

Quant. Comp. HW - 2

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1 Is Valid Superposition?

Determine if the state $|\phi\rangle$ is a valid superposition

$$|\phi\rangle = \frac{1}{1+i}|0\rangle + \frac{1}{1-i}|1\rangle$$

The state is valid provided that it is length normalized to one.

$$\begin{aligned} & \left| \frac{1}{1+i} \right|^2 + \left| \frac{1}{1-i} \right|^2 \\ &= \frac{1}{(1+i)(1-i)} + \frac{1}{(1-i)(1+i)} \\ &= (1/2) + (1/2) = 1 \end{aligned}$$

Therefore the state is **valid**.

2 Find valid superposition

Given $|\phi\rangle = (1/2)|00\rangle + \frac{x}{2\sqrt{2}}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle + \frac{1}{2}|11\rangle$, what values of x would make this a valid superposition?

Again the normalization condition is applied:

$$\begin{aligned} & \left| \frac{1}{2} \right|^2 + \left| \frac{x}{2\sqrt{2}} \right|^2 + \left| \frac{1}{2\sqrt{2}} \right|^2 + \left| \frac{1}{2} \right|^2 = 1 \\ & (1/4) + \frac{|x|^2}{8} + (1/8) + (1/4) = 1 \end{aligned}$$

$$\frac{(5 + |x|^2)}{8} = 1$$

$$|x|^2 = 3$$

Since x can be imaginary, all we know is that the real part must be equal to $\sqrt{3}$ Therefore:

$$\boxed{x = \sqrt{3} + bi}$$

3

Let $|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$ Find:

$$\begin{aligned} |\phi\rangle &= (H \otimes H)|\psi\rangle \\ &= \frac{1}{\sqrt{6}}[(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ &\quad + (|00\rangle - |01\rangle + |10\rangle - |11\rangle) \\ &\quad + (|00\rangle + |01\rangle - |10\rangle + |11\rangle)] \\ &= \frac{1}{\sqrt{6}}[3|00\rangle + |01\rangle + |10\rangle + |11\rangle] \\ &= \frac{1}{2}|00\rangle + \frac{1}{6}|1\rangle + \frac{1}{6}|2\rangle - \frac{1}{6}|3\rangle \end{aligned}$$

4 Deutsch Problem

Suppose we take U_f from the Deutsch problem and compute

$$(H \otimes 2)U_f(H \otimes 2)|11\rangle$$

Recall that for the Deutsch function:

First let's apply the first hadamard gate and calls this intermediate state $|a\rangle$:

$$\begin{aligned} |a\rangle &= (H \otimes H)|11\rangle \\ &= \frac{1}{2}(|0\rangle - |1\rangle)(|0\rangle - |1\rangle) \end{aligned}$$

	X=0	X=1
f_0	0	0
f_1	0	1
f_2	1	0
f_3	1	1

$$= \frac{1}{2}(|0\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|0\rangle - |1\rangle|1\rangle)$$

In computing the Hadamard on this next, a general equation can be set up for the four 'f' functions. Since anything XOR'ed with itself is 0, and anything XOR'ed with 1 is it's complement, we can write $U_f|a\rangle$ as:

$$U_f|a\rangle = \frac{1}{2}[|0\rangle|f(0)\rangle - |0\rangle|\bar{f}(0)\rangle - |1\rangle|f(1)\rangle - |1\rangle|\bar{f}(1)\rangle]$$

Now let's consider the two cases $f(0) = f(1)$ and $f(0) \neq f(1)$:

Case 1: $f(0) = f(1)$

This implies we are using either f_0 or f_3 .

f_0 :

$$\begin{aligned} HU_{f_0}|a\rangle &= H\frac{1}{2}[|00\rangle - |01\rangle - |10\rangle - |11\rangle] \\ &= \frac{1}{2}[-|00\rangle + |11\rangle + |11\rangle + |01\rangle + |10\rangle] \\ &= \frac{1}{2}[-|00\rangle + 2|11\rangle + |01\rangle + |10\rangle] \quad (f_0) \end{aligned}$$

f_3 :

$$\begin{aligned} HU_{f_3}|a\rangle &= H\frac{1}{2}[|01\rangle - |00\rangle - |11\rangle - |10\rangle] \\ &= \frac{1}{4}[(|00\rangle + |01\rangle - |10\rangle + |11\rangle) - (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ &\quad - (|00\rangle - |01\rangle - |10\rangle - |11\rangle) - (|00\rangle + |01\rangle - |10\rangle - |11\rangle)] \\ &= \frac{1}{4}[-2|00\rangle + 2|11\rangle] \\ &= \frac{1}{2}[|00\rangle + |11\rangle] \quad (f_3) \end{aligned}$$

Now for the case of $f(0) \neq f(1)$, so we look at f_1 and f_2 . Note that applying U is easy here too, because we are always XOR a bit with it's complement, which always yields 1. Since the input gates will be the same for f_1 and f_2 , then each should return the same state:

$f_1 :$

$$\begin{aligned} HU_{f_1}|a\rangle &= H\frac{1}{2}[|00\rangle - |01\rangle - |11\rangle - |10\rangle] \\ &= \frac{1}{4}[2|01\rangle + 2|11\rangle - 2|00\rangle + 2|10\rangle + 2|11\rangle] \\ &= \frac{1}{2}[-|00\rangle + |01\rangle + |10\rangle + |11\rangle] \quad (f_1) \end{aligned}$$

