Question 5

a)

1.12.2) Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

b)

$$\begin{array}{l} p \to (q \land r) \\ \neg q \\ \hline \therefore \neg p \end{array}$$

1.	$\neg q$	*Hypothesis*
2.	$(\neg q \lor \neg r)$	*Addition **1***
3.	$\neg(q \land r)$	*De Morgan's law **2***
4.	$p \to (q \land r)$	*Hypothesis*
5.	$\neg p$	*Modus tollens **3,4***

e)

$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \neg q \\ \hline \therefore r \end{array}$$

1.	$p \lor q$	*Hypothesis*
2.	$\neg p \lor r$	*Hypothesis*
3.	$q \lor r$	*Resolution **1,2***
4.	$\neg q$	*Hypothesis*
5.	r	*Disjunctive syllogism **3,4***

1.12.3) Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic.

c)

$$\begin{array}{l} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$$

Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive syllogism. (Hint: you will need one of the conditional identities from the laws of propositional logic).

1.	$p \vee q$	*Hypothesis*
2.	$\neg p \to q$	*Conditional identity **1***
3.	$\neg p$	*Hypothesis*
4.	$\neg\neg p$	*Double negation law **3***
5.	q	*Modus pollens **2,3***

1.12.5) Give the form of each argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

c)

I will buy a new car and a new house only if I get a job.
I am not going to get a job.

∴ I will not buy a new car.

***c**: I will buy a new car*

***h**: I will buy a new house*

***j**: I got a job*

$$\begin{array}{c} c \wedge h \to j \\ \underline{\neg j} \\ \therefore \neg c \end{array}$$

c	h	j	$c \wedge h$	$c \wedge h \to j$	$\neg j$	$\neg c$
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	F	T	F	F
T	F	F	F	T	T	F
F	T	T	F	T	F	T
F	T	F	F	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	T	T

Invalid argument; c = T, h = F, j = F

d)

I will buy a new car and a new house only if I get a job.
I am not going to get a job.
I will buy a new house.

∴ I will not buy a new car.

c: *I will buy a new car*

h: *I will buy a new house*

j: *I got a job*

$$\begin{array}{c} c \land h \to j \\ \neg j \\ h \\ \hline \therefore \neg c \end{array}$$

c	h	j	$c \land h$	$c \land h \to j$	$\neg j$	$\neg c$
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	F	T	F	F
T	F	F	F	T	T	F
F	T	T	F	T	F	T
F	T	F	F	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	T	T

Valid argument

1.	$\neg j$	*Hypothesis*
2.	$c \land h \to j$	*Hypothesis*
3.	$\neg(c \land h)$	*Modus tollens* ***1, 2***
4.	$\neg c \lor \neg h$	*De Morgan's law* ***1***
5.	$\neg h \lor \neg c$	*Commutative law* ***4***
6.	h	*Hypothesis*
7.	$\neg\neg h$	*Double negation law* ***6***
8.	$\neg c$	*Disjunctive syllogism* ***5, 7***

b)

1.13.3) Show that the given argument is invalid by giving values for the predicates P and Q over the domain $\{a, b\}$

b)

$$\begin{array}{l} \exists x(P(x) \vee Q(x)) \\ \exists x \neg Q(x) \\ \hline \therefore \exists x P(x) \end{array}$$

	P	Q
a	F	T
b	F	F

1.13.5) Prove whether each argument is valid or invalid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. If the argument is valid, then use the rules of inference to prove that the form is valid. If the argument is invalid, give values for the predicates you defined for a small domain that demonstrates the argument is invalid.

The domain for each problem is the set of students in a class.

M(x): x missed class
A(x): x got an A
D(x): x got a detention

d)

Every student who missed class got a detention.
Penelope is a student in the class.
Penelope did not miss class.

Penelope did not get a detention.

$$\begin{array}{c} \forall x(M(x) \rightarrow D(x)) \\ \text{Penelope is a particular element} \\ \neg M(\text{Penelope}) \\ \hline \therefore \neg D(\text{Penelope}) \end{array}$$

x	$M(x)$	$D(x)$
Penelope	F	T

Invalid argument

e)

Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.

