Question 3

4.1.3 Which of the following are functions from $\mathbb{R}$ to $\mathbb{R}$? If f is a function, give its range.

b) $f(x) = \frac{1}{x^2-4}$

$\boldsymbol{f}$ **is not well defined when x** $= \pm 2$

c) $f(x) = \sqrt{x^2}$

$\boldsymbol{f\ is\ well\ defined; range: all\ non-negative\ real\ numbers}$

4.1.5 Express the range of each function using roster notation.

b) Let $\boldsymbol{A} = \{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}, \boldsymbol{f}: \boldsymbol{A} \to \mathbb{Z}, such\ that\ \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^{\mathbf{2}}$

$\{\mathbf{4}, \mathbf{9}, \mathbf{16}, \mathbf{25}\}$

d) $\boldsymbol{f}: \{\mathbf{0}, \mathbf{1}\}^{\mathbf{5}} \to \mathbb{Z}$. For $\boldsymbol{x} \in \{\mathbf{0}, \mathbf{1}\}^{\mathbf{5}}, \boldsymbol{f}(\boldsymbol{x})$ is the number of $\mathbf{1}$'s that occur in x. For example, $\boldsymbol{f}(\mathbf{01101}) = \mathbf{3}$ because there are three $\mathbf{1}$'s in the string **"01101".**

$\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$

h) Let $\boldsymbol{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$. $\boldsymbol{f}: \boldsymbol{A} \times \boldsymbol{A} \to \mathbb{Z} \times \mathbb{Z}$, where $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{y}, \boldsymbol{x})$.

$\{(\mathbf{1}, \mathbf{2}), (\mathbf{1}, \mathbf{3}), (\mathbf{2}, \mathbf{3}), (\mathbf{2}, \mathbf{1}), (\mathbf{3}, \mathbf{1}), (\mathbf{3}, \mathbf{2}), (\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3})\}$

i) Let $\boldsymbol{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$. $\boldsymbol{f} = \boldsymbol{A} \times \boldsymbol{A} \to \mathbb{Z} \times \mathbb{Z}$, where $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y} + \mathbf{1})$.

$\{(\mathbf{1}, \mathbf{2}), (\mathbf{1}, \mathbf{3}), (\mathbf{1}, \mathbf{4}), (\mathbf{2}, \mathbf{2}), (\mathbf{2}, \mathbf{3}), (\mathbf{2}, \mathbf{4}), (\mathbf{3}, \mathbf{2}), (\mathbf{3}, \mathbf{3}), (\mathbf{3}, \mathbf{4})\}$

l) Let $\boldsymbol{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$. $\boldsymbol{f}: \boldsymbol{P}(\boldsymbol{A}) \to \boldsymbol{P}(\boldsymbol{A})$. For $\boldsymbol{X} \subseteq \boldsymbol{A}, \boldsymbol{f}(\boldsymbol{X}) = \boldsymbol{X} - \{\mathbf{1}\}$

$\{\emptyset, \{\mathbf{2}\}, \{\mathbf{3}\}, \{\mathbf{2}, \mathbf{3}\}\}$

Question 4

i) a) **4.2.2** For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

c. $\boldsymbol{h}: \mathbb{Z} \rightarrow \mathbb{Z}.\ \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{x}^{\mathbf{3}}$

Not onto; no x $\in \mathbb{Z}$ *maps to 10.*

One-to-one

g. $\boldsymbol{f}: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z},\ \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x} + \mathbf{1}, \mathbf{2}\boldsymbol{y})$

Not onto; no $(\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{Z}^{+} \times \mathbb{Z}^{+})$ *maps to (0, 11).*

One to one

k. $\boldsymbol{f}: \mathbb{Z}^{+} \times \mathbb{Z}^{+} \rightarrow \mathbb{Z}^{+},\ \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{2}^{\boldsymbol{x}} + \boldsymbol{y}$

Not onto; no $(\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{Z}^{+} \times \mathbb{Z}^{+})$ *maps to 1.*

Not one to one; $\boldsymbol{f}(\mathbf{4}, \mathbf{2}) = \mathbf{18}$ *and* $\boldsymbol{f}(\mathbf{3}, \mathbf{10}) = \mathbf{18}$

b) **4.2.4** For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

b. $\boldsymbol{f}: \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}} \rightarrow \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}}$. The output of $\boldsymbol{f}$ is obtained by taking the input string and replacing the first bit by $\mathbf{1}$, regardless of whether the first bit is $\mathbf{0}$ or $\mathbf{1}$. For example, $\boldsymbol{f}(\mathbf{001}) = \mathbf{101}$ and $\boldsymbol{f}(\mathbf{110}) = \mathbf{110}$

