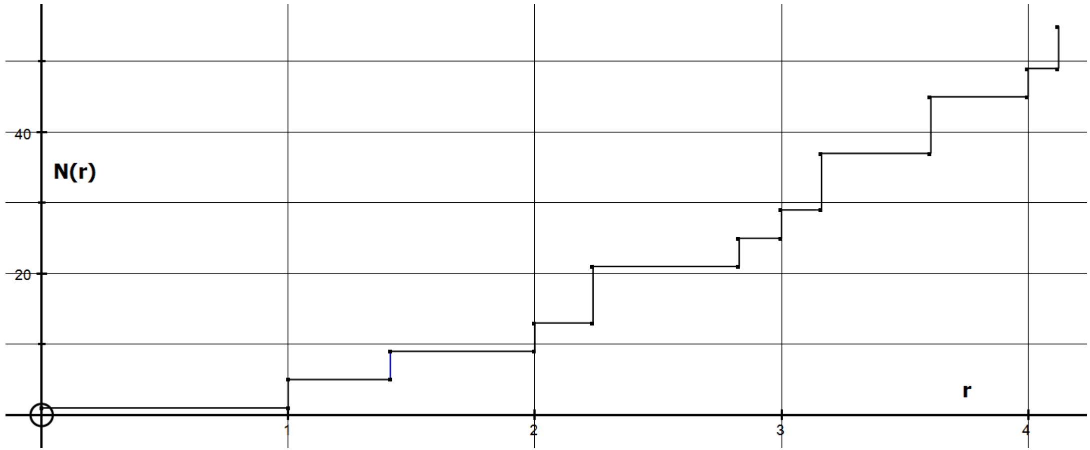
Status: unsolved ***Name of problem: The Gauss Circle Problem***

Level 1: The count of lattice points goes from 1 ($0 < r < 1$) to 5($1 < r < \sqrt{2}$) to 9 ($\sqrt{2} < r < 2$).

Level 2:

r just greater than;	**0**	**1**	**$\sqrt{2}$**	**2**	**$\sqrt{5}$**	**$2\sqrt{2}$**	**3**
N(r)	**1**	**5**	**9**	**13**	**21**	**25**	**29**

Level 3: We're aiming to come up with a formula in terms of r that gives the number of lattice points inside the circle (let's call this N(r)).



The jagged line above shows the progress in N(r) as r increases starting with small r. We can expect that the line will ‘become smoother’ as r gets larger, when there will be more and more combinations of lengths that will push N(r) higher.

This is such an innocent-looking challenge, yet it has proved too much for everyone so far. We can come up with a rough result. We know that each square unit on average contributes one lattice point to our count. The area of the circle is πr^2, so the circle will roughly enclose πr^2 lattice points. It’s that little extra bit, what mathematicians call ‘the error term’, that has proved so elusive.

We have

$$N(r) = \pi r^2 + E(r).$$

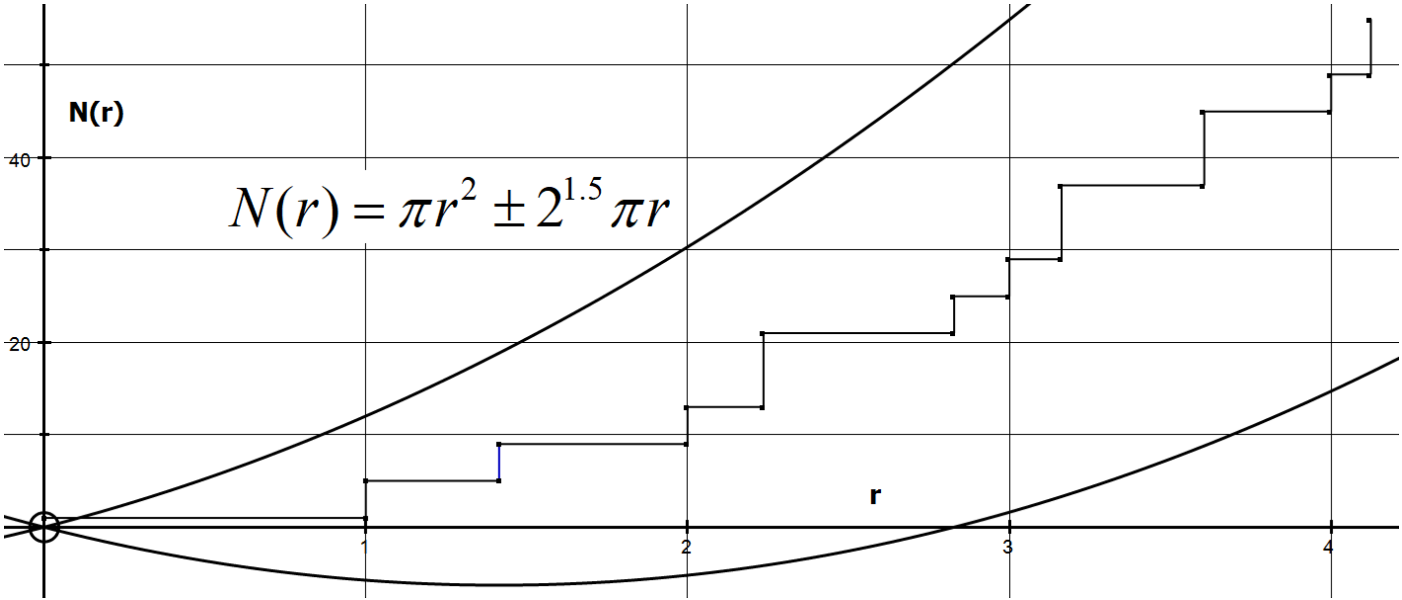
The best current thinking says that $|E(r)| \leq Cr^t$, for some constant C and for some constant t. Finding C and t is the task that faces us.

The first person to make progress with this was Carl Frederick Gauss, hence the problem's name. If anyone was likely to lay this problem to rest, it would be Gauss, one of everyone's top five mathematicians of all time.

Gauss, the mathematical giant, who from his lofty heights embraces in one view the stars and the abysses. **Farkas Bolyai**

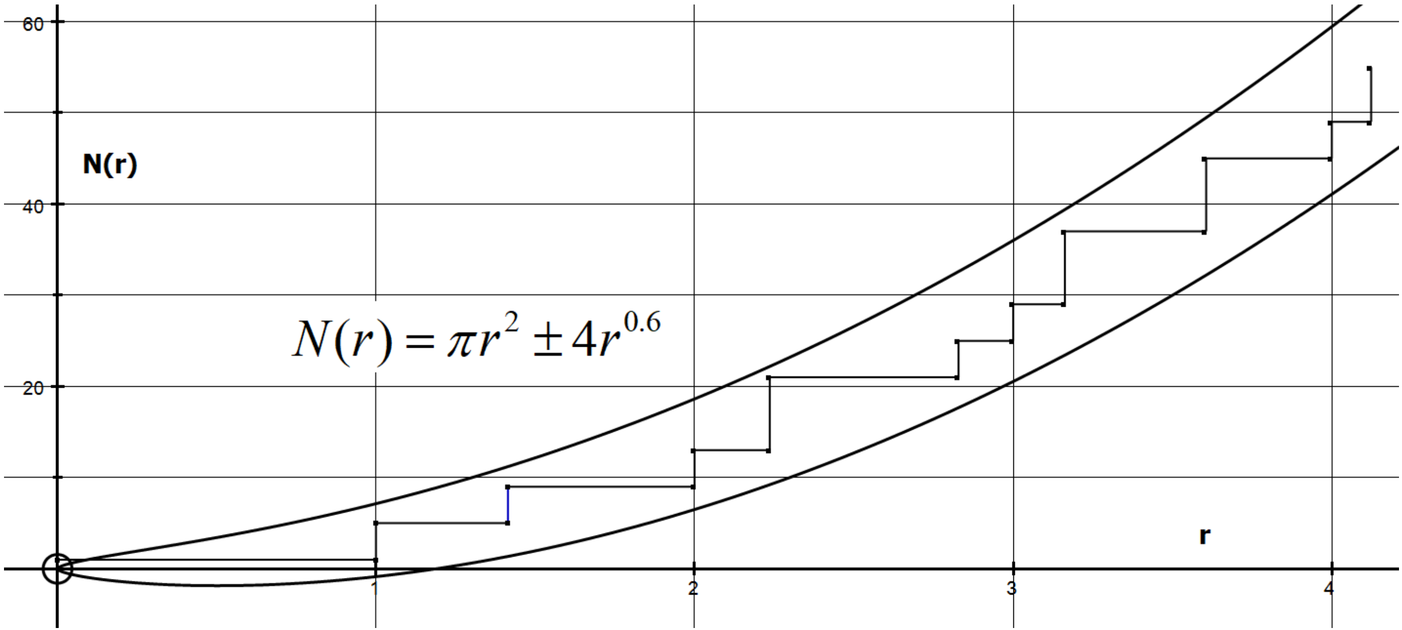
(*Describing Gauss's writing style*) *He is like the fox, who effaces his tracks in the sand with his tail.* **Niels Abel**

Gauss proved that $|E(r)| \leq 2\sqrt{2}\pi r$, which gives us the values $C = 2\sqrt{2}\pi, t = 1$. These bounding lines are shown on the graph below.



Hardy and Landau in 1915 found a lower bound for t, showing that 1/2 < t.

Huxley in 2000 showed that t $\leq$ 131/208 = 0.6298…. How much bigger than 1/2 is t?



The graph here puts C = 4, and t = 0.6. You can see that the bounds appear tighter than in Gauss's formulation, but we must remember the graph above is only for $0 \leq r \leq 4$, and infinity is a long way away!

There are other problems that suggest themselves from this one, all equally problematic. Suppose we replace the circle with an ellipse - can we find the error term now?

Suppose we go into three dimensions, and examine a sphere centred at the origin; how many three-dimensional lattice points are enclosed?

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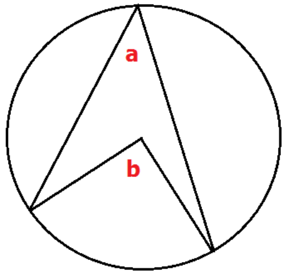
Axioms

Here's another geometrical problem.

Draw a circle and pick two points on the circumference.
Now join these points to some point elsewhere on the circumference,
and then join them to the centre of the circle.



Find a protractor, and measure a and b. What do you find?

Will this always be true? Can you prove it?

Home Notes

10 Notes: The Is-Every-Number-a-Fraction Problem

Status: solved

Level 1 and **Level 2:** We can’t always find a whole number between two given whole numbers (e.g. 5 and 6). But it is certainly true that given any two fractions, we can always find another one between them, since

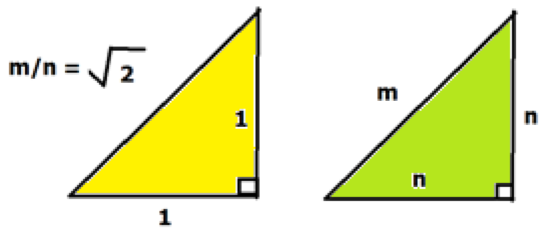
$$\frac{a}{b} < \frac{c}{d} \quad \Rightarrow \quad \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} \text{ (for a, b, c, d all positive).}$$

(Can you prove this?) So if we put in all possible fractions, there won’t be any ‘gaps’ in the sense of intervals where there are no rational numbers (we say the rationals are ***dense***). That’ll fill up the line completely, won’t it?

Level 3: Once upon a time, that’s what mathematicians thought. But then someone came up with a brilliant argument to show that not all numbers on the number line were rational; in fact, there were many more irrational numbers than rational ones. This was such a shock to the mathematics of the time that the mathematician involved (so the legend goes) was rewarded for his (or her) efforts by being put to death. Being brilliant sometimes comes at a cost.

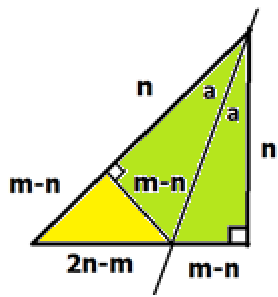
Let's look at one argument then, that tells us that the square root of two cannot be written as a fraction. We will use proof by contradiction.

Suppose that we can write √2 as a fraction, say m/n. By Pythagoras's Theorem, we know that √2 is the diagonal of an isosceles right-angled triangle with two sides of length 1.



So if we scale up by multiplying the sides by n, we end up with an isosceles right-angled triangle with integer sides, m, n and n.

Now bisect the top angle,
and drop a perpendicular
from where it meets the bottom side
to the hypotenuse.



The new lengths are clear if we use the symmetry of the diagram, and they must be all whole numbers.

Now look at the yellow triangle in the bottom left corner. It's an isosceles right-angled triangle, so

$$\sqrt{2} = \frac{2n-m}{m-n}.$$

So we have another fraction for √2, but this time using smaller whole numbers, since

2n - m < m and **m - n < n** (we know this since **2n < 2m,** and **m < 2n**).

We can do this again, and again, getting smaller whole numbers each time...

But now we have a problem, because however big m and n are to start with, we are heading towards 0, and sooner or later we will run out of whole numbers! This method of proof is called *descent*, and was used extremely effectively by Fermat in the 17th Century.

Here's an alternative proof, using again the method of contradiction. We want to show that the square root of 2 is irrational. So for a proof by contradiction, we say, suppose √2 = a/b, where a and b are positive whole numbers. We can also add that a and b should have no common factor (if they do, we can cancel this factor from top and bottom in a/b before we start).

So we have √2 = a/b, which when squared gives us

$$2 = a^2/b^2 \Rightarrow a^2 = 2b^2.$$

This tells us that a^2 is an even number, since it is of the form 2n.

But if we square an odd number, we always get an odd number, so a must be even.

So we can write a = 2c for some whole number c.

Now $a^2 = 2b^2 \Rightarrow (2c)^2 = 2b^2 \Rightarrow 4c^2 = 2b^2 \Rightarrow 2c^2 = b^2.$

But we have been here before; a ghostly repetition takes place. We know that b^2 is even, which we already know implies that b is even.

This is, in fact, as far as we need to go - where's the contradiction that we're looking for?

We have that a is even and b is even, so they have a common factor of 2.

But at the start we insisted, entirely reasonably, that a and b should have no common factor. We've now deduced the opposite.

The only way out, scary as such an idea may be, is to say that our initial supposition must be untrue, and so $\sqrt{2}$ is irrational.

There is a kind of remorselessness to this proof - it’s a mathematical whodunit of sorts, with a satisfying denouement that Sir Arthur Conan Doyle himself would have been proud to engineer.

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Tables

Sometimes we lose sight of why we’re studying or teaching maths. I offer you this fable now by Professor Ronnie Brown, called ***Carpentry.***

Recently I attended a carpentry course. It was pretty tough. All the students (or almost all) were eager to learn. The first three weeks we learned to drill holes. We found out about curious kinds of drills, and how to make holes at odd angles. We got pretty good and accurate at drilling holes.

The next six weeks were involved in cutting wood. We used all kinds of saws, found out how they interacted with different kinds of wood, and learned to cut accurately and smoothly. I got pretty good at cutting wood.

The next four weeks we learned to plane wood. We used all kinds of planes, on many different kinds of wood. I got pretty good at planing wood.

'Joints' was a difficult course. It took eight weeks, and we learned many kinds of joints. I was quite good at making joints.

We did courses on other things too: sanding, turning, polishing, gluing, and so on.

Finally, we had an examination. We had to use all these skills. I did reasonably well, and came fifth in the class.

After the course ended, I went to see the Director. I told him I quite liked the course in a way, though some of the students were turned off by it all. But really, I said, I took the course because I wanted to make a table.

He said that only the top two or three went on to do things like that. I began to get mad. I said: 'What did we learn all that stuff for?'

He said: 'Our course prepares people to make tables.' His face got larger and larger. He began to fill the room. I got scared. Then I woke up.

This was worrying. I discussed it with my colleagues. A psychiatrist took me back to my childhood. But no-one could explain why a professor of mathematics should have a nightmare like that.

Ronnie Brown
School of Mathematics, University of Wales, Bangor
MATHEMATICAL INTELLIGENCER VOL. 11, NO. 4, 1989, p.37

My first attempt at studying university mathematics felt, to me at least, alarmingly similar to that described in the fable above (I should say many of my colleagues were perfectly happy with the sanding and polishing and gluing we were asked to do.)

Mathematics is a game played according to certain simple rules with meaningless marks on paper.
David Hilbert

As I read David Hilbert here, he was saying that it would be desirable to formalise mathematics in such a way that it can be seen to be both consistent and complete (Kurt Godel eventually proved this dream to be impossible). But some maths courses to some students, at every level, can feel like the meaningless manipulation of symbols, which induces fear. The antidote to that is curiosity, and that often begins with a surprise.

This book assumes you want to make tables from the start. It contains unfinished blueprints for tables, designs for furniture that nobody on the planet knows how to finish. Thankfully too there are easier, more finishable tables in here too. All the techniques this book passes on will hopefully help you with making your own tables, however easy or hard their designs might be.

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11 Notes: The Handshake Problem

Status: solved

Level 1: The maximum number of handshakes for five people is the number of ways of choosing two people from 5, or $\binom{5}{2} = 10$. (Or they all shake hands four times, which means there are 4 x 5/2 = 10 handshakes altogether.)

Level 2: There are six handshakes shown in the diagram. In general, the total of all the handshake counts divided by 2 gives the number of handshakes.

Level 3: One line of attack for this problem uses the **Pigeonhole Principle**. This is stunning in its simplicity, and it's hard to believe that it would be of much use to a research mathematician, but there are high-powered proofs out there that rely completely on this. You have n pigeonholes and n + 1 letters to post. The Pigeonhole Principle tells you that once you have done this, there will be at least one pigeonhole containing more than one letter. Reasonable?

So could the Pigeonhole Principle help us with our Handshake Problem? There are two cases:

1. **Someone shakes everybody else's hand.**

2. **Nobody shakes everybody else's hand.**

If somebody shakes everybody else's hand, then the possible handshake-scores around the room are

1, 2, ..., n – 1 (no one can score 0, since everybody's hand has been shaken.)

We have n – 1 possible scores therefore, and n people, so by the Pigeonhole Principle, two people must have the same score.

If nobody shakes everybody else's hand, then the possible handshake-scores around the room are

0, 1, 2, ..., n – 2 (the value n – 1 is impossible, since no one shakes everybody else's hand.)

Once again we have n – 1 possible scores and n people, so by the Pigeonhole Principle once more, two people must have the same score.

Pierre de Fermat and Andrew Wiles

Mathematics has thrown up all sorts of problems for its adherents to tackle down the years, some of which have become legends. It is, of course, one thing to state a problem, and quite another to solve it; there can be many years between the two. The Greeks asked, ‘Can we, using just a straight edge and compasses, construct a square with the same area as a given circle?’ This took two thousand years to resolve; the answer is, ‘No, it can’t be done.’

Another famous gap between conjecture and resolution lies in the time it took to solve Fermat’s Last Theorem. Fermat was a French lawyer in the 1600s who spent all the time he could spare on maths, becoming an extraordinary mathematician in the process. He stated the problem that would make him a household name in 1637; that the equation $x^n + y^n = z^n$ has no solution in whole numbers when n is bigger than 2 (he even claimed he had a solution, but this looks unlikely now). The problem was finally beaten in 1994, when Andrew Wiles, an English mathematician, succeeded in proving Fermat right. That’s 360 years from posing to resolving, years that tested some of the finest mathematicians that the world has seen, and which produced some superb mathematics as a by-product of seeking a solution.

Wiles talks of his early interest in the problem. As a ten-year-old boy, he read of Fermat’s challenge, and was amazed that something he could understand at such a young age could have defeated every adult on the planet. Here I present 20 problems that remain unsolved, but like Fermat’s Last Theorem, they are problems that are understandable by a ten-year-old boy. It’s my hope that this book could spark the same ambition in fledgling mathematicians today.

By and large it’s pure mathematicians who prove things. Applied mathematicians do too, but their methods may be less rigorous than those of their pure partners. Approximate solutions may be more acceptable, and finding ‘something that works’ may be more important than finding something logically water-tight.

It was as though applied mathematics was my spouse, and pure mathematics was my secret lover.
Edward Frenkel

Lots of things work in practice for which the laboratory has never found proof. **Martin H. Fischer**

The path to a proof may be rocky for the pure mathematician. It’s easy to overlook some tiny exception to your argument, or to claim just a little too much without a full justification. This is what happened to Andrew Wiles. He announced that he had his proof before it’d been rigorously checked by his peers. When his friend Nick Katz sat down to work it through, he spotted a problem, one that was not trivial.

The highest moments in the life of a mathematician are the first few moments after one has proved the result, but before one finds the mistake. **Anon**

The mistakes and unresolved difficulties of the past in mathematics have always been the opportunities of its future. **E. T. Bell**

The message from this is that a proof may evolve; it may start as something informal, which can be tightened into something more conventional. Imre Lakatos, in his book *Proofs and Refutations,* tells the story of Conjecture Y (one of the problems in this book). As Lakatos tells the story, you learn how the proof of Y is refined in the light of fresh examples (that he calls ‘monsters’) that challenge the initial statement of the theorem. It raises the question of whether a proof is ever really finished or not.

It is the story that matters, not just the ending. **Paul Lockhart**

Wiles found himself in a tricky situation; the wider maths community wanted to see his proof in detail, but he was making them wait. The strain was intense. But the story has a happy ending; in a moment of revelation, Wiles saw what would make his proof work. The rest is history, and he has his place amongst the great mathematicians of this or any age.

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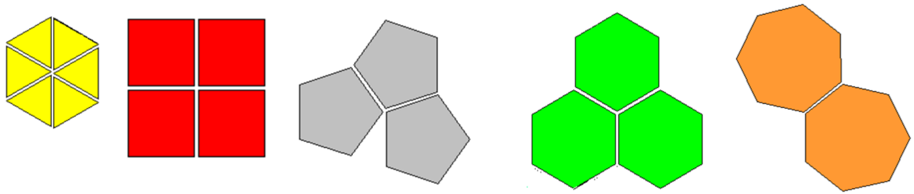
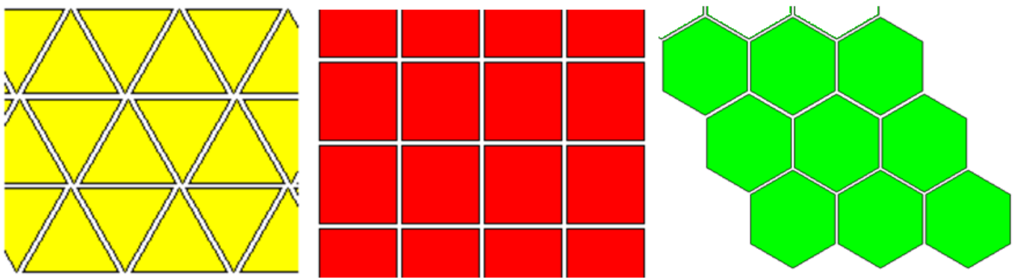
12 Notes: The How-Many-Tilings Problem

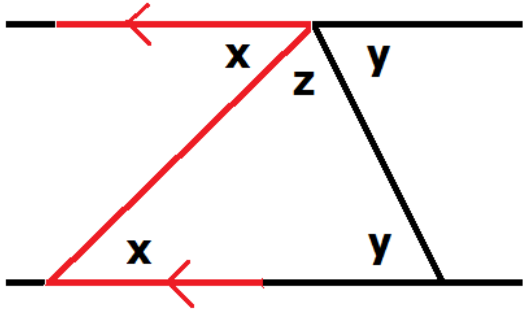


Status: solved

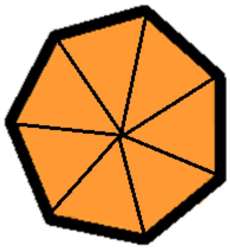
Level 1: The diagram above should convince us it's only regular polygons with 3, 4 or 6 sides that can possibly tile the plane on their own, and when we check, they do.



