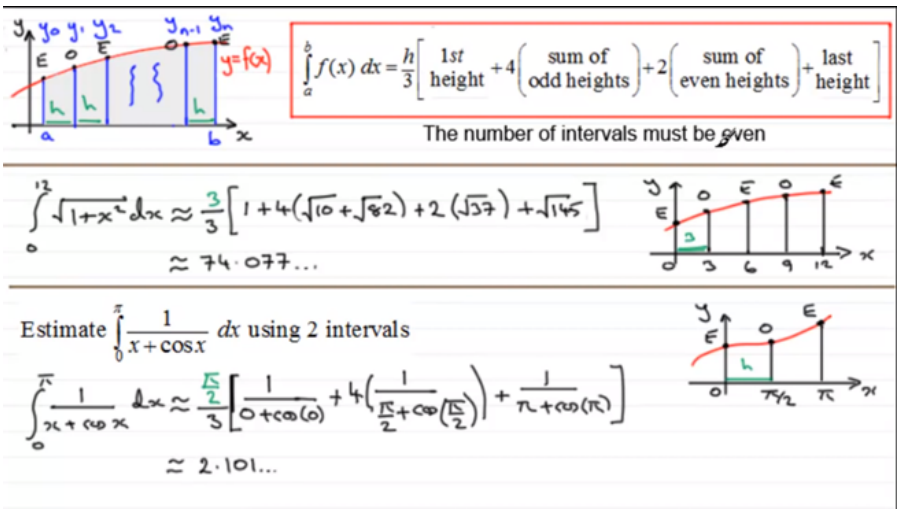
1. How does Simpson's rule work?



Key point

If the area required is divided into n strips of equal thickness, **where n is even,**

then $\int_{x_0}^{x_n} f(x)\,dx \approx \frac{d}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots+y_{n-1})+2(y_2+y_4+y_5+\cdots+y_{n-2})\right]$

This means you require 5 ordinates.

Kinda useless really?

2. When are equations inconsistent in 'systems of equations'?

When they don't have a unique point of intersection.

Delete?

3. What possible particular integrals can you have? How do these depend on the complementary function?

$f(x) = ce^{\lambda x}$	y_c - contains $ae^{\lambda x}$ but **not** $axe^{\lambda x}$	$y = axe^{\lambda x}$
	y_c - contains $axe^{\lambda x}$	$y = ax^2e^{\lambda x}$
	y_c - does not contain $ae^{\lambda x}$ *or* $axe^{\lambda x}$	$y = ae^{\lambda x}$
$f(x) = ccos\lambda x$ or $f(x) = csin\lambda x$	y_c - $Acos\lambda x + Bsin\lambda x$	$y = axsin\lambda x$ if $f(x) = ccos\lambda x$ $y = axcos\lambda x$ if $f(x) = csin\lambda x$
	y_c - does **not** contain $Acos\lambda x + Bsin\lambda x$	$y = acos\lambda x + bsin\lambda x$
f(x) polynomial f degree n		$y = ax^n + bx^{n-1} + \cdots$

You need to ensure you don't have the same constants on either side of the complementary function. Eg,

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, general solution $y = C.F. + P.I$

Find the general solution to $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 2$ ①

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$

Auxillary Equation

$m^2 - 4m = 0$

$\therefore m(m-4) = 0$

$\therefore m = 0$ or $m = 4$

$\therefore$ C.F. $y = A + Be^{4x}$

$\therefore$ P.I. $y = \lambda x$

$\therefore \frac{dy}{dx} = \lambda$, $\frac{d^2y}{dx^2} = 0$ Sub into ①

$\therefore -4\lambda = 2$

$\therefore \lambda = -\frac{1}{2}$

$\therefore$ P.I. is $-\frac{1}{2}x$

$\therefore$ General solution:

$y = A + Be^{4x} - \frac{1}{2}x$

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, general solution $y = C.F. + P.I$

Find the general solution to $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$ ①

$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Auxillary Equation

$m^2 - 3m + 2 = 0$

$\therefore (m-2)(m-1) = 0$

$\therefore m = 2$, $m = 1$

$\therefore$ C.F. $y = Ae^{2x} + Be^{x}$

$\therefore$ P.I. let $y = \lambda xe^{2x}$

$\therefore \frac{dy}{dx} = \lambda e^{2x} + 2\lambda xe^{2x}$

$\therefore \frac{d^2y}{dx^2} = 2\lambda e^{2x} + 2\lambda e^{2x} + 4\lambda xe^{2x}$

$= 4\lambda e^{2x} + 4\lambda xe^{2x}$

Sub. into ①

$4\lambda e^{2x} + 4\lambda xe^{2x} - 3\lambda e^{2x} - 6\lambda xe^{2x} + 2\lambda xe^{2x} = e^{2x}$

$\therefore \lambda e^{2x} = e^{2x} \Rightarrow \lambda = 1 \Rightarrow$ P.I. is xe^{2x}

General solution

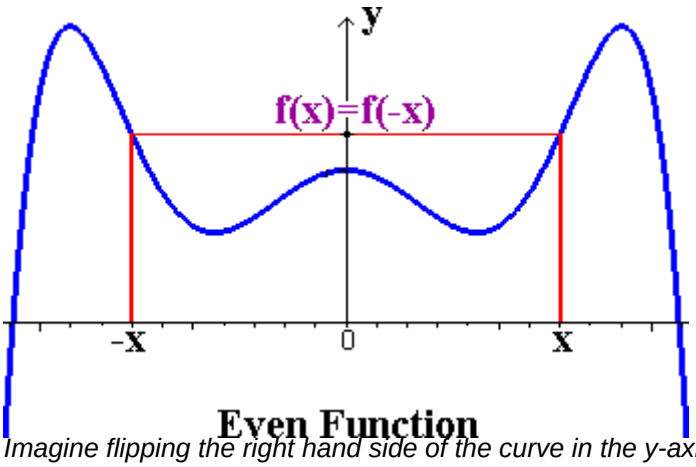
$y = Ae^{2x} + Be^{x} + xe^{2x}$

4. What are invariant points?

Points which remain unchanged under a transformation.

5. What is an even function (with an example)?

A function where f(x) = f(-x). An example is y = cos x (not shown below).



Imagine flipping the right hand side of the curve in the y-axis.

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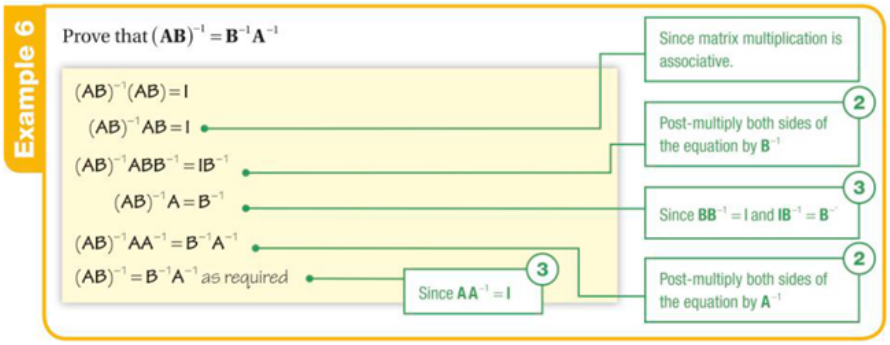
6. What is $(\mathbf{AB})^{-1}$ equal to? Why? When is this the case?

$\mathbf{B}^{-1}\mathbf{A}^{-1}$ since…



- This is true when **A** and **B** are non-singular so have inverses.

---

7. How is diagonalisation of a matrix useful? (with derivation)

You can use the diagonalised form of a matrix to calculate powers of a matrix more easily.

If $\mathbf{A} = \mathbf{PDP}^{-1}$ then $\mathbf{A}^n = (\mathbf{PDP}^{-1})^n$

$$= (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\ldots(\mathbf{PDP}^{-1})$$
$$= (\mathbf{PD})(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}\ldots(\mathbf{P}^{-1}\mathbf{P})(\mathbf{DP}^{-1})$$
$$= (\mathbf{PD})\mathbf{DD}\ldots\mathbf{D}(\mathbf{DP}^{-1})$$
$$= \mathbf{PD}^n\mathbf{P}^{-1}$$

Since matrix multiplication is associative.

Since $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$

Key point

If $\mathbf{A} = \mathbf{PDP}^{-1}$ then $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$

You can use this to solve problems involving $\mathbf{A}^n$, since

$$\mathbf{D}^n = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$$

You can use proof by induction to prove this result.

---

8. What results should you know for the roots topic?

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$$

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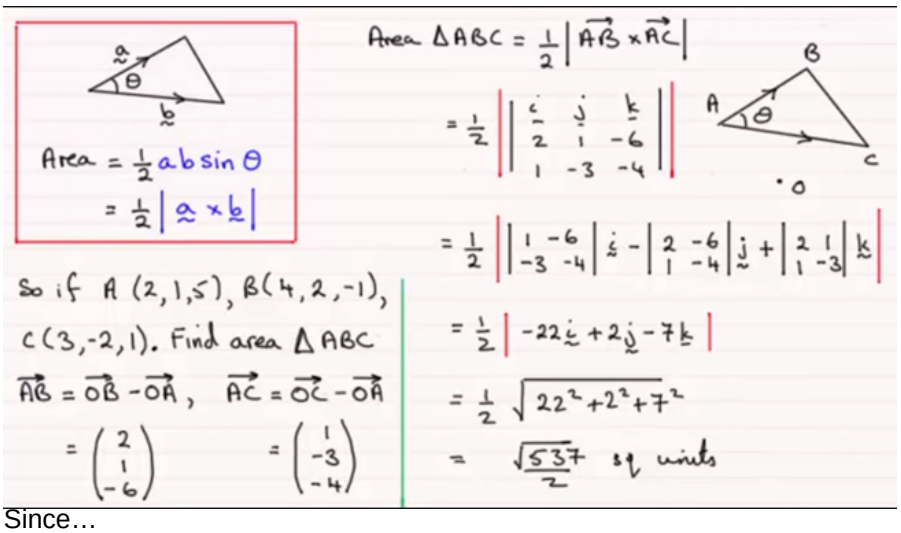
9. How can you work out the area of a triangle using cross product?



Since…

$$\mathbf{a} \times \boldsymbol{b} = \mathbf{ab} \sin \theta \hat{\mathbf{n}}$$

and

$$\mathbf{a} \times \boldsymbol{b} = \mathbf{n}$$

so…

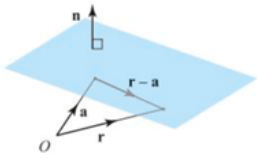
$$\frac{\mathbf{a} \times \boldsymbol{b}}{\hat{\mathbf{n}}} = |\mathbf{a} \times \boldsymbol{b}|$$

10. What is the dot/scalar product equation of a plane?

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = p$$

Since…

Consider a plane containing the point with position vector **a** and with a perpendicular vector **n**. Any vector on this plane will be perpendicular to the vector **n**. Using the definition of the scalar product, any point on the plane with position vector **r**, satisfies $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$. Expanding the bracket gives $\mathbf{r}\cdot\mathbf{n}-\mathbf{a}\cdot\mathbf{n}=0$, which you can rearrange to give $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$

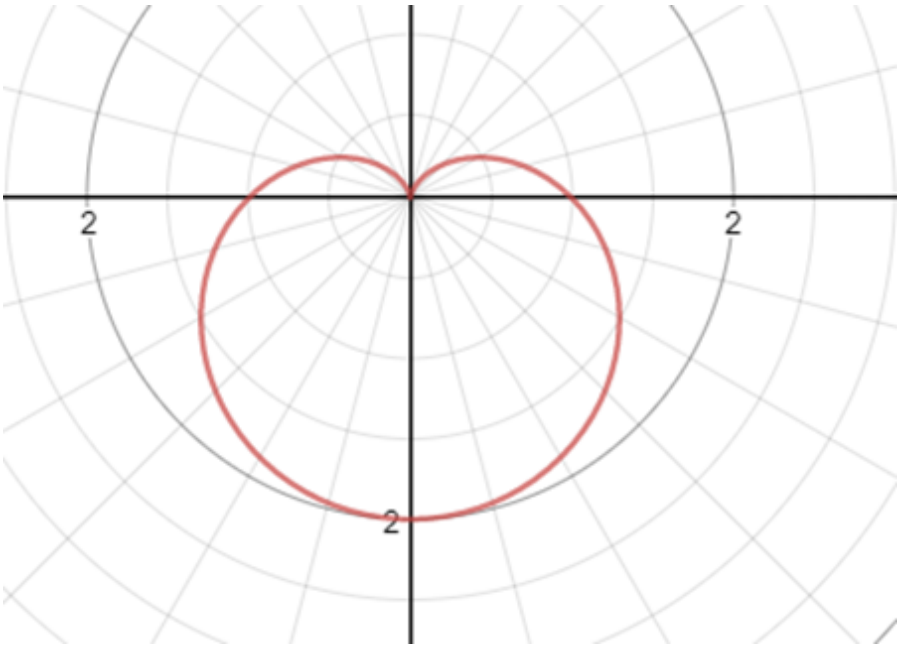


Key point

The scalar product equation of the plane perpendicular to the vector **n** and passing through the point with position vector **a** is $\mathbf{r}\cdot\mathbf{n}=p$, where $p=\mathbf{a}\cdot\mathbf{n}$

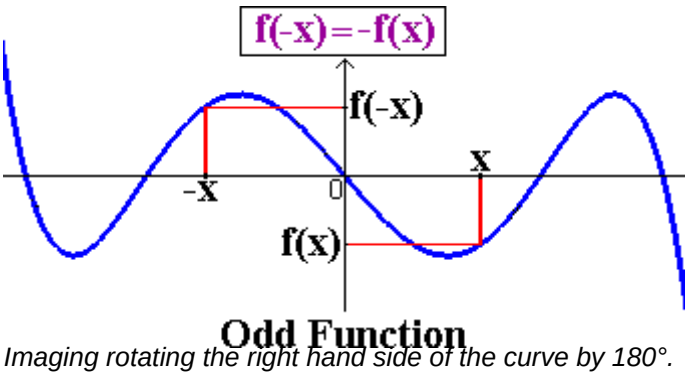
11. How do tangents work under polar coordinates?

If $r \to 0$ as $\theta \to \alpha$, the line $\theta = \alpha$ is a tangent to the curve at this pole.
Example:



12. What is an odd function (with an example)? How can this become an even function?

- A function where f(-x) = -f(x). An example is y = sin x (not shown below).
- By raising it to an even power.



Imaging rotating the right hand side of the curve by 180°.

13. How can knowing whether a function is odd or even help with integration?

If even:

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$

If odd:

$$\int_{-a}^{a} f(x)\ dx = 0$$

For any function f(x) continuous on [-a, a].

14. What 2 limit results do you need to know for the integration topic?

Key point

For any real number k, $x^k e^{-x} \to 0$ when $x \to \infty$

Key point

For any real number k, $x^k \ln x \to 0$ when $x \to 0+$ (this means that x approaches zero from above as x must be positive for $\ln x$ to be defined).

Where k > 0.

---

15. What does De Moivre's Theorem state?

If z = r(cosθ + isinθ) then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ where n is an integer.

---

16. What row operations can be carried out on a matrix determinant? What effect can these have?

No effect on determinant value:
- Adding or subtracting any multiple of a row to another row or column to another column.

Changes sign of determinant value:
- Swapping two rows or two columns.

Changes determinant value by scalar:
- Multiplying/dividing a row or column by a scalar will multiply/divide (respectively) the determinant by that same scalar.

Thus to ensure the value isn't changed, you will have to add a minus if swapping or multiply when dividing or dividing when multiplying.

---

17. When can L'Hopital's rule be used? And with what?

- Provided f(c) = g(c) = 0 or f(c) = g(c) = ±∞ below:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

c can be any value, not necessarily zero or infinity

- This may be used alongside Maclaurin expansion.

