The Precession of a Symmetric Spinning Top and the Relation to Liouville's Theorem

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I. INTRODUCTION

In many fields of physics, objects can be modeled as rigid tops [1]. Some of these objects include toy spinning tops, gyroscopes and celestial bodies such as planets [1]. A unique case is the symmetric top in which the principle moments about axis $\mathbf{\hat{x}_1}$ and $\mathbf{\hat{x}_2}$ are the same and the principle moment about the third axis $\mathbf{\hat{x}_3}$ differs [1]. This allows us to simplify our analysis of the spinning top such that we can solve for its rotational behaviour. With current techniques, this problem can only be analytically solved in limiting cases, so a numerical approach must be taken in order to solve for the tops motion. In this report we will derive Hamilton's equations for a symmetric spinning top in terms of the Euler angles and numerically solve for the nutation (motion in θ) of the spinning top. This is done using the scientific python package (scipy) along with the numerical python package (numpy) and the matplotlib package.The primary motivation of this calculation is to investigate Liouville's theorem which states that the density of systems in phase space stays constant in time [1].

II. HAMILTONIAN ANALYSIS

The classical Hamiltonian is found by

$$\sum_{i=1}^{n} \dot{q}_i p_i - L \tag{1}$$

where L is the Lagrangian, $\dot{q}_i$ is a speed and p_i is a canonical momenta, both corresponding to the same generalized coordinate [1]. In this problem, we are describing the rotation of the top by the angular velocity in the Euler angle basis. That is

$$\vec{\omega} = \vec{\dot{\phi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}}. \tag{2}$$

These angular velocities are described by a set of rotations about the standard Cartesian axes, allowing us to write the angular velocity in terms of the three Euler angles:

$$\vec{\omega} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix} [2]. \tag{3}$$

Now that we understand how the top rotates with respect to each Euler angle, this allows us to write the Lagrangian and thereby the Hamiltonian of the symmetric

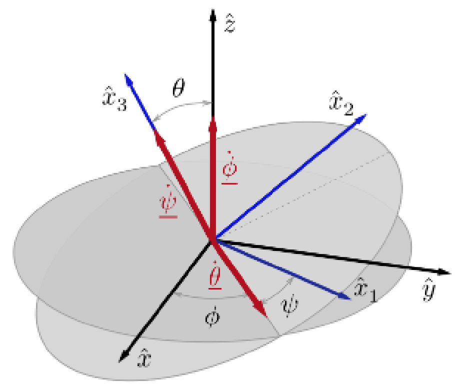

FIG. 1. Figure showing the direction of each component of the angular velocity which are being expressed in terms of Euler angles θ, ϕ, ψ [2].

spinning top:

$$H = \frac{p_\theta^2}{2I} + [\frac{(p_\phi + p_\psi\cos\theta)^2}{2I\sin^2\theta} + MgR\cos\theta] + C \tag{4}$$

where I is the moment of inertia about the $\mathbf{\hat{x}_1}$ and $\mathbf{\hat{x}_2}$ axes, M is the mass of the top, g is the acceleration due to gravity, R is the distance to the center of mass from the support point about which the top rotates and θ is the angle measured from the z-axis to the $\mathbf{\hat{x}_3}$ axis. C represents any constants which could be present and not have any affect on our solutions to Hamilton's equations.

We are interested in determining the nutation of the symmetric spinning top. We do so by considering Hamilton's equations:

$$\vec{\dot{q}}_i = \frac{\partial H}{\partial p_i} \tag{5}$$

$$\vec{\dot{p}}_i = -\frac{\partial H}{\partial q_i}. \tag{6}$$

Using Hamilton's equations we see that p_ϕ and p_ψ are conserved (that is their time derivatives are zero), allowing us to treat them as constants. We also get the two following equations to describe the tops nutation:

$$\dot{\theta} = \frac{p_\theta}{I} \tag{7}$$

$$\begin{aligned} \dot{p}_\theta = MgR\sin\theta + \frac{(p_\phi^2 + p_\psi^2)}{I}\csc^2\theta\cot\theta + \cdots \\ \cdots + \frac{p_\phi p_\psi(\csc\theta - 2\csc^3\theta)}{I}. \end{aligned} \tag{8}$$