Level 2 and **3:** We have to think about the angle in a regular polygon. We'll start by saying there are 360 degrees (360°) in a whole turn, and so 180° in a half turn. So in the diagram below, x + y + z = 180°.



If you are happy with angles in the red 'parallel zed' above being equal, this diagram convinces us that the angles in a triangle always add to 180° (at least in standard Euclidean geometry).



For an n-sided regular polygon, we can create n triangles from a point at the centre, as in the diagram above.

The angles in these triangles add to 180n°, so if we take off the central 360°, the n interior angles of the polygon add to (180n - 360)°, and so each interior angle is (180 - 360/n)°.

We can now draw up a table;

Sides	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Angle	**60°**	**90°**	**108°**	**120°**	**128.5..°**	**135°**	**140°**	**144°**	**147.2..°**	**150°**
Sides	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**
Angle	**152.3..°**	**154.2..°**	**156°**	**157.5°**	**158.8°**	**160°**	**161.0..°**	**162°**	**162.8..°**	**163.6..°**

We need to find combinations that add to 360°. If we check carefully and logically, we end up with the seventeen results in the following table:

3 tiles around a point

3	7	42
3	8	24
3	9	18
3	10	15
3	12	12
4	5	20
4	6	12
4	8	8
5	5	10
6	6	6

4 tiles around a point

3	3	4	12
3	3	6	6
3	4	4	6
4	4	4	4

5

3	3	3	3	6
3	3	3	4	4

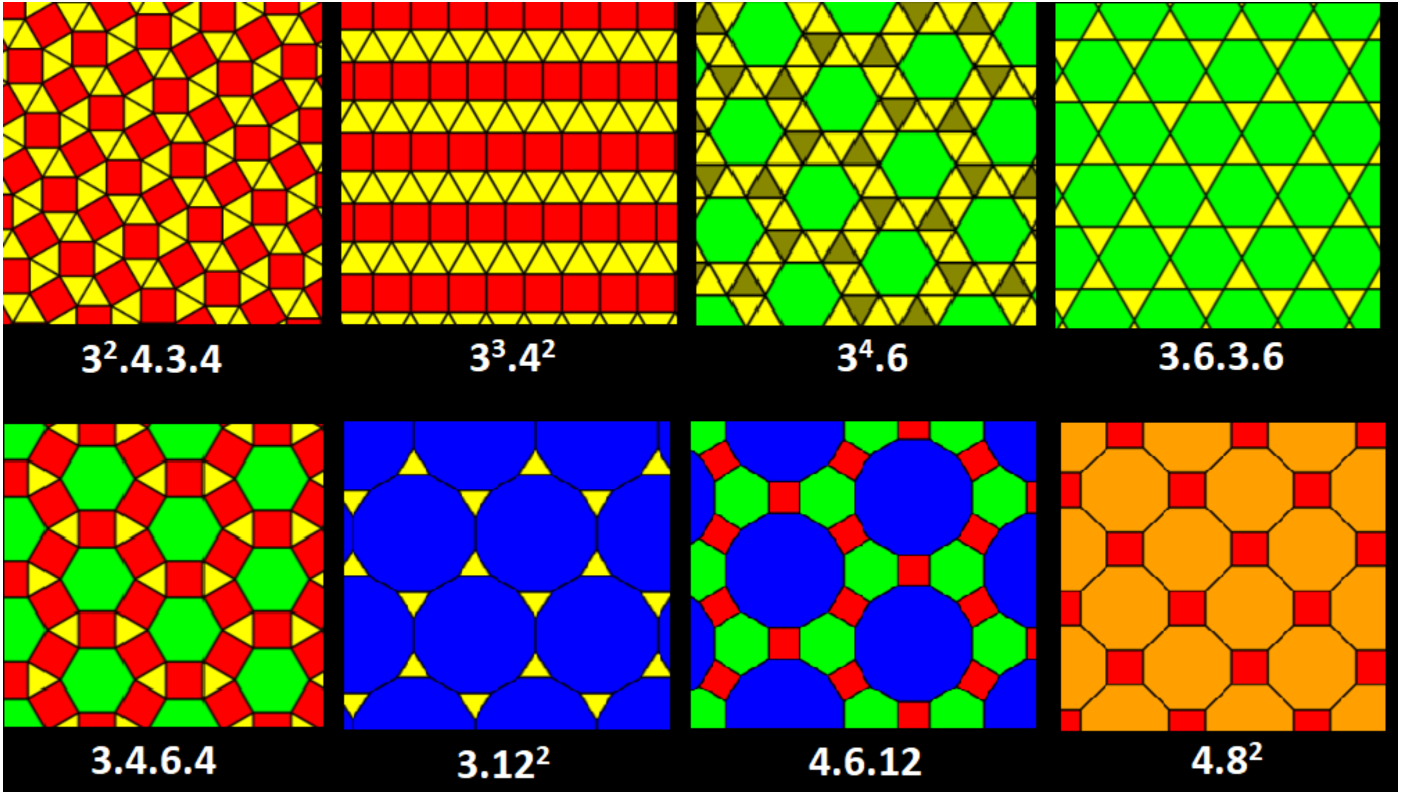
6

3	3	3	3	3	3

Try as we might, these are the only combinations that are possibly going to work, and given that we have checked in a logical way, there can be no others.

But we have more checking to do, because some of these combinations can't be extended to give full tilings of the plane.

We need to remember too that three triangles and two squares around a point could be (3, 3, 3, 4, 4) or (3, 3, 4, 3, 4). In the notation below, these become $3^3.4^2$ and $3^2.4.3.4$.


$3^2.4.3.4$
$3^3.4^2$
$3^4.6$
$3.6.3.6$
$3.4.6.4$
3.12^2
$4.6.12$
4.8^2

Once our proof by exhaustion is over, and we have checked what is possible, we have the tilings above, the eight ***semi-regular tessellations***.

There are lots of further questions that we can ask; what if we allow there to be more than one combination around corner points? What if we move into three dimensions; what tessellations are possible there?

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Lists

This book is in part a collection of unresolved conjectures, and mathematicians are fond of coming up with such lists. In 1900, David Hilbert, the master mathematician of the age, listed 23 questions that he predicted would define 20th Century mathematics. The top ten he presented at a conference in the Sorbonne in 1900. As the century unfolded, these questions occupied mathematicians much as Hilbert had predicted. Many succumbed, but the most famous Hilbert problem, the Riemann Hypothesis, defied solution.

The mathematician Landau in 1912 presented four problems from number theory, more precisely about prime numbers. All feature in this book; they are all still unsolved. He said they were 'unattackable' using maths as it currently stood, and he probably didn't realise how right he was.

A similar list to mark the dawning of the year 2000 seemed apropos. In 1998, the mathematician Steve Smale listed 18 problems for the impending millennium to consider. A shorter list of seven problems were posed, or re-posed, to mark the date 2000 by the Clay Institute. One of these was Riemann's Hypothesis once more, a testament to its enduring importance and its unyielding nature.

How have these seven problems fared so far? One, the Poincaré conjecture, has fallen, to a mysterious and reclusive mathematician called Grigori Perelman, who refused to accept the prize money and barricaded himself into the flat he lives in with his mother and sister to evade any publicity. The other six remain open as I type. None of these can be regarded as easy-to-understand; they each involve technical ideas that are tough, even for well-qualified mathematicians. But it remains gripping to wonder what the 2100 list will be, and how many of the Clay choices will still be on it.

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13 Notes: The Lonely Runner Problem

Status: unsolved

Name: the Lonely Runner Conjecture,
first posed in 1967 by J. M. Mills, and named by L. Goddyn in 1998.

Level 1: Suppose we have 8 runners, and the track is of length 1. Suppose their speeds are **[1, 2, 4, 5, 7, 8, 9, 11].** When t = 0.264, the runners **[1, 2, 3, 4, 5, 6, 7, 8]** are at **[0.264, 0.528, 0.056, 0.320, 0.848, 0.112, 0.376, 0.904]** on the track.

Level 2: The pair [3, 6] are close (within 1/8), as are [1, 4, 7] and [5, 8]. No runner, however, is within 1/8 of Runner 2 with speed 2, who is thus lonely.

Level 3: Is it true that every runner is lonely at some time? Simple as that question sounds, no one has been able to resolve it for sure. But progress has been made for small numbers of runners, as shown in the table below.

n	*Year proved*	*Proved by*
1	Trivial	Trivial
2	Trivial	Trivial
3	1972	Proved by n = 4
4	1972	Betke and Wills; Cusick
5	1984	Cusick and Pomerance;Bienia et al
6	2001	Bohman, Holzman, Kleitman; Renault
7	2008	Barajas and Serra

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The Christmas Project

It was Christmas 1976, and Steve, one of my schoolteachers, decided to set us an extra-curricular challenge.

'You all know that the history of mathematics is my bag; I challenge you to come up with a project on that theme. You've got plenty of areas to choose from, and there'll be a prize!'

After a little thought, I picked my title; **'Unsolved Problems in Mathematics.'** I hit the deadline, and was left to wonder what Steve made of my work. A week later, he called me over, looking pensive; I feared the worst. Had I chosen my problems badly?

'Jonny, your third problem was the X Conjecture,' he said (I'm hiding its name since it's one of the forty here). 'I have some news for you - a proof has just been announced!' he smiled. 'It's now the X Theorem. But that doesn't affect your project. It just goes to show that unsolved problems eventually become solved problems.'

The timing still strikes me as so unlikely as to be some kind of wind-up, but then, I can look up the date that these particular mathematicians achieved their wizardry, and 1976 it is. This coincidence at least showed me definitively that maths was alive and kicking. So many people see maths as over, finished, written up in textbooks and impossible to add to. The truth is that there've never in human history been as many theorems proved as there are today. One estimate I heard recently was of 100 000 newly published theorems in a year.

I won the prize (modesty compels me to tell you that my fellow students were busy that Christmas and I was the only entry). I don't possess the pages I handed in now, but this ebook is proof that my fascination with unsolved problems remained.

I did not become a mathematical researcher in the classical sense, working in a university at the outer boundaries of the subject - I became a sixth-form college teacher, yet I found that 'mathematical research' was still part of my life. Up at the board as I taught my students, I ran into fresh questions, conjectures and problems on a daily basis. Every time I paused mid-sentence, I thought afterwards about why that had happened; there was usually a glitch in my understanding that needed to be explored, to be written about, without any pressure to get anything published. I would say that every mathematics teacher is a researcher every day, into mathematics and mathematics education, whether they realise it or not. And maybe that is also true for every mathematics student. Who can look at primary pupils engrossed with their Cuisenaire rods without thinking, 'This is mathematical research too!'

Some of these problems from my classroom I've lived with for a long time; some remain unsolved even now, although perhaps a better mathematician than I am would make short work of them. I'm 57, and I'm coming to terms with the idea that I may not solve these questions before I die. There'll be maybe no one else who ever loves these problems as much as I do; perhaps these questions will die with me. Thoughts such as these add urgency to your mathematical life! With luck, mathematics will be a substantial part of life beyond the grave, and I’ll have time to finish them there; certainly the array of helpers on offer should be spectacular…

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14 Notes: The Odd Perfect Number Problem

Status: unsolved

Perfect numbers, like perfect men, are very rare. **René Descartes**

Level 1:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
S(n)	0	1	1	3	1	6	1	7	4	8	1	16	1	10	9
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
N	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
S(n)	15	1	21	1	22	11	14	1	36	6	16	13	28	1	42

Level 2: An odd number n with lots of small, odd, prime factors will have many different factors, so s(n) > n looks possible.

The number 3.5.7 = 105 seems a good bet; **s(105) = 1 + 3 + 5 + 7 + 3.5 + 3.7 + 5.7 = 87.**

How about $3^2.5.7 = 315$? We have

$$s(315) = 1 + 3 + 5 + 7 + 9 + 3.5 + 3.7 + 5.7 + 3^2.5 + 3^2.7 + 3.5.7 = 309$$

Close!

Will $3^3.5.7 = 945$ work? We have

$$s(945) = 1 + 3 + 5 + 7 + 9 + 3.5 + 3.7 + 5.7 + 3^2.5 + 3^2.7 + 3.5.7 + 3^3.5 + 3^3.7 + 3^2.5.7 = 948$$

Yes!

The number 945 is, in fact, the smallest odd number so that s(n) > n. Such odd numbers get more common the further down the number line we go.

In the table above, only 6 and 28 satisfy s(n) = n, making them the only perfect numbers under 30.

We have $496 = 2^4(31)$, and so

$$s(496) = 1 + 2 + 4 + 8 + 16 + 31(1 + 2 + 4 + 8) = 496$$

and so 496 is perfect too.

Level 3: Here are the Mersenne numbers for n = 1 to 10. (Mersenne was a monk in the seventeeth century who was a marvellous mathematical enabler, writing letters tirelessly and circulating results amongst the mathematicians of the day.)

n	**2^n-1**	**Prime?**
1	1	No
2	3	Yes
3	7	Yes
4	15	No
5	31	Yes
6	63	No
7	127	Yes
8	255	No
9	511	No
10	1023	No

This suggests a conjecture; we only have ten examples to go on, but it looks as if the Mersenne number 2^n-1 is prime if and only if n is prime.

But... $2^{11} - 1 = 2047 = 23 \times 89.$

A counterexample! We can say something, however; notice that

$$2^6 - 1 = (2^2 - 1)(2^4 + 2^2 + 1).$$

We can factorise $2^n - 1$ like this whenever n is a composite number. So we have

2^n - 1 is prime ⇒ n is prime, but the converse is not true.

In 1876, a mathematician called Lucas proved that $2^{127} - 1$ was prime, and this remained the highest known prime for 70 years. Nowadays the search for really large primes centres on Mersenne numbers. Most of the biggest primes we know are all of this type. By February 2018, 50 Mersenne primes were known.

There is a rather remarkable connection here; it turns out that every Mersenne prime generates a perfect number. Suppose that $m = 2^n - 1$ is prime, and think about the number $2^{n-1}(2^n-1)$. What are the factors of this number?

We have **$1, 2, 4, \dots 2^{n-1}$** from the 2s at the front.

We also have **$(2^n - 1), 2(2^n - 1), 4(2^n - 1).,\dots 2^{n-2}(2^n - 1).$**

So **$s(m) = 1 + 2 + 4 + \dots + 2^{n-1} + (2^n-1) + 2(2^n-1) + 4(2^n-1) + \dots + 2^{n-2}(2^n-1)$**

$= (1 + 2 + 4 + \dots + 2^{n-2})(1 + 2^n - 1) + 2^{n-1} = (2^{n-1}-1)2^n + 2^{n-1}$

$= (2^n-2)2^{n-1} + 2^{n-1} = 2^{n-1}(2^n-1),$ and so m is perfect.

It's been proved that all even perfect numbers are of this type. The 50 Mersenne primes mentioned above give us 50 (even) perfect numbers. What's the largest Mersenne prime to date (February 2018)? It's

$2^{77\,232\,917} - 1$

The perfect number associated with this is a lot bigger than this unimaginably huge prime.

The Lenstra–Pomerance–Wagstaff conjecture asserts that there are infinitely many Mersenne primes; nobody knows if this is true or not.

So we know what an even perfect number looks like. What about odd numbers? Are there any odd perfect numbers? None have been found. If we start to look with the help of computer, try as we might, achieving s(n) = n for an odd number eludes everyone. The mathematician Carl Pomerance has given an *heuristic* argument that an odd perfect number is impossible, but it's not rigorous enough to completely satisfy the mathematical community. If there IS an odd perfect number, it'll be vast.

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Elegance

There might be many ways to prove something. The one theorem that most people know is Pythagoras's, which can be proved in scores of ways. If that's the case, then is there some way of choosing between these proofs?

Certainly any proof claiming to be 'the best' must be logically watertight. But beyond that? This is where proofs can lay claim to the mysterious quality of ***elegance***. Proofs can embody a delicious twist in their construction. Some proofs take fewer and less involved lines of argument than others; elegant proofs tend to be the shorter ones, and they are likely to be aesthetically pleasing. Mathematicians tend to pick the most beautiful proof available when choosing to present their results. A golden rule for all apprentice provers: may our proofs be as elegant as possible.

Beauty is the first test: there is no permanent place in the world for ugly mathematics.
G. H. Hardy

It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is. **Paul Erdős**

This is a one line proof, if we start sufficiently far to the left. **Anon**

The elegance of a mathematical theorem is directly proportional to the number of independent ideas one can see in the theorem, and inversely proportional to the effort it takes to see them. **George Pólya**

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15 Notes: The Dots and Area Problem

Status: solved ***Name: Pick's Theorem***

Levels 1 and **2:** Let's draw up the table.

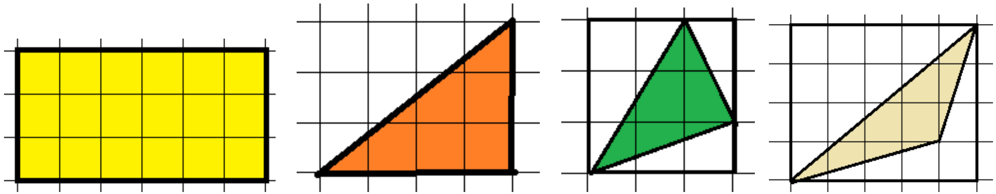
Shape	I	P	Area
A	**10**	**5**	**11.5**
B	**4**	**12**	**9**
C	**3**	**12**	**8**
D	**1**	**7**	**3.5**

If we try a few more shapes, a formula hopefully suggests itself;

$$\text{Area} = I + P/2 - 1$$

Level 3: This works for all the above. Can we prove this formula to be true?

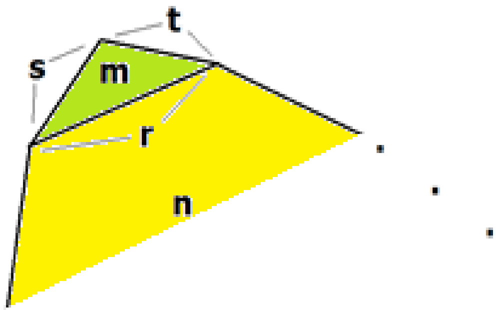
We could start by proving this formula for some basic shapes; a rectangle, a right-angled triangle with two sides along gridlines, an acute-angled triangle, and an obtuse-angled triangle.



It's relatively easy (although a bit of a slog) to prove the formula works for these shapes above, when taken in order from left to right, and using previous results as we go.

Now we'll try our ***proof by induction*** technique on this. First, we can say that the formula works for all triangles (k = 3; we have shown this).

Next, let's suppose the formula works for all polygons with k sides or fewer. Take a k+1-sided polygon, and cut off a triangle from this to make a k-sided polygon, like so:



The values r, s and t count the lattice points on the lines indicated, but NOT including the endpoints.

Say the perimeter of the k-sided polygon, not counting r, has Q lattice points on it, so the original k+1-sided polygon here has

$$\mathbf{I = n + m + r,} \text{ and } \mathbf{P = Q + s + 1 + t.}$$

Since we are assuming the formula works for k-sided shapes,
the area of the k-sided polygon created is **n + 1/2(Q + r) - 1.**

Since we know the formula works for $k = 3$,

the area of the triangle is **$m + 1/2(s + r + t + 3) - 1$.**

So the area of the whole $k+1$-sided shape is

$m + n + r + Q/2 + s/2 + t/2 - ½ = (m + n + r) + 1/2(Q + s + t + 1) - 1$,

which is what our formula predicts.

So by induction, our formula always works,

and we can always say for any such polygon

Area = $I + P/2 - 1$.

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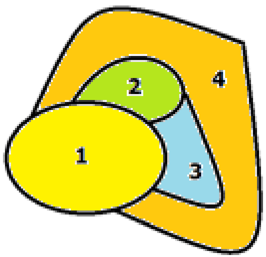
Sharp

'Sharp' is a word that mathematicians use about theorems or conjectures. Let's examine for a moment the Map-Colouring Problem: how many colours are needed to colour any flat map so that no two neighbouring countries share the same colour?

Proving that FIVE colours are enough is relatively straightforward. But this Five Colour Theorem does not appear to be sharp, because we can't seem to demonstrate a map that HAS to use five colours.



We CAN, however, demonstrate a map that HAS to use four colours, so the Four Colour Conjecture IS sharp. A sharp statement of a theorem or conjecture 'goes right up to the edge'; there's no wriggle room.

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16 Notes: The How-Many-Primes-Are-There Problem

Status: solved

Level 1: *A list of primes from 1 to 100 (there are 25):*

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

Level 2:

Range	**1-100**	**1001-1100**	**2001-2100**	**3001-3100**	**4001-4100**	**5001-5100**
# of primes	25	16	14	12	15	12

The trend is downwards, but the exact pattern is hard to predict.

Level 3: The list of primes goes on forever. For every apprentice prover, this is something you just have to see.

Suppose we want to find a number that gives a remainder of 1 when we divide by either 3 or 5 or 7.

It turns out that the smallest such example is 106. How did I find this? We have

$$\mathbf{3 \times 5 \times 7 + 1 = 106.}$$

We can apply this technique to primes more generally. If we are given a list of primes, how can we find another prime that is not in our list? Suppose our starting list is 2, 3, 7, 11. Then let's think about the number

$$\mathbf{2.3.7.11 + 1 = 463.}$$

It's clear that none of 2, 3, 7 or 11 divide into 463 exactly, because they each leave a remainder of 1. In fact, 463 is prime, so we've found a prime that is not in our list.

This doesn't always happen quite so conveniently. Try 2, 5, 7, 11 as our list now. This time

$$\mathbf{2.5.7.11 + 1 = 771}$$

which is not prime, since 771 = 3 x 257. But we've now found two primes that are not in our list, 3 and 257 (they can't be on our list because they divide exactly into 771). In this situation we're guaranteed to find at least two new primes not in our list, maybe more.

So here's the proof. Suppose we have a finite list of n primes, p_1, p_2, p_3..., p_n. Consider the number

$$P = p_1 p_2 p_3 \ldots p_n + 1.$$

None of the primes in our list go into P exactly, so either P is prime (and we have an addition to the list) or it's the product of primes that are not in our list (and so these are also additions to our list).

So given a list of primes, we can always find at least one other, which means there is an infinite number of primes. Euclid came up with this proof over 2 000 years ago, and it's still as fresh as ever.

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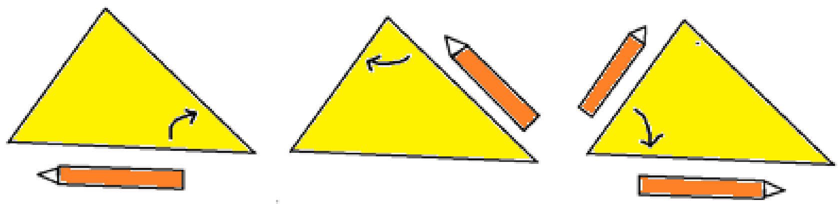
Heuristic

Another helpful proof-word is *heuristic*, which roughly stands for ‘rule of thumb’. This will be a non-rigorous but nonetheless compelling argument that might well include a calculation of likelihoods (an appeal to probabilities) along the way. It is likely to be attractive to intuition, but will be regarded as speculative, a pointer to truth or falsity that will need tidying up before it can be regarded as secure.

Mathematics has been most advanced by those who distinguished themselves by intuition rather than by rigorous proofs. **Felix Klein**

What might a heuristic proof look like? We know that the angle sum of a triangle is 180^{o} or half a turn. We could establish this by moving a pencil around the edges of a triangle.