---

18. What substitutions do you need to know for what what integrals?

$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)+c \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right)$ or $\ln\{x+\sqrt{x^2-a^2}\}+c \quad (x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right)$ or $\ln\{x+\sqrt{x^2+a^2}\}+c$

For integrals involving … try the substitution ...

$$\sqrt{a^2-x^2},\ \text{try } x = a\sin u$$

$$a^2+x^2,\ \text{try } x = a\tan u$$

$$\sqrt{x^2+a^2},\ \text{try } x = a\sinh u$$

$$\sqrt{x^2-a^2},\ \text{try } x = a\cosh u$$

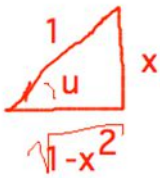
Examples:

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$$

Would involve using x = sin(u).
And so would the one below:

$$\int \sqrt{1-x^2}\,dx \qquad x = \sin u \qquad \text{or } x = \cos u$$
$$dx = \cos u\,du \qquad dx = -\sin u\,du$$

$$= \int \sqrt{1-\sin^2 u}\,\cos u\,du$$



$$= \int \cos^2 u\,du$$

$$= \int \frac{1+\cos 2u}{2}\,du$$

$$= \frac{1}{2}\int 1+\cos 2u\,du$$

$$= \frac{1}{2}\int du + \frac{1}{2}\int \cos 2u\,du$$

$$v = 2u$$
$$dv = 2du$$

$$= \frac{u}{2} + \frac{1}{4}\sin 2u + C = \frac{\sin^{-1}x}{2} + \frac{\sin u\cos u}{2} + C$$
$$= \frac{\sin^{-1}x}{2} + x\sqrt{1-x^2}/2 + C$$

Yet, this particular integral can also be done by parts:

$$\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$\implies \int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx$$

$$\implies 2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \arcsin x + C$$

$$\implies \int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + C.$$

19. What is the general solution for a 2nd order differential equation made up of? Define both parts

- General Solution = Complementary Function + Particular Integral.
- GS = CF + PI
- Complementary function - solution when 2nd order differential equation = 0.
- Particular integral - particular solution for what it equals.

20. What are the 3 possible complementary functions?

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ 2nd order linear differential equation

The equation has a solution based on the form $y = Ae^{mx}$

where $am^2 + bm + c = 0$ (known as the Auxillary Equation)

Roots are real and different $(b^2 - 4ac > 0)$ $m = m_1, m_2$

General Solutions $y = Ae^{m_1 x} + Be^{m_2 x}$

Roots are equal $(b^2 - 4ac = 0)$ $m = m_1, m_1$

General Solutions $y = e^{mx}(Ax + B)$

Roots are imaginary $(b^2 - 4ac < 0)$ $m = p \pm iq$

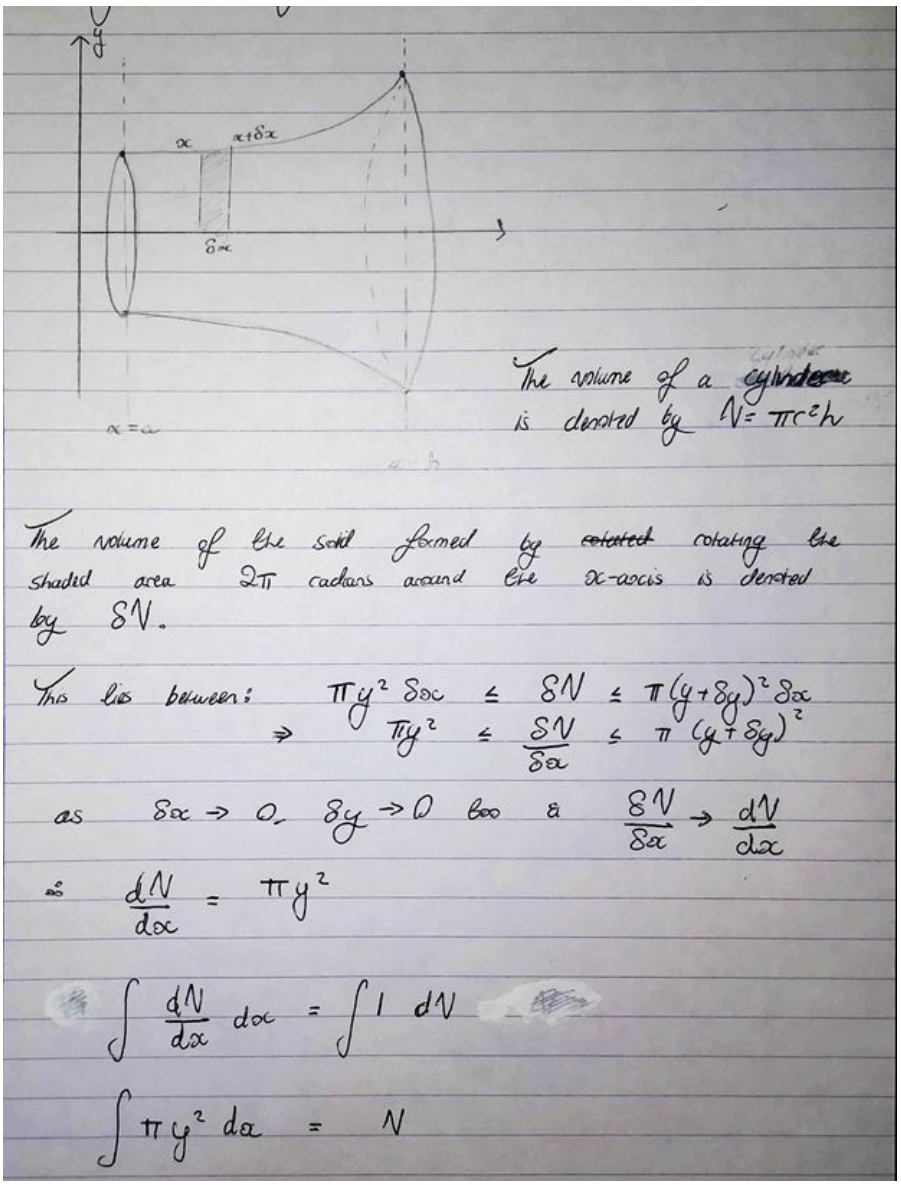
General Solutions $y = e^{px}(A\cos qx + B\sin qx)$

Also shown below:

Table 1

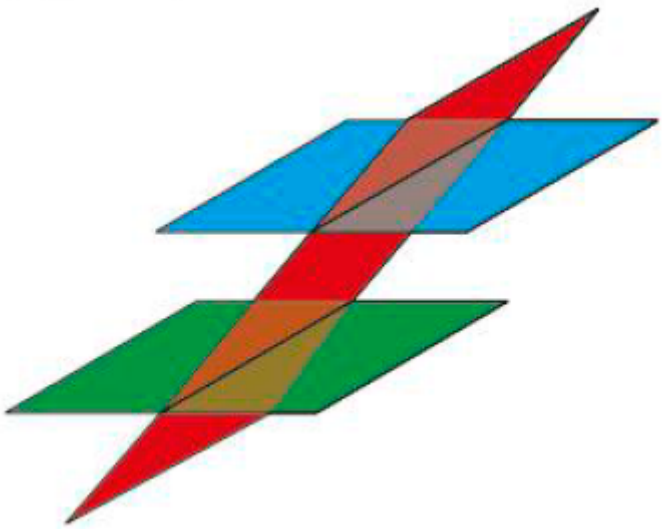
Roots of auxiliary equation $ax^2 + bx + c = 0$	General solution
Real distinct roots, m_1 and m_2	$y = Ae^{m_1 x} + Be^{m_2 x}$
Repeated root, m	$y = (Ax + B)e^{mx}$
Complex roots $m \pm in$	$y = e^{mx}(A\cos nx + B\sin nx)$

21. Derive the volumes of revolution formula



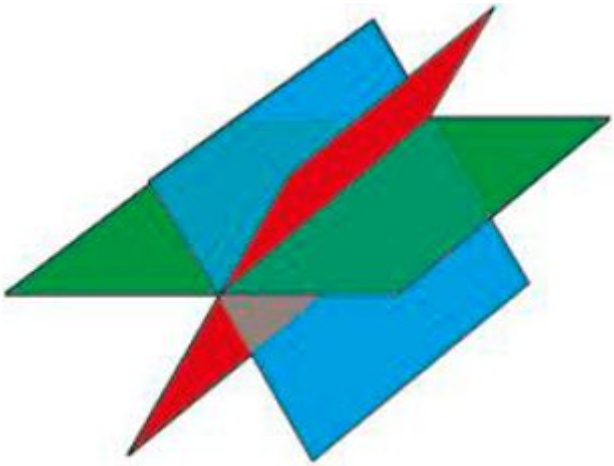
22. Give the 3 cases where systems of equations have no unique solution and the conditions of each

1.When two of the three planes are parallel.



Easy to check if any 2 are parallel.
2. Form a triangular prism:

Eliminating 1 variable will show they are inconsistent.
3. They form a sheaf:



Eliminating 1 variable will show they are consistent.

23. How can conics be translated?

-Replacing f(x) by f(x - c) in an equation translates the curve c units in the x-direction. Similarly, replacing g(y) by g(y - c) translates the curve c units in they-direction.
-Replacing f(x) by f(x/k) in an equation will stretch the curve by scale factor k in the x-direction. Similarly, replacing g(y) by g(y/k) will stretch the curve by scale factor k in they-direction.

Examples of this includes:

Key point

A parabola with equation $(y-y_1)^2 = 4a(x-x_1)$ will have its vertex on the point (x_1, y_1)

Key point

An ellipse with equation $\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} = 1$ will be centred on the point (x_1, y_1) and have radius of a in the x-direction and b in the y-direction.

Key point

A rectangular hyperbola with equation $(x-x_1)(y-y_1) = c^2$ will be centred on the point (x_1, y_1) and have asymptotes $x = x_1$, $y = y_1$

Key point

A hyperbola with equation $\frac{(x-x_1)^2}{a^2} - \frac{(y-y_1)^2}{b^2} = 1$ will be centred on the point (x_1, y_1) and have asymptotes $y - y_1 = \pm\frac{b}{a}(x-x_1)$

---

24. When does a body undergo SHM?

Key point

Motion that satisfies a differential equation of the form $\frac{d^2x}{dt^2} = -\omega^2 x$ is called simple harmonic motion (SHM).

This is a requirement and gives the following solution:

$$\frac{dx^2}{(dt)^2} = -\omega^2 x$$

$$\frac{dx^2}{(dt)^2} + \omega^2 x = 0$$

$$m^2 + \omega^2 = 0$$

$$m = \pm\omega i$$

$$x = Pcos(\omega t) + Qsin(\omega t)$$

$$= asin(\omega t + \varepsilon)$$

---

25. How can you derive the standard results for SHM?

There are certain standard results that apply to SHM. These are derived below.

Using $\frac{d^2x}{dt^2} = v\frac{dv}{dx}$ gives $v\frac{dv}{dx} = -\omega^2 x$

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = \frac{dv}{dx} \times v$$

So $\int v\,dv = -\int \omega^2 x\,dx$

Separate the variables.

$$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + c \qquad (1)$$

Let the amplitude of the motion be a, then

$$\frac{0^2}{2} = -\frac{\omega^2 a^2}{2} + c \Rightarrow c = \frac{\omega^2 a^2}{2}$$

Use the fact that when $x = a$ the velocity is zero.

So $\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{\omega^2 a^2}{2}$

Substitute for c into (1)

i.e. $v^2 = \omega^2(a^2 - x^2)$ or $v = \omega\sqrt{a^2 - x^2}$

Standard results

$$\frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow v = \omega\sqrt{a^2 - x^2}$$

$$x = a\cos\omega t \quad \text{or}$$

$$x = a\sin\omega t$$

$$T = \frac{2\pi}{\omega}$$

It is the starting conditions that determine which form of x to use.

where a is the amplitude and T is the period.

You can have 2 different values for x, it depends on when the clock starts ticking.

The value of ε depends on when the clock starts ticking, i.e. where we measure $t = 0$ from.

- If $t = 0$ when $x = 0$, then $\varepsilon = 0$, and $x = a\sin\omega t$
- If $t = 0$ when $x = a$, then $\varepsilon = \frac{\pi}{2}$, and $x = a\sin\left(\omega t + \frac{\pi}{2}\right) = a\cos\omega t$

The period of a simple $\sin t$ or $\cos t$ function is 2π. Hence the period of $x = a\sin(\omega t + \varepsilon)$ is $\frac{2\pi}{\omega}$

26. Derive arccosh x

We must restrict x ≥ 0 for a one-to-one mapping.

$$
\begin{aligned}
y &= \cosh^{-1} x \\
x &= \cosh y \\
x &= \frac{e^y + e^{-y}}{2} \\
2x &= e^y + e^{-y} \\
2xe^y &= e^{2y} + 1 \\
e^{2y} - 2xe^y + 1 &= 0 \\
e^y &= \frac{2x \pm \sqrt{(-2x)^2 - 4}}{2} \\
&= x \pm \sqrt{x^2 - 1} \\
y &= \ln\left(x \pm \sqrt{x^2 - 1}\right) \\
\cosh^{-1} x &= \ln\left(x + \sqrt{x^2 - 1}\right)
\end{aligned}
$$

We take the solution with the positive as now y ≥ 0 so e^y ≥ 1 for all y, and since x - \sqrt(x^2 - 1) fails to exceed this for some x (eg, -1), we discard it.

27. How can sin(nθ) and cos(nθ) be expressed in terms of z = cos(θ) + isin(θ) and e's?

[redacted]

Thus…

$$\cos x = \mathrm{Re}\left(e^{ix}\right) = \frac{e^{ix} + e^{-ix}}{2},$$

$$\sin x = \mathrm{Im}\left(e^{ix}\right) = \frac{e^{ix} - e^{-ix}}{2i}.$$

This can be shown using De Moivre's and the fact that cosine is odd and sine is even.

---

28. What symmetries are there under polar coordinates?

1.If r is a function of cos only - symmetrical about initial line.
2.If r is a function of sin only - symmetrical about θ = π/2.

---

29. Describe the motion of heavy damping

Returns slowly to its equilibrium position.

---

30. How can you express $\sin^n\theta$ OR $\cos^n\theta$ in terms of $\sin(n\theta)$ or $\cos(n\theta)$?

Express $\cos^3\theta$ in the form $a\cos 3\theta + b\cos\theta$

$z^n + z^{-n} = z^n + \frac{1}{z^n} = 2\cos n\theta$ $\quad$ $z + z^{-1} = z + \frac{1}{z} = 2\cos\theta$

$(a+b)^n \equiv {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots + {}^nC_n a^0 b^n$

$$(2\cos\theta)^3 \equiv (z+z^{-1})^3$$
$$\equiv {}^3C_0 (z)^3 (z^{-1})^0 + {}^3C_1 (z)^2 (z^{-1})^1 + {}^3C_2 (z)^1 (z^{-1})^2 + {}^3C_3 (z)^0 (z^{-1})^3$$
$$\equiv z^3 + 3z + 3z^{-1} + z^{-3}$$
$$\equiv (z^3 + z^{-3}) + 3(z + z^{-1})$$
$$\therefore 8\cos^3\theta \equiv 2\cos 3\theta + 3(2\cos\theta)$$
$$\therefore \cos^3\theta \equiv \frac{2}{8}\cos 3\theta + \frac{6}{8}\cos\theta$$
$$\therefore \cos^3\theta \equiv \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

1. Use the fact that $2\cos\theta = z + z^{-1}$ or $2i\sin\theta = z - z^{-1}$
2. Raise both sides the required power
3. Use the fact that $2\cos(n\theta) = z^n + z^{-n}$ or $2i\sin(n\theta) = z^n - z^{-n}$

Express $\sin^4\theta$ in terms of the form $\cos k\theta$

$z^n + z^{-n} = z^n + \frac{1}{z^n} = 2\cos n\theta$ $\quad$ $z + z^{-1} = z + \frac{1}{z} = 2\cos\theta$

$z^n - z^{-n} = z^n - \frac{1}{z^n} = 2i\sin n\theta$ $\quad$ $z - z^{-1} = z - \frac{1}{z} = 2i\sin\theta$

$(a+b)^n \equiv {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots + {}^nC_n a^0 b^n$

$$(2i\sin\theta)^4 \equiv (z - z^{-1})^4$$
$$\equiv {}^4C_0 (z)^4 (-z^{-1})^0 + {}^4C_1 (z)^3 (-z^{-1})^1 + {}^4C_2 (z)^2 (-z^{-1})^2 + {}^4C_3 (z)^1 (-z^{-1})^3 + {}^4C_4 (z)^0 (-z^{-1})^4$$
$$\equiv z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$$
$$\equiv (z^4 + z^{-4}) - 4(z^2 + z^{-2}) + 6$$
$$16\sin^4\theta \equiv 2\cos 4\theta - 8\cos 2\theta + 6$$
$$\therefore \sin^4\theta \equiv \frac{2}{16}\cos 4\theta - \frac{8}{16}\cos 2\theta + \frac{6}{16}$$
$$\therefore \sin^4\theta \equiv \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

31. How is the half-line loci written using complex numbers?

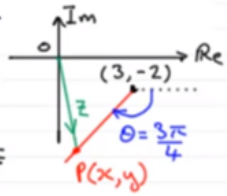
[redacted]

$\text{Arg}(z - z_1) = \theta$

$-\pi < \theta \leq \pi$

$\arg(z - z_1) = \theta$ represents a half-line from the fixed point z_1 making an angle θ with a line from the fixed point z_1 parallel to the real axis.