Not onto; no x $\in \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}}$ *maps to 011.*

Not one to one; $\boldsymbol{f}(\mathbf{001}) = \mathbf{101}$ *and* $\boldsymbol{f}(\mathbf{101}) = \mathbf{101}$*.*

c. $\boldsymbol{f}: \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}} \rightarrow \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}}$. The output of $\boldsymbol{f}$ is obtained by taking the input string and reversing the bits. For example, $\boldsymbol{f}(\mathbf{011}) = \mathbf{110}$.

Onto and one-to-one

d. $\boldsymbol{f}: \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}} \rightarrow \{\mathbf{0}, \mathbf{1}\}^{\mathbf{4}}$. The output of $\boldsymbol{f}$ is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, $\boldsymbol{f}(\mathbf{100}) = \mathbf{1001}$.

Onto and one-to-one

g. Let $\boldsymbol{A}$ be defined to be the set $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\}$ and let $\boldsymbol{B} = \{\mathbf{1}\}$. $\boldsymbol{f}: \boldsymbol{P}(\boldsymbol{A}) \to \boldsymbol{P}(\boldsymbol{A})$. For $\boldsymbol{X} \subseteq \boldsymbol{A}, \boldsymbol{f}(\boldsymbol{X}) = \boldsymbol{X} - \boldsymbol{B}$. Recall that for a finite set ***A***, ***P(A)*** denotes the power set of ***A*** which is the set of all subsets of ***A***.

Not onto; no X $\in \boldsymbol{P}(\boldsymbol{A})$ ***maps to {1, 2, 3}***

Not one-to-one; $\boldsymbol{f}(\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}) = \{\mathbf{2}, \mathbf{3}\}$ ***and*** $\boldsymbol{f}(\{\mathbf{2}, \mathbf{3}\}) = \{\mathbf{2}, \mathbf{3}\}$

ii) Give an example of a function from the set of integers to the set of positive

a. One-to-one, but not onto

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{cases} \mathbf{5}\boldsymbol{x} & \boldsymbol{if}\ \boldsymbol{x} \geq \mathbf{0} \\ \mathbf{5}|\boldsymbol{x}| + \mathbf{1} & \boldsymbol{if}\ \boldsymbol{x} < \mathbf{0} \end{cases}$$

b. Onto, but not one-to-one

$$\boldsymbol{f}(\boldsymbol{x}) = |\boldsymbol{x}|$$

c. One-to-one and onto

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{cases} \mathbf{2}\boldsymbol{x} & \boldsymbol{if}\ \boldsymbol{x} \geq \mathbf{0} \\ \mathbf{2}|\boldsymbol{x}| + \mathbf{1} & \boldsymbol{if}\ \boldsymbol{x} < \mathbf{0} \end{cases}$$

d. Neither one-to-one nor onto

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^{\mathbf{2}}$$

Question 5

a) **4.3.2** For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}

c. $\boldsymbol{f}: \mathbb{R} \to \mathbb{R}.\ \boldsymbol{f(x) = 2x + 3}$

Onto and one-to-one (well-defined inverse).

$\boldsymbol{f^{-1}(y) = \frac{y-3}{2}}$

d. Let $\boldsymbol{A}$ be defined to the set $\boldsymbol{\{1, 2, 3, 4, 5, 6, 7, 8\}}$ $\boldsymbol{f}: \boldsymbol{P(A)} \to \boldsymbol{\{0, 1, 2, 3, 4, 5, 6, 7, 8\}}$. For $\boldsymbol{X \subseteq A}$, $\boldsymbol{f(X) = |X|}$. Recall that for a finite set ***A, P(A)*** denotes the power set of $\boldsymbol{A}$ which is the set of all subsets of ***A.***

Onto but not one-to-one (not well-defined inverse)

g. $\boldsymbol{f}: \boldsymbol{\{0, 1\}^3} \to \boldsymbol{\{0, 1\}^3}$. The output of $\boldsymbol{f}$ is obtained by taking the input string and reversing the bits. For example, $\boldsymbol{f(011) = 110}$.

