SC 625 - Systems Theory
Practice Sheet - II, 2025

Question 1

For the multivariable function $g(x, y) = e^{xy} + \ln y$, compute the linear approximation around the point $(x_0, y_0) = (1, 1)$. Then, estimate $g(1.05, 0.95)$ and explain how this linearization relates to the total differential.

Question 2

Consider the pendulum equation $\theta'' + \frac{g}{L}\sin\theta = 0$. Linearize it around the stable equilibrium $\theta = 0$ to obtain the simple harmonic oscillator approximation. Then, solve the linearized equation with initial conditions $\theta(0) = \theta_0$, $\theta'(0) = 0$ (for small θ_0), and discuss the limitations of this approximation for larger angles.

Question 3

Linearize the reaction-diffusion PDE $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + u(1-u)$ around the uniform equilibrium solution $u = 1$. Analyze the stability by solving the resulting linear PDE for small perturbations, assuming periodic boundary conditions on $[0, L]$.

Question 4

Let J be an $n \times n$ Jordan block with eigenvalue $\lambda \neq 0$ over the complex numbers. Prove that there exists a matrix B such that $B^2 = J$. Extend this result to show that every non-singular complex matrix has a square root, and discuss why the non-singularity condition is necessary.

Question 5

Let A be an $n \times n$ nilpotent matrix over $\mathbb{C}$ such that $A^{n-1} \neq 0$ but $A^n = 0$. Prove that the Jordan canonical form of A consists of exactly one Jordan block of size n. Then, generalize this to determine the possible Jordan forms when the index of nilpotency is $k < n$, and provide an example for $n = 4$, $k = 3$.