Penelope did not get a detention.

$$\forall x((M(x) \lor D(x)) \to \neg A(x))$$
$Penelope$ is a particular element
$$A(Penelope)$$

$$\therefore \neg D(Penelope)$$

1.	$\forall x((M(x) \lor D(x)) \to \neg A(x))$	*Hypothesis*
2.	$Penelope\ is\ a\ particular\ element$	*Hypothesis*
3.	$(M(Penelope) \lor D(Penelope)) \to \neg A(Penelope)$	*Universal instantiation* ***1,2***
4.	$A(Penelope)$	*Hypothesis*
5.	$\neg\neg A(Penelope)$	*Double negation law* ***4***
6.	$\neg(M(Penelope) \lor D(Penelope))$	*Modus tollens* ***3, 5***
7.	$\neg M(Penelope) \land \neg D(Penelope)$	*De Morgan's law*
8.	$\neg D(Penelope)$	*Simplification*

Question 6

2.4.1) Each statement below involves odd and even integers. An odd integer is an integer that can be expressed as $2k + 1$, where k is an integer. An even integer is an integer that can be expressed as $2k$, where k is an integer.

Prove each of the following statements using a direct proof.

d) The product of two odd integers is an odd integer.

$$Let\ \boldsymbol{x}\ and\ \boldsymbol{y}\ be\ odd\ integers. We\ wil\ prove\ \boldsymbol{xy}\ is\ an\ odd\ integer.$$

$$Since\ x\ is\ odd, then\ \boldsymbol{x} = \boldsymbol{2k+1}\ where\ \boldsymbol{k}\ is\ some\ integer. Since\ \boldsymbol{y}\ is\ odd, then\ \boldsymbol{y} = \boldsymbol{2j+1}, where\ \boldsymbol{j}\ is\ some\ integer.$$

$$Plugging\ \boldsymbol{2k+1}\ and\ \boldsymbol{2j+1}\ in\ for\ \boldsymbol{x}\ and\ \boldsymbol{y}\ into\ \boldsymbol{xy}\ gives: \boldsymbol{xy} = (\boldsymbol{2k+1})(\boldsymbol{2j+1})$$

$$Expanding\ the\ righthand\ side\ gives: \boldsymbol{xy} = \boldsymbol{4jk+2k+2j+1}$$

$$Which\ can\ be\ expressed\ as: \boldsymbol{xy} = \boldsymbol{2}(\boldsymbol{2jk+k+j})+\boldsymbol{1}.$$

$$Since\ \boldsymbol{j}\ and\ \boldsymbol{k}\ are\ integers, \boldsymbol{2jk+k+j}\ is\ also\ an\ integer.$$

$$Since\ \boldsymbol{xy}\ is\ \boldsymbol{2m+1}, where\ \boldsymbol{m} = \boldsymbol{2jk+k+j}\ is\ an\ integer, \boldsymbol{xy}\ is\ odd.$$

2.4.3) Prove each of the following statements using a direct proof.

b) If $\boldsymbol{x}$ is a real number and $\boldsymbol{x \leq 3}$, then $\boldsymbol{12 - 7x + x^2 \geq 0}$.

$$Let\ x\ be\ a\ real\ number\ and\ \boldsymbol{x \leq 3}. We\ will\ prove\ that\ \boldsymbol{12 - 7x + x^2 \geq 0}.$$

Subtract 3 *from both sides of* $\boldsymbol{x \leq 3}$ *to get:*

$$\boldsymbol{x - 3 \leq 0}$$

Multiply both sides by -1 to get:

$$\boldsymbol{-x + 3 \geq 0}$$

Which can be rearranged as $\boldsymbol{x - 3 \geq 0}$*.*

Since $3 < 4$ *and* $\boldsymbol{x}$ *is a real number and* $\boldsymbol{x \leq 3}$*,* $\boldsymbol{x \leq 4}$ *is also true.*

Subtract 4 *from both sides of* $\boldsymbol{x \leq 4}$ *to get:*

$$\boldsymbol{x - 4 \leq 0}$$

Multiply both sides by -1 to get:

$$\boldsymbol{-x + 4 \geq 0}$$

Which can be rearranged as $\boldsymbol{x-4 \geq 0}$.

Since $\boldsymbol{x-4}$ *and* $\boldsymbol{x-3}$ *are non-negative real numbers, their product will be non-negative real numbers, or in other words:*

$$\boldsymbol{(x-4)(x-3) \geq 0}$$

Which can be expanded to get: $\boldsymbol{x^2 - 7x + 12 \geq 0}$

Which can be arranged to get $\boldsymbol{12 - 7x + x^2 \geq 0}$.

$Therefore\ if\ \boldsymbol{x}\ is\ a\ real\ number\ and\ \boldsymbol{x \leq 3}, then\ \boldsymbol{12 - 7x + x^2 \geq 0}.$

Question 7

2.5.1d) For every integer $\boldsymbol{n}$, if $\boldsymbol{n^2 - 2n + 7}$ is even, then $\boldsymbol{n}$ is odd.