Onto and one-to-one (well-defined inverse)

$\boldsymbol{f^{-1}(y) = reversing\ the\ bits\ in\ y}$

i. $\boldsymbol{f}: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, \boldsymbol{f(x, y) = (x + 5, y - 2)}$

Onto and one-to-one (well-defined inverse). $\boldsymbol{f^{-1}(x, y) = (x - 5, y + 2)}$

b) **4.4.8**

The domain and target set of functions $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{h}$ are $\mathbb{Z}$. The functions are defined as:

- $\boldsymbol{f(x) = 2x + 3}$
- $\boldsymbol{g(x) = 5x + 7}$
- $\boldsymbol{h(x) = x^2 + 1}$

Give an explicit formula for each function given below.

c. $\boldsymbol{f \circ h}$

$f(h(x)) = 2(x^2 + 1) + 3$

$= 2x^2 + 2 + 3$

$= \boldsymbol{2x^2 + 5}$

d. $\boldsymbol{h \circ f}$

$$h(f(x)) = (2x+3)^2 + 1$$

$$= (4x^2 + 12x + 9) + 1$$

$$= \mathbf{4x^2 + 12x + 10}$$

c) **4.4.2** Consider three functions $\boldsymbol{f}, \boldsymbol{g}$, and $\boldsymbol{h}$, whose domain and target are $\mathbb{Z}$. Let

$f(x) = x^2 \qquad g(x) = 2^x \qquad h(x) = \left\lceil \frac{x}{5} \right\rceil$

b. Evaluate $(\boldsymbol{f \circ h})(\mathbf{52})$

$$h(52) = \left\lceil \frac{52}{5} \right\rceil = 11$$

$$f(11) = 11^2 = \mathbf{121}$$

c. Evaluate $(\boldsymbol{g \circ h \circ f})(\mathbf{4})$

$$f(4) = 4^2 = 16$$

$$h(16) = \left\lceil \frac{16}{5} \right\rceil = 4$$

$$g(4) = 2^4 = \mathbf{16}$$

d. Give a mathematical expression for $\boldsymbol{h \circ f}$

$$\left\lceil \frac{\boldsymbol{x^2}}{\mathbf{5}} \right\rceil$$

d) **4.4.6** Define the following functions $\boldsymbol{f}, \boldsymbol{g}$, and $\boldsymbol{h}$:

- $\boldsymbol{f}: \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}} \rightarrow \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}}$**.** The output of $\boldsymbol{f}$ is obtained by taking the input string and replacing the first bit by **1**, regardless of whether the first bit is a **0** or **1.** For example, $\boldsymbol{f}(\mathbf{001}) = \mathbf{101}$ and $\boldsymbol{f}(\mathbf{110}) = \mathbf{110}$
- $\boldsymbol{g}: \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}} \rightarrow \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}}$**.** The output of $\boldsymbol{g}$ is obtained by taking the input string and reversing the bits. For example, $\boldsymbol{g}(\mathbf{011}) = \mathbf{110}$**.**
- $\boldsymbol{h}: \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}} \rightarrow \{\mathbf{0}, \mathbf{1}\}^{\mathbf{3}}$**.** The output of $\boldsymbol{h}$ is obtained by taking the input string and replacing the last bit with a copy of the first bit. For example, $\boldsymbol{h}(\mathbf{011}) = \mathbf{010}$**.**

c. What is $(\boldsymbol{h \circ f})(\mathbf{010})$?

$$f(010) = 110$$

$$h(110) = \mathbf{111}$$

d. What is the range of $\boldsymbol{h \circ f}$?

$\{\mathbf{101}, \mathbf{111}\}$

e. What is the range of $\boldsymbol{g} \circ \boldsymbol{f}$**?**

{001, 011, 101, 111}

Extra credit

4.4.4 Let $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}: \boldsymbol{Y} \to \boldsymbol{Z}$ be two functions.

c. Is it possible that $\boldsymbol{f}$ is not one-to-one and $\boldsymbol{g} \circ \boldsymbol{f}$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for $\boldsymbol{f}$ and $\boldsymbol{g}$.

Yes. Let $\boldsymbol{f}: \{-\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{3}\} \to \{\mathbf{1}, \mathbf{4}, \mathbf{9}\}$, $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^2$ ***and*** $\boldsymbol{g}: \{\mathbf{1}, \mathbf{4}, \mathbf{9}\} \to \{\mathbf{2}, \mathbf{5}, \mathbf{10}\}, \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x} + \mathbf{1}$

d. Is it possible that $\boldsymbol{g}$ is not one-to-one and $\boldsymbol{g} \circ \boldsymbol{f}$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for $\boldsymbol{f}$ and $\boldsymbol{g}$.

No