After rotating the pencil clockwise through the three angles, it faces in the opposite direction - it’s been rotated through exactly half a turn.

We might be completely happy with this, but others might say that our working needs to be part of a systematic structure like Euclid’s to make it rigorous. The interplay between intuition and rigour is a leitmotiv in mathematics.

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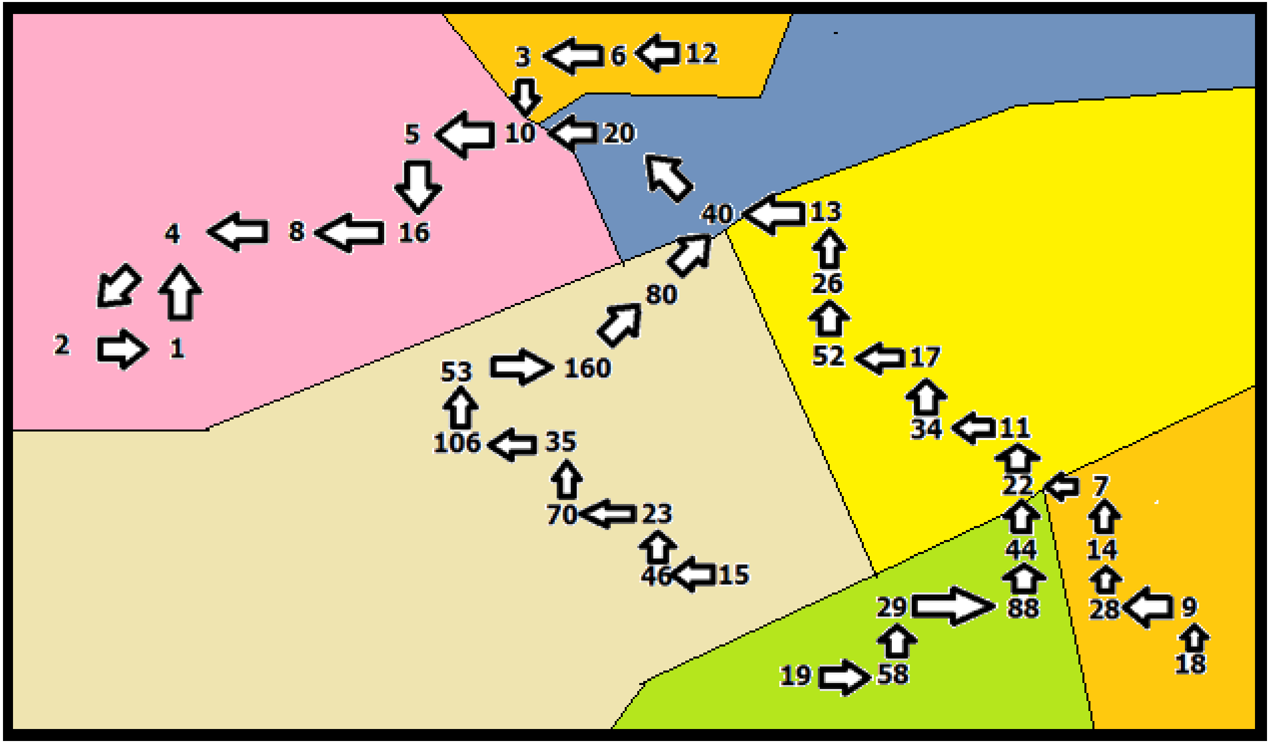
17 Notes: The HOTPO Problem

Status: unsolved ***Name: the Collatz conjecture***

Apologies to those of you that hate acronyms. This problem has assumed a range of names; the Collatz conjecture (Lothar Collatz is said to have first posed the problem in 1937), the Ulam conjecture, the Thwaites conjecture, the list goes on.

Level 1 and **2:** We can try the first 20 whole numbers and see what happens to them. The results are in the diagram below;


3
6
12
5
10
20
40
13
4
8
16
26
80
2
1
52
17
53
160
34
11
106
35
22
7
70
23
44
14
46
15
29
88
28
9
18
19
58

Level 3: We can see how there'll be infinitely long strands reaching out from the 1-2-4 cycle; the powers of 2, for example, will form one such strand. But it's not inconceivable that there might be some loop out there like the 1-2-4 one, only with much larger numbers. Once in such a cycle, the rule would mean any strand that led to a number in the loop would cycle around endlessly, with no possibility of reaching 1, and the computer would run forever (unless the plug were to be pulled out). All we can say is that no such cycle has been found.

Looking at the diagram above, we can see that certain numbers, 10, 40, 22, for example, are branching points. The diagram below helps to explain.

even parent	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
offspring	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
odd parent				1						3						5

The numbers in blue here are those offspring of the form 3n + 1. Every number has at least one HOTPO parent; clearly 2n will always lead to n when the HOTPO rule is applied. But the branching points (like 4 and 10), those with two parents, are less common than those with one. The number 7 is of the form 3n + 1, but n here is 2, so 2 cannot parent 7, for it has to obey the halving rule. The branching points are those of the form ‘3 times an odd number plus 1’, or 3(2k+1) + 1, or 6k + 4, numbers like 4 and 10.

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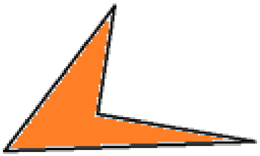
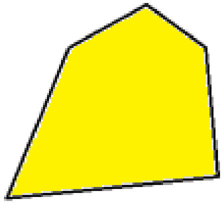
Convexity

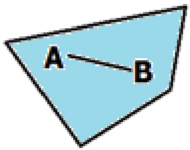


Concave

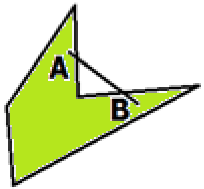


Convex

This is a simple idea; shapes that have indentations are called concave, those that don’t are called convex.



Convex



Concave

More precisely, if we have a convex shape, for all choices of two points A and B in the shape, every point on the straight line AB is also in the shape.

The same does not apply to the concave shape, where we can pick points A and B so that the line AB is outside the shape for some of the time.

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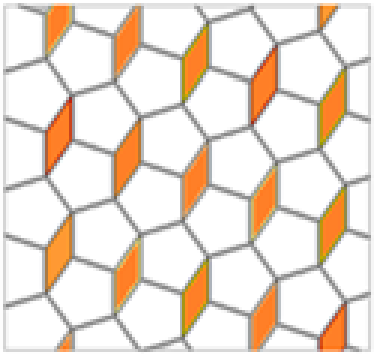
18 Notes: The One Tile Problem

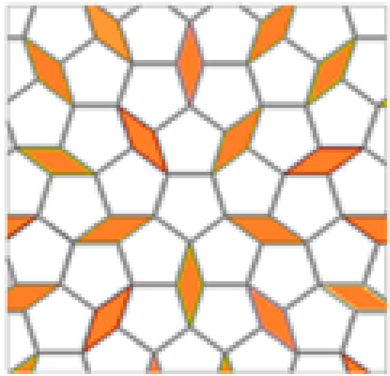
Status: unsolved

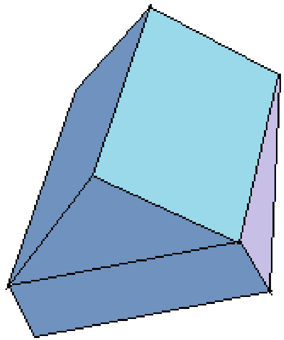
Name; the einstein problem (not because Einstein dreamt it up, but because 'ein stein' means 'one tile' in German)

Level 1:



Level 2:





Level 3: The curious thing about this problem is that it's solved for three dimensions, but not for two. The Schmitt-Conway-Danzer tile above only tiles non-periodically in three dimensions. It's called a biprism, since it's formed by gluing two prisms together. It's convex, and it was found in 1993. The two-dimensional problem, however, remains unsolved.

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Elementary

You'll see sometimes a proof described by a mathematician as ***elementary***. That doesn't mean quite what it appears to mean; some elementary proofs can be seriously involved, and require plenty of effort to understand. It means ***not resorting to advanced mathematics***. The logic may be lengthy, but the heavy artillery has been left at home. The language of 'differentiable manifolds' and ' Zermelo–Fraenkel set theory', and 'Galois representations' will not feature. Pre-university maths will be to the fore.

In particular, one definition of an elementary proof promises to make no use of complex numbers. These arise when one decides to embrace the idea of the square root of a negative number, rather than outlaw it. If we say **$\sqrt{(-1)} = i$**, then we can invent the number **$a + bi$**, where a and b are real numbers. The number **$a + ib$** is called a ***complex number***. These turn out to be staggeringly useful in advanced mathematics (they appear in the Types of Number diagram).

Friedman has conjectured, 'If you can state it simply, you can prove it simply'. If true, this will mean there's an elementary proof of Fermat's Last Theorem (Andrew Wiles's proof would certainly not count as elementary). For the mathematician Hardy, some results were too deep to have an elementary proof.

Certain theorems that only have advanced proofs are challenges to those committed to elementary methods. The Prime Number Theorem, describing the distribution of prime numbers, could initially only be proved via advanced techniques. In 1948, however, Erdős and Selberg found an elementary proof.

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19 Notes: The Are-The-Infinities-Equal Problem

Status: solved ***Name: Cantor's diagonal argument***

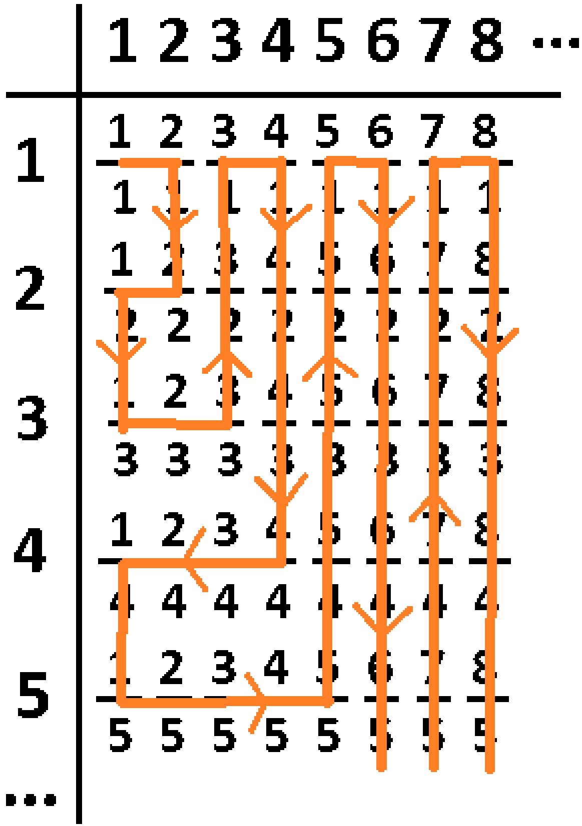
Level 1: Are there more integers than natural numbers? No, since we can pair off the two sets of numbers successfully.

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots \end{array}$$

We say that the integers are ***countable.***

Level 2: The rational numbers are also countable - they CAN be put into a list, like this:

	1	2	3	4	5	6	7	8	…
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$	$\frac{8}{1}$	
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$	
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	
4	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$	
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	
…									



The arrows on the right-hand diagram give you the order for the list, which you can see contains every positive fraction (we can simply cross off equivalent fractions as they crop up).

Level 3: Georg Cantor spent much of his life thinking about infinity. He found a wonderful argument to show that the infinity of numbers between 0 and 1 is greater than the infinity of counting numbers.

Suppose that the numbers from 0 to 1 are countable. Then we can put them into a list, describing them by their decimal expansion:

1. $0.\ a_1\ a_2\ a_3\ a_4\ \ a_5\ \ a_6 \ldots$
2. $0.\ b_1\ b_2\ b_3\ b_4\ b_5\ b_6 \ldots$
3. $0.\ c_1\ c_2\ \ c_3\ c_4\ \ c_5\ \ c_6 \ldots$
4. $0.\ d_1\ d_2\ d_3\ d_4\ d_5\ d_6 \ldots$
5. $0.\ e_1 \ldots\ldots\ldots\ldots\ldots\ldots$

Form a new number x, where

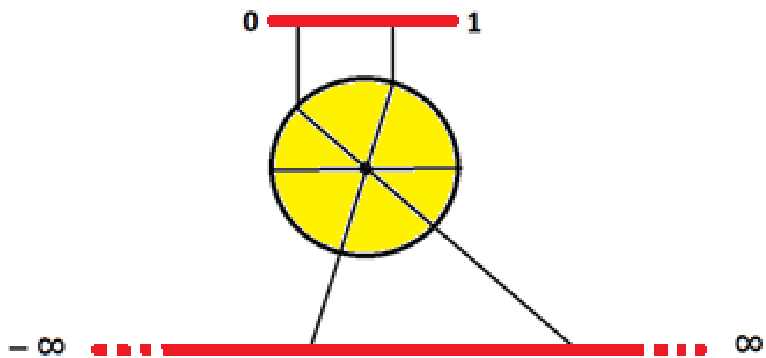
$$\mathbf{x = 0.a_1'b_2'c_3'd_4'\ldots}$$

and where a_1' differs from a_1, b_2' differs from b_2, and so on.

Now think about where x appears in our list. It cannot appear in position 1, since the first digits disagree. Nor can it appear in position 2, for the same reason. Indeed, it disagrees with the n^{th} number in the list at its n^{th} digit, but it's certainly a number between 0 and 1.

The only possible conclusion is that we were wrong to suppose that these numbers can be put into a list, and so the infinity of numbers between 0 and 1 is a bigger infinity that that of the counting numbers.

The diagram below shows that the infinity of numbers between 0 and 1 (with 0 and 1 excluded) is the same as the infinity of numbers on the number line between -∞ and ∞. Given any starting number between 0 and 1, we can find a unique number between -∞ and +∞ by crossing the yellow circle through its centre, and vice versa.



Another question comes out of this; is there a level of infinity between that of the natural numbers and that of the numbers between 0 and 1? This question (called the Continuum Hypothesis) absorbed mathematicians for many years (it was the first of Hilbert's list of problems in 1900). It's been resolved, by Kurt Godel and Paul Cohen. They found that it's possible to build a coherent mathematics that DOES include such an infinity, yet it's also possible to build another version of mathematics that does NOT include such an infinity, which works just as well. We are free to include the existence of this infinity as an axiom one way or the other.

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Mathematics and Music

I'm walking in the park with my friend James, who researches into mathematics at a nearby university. In his area, he's regarded as one of the top mathematicians in the country.

Jonny: Would you say, James, that doing original pure maths is like composing music?

James: Sounds promising, Jonny - can you elaborate on that?

Jonny: Okay - I'd say that when it comes to mathematical research, you're improvising concert piano to a full house at the Festival Hall, while I'm crashing out grade seven pieces at home with my partner telling me to keep it down in the background.

James: But if they're GOOD grade seven pieces…

Jonny: I’m saying that your mathematical technique and mine cannot sensibly be compared. But then, my mathematical intuition might one day turn out to be better than expected; I can at least dream!

James: I like your analogy between composing and proving, but I really don’t think I’m playing in big concert halls. For me, doing research mathematics is a much more private thing.

Jonny: So it’s not as glamorous as it sounds?

James: No, it’s quite introverted. The professional mathematics research community is like an intensely close network of modernist composers. It’s certainly true that we all know a lot of music, and performing might be something some of us are good at, it might not, but really what we do…

Jonny: You’re sounding gloomy, James!

James: We like writing new music. That bit's fine, but some might say that we like writing music that is so inaccessible that no one likes listening to it except us. And sometimes, I'm not even sure that we do.

Jonny: That can't be true all the time?

James: Oh, a mathematician like Andrew Wiles comes along once a century. But there's almost a snobbery here; the more accessible your music is, the less deep it must be.

Jonny: But is some of this music seen as better than some other music?

James: Yes, we might manage to get our other composer friends to agree that we've created something worth listening to. We're all trying to create something that will still be around and listened to when we're gone, but there's no guarantee of that. In fact, we all know that it's only a tiny percentage of what we produce between us that will be listened to and built on in a hundred years' time. Most mathematics, most music, will be forgotten. Including mine, unless I'm really lucky.

Jonny: Is the truth more, James, that you just had a bad day composing yesterday?

James: You’ve rumbled me, Jonny! Shall we go and feed the ducks?

Home

20 Notes: The Even = Prime + Prime Problem

Status: unsolved ***Name: The Strong Goldbach Conjecture***

Level 1: **20 = 13 + 7, 22 = 19 + 3, 24 = 13 + 11, 26 = 13 + 13, 28 = 11 + 17, 30 = 17 + 13.**

Level 2: **100 = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53** (six possible ways).

Level 3: Of all the unsolved problems in mathematics, Goldbach's Conjecture is one of the oldest and most famous. Most people would claim that Goldbach's conjecture says this:

Every even number bigger than 2 is the sum of two prime numbers.
(We'll call this **Statement A**.)

In fact, to call **Statement A** 'Goldbach's Conjecture' is not quite historically accurate. Goldbach's original conjecture, posed in a letter to Euler in 1742, said this;

Every whole number greater than 5 is the sum of three prime numbers. (**Statement B**.)

Euler in a reply to Goldbach gave the **Statement A** formulation, which is the one most people remember today.

In fact, Euler's conjecture is stronger than Goldbach's (stronger here means 'implying more mathematically', and it also generally means 'harder to prove').

Let's assume that **Statement A** is true; given any number n we can write it as the sum of three primes.

If n is even, then n - 2 is even, and we can write that as the sum of two primes, since we are assuming **Statement A**.

If n is odd, then n - 3 is even, and we can similarly write that as the sum of two primes; either way, we can write n as the sum of three primes (if n is bigger than 6), and Statement B is true.

So we have that **Statement A** implies **Statement B**; is the converse true? Does **Statement B** imply **Statement A**?

The answer is, 'No'. Given an even number n, consider m > n. Now we know m splits into three primes (and it might do this is lots of different ways), but we require two of the primes in one of these three-prime-collections to add to n, and there is no guarantee of that. Okay, if it doesn't work for m, then we can try m + 1, m + 2, m + 3 ... splitting each of these into three primes in as many different ways as we can. It seems extremely likely that eventually we'll find a pair of primes that will add to n, but we can't rely on it.

Both **Statements A** and **B** have defied proof until recently, when it was announced, to great fanfare, that a proof of the Weak Goldbach Conjecture had been claimed. Tom Ward, one of my MSc supervisors, was jubilant; 'How fortunate we are that this has happened in our lifetimes.' The proof has now been checked, and we can say definitively that Harald Helfgott in 2013 proved the Weak Goldbach Conjecture. Goldbach glory-hunters only have the Strong Conjecture to pursue now.

It should be said that most mathematicians believe that Goldbach's Strong Conjecture is true; there's just too much circumstantial evidence suggesting that's the case. The graphs all point one way. Certainly any counterexample will be large; computers have checked every even number up to 4×10^{18}. We would need good luck to find one...

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Ambivalent Thoughts on Proof

Mathematicians like talking about proof, and their comments are not always sanguine. If you spend many hours with a problem on a particular line of attack without success, the frustration can become intense. ‘Why do we do it?’ implored Graham Everest, another of my MSc professors. Certainly the word ‘torture’ features often in proof quotations.

Proof is an idol before whom the pure mathematician tortures himself.
Arthur Stanley Eddington

Torture numbers, and they'll confess to anything. **Gregg Easterbrook**

Young man, in mathematics you don't understand things, you just get used to them.
John von Neumann

Now another reason to be disappointed mathematically; there are some things we desire to be true with all our hearts, but they are sadly false.

A tragedy of mathematics is a beautiful conjecture ruined by an ugly fact. **Anon**

Hastiness in claiming proofs is also ill-advised.

The Golden Rule of Proof: ***Never trust any result that was proved after 11 pm.*** **Anon**

Sometimes mathematics itself may have limitations that we can’t argue with. Kurt Gödel famously proved in 1930 that in any mathematical system complicated enough to include arithmetic, there would be true results that could not be proved. He hoped additionally to prove mathematics to be consistent (contradiction-free) but this proved to be impossible within mathematics itself.

God exists since mathematics is consistent, and the Devil exists since we cannot prove it. **André Weil**

And finally,
We often hear that mathematics consists mainly of 'proving theorems.' Is a writer's job mainly that of 'writing sentences’? **Gian-Carlo Rota**

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21 Notes: The Dot-Line-Region Problem

Status: solved ***Name: Euler's Theorem (warning; Euler has lots of theorems!)***

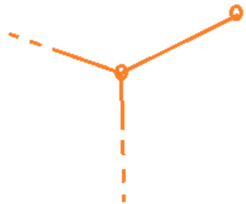
Level 1 and **2:** You've drawn up a table and counted vertices, regions and edges carefully. You should find that V + R - E = 2, always. This is Euler's formula, one of the most profoundly helpful theorems in mathematics.

Level 3: Can you prove it? Let's try a proof by contradiction. Suppose that Euler's formula is untrue. We could first point out that it's certainly true for all connected graphs with one edge.

V + R - E = 2 + 1 - 1 = 2

Now we suppose that our conjecture is false, and that we have before us the smallest graph to show the conjecture to be untrue (we could call this graph 'the minimal criminal'). If this graph has E + 1 edges, then we know that any graph with E edges or less will obey the **V + R - E = 2** rule. From our work above, we know E is 1 or bigger.

Now let's make this 'minimal criminal' graph smaller. If there's a vertex-edge combination like the one below somewhere in the graph (where the degree of the vertex is 1), remove the edge and the vertex, reducing V by 1 and E by 1.



The value of **V + R - E** will be unaffected.

If there's no vertex of degree 1, then remove any edge that does not disconnect the graph into two separate parts (any extended path will either create a cycle, where we can remove any edge, or end in a degree 1 vertex). This will not isolate any vertex, but will have the effect of reducing the number of regions by 1. Again, the value of **V + R - E** will be unaffected.

Either way, we now have a graph with one less edge, that is with E edges, so we know **V + R - E = 2,** which must therefore be also true for our original graph with E + 1 edges. So we have a contradiction, and the only way out is to say that our initial assumption that there is a 'minimal criminal' graph where **V + R - E** is not 2 must be at fault, and so we are done.

If you want to quibble with this proof, there are some awkward questions you might ask. But if we're not too wrapped up with the question of rigour, this argument hopefully satisfies our intuition.

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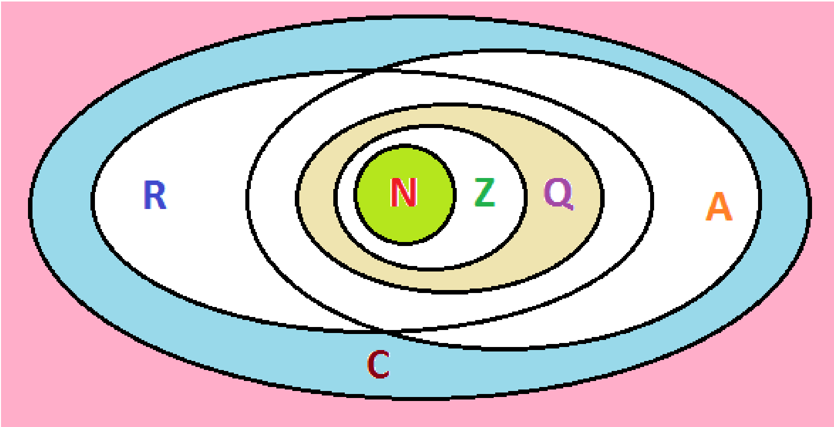
Types of Number

Equation	Solution	Type of number	Set	Name
$x + 1 = 2$	$x = 1$	Natural	{0, 1, 2, 3...}	$\mathbb{N}$
$x + 2 = 1$	$x = -1$	Integer	{...-2, -1, 0, 1, 2...}	$\mathbb{Z}$
$2x + 1 = 0$	$x = -1/2$	Rational	{a/b, where a, b are in $\mathbb{Z}$, b non-zero}	$\mathbb{Q}$
$x^2 - 2 = 0$	$x = \pm\sqrt{2}$	Algebraic	Solutions to polynomials with integer coefficients	$\mathbb{A}$
$x^x - 2 = 0$	$x = ?$	Real	Any number on the number line	$\mathbb{R}$
$x^2 + 1 = 0$	$x = i = \sqrt{-1}$	Complex	{a + bi, a, b in $\mathbb{R}$}	$\mathbb{C}$



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22 Notes: The Three Averages Problem

Status: solved

Level 1: (3+6)/2 = 4.5, √(3.6) = 4.24…, 2.3.6/(3+6) = 4, so AM > GM > HM here.

