Question 3

a) 8.2.2 Give complete proofs for the growth rates of the polynomials below. You should provide specific values for $\boldsymbol{c}$ and $\boldsymbol{n_0}$ and prove algebraically that the functions satisfy the definitions for $\boldsymbol{O}$ and $\boldsymbol{\Omega}$

b. $\boldsymbol{f(n) = n^3 + 3n^2 + 4}$. Prove that $\boldsymbol{f = \Theta(n^3)}$.

$$Claim: \boldsymbol{f = O(n^3)\ and\ f = \Omega(n^3)}$$

$$Select\ c = 8\ and\ n_0 = 1. We\ will\ show\ that\ for\ any\ n \geq 1, f(n) \leq 8n^3$$

$$For\ n \geq 1, n^3 > n^2\ \ and\ n^3 \geq 1\ so$$

$$n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 4n^3 = 8n^3$$

$$Therefore, f(n) \leq 8n^3\ which\ means\ f = O(n^3)$$

$$Select\ c = 1\ and\ n_0 = 1. We\ will\ show\ that\ for\ any\ n \geq 1, f(n) \geq n^3.$$

$$For\ n \geq 1, 3n^2 + 4 \geq 0\ so$$

$$n^3 \leq n^3 + 3n^2 + 4$$

$$Therefore, f(n) \geq n^3\ which\ means\ f = \Omega(n^3)$$

$$Since\ f(n^3) = O(n^3)\ and\ f = \Omega(n^3)$$

b) 8.3.5

MysteryAlgorithm

Input: $a_1, a_2, \ldots, a_n$
n, the length of the sequence.
p, a number.
Output: ??

$i := 1$
$j := n$

While ($i < j$)
    While ($i < j$ and $a_i < p$)
        $i := i + 1$
    End-while
    While ($i < j$ and $a_j \geq p$)
        $j := j - 1$
    End-while
    If ($i < j$), swap a_i and a_j
End-while

Return($a_1, a_2, \ldots, a_n$)

a) Describe in English how the sequence of numbers is changed by the algorithm. (Hint: try the algorithm out on a small list of positive and negative numbers with p=0)

The mystery algorithm compares the numbers in the sequence starting from the ends and moves inwards, swapping numbers on the left that are greater than or equal to p with the numbers on the right that are less than p.

b) What is the total number of times that the lines "$i := i + 1$" or "$j := \ j - 1$" are executed on a sequence of length *n*? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the two lines are executed.

Lines "$i := i + 1$" and "$j := \ j - 1$" execute as many times as there are numbers on the left side of the sequence that are greater than or equal to p and numbers on the right side that are less than p and the numbers do not overlap.

c) What is the total number of times that the swap operation is executed? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the swap is executed.

The total swap operation is executed as many times as i or j is incremented/decremented without i equaling j.

d) Give an asymptotic lower bound for the time complexity of the algorithm. Is it important to consider the worst-case input in determining an asymptotic lower bound (using Ω) on the time complexity of the algorithm? (Hint: argue that the number of swaps is at most the number of times that **i** is incremented or **j** is decremented).

For the input where all the numbers in the sequence are larger than p the outer loop will execute n – 1 times. For each outer loop execution, the first loop will execute 1 time while the second inner loop will execute n – 1 times. Therefore, the number of operations performed is $\Omega(n^2)$.

e) Give a matching upper bound (using ***O***-notation) for the time complexity of the algorithm.

The outer loop can execute at most n – 1 times and during each outer loop iteration, the inner loops execute a combined total of n times. The total number of operations executed in the entire nested loop is at most cn^2 *for some constant c. Let d be the number of operations executed before and after the nested loop. The total number of operations for any input of length n is at most* $c + n^2 + d$ *which is* $O(n^2)$.

Question 4

a) **5.1.2** Consider the following definitions for sets of characters:

- Digits = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
- Letters = {a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}
- Special characters = {*, &, $, #}

Compute the number of passwords that satisfy the given constraints.

b) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.

$$(10+26+4)^7 + (10+26+4)^8 + (10+26+4)^9$$

$$= 40^7 + 40^8 + 40^9$$

$$= \mathbf{2.69 \cdot 10^{14}}$$

c) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

$$(10+4) * (10+4+26)^6 + (10+4) * (10+4+26)^7 + (10+4) * (10+4+26)^8$$

$$= 14 * 40^6 + 14 * 40^7 + 14 * 40^8$$

$$= \mathbf{9.41 \cdot 10^{13}}$$

b) **5.3.2a** How many strings are there over the set **{a, b, c}** that have length **10** in which no two consecutive characters are the same?

$$3 \cdot 2^9 = \mathbf{1536}$$

c) **5.3.3** License plate numbers in a certain state consist of seven characters. The first character is a digit. The next four characters are capital letters, and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form: Digit-Letter-Letter-Letter-Letter-Digit-Digit

b) How many license plate numbers are possible if no digit appears more than once?

$$P(10,3) * 26^4 = \mathbf{329{,}022{,}720}$$

c) How many license plate numbers are possible if no digit or letter appears more than once?