Sketch the locus of z for which $\arg(z - 3 + 2i) = -\frac{3\pi}{4}$ and find the cartesian equation.



$$\therefore \text{Arg}[z - (3 - 2i)] = -\frac{3\pi}{4}$$
let $z = x + iy$
$$\therefore \text{Arg}[x + iy - (3 - 2i)] = -\frac{3\pi}{4}$$
$$\therefore \text{Arg}[(x-3) + i(y+2)] = -\frac{3\pi}{4}$$
$$\therefore \tan\left(-\frac{3\pi}{4}\right) = \frac{y+2}{x-3}$$
$$\therefore 1 = \frac{y+2}{x-3} \Rightarrow x - 3 = y + 2$$
$$\therefore y = x - 5, \text{ for } x < 3$$

32. How do you work out the distance between a point and line (or 2 parallel lines) using 2 different methods?

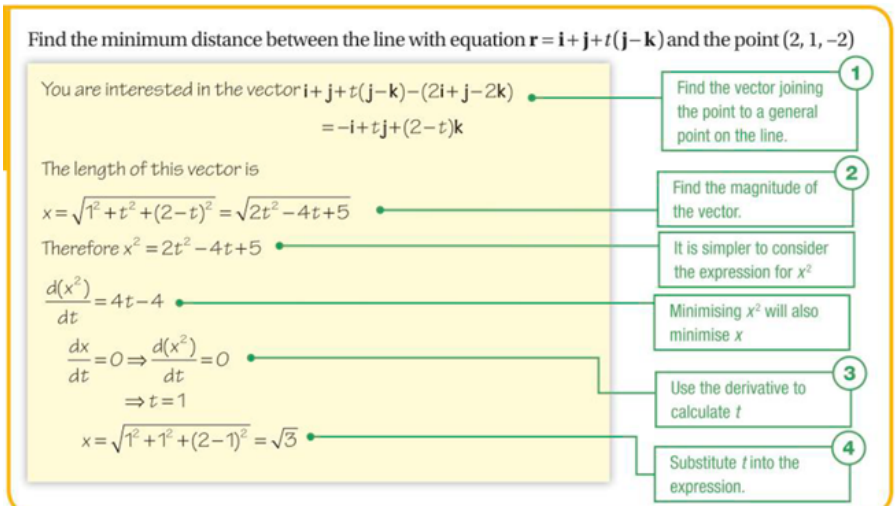
First Method:

1.Calculate the vector between a general point on the line and the point in question (P).

2.Dot this vector with the direction vector of the line.

3.Calculate its magnitude.

Second Method:



33. How is the integrating factor used to solve 1st order linear differential equations?

1. Get the differential equation into the following form:

$$\frac{dy}{dx} + Py = Q$$

2. Find the integrating factor:

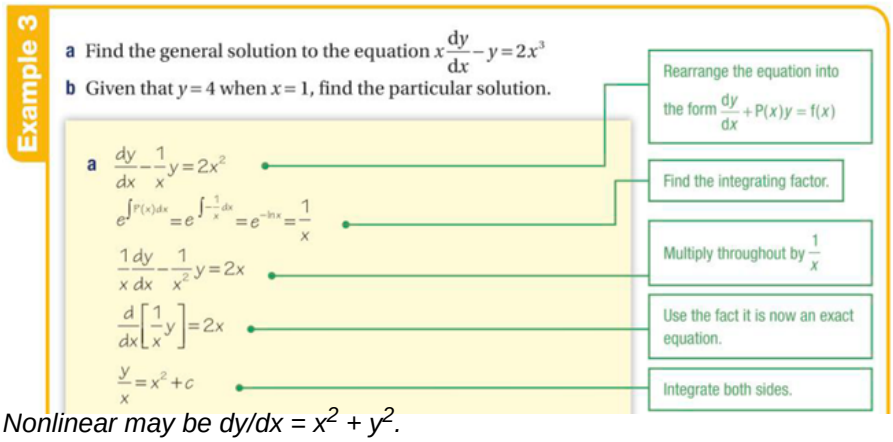
$$I = e^{\int P dx}$$

3. Multiply the equation in (1) by the integrating factor.

4. Use the product rule in reverse.

5. Integrate both sides.

Example:



Nonlinear may be dy/dx = x^2 + y^2.

34. How can you evaluate an improper integral? What outcomes can you have?

1. By replacing the limit with a variable and taking the limit of that variable.
2. You can have convergent (if the limit exists) or divergent (if the limit does not exist) integral.

Key point

To evaluate an improper integral with a limit of $\pm\infty$ use

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx \quad \text{or} \quad \int_{\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$$

If the limit exists, then the improper integral is called **convergent**.

If the limit does not exist, then the improper integral is called **divergent**.

Examples:

When splitting the integral, you need to use different variables for ∞ and $-\infty$
This is because both parts of the integral must be convergent for the original integral to exist.

Example 2

a Find the value of $\int_{-\infty}^{\infty} \frac{x}{e^{x^2}}\,dx$

b Show that the improper integral $\int_{-\infty}^{\infty} e^x$ is divergent.

a $\int_{-\infty}^{\infty} \frac{x}{e^{x^2}}\,dx = \lim_{a\to-\infty}\int_a^0 xe^{-x^2}\,dx + \lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx$

Choose to split the integral at 0

$= \lim_{a\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_a^0 + \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^b$

Since $\frac{d}{dx}(e^{-x^2}) = -2xe^{-x^2}$ using the chain rule.

$= \lim_{a\to-\infty}\left(-\frac{1}{2}e^0 - -\frac{1}{2}e^{-a^2}\right) + \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} - -\frac{1}{2}e^0\right)$

$= \lim_{a\to-\infty}\left(-\frac{1}{2} + \frac{1}{2}e^{-a^2}\right) + \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}\right)$

Since $e^0 = 1$

$= \left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)$

Both limits exist, therefore the improper integral is convergent.

Since as $a \to -\infty$, $e^{-a^2} \to 0$ and as $b \to \infty$, $e^{-b^2} \to 0$

So the improper integral $\int_{-\infty}^{\infty} \frac{x}{e^{x^2}}\,dx = -\frac{1}{2} + \frac{1}{2} = 0$

Example 3

Find the value of $\int_0^4 \frac{2}{\sqrt{x}}\,dx$

$\int_0^4 \frac{2}{\sqrt{x}}\,dx = \lim_{t\to 0}\int_t^4 x^{-\frac{1}{2}}\,dx$

Replace 0 with t since $\frac{1}{\sqrt{x}}$ is undefined at $x = 0$

$= \lim_{t\to 0}\left[2x^{\frac{1}{2}}\right]_t^4$

$= \lim_{t\to 0}(4 - 2\sqrt{t})$

$= 4$

Since $2\sqrt{t} \to 0$ as $t \to 0$, the value of the integral is 4

35. What is the vector equation of a plane?

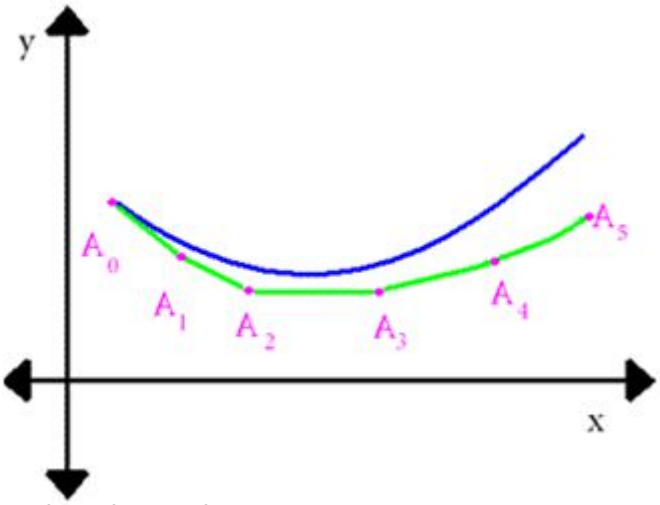
$$\mathbf{r} = \mathbf{a} + \alpha\mathbf{s} + \beta\mathbf{t}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \alpha \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \beta \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

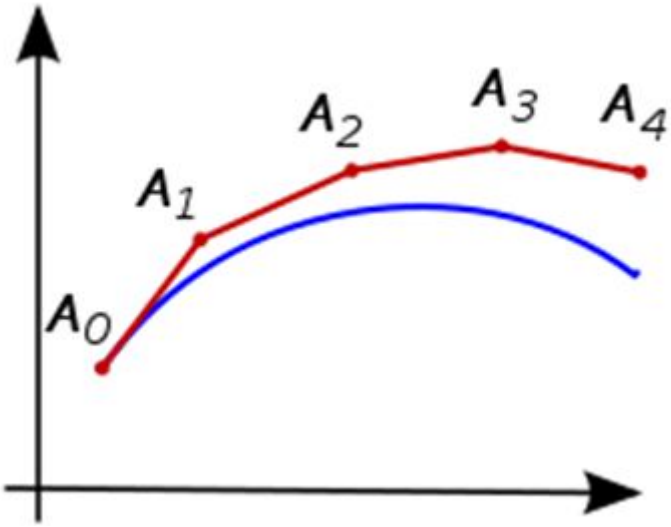
*Where **a** is some point on the plane and where **s** and **t** are 2 non-parallel vectors used to define the plane.*

36. When will Euler's Method be an underestimate or an overestimate?

Underestimate when convex:



Underestimate when concave:



37. When is a matrix singular? When is this useful?

When…

$$det(A) = 0$$

Can be used to see whether a system of equations has a solution.

38. Prove $|z_1z_2| = |z_1||z_2|$ and $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

$$(1)\ |z_1z_2| = |z_1||z_2|$$
$$\arg(z_1z_2) = \arg z_1 + \arg z_2$$

NOTE: Multiplication rotates z_1 by $\arg z_2$

Proof: let $z_1 = r_1 cis\,\theta_1$ and $z_2 = r_2 cis\,\theta_2$

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\times r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 - \sin\theta_1\sin\theta_2) \\ &= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\ &= r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} \end{aligned}$$

$$\therefore |z_1z_2| = r_1r_2 = |z_1||z_2| \qquad \arg(z_1z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

And note, it follows that…

$$|z_1z_2z_3\ldots z_n| = |z_1||z_2||z_3|\ldots|z_n|$$
$$\arg(z_1z_2z_3\ldots z_n) = \arg z_1 + \arg z_2 + \arg z_3 + \ldots + \arg z$$

$$\left|\frac{z_1z_2}{z_3z_4}\right| = \frac{|z_1||z_2|}{|z_3||z_4|}$$
$$\arg\left(\frac{z_1z_2}{z_3z_4}\right) = \arg z_1 + \arg z_2 - \arg z_3 - \arg z_4$$

$$(3)\ |z^n| = |z|^n$$
$$\arg(z^n) = n\arg z$$

---

39. What 2 results arise from the cube roots of unity?

1. The sum of all roots to zero (by symmetry of vectors)
2. If $z_2 = \omega$ then multiplying it by itself gives $\omega^2 = z_3$

Since it rotates the complex number (keeping magnitude the same) as $\arg(z_n) = n \times \arg(z)$

$$\begin{aligned} z^3 &= 1 \\ &= \cos 0 + i\sin 0 \\ &= \cos(0+2k\pi) + i\sin(0+2k\pi) \\ &= \cos(2k\pi) + i\sin(2k\pi) \end{aligned}$$

or $z^3 = e^{i(2k\pi)}$

$$\therefore z = \sqrt[3]{1} = 1^{1/3} = [\cos(2k\pi) + i\sin(2k\pi)]^{1/3} = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$$

$$\therefore z = [e^{i(2k\pi)}]^{1/3} \qquad \therefore z = e^{i\frac{2k\pi}{3}}$$

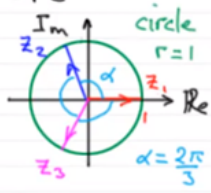
when $k=0$, $z_1 = 1$

when $k=1$, $z_2 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

when $k=-1$, $z_3 = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

(2)

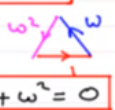
$z = r(\cos\theta + i\sin\theta)$
$= re^{i\theta}$



If $z_2 = \omega$ then $z_3 = \omega^2$

and $\omega^* = \omega^2$

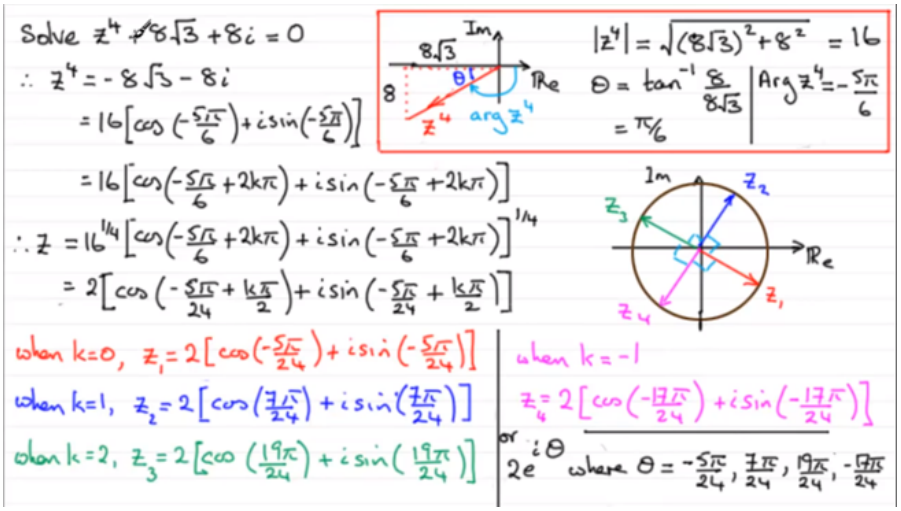
Also



$1 + \omega + \omega^2 = 0$

40. How do you find the nth root of a complex number?

1. Set z^n = (complex number)
2. Add $2k\pi$ (or $2k\pi i$ if Euler's Formula is used) to the complex number
3. nth root both sides
4. Find the value for z for different values of k



41. How can you find the distance between skew lines?

1.Consider a general point on both lines.
2.Find a vector joining these 2 general point.
3.Dot this vector with the direction vector of both lines.
4.Use the newly found vector and calculate its magnitude.

Find the minimum distance between the skew lines with equations

$\mathbf{r}=2\mathbf{i}+\mathbf{j}+\lambda(-\mathbf{j}+2\mathbf{k})$ and $\mathbf{r}=\mathbf{j}-2\mathbf{k}+\mu(\mathbf{i}+2\mathbf{j})$

A general point, P, on the first line has position vector, $\overrightarrow{OP}=\begin{pmatrix}2\\1-\lambda\\2\lambda\end{pmatrix}$.

A general point, Q, on the second line has position vector

$\overrightarrow{OQ}=\begin{pmatrix}\mu\\1+2\mu\\-2\end{pmatrix}$.

Therefore, the vector joining these two points is

$\overrightarrow{PQ}=\begin{pmatrix}\mu\\1+2\mu\\-2\end{pmatrix}-\begin{pmatrix}2\\1-\lambda\\2\lambda\end{pmatrix}=\begin{pmatrix}\mu-2\\2\mu+\lambda\\-2-2\lambda\end{pmatrix}$

Use $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$

The distance PQ will be minimal when the vector $\overrightarrow{PQ}$ is perpendicular to both lines.

$\begin{pmatrix}\mu-2\\2\mu+\lambda\\-2-2\lambda\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\2\end{pmatrix}=0$

Since $\overrightarrow{PQ}$ is perpendicular to the first line.

So $-(2\mu+\lambda)+2(-2-2\lambda)=0$

$\Rightarrow -2\mu-5\lambda=4$ (1)

$\begin{pmatrix}\mu-2\\2\mu+\lambda\\-2-2\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\0\end{pmatrix}=0$

Since $\overrightarrow{PQ}$ is perpendicular to the first line.

So $(\mu-2)+2(2\mu+\lambda)=0$

$\Rightarrow 5\mu+2\lambda=2$ (2)

Solve the equations simultaneously (you could use your calculator).

Solving equations (1) and (2) simultaneously gives, $\mu=\frac{6}{7}, \lambda=-\frac{8}{7}$

$\overrightarrow{PQ}=\begin{pmatrix}\left(\frac{6}{7}\right)-2\\2\left(\frac{6}{7}\right)+\left(-\frac{8}{7}\right)\\-2-2\left(-\frac{8}{7}\right)\end{pmatrix}=\begin{pmatrix}-\frac{8}{7}\\\frac{4}{7}\\\frac{2}{7}\end{pmatrix}$

Substitute the values of μ and λ into the vector $\overrightarrow{PQ}$

$\overrightarrow{PQ}=\sqrt{\left(-\frac{8}{7}\right)^2+\left(\frac{4}{7}\right)^2+\left(\frac{2}{7}\right)^2}=\frac{2\sqrt{21}}{7}$ (1.31 to 3 sig. fig.)