Level 2: Let the distance from A to B be d, let the time taken to travel from A to B be t,

and let the time taken to travel from B to A be T.

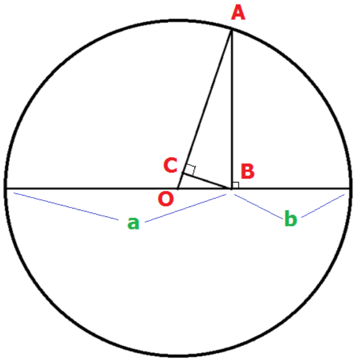
Time = distance/speed, so t + T = d/x + d/y.

Average speed for whole trip = total distance/total time

$$= \frac{2d}{\frac{d}{x}+\frac{d}{y}} = \frac{1}{\frac{\left(\frac{1}{x}+\frac{1}{y}\right)}{2}}$$ = harmonic mean (HM) of x and y = $\frac{2xy}{x+y}$.

Notice the distance from A to B does not come into the solution; it cancels out.

Level 3: There is a nice geometrical way to look at this problem. Take your two numbers a and b, and draw them like this.



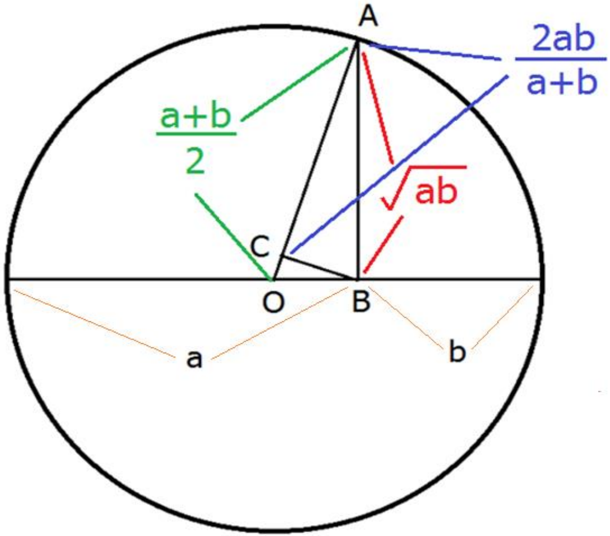
Clearly from the diagram, OA $\geq$ AB $\geq$ AC, since the hypotenuse of a right-angled triangle has to be the longest side.

But OB is $\frac{a-b}{2}$, and OA is $\frac{a+b}{2}$. By Pythagoras,

$$AB^2 = OA^2 - OB^2 = \frac{a^2}{4} + \frac{ab}{2} + \frac{b^2}{4} - \left(\frac{a^2}{4} - \frac{ab}{2} + \frac{b^2}{4}\right) = ab \Rightarrow AB = \sqrt{ab}.$$

By similar triangles, BC/AB = OB/OA, so BC = $\frac{(a-b)\sqrt{ab}}{a+b}$,

and so AC is (by Pythagoras) $\sqrt{ab - \frac{ab(a-b)^2}{(a+b)^2}} = \frac{2ab}{a+b}$.



So we have proved AM ≥ GM ≥ HM for all positive number pairs a and b, and we can see from the diagram that equality only holds when a = b, when A is at the top of the circle.

Using a Computer - 1

You may never have done any computer programming, but when you're searching for solutions to an equation, a program can be of enormous benefit. It really is (honestly!) easy to get started with coding, and once you've started, there is no telling where you might finish.

I'm going to assume you have access to the Microsoft Excel program. Lots of people use Excel without knowing that it has the programming language Visual Basic built into it. I'll show you how it works.

Open a spreadsheet, and go to the Developer tab; if you don't have that showing, searching the internet with

developer tab excel

for advice should help. Now click on Insert, and in the ActiveX controls section, click on the Command Button (if you hover over the options, it will tell you which is which).

Note: if Command Buttons are not enabled on your computer, search the net for help.

Now click somewhere on the spreadsheet, and the Command Button should appear. Double click on it, and you get a page where you can enter your Visual Basic code.

Private Sub CommandButton1_Click()

Space for code

End Sub

A simple program to be inserted into the 'Space for code' might say:

```
s=0
'puts the initial value of s to 0
For a = 1 to 10
'starts with a = 1, increasing a by 1 each time you reach 'Next a', finishing when a gets to 10
s = s + a^2
'the new value for s is the old value for s + a squared.
Next a
'move on to the next value of a, going back to the 'For' statement.
Cells(1,1) = s
'insert the final value for s into Row 1, Column 1.
```

The green lines above starting with an apostrophe are simply comments, and explain what each line does without having any effect on the program. Can you tell what the program does?

Try typing out the program (you don't need to put in the comments) into the space provided, then click on Design Mode on the Excel toolbar to come out of this.

Now – very important – save your file, which includes your program, as an .xlsm file! The danger is that if something goes wrong when you run the program, you might lose your code.

Now run the program, by clicking on your Command Button. This program adds together the first 10 square numbers, so you should end up with 385 in Cell A1.

Note; if your program goes awry and the screen seems to freeze, try pressing Esc.

You may not need much more than For-Next loops, the Cells(1, 1) notation, and the If-then statement to build some handy programs. A nested pair of For-Next loops might look like this:

```
s = 0
For a = 1 to 10
For b = 1 to 10
s = s + a*b
Next b
Next a
Cells(1, 1) = s
```

Can you see what this program does?

For each of the ten values for a, b runs through 1 to 10, which makes 100 calculations of s in total. This program adds together all the numbers in the white area in the grid below. The answer 3025 will hopefully appear in Cell A1.

x	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Using a Computer – 2

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23 Notes: The How-Many-Powers Problem

Status: unsolved ***Name: The Waring Problem***

Level 1:

0	0			
1	1			
2	1	1		
3	1	1	1	
4	4			
5	4	1		
6	4	1	1	
7	4	1	1	1
8	4	4		
9	9			
10	9	1		

11	9	1	1	
12	4	4	4	
13	9	4		
14	9	4	1	
15	9	4	1	1
16	16			
17	16	1		
18	9	9		
19	9	9	1	
20	4	16		

The numbers 0, 1, 4, 9 and 16 need just one square, 2, 5, 8, 10, 13, 17, 18 and 20 require two, while 3, 6, 11, 12, 14 and 19 need three, and 7 and 15 require four.

So if we use 0^2 to fill in the gaps, we can say that every whole number from 0 to 10 can be written as the sum of four squares.

Level 2:

1000	900	100		
1001	900	100	1	
1002	900	100	1	1
1003	27^2	15^2	7^2	
1004	900	100	4	
1005	28^2	14^2	5^2	
1006	31^2	6^2	9	
1007	31^2	6^2	9	1
1008	900	100	4	4
1009	28^2	15^2		
1010	31^2	7^2		

There are two numbers that insist on four non-zero squares to construct them, 1007 and 1008. We could justifiably come up with this conjecture:

Every positive integer can be written as the sum of four squares.

Or are there any numbers out there that need more than four squares to construct them?

Level 3: This is Waring's conjecture, proposed in 1770:

Given k, we can find g(k) so that every positive integer is the sum of g(k) k^{th} powers.

David Hilbert proved Waring's conjecture to be true in 1909. The first few values of g(k) are in the table below.

k	1	2	3	4	5	6	7	8	9
g(k)	1	4	9	19	37	73	143	279	548

You can see that four squares are indeed enough to represent any natural number. This question was resolved by the mathematician Lagrange in 1770.

The table tells us that 9 cubes are enough to represent any natural number. The smallest number that actually needs 9 cubes is 23, and the smallest number after that is 239 (in fact, these are the only two numbers to need 9 cubes).

From working on the problem, you can see that there are fewer possible choices for the powers with small numbers; once they get bigger, it's easier to find a combination that'll work. It makes sense, then, to ask, how many k^{th} powers do large numbers need?

How large is large? As large as we like. So instead of thinking about g(k), we can switch our attention to G(k), which is

the smallest number of k^{th} powers needed to represent every whole number beyond a certain point.

This must means that $G(k) \leq g(k)$ for all k.

The value of G(2) is definitely 4, since it can be shown that any number that gives a remainder of 7 when divided by 8 cannot be the sum of 3 squares.

How? Write any number m as 8n + j, with $0 \le j < 8$ (so j is the remainder when we divide m by 8). This means that

$$m^2 = (8n + j)^2 = (\text{multiple of } 8) + j^2.$$

Now **j^2 = 0, 1, 4, 9, 16, 25, 36, or 49.**

When we look at the remainders here when we divide by 8, these are **0, 1, 4, 1, 0, 1, 4, 1.**

So if we could write 8n + 7 as the sum of three squares, three of 0, 1 and 4 (with repeats) will add to 7, but this can't be done.

The value G(3) is known to be 7 or less. The value of G(4) is 16; this was proved by Davenport in 1939, but no other values for G(k) are known.

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A rational close to an irrational

The rational numbers are the fractions, those numbers a/b where a and b are whole numbers, and where b is not 0.

If we are given an irrational number, we can find a rational number that is as close as we like to it. You doubt me? Write your irrational number out as a decimal, say

1.2491763978051...

If the expansion starts to recur, we can sure that the number is, in fact, rational. Why? An example; suppose

x = 1.2345234523452345... is a recurring decimal,

so **10 000 x = 12345.234523452345....**

We can now subtract these two equations, so **10000x - x = 12344,**

which means **9999x = 12344,** which means **x = 12344/9999,** and so x is rational.

So any irrational number has a non-recurring decimal.

Returning to **1.2491763978051...**

Think about the sequence **1, 1.2, 1.24, 1.249, 1.2491, ...**

All of these numbers are rational, since for example, **1.249 = 1249/1000.** So we have a sequence of rational numbers getting ever closer to the value of the irrational number.

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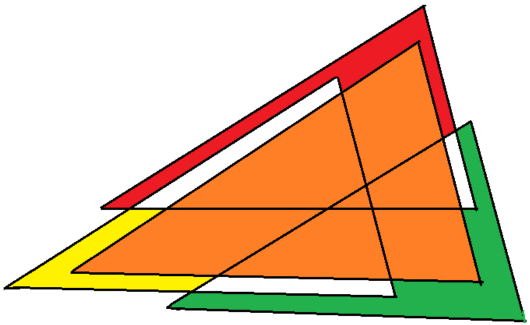
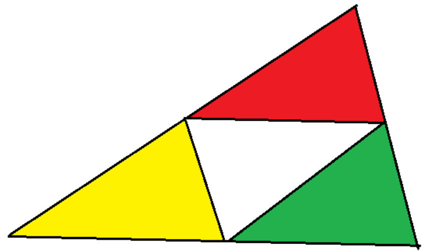
24 Notes: The Shape-Covering Problem

Status: unsolved ***Name: The Levi-Hadwiger Shape Covering Conjecture***

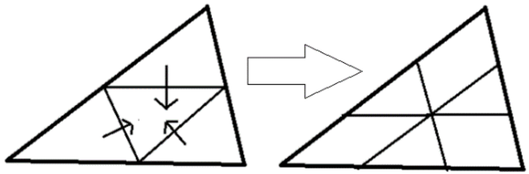
Level 1: When it comes to covering Triangle A with Triangle Bs, you should be able to make a Triangle B so that the answer is 3. Each corner has to be covered, and no one triangle can cover two corners, so you certainly can't improve on 3 (below left).





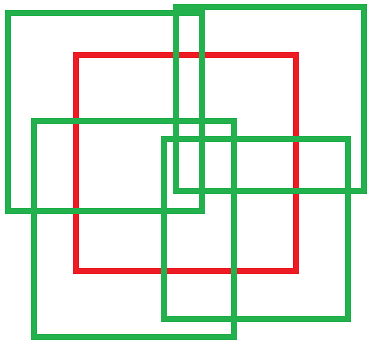
What is the smallest that Triangle B can be for this to work? If the Triangle B lengths are half those in Triangle A, three Triangle Bs would leave a gap in the middle (above right).

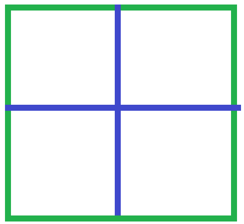
We could imagine expanding the three edge triangles here until Triangle A is covered:



In this case Triangle B's edges are 2/3 the size of the starting triangle's.

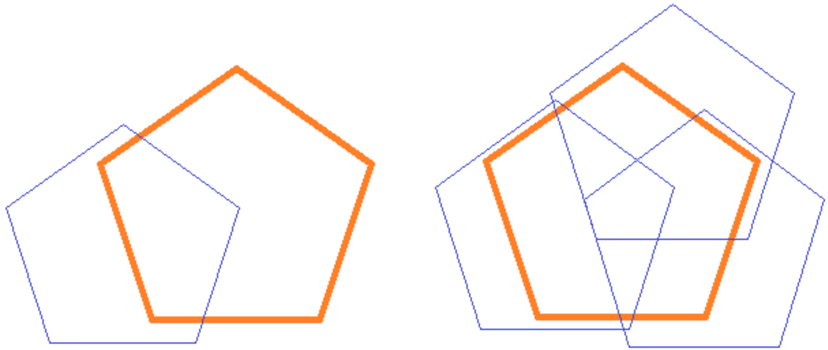
Level 2: How would this work for a square? Once again, each corner needs one of the smaller square copies to cover it, and once again, each of these can only cover one corner at most. So we definitely need four smaller square copies to cover the bigger square.





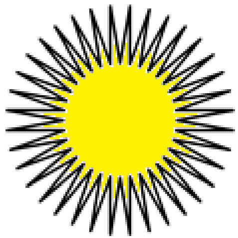
The smallest possible squares that will still cover the big one as a foursome are half the side-length of the original.

With the pentagon, things change. Here a smaller pentagon can cover two corners, and it’s clear that three pentagons will do the job.

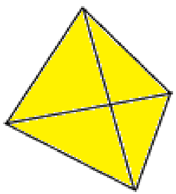


So we know what happens for squares - what about rhombuses? Kites? Parallelograms? Trapezia? Rectangles? We can manage with three copies for kites with different side lengths, and irregular trapezia, but parallelograms, which include squares, rectangles, and rhombuses, require four copies.

We do have a problem with 'pointy' shapes with lots of corners. It's far from certain that every corner can be covered by a small number of copies. It makes sense therefore to restrict our attention to shapes that are *convex*, which effectively means, *without indentations*.



Level 3: In three dimensions, the simplest object is a tetrahedron.



Once again, each corner needs a dedicated smaller tetrahedron to cover it, so the best we can do here is 4. A cube? This has eight corners, and a smaller cube can only cover one of them, so eight cubes are needed.
If we get beyond the cube, then it seems that things start to get easier again, as they did in the 2 dimensional case; we need fewer copies.

This conjecture now seems plausible:

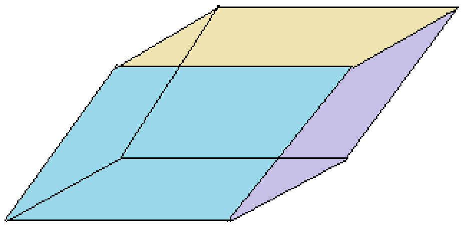
If we're working in n dimensions,
then 2^n smaller copies of a convex Shape A will always be enough to cover Shape A.

We have been working in 2 and 3 dimensions; what if feels like to work in more than 3 dimensions takes a bit of imagination!

In fact, it's possible for us to sharpen the above conjecture up.

The upper bound of 2^n is necessary if and only if the shape is a parallelepiped.

A parallelepiped is the higher-dimension version of the parallelogram; its opposite faces are parallel. In three dimensions, a parallelepiped has six faces, in three parallel pairs, where each face is a parallelogram.



So how does this conjecture currently stand? In two dimensions, the question was settled by the mathematician Levi in 1955. Three copies are enough to cover any shape EXCEPT for parallelograms; these need four.

There has been a proof for simplexes (a ***simplex*** is an n-dimensional version of a two-dimensional triangle; so for example, a simplex in three dimensions is a tetrahedron). A simplex can always be covered by n + 1 copies of itself (we checked this above for n = 2 and 3) and the scaling required is n/(n+1) (we showed this for n = 2).

But for all convex shapes in three dimensions, the question is still unresolved. It's been proved that 16 copies are enough to cover any convex three-dimensional shape, but that's a long way away from the eight copies that the conjecture suggests should be enough.

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Deep

Deep theorems are what every mathematician dreams of proving. They're fundamental, ground-breaking ones that link together previously disparate parts of the subject. They may be stunningly general in their application; they may be deceptively simple to state, but really tough to prove. Even conceiving of a deep conjecture is hard; this tends to require years of immersion. Other descriptions; a deep proof is difficult, ubiquitous and influential. They are far-ranging, and involve fundamentally new ways of thinking. The kind of work (by Taniyama, Shimura and Wiles) underlying the proof of Fermat's Last Theorem qualifies, that much is agreed. Today there's a group founded by the mathematician Robert Langlands that aims for such deep theorems in the area of number theory. Even extremely accomplished mathematicians are often likely to find its work impenetrable.

Will any of the unproved conjectures offered in this book class as deep theorems once proved? Perhaps we'll only know once that happens. Or perhaps they'll turn out be footnotes to other, much deeper, theorems.

The law of conservation of difficulties: *there is no easy way to prove a deep result.* **Anon**

‘Deep’ usually means, ‘I do not understand its proof.’ **Anton Petrunin**

A result is deep if it depends on a breakthrough idea, that depends on a breakthrough idea, that depends on a breakthrough idea, that depends on a breakthrough idea, that…
Tim Gowers

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25 Notes: The Four-in-a-Bag Triple Problem

Status: unsolved

One day in 2002, I was sitting at a desk trying to write a worksheet for my GCSE resit class. The topic was equations, and I wanted something to link the problems together, rather than setting a dry list of unrelated exercises. I came up with the 'Four-in-a-Bag' task.

Level 1: Suppose the numbers in the bag are a, b, c and d. We can order these in 24 different ways into the circles, but ax + b = cx + d is the same equation as cx + d = ax + b, so there can be at most 12 different solutions. We can certainly get 12 different solutions; if 1, 2, 3, 8 are in the bag, then we get the set of solutions

S = {3, -3, 1/3, -1/3, 5, -5, 1/5, -1/5, 7,- 7, 1/7, -1/7}.

This suggests that if k is a possible solution, then -k, 1/k and -1/k will also be possible solutions, and that's easy to show.

$$ak + b = ck + d \Rightarrow a + b/k = c + d/k,$$
$$ak + b = ck + d \Rightarrow (-k)c + b = (-k)a + d,$$
$$ak + b = ck + d \Rightarrow a + d(-1/k) = c + b(-1/k).$$

Level 2: Only one of k, -k, 1/k, -1/k can be an integer bigger than 1. Given that our solution set is three such quartets, then S can contain at most three whole numbers a, b and c all bigger than one.

Our Level 1 example above shows that S can contain three positive integers greater than 1 (7, 5 and 3 in this case).

If this is what happens, then I've called these three natural numbers a Hikorski Triple, or an HT. (To explain, I was playing a character called Max Hikorski in a college musical the day I thought of these things.) So we have from our example that (7, 5, 3) is an HT.

How are 7, 5, and 3 related? A Pythagorean Triple is given by (a, b, c) where $c = \sqrt{(a^2 + b^2)}$. If (p, q, r) is an HT, is r some function of p and q?

If ax + b = cx + d, then we have x = $\frac{d-b}{a-c}$.

So if a < b < c < d, and if we want one positive solution from each of our three quartets, we could choose $p = \frac{d-a}{c-b}, r = \frac{d-b}{c-a}, q = \frac{d-c}{b-a}$.

If we now eliminate a, b, c and d, they cancel obligingly to give $r = \frac{pq+1}{p+q}$.

So to check then; if p = 7, and q = 5, does $3 = \frac{7 \times 5 + 1}{7+5}$? Yes, it does!

So $\left(p, q, \frac{pq+1}{p+q}\right)$ is an HT iff p ≥ q ≥ $\frac{pq+1}{p+q}$ ≥ 1 are all natural numbers. We can see that

(n, 1, 1) is always of the form $\left(p, q, \frac{pq+1}{p+q}\right)$; these are *the trivial HTs*.

How common are HTs? There are lots; they are, for example, more plentiful than Pythagorean Triples.

Level 3: If you multiply the three elements of an HT together, you never seem to get the same number.

a	5	7	11	9	19	11	13
b	3	5	4	7	5	9	8
c	2	3	3	4	4	5	5
abc	30	105	132	252	380	495	520

a	17	13	29	15	19	23	41
b	7	11	6	13	11	10	7
c	5	6	5	7	7	7	6
abc	595	858	870	1365	1463	1610	1722

So the HT Uniqueness Conjecture is:

if (a, b, c) and (p, q, r) are non-trivial HTs so that abc = pqr, then (a, b, c) = (p, q, r).

Now the curve $xy\frac{xy+1}{x+y}=k$ is an interesting one; it's an example of an ***elliptic curve*** (a key tool for Andrew Wiles in proving Fermat's Last Theorem). There's a famous theorem that says the number of integer points (points where both coordinates are integers) on such a curve is finite. The HT uniqueness conjecture would seem to imply that as k varies, we get a family of elliptic curves that all have an identical number of easily-locatable integer points. This is highly unlikely, and observers say the HT Uniqueness conjecture is therefore probably false. On the other hand, the rewards for proving it to be true if it were would be substantial!

There's another unsolved uniqueness problem generated by an integer triple, and that's to do with Markov Triples. A Markov Triple (a, b, c) (called an MT) is three positive whole numbers that satisfy the equation

$a^2 + b^2 + c^2 = 3abc.$

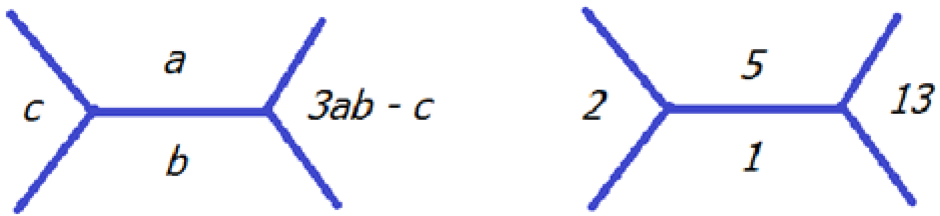
Early MTs are **(1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 5, 13), (2, 5, 29),(1, 13, 34), (1, 34, 89), (2,29,169)…**

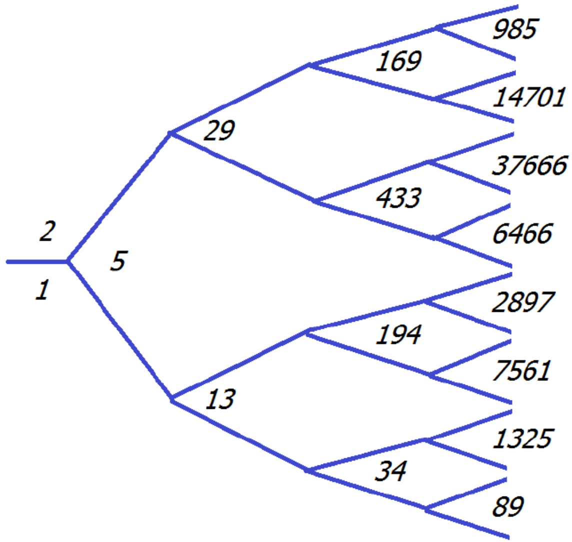
A little calculation confirms that if (a, b, c) is an MT, then (a, b, 3ab - c) will be one too, since

$$a^2 + b^2 + (3ab - c)^2 = a^2 + b^2 + 9a^2b^2 - 6abc + c^2 = 9a^2b^2 - 6abc + 3abc = 3ab(3ab - c).$$

So if we start with (1, 1, 1) this generates (1, 1, 2). We can put all the Markov numbers (any number that appears in an MT is said to be a Markov number) into a tree pattern.