$$P(10,3) * P(26,4) = \mathbf{258{,}336{,}000}$$

d) **5.2.3** Let $\boldsymbol{B} = \{\mathbf{0}, \mathbf{1}\}$. $\boldsymbol{B^n}$ is the set of binary strings with $\boldsymbol{n}$ bits. Define the set $\boldsymbol{E_n}$ to be the set of binary strings with $\boldsymbol{n}$ bits that have an even number of **1's**. Note that zero is an even number, so a string with zero **1's** (i.e., a string that is all **0**s) has an even number of **1**s.

a) Show a bijection between $\boldsymbol{B^9}$ and $\boldsymbol{E_{10}}$. Explain why your function is a bijection.

$$f: B^9 \to E^{10}$$

$$f(x) = \begin{cases} x0 & \textit{when } x \textit{ has an even number of } 1s \\ x1 & \textit{when } x \textit{ has an odd number of } 1s \end{cases}$$

This is a bijection because all elements of B^9 map to exactly one element of E^{10} (one-to-one) and all elements of E^{10} map to exactly one element of B^9 (onto).

b) What is $|\boldsymbol{E_{10}}|$?

$$|E_{10}| = |B^9| = 2^9 = \mathbf{512}$$

Question 5

a) **5.4.2** At a certain university in the U.S., all phone numbers **7**-digits long and start with either **824** or **825**.

   a. How many different phone numbers are possible?

$$\mathbf{2 * 10^4}$$

   b. How many different phone numbers are there in which the last four digits are all different?

$$2 * P(10,4) = \mathbf{10080}$$

b) **5.5.3** How many 10-bit strings are there subject to each of the following restrictions?

   a. No restrictions

$$\mathbf{2^{10} = 1024}$$

   b. The string starts with **001**.

$$\mathbf{2^7 = 128}$$

   c. The string starts with **001** or **10.**

$$\mathbf{2^7 + 2^8 = 384}$$

   d. The first two bits are the same as the last two bits.

$$\mathbf{2^6 + 2^6 = 128}$$

   e. The string has exactly six **0**'s.

$$\boldsymbol{\binom{10}{6} = 210}$$

   f. The string has exactly six **0**'s and the first bit is **1**.

$$\boldsymbol{\binom{9}{6} = 84}$$

   g. There is exactly one **1** in the first half and exactly three **1**'s in the second half.

$$\boldsymbol{\binom{5}{1} * \binom{5}{3} = 50}$$

c) **5.5.5a** There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?

$$\boldsymbol{\binom{30}{10} * \binom{35}{10} = 5.52 \cdot 10^{15}}$$

d) **5.5.8** This question refers to a standard deck of playing cards. If you are unfamiliar with playing cards, there is an explanation in "Probability of an event" section under the heading "Standard playing cards." A five-card hand is just a subset of 5 cards from a deck of 52 cards.

c. How many five-card hands are made entirely of hearts and diamonds?

$$\binom{8}{5} = 56$$

d. How many five-card hands have four cards of the same rank?

$$\binom{4}{4} * \binom{48}{1} * 13 = 624$$

e. A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. How many five-card hands contain a full house?

$$13 * \binom{4}{2} * 12 * \binom{4}{3} = 3744$$

f. How many five-card hands do not have any two cards of the same rank?

$$\binom{13}{5} * 4^5 = 1,317,888$$

e) **5.6.5** A group of five friends goes to a restaurant for dinner. The restaurant offers 20 different main dishes.

a. Suppose that the group collectively selects five different dishes to share. The waiter just needs to place all five dishes in the center of the table. How many different possible meals are there for the group?

$$\binom{20}{5} = 15,504$$

b. Suppose that each individual selects a main course. The waiter must remember who selected which dish. It's possible for more than one person to select the same dish. How many different possible meals are there for the group?

$$20^5 = 3,200,000$$

Question 6

a) **5.7.2** A 5 card hand is drawn from a deck of standard playing cards.
   a. How many 5 card hands have at least one club?

$$\binom{52}{5} - \binom{39}{5} = \mathbf{2{,}023{,}203}$$

   b. How many 5 card hands have at least two cards with the same rank?

$$\binom{52}{5} - \binom{13}{5} * 4^5 = \mathbf{1{,}281{,}072}$$

b) **5.8.4** 20 different comic books will be distributed to five kids.
   a. How many ways are there to distribute the comic books if there are no restrictions on how many go to each kid (other than the fact that all 20 will be given out)?

$$\mathbf{5^{20} = 9.54E13}$$

   b. How many ways are there to distribute the comic books if they are divided evenly so that 4 go to each kid?

$$\frac{\mathbf{20!}}{\mathbf{4!\,4!\,4!\,4!\,4!}} = \mathbf{3.06E11}$$

Question 7

How many one-to-one functions are there from a set with five elements to set with the following number of elements?

a) $\mathbf{0}$
b) $5! = \mathbf{120}$
c) $P(6,5) = \mathbf{720}$
d) $P(7,5) = \mathbf{2520}$