Calculate the length of $\overrightarrow{PQ}$

So the perpendicular distance between the two lines is 1.31 units

42. What is separation of variables?

$$\frac{dy}{dx} = 3x^2 y$$
$$\frac{1}{y}\frac{dy}{dx} = 3x^2$$
$$\int \frac{1}{y}\frac{dy}{dx}dx = \int 3x^2 dx$$
$$\int \frac{1}{y}dy = \int 3x^2 dx$$
$$\ln|y| = x^3 + c$$
$$y = Ae^{x^3}$$

---

43. How is an exact first order differential equation solved?

1. Use the product rule in reverse.
2. Integrate both sides.

$\frac{d}{dx}[f(x)y] = f(x)\frac{dy}{dx} + f'(x)y$

So if

$f(x)\frac{dy}{dx} + f'(x)y = g(x)$

$\therefore \frac{d}{dx}[f(x)y] = g(x)$

$\therefore f(x)y = \int g(x)\,dx$

Example 1

$x^4\frac{dy}{dx} + 4x^3 y = e^{2x}$

$\therefore \frac{d}{dx}[x^4 y] = e^{2x}$

$\therefore x^4 y = \int e^{2x}dx$

$\therefore x^4 y = \frac{1}{2}e^{2x} + c$

Example 2

$\cos x\frac{dy}{dx} - y\sin x = x^2$

given $y=1$, when $x=0$

$\frac{d}{dx}(y\cos x) = x^2$

$\therefore y\cos x = \int x^2 dx$

$\therefore y\cos x = \frac{1}{3}x^3 + c$

when $y=1, x=0$

$\therefore 1 = c \Rightarrow$

May require some manipulation to get it into the form required.

---

44. What is a line of invariant points?

A whole line of points, each which remains unchanged under a transformation.

---

45. How do the roots relate to coefficients for quadratics, cubics, and quartics?

For $ax^2+bx+c=0$ with roots α and β then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

For $ax^3+bx^2+cx+d=0$ with roots α, β and γ then $\sum\alpha=\alpha+\beta+\gamma=-\frac{b}{a}$, $\sum\alpha\beta=\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$ and $\alpha\beta\gamma=-\frac{d}{a}$

For $ax^4+bx^3+cx^2+dx+e=0$ with roots α, β, γ and δ then $\sum\alpha=-\frac{b}{a}$, $\sum\alpha\beta=\frac{c}{a}$, $\sum\alpha\beta\gamma=-\frac{d}{a}$ and $\alpha\beta\gamma\delta=\frac{e}{a}$

The proof for the quadratic roots is shown below:

Quadratic equations

Let the quadratic equation $ax^2+bx+c=0$ have roots $x=\alpha$ and $x=\beta$

Dividing through by a gives

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Since $x=\alpha$ and $x=\beta$ are the roots of this quadratic, you can write the equation in the form

$$(x-\alpha)(x-\beta)=0$$

Expanding the brackets gives

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

Comparing the two versions of the quadratic equation gives

$$x^2+\frac{b}{a}x+\frac{c}{a}\equiv x^2-(\alpha+\beta)x+\alpha\beta=0$$

So, comparing the coefficients for x and the constant gives $(\alpha+\beta)=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

The proof for the cubic and quartics are similar.

---

46. What is the vector cross product form of a line?

$$(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$$

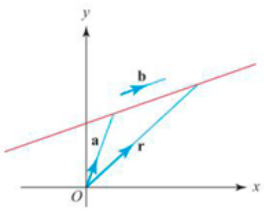
Since…

Two forms for the equation of a line in 3D are the Cartesian form $\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}$ and the vector form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$

A third possibility is the **vector product form**.

Key point

A straight line passing through the point with position vector $\mathbf{a}$ and parallel to the vector $\mathbf{b}$ has equation $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$



This follows from the fact that any vector on the line will be parallel to the vector $\mathbf{b}$ and the vector product of two parallel vectors is zero.

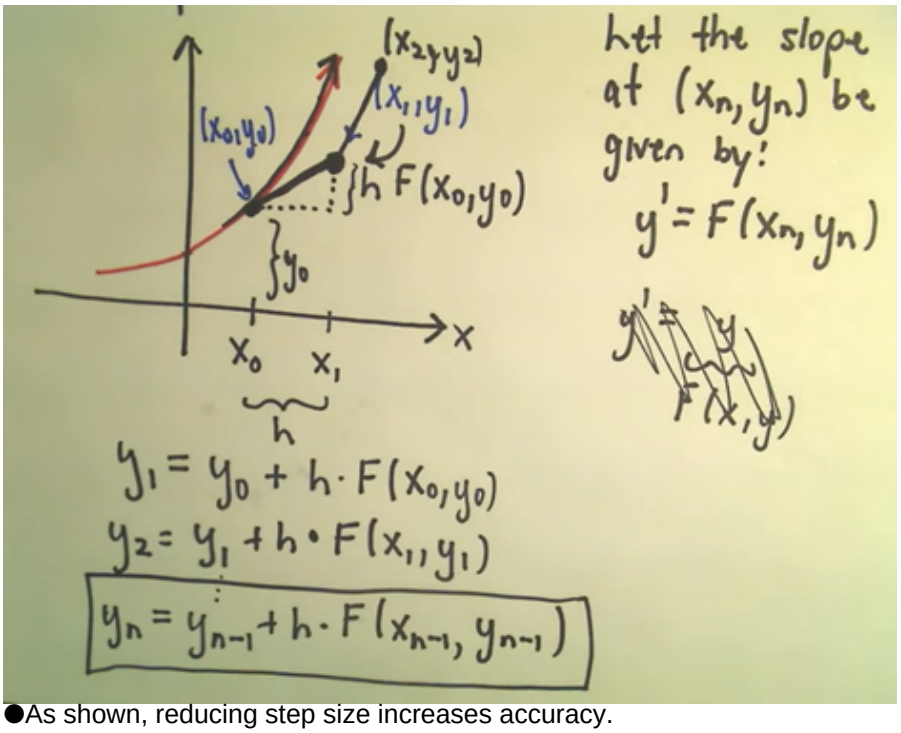
You can expand the brackets of $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$ to give $\mathbf{r}\times\mathbf{b}-\mathbf{a}\times\mathbf{b}=\mathbf{0}$ which rearranges to $\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$

You should calculate the result of $\mathbf{a}\times\mathbf{b}$ using the vectors given.

---

47. When is Euler's Method usually used? How does it work geometrically? What should you note?

●To solve nonlinear differential equations (eg, $dy/dx = x^2 + y^2$).



●As shown, reducing step size increases accuracy.

48. How can you find the line of intersection of 2 planes?

1. Eliminate one of the variables.
2. Let another variable equal λ.
3. Find the remaining variables that define the line.

Example:

(A), x + y + z = -1

(B), x + 2y + 3z = -4

(B) - (A) gives (C), y + 2z = -3 which is true for all points on the line.

Let z = λ so y = -2λ - 3 from (C) and x = 2 + λ from (A). These all define the line of intersection.

49. What are the 2 equations for the scalar/dot product?

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = a_1a_2 + b_1b_2 + c_1c_2$$

[redacted]

Where θ is the acute angle between the 2 vectors.

50. How is a shear parallel to one axis represented?

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Left is a shear parallel to the x-axis. Right is a shear parallel to the y-axis.

---

51. How can you express sin(nθ) and cos(nθ) in powers of sinθ and cosθ?

1. Rewrite sin(nθ) and cos(nθ) as [sin(θ) and cos(θ)]^n using De Moivre's Theorem.
2. Expand the brackets.
3. Equate real and imaginary parts.
4. Use any identities, if necessary.

Express $\cos 4\theta$ in terms of $\cos\theta$

de Moivre's Theorem
if $z = (\cos\theta + i\sin\theta)$
then $z^n = (\cos n\theta + i\sin n\theta)$

$$\cos 4\theta + i\sin 4\theta \equiv (\cos\theta + i\sin\theta)^4$$
$$\equiv (c + is)^4$$
$$\equiv c^4 + 4c^3(is)^1 + 6c^2(is)^2 + 4c^1(is)^3 + (is)^4$$
$$\equiv c^4 - 6c^2s^2 + s^4 + i(4c^3s - 4cs^3)$$
$$\equiv c^4 - 6c^2(1-c^2) + (1-c^2)^2 + i(4c^3s - 4cs^3)$$
$$\equiv c^4 - 6c^2 + 6c^4 + 1 - 2c^2 + c^4 + i(4c^3s - 4cs^3)$$
$$\equiv 8c^4 - 8c^2 + 1 + i(4c^3s - 4cs^3)$$

Compare Re: $\cos 4\theta \equiv 8\cos^4\theta - 8\cos^2\theta + 1$

Note! Im: $\sin 4\theta \equiv 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

---

52. How can you split nonlinear partial fractions?

$\dfrac{f(x)}{(\alpha x^2+\beta)(\gamma x+\delta)}$ can be split into partial fractions of the form $\dfrac{Ax+B}{\alpha x^2+\beta} + \dfrac{C}{\gamma x+\delta}$

An example is below…

Example 2

the denominator yourself.

Work out $\int \dfrac{2x+12}{(x+1)(x^2+9)}\,dx$

$$\frac{2x+12}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$2x+12 = A(x^2+9) + (Bx+C)(x+1)$

$= Ax^2 + 9A + Bx^2 + Bx + Cx + C$

Multiply both sides by $(x+1)(x^2+9)$

Equating coefficients

$x^2: 0 = A + B$ so $A = -B$

$x: 2 = B + C$ (equation 1)

$1: 12 = 9A + C$ (equation 2)

Or use an alternative method to find the values of A, B and C

Subtract equation 1 from equation 2 to give: $9A - B = 10$

$A = -B \Rightarrow -9B - B = 10 \Rightarrow B = -1$

Solve the three equations simultaneously.

So $A = 1$ and $C = 3$

53. What is the form of a differential equation under damping? How does it link to the different types of damping?

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$$

Something in this form with k in front of dx/dt. The larger k is, the stronger resistance to motion.

If the discriminant of the auxiliary equation is…

- > 0, heavy damping
- = 0, critical damping
- Here, the amplitude reduces as fast as possible.
- < 0, light damping.

---

54. When is orientation preserved under a transformation? What does it mean when it isn't?

When…

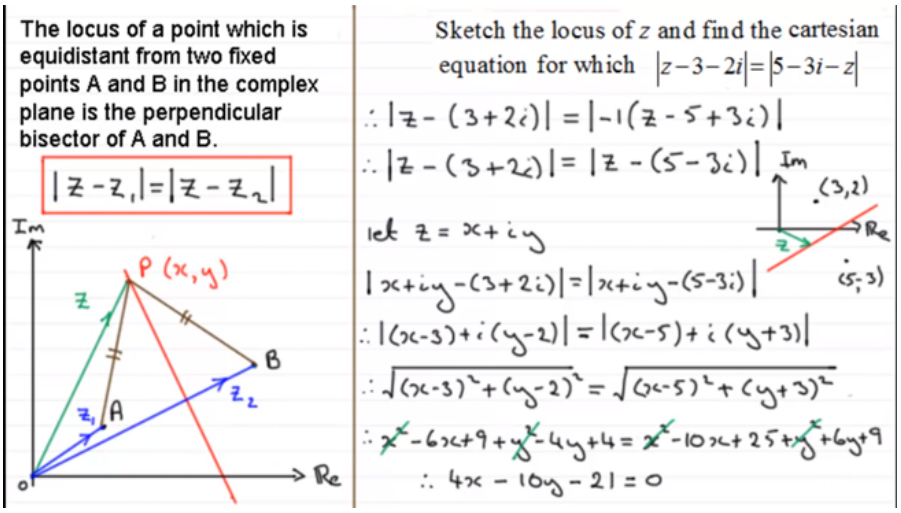
$$det(T) > 0$$

Less than 0 means some reflection is involved in the transformation.

---

55. How is a perpendicular bisector loci written using complex numbers?

[redacted]



56. Derive arctanh x

$$
\begin{aligned}
y &= \tanh^{-1} x \\
x &= \tanh y \\
x &= \frac{e^y - e^{-y}}{e^y + e^{-y}} \\
&= \frac{e^y - e^{-y}}{e^y + e^{-y}} \cdot \frac{e^y}{e^y} \\
&= \frac{e^{2y} - 1}{e^{2y} + 1} \\
xe^{2y} + x &= e^{2y} - 1 \\
e^{2y} - xe^{2y} &= 1 + x \\
e^{2y} &= \frac{1+x}{1-x} \\
2y &= \ln\left(\frac{1+x}{1-x}\right) \\
y &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)
\end{aligned}
$$

57. How does z and z* relate on an Argand Diagram and in mod-arg form?

- z and z* are mirror images in the real axis.
-

$$z = x + iy, \;\; z^* = x - iy$$
$$\Rightarrow z = \cos(\theta) + i\sin(\theta)$$
$$\Rightarrow z^* = \cos(-\theta) + i\sin(-\theta)$$
$$= \cos(-\theta) + i\sin(-\theta)$$
$$= \cos(\theta) - i\sin(\theta)$$

---

58. How can you find maximums and minimums of graphs without calculus?

1. Say f(x) intersects y = k so f(x) = k.
2. Consider the discriminant.

The function $f(x) = \frac{x^2 - 4x + 4}{x}$ intersects the straight line $y = k$

a Form a quadratic equation in x and k

b Hence find the values of k for which $f(x)$ has real roots.

c Use a graphical calculator or graph sketching software on your computer to confirm your answer.

a $k = \frac{x^2 - 4x + 4}{x}$

$kx = x^2 - 4x + 4$

$x^2 - (k+4)x + 4 = 0$

(1) Multiply both sides by x and simplify to form a quadratic equation in x

b The function has real roots if $b^2 - 4ac \geq 0$

The discriminant must be ≥ 0

$(k+4)^2 \geq 4 \times 1 \times 4$

Substitute values and solve the inequality.

$k^2 + 8k + 16 - 16 \geq 0$

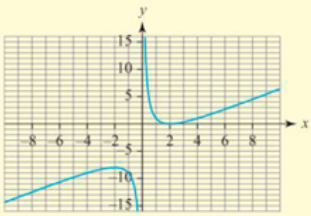
$k^2 + 8k \geq 0$

$k(k+8) \geq 0$

$k = \frac{x^2 - 4x + 4}{x}$ has real roots if $k \geq 0$ or $k \leq -8$

(2)(3) k and $(k+8)$ must either both be positive or both be negative.

c



(2) The graph of $y = \frac{x^2 - 4x + 4}{x}$ shows that x does not exist if $-8 \leq y \leq 0$

So x can only have real roots if $y \geq 0$ or $y \leq -8$

---

59. How should you deal with double inequalities?

Consider each case seperately.

Example 7

Find the values of x for which $0 \le \frac{4+x}{3-x} < 1$

Consider $0 \le \frac{4+x}{3-x}$

Critical values are $x = -4$ and $x = 3$

For $x < -4$, $\frac{4+x}{3-x}$ is negative.

For $-4 < x < 3$, $\frac{4+x}{3-x}$ is positive.

For $x > 3$, $\frac{4+x}{3-x}$ is negative.

So $\frac{4+x}{3-x} \ge 0$ when $-4 \le x < 3$

Now consider $\frac{4+x}{3-x} < 1$

When $x < 3$

$4 + x < 3 - x$

$x < -\frac{1}{2}$

When $x > 3$

$4 + x > 3 - x$

$x > -\frac{1}{2}$, but we have the more restrictive condition $x > 3$

So $\frac{4+x}{3-x} < 1$ when $x < -\frac{1}{2}$ or $x > 3$

Full solution: $0 \le \frac{4+x}{3-x} < 1$ when $-4 \le x < -\frac{1}{2}$

2 Consider the signs of the numerator and denominator for each set of values.

Do not include 3 as $\frac{4+x}{3-x}$ is not defined at $x = 3$

$(3 - x)$ is positive.

1 $(3 - x)$ is negative so reverse the inequality.