Example





So where's the unsolved problem here? It's this; can we show that no Markov number appears twice in this tree? A lot of people have tried, but no one has succeeded.

The How-Many-Ways Numbers

Suppose we have four cards labelled A, B, C and D. In how many ways can we pick two of them?

We have AB, AC, AD, BC, BD and CD (the order of picking doesn't matter here) so the answer is 6 ways.

We can say, **the number of ways of picking r things from n things is ^nC_r.**

So in our example, we've been looking at $^4C_2 = 6$.

You may have noticed a button on your calculator for ^nC_r, and most useful it is too.

I need to introduce the *factorial* function at this point, another button on your calculator (this time labelled !). We have **$n! = n \times (n - 1) \times (n - 2) \dots \times 1$,** so to get n! we have to multiply all the whole numbers less than or equal to n together.

Thus we have **4! = 4.3.2.1= 24, 5! = 120** and so on. The factorial function gets very big very quickly. If you try 100! on your calculator, it'll probably turn you down.

There's a formula for ^nC_r that uses factorials, which is $\mathbf{^nC_r} = \dfrac{n!}{r!(n-r)!}$, so $\mathbf{^4C_2} = \dfrac{4!}{2!2!} = \mathbf{6}$.

An alternative way to write $\mathbf{^nC_r}$ is $\dbinom{n}{r}$.

There are some useful facts about our ^nC_r numbers (these are often called ***binomial coefficients***). The number $^nC_0 = 1$ for all n, and $^nC_1 = n$ for all n; these follow straight from the definition.

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & \cdots\cdots & & & & &
\end{array}
$$

The number $^nC_r = {}^nC_{(n-r)}$ for all n, since choosing r things from n is the same as not choosing n - r things from n. There's a symmetry here. 'Symmetry' might remind us of Pascal's Triangle. If we put our binomial coefficients into a triangle as on the left, then this arrangement turns out to be Pascal's Triangle.

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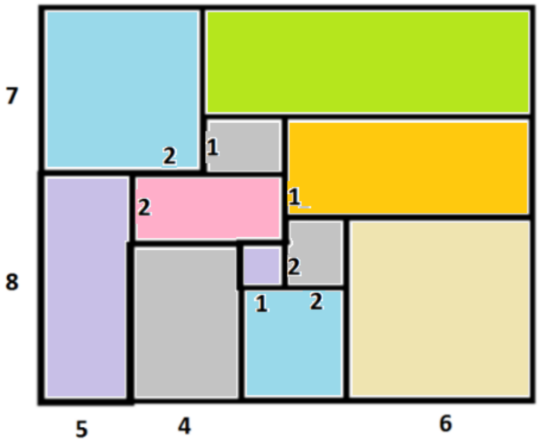
26 Notes: The Squaring the Square Problem

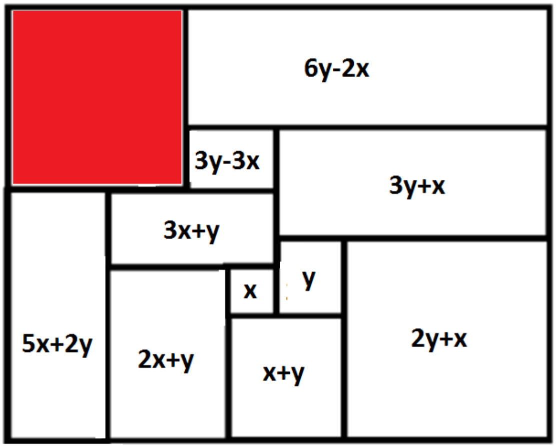
Status: solved

Level 1: Given a sizeable rectangle with integer sides, cutting it into smaller rectangles with integer sides that are all different is usually not too hard (although an n×1 rectangle is tricky!)



If we draw a rough dissection of a rectangle (we may have to experiment to find one that works), we are likely to have plenty of freedom over how to assign the side lengths. Any sensible choice of the numbers chosen on the left will give a solution.

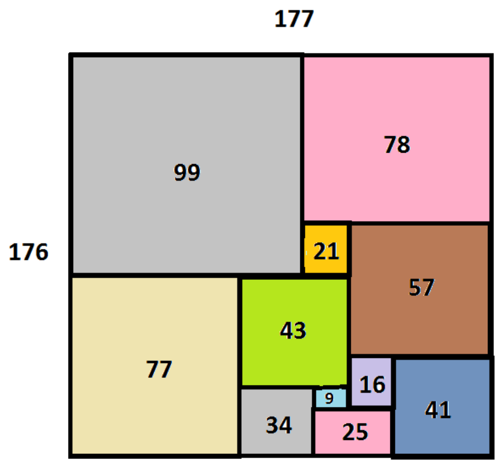
Level 2: What happens if we try to insist these are all squares? Suddenly we have a lot less freedom. Call the side of the smallest rectangle-square x, and the next smallest y.



Roughly drawn

We can deduce the sides of the other squares in terms of x and y, until we get to the red square above, that has one side equal to 11x, and the other equal to 9y - 5x. If this shaded region is to be a square, we must put these equal to each other.

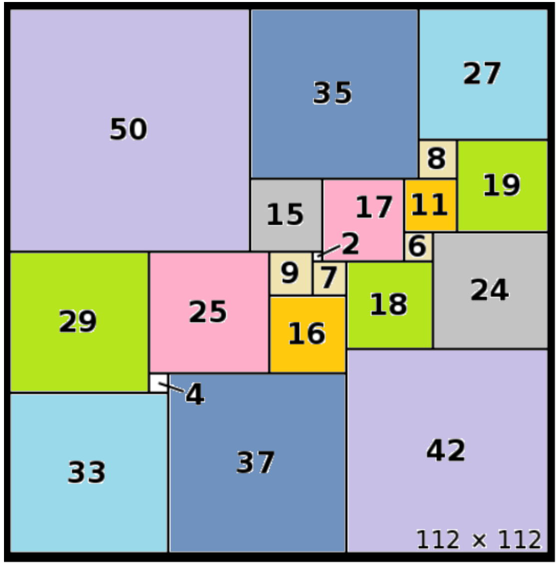
$11x = 9y - 5x$ means **$16x = 9y$**, so one solution is **$x = 9$** and **$y = 16$**.



Roughly drawn

Level 3: this is very close to being a squared square! But if we miss a solution by 1, that's still a miss.

The story of how four undergraduates at Cambridge solved this problem is a mathematical thriller - it appears in *More Mathematical Puzzles and Diversions* by Martin Gardner. They found that it IS possible to square the square, searching for solutions using an adaptation of electrical circuit theory. The example here is one found subsequently by A. J. W. Duijvestijn, using a computer search.



Transcendental Numbers

You may already have met rational numbers, those that can be written as one whole number over another. Maybe you have also seen that irrational numbers exist by examining √2. But √2 is additionally an ***algebraic number***. To explain this, you need to know what a polynomial is.

I had a polynomial once. My doctor removed it. **Michael Grant**

A polynomial in x is a sum of non-negative integer powers of x. So $\mathbf{x^2 + x + 1}$ is a polynomial, and so is $\mathbf{4x^{10} - x^4 + 2}$ (remember that $x^0 = 1$, so a constant term counts as a power of x). We concentrate here on the cases where the coefficients are whole numbers. If we put a polynomial equal to 0, we get a polynomial equation. Some of these are easy to solve:

$\mathbf{x - 2 = 0}$ solves to give $\mathbf{x = 2}$**.**

How about $\mathbf{x^2 + 3x + 2 = 0}$? This can be factorised into $\mathbf{(x + 1)(x + 2) = 0}$, and so the roots of the equation are $\mathbf{x = -1}$ and $\mathbf{x = -2}$.
How about $\mathbf{x^2 + 3x - 2 = 0}$? This doesn't factorise, but there's a formula for polynomial equations that have 2 as the highest power (these are called ***quadratic equations***). The formula says

if $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

So in the case of $x^2 + 3x - 2 = 0$, $a = 1$, $b = 3$, $c = -2$, and we have $x = \dfrac{-3 \pm \sqrt{17}}{2}$.

That $\sqrt{17}$ assures us that we have irrational roots here.

What if 3 is the highest power of x in our polynomial equation? (This is called a ***cubic equation***.) Is there a formula we can use here? It gets more complicated, but yes, there is.

Similarly, we have a formula for quartic equations (when the highest power is 4). But when we get to 5 (quintic equations) and higher, something strange happens; no general formula exists for solving such polynomial equations, and they need to be tackled in some other way. This was proved by the brilliant young Norwegian mathematician Abel in 1824.

We can now say what an algebraic number is; it's a number that is the solution to some polynomial equation with integer coefficients. So is every irrational number algebraic? The answer is, 'No'; there are irrational numbers that can't be expressed as the root of a polynomial equation, and we call such numbers ***transcendental***.

How many transcendental numbers are there? It's possible to put all algebraic numbers into an infinite list (they are *countable*), but that's impossible with transcendental numbers, so the vast majority of numbers on the number line are transcendental.

If equations are trains threading the landscape of numbers, then no train stops at pi. **Richard Preston**

The most famous transcendental numbers are π and e. The number π was proved to be so by Lindemann in 1882, following Hermite's proof that e was transcendental in 1873.

Proving a number to be transcendental is hard. It's easy to come up with a list where the matter remains unresolved. Almost any way of combining π and e gives a number in this camp. The status of all of the numbers $\pi + e$, $\pi - e$, πe, π/e, π^e, π^π, e^e remains undecided, although it would be a surprise to find that any one of these numbers is not transcendental.

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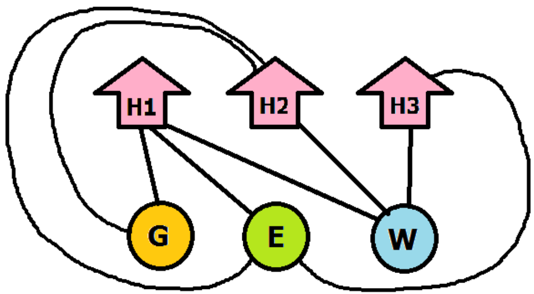
27 Notes: The Factor Graph Problem

Status: solved

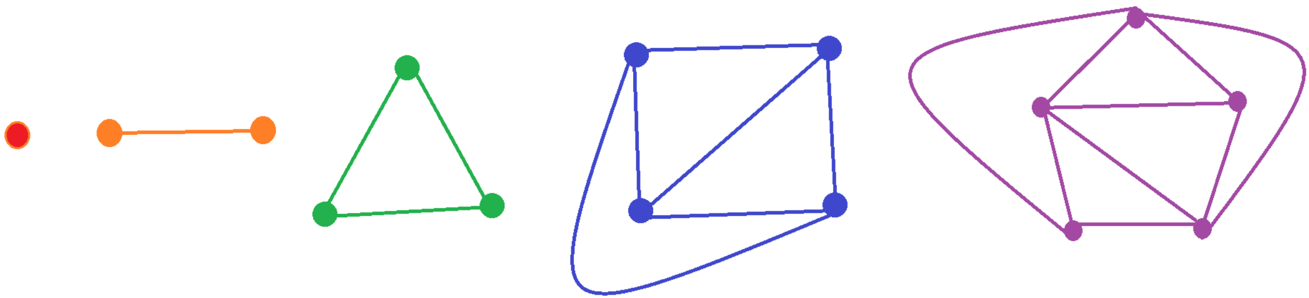
The idea of a graph, as a number of dots (vertices) connected by lines (edges) is useful. Graph theory is a relatively new branch of mathematics, but its importance grows by the day; taxing real-life problems can be modelled effectively by graphs. The Travelling Salesman problem, that asks how to visit n towns using the quickest route possible while arriving back at the starting town, is a graph theory problem (finding an efficient algorithm to apply in general remains the subject of ongoing research).



Level 1: Can we connect the three houses to the three utilities? You should find that however you arrange the pipes and the cables, you can never connect everything so that lines don't cross. The above diagram is the best we can do; sadly, the gas can't be connected to H3 without a crossing.

We're attempting to draw what's called a *bipartite graph*, where we have two groups of dots (the houses, on the one hand, and the utilities on the other) with edges running only from one group to the other. Moreover, we're here trying to draw every line from one group to the other, which makes 3 lots of 3 = 9 lines. This is called a *complete bipartite* graph. The complete bipartite graph for two groups of three points is called $\mathbf{K_{3,3}}$**.** If a graph can be drawn on a flat piece of paper so that none of the edges cross, we say it's *planar*. What we've discovered in attempting this problem is that $\mathbf{K_{3,3}}$ is not planar.

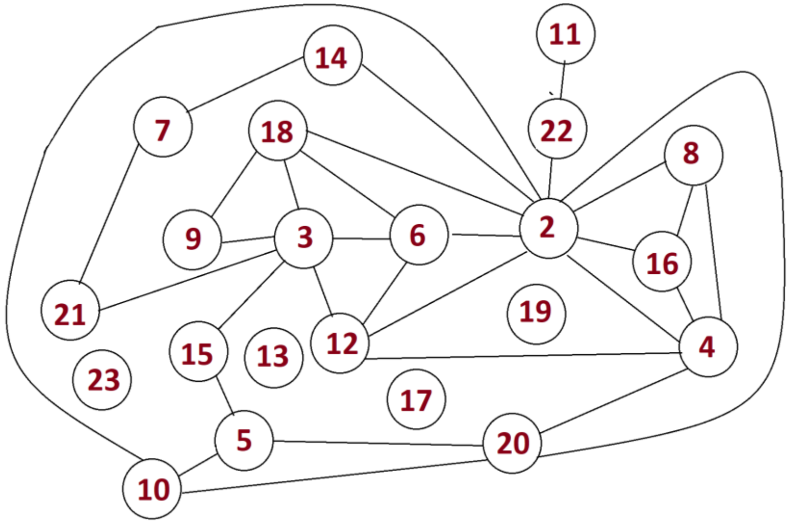
Level 2: The symbol $\mathbf{K_n}$ is used to denote the complete graph with n dots, where every dot is connected to every other. You should find that $\mathbf{K_1}$**,** $\mathbf{K_2}$**,** $\mathbf{K_3}$ and $\mathbf{K_4}$ are all planar, but $\mathbf{K_5}$ (and so $\mathbf{K_n}$ for n larger than 5) is not.



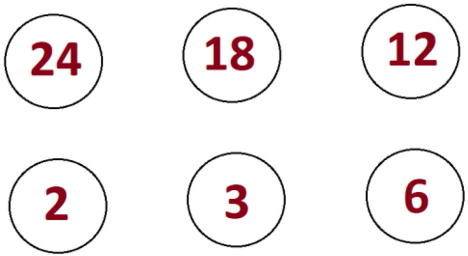
Once again, in the last diagram we are left with a single line that we can't draw without a crossing.

Here's a theorem that HAS been proved. Roughly speaking, all non-planar graphs, ones that can only be drawn on the page if lines are allowed to cross, contain either a **K_5** or a **$K_{3,3}$**. (This is Kuratowski's Theorem). In other words, given a non-planar graph we can always rub out edges and dots to leave either a **K_5** or a **$K_{3,3}$.** (It is possible that other points lie on the paths connecting the five vertices in the K_5, or the six vertices in the $K_{3,3}$.) These two non-planar graphs are somehow the most elemental; all other non-planar graphs are elaborations of these two basic ones.

Now to our factor graph problem. If we're careful, we can get up to 23 without having to cross lines.



But if we try to add 24...



We need to join all of the top three circles to all of the bottom three circles. We’re back to the houses and utilities problem, which we know cannot be drawn as a planar graph. So 23 is as far as we can get.

Home Level 1 Level 2 Level 3

Problem Proof 1

Proof's a wonderful thing, but it can go wrong. Sometimes a 'proof' can come up with something so ludicrous that it takes us aback; how can such a reasonable chain of logic bring us to this point of incredulity? Have a look at this proof; does it convince you?

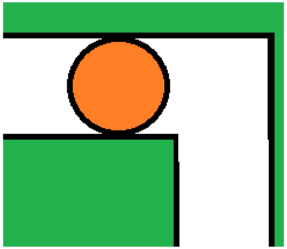
Suppose $a = b$, then multiplying both sides by a gives $a^2 = ab$.
Subtract b^2 from both sides to give $a^2 - b^2 = ab - b^2$.
Now factorise, which gives us $(a - b)(a + b) = b(a - b)$.
Now divide both sides by $a - b$, to give $a + b = b$.
But $a = b$, so we have $b + b = b$, or $2b = b$.
Dividing by b now gives $2 = 1$.

What's gone wrong? Clearly something has! Can you diagnose the difficulty?

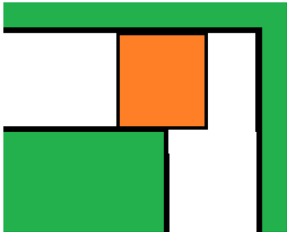
Home Notes

28 Notes: The Moving Sofa Problem

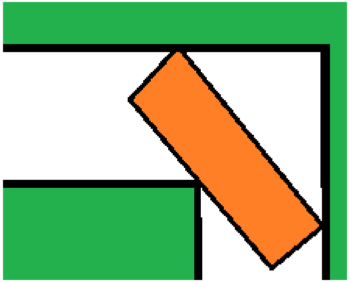
Status: unsolved **Proposed by Leo Moser in 1966**



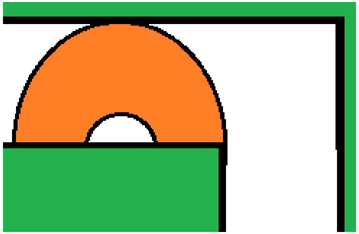
Level 1: We can certainly push a circle (left) of unit diameter (okay, it's an unusual sofa) around the corner. This will have an area of $\pi/4 = 0.785$.



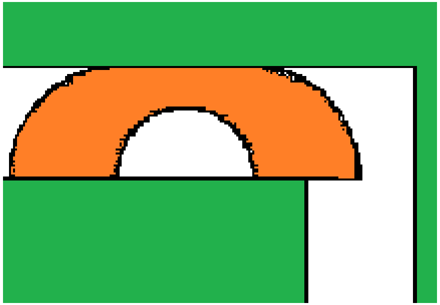
Level 2: In fact, a unit square (right) will be no problem. This improves our answer to 1.



We might think we can improve things by choosing a rectangle (left), but in fact, this keeps us with 1 as our best possible answer for the sofa area.



Level 3: We might decide that curves are the way to make progress. How about a semicircle of radius 1 with a small semicircle removed at the centre, to allow an easier passage around the sharp corner? If the radius of the small circle is 1/2, this now gives us 1.18 for the area of the sofa.



This last technique can be refined to arrive at the **Hammersley sofa** (right), which weighs in with a remarkable area of 2.2074 square units. This is known, however, not to be the best possible solution.

We are actually searching here for what is called the ***sofa constant***; its true value is unknown.

Home Level 1 Level 2 Level 3

Mathematics Coming Alive

My first real joy in learning sparked up as I started to study harder maths in my mid-teens. Until then, I'd been your average competitive schoolboy, just trying to score as well as I could whatever the subject, and lacking the imagination to view study in any other way. But then everything changed. Suddenly our maths teachers appeared to enjoy our company. They began to call us by our first names, and they acknowledged us as 'mathematical equals', or at least they seemed to silently acknowledge we might become so at some time in the future. They loved their mathematics, and they wanted to communicate that to us.

Two teachers in particular stand out in my memory; there was Ray, the brilliant head of department and mechanics whizz, and there was Steve, a man with an engrossing teaching style who researched into a mathematician called Bolzano in his spare time. Steve in particular went out of his way to involve us in off-piste mathematics, where the syllabus was incidental and where we could allow our minds to slalom freely downhill.

One day, at a whim, Steve stood up at the board sketching out the first few terms of a recurrence relation (a sequence where each term is defined in terms of previous terms).

This (remarkably) proved to be periodic (the terms repeated after applying the rule five times). It was no coincidence that when I came to choose a topic for my MSc by Research thirty years later, the foundation of my thesis was Steve's sequence. An off-the-syllabus five-minute exposition in 1976 turned into a three-year research project in 2009.

The University of East Anglia gave me another chance at university mathematics, and I found that each day I was encouraged to make fresh designs for my own tables, and then make them. It turned into the most intellectually intoxicating time. Paul Lockhart describes the experience brilliantly:

***To do mathematics is to engage in an act of discovery and conjecture, intuition and inspiration; to be in a state of confusion—not because it makes no sense to you, but because you* gave *it sense and you still don't understand what your creation is up to; to have a break-through idea; to be frustrated as an artist; to be awed and overwhelmed by an almost painful beauty; to be* alive*, damn it.* Paul Lockhart**

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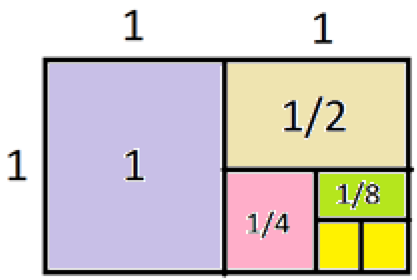
29 Notes: The Infinite Sum of Fractions Problem

Status: solved



Level 1: This diagram shows that S_n gets closer and closer to 2 as n heads off towards infinity. We say that S_n ***converges*** to 2.

Level 2: On the other hand,

$$T_n = \frac{1}{1} + \frac{5}{8} + \frac{4}{8} + \frac{11}{27} + \frac{10}{27} + \frac{9}{27} + \frac{19}{64} + \frac{18}{64} + \frac{17}{64} + \frac{16}{64} + \frac{29}{125} + \ldots + \frac{n^2}{n^3}$$

$$> \frac{1}{1} + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}\right)$$ **= 1 + 1 + 1 +… + 1 = n.**

So as n gets bigger, T_n gets bigger than any positive number we can possibly think of, even though the terms in T_n are decreasing in size. We say that T_n ***diverges*****.**

Level 3: we can use a similar trick with

$$U_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n}$$

$$> \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^m} + \ldots + \frac{1}{2^m}\right)$$ **(where $2^m \leq n \leq 2^{m+1}$)**

$$= \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} = \frac{m}{2}$$

So as n gets bigger, U_n also diverges, albeit very slowly.

What about $V_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{p_n}$?

These terms lie between S_n and U_n. We know that S_n converges to 1, and that U_n diverges; is V_n like the former or the latter?

It turns out (remarkably) that V_n diverges (proved by Euler in 1737), but even more slowly than U_n does. Adding together the reciprocals of all the known primes gives you a number less than 4, but V_n does still head off to infinity!

Home Level 1 Level 2 Level 3

Notes: Problem Proof 1

What's the problem here? It's the step where we divide by a - b.