$f_2 :$

$$\begin{aligned} HU_{f_2}|a\rangle &= H\frac{1}{2}[|01\rangle - |00\rangle - |11\rangle - |10\rangle] \\ &= \frac{1}{4}[-2|01\rangle - 2|11\rangle - 2|00\rangle + 2|10\rangle + 2|11\rangle] \\ &= \frac{1}{2}[-|00\rangle + |01\rangle + |10\rangle] \quad (f_2) \end{aligned}$$

5 Bernstein-Verizani

Take $f(x)$ from the Bernstein-Verizani problem and compute:

$$\begin{aligned} U_f(H^{n \otimes 1})(|0\rangle_n |1\rangle_1) \\ = U_f\left(\frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle\right) \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

Since $U_f|x\rangle_n \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = (-1)^{f(x)}|x\rangle_n \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$= \frac{1}{2^{n/2}} \left(\sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle \right) \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{1}{2^{(n+1)/2}} \left(\sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle \right) (|0\rangle - |1\rangle)$$

Recall that the function $f(x)$ in the Bernstein-Verizani is $a \cdot x$. Therefore, $f(x)$ is always either 0 or 1, and so each value in the sum is either $|x\rangle$ or $-|x\rangle$. The possible values of the input register are:

$$\frac{1}{2^{n/2}} \left(\sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle \right)$$

The only possible value of the output register is the superposed state:

$$\frac{1}{2^{(n+1)/2}} (|0\rangle - |1\rangle)$$

6 Derive State

Derive the state from problem 3 using any 1-bit gates and the 2-bit gate cNOT starting from $|00\rangle$. Just applying a hadamard to expand leaves a useless state, since all other gates applied maintain the state. So first apply a NOT gate to the first bit:

$$(X \otimes 1)|00\rangle = |10\rangle$$

Applying a hadamard to this yields:

$$H(X \otimes 1)|00\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$$

Applying cNOT gates:

$$C_{21}H(X \otimes 1)|00\rangle = \frac{1}{2}(|00\rangle + |11\rangle - |10\rangle - |01\rangle)$$

And another cNOT:

$$C_{12}C_{21}H(X \otimes 1)|00\rangle = \frac{1}{2}(|00\rangle + |10\rangle - |11\rangle - |01\rangle)$$

Applying a X to the second bit:

$$(X \otimes 2)C_{12}C_{21}H(X \otimes 1)|00\rangle = \frac{1}{2}(|01\rangle + |11\rangle - |10\rangle - |00\rangle)$$

Applying a Hadamard to this:

$$\begin{aligned} H(X \otimes 2)C_{12}C_{21}H(X \otimes 1)|00 \rangle &= \frac{1}{2}(|00 \rangle - |10 \rangle - |11 \rangle - (|00 \rangle + |01 \rangle)) \\ &= \frac{1}{2}[-|10 \rangle - |01 \rangle - |11 \rangle] \end{aligned}$$

Now I can just NOT each bit

$$\begin{aligned} (X \otimes 1)(X \otimes 2)H(X \otimes 2)C_{12}C_{21}H(X \otimes 1)|00 \rangle \\ = \frac{1}{\sqrt{3}}[|00 \rangle + |01 \rangle + |10 \rangle] \end{aligned}$$

7 Numbered formulae

Useful Hadamards:

$$(H \otimes H)|00 \rangle = \frac{1}{2}(|00 \rangle + |01 \rangle + |10 \rangle + |11 \rangle) \quad (1)$$

$$(H \otimes H)|01 \rangle = \frac{1}{2}(|00 \rangle - |01 \rangle + |10 \rangle - |11 \rangle) \quad (2)$$

$$(H \otimes H)|10 \rangle = \frac{1}{2}(|00 \rangle + |01 \rangle - |10 \rangle - |11 \rangle) \quad (3)$$

$$(H \otimes H)|11 \rangle = \frac{1}{2}(|00 \rangle - |01 \rangle - |10 \rangle - |11 \rangle) \quad (4)$$

Use the *equation* environment to get numbered formulae, e.g.,

$$y_{i+1} = x_i^{2n} - \sqrt{5}x_{i-1}^n + \sqrt{x_{i-2}^7} - 1 \quad (5)$$

$$\frac{\partial u}{\partial t} + \nabla^4 u + \nabla^2 u + \frac{1}{2}|\nabla u|^2 = c^2 \quad (6)$$

8 Acknowledgments

I'm now going to assume I have a C_H , which is a controlled hadamard gate, that functions just like a cNOT gate except it applies the hadamard operator instead of the NOT operator to the non-control bit.

$$\begin{aligned}
 C_{H_{12}}H|00\rangle &= \frac{1}{2}[|00\rangle + |01\rangle + |1\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + |1\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)] \\
 &= \frac{1}{2}[|00\rangle + |01\rangle + \frac{1}{\sqrt{2}}(|10\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{\sqrt{2}}|10\rangle - \frac{1}{\sqrt{2}}|11\rangle)] \\
 &= \frac{1}{2}[|00\rangle + |01\rangle + \frac{2}{\sqrt{2}}|10\rangle]
 \end{aligned}$$

Thanks to my buddies Æschylus and Chloë, who helped me define the macro `\piRsquare` which is πr^2 . The end.