3 Find the values which satisfy both inequalities.

60. Derive arcsinh x

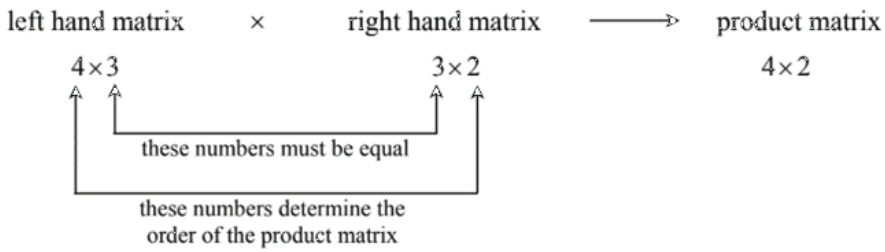
$$
\begin{aligned}
y &= \sinh^{-1} x \\
x &= \sinh y \\
x &= \frac{e^y - e^{-y}}{2} \\
2x &= e^y - e^{-y} \\
2xe^y &= e^{2y} - 1 \\
e^{2y} - 2xe^y - 1 &= 0 \\
e^y &= \frac{2x \pm \sqrt{(-2x)^2 + 4}}{2} \\
&= x \pm \sqrt{x^2 + 1} \\
y &= \ln\left(x \pm \sqrt{x^2 + 1}\right) \\
\sinh^{-1} x &= \ln\left(x + \sqrt{x^2 + 1}\right)
\end{aligned}
$$

We take the positive log at the end.

---

61. When is matrix multiplication allowed?

The column number of the left matrix = row number right matrix.

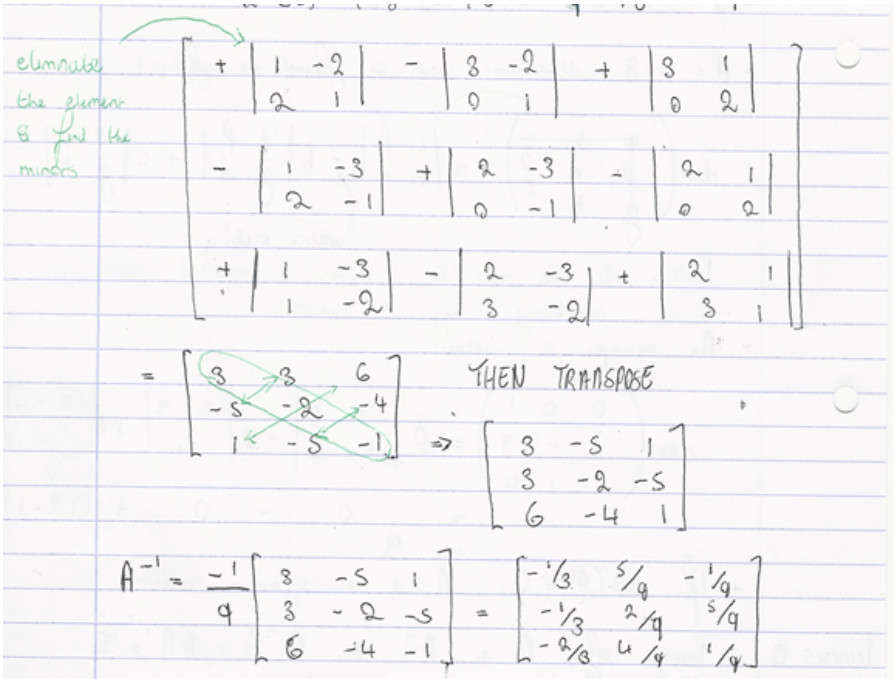


You say they're **conformable to multiplication**.

---

62. How do you find the inverse of a 3x3 matrix?

1.Calculate the determinant.
2.Find the minor of each value with alternating +'s and -'s.
3. Transpose this matrix of cofactors.
4.Divide by the determinant.



Cofactors are denoted by capital letters:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{'s cofactor matrix is} \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

Hence why | matrix | = $a_1A_1 + b_1B_1 + c_1C_1$ yet B_1 is negative.

63. What is the formulae for ‘Volumes of Revolution’?

$$\pi \int_a^b y^2 dx$$

For rotation around the x-axis.

$$\pi \int_a^b x^2 dy$$

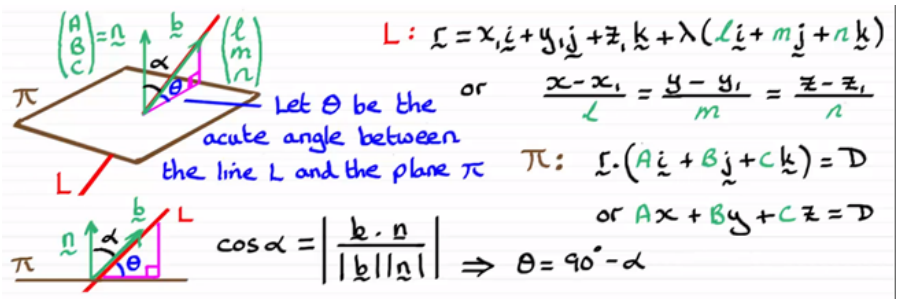
For rotation around the y-axis.

This is because, when you have a small change in x then the radius of the cylinders are y and following πr^2 you get the top formula with limits a b (both x-values with b - a representing the total height). Whereas, rotating around the y-axis means a small change in y with a radius x. This gives you the bottom formula with limits a b (both y-values).

---

64. How can you work out the angle between a line and plane?

1.Find the angle between the direction vector of the line and the normal of the plane using the dot product.
2.Do 90 - (angle above) (aka finding the complement).



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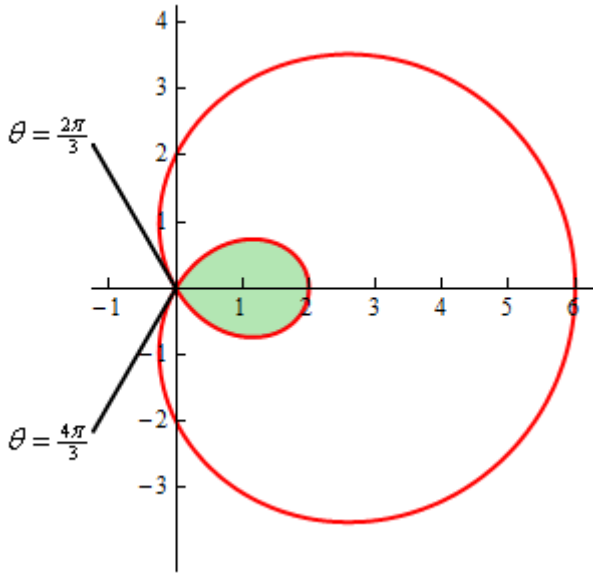
65. What 2 things should you note when integrating polar curves?

1.You may end up with negative areas.
2.You can use symmetries to help find areas.



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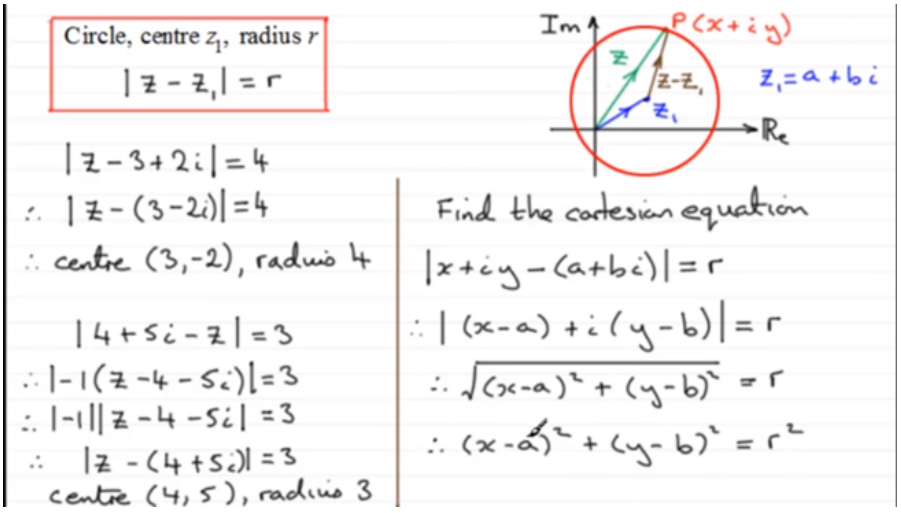
66. What is det(**AB**) equal to?

det(**A**) x det(**B**)

---

67. How a circle loci written using complex numbers?

[redacted]



---

68. What is $(\mathbf{AB})^T$ equal to?

$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

---

69. How is a matrix diagonalised? What is important here?

Key point

A matrix, **M**, can be **diagonalised** by finding **P** and **D** such that $\mathbf{M} = \mathbf{PDP}^{-1}$. It can be shown that:

- **D** is a diagonal matrix with the eigenvalues of M along the leading diagonal
- **P** is a matrix where the columns are the eigenvectors of M

The eigenvectors in the columns of **P** must occur in the same order as their corresponding eigenvalues in **D**

The equation above comes from $\boldsymbol{P^{-1}MP} = \boldsymbol{D}$*. This works because we first apply transformation P which turns our basis vectors into the eigenvectors. Now these eigen basis vectors are scaled using the transformation* ***M*** *(which is responsible for the eigenvectors in* ***P****). Their direction doesn't change. Now they are transformed back into our basis vectors of [(1, 0) (0,1)]. This is identical to purely having the eigenvalues (the values by which they're scaled) as the basis vectors.*

---

70. How is Simpson's Rule better than the Midordatine rule?

By minimising the differences at the top and bottom of each rectangle by replacing the mid ordinates with a quadratic curve.

---

71. What is the characteristic equation for an eigenvector? And how is it derived?

You can rearrange the equation $\mathbf{Ax} = \lambda\mathbf{x}$ to give $\mathbf{Ax} - \lambda\mathbf{Ix} = 0$ since $\mathbf{x} = \mathbf{Ix}$

then factorise to give $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$

Since **x** is a non-zero vector, it must be the case that the matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular.
Therefore $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Key point

The equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is the **characteristic equation** of **A** and is used to find the eigenvalues.

---

72. What is the determinant of a 2x2 matrix?

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

---

73. How do you find the inverse of a 2x2 matrix?

If…

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then…

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

---

74. Describe the order of composite transformations

$$C \circ B \circ A$$

Described as A followed by B followed by C.

---

75. What does the determinant represent?

The change in area/volume of a shape under the transformation.

---

76. What is the determinant of a 3x3 matrix?

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

You can also used any other row / column yet have to follow these +'s and -'s.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

---

77. Describe what happens to invariant lines under a transformation

Every point on the line is mapped to **ANOTHER** point on the same line.

---

78. When are two vectors perpendicular?

When **a**.**b** = 0

---

79. Give 3 different forms of a complex number

$$\begin{aligned} z &= x + yi \\ &= r(\cos(\theta) + i\sin(\theta)) \\ &= re^{i\theta} \end{aligned}$$

---

80. What is det($\mathbf{A}^{-1}$) equal to? How can this be proved?

Key point

For any square, non-singular matrix **A**: $\det(\mathbf{A}^{-1}) \equiv \frac{1}{\det(\mathbf{A})}$

Can be proved using the fact det($\mathbf{AA}^{-1}$) = det(**I**).

---

81. What is det($\mathbf{M}^{T}$) equal to?

det(**M**)

---

82. What is an eigenvector and an eigenvalue?

- Eigenvector - a vector whose direction is maintained under a transformation.
- Eigenvalue - the value by which the eigenvector is scaled under that transformation.

This satisfies the equation **Ax** = λ**x**.

---

83. How do eigenvectors relate to lines of invariant points?

From **Tx** = λ**x** ...

If a transformation given by a matrix **T** has an eigenvalue of 1 then the corresponding eigenvectors determine the direction of a line of invariant points through the origin.

---

84. How can you transform linear and reciprocal roots?

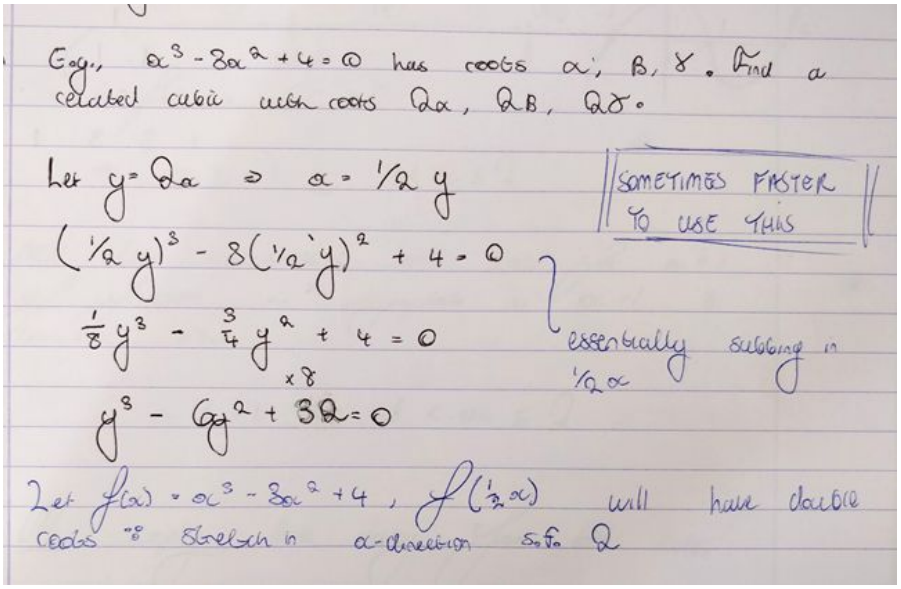
Key point

If the roots are transformed in a linear way, so that $y = mx + c$,

then you transform the equation by substituting $x = \frac{y-c}{m}$

If the new roots are reciprocals, so that $y = \frac{1}{x}$, then you transform the equation by substituting $x = \frac{1}{y}$

Example:



E.g., $x^3 - 8x^2 + 4 = 0$ has roots α, β, γ. Find a related cubic with roots $2\alpha, 2\beta, 2\gamma$.

Let $y = 2\alpha \Rightarrow \alpha = \frac{1}{2}y$

SOMETIMES FASTER TO USE THIS

$(\frac{1}{2}y)^3 - 8(\frac{1}{2}y)^2 + 4 = 0$

$\frac{1}{8}y^3 - \frac{3}{4}y^2 + 4 = 0$

$\times 8$

$y^3 - 6y^2 + 32 = 0$

essentially subbing in $\frac{1}{2}\alpha$

Let $f(x) = x^3 - 8x^2 + 4$, $f(\frac{1}{2}x)$ will have double roots if stretch in x-direction s.f. 2

85. How can you rotate conics by multiples of 90°?

Key point

To rotate a conic by $\frac{\pi}{2}$ radians, replace x by y and y by $-x$

To rotate a conic by π radians, replace x by $-x$ and y by $-y$

To rotate a conic by $\frac{3\pi}{2}$ radians, replace x by $-y$ and y by x

This relates nicely to matrices.

86. When is an integral improper?

Key point

An **improper integral** is a definite integral where either:

- one or both of the limits is $\pm\infty$
- the integrand (expression to be integrated) is undefined at one of the limits of the integral
- the integrand is undefined at some point between the limits of the integral.

87. How can you calculate the mean value of a function?

[redacted]

88. What are the derivatives of sinh and cosh?

$$\frac{d(\sinh x)}{dx} = \cosh x \qquad \frac{d(\cosh x)}{dx} = \sinh x$$

89. How should you differentiate inverse trigonometric and hyperbolic functions?

1.Let y = (inverse function).
2.Take the function of both sides.
3.Use the chain rule.
4.Use an identity.
5.Substitute either x or y back in.

Example 3

Differentiate $\operatorname{arcosh} x$ with respect to x

Let $y = \operatorname{arcosh} x$

Then $x = \cosh y$

$\frac{dx}{dy} = \sinh y$

So $\frac{dy}{dx} = \frac{1}{\sinh y}$

$= \frac{1}{\sqrt{\cosh^2 y - 1}}$

$= \frac{1}{\sqrt{x^2 - 1}}$

Use the chain rule: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Use the identity $\cosh^2 y - \sinh^2 y \equiv 1$

Use the identity $\cosh^2 x - \sinh^2 x \equiv 1$

90. What is a reduction formula and how can you find it?

A reduction formula for I_n is an equation that relates I_n to I_{n-1} and/or I_{n-2} which is used to reduce an integral.