Since a = b, a - b is zero, and once you allow dividing by zero, anything equals anything. We might as well say:

45 times 0 equals 13 times 0 (true) so therefore **45 = 13 (untrue).**

The rule 'dividing by zero is not allowed' should be taken as read in mathematics, or at the very least, 'dividing by zero' should be handled with extreme care! 'Dividing something finite by infinity gives zero' is a much safer bet, you'll be glad to hear.

Home Back to problem

Problem Proof 2

Here's another worrying proof, where the difficulties arise from the fact that infinity is a tricky customer.

The infinite in mathematics is always unruly unless it is properly treated. **James Newman**

Mathematics has been forced to think harder about infinity than any other subject it covers, and it still throws up something counter-intuitive sometimes. Take the following 'proof';

What's the sum **1 + 2 + 3 + 4 ....** to infinity? Let's call this infinite sum S.

We will call T the infinite sum **1 - 1 + 1 - 1 + 1 ...** and U the infinite sum **1 - 2 + 3 - 4 + 5 - 6 ...**

What is T? We can write the following:

$$\begin{array}{rlllllll} & T = 1 - 1 + 1 - 1 + 1 - 1 + \ldots\ldots & \\ + & T = \quad\;\; 1 - 1 + 1 - 1 + 1 - 1 + \ldots\ldots \\ \hline & 2T = 1 + 0 + 0 + 0 + 0\ldots \end{array}$$

So if 2T = 1, T = 1/2, surely? Now we have

$$\begin{array}{rl} & T = 1 - 1 + 1 - 1 + 1 - 1 + \ldots\ldots \\ - & U = 1 - 2 + 3 - 4 + 5 - 6 \ldots \\ \hline & T - U = 0 + 1 - 2 + 3 - 4 + 5 \ldots \end{array}$$

So T - U = U, and U = T/2 = 1/4. Finally we have

$$\begin{array}{rl} & S = 1 + 2 + 3 + 4 + 5 + 6\ldots \\ - & U = 1 - 2 + 3 - 4 + 5 - 6 \ldots \\ \hline & S - U = 0 + 4 + 0 + 8 + 0 + 12 + \ldots \end{array}$$

So S - U = 4S, so S = -U/3 = -1/12.

Look where all this impeccable logic has brought us; the infinite sum

1 + 2 + 3 + 4 ... = -1/12.

Hmmm.

The same result can be found in the brilliant mathematician Ramanujan’s notebooks.

Another way of finding the constant is as follows — 41
Let us take the series $1+2+3+4+5+\&c$. Let C be its constant. Then

$$C = 1+2+3+4+\&c$$
$$\therefore 4C = \quad 4 \quad +8 \quad +\&c$$
$$\therefore -3C = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$$
$$\therefore C = -\frac{1}{12}.$$

We’ll just have to remember that when our mathematics wanders into these areas, we are entering a minefield!

Home

30 Notes: Second Triomino Tiling Problem

Status: solved

Level 1: The standard chessboard has 64 squares = 63 + 1. So our best possible tiling leaves one square left blank. This is certainly possible, and it appears we can place the blank square anywhere.

What happens if we try our colouring argument that we used for The First Triomino Problem here again? It breaks down; why?

The difficulty is that the three-in-straight-line triomino tile must cover one each of blue, yellow and grey. With the L-shaped tile, we could cover two greys and a blue, or two yellows and a grey; there's no consistency.

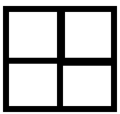
Strangely, we might end up with more freedom with the L-shaped tile than with the three-in-a-row one. Can we prove the empty square can be anywhere?

Level 2: Dealing with all 2^n x 2^n grids now, does there always have to be at least one empty square? The total number of squares in the grid is $\mathbf{2^n \times 2^n}$, which is $\mathbf{2^{2n}}$ **squares,** and $\mathbf{2^{2n} = (3 - 1)^{2n}}$ **squares.**

If we expand $(3 - 1)^{2n}$, we get lots of multiples of 3 together first (these are the L-shaped tiles) with a final term that is $(-1)^{2n}$, which is 1. So yes, it's guaranteed that any best possible tiling will leave one square blank.

Level 3: Now let's try our proof by induction technique (this proof was first given for this problem by Solomon Golomb). Our statement about whole numbers S_n is this;

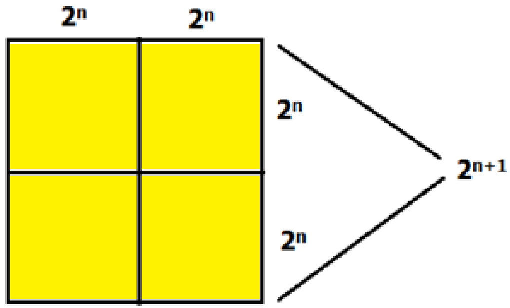
You can tile any 2^n by 2^n square grid with 3-square L-shaped tiles so that there is exactly one empty square left over that can appear anywhere.



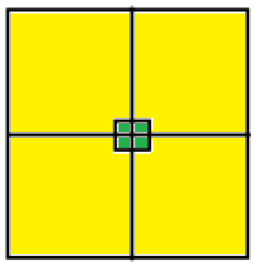
S_1 says that on this grid, the empty square can be anywhere. That's clear - we can use one L-shaped tile, and there are four positions for it, each of which leaves the empty square in a different place, and so all squares on the grid are candidates for being the empty one.

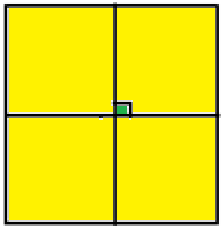
Now the second part of our induction. Assume that S_n is true; can we deduce S_{n+1} from that?

Our statement S_n tells us that we can tile any 2^n by 2^n square grid with 3-square L-shaped tiles so that there is one empty square that can be anywhere. Consider a 2^{n+1} by 2^{n+1} grid. We can cut this into four identical 2^n by 2^n grids like this:



We get going with our L-shaped tiling now. We know from assuming S_n that we can tile each of the four 2^n by 2^n squares so that the empty space appears anywhere. Let's now tile in such a way that for each of our four 2^n by 2^n squares, we leave the empty space at the centre.





Now we can add one more L-shaped tile, which will leave any one of those four central ones as the empty square for the 2^{n+1} by 2^{n+1} square. Let's suppose the top right square for the green square is left empty.

But now we already know (from assuming S_n) that we can tile that top right 2^n by 2^n square so that that empty square appears anywhere.

So we can certainly have the empty square anywhere in the top-right quarter. But we could have chosen any of the other quarters, so the empty square could be anywhere.

We have that S_n implies S_{n+1}, and our proof by induction is finished.

Home Level 1 Level 2 Level 3

Exhaustion: Notes

Proof: Notice that the last two digits of a square only depend on the last two digits of the starting number. For example, $1524^2 = (1500 + 24)^2 = (\text{multiple of } 100) + 24^2$, and so the end two digits are 76, since $24^2 = 576$.

So to prove our conjecture, we only have to try out the 100 possible pairs of end digits and see what happens when we square them.

The good news is that we can reduce our workload further. To end in a 4, the final digit must be 2 or 8.

02^2 ends in 04, 12^2 ends in 44, 22^2 ends in 84, 32^2 ends in 24, 42^2 ends in 64, 52^2 ends in 04, 62^2 ends in 44, 72^2 ends in 84, 82^2 ends in 24 and 92^2 ends in 64.

08^2 ends in 64, 18^2 ends in 24, 28^2 ends in 84, 38^2 ends in 44, 48^2 ends in 04, 58^2 ends in 64, 68^2 ends in 24, 78^2 ends in 84, 88^2 ends in 44 and 98^2 ends in 04.

So no perfect square can end in a 14, or indeed in 34, 54 or 74.

This is proof by exhaustion in action; you have a finite list of possibilities, and you check them out one by one. It’s unlikely, I should add, that a proof by exhaustion will win any prizes for elegance. But it might suggest to you a better and more elegant proof along the way.

In our case, if n^2 ends in 4, then

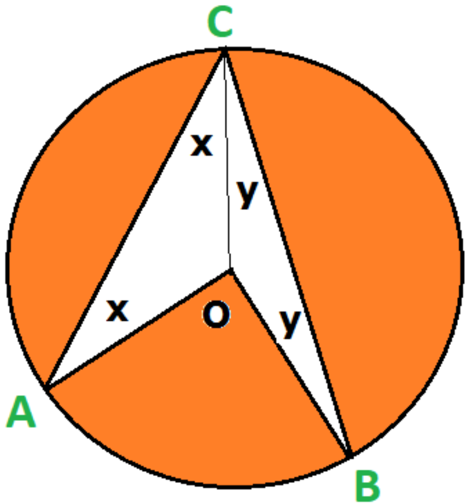
$$\mathbf{n^2 = (100a + 10b \pm 2)^2 = (multiple\ of\ 100) \pm 40b + 4.}$$

It’s clear from this that whatever digit b is, the penultimate digit of n^2 can never be odd.

Home

Axioms: Notes

Conjecture: it seems that b, the angle at the centre of the circle, is always exactly twice a, the angle at the circumference. How might we prove this?



Suppose we label the points A, B, C and O as above.

Now OA, OB, and OC are all the radius of the circle, and are thus all equal.

That means that the triangles OAC and OBC are both isosceles, that is, they both have two equal sides.

Now it's well-known that the base angles of an isosceles triangle are equal.

We can label these as x and x for AOC and y and y for BOC. That means the angle AOC is $180^{o} - 2x$ (since the angles in a triangle add to 180 degrees) and the angle BOC is $180^{o} - 2y$.

So what is the angle AOB in degrees now? It must be

$$\mathbf{360^{o} - (180^{o} - 2x) - (180^{o} - 2y) = 2x + 2y.}$$

So ACB is x + y, and AOB is 2x + 2y, and we've proved our theorem.

You notice here that we recruited simpler theorems (about isosceles triangles, and the angle sum in a triangle) to prove our harder theorem. This is where we owe Euclid, a Greek mathematician of about 300 BC, a major debt of thanks. He was the first person (at least many people think so) to introduce the axiomatic method, where we choose some simple statements called axioms (as obviously truthful as we can make them, and as few in number as possible), and we deduce from them more complicated statements called theorems. If your axioms are true and consistent with each other, and your logic is sound, you guarantee that your theorems will be true also.

Euclid built a wonderfully systematic structure from scratch in this way, numbering his theorems as he went (Pythagoras's theorem came in at number 47).

Sometimes the axiomatic method can get in the way of our proving. We have a nice argument, and attempting to place it into a rigorous structure along Euclid's lines feels unnecessary. Sometimes too the axiomatic method begins with such glaringly obvious theorems that we lose the will to live.

The effect of such a production being made over something so simple is to make people doubt their own intuition. Calling into question the obvious by insisting that it be 'rigorously proved' is to say to a student, 'Your feelings and ideas are suspect. You need to think and speak our way.' **Paul Lockhart**

Mathematics brought rigour to economics. Unfortunately it also brought mortis.
Kenneth E. Boulding

But, on the other hand, maths is full of things that appear self-evident, yet when you poke them about a bit, they turn out to be less obvious than you might think.

'Obvious' is the most dangerous word in mathematics. **E. T. Bell**

Home

Using a Computer - 2

Programs can help us with searching for an answer to a problem. I've chosen the Fermat-Catalan conjecture to show this, the result of merging Fermat's Last Theorem with Catalan's Theorem.

Fermat's Last Theorem says the equation $\mathbf{x^n + y^n = z^n}$ has no whole number solutions for x, y and z if n is greater than 2 (proved by Andrew Wiles in 1997).

Catalan's Theorem (suggested in 1844) states that $\mathbf{x^a - y^b = 1}$ has only one solution in whole numbers,

$3^2 - 2^3 = 1$ (this was proved by Preda Mihailescu in 2002).

So what might the love child of these two great theorems look like? With almost any conjecture, it's possible to make it more general, and that creates something harder to prove.

We're going to try to find solutions to the equation $\mathbf{a^m + b^n = c^k}$. (I'll call this 'the key equation.')

How many whole number solutions can we find to the key equation? What happens if one or more of m, n and k is 1? What if m, n and k are all 2? What if two of m, n and k are 2? What if m, n and k are all 3? If we insist that **$1/m + 1/n + 1/k < 1$,** then we guarantee that m, n and k are not all small. Can we find a solution for ***(a, b, c, m, n, k)*** with this restriction? How could we find other solutions?

Does the key equation have infinitely many solutions? Or finitely many? We can surely insist that a, b and c have no common factor. So how could we code a program to tackle the Fermat-Catalan conjecture?

```
Private Sub CommandButton1_Click()
s = Cells(1, 7)
j = 1
For a = 1 To s
For b = a + 1 To s + 1
For c = 1 To 150
For m = 2 To 10
For n = 2 To 10
For k = 2 To 10
If 1 / m + 1 / n + 1 / k < 1 Then
If a ^ m + b ^ n = c ^ k Then
Cells(j, 1) = a
Cells(j, 2) = b
Cells(j, 3) = c
Cells(j, 4) = m
Cells(j, 5) = n
Cells(j, 6) = k
j = j + 1
End If
End If
Cells(1, 8) = a
Next k
Next n
Next m
Next c
Next b
Next a
End Sub
```

A simple search for solutions might look like the above. This program employs six nested For-Next loops. Clicking out of Design Mode and then clicking the Command Button runs the program. Putting s = 20 generates the output below.

1	2	3	7	3	2	20	5
1	2	3	8	3	2	CommandBut	
1	2	3	9	3	2		
1	2	3	10	3	2		
2	4	2	4	2	5		
2	4	2	6	3	7		
2	4	2	8	4	9		
2	4	8	8	4	3		
2	4	24	9	3	2		
2	7	3	5	2	4		
2	8	2	6	2	7		
2	8	2	9	3	10		
2	8	4	9	3	5		
2	8	32	9	3	2		
2	16	2	8	2	9		
2	16	8	8	2	3		
2	17	71	7	3	2		
3	6	3	3	3	5		
3	6	45	6	4	2		
3	9	54	7	3	2		
3	11	122	5	4	2		
3	18	3	6	3	8		
3	18	9	6	3	4		
4	8	2	3	2	7		
4	16	2	4	2	9		
4	16	8	10	5	7		
4	16	128	10	5	3		

This at least is the start of the output, which I cut short here by pressing Esc. You can see that the first four solutions have been found (the black rows) along with other solutions that are repeats or which have common factors.

It takes Excel a while to run this. Every time you add another nested loop, the number of calculations increases exponentially (the idea of cell H1 is to keep track of how far the program has run). Sometimes you need to leave a computer running overnight to arrive at an answer. Once you've experienced the thrill of waking to remember that your machine is waiting for you with an answer, you'll never feel the same way about maths or computing again.

Excel can go up to 16 significant figures; larger numbers than this are going to end in approximate results.

These ten red solutions to the key equation are the only ones known. Our program found four of these, as far as it was allowed to run. You'll notice that in each of these ten solutions, one of m, n and k is 2. So we can add to the unsolved problem above Beal's Conjecture, which states that if we have a solution to the key equation, then one of m, n and k has to be 2. Beal is a rich man, and he currently offers a prize of a million US dollars to anyone resolving his conjecture one way or the other. This book could be the wisest investment you've ever made…

$1^m+2^3=3^2$

$2^5+7^2=3^4$

$13^2+7^3=2^9$

$2^7+17^3=71^2$

$3^5+11^4=122^2$

$33^8+1549034^2=15613^3$

$1414^3+2213459^2=65^7$

$9262^3+15312283^2=113^7$

$17^7+76271^3=21063928^2$

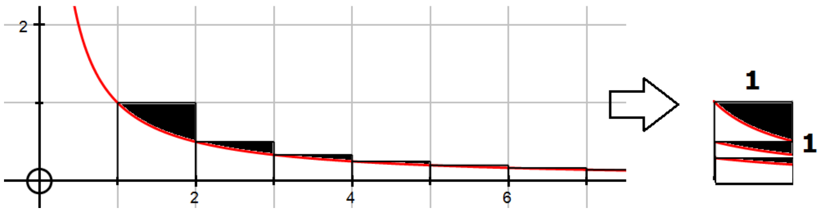
$43^8+96222^3=30042907^2$

If you're going to investigate conjectures, using a computer is common practice. It can save you time by finding you a quick counterexample, and stopping you from disappearing up blind alleys. But mathematicians disagree on how helpful they are; Andrew Wiles says he never uses them, and it certainly hasn't troubled his career. We need to beware too of leaping onto a computer to resolve something when a little hard thought would answer the question for us immediately. Sometimes a pencil and paper are all we need.

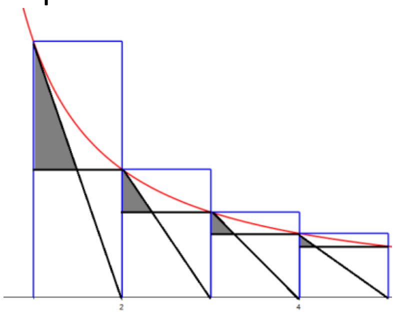
Home

31 Notes: The y = 1/x Number problem

Status: unsolved



Level 1: The diagram below shows that γ must be bigger than 1/2, but less than 1. The sum of the black areas γ is also called the Euler-Mascheroni constant. It seems a rather left-field way to define a number, so it comes as a surprise to find that γ crops up all over mathematics, rather in the way that π does. The list of theorems, conjectures and methods where γ comes in is impressive.



Level 2: We could reduce the upper bound for γ by taking off the area of the grey triangles on the left. The n[th] grey triangle has area

$$0.5\frac{1}{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

Summing these gives $1-\frac{\pi^2}{12}$, and so we can say 0.5 < γ < 0.8225…

Level 3: The number γ to 30 decimal places is **0.577215664901532860606512090082…**

The problem is to establish whether γ is irrational or not; no one has been able to do this. If we could prove it was, we'd still not necessarily know if it were algebraic or transcendental. If γ turns out to be the rational number a/b in its lowest form, then it has been proved that b must be greater than 10^{242080}.

Home Level 1 Level 2 Level 3

The n n^{th} Powers Equation (Bonus) Problem

What do you think of these equations?

$$y^2 = z^2, \qquad x^3 + y^3 = z^3, \qquad w^4 + x^4 + y^4 = z^4, \qquad v^5 + w^5 + x^5 + y^5 = z^5$$

and so on. Do they have any whole number solutions?

The first involves two squares and is easy to solve; we have (trivially) lots of whole number solutions here.

The second one is impossible in integers, as proved by Euler in the 1700s.

What about the equations above with four fourth powers, and five fifth powers; do these equations have any integer solutions, would you say?

What does your intuition suggest? Can you prove it one way or the other?

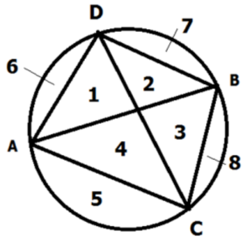
Home Notes

32 Notes: The Circle-Region Problem

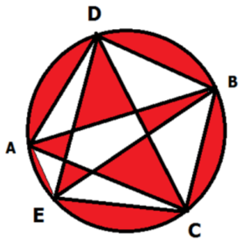
Status: solved **Name: Moser's circle problem**

Level 1:

Four dots:
eight regions

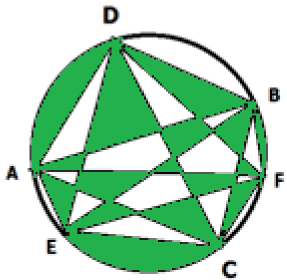


Level 2:



Five dots:
sixteen regions

This problem is remarkable because the initial pattern breaks down; the first five terms of the sequence are powers of 2, but the sixth is not.



Try as we might, we cannot beat 31 for six dots. The sequence starts:

N	1	2	3	4	5	6	7	8	9	10
Circle regions	1	2	4	8	16	31	57	99	163	256
Powers of 2	1	2	4	8	16	32	64	128	256	512

Level 3: So how can we come up with a formula to count the regions sensibly?

Think what happens when we add a chord. How many extra regions do we create? One for every chord that's crossed, plus one. Or, the number of new internal intersection points, plus one.

Think what happens as we add the n chords one by one. The number of extra regions added is 'the total number of internal intersection points' plus 'the number of chords' (that's adding the 'ones' together).

Now each internal intersection point is where two chords meet. So the total number of internal intersection points is the number of ways of picking two chords. This is the number of ways of picking four endpoints from the n points, or $\binom{n}{4}$. The number of chords is the number of ways of picking 2 points from the n points, or $\binom{n}{2}$.To finish, we remember that this is the number of *extra* regions, and we had one to start with, so we must add 1. This gives us that the total number of regions for n points on the edge of the circle is $1 + \binom{n}{2} + \binom{n}{4}$. This formula gives us the values in our table.

Composing: a Story

Once upon a time, a young and gifted boy set out to become a musician. At the start of his journey, he was absorbed in performing classical music. He was taken on by one of our great cathedrals, and became a successful choir boy.

He won a music scholarship to a top university, and led one of the best college choirs while studying the mechanics of music to a high level. Harmony, counterpoint, musical history, he took it all in his stride.

The only thing missing for him was composition, by which I mean real composition, not the exercises in writing formulaic examples that were necessary to pass his exams. By composition, I mean writing something original from the heart. That was something to be left to the greats, to Bach, to Brahms, to Prokofiev.

He left university, and his tastes in music widened; he formed a group to perform classic popular music from previous decades. His brilliant arranging skills made sure that the band always sounded fresh and attractive. They were a hit: but still, no composition.

But then someone who liked the group rang, to ask if the young man could write a piece of music for a special event. The young man was thrown into turmoil by this. He was classically trained. Composition was not something that he did; it felt like arrogance to even try.

Thirty years later, and this young man has become one of the foremost composers in the country. He's not Brahms, or Bach or Prokofiev, but nor would he want to be. He is himself. He dared to sit down at the piano, starting with nothing, and threw off his past to make something new.

Home

33 Notes: The Transversals in a Latin Square Problem

Status: unsolved **Proposed by Ian Wanless, 2003**

A	B	C
B	C	A
C	A	B

A	B	C
B	C	A
C	A	B

A	B	C
B	C	A
C	A	B

Level 1: There is only one 3 by 3 latin square (counting rotations and reflections as the same), with three different transversals.

		Year 1	Year 2	Year 3	Year 4
Field	1	W	C	B	T
	2	T	W	C	B
	3	B	T	W	C
	4	C	B	T	W

Level 2: I remember a school history lesson about the way in which crop rotation revolutionised agriculture in the 18th century. There were four crops, wheat, turnips, barley and clover, to be rotated on four fields. This produces a four by 4 latin square. It's been constructed in the simplest way possible, that is, to simply cycle W, T, B and C around in the columns. This will always produce a latin square, no matter what n is, so clearly you can have at least one latin square of any size.

C	W	B	T
W	C	T	B
B	T	C	W
T	B	W	C

If we try to find a transversal for the latin square above, we will struggle. It's a frustrating task, one that turns out to be impossible. But if we choose a different 4 by 4 Latin Square with four crops, a transversal is possible.

Level 3: So what's the maximum number of transversals (let's call this T(n)) that an n by n latin square can contain? Nobody knows. Wanless, McKay and McLeod have come up with lower and upper bounds for T(n), showing

$$c^n < T(n) < d^n n!,$$

where $c > 1$ and d is about 0.6. Taking $c = 1$, $d = 0.6$ and $n = 5$ gives $\mathbf{1 < T(5) < 9.33.}$

We could also ask, what is the **minimum** number of transversals for a Latin square? It turns out that this too is also an open problem. The mathematician Ryser has conjectured that every Latin square of odd order has at least one transversal.

Problem Proof 3

Proof by Induction can go wrong too. What's the problem here?