Let $\boldsymbol{n}$ *be an even integer. We will prove* $\boldsymbol{n^2 - 2n + 7}$ *is odd.*

Since $\boldsymbol{n}$ *is even, then* $\boldsymbol{n = 2k}$ *for some integer* $\boldsymbol{k}$.

Substituting $\boldsymbol{n = 2k}$ *into* $\boldsymbol{n^2 - 2n + 7}$*:*

$$\boldsymbol{(2k)^2 - 2(2k) + 7}$$

Simplifying gives: $\boldsymbol{4k^2 - 4k + 7}$

Which can be rewritten as: $\boldsymbol{2(2k^2 - 2k + 3) + 1}$

Since k is an integer, $\boldsymbol{2k^2 - 2k + 3}$ *is also an integer.*

Since $\boldsymbol{n^2 - 2n + 7 = 2m + 1}$*, where* $\boldsymbol{m = 2k^2 - 2k + 3}$ *is an integer,* $\boldsymbol{n^2 - 2n + 7}$ *is odd.*

2.5.4a) For every pair of real numbers $\boldsymbol{x}$ and $\boldsymbol{y}$, if $\boldsymbol{x^3 + xy^2 \leq x^2y + y^3}$, then $\boldsymbol{x \leq y}$.

Let x and y be real numbers where $\boldsymbol{x > y}$*. We will prove* $\boldsymbol{x^3 + xy^2 > x^2y + y^3}$.

Since $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are real numbers,* $\boldsymbol{x^2}$ *and* $\boldsymbol{y^2}$ *are also real numbers.*

Since $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are positive and the square of a positive number is always positive,* $\boldsymbol{x^2 > 0}$ *and* $\boldsymbol{y^2 > 0}$.

Since $\boldsymbol{x^2}$*and* $\boldsymbol{y^2}$ *are positive real numbers,* $\boldsymbol{x^2 + y^2}$ *is also a positive real number.*

Since $\boldsymbol{x^2 + y^2 > 0}$*, we can multiply* $\boldsymbol{x^2 + y^2}$ *to both sides of* $\boldsymbol{x > y}$ *to get:*

$$\boldsymbol{x(x^2 + y^2) > y(x^2 + y^2)}$$

Simplifying gives: $\boldsymbol{x^3 + xy^2 > x^2y + y^3}$

Therefore $\boldsymbol{x^3 + xy^2 > x^2y + y^3}$ *when* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are real numbers.*

2.5.4b) For every pair of real numbers $\boldsymbol{x}$ and $\boldsymbol{y}$, if $\boldsymbol{x}+\boldsymbol{y}>\mathbf{20}$, then $\boldsymbol{x}>\mathbf{10}$ or $\boldsymbol{y}>\mathbf{10}$.

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be real numbers. We will assume it is not true that $\boldsymbol{x}>\mathbf{10}$ or $\boldsymbol{y}>\mathbf{10}$. We will prove $\boldsymbol{x}+\boldsymbol{y}\leq \mathbf{20}$.

The assumption that it is not true that $\boldsymbol{x}>\mathbf{10}$ or $\boldsymbol{y}>\mathbf{10}$ is equivalent to the condition that $\boldsymbol{x}\leq\mathbf{10}$ and $\boldsymbol{y}\leq\mathbf{10}$ by De Morgan's Law.

Adding the inequalities $\boldsymbol{x}\leq\mathbf{10}$ and $\boldsymbol{y}\leq\mathbf{10}$ gives $\boldsymbol{x}+\boldsymbol{y}\leq\mathbf{20}$

Therefore, $\boldsymbol{x}+\boldsymbol{y}\leq\mathbf{20}$ when $\boldsymbol{x}$ and $\boldsymbol{y}$ are real numbers.

2.5.5c) For every non-zero real number $\boldsymbol{x}$, *if $\boldsymbol{x}$ is irrational, then $\frac{\mathbf{1}}{\boldsymbol{x}}$ is also irrational.*

Let $\boldsymbol{x}$ be a non $-$ zero real number. We will assume $\frac{\mathbf{1}}{\boldsymbol{x}}$ is rational. We will prove $\boldsymbol{x}$ is rational.

Since $\frac{\mathbf{1}}{\boldsymbol{x}}$ is rational, $\frac{\mathbf{1}}{\boldsymbol{x}}=\frac{\boldsymbol{a}}{\boldsymbol{b}}$ for some integers $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{b}\neq\mathbf{0}$.

Since $\boldsymbol{x}$ is non-zero and $\frac{\mathbf{1}}{\boldsymbol{x}}$ is rational, $\boldsymbol{a}\neq\mathbf{0}$.