Start by setting:

$$I_n = \int \cos^n x \, \mathrm{d}x.$$

Now re-write as:

$$I_n = \int \cos^{n-1} x \cos x \, \mathrm{d}x,$$

Integrating by this substitution:

$$\cos x \, \mathrm{d}x = \mathrm{d}(\sin x),$$

$$I_n = \int \cos^{n-1} x \, \mathrm{d}(\sin x).$$

Now integrating by parts:

$$\begin{aligned}
\int \cos^n x \, \mathrm{d}x &= \cos^{n-1} x \sin x - \int \sin x \, \mathrm{d}(\cos^{n-1} x) \\
&= \cos^{n-1} x \sin x + (n-1) \int \sin x \cos^{n-2} x \sin x \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, \mathrm{d}x - (n-1) \int \cos^n x \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n,
\end{aligned}$$

Reduction Formula

$\int x^n e^x \, dx$ for any positive integer n

$$\int \underbrace{x^n}_{u} \underbrace{e^x \, dx}_{dv} = x^n e^x - n \int x^{n-1} e^x \, dx$$

EX 2.6 Using a Reduction Formula

Evaluate the integral $\int x^4 e^x \, dx$

$$n = 4 \qquad \int x^4 e^x \, dx = x^4 e^x - 4 \int x^{4-1} e^x \, dx = x^4 e^x - 4 \int x^3 e^x \, dx$$

$$n = 3 \qquad \int x^4 e^x \, dx = x^4 e^x - 4 \left(x^3 e^x - 3 \int x^2 e^x \, dx \right)$$

$$\vdots$$

$$\int x^4 e^x \, dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24 e^x + c$$

91. What is the vector/cross product?

●**a** x **b** = **n** where n is a vector normal to both a and b.

$$\mathbf{a}\times\mathbf{b}=\left|\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}\right|$$

●From the textbook:

Key point

The **vector product** $\mathbf{a}\times\mathbf{b}$ of vectors **a** and **b** is defined as $\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\hat{\boldsymbol{n}}$ where θ is the angle between the vectors **a** and **b** and $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to both **a** and **b**

In 2D, the cross product gives the area of a parallelogram made of the 2 vectors. This is since you can express the 2 vectors as a matrix and the determinant (change in area from a unit square) is its area. Whereas in 3D, the cross product of 2 vectors is another vector whose length is the area of the parallelogram.
You can ensure you get the right direction by this: https://www.youtube.com/watch?v=zGyfiOqiR4s.

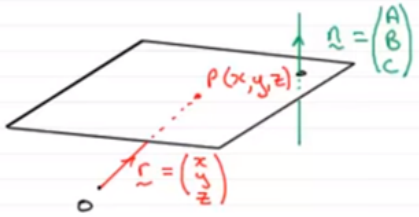
---

92. When are 2 vectors parallel?

When **a** x **b** = **0**

---

93. What is the Cartesian equation of a plane and how does it relate to another form?



$$\underset{\sim}{r}\cdot\underset{\sim}{n}=D$$

$$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}A\\B\\C\end{pmatrix}=D$$

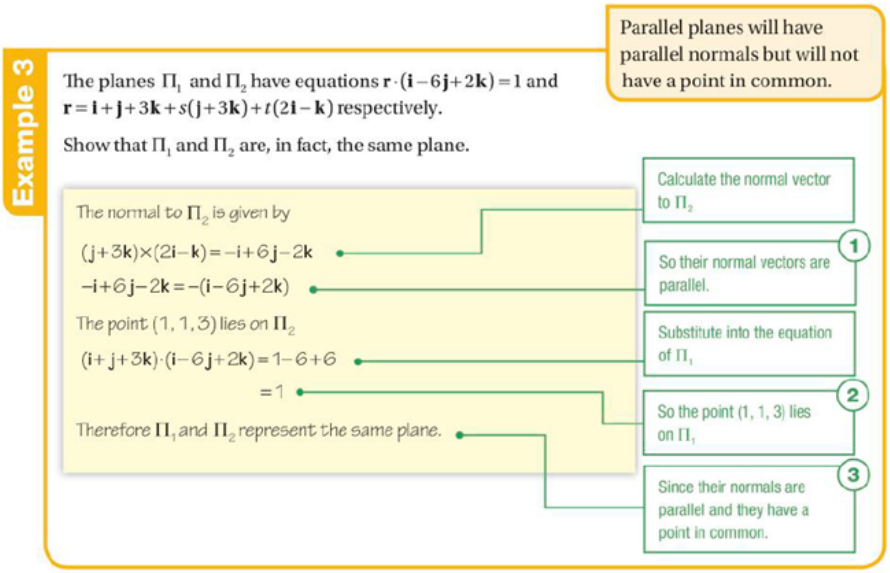
$$\therefore \boxed{Ax+By+Cz=D}$$

Find the cartesian equation of a plane that passes through the point $(2,-4,-5)$ and is perpendicular to the vector $3\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

---

94. How do you show 2 planes are the same?

1.Show they have the normals which are parallel (so planes are parallel).
2.Check if the point of one lies on the other.



95. How can you work out the shortest distance from a point to a plane?

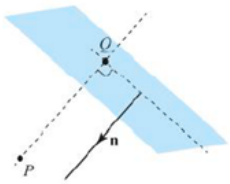
1.Form a line from that point (P) perpendicular to the plane.
2.Find where it intersects the plane (Q).
3.Calculate the distance PQ.
From the textbook:

You can find the shortest distance from a point to a plane.

Suppose you have the plane Π with equation $\mathbf{r}\cdot\mathbf{n}=p$ and the point P The shortest distance from P to the plane will be perpendicular to the plane. Therefore you need the length of $\overrightarrow{PQ}$

Q is the point of intersection of the plane and the line through P and Q. You can easily write down the equation of this line using the fact that it passes through P and has direction vector parallel to the vector $\mathbf{n}$ which is normal to the plane.

Once you've found the point of intersection Q you can find the length of the line segment PQ and hence the shortest distance from the point P to the plane Π



96. How can you find the angle between 2 planes?

Find the angle between their normals.

97. How can you convert from polar to Cartesian and vice versa?

Key point

To convert from polar to Cartesian, use

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

$$\cos\theta = \frac{x}{r} = \pm\frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\theta = \frac{y}{r} = \pm\frac{y}{\sqrt{x^2+y^2}}$$

[Choose + or – depending on whether r is positive or negative]

To convert from Cartesian to polar, use

$$x = r\cos\theta$$

$$y = r\sin\theta$$

---

98. What is a homogeneous differential equation?

When it equals **ZERO**.

$$\frac{d^2y}{dt^2} + P_1\frac{dy}{dt} + P_0 y = Q(t)$$

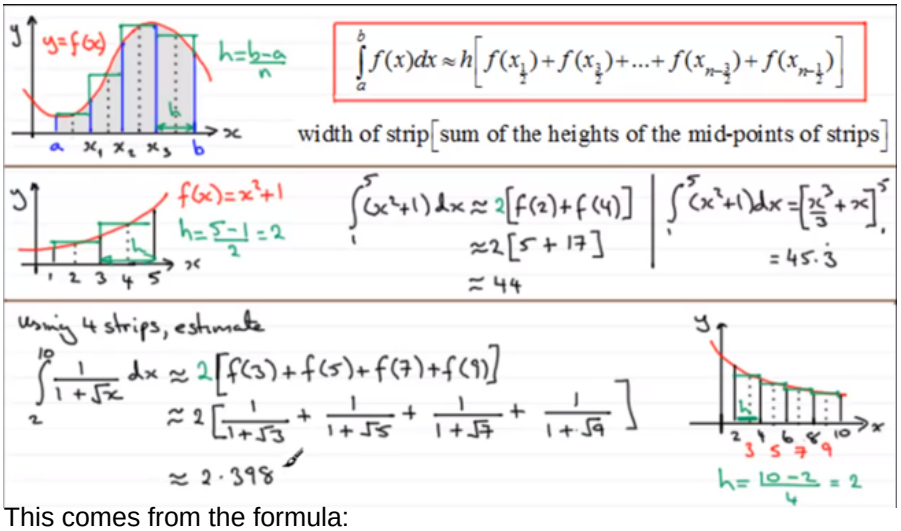
Non - homogeneous equation

$$\frac{d^2y}{dt^2} + P_1\frac{dy}{dt} + P_0 y = 0$$

Homogeneous equation

---

99. How does the mid-ordinate rule work?



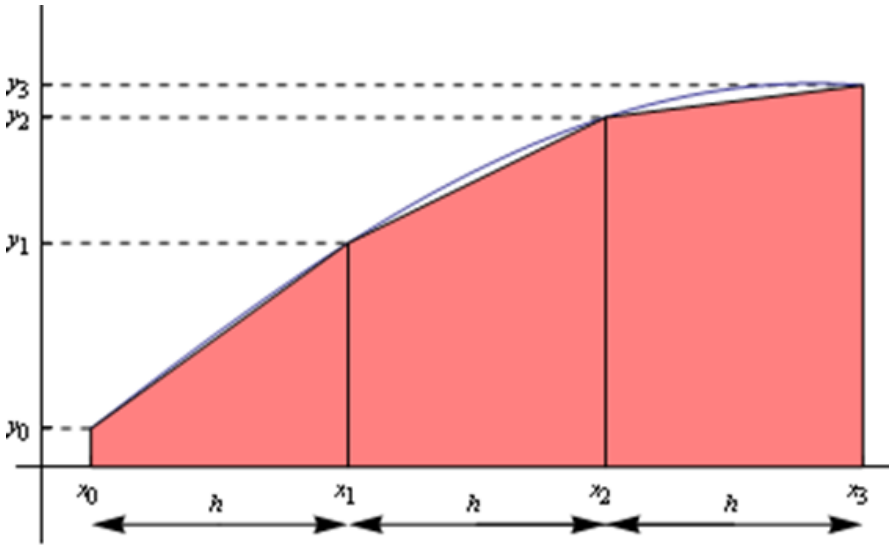
This comes from the formula:

$$\int_a^b y\,dx \approx h\left(y_{\frac{1}{2}} + y_{\frac{3}{2}} + \ldots + y_{n-\frac{3}{2}} + y_{n-\frac{1}{2}}\right), \text{ where } h = \frac{b-a}{n}$$

---

100. What is an ordinate? (with example)

- A y-value.
- The following has 4 ordinates (y-values) starting from y_0 to y_3.



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101. How can you that something is a factor of a determinant?

To show (x - y) is a factor of the determinant, we substitute x = y into it and show that the determinant is now equal to zero.

$$\begin{vmatrix} x & y & z \\ yz & zx & xy \\ y+z & x+z & x+y \end{vmatrix}$$

Becomes...

$$\begin{vmatrix} x & y & z \\ yz & yz & y^2 \\ y+z & y+z & 2y \end{vmatrix}$$

Which becomes (when col 1 - col 2):

$$\begin{vmatrix} 0 & y & z \\ 0 & yz & y^2 \\ 0 & y+z & 2y \end{vmatrix} = 0$$

Hence, we've also shown when 2 columns or rows are equal, the determinant is always 0.

---

102. How can you find invariant points / lines of invariant points OR invariant lines?

Invariant points / lines of points:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Invariant lines:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ mx + c \end{bmatrix} = \begin{bmatrix} x' \\ mx' + c \end{bmatrix}$$

---

103. How does modular arithmetic work using negatives?

-8 (mod 5) ≡ 2 (mod 5) and **NOT** ≡ 3 (mod 5)
Since you count up.

---

104. What is a pure strategy?

When both players play their play-safe strategy every time **THUS** minimising potential losses for each player.

---

105. How is the Simplex Algorithm performed? (with example)

1.Write the constraints and objective functions as equations in standard form using slack variables.
2.Transfer (1) to a simplex tableau where the slack variables form the basis.
3.Choose column with most negative coefficient in object row. This is the **PIVOT COLUMN**.
■This generally tells you the corresponding variables have to be increased to the optimal solution.
4.Choose the row giving the **smallest** θ-value. This is the **PIVOT**.
■This is found by dividing the each value by the **positive** numbers in the pivot column.
5.Divide the pivot row by the pivot
6.Combine suitable multiples of the new row with other rows to make all other values zeros in the pivot column.
■This replaces the basic variable, eg, it may go from s to x.
7.If there are **NO NEGATIVE** coefficients in the object row, the solution is optimal. Otherwise go to step 3.
A DETAILED EXAMPLE IS SHOWN BELOW:

$$\begin{aligned} &\text{Maximise} && P = 6x + 8y \\ &\text{subject to} && 4x + 3y \le 1500 \\ & && x + 2y \le 500 \\ & && x \ge 0, y \ge 0 \end{aligned}$$

Becomes…

$$\begin{aligned} &\text{Maximise} && P = 6x + 8y \\ &\text{subject to} && 4x + 3y + s = 1500 \\ & && x + 2y + t = 500 \\ & && x \ge 0,\ y \ge 0,\ s \ge 0,\ t \ge 0 \end{aligned}$$

Hence...

$$P - 6x - 8y - 0s - 0t = 0$$
$$4x + 3y + s + 0t = 1500$$
$$x + 2y + 0s + t = 500$$

and the simplex tableau is

Basic variable	P	x	y	s	t	Value	Row
	1	−6	−8	0	0	0	R1
s	0	4	3	1	0	1500	R2
t	0	1	2	0	1	500	R3

And thus the pivot is chosen…

Pivot column

Basic variable	P	x	y	s	t	Value	Row
	1	−6	−8	0	0	0	R1
s	0	4	3	1	0	1500	R2
t	0	1	2	0	1	500	R3

Pivot row

Pivot

Since the θ-value for s = 1500 / 3 = 500 and t = 500 / 2 so the final row is chosen.
Dividing by the pivot row by the pivot and getting rid of the remaining numbers in the pivot column yields…

Basic variable	P	x	y	s	t	Value	Row
	1	−2	0	0	4	2000	$R4 = R1 + 8 \times R6$
s	0	$2\frac{1}{2}$	0	1	$-1\frac{1}{2}$	750	$R5 = R2 - 3 \times R6$
y	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	250	$R6 = \frac{R3}{2}$

Repeating the change of basis for the x-column. We get…

Basic variable	P	x	y	s	t	Value	Row
	1	0	0	$\frac{4}{5}$	$2\frac{4}{5}$	2600	$R7 = R4 + 2 \times R8$
x	0	1	0	$\frac{2}{5}$	$-\frac{3}{5}$	300	$R8 = \frac{R5}{2\frac{1}{2}}$
y	0	0	1	$-\frac{1}{5}$	$\frac{4}{5}$	100	$R9 = R6 - \frac{1}{2} \times R8$

You now have $x = 300$, $y = 100$, $s = 0$, $t = 0$, giving $P = 2600$

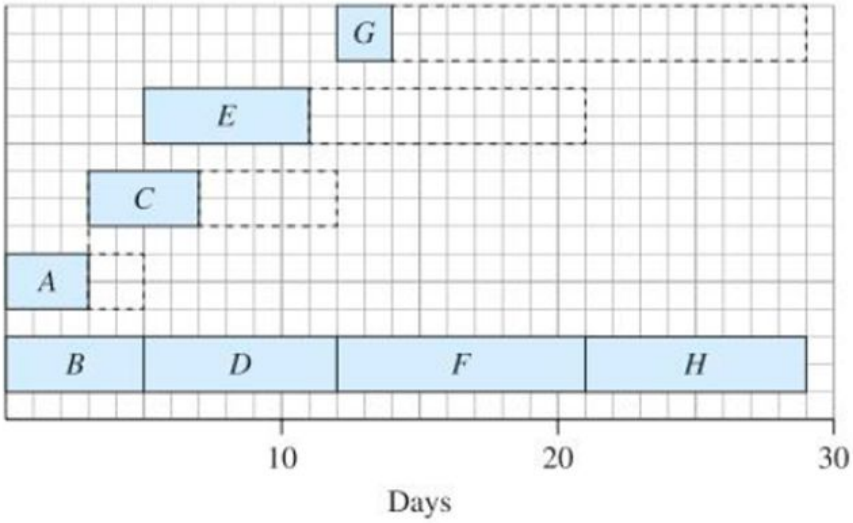
106. What are the limitations of critical path analysis?

1.Estimating competition times can be difficult - especially for new projects.
2.Defining links for large projects becomes complicated very quickly.

---

107. How are Gantt / Cascade Charts constructed?

1.Place the critical activities first.
2.Add every other activity on its own row **WITH** its earliest start and latest finished times in boundaries (within which they can be moved).
3.Put a ‘fence’ on the chart to show precedence, such as C cannot begin until A is complete.



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108. How should you set up a game theory problem involving Simplex?

1.Ensure the value you’re maximising is less than all the probability constants (eg, $v \leq p_2 + 2p_3$).
2.Ensure all probabilities sum to ≤ 1 (you can add a slack variable which will then equal zero).
3.All the variables are ≥ 0 as shown.

Let A play A_1, A_2 and A_3 with probabilities p_1, p_2 and p_3

Maximise $P = v$

subject to
$$v \leq 3p_1 + 2p_2 + p_3$$
$$v \leq p_2 + 2p_3$$
$$p_1 + p_2 + p_3 \leq 1$$
$$v, p_1, p_2, p_3 \geq 0$$

$$P - v = 0$$
$$v - 3p_1 - 2p_2 - p_3 + s = 0$$
$$v - p_2 - 2p_3 + t = 0$$
$$p_1 + p_2 + p_3 + u = 1$$

A's expected profit, P, is the value, v, of the game.

You replace $p_1 + p_2 + p_3 = 1$ with the inequality shown so that you can introduce a slack variable for the simplex tableau. The slack variable will be found to be zero.