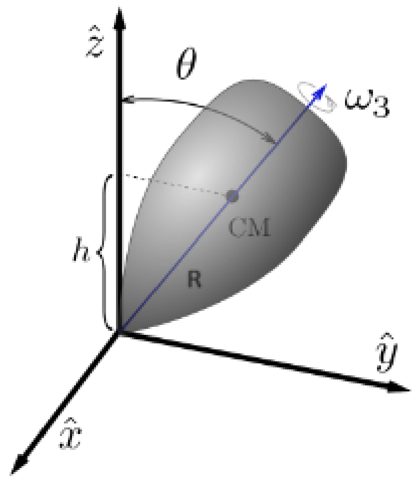

FIG. 2. Diagram of a symmetric spinning top and the Euler angle θ [2]. CM represents the center of mass, R is the distance to the center of mass along $\mathbf{\hat{x}_3}$, h is the height of the center of mass above the support point, and ω_3 is the angular velocity about the $\mathbf{\hat{x}_3}$ axis.

This is a set of coupled differential equations which cannot be analytically solved. We must now turn to computational methods to continue our analysis of the nutation of a symmetric spinning top.

III. LIOUVILLE'S THEOREM

Liouville's theorem states that as a system contained in a tiny region of phase space evolves according to the laws of mechanics, the volume the system occupies remains constant [1]. Since the volume is constant the phase space density remains the same as well. We will look at a two-dimensional phase space by plotting p_θ against θ. This means that we should find the area density of phase space to be constant rather than the volume density. This will be done by solving equations 7 and 8 for several initial conditions, and then plotting each solution at different times to see if area is conserved. Investigating Liouville's theorem is the primary motivation for the following analysis.

IV. COMPUTATION AND INITIAL CONDITIONS

In order to numerically solve equations 7 and 8 the programming language python was implemented. This was done using the scientific python package (scipy) along with the numerical python package (numpy) and the matplotlib package for data visualization. Before solving we must consider what physical constants we will use for our symmetric spinning top. These values are chosen arbitrarily to be $M = 100$ kg, $g = 9.81$ m/s^2, $R = 10$ m, and $I = 100$ kgm^2. This means that our spinning top is quite long, with its mass tightly distributed about the center of mass axis (relative to its length). The constants related to the rotation of the top in the other angular directions are chosen to be $p_\psi = 1000$ kgm^2/s and $p_\phi = 2000$ kgm^2/s are chosen to be quite high. This roughly indicates to us that the top is rotating quickly in the ψ and ϕ directions. These constants remain the same for several choices of initial conditions θ_0 and $p_{\theta,o}$ which are shown in table I.

θ_0 (rads)	$p_{\theta,o}$ (kgm^2/s)
0.9	0
1.1	0
1.1	0
1.2	0

TABLE I. Table of initial conditions for which Hamilton's equations for generalized coordinate θ were numerically solved.

After setting these constants and initial conditions, the equations were solved using Odeint, a sub-package of scipy. This is done by putting equations 7 and 8 together in a vector, and then solving them simultaneously as seen in equation 9. Solutions were calculated for 1000 evenly spaced points over a 2 s time interval.

$$\begin{pmatrix} \dot{\theta} \\ \dot{p}_\theta \end{pmatrix} = \begin{pmatrix} \frac{p_\theta}{I} \\ MgR\sin\theta + \frac{(p_\phi^2 + p_\psi^2)}{I}\csc^2\theta\cot\theta + \frac{p_\phi p_\psi(\csc\theta - 2\csc^3\theta)}{I} \end{pmatrix} \tag{9}$$

Odeint uses the lsoda algorithm from the FORTRAN library odepack to solve first order ordinary differential equations [3]. All code written for this analysis can be found on my personal GitHub page [4].