Since $\boldsymbol{x}$ is a real number, we can multiply $\boldsymbol{x}$ to both sides of $\frac{\mathbf{1}}{\boldsymbol{x}}=\frac{\boldsymbol{a}}{\boldsymbol{b}}$ to get:

$$\mathbf{1}=\frac{\boldsymbol{ax}}{\boldsymbol{b}}$$

Since $\boldsymbol{b}$ is a real number, we can multiple $\boldsymbol{b}$ to both sides to get:

$$\boldsymbol{b}=\boldsymbol{ax}$$

Since $\boldsymbol{a}$ is a real number, we can multiple $\boldsymbol{a}$ to both sides to get:

$$\frac{\boldsymbol{b}}{\boldsymbol{a}}=\boldsymbol{x}$$

Which can be rearranged to get: $\boldsymbol{x}=\frac{\boldsymbol{b}}{\boldsymbol{a}}$.

Therefore, $\boldsymbol{x}$ is rational since $\boldsymbol{x}=\frac{\boldsymbol{b}}{\boldsymbol{a}}$ and $\boldsymbol{a}$ and $\boldsymbol{b}$ are integers where $\boldsymbol{a}\neq\mathbf{0}$

Question 8

2.6.6) Proofs by contradiction

c) The average of three real numbers is greater than or equal to at least one of the numbers.

Suppose there are three real numbers, ***x****,* ***y****, and* ***z****, such that their average is less* ***x****,* ***y****, and* ***z.***

In other words, $\frac{(x+y+z)}{3} < x$, $\frac{(x+y+z)}{3} < y$, *and* $\frac{(x+y+z)}{3} < z$.

Multiplying 3 to both sides for each inequality becomes:

$$(x + y + z) < 3x, (x + y + z) < 3y, (x + y + z) < 3z$$

Adding all the inequalities gives:

$$(3x + 3y + 3z) < (3x + 3y + 3z)$$

Since ***x, y,*** *and* ***z*** *are real numbers,* ***3x+3y+3z*** *is also real.*

$(3x + 3y + 3z) < (3x + 3y + 3z)$ *is a contradiction since a real number cannot be less than itself. Therefore, the average of three real numbers must be greater than or equal to at least one of the numbers.*

d) There is no smallest integer

Suppose there is a smallest integer x.

Since ***x*** *is an integer,* ***x – 1*** *is also an integer.*

Since both are integers, $x - 1 < x$ *must be true.*

This contradicts the assumption that ***x*** *is the smallest integer.*

Question 9

2.7.2b) If integers $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *have the same parity, then* $\boldsymbol{x + y}$ *is even. The parity of a number tells whether the number is odd or even. If* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *have the same parity, they are either both even or both odd.*

Case **1**: $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are even*

Since $\boldsymbol{x}$ *is even,* $\boldsymbol{x = 2k}$ *for some integer* $\boldsymbol{k}$*. Since* $\boldsymbol{y}$ *is even,* $\boldsymbol{y = 2j}$ *for some integer* $\boldsymbol{j}$*.*

Substituting $\boldsymbol{x = 2k}$ *and* $\boldsymbol{y = 2j}$ *into* $\boldsymbol{x + y}$ *gives:*

$$\boldsymbol{2k + 2j}$$

Which can be rewritten as:

$$\boldsymbol{2(k + j)}$$

Since $\boldsymbol{k}$ *and* $\boldsymbol{j}$ *are integers,* $\boldsymbol{(k + j)}$ *are also integers.*

Since $\boldsymbol{x + y = 2m}$ *where* $\boldsymbol{m = (k + j)}$ *is an integer,* $\boldsymbol{x + y}$ *is even. Therefore,* $\boldsymbol{x + y}$ *is even when* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are even.*

Case **2**: $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are odd*

Since $\boldsymbol{x}$ *is odd,* $\boldsymbol{x = 2k + 1}$ *for some integer* $\boldsymbol{k}$*. Since* $\boldsymbol{y}$ *is odd,* $\boldsymbol{y = 2j + 1}$ *for some integer* $\boldsymbol{j}$**.**

Substituting $\boldsymbol{x = 2k + 1}$ *and* $\boldsymbol{y = 2j + 1}$ *into* $\boldsymbol{x + y}$ *gives:*

$$\boldsymbol{(2k + 1) + (2j + 1) = 2k + 2j + 2 = 2(k + j + 1)}$$

Since $\boldsymbol{k}$ *and* $\boldsymbol{j}$ *are integers,* $\boldsymbol{k + j + 1}$ *is also an integer.*

Since $\boldsymbol{x + y = 2m}$ *where* $\boldsymbol{m = (k + j + 1)}$ *is an integer,* $\boldsymbol{x + y}$ *is even. Therefore,* $\boldsymbol{x + y}$ *is even when* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are even.*