Introduce slack variables.

This is because simplex requires us to be working in the positive region of a graph.

---

109. How can you make an initial flow for a network with minimum flows?

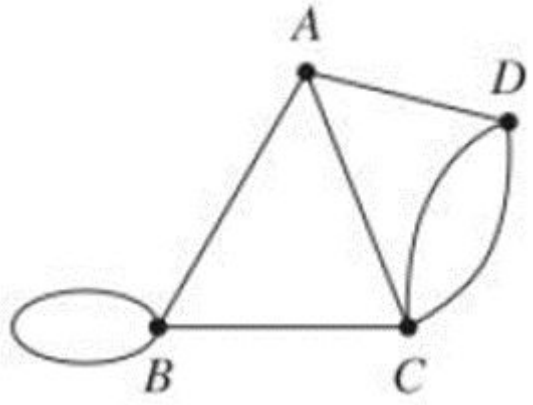
- Set the flow through each arc to the minimum.
- Adjust accordly so it works (in’s = out’s).

110.	What are the upper and lower bounds for the Travelling Salesperson Problem?	-The upper bound - a total distance for which there **DEFINITELY** exists a Hamiltonian cycle. -The lower bound - a minimum total distance for which a Hamiltonian cycle **MAY** exist.
111.	Define semi-Eulerian graph	A **CONNECTED** graph with **EXACTLY TWO ODD VERTICES**. *This means if you start at an odd degree vertex, you can visit every vertex yet you'll end up at the other odd degree vertex.*
112.	What is an inverse element?	$a * a^{-1} = e$ When value which combined with its 'inverse' gives the identity element. *For addition, where e = 0, the inverse of 5 is -5. For multiplication, where e = 1, the inverse of 5 is ■.*
113.	Define isomorphic graph	A graph that has the same number of vertices connected in the same way as another.
114.	Define Eulerian trial	A route which visits every edge **EXACTLY ONCE**.
115.	What is a play-safe strategy?	A strategy that gives the best guaranteed outcome regardless of what the opponent does. Or the strategy that minimises the potential loss for each player.

116. Define adjacency/incidence matrix

A representation of the numbers of routes between the vertices of a graph in a matrix.

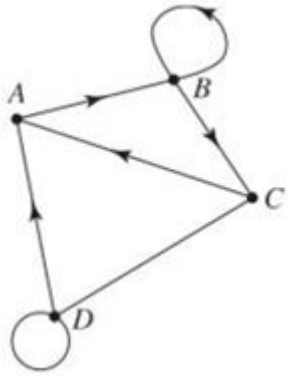
It is routes because…



	A	B	C	D
A	0	1	1	1
B	1	2	1	0
C	1	1	0	2
D	1	0	2	0

Has 2 direct routes of getting from B to B.

Whereas…



		To			
		A	B	C	D
From	A	0	1	0	0
	B	0	1	1	0
	C	1	0	0	1
	D	1	0	1	2

Only has 1 direct route of getting from B to B.

117. What does the maximum-flow minimum-cut theorem state?

That the flow through any network cannot exceed the value of any cut and so the maximum flow is equal to the minimum cut.

---

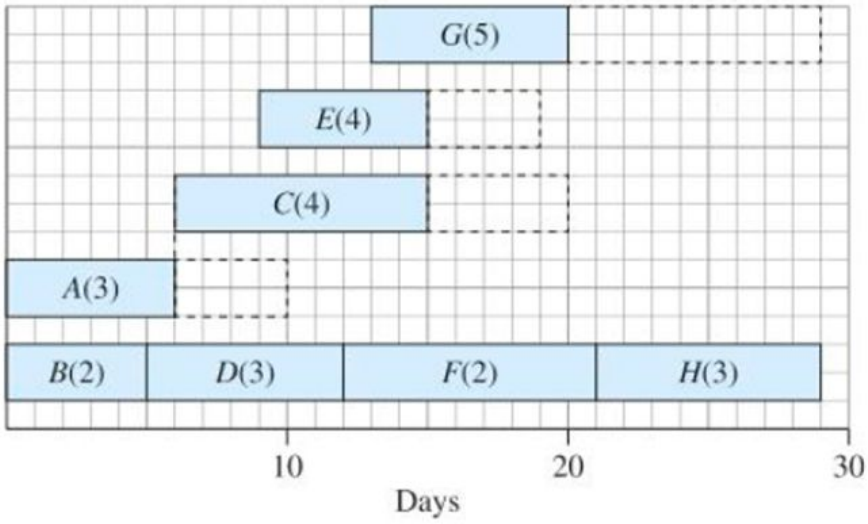
118. What is a subgroup?

A group that has the same binary operation as the parent group **AND** each element of the subgroup has its inverse in the subgroup.
A group is a subgroup of itself just like how 7 has a factor of 7.

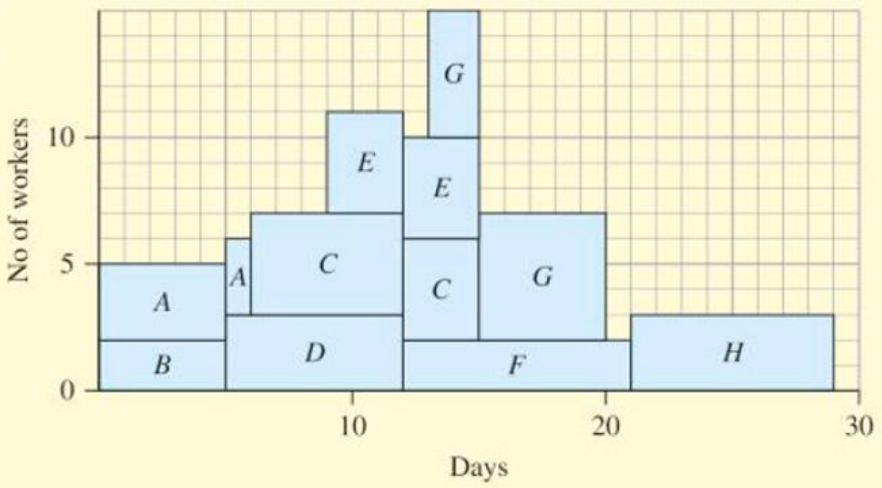
---

119. How are Resource Histograms levelled?

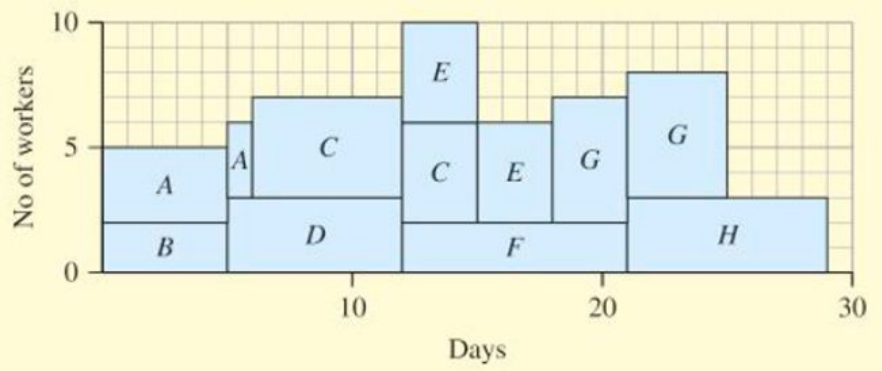
By looking at both your Gantt chart and resource histogram to see what can be moved.



Becomes...



Which can be levelled to...



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120. What is the aim of the Travelling Salesperson Problem?

To travel to **EVERY VERTEX** (to sell items in towns and cities) and return to the starting vertex in the least possible distance.
This is simply trying to find a Hamiltonian cycle of minimum length.

---

121. What is the value of a game to a player?

The payoff to the player if they use their best strategy.
This best strategy may be their play-safe strategy or an optimal mixed strategy.

---

122. What should you make sure of when finding an optimal mixed strategy using Simplex?

●All the entries are positive (as simplex requires all variables to be non-negative).
●If they aren't, add some constant to each entry then subtract this from the value once you've found it.
<u>This is demonstrated below:</u>

		B	
		B_1	B_2
A	A_1	1	−2
	A_2	0	−1
	A_3	−1	0

Add 2 to each table entry.

		B	
		B_1	B_2
A	A_1	3	0
	A_2	2	1
	A_3	1	2

All entries must be non-negative.

Let A play A_1, A_2 and A_3 with probabilities p_1, p_2 and p_3

Maximise $P = v$

subject to $v \le 3p_1 + 2p_2 + p_3$

$v \le p_2 + 2p_3$

$p_1 + p_2 + p_3 \le 1$

$v, p_1, p_2, p_3 \ge 0$

$P - v = 0$

$v - 3p_1 - 2p_2 - p_3 + s = 0$

$v - p_2 - 2p_3 + t = 0$

$p_1 + p_2 + p_3 + u = 1$

A's expected profit, P, is the value, v, of the game.

You replace $p_1 + p_2 + p_3 = 1$ with the inequality shown so that you can introduce a slack variable for the simplex tableau. The slack variable will be found to be zero.

Introduce slack variables.

Perform Simplex…

This gives $v = 1.5$ when $p_1 = \frac{1}{4}$, $p_2 = 0$ and $p_3 = \frac{3}{4}$

So the value of the game is −0.5

Subtract the 2 you added initially.

A's optimal strategy is to play A_1 and A_3 with probabilities 0.25, 0.75 and never play A_2

The value of the game to B is 0.5. Suppose B plays B_1 and B_2 with probabilities q_1 and q_2

If A plays A_1, the value to B is $-q_1 + 2q_2$, so $-q_1 + 2q_2 = 0.5$ [1]

Similarly, for A_2 $q_2 = 0.5$ [2]

and A_3 $q_1 = 0.5$ [3]

[1], [2] and [3] are all satisfied by $q_1 = 0.5$ and $q_2 = 0.5$, so B plays B_1 and B_2 with equal probability.

You may want to find the optimal mixed strategy for the other player too.

123.	Define Hamiltonian cycle (with example)	A route that... - Visits every vertex exactly once - Returns to the starting vertex - Without repeating any edges Example: ABCDEA for a pentagonal graph
124.	What is the general rule for multiplication in modular arithmetic?	$a \equiv c \pmod{\mathrm{m}}$ $b \equiv d \pmod{\mathrm{m}}$ $a \cdot b \equiv c \cdot d \pmod{\mathrm{m}}$
125.	How should you structure a linear programming problem?	1.State the the variables and what they represent. ■Eg, “let x = the number of frogs, y = the number of rocks” 2.Find constraints (including x, y ≥ 0). 3.Find the objective function. 4.Write the conclusion. ■ “Maximise/minimise [objective function] subject to [constraints]”
126.	What are cyclical groups? How do they relate to dihedral groups?	●Groups generated by a single element. ●For example, if only rotational groups are considered. The dihedral group is a cyclic group.
127.	Define simple graph	A graph with no loops and no repeated edges.
128.	What is the definition for zero-sum games?	For each pair of strategies (1) X’s gains + Y’s gains = 0 (1).
129.	When can the standard Simplex Algorithm be used?	1.You need to maximise the object function. 2.Every non-trivial constraint is an inequality using ≤. 3.All variables are ≥ 0 (including the slack). 4.The origin is a vertex of the feasible region.

130. What techniques can you use to show isomorphism between groups? What should you check for beforehand?

- Ensure the groups have the **same order**, they are either **both finite or not**, **both albeian or not**.
1. **Identity Element:** Identify the identity element in each group (this is one mapping).
2. **Order each element:** Corresponding elements will have the same order, check using a Cayley Table.
3. **Self-inverse elements:** Find self-inverse elements from each group.
4. Suggest a mapping and check.

Example:

A group G under the binary operation $\bullet$ has the Cayley table

$\bullet$	p	q	r	s
p	p	q	r	s
q	q	p	s	r
r	r	s	q	p
s	s	r	p	q

A second group $H = (\{i, -i, 1, -1\}, \times)$

Show that H is isomorphic to G and state the corresponding elements in each group.

The Cayley table for H is

$\times$	i	–i	1	–1
i	–1	1	i	–i
–i	1	–1	–i	i
1	i	–i	1	–1
–1	–i	i	–1	1

The identity element for H is 1 since $1 \times a = a \times 1 = a$

Elements 1 and –1 are self-inverse, so 1 corresponds to p and –1 to q

Reorder the columns and rows in the Cayley table:

$\times$	1	–1	i	–i
1	1	–1	i	–i
–1	–1	1	–i	i
i	i	–i	–1	1
–i	–i	i	1	–1

Hence i corresponds to r and –i corresponds to s

The groups are isomorphic.

The corresponding elements for each group are

$1 \leftrightarrow p$, $-1 \leftrightarrow q$, $i \leftrightarrow r$ and $-i \leftrightarrow s$

By reordering the columns and rows in the Cayley table, you can see that the pattern of entries here is the same as in the Cayley table for G

Note that this correspondence is not unique. Interchanging i and –i would also give the same pattern of entries as in G

This is used as they may not have the rows and columns arranged neatly to determine it.

---

131. How can you solve m x n zero-sum games?

1. Look for a stable solution. If none then mixed strategy.
2. Look for dominance and reduce size.
3. If m x 2 or 2 x n, sketch graphs. Otherwise use LP.

---

132. What is Lagrange's theorem?

For any finite group, G, the order of every subgroup of G divides the order of G.
This can be used to eliminate many potential subgroups.

---

133. What is the period (or order) of an element, x, of a group?

The smallest non-negative integer such that $x^n = e$ where e is the identity element (thus $n \in \mathbb{N}$).
Eg, an element may have period 2 since $a^2 = e$ and element c has period 3 since $c^3 = c(cc) = cd = e$.

---

134. What is Kuratowski's Theorem?

Graph is planar ⇔ **<u>does not</u>** contain a subgraph that is a subdivision of K_5 or of $K_{3,3}$.

---

135. What are the 4 group axioms?

1.Closure - the group is a set closed under the binary operation.
2.Has an identity element.
3.Each element has in inverse.
4.Binary operation is associative on all combinations of elements in the group.
Formal notation is…

Key point

A group $(S, \circledcirc)$ is a non-empty set S with a binary operation $\circledcirc$ such that:

$\circledcirc$ is closed in S

$\circledcirc$ is associative

there is an identity element such that $x \circledcirc e = e \circledcirc x = x$ for all x

each element has an inverse x^{-1} such that $x \circledcirc x^{-1} = x^{-1} \circledcirc x = e$

---

136. Define tree and a spanning tree

- A tree is a simple graph with no cycles.
- A spanning tree is a **<u>CONNECTED</u>** tree.

---

137. What's the **USUAL** aim of the Chinese Postman Problem?

To travel along **EVERY EDGE** (to deliver letters) and return to the starting vertex in the least possible distance.

---

138. How do you find the upper and lower bound of the Travelling Salesperson Problem?

By using the Nearest Neighbour Algorithm and Lower Bound Algorithm respectively.

---

139. How are solutions for the Simplex Algorithm given?

With the basis variables equalling the value of the row and the rest equalling 0 as shown:

Step 6

B.V.	P	x	y	z	s	t	u	Value	Row
	1	0	0	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	0	44	$R9 = R5 + \frac{R11}{3}$
y	0	0	1	$\frac{4}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	0	16	$R10 = R6 - \frac{R11}{3}$
x	0	1	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	0	12	$R11 = \frac{R7}{1\frac{2}{3}}$
u	0	0	0	$3\frac{2}{5}$	$\frac{1}{5}$	$-\frac{3}{5}$	1	18	$R12 = R8 - R11$

Step 7 There are no negative numbers in the top row so the solution is optimal.

The optimal solution is $y = 16, x = 12, u = 18, z = 0, s = 0, t = 0$, giving $P = 44$

140. Define non-Eulerian graph

A **CONNECTED** graph with **MORE THAN TWO ODD VERTICES**.

141. How can you apply the standard Simplex Algorithm when 'the objective function is being minimised' or 'an inequality involves ≥ instead'?

1.

Key point

Minimising the objective function C is equivalent to maximising the objective function $P = -C$

2. You rewrite the inequality using ≤ by multiply through by -1.
Both of these are applied below:

Example 3

Minimise $C = 3x - 4y - 3z$

subject to $x + y - z \geq -2$

$x + 2y + z \leq 3$

$x \geq 0, y \geq 0, z \geq 0$

Maximise $P = -3x + 4y + 3z$

subject to $-x - y + z \leq 2$

$x + 2y + z \leq 3$

$x, y, z \geq 0$

First restate the problem.

$P + 3x - 4y - 3z = 0$

$-x - y + z + s = 2$

$x + 2y + z + t = 3$

$x, y, z, s, t \geq 0$

Write in standard form with slack variables.