Suppose our statement $\mathbf{S_n}$ says

every set of n natural numbers will all have the same value.

If we think about this for a second, this is clearly nonsense.

$\mathbf{S_1}$ states that a set of one number contains numbers that are all the same, which is true. Assume $\mathbf{S_k}$**.** Let's consider a set A of k + 1 natural numbers. Pick any two numbers from the set, call them a and b. We need to show that a = b.

Take the set of numbers that is A without a. This is of size k, so we know all the numbers in this set are equal in size. Pick a number other than b in this set (b must be in there) and call it c, so b = c.

Now consider the set A without b. This again is of size k, so all its members must be equal, and so a = c. But now we have a = c and b = c, so a = b, and all the numbers in A are equal. Our proof by induction is complete, and we have proved something crazy! What went wrong?

The problem is that we identified three distinct elements for our argument, a, b and c. That means k has to be at least 2, so that k + 1 can be at least 3. We prove S_1, we prove that S_2 implies S_3, S_3 implies S_4, and so on, but we don't ever prove S_2 - we can't, it's false! This little gap in the proof scuppers it completely.

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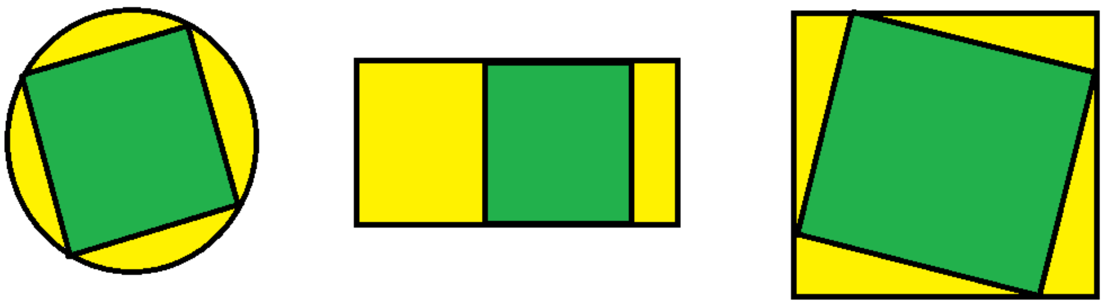
34 Notes: The Inscribed Square Problem

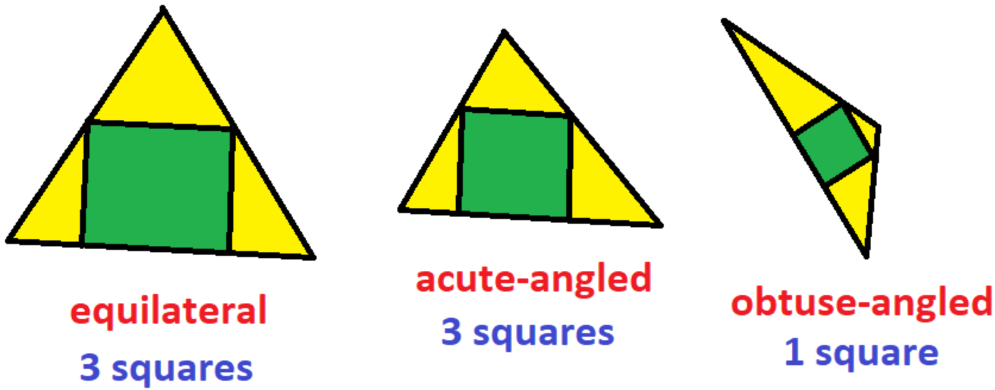
Status: unsolved ***Name: the Toeplitz Conjecture, proposed 1911***

So does every closed curve that does not cross itself contain the four vertices of a square? This is unsolved, except for certain special cases - if the curve is convex, for example, then it's true (there are other special cases also).

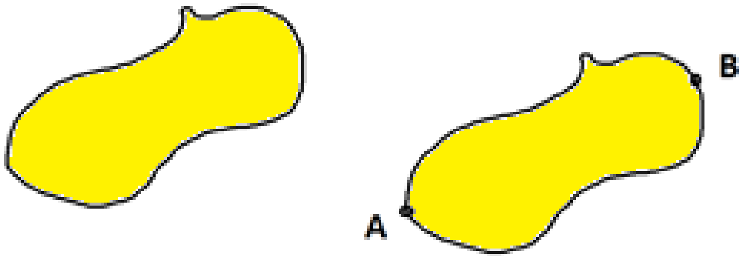
Level 1: Certainly a circle has an infinite number of squares with corners on its perimeter, as does a rectangle, as does a square.



Level 2: With a triangle, two of the four corners must be on one of the sides (by the Pigeonhole Principle). For an equilateral triangle, that gives us three possible squares and the same goes for any acute-angled triangle. Obtuse-angled triangles, however, can only give one inscribed square, with its base on the longest side.



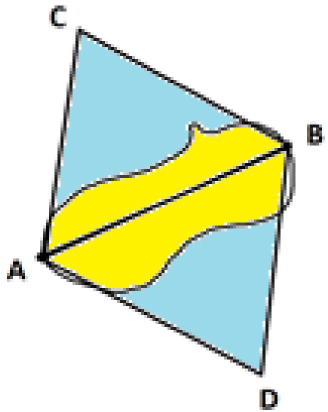
Level 3: Often with a problem it helps to consider an amended version that might be easier. The hope is that experience with the revision might suggest things for the original version. How about changing ‘inscribed square’ to ‘inscribed equilateral triangle’ here?



Take a closed loop that does not cross itself. We can now pick the two points A and B that are both on the curve, but which are as far apart as they can be (AB is the full width of the curve).

Now draw the two equilateral triangles that have AB as one side. The points C and D can’t both be inside the curve, or else we’d be able to find two points on the curve further apart than A and B.

Suppose that C is outside the curve. Now move B along the curve towards A in the direction of D, adjusting C at the same time so that ABC remains an equilateral triangle.



When B is close to A, this will be a small equilateral triangle, as small as we like, in fact, and so C will now be inside the curve.

So C has crossed from being outside the curve to being inside it; there must have been a point where C was on the curve. So we can find three points on the curve that make an equilateral triangle.

This is a rough heuristic proof, but hopefully it works as such. Sadly, the square is a lot harder!

Grief and Nostalgia

When a great problem that's tormented mathematicians for years falls, there's a great rejoicing, but there's an element of grieving as well.

Professor John Conway, brilliant exponent of mathematics and committed to exploring its most beautiful areas, sighed sadly after watching Fermat's Last Theorem succumb to the work of his colleague Andrew Wiles. A large part of that sigh was admiration, but not all.

'This conjecture has stimulated so much wonderful mathematics,' he said. 'What shall we find to replace it?'

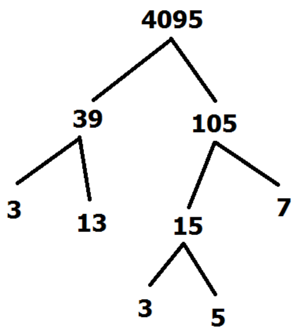
Fermat's conqueror himself said something similar.

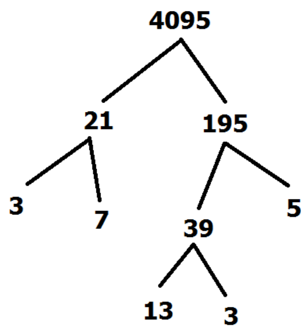
We've lost something that's been with us for so long, and something that drew a lot of us into mathematics. But perhaps that's always the way with math problems, and we just have to find new ones to capture our attention. **Andrew Wiles**

Home

35 Notes: The Same-Prime Trees Problem

Status: solved ***Name: The Fundamental Theorem of Arithmetic***





$4095 = 3^2.5.7.13.$

Level 1 and **2:** Notice here that the prime numbers at the end of the branches are the same for both trees, two 3s, a 5, a 7 and a 13. This always seems to happen. A conjecture; there's only one way of writing any number as a product of primes.

Level 3: Can we prove this? Let's try a proof by contradiction. Suppose some number n can have two different prime factorisations, so $n = p_1p_2p_3\ldots p_r = q_1q_2q_3\ldots q_s$, where r and s are the smallest possible values for which this happens.

We can't have that one of the p_i equals one of the q_j,
because then we could divide both sides by p_i and get a smaller example.

The q_js are prime, so p_1 does not divide any of them.

That means p_1 cannot divide their product, $q_1q_2q_3...q_s$.

But p_1 DOES divide the left hand side, so it must divide the right-hand side.

We have a contradiction. Thus no number can have two different prime factorisations.

If we class 1 as a prime number, then this is not true:

$$\mathbf{4095 = 1^3.3^2.5.7.13 = 1^5.3^2.5.7.13,}$$ for example.

This is a good reason for NOT having 1 as a prime number.

You notice this is called **the Fundamental Theorem of Arithmetic,** which suggests this is a big deal, and it is; it's one of the cornerstones of number theory.

Once again, this is a proof from Euclid, and so it's been around for a long time.

Home Level 1 Level 2 Level 3

Mathematicians 1

Home

A list of some of the mathematicians who've appeared in this book.

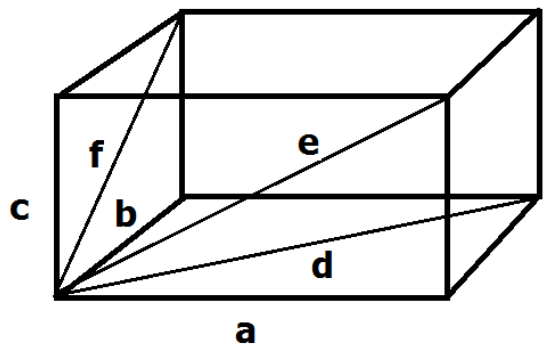
Given	Surname	Nationality	Born	Died	Given	Surname	Nationality	Born	Died
Niels	Abel	Norwegian	1802	1829	Peter	Dirichlet	German	1805	1859
Kenneth	Appel	American	1932	2012	Gregg	Easterbrook	American	1953	
Emil	Artin	Austrian/American	1898	1962	Arthur	Eddington	British	1882	1944
Andrew	Beal	American	1952		Albert	Einstein	German	1879	1955
Eric	Bell	Scottish	1883	1960	Paul	Erdős	Hungarian	1913	1996
Farkas	Bolyai	Hungarian	1775	1856		Euclid	Greek	c 350BC	c 250BC
Bernard	Bolzano	Bohemian	1781	1848	Leonhard	Euler	Swiss	1690	1764
Ronnie	Brown	English	1935		Istran	Fary	Hungarian	1922	1984
Georg	Cantor	German	1845	1918	Pierre de	Fermat	French	1601	1665
Eugene	Catalan	French/Belgian	1814	1894	Leonardo	Fibonacci	Italian	1170	1250
Pafnuty	Chebyshev	Russian	1821	1894	Abraham	Fraenkel	German/Israeli	1891	1965
Paul	Cohen	American	1934	2007	Philip	Franklin	American	1898	1965
Lothar	Collatz	German	1910	1990	Edward	Frenkel	Russian	1968	
John	Conway	English	1939		Carl	Gauss	German	1777	1855
Richard	Courant	German	1888	1972	Alexander	Gelfond	Russian	1906	1968
Tobias	Dantzig	German/Russian	1884	1956	Sophie	Germain	French	1776	1831
Harold	Davenport	English	1907	1969	Kurt	Gödel	Austrian/American	1906	1978
Alphonse	de Polignac	French	1828	1863	Christian	Goldbach	German	1690	1764
René	Descartes	French	1596	1650	Francis	Guthrie	South African	1881	1899

36 Notes: The Perfect Box Problem

Status: unsolved

Level 1: If a cuboid has sides 3, 4 and 5 units, then by Pythagoras's theorem, the lengths of the face diagonals will be $\sqrt{(3^2 + 4^2)} = 5$, $\sqrt{(3^2 + 5^2)} = 5.831$, $\sqrt{(5^2 + 4^2)} = 6.403$.

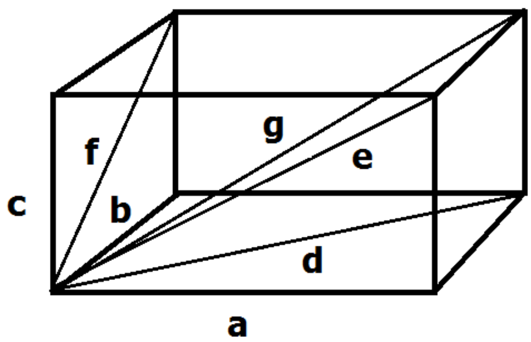
Remember, an Euler Brick is one where the three lengths for the sides, and the three lengths for the face diagonals, are all whole numbers.



So in the diagram, a, b, c, d, e and f are all integers. (Once again, if a, b, c, d, e, f are not all integers but are all rational, then we can magnify the box by some number that turns all the edges into integer lengths - so it doesn't matter whether we work with integers or rational numbers here).

Level 2: Once again, we can turn to Pythagoras's Theorem. We're seeking for integer solutions to the three equations, $a^2 + b^2 = d^2$, $a^2 + c^2 = e^2$, and $b^2 + c^2 = f^2$. If we can find whole numbers that make this system work, we are done. We have six numbers to play with, and three equations, so you might think that there would be plenty of sets of numbers that would work. In fact, things are harder than that.

The smallest Euler brick is **(a, b, c, d, e, f) = (44, 117, 240, 125, 244, 267).** How small is this? All lengths are less than 300, so maybe it does qualify as small. This was found a long time ago, in 1719, by Paul Halke. There are four more Euler bricks with all lengths less than 1000.



Level 3: Could we search for a brick that's even more special? We could ask for the space diagonal to be a whole number too. From the diagram above we can see that

$$\mathbf{a^2 + b^2 = d^2}, \text{ and } \mathbf{d^2 + c^2 = g^2}, \text{ so } \mathbf{g^2 = a^2 + b^2 + c^2.}$$

To find a perfect cuboid, we need to add this equation to the three above, and that extra equation makes the system extremely difficult to satisfy.

So difficult, in fact, that no perfect box has ever been found. A large amount of computer-time has been devoted to looking for one, but there's been no joy so far. Imagine the statement, 'no perfect box exists', and imagine trying to search for a counterexample from scratch. Work so far tells us that our first one could not appear until we started checking those with edge lengths longer than 10^{10}. I wondered earlier if disproving things with a counterexample is easier than proving things directly; I might have to take that back.

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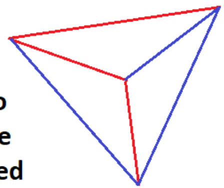
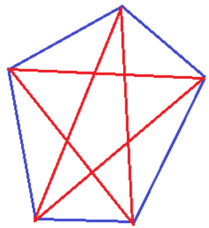
37 Notes: The Friends and Strangers Problem

Status: unsolved

Level 1: Trying out small values for n, we see 4 is too few people to guarantee an all-friends or an all-strangers trio, and so is 5.

No trio all blue or all red

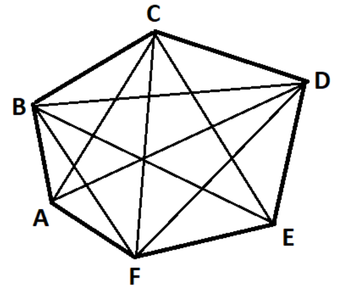




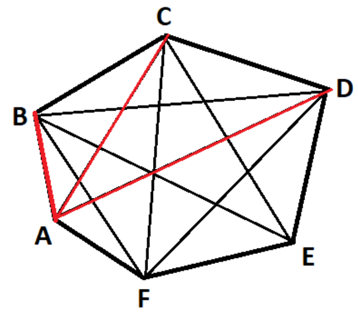
No trio all blue or all red

Red here stands for ‘knows’, and blue for ‘does not know.’ In each case, there is no all red or all blue triangle.

Level 2: What happens when n = 6?



There are five edges coming from A. At least three of these must be the same colour, because we only have two colours to choose from. Let’s say AB, AC, and AD are red.



Now if BC or CD or BD is red, we have an all red triangle. But if BC, CD, and BD are all blue, then BCD is an all blue triangle. Either way, you have a trio that are all strangers or all friends, and so 6 guests are all that's needed.

We have shown here (if we invent some new notation) that **R(3, 3)** is 6. The number **R(n, m)** is the smallest number of dots required to guarantee there is either a K_n (complete graph with n vertices) of one specified colour or a K_m of the other, and is named after Frank Ramsey. A similar argument to the one above shows that **R(3, 4)** = **R(4, 3)** = 9, and from there we can show that **R(4, 4)** = 18. So what is **R(5, 5)**? The best results so far have shown 42 ≤ **R(5, 5)** ≤ 55 – the exact value remains unresolved. We know, however, that R(m, n) is finite for all m and n, and we have bounds on R(m, n) – to this extent, we can say the problem is partially solved.

Home Level 1 Level 2 Level 3

Mathematicians 2 – more mathematicians from this book.

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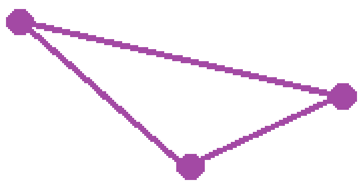
Given	Surname	Nationality	Born	Died	Given	Surname	Nationality	Born	Died
Wolfgang	Haken	German	1928		Grigori	Perelman	Russian	1966	
John	Hammersley	Scottish	1920	2004	Georg	Pick	Austrian	1859	1942
Heiko	Harborth	German	1938		Henri	Poincare	French	1854	1912
Godfrey	Hardy	English	1877	1947	George	Pólya	Hungarian	1887	1985
Harald	Helfgott	Peruvian	1977		Carl	Pomerance	American	1944	
Heinrich	Hertz	German	1857	1894	Richard	Preston	American	1954	
David	Hilbert	German	1862	1943		Pythagoras	Greek	571BC	495BC
Alfred	Kempe	English	1849	1922	Srinivasa	Ramanujan	Indian	1887	1920
Felix	Klein	German	1849	1925	Frank	Ramsey	English	1903	1930
Joseph-Louis	Lagrange	Italian	1736	1813	Bernhard	Riemann	German	1826	1866
Imre	Lakatos	Hungarian	1922	1974	Gian-Carlo	Rota	Italian/American	1932	1999
Edmund	Landau	German	1877	1938	Herbert	Ryser	American	1923	1985
Robert	Langlands	Canada	1936		Theodor	Schneider	German	1911	1988
Adrien-Marie	Legendre	French	1752	1833	Goro	Shimura	Japanese	1930	
Friedrich	Levi	German	1888	1966	David	Singmaster	American	1939	
Ernst	Mach	Austrian	1838	1916	Steve	Smale	American	1930	
Andrey	Markov	Russian	1856	1922	George	Szekeres	Hungarian/Australian	1911	2005
Francesco	Maurolica	Italian	1494	1575	Otto	Toeplitz	German	1881	1940
Marin	Mersenne	French	1588	1648	Stanislav	Ulam	Polish/American	1909	1984
Preda	Mihailescu	Romanian	1955		Edward	Waring	English	1736	1798
Leo	Moser	Austrian/Canadian	1921	1970	André	Weil	French	!906	1998
John von	Neumann	Hungarian/American	1903	1957	Andrew	Wiles	English	1953	
James	Newman	American	1907	1966	Ernst	Zermelo	German	1871	1953
Blaise	Pascal	French	1623	1662					

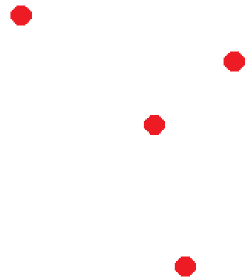
38 Notes: The Minimum-Number-of-Points Problem

Status: unsolved

Name: The Happy End problem
(two of the early researchers on this question eventually got married)

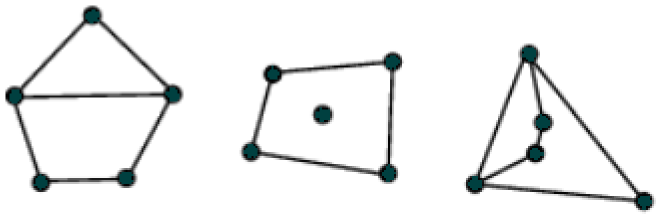
Level 1: Three dots (below left) guarantee that you make a triangle, and that must be convex, so g(3) = 3.

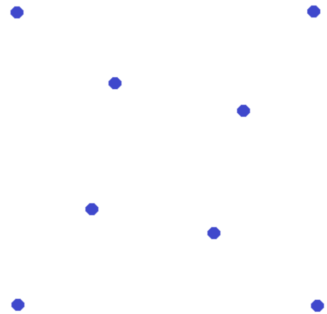




The four dots on the right above clearly fail to make a convex quadrilateral, so g(4) > 4.

Level 2: If you have five dots, however, then they must fall into one of the three cases on the left below, and a convex quadrilateral is guaranteed. So g(4) = 5.





What about g(5)? The diagram of eight dots on the right above has no convex pentagons, and so g(5) > 8.

It has been proved, in fact, by the mathematician Makai, that g(5) = 9.

We also have g(6) = 17, proved with the help of a computer by Szekeres and Peters (2006).

Level 3: It's something of a relief to learn that g(n) is finite for any n, which was proved by Erdős and Szekeres in 1935. They also found lower and upper bounds for g(n), stating

$$2^{n-2} + 1 \le g(n) \le \binom{2n-4}{n-2} + 1.$$

The upper bound here is not that tight:

n	3	4	5	6
$2^{n-2} + 1$	3	5	9	17
g(n)	3	5	9	17
$\binom{2n-4}{n-2}+1$	3	7	20	71

For n > 6, the value of g(n) is not known.

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Can an irrational number to an irrational power ever be rational?

A compelling question. Well, we've shown $\sqrt{2}$ to be irrational; why don't we have a look at $\sqrt{2}^{\sqrt{2}}$ to get us started? This is, to be sure, an irrational number to the power of an irrational number. There are two possibilities; it could be rational, or irrational.

If it's rational, then we are done - we've successfully found an example of an irrational number to the power of an irrational number being rational.

On the other hand, suppose it's irrational. Now think about ($\sqrt{2}^{\sqrt{2}}$) to the power $\sqrt{2}$. This is an irrational to the power of an irrational. Using the laws of indices, this is

$$((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^{(\sqrt{2}\sqrt{2})} = (\sqrt{2})^2 = 2,$$

which is rational. So again, we've found an example of an irrational number to the power of an irrational number being rational.

So the answer to the question in our title must be, ‘Yes’.

Notice the quirky bit here; we’ve shown that such a number exists, but we can’t say exactly what it is, since we don’t know if $\sqrt{2}^{\sqrt{2}}$ is irrational or not. We’ve given an ***existence proof***; a solution exists, and we might even be able to give some hints as to where it is, but we can’t say what it is.

In this case, in fact, the issue has now been resolved; $\sqrt{2}^{\sqrt{2}}$ has been shown to be irrational, thanks to the Gelfond-Schneider Theorem.

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39 Notes: The Cubes + Cubes = Cubes Problem

Status: solved

Level 1: Certainly $\mathbf{1\times2^3 + 1\times3^3 = 35\times1^3}$**.**

Level 2: It's not difficult to find solutions to $ax^3 + by^3 = cz^3$ if we let x or y or z be 1. If we say that none of x, y, z can be 1, with no common factor, it's harder, but even then, solutions like (1, 1, 7, 4, 5, 3) can be found.