V. RESULTS

We were able to numerically solve equations 7 and 8 for each pair of initial conditions specified in table I. As seen in figures 3 and 4 the motion in θ and p_θ is oscillatory. We

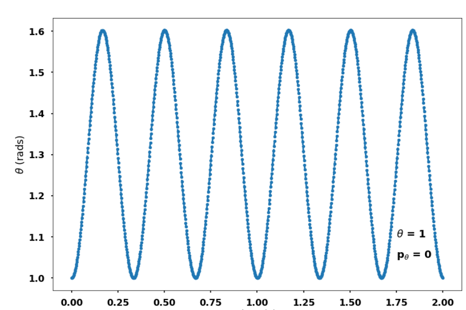

FIG. 3. Plot of θ (rads) vs time (s). Initial conditions for θ and p_θ specified on the plot.

chose p_ϕ to be positive, and further we chose $p_\phi > p_\psi \cos\theta$

always. This means that the path of the top oscillates between some angles θ_1 and θ_2, which are the minimum and maximum values of figure 3 [1]. We see that p_θ oscillates over time as well, and its maximum absolute value is an order of magnitude smaller than the constant values of p_ψ and p_ϕ that were set. Using these figures, and those shown in section VII, new arrays were created so we could try to verify Liouville's theorem. Data points were taken at specific times for each of the 4 solutions for θ and p_θ (corresponding to 4 pairs of initial conditions). I chose to take points at times $t = 0.002$ s, 1.001 s, 1.501 s, and 2.0 s from each solution. This means that when we plot p_θ versus θ we will have 4 data points for each specified time.

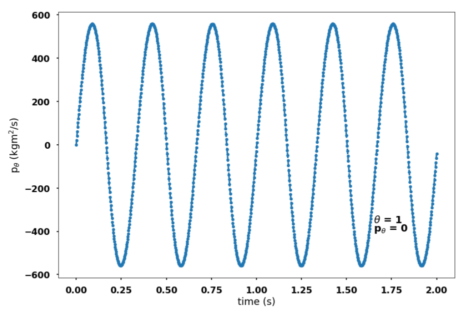

FIG. 4. Plot of p_θ (kgm^2/s) vs time (s). Initial conditions for θ and p_θ specified on the plot.

Looking at figure 5 it is hard to say that the area filled out by each group of points is conserved over time. With the exception of the red data points forming a line, the colours appear to make the outline of a somewhat similar shape. Despite this, we can't qualitatively say that the area they fill out is conserved over time. In the future, a better conclusion could be drawn by considering a far greater number of solutions (initial conditions) and seeing the area they fill at each selected time.

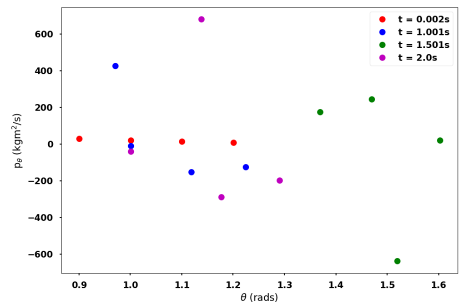

FIG. 5. Plot of p_θ (kgm^2/s) vs θ (rads). Each colour corresponds to the value of a given solution of Hamilton's equations for a set of specific initial conditions. Each colour has a data point for each of 4 pairs of initial conditions.

VI. CONCLUSION

We were successful in numerically solving Hamilton's equations for θ and p_θ for a variety of initial conditions and one set of constants that physically describe our symmetric spinning top and how it initially rotates in the ϕ and ψ directions. The numerical solution was calculated in python using odeint, a sub-package of the scientific python (scipy) package. We were unable to make any qualitative conclusions that verify Liouville's theorem which states that the density of phase space remains constant over time. This could be due to a lack of data points being considered at each point in time in our plot of p_θ versus θ (figure 5). Further analysis must be conducted before any definitive conclusions can be drawn about Liouville's theorem and its application to the nutation of a symmetric spinning top.

[1] Andrew Frey. Phys-3203 lecture notes. 2024.
[2] Alexei Gilchrist. Heavy symmetrical top lecture notes. *Macquarie University*.
[3] scipy 1.13.0. Odeint documentation. 2024.
[4] Thomas Hepworth (thepworth3). Self developed analysis code. 04 2024.