Level 3: So what about the equation $\mathbf{x^3 + 2y^3 = 4z^3}$**?**

Suppose we have a natural number solution (x, y, z). Then $2y^3$ is even, and $4z^3$ is even, and so x^3 is even. But if x^3 is even, x must be even; let's say x = 2u. If we now substitute this in for x in our equation, we get $8u^3 + 2y^3 = 4z^3$. Dividing by 2 gives us $\mathbf{4u^3 + y^3 = 2z^3}$**.**

But now, using the same logic again, y^3 must be even, and so y is even; let's say y = 2v. Substituting in gives $4u^3 + 8v^3 = 2z^3$, or $\mathbf{2u^3 + 4v^3 = z^3}$**.**

But now we have to say z is even, let's say z = 2w. Substituting in gives us **$2u^3 + 4v^3 = 8w^3$**, which is **$u^3 + 2v^3 = 4w^3$** on dividing by 2.

This is identical to our starting equation. So starting from an imaginary solution to our equation, we've ended up with a smaller solution. What happens if we try to repeat this again and again? We run out of natural numbers, since there are only a finite number of these below x, y and z.

So we now have a contradiction. We have to say that supposing we had a solution (1, 2, 4, x, y, z) to start with must be an error - there can't be any such solution. Thus (1, 2, 4, 0, 0, 0) is the only possible solution. This helpful proof method is called **descent**.

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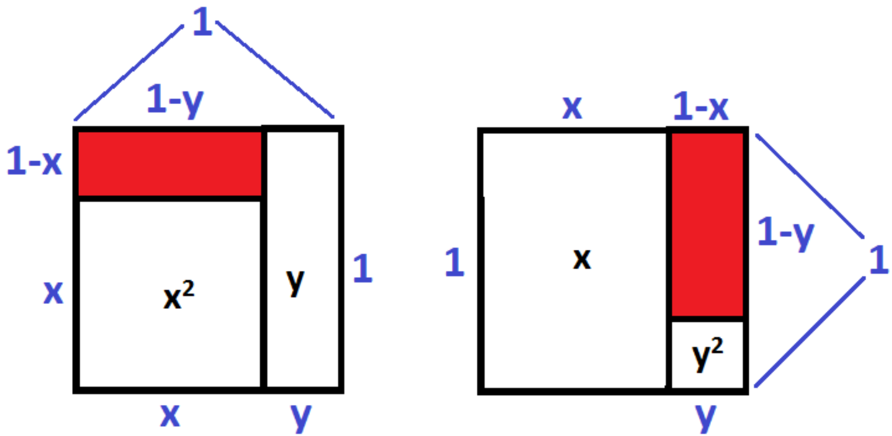
Notes: The Square-and-add-the-Other problem

We pick the numbers x and y, with $x + y = 1$. So we can write our two numbers as x and $1 - x$.

Squaring the first and add the second gives $\mathbf{x^2 - x + 1.}$ Squaring the second and add the first gives

$\mathbf{(1 - x)^2 + x = x^2 - 2x + 1 + x = x^2 - x + 1,}$ So we get the same result either way.



It's easy to see that the shaded area is $(1 - x)(1 - y)$ in both diagrams.

Or assume $\mathbf{x \neq y}$ (if $x = y$, the result is obvious). Then

$$\mathbf{x^2 + y = y^2 + x \Leftrightarrow x^2 - y^2 = x - y \Leftrightarrow (x + y)(x - y) = x - y \Leftrightarrow x + y = 1.}$$

Using the left-to-right arrows here would not be enough to prove the whole result, but since we can use the double-arrows, we have a complete proof.

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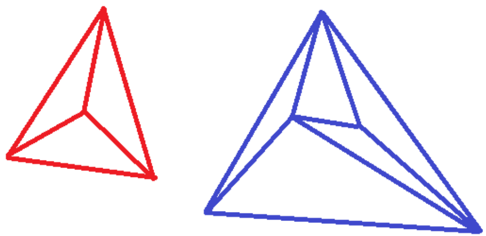
40 Notes: The Integer Edge-Length Problem

Status: unsolved ***Name of problem: Harborth's conjecture***

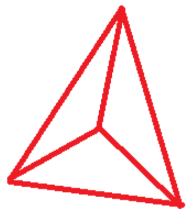
Level 1: We can convert both graphs into ones where the edges are straight lines.



Level 2: It turns out that any planar graph can be drawn with straight line edges. This is known as Fáry's theorem. Planar graphs sometimes appear complicated, indeed, we might wonder if they can be as complicated as we wish; this theorem tells us that there are limits.

Level 3: Harborth's conjecture takes Fáry's theorem one step further; every planar graph can be drawn with straight line edges, but is it possible for all of those edges to be of integer length?

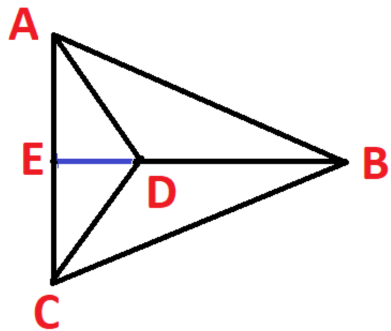
To see the extra restriction this places on our drawing of the graph, let's examine **K_4**.



We can certainly draw **K_4** using just straight lines, but can we make them all of integer length? Suppose the outer three points form an equilateral triangle of side length n, where n is a whole number. Suppose also that the inside point is at the centre of the equilateral triangle. Using Pythagoras, we find the distance from the centre dot to an outside dot is $\frac{n}{\sqrt{3}}$. Now √3 is irrational, so if n is a whole number, $\frac{n}{\sqrt{3}}$ cannot be rational.

Notice that if we can show that all sides can be rational, we are done; we can just put n equal to the lowest common multiple of all the denominators of the additional lengths, and multiply all lengths in the diagram by n, which has the effect of scaling the graph and turning all the edge lengths into whole numbers.

Let’s adjust our diagram to look something like this:

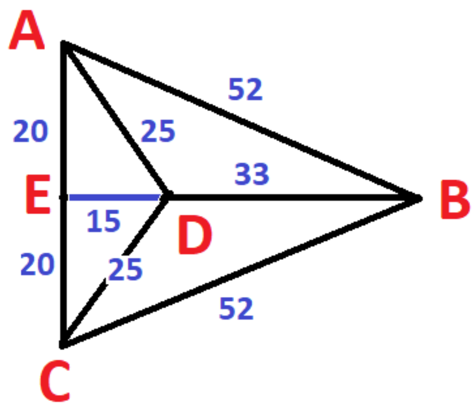


(The blue line has been added purely as a visual aid). We can use Pythagoras in the right-angled triangles ABE and ADE here. There are an infinite number of solutions (a, b, c) for $a^2 + b^2 = c^2$ that are whole numbers, (3, 4, 5) and (5, 12, 13) for example.

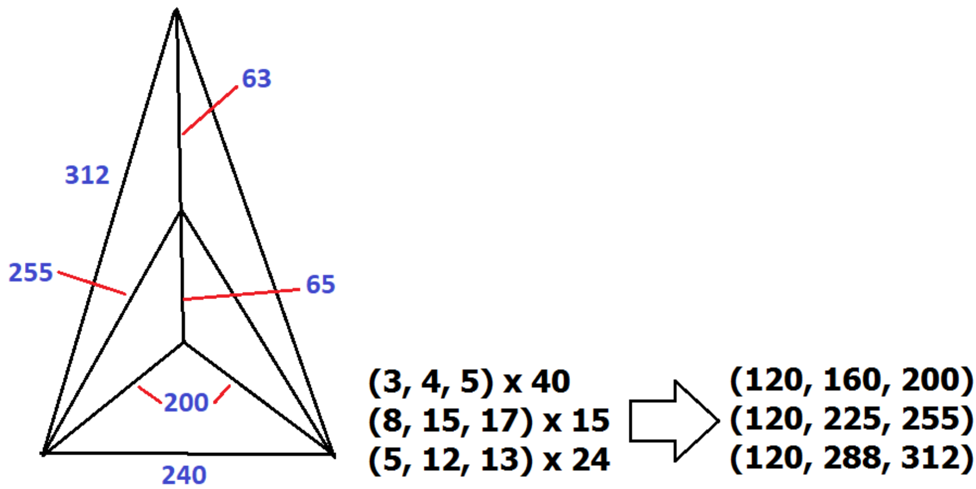
We have $3^2 + 4^2 = 9 + 16 = 25 = 5^2$, while $5^2 + 12^2 = 13^2$. The triples (3, 4, 5) and (5, 12, 13) are the two smallest with sides that have no common factors, and we can use them here.

Clearly if (a, b, c) is a right-angled triangle with integer sides, then ka, kb, kc will be too (think Pythagoras). Let’s try to turn ADE into a 3-4-5-type triangle, while making ABE a 5-12-13-type triangle at the same time.

Let's say AE is going to be the 4-type-side in ADE. This will have to be the 5-type-side in ABE. What's the smallest number that 4 and 5 both go into? That's 20, so we can multiply by 5 to make ADE into a 15-20-25 triangle, and multiply by 4 to make ABE into a 20-48-52 triangle.

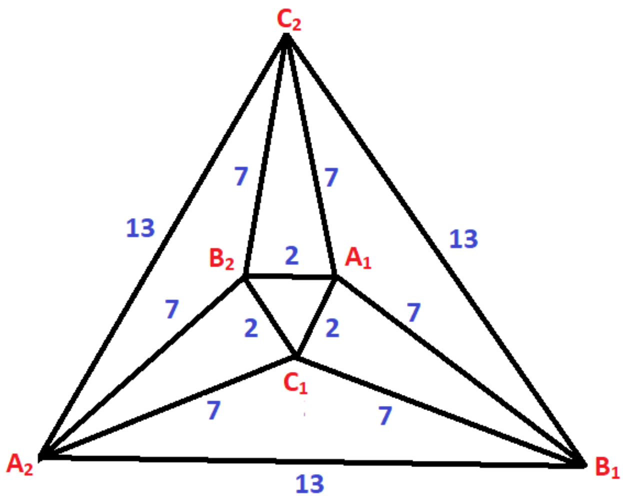


What about K_5 less one edge, which is planar? We can employ the same trick again, adding an (8, 15, 17) triangle to the two we used above this time.



So we have verified Harborth's Conjecture for K_4 and for K_5-less-one-edge. It's a small start, but it is a start.

The graph $K_{2,2,2}$ consists of three sets A, B, C, each containing two vertices, with all possible edges between sets A, B and C drawn. Below $K_{2,2,2}$ is drawn with integer sides in a similar way to K_4 and K_5-less-one-edge above.



Look at the planar graph for the answer to the Factor Graph problem; we are saying that these edges can be made straight, and then these straight lines lengths can all be tweaked or magnified to become integers. And remember, if Harborth's conjecture is true, our planar graph can be made arbitrarily larger; we can add as many further points and edges as we wish.

From one point of view, Harborth's conjecture seems likely. Fáry's theorem tells us that we can always draw our planar graph with straight line edges. Now if some of these edges are irrational in length, can we not just tweak them ever so slightly into being rational lengths? Then we're done. In other words, the tweak on our planar graph diagram required to make our lengths rational could be something so slight that we might not even notice it. The trouble is that tweaking one length, in however tiny a way, might throw out another from being rational to irrational. Somehow a proof has to be found that deals with all possible lengths simultaneously.

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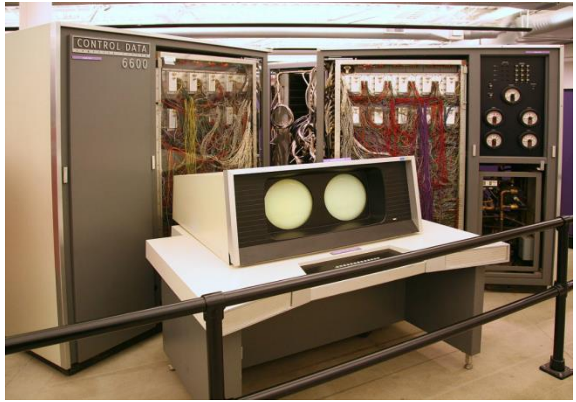
Notes for the n n^{th} Powers Equation (Bonus) Problem

Euler made the conjecture that if n is greater than 2, at least n n^{th} powers are required to sum to an nth power. He was wrong!
Lander and Parkin were able to come up with a counterexample in 1966 using the help of a computer called the CDC 6600 (shown right).



Their finding was that

$$\mathbf{27^5 + 84^5 + 110^5 + 133^5 = 144^5.}$$

Hard to find at the time, but once found, easy to check!

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Postscript

When David Hilbert gave his list of unsolved problems in 1900, he shared his excitement over the future course of his subject. Here he reveals his yearning to glimpse some of the mathematics that his colleagues would create in the years ahead.

Who among us would not be happy to lift the veil behind which is hidden the future; to gaze at the coming developments of our science and at the secrets of its development in the centuries to come? What will be the ends toward which the spirit of future generations of mathematicians will tend? What methods, what new facts will the new century reveal in the vast and rich field of mathematical thought?
David Hilbert

Fifteen years into the twenty-first century, we can all echo his words. One thing is certain, mathematics will evolve, and as it does, the unsolved problems in this book will succumb, perhaps to the attentions of mathematicians yet to be born, to be replaced by fresh problems to tax the mathematical community. New proofs will be discovered and refined, becoming more attractive and more accessible in the process. Even the deepest proof will become more mainstream, stimulating novel areas for research that will lead to more original conjectures. I hope I speak for all of us when I say, it'd be a beautiful privilege to play a part, however tiny, in that process.

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My better two-thirds Meg quietly allowed me the time to work on this, as she has with so many of my hare-brained schemes, and my warm thanks goes to her.

Much of this book comes from scouring the internet for information, which means I’m truly indebted to all those who devote their time to putting this information out there for free. It seems to me that if you want

to turn to the internet for free help, then you should try to donate some of your work freely in return, and I hope my maths education sites at

www.risps.co.uk, **www.making-statistics-vital.co.uk**, and **www.carom-maths.co.uk**

go some of the way to balancing things up. Sites that have been especially helpful to me for this ebook include Wikipedia, Wolfram MathWorld, and Plus magazine. Special thanks goes to Tomruen for the semi-regular tessellation pictures and to ArnoldReinhold for the CDC 6600 picture.

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The status of the problems – a table to print off and fill in

Some of these theorems have been proved in my lifetime; how many more will be proved in yours?

	Name	*Status*	*Proposed by*	*Date*	*Solved by*	*Date*
1	The Lines-into-Triangles Problem	Unsolved				
2	The Partition Problem	Solved				
3	The Twin Prime Problem	Unsolved				
4	Arithmetic Simultaneous Equation Problem	Solved				
5	Sum of single powers, Sum of cubes	Solved				
6	The Triangle of 1s Problem	Unsolved				
7	The First Triomino Tiling Problem	Solved				
8	The Map-Colouring Problem	Solved				
9	The Points-in-a-Circle problem	Unsolved				
10	The Is-Every-Number-a-Fraction Problem	Solved				
11	The Handshake Problem	Solved				
12	The How-Many-Tilings Problem	Solved				
13	The Lonely Runner Problem	Unsolved				
14	The Odd Perfect Number Problem	Unsolved				
15	The Dots and Area Problem	Solved				
16	The How-Many-Primes-Are-There Problem	Solved				
17	The HOTPO Problem	Unsolved				
18	The One Tile Problem	Unsolved				
19	The Are-the-Infinities-Equal Problem	Solved				
20	The Even = Prime + Prime Problem	Unsolved				

21	The Dot-Edge-Region Problem	Solved				
22	The Three Averages Problem	Solved				
23	The How-Many-Powers Problem	Unsolved				
24	The Shape-Covering Problem	Unsolved				
25	The Four-in-a-Bag Triples Problem	Unsolved				
26	The Squaring-the-Square Problem	Solved				
27	The Factor Graph Problem	Solved				
28	The Moving Sofa Problem	Unsolved				
29	The Infinite Sum of Fractions Problem	Solved				
30	The Second Triomino Tiling Problem	Solved				
31	The y = 1/x Number Problem	Unsolved				
32	The Circle-Region Problem	Solved				
33	The Transversals-in-a-Latin-Square Problem	Unsolved				
34	The Inscribed Square Problem	Unsolved				
35	The Same-Prime Trees Problem	Solved				
36	The Perfect Box Problem	Unsolved				
37	The Friends and Strangers Problem	Unsolved				
38	The Minimum-Number-of-Points Problem	Unsolved				
39	The Cubes + Cubes = Cubes Problem	Solved				
40	The Integer Edge-length Problem	Unsolved				

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Glossary

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algebra	The art of dealing with letters rather than numbers. See here.
algebraic	A number that can be written as the solution to an integer polynomial equation.
arithmetic	Describes a sequence that goes up (or down) in steady jumps.
arithmetic mean of a and b	(a+b)/2 for two positive numbers a and b.
axiom	A simple statement assumed to be true without proof.
binomial coefficient	${}^nC_r$ – the number of ways of choosing r things from n things. See here.
bipartite	A graph that splits into two sets of dots, where edges only join dots in different sets.
bound	Upper or lower; a value that a sequence can't go above or below.
brick	See cuboid.
ceiling(x)	the smallest whole number greater than or equal to x.
chord	A line joining two points on a circle.
complete	Describes a graph with n vertices with all ${}^nC_2$ possible edges drawn.
complex	Describes a number of the form a + bi, where a and b are real and i = $\sqrt{-1}$.
complete	Describes a mathematical system where all true statements can be proved.
composite	Describes a whole number that is not prime.
concave	Roughly, with indentations. See here.
conjecture	A mathematical claim that might or might not be true.
consistent	Describes a mathematical system that contains no contradictions.
contradiction	A situation where A and not-A have to be true together.

converges	Describes a sequence that homes in on a number.
converse	The converse of the statement 'A implies B' is 'B implies A'.
convex	Roughly, without indentations. See here.
coordinate geometry	A system that involves setting up axes to specify a point's position.
countable	Describes a set that can be placed into an infinite list.
counterexample	An example that disproves a conjecture.
cube root(x)	When multiplied by itself, and then by itself again, this number gives x.
cubic	Describes a polynomial with 3 as the highest power.
cuboid	A three-dimensional box with rectangles for faces.
cycle	A loop.
decimal part	The base ten fractional part of a number.
deep	Profound. See here.
degree	A unit for measuring angles (360^o = a full turn).
degree (of a vertex)	The number of edges coming out of a vertex in a graph.
descent	A method of proof by finding a smaller solution from a larger one. See here.
diverges	Describes a sequence that does not converge.
e	Transcendental number = 2.71818… that occurs naturally with great frequency in mathematics.
edge	A line connecting vertices in a graph.
elliptic curve	Roughly, a cubic curve without any singularities.
equilateral	Describes a triangle with all three sides equal.
even	An integer that has remainder 0 on dividing by 2.

Euclidean	Describes our usual plane geometry as formulated by Euclid.
expand	To take brackets out of an algebraic expression.
factor	A whole number that goes into a whole number.
factorial(n)=n!	The product of all integers from n down to 1 (note that 0! = 1).
factorise	Put into brackets.
Fermat's Last Theorem	$a^n + b^n = c^n$ has no whole number solutions, n > 2.
Fibonacci number	A number from the sequence 1, 1, 2, 3, 5, 8, 13...
flowchart	A box/arrow diagram showing how a program works.
formula	A rule for getting one number from another.
For-next	A type of loop in coding.
fraction	One integer over another non-zero integer.
generalise	To move from examples to the case for n.
geometric mean	$\sqrt{ab}$ for two positive numbers a and b.
graph	A collection of dots connected by lines.
harmonic mean	2ab/(a+b) for two positive numbers a and b.
heuristic	Rule of thumb.
HT	Hikorski Triple. See here.
hypotenuse	The longest side in a right-angled triangle.
iff	Short for ‘if and only if,’ or ‘implies and is implied by’.
If-then	A conditional statement in coding.
implies	If A implies B, then B follows as a consequence from A.

induction	**A proof method modelled on dominoes falling.**
infinite	**When describing a sequence, this means ‘continuing for ever’.**
infinity	**Roughly, a number larger than any finite number.**
integer	**A whole number, possibly negative or zero.**
irrational	**Describes a real number that is not rational – it cannot be written as a fraction.**
isosceles	**Describes a triangle with two equal sides.**
K_n	**The complete graph with n vertices.**
kite	**4-sided two-dimensional shape with a diagonal as a line of symmetry.**
latin square	**A Sudoku-like grid - see here.**
lattice	**A grid.**
lattice point	**An intersection point on a grid.**
limit	**A value that a sequence tends towards as the number of terms increases.**
Mersenne number	**A number of the form 2^n-1.**
minimal criminal	**The smallest possible counterexample.**
MT	**A Markov Triple - see here.**
natural number	**One of the set {0, 1, 2, 3,...}**
non-periodic	**A tiling of this type cannot be lifted, shifted and placed back onto itself exactly.**
odd	**Has a remainder 1 when divided by 2.**
parallel	**Describes two straight lines in two dimensions that never meet.**
parallelogram	**4-sided shape in two dimensions with two pairs of parallel sides.**
parallelepiped	**Three-dimensional shape with six parallelograms as faces.**

partition	**A way to divide up a whole number into smaller whole numbers.**
Pascal's Triangle	**Formed by the values of ^nC_r.**
perfect number	**Any number whose factors (not including itself) add to the number.**
perimeter	**The border, or length of border, for a shape.**
periodic	**Describes a tessellation that can be lifted, shifted, and placed back on itself.**
Pigeon Hole Principle	**n + 1 letters into n pigeon-holes implies one pigeon hole has two letters.**
planar	**a graph that can be drawn on a page without edges crossing.**
plane	**A two-dimensional flat surface.**
polygon	**A shape with straight line sides.**
polynomial	**A sum of powers of x.**
power	**x^y is y lots of x multiplied together.**
prime	**A natural number with only two factors, itself and 1.**
Prime Number Theorem	**Describes the approximate distribution of the primes.**
Pythagoras's Theorem	**Tells us that in a right-angled triangle, $a^2+b^2=c^2$.**
quadratic	**A polynomial with 2 as the highest power.**
quadrilateral	**A 4-sided polygon in two dimensions.**
quartic	**A polynomial with 4 as the highest power.**
quintic	**A polynomial with 5 as the highest power.**
rational	**A number that can be written as a fraction.**
real	**Describes any number on the number line.**

recurrence relation	A sequence where terms are defined in terms of previous terms (eg $u_{n+1}=u_n+u_{n-1}, u_1=1, u_2=2$)
reductio ad absurdum	Proof by contradiction.
region	An area bounded by dots and edges in a graph
regular	Describes a shape with all sides equal, all angles equal.
rhombus	4-sided shape with four equal sides.
s(n)	Sum of factors of n (not including n).
sharp	Describes a theorem whose bounds are precise.
simplex	An n-dimensional version of the triangle.
simultaneous	In mathematics, describes equations that are true together.
square	A number with an integer square root.
square root of x	When multiplied by itself, this number gives the number x.
squared square	A square cut into smaller squares that are all different.
tessellation	A tiling of the plane or space.
tetrahedron	A solid with four triangles for faces.
transcendental	A number that cannot be expressed as the solution to a polynomial equation.
transversal	See here.
trapezium	A 4-sided shape in two dimensions with a pair of parallel sides.
tree	One of a range of objects that possess a branching structure.
triangle number	A number in the sequence 1, 3, 6, 10, 15, 21,...
triomino	Three squares stuck together so edges match (there are only two).
twin primes	Two primes that are 2 apart.

verify	**To check a result or a particular example rather than to prove.**
vertex	**Node or dot.**
x-axis	**The horizontal axis in coordinate geometry.**
y-axis	**The vertical axis in coordinate geometry.**

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The Proving Ground – an introduction to mathematical proof

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Association of Teachers of Mathematics
2A Vernon Street
Vernon House
Derby
DE11FR

Tel: 01332 977891

Email: admin@atm.org.uk

www.atm.org.uk