VII. APPENDIX

This appendix contains all plots of θ and p_θ vs time not included in section V.

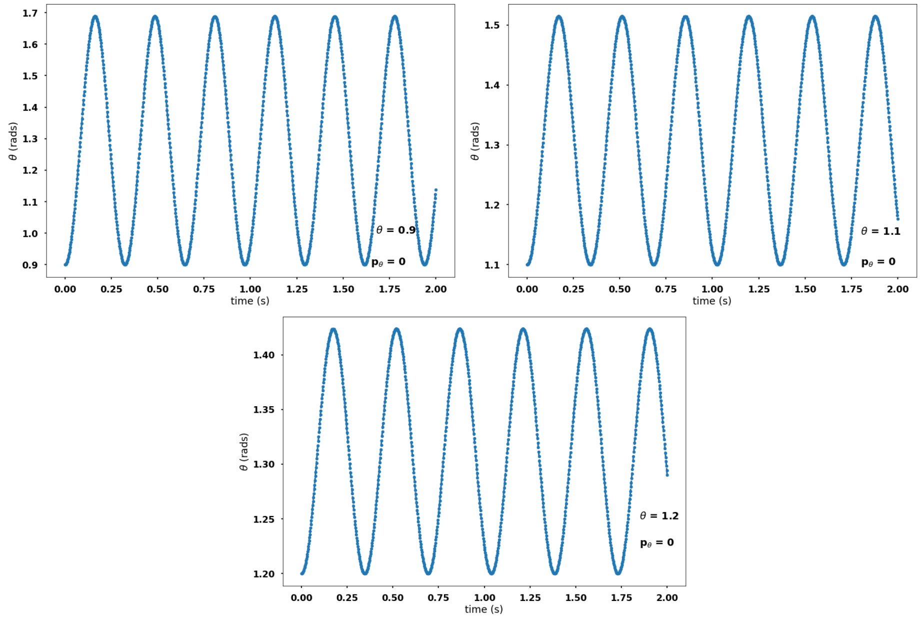

FIG. 6. Plot of θ (rads) vs time (s). Initial conditions for θ and p_θ specified on each plot.

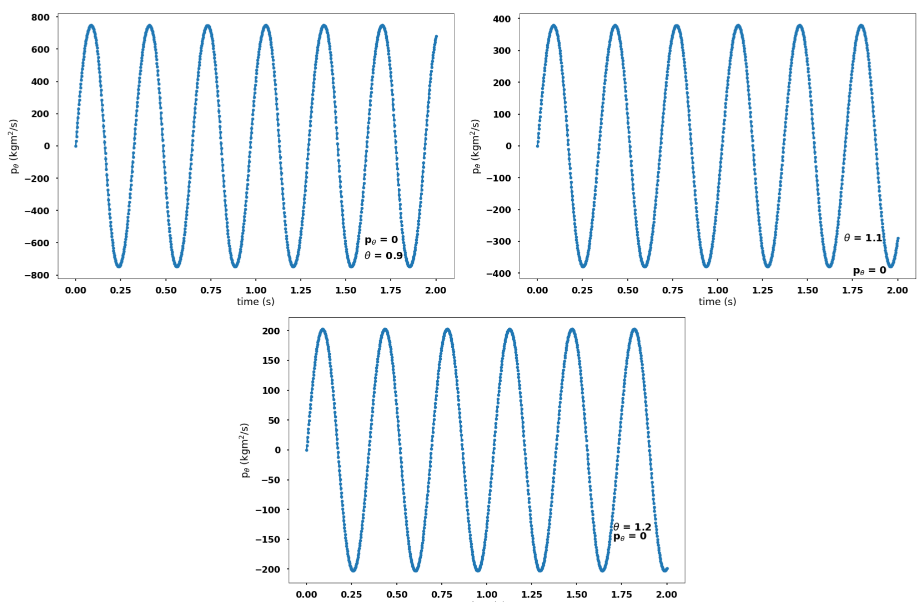

FIG. 7. Plot of p_θ (kgm^2/s) vs time (s). Initial conditions for θ and p_θ specified on each plot.