P	x	y	z	s	t	Value	Row
1	3	−4	−3	0	0	0	R1
0	−1	−1	1	1	0	2	R2
0	1	2	1	0	1	3	R3

Complete the simplex tableau.

142. How can you show there is no stable solution? If so, what must be done?

		B				
		K	Q	J	Row minimum	
A	K	5	−4	2	-4	
	Q	3	1	4	1	← max = 1
	J	2	3	−1	-1	
Column maximum		5	3	4		
			↑ min = 3			

●By showing that the minimax (worst case for A) ≠ maximin (worst case for B).
■ Since stable solution exists ⇔ maximin = minimax.
●Use an optimal mixed strategy where you play each move with some probability.
This stable solution would mean that neither player can gain by changing their play-safe strategy.

---

143. Give a proof for the addition rule of modular arithmetic

$$a \equiv b \pmod{n}, c \equiv d \pmod{n}$$

$$\Rightarrow a = b + kn, c = d + mn$$

$$\Rightarrow a + c = (b + kn) + (d + mn)$$

$$= (b + d) + (k + m)n$$

$$= b + d \pmod{n}$$

Where k, m, n are some integers.

---

144. No. of Hamiltonian Cycles

- For distinct cycles in which you consider ABCA to equal ACBA, it is:

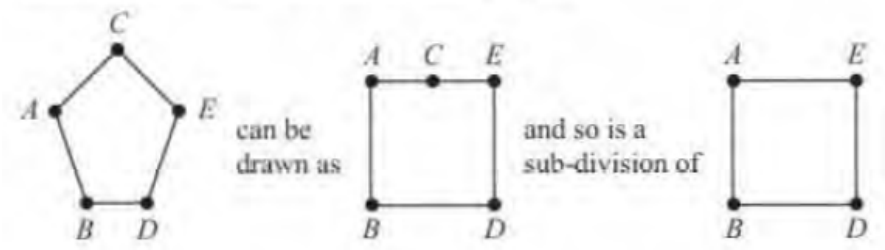
$$\frac{(n-1)!}{2}$$

- So for non-distinct, it is:

$$(n-1)!$$

*This is because if you had 4 vertices (in ABCDA), you would have 3! ways of arranging (A**BCD**A). However, for any start it would be 4!.*
Both ABCDA and ADCBA have the same shape but opposite directions. If you treat them as equal, you'll get half as many cycles.

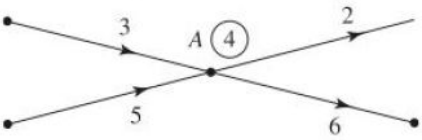
145.	What 3 things should you remember when doing Prim's Algorithm on a table?	1.To delete vertices as you as along so you don't form a cycles. 2.Numbering the order of vertices as you go along. 3.Looking down/across all columns/rows after adding one.
146.	When is and isn't a graph Hamiltonian-connected?	- Is when there exists a Hamiltonian cycle for every vertex. - Isn't when you're forced through one vertex twice or thrice. *If there exists a cycle for one then there must exist a cycle for them all (as you can merely reorder the letters).*
147.	Define edge/arc	A line connecting two vertices.
148.	Define network/weighted graph	A graph with weighted edges (i.e., edges with numbers).
149.	Define order/degree of vertex	The number of edges starting and finishing at the vertex.
150.	Define loop	An edge starting and finishing at the same vertex.
151.	Define weight	A real world value assigned to an edge (e.g., metres).
152.	Define cycle	A closed path.
153.	TEPEV	**T**rial **E**dge **P**ath **E**dge **V**ertex. A trial doesn't repeat any edge. A path doesn't repeat any edge or vertex.
154.	Define connected graph	Every node is connected to the graph. *Therefore, it is possible to get from any node to any other node (however not necessarily directly).*
155.	Define complete graph (K_n)	A graph where every vertex is connected to every other vertex directly.
156.	What is the no. of edges on a complete graph?	The number of edges on a complete graph is defined by: $$\frac{n(n-1)}{2}$$ *A complete graph has n - 1 outgoing edges and n edges in total thus you'd be tempted to say there are n(n - 1) edges. However, every edge is counted twice because every edge going out a vertex is going into another thus you divide by 2.*

157.	Define Eulerian graph	A **CONNECTED** graph with **ONLY EVEN VERTICES**. *This means you can start and end at the same point (because every vertex has an edge going into and out of it).*
158.	Define digraph	A graph with one or more directed edges.
159.	Define bipartite graph	Two sets of vertices with edges that can only connect from one set to the other.
160.	Define minimum spanning tree	A spanning tree of minimum weight.
161.	Define planar graph	A graph that can be (re)drawn so that none of its arcs cross. *This can be useful for insulated wires on microchips.*
162.	Define complement of a graph	A graph of all missing edges required to make another graph complete.
163.	Define subgraph	Part of a graph.
164.	Define subdivision	Where you split an edge into two edges by adding an extra vertex. 
165.	What is Euler's Formula and when does it hold true?	Holds true for any **CONNECTED PLANAR** graph (e.g., many 3D solids). $$V + F - E = 2$$ *This can be remembered by using the vertices, faces, and edge of a cube.*
166.	What are saturated arcs?	Arcs that are at full capacity.
167.	What should you do when there are multiple sinks and sources?	Create super sinks and super sources with sensible capacities.

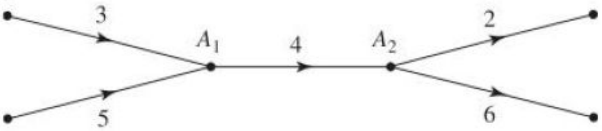
168. What should you do if you come across a node with a restricted capacity?

Add in an extra capacitated arc and two nodes to replace the node.

For example, if this node has a capacity of 4



you redraw it as



Notice that all inflows to A go to the first node A_1 and all outflows emerge from the second node A_2

169. What is the value of the cut for a network with minimum flows?

The sum of the upper capacities from S to T minus the sum of the lower capacities from T to S.

170. What are the steps of Prim’s Algorithm?

1.Choose a starting vertex.
2.Connect this vertex to the nearest unconnected vertex by the shortest edge.
3.For any already connected vertex, add the shortest edge that connects a new vertex.
4.Repeat step 3 until all the vertices are connected.

171. What are the steps of Kruskal's Algorithm?

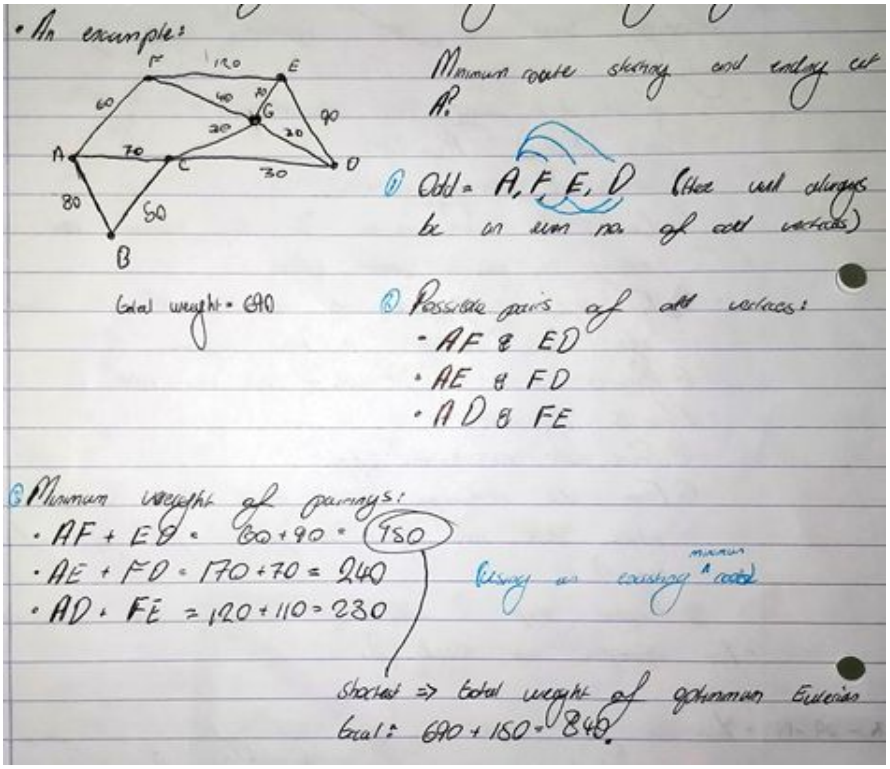
1.Start with the shortest edge.
2.From the remaining edges, select the next shortest edge that does not form a cycle.
3.Repeat step 2 until all the vertices are connected.
Or this can be done by listing the arcs in ascending order of weight then following the method above.

172. What does the Chinese Postman Problem usually require?

The graph to be Eulerian.

173. What are the steps of the Chinese Postman Problem?

1.Write down the odd vertices.
2.Pair the odd vertices and find the shortest distance between each pair (n.b. to focus here!)
3.Find the shortest odd pairings to make the graph Eulerian.



174. What can you split the Travelling Salesperson Problem into?

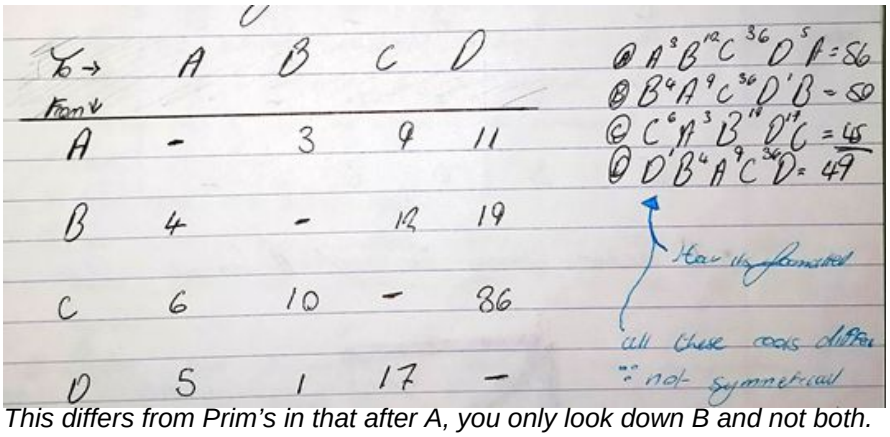
Finding a upper bound and a lower bound between which the optimal solution lies.

175. What must the graph be in the Travelling Salesperson Problem and how can you make it this?

-Complete.
-By adding indirect edges of minimum length (like in the Chinese Postman Problem).

176. What are the steps for the Nearest Neighbour Algorithm?

1.Choose a starting vertex on the table.
2.From this current vertex, go to the nearest unvisited vertex.
3.Repeat step 2 until all the vertices have been visited.
4.Return to the starting vertex.



To → / From ↓	A	B	C	D
A	-	3	9	11
B	4	-	12	19
C	6	10	-	36
D	5	1	17	-

This differs from Prim's in that after A, you only look down B and not both.

177.	What are the steps for the Lower Bound Algorithm?	1.Delete the stated vertex and all the edges connected to it. 2.Find the minimum spanning tree of the remaining vertices (using Prim's from a table at any start). 3.Connect the two shortest edges from the deleted vertex to the minimum spanning tree. *You have to connect two edges from the deleted vertex to try and make it a Hamiltonian cycle.*
178.	What should you be note for the Nearest Neighbour Algorithm?	Starting at different vertices may give different no.'s which is why you may have different upper bounds.
179.	How many edges are there in a spanning tree of n vertices and why?	n - 1 edges. For each new vertex you add to the spanning tree, you increase the original tree (of 2 vertices and 1 edge) by 1 edge.
180.	When is a set closed under a binary operation?	If for any two no.'s from the set, the result * returns a no. of the set.
181.	What is the set N?	The set of counting numbers: {1, 2, 3, ...}
182.	What is the set Z?	The set of real integers: {..., -1, 0, +1, +2, ...}
183.	What is the set Q?	The set of rational numbers.
184.	What is commutative in binary operations?	When a * b = b * a. *Things such as addition are commutative where the order of operation doesn't matter.*
185.	What is associative in binary operations?	When (a * b) * c = a * (b * c).
186.	What is an identity element?	e * a = a * e = a A value for which under an operation * leaves the output value unchanged from the input. *For addition, e = 0. For multiplication, e = 1.*

187. What are the general rules for addition and subtraction in modular arithmetic?

$$a \equiv c \pmod{\mathrm{m}}$$
$$b \equiv d \pmod{\mathrm{m}}$$
$$a + b \equiv c + d \pmod{\mathrm{m}}$$

Use the opposite signs for subtraction. Essentially, their residues are combining

---

188. What is the general rule for exponentiation in modular arithmetic?

$$a^e \equiv b^e \pmod{\mathrm{m}}$$

E.g., $7 \equiv 1 \pmod 3$ so $7^6 \equiv 1^6 \equiv 1 \pmod 3$.
Think of this as repeated multiplication. Multiply both sides by the same thing.

---

189. Define Eulerian cycle

A path which visits every edge **EXACTLY ONCE** and returns to the starting vertex.

---

190. What is the order of a group?

The number of elements within the group.

---

191. What are dihedral groups? How are they denoted?

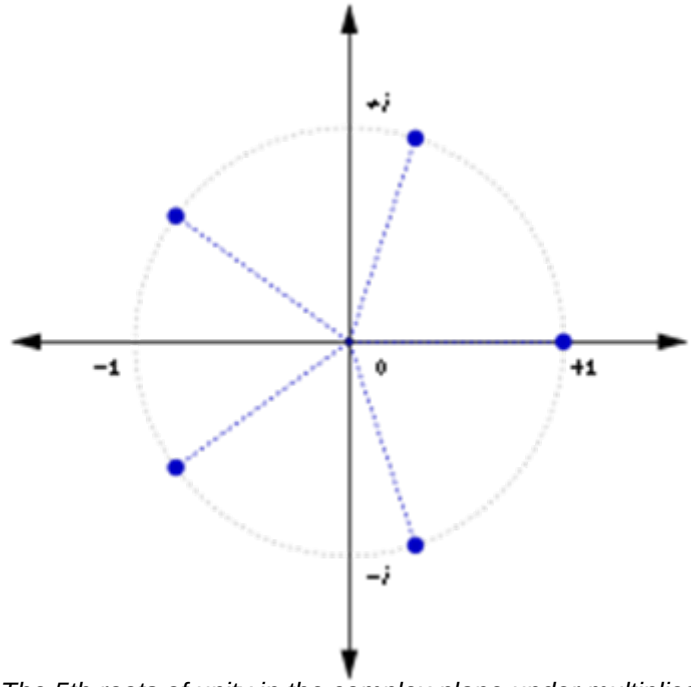
- A group of all symmetries of a regular n-gon.
- Denoted by D_n where n is number of sides in the regular n-gon.

These symmetries may include rotational or mirror.

---

192. What is a generator?

●An element that when applied to itself, can generate all other elements in the group.
●This group is a cyclic group.



The 5th roots of unity in the complex plane under multiplication form a group of order 5. Each non-identity element by itself is a generator for the whole group.

---

193. What is a proper subgroup?

Any subgroup that is not the parent group itself.

---

194. What is a trivial subgroup?

A group containing only the identity element of the parent group.

---

195. What is an abelian group?

A group that also has a commutative binary operation.

---

196. When are two groups isomorphic? How is this denoted?

●When there is a **one-to-one mapping** which associates the elements of the first with those of the second.
●A ≅ B.
This is useful since the results proved true for the first holds true for all isomorphic groups.

---

197. What is the triangle inequality?

Key point

The triangle inequality states that the sum of the lengths of any two sides of a triangle cannot be less than the length of the third side.

$AB + BC \geq AC$

---

198. What is a basic variable under simplex?

A variable that has a 1 in its row **WHERE** the rest of the column of this 1 cell is filled with 0's.
An example is shown below:

Basic variable	P	x	y	s	t	Value	Row
	1	−2	0	0	4	2000	$R4 = R1 + 8 \times R6$
s	0	$2\frac{1}{2}$	0	1	$-1\frac{1}{2}$	750	$R5 = R2 - 3 \times R6$
y	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	250	$R6 = \frac{R3}{2}$

199. What is a critical activity and a critical path?

●Critical activity - an activity with 0 float.
●Critical path - a sequence of critical activities.
There may be more than 1 critical path.

200. Give a problem with the Travelling Salesperson Heuristic Solution

Doesn't account for if you wanted to visit a vertex twice.

201. What are the best upper and lower bounds for the Travelling Salesperson Problem and why?

-The lowest upper bound because it will definitely exist.
-The greatest lower bound because as it cancels all the other lower bounds.
This is because if a lower bound forms a tour then that tour will be the optimal solution. If none of the possible lower bounds form a tour, the optimal tour must be greater than the best lower bound.

202. How can you find the P(Type II Error)?

1. Find the critical values.
2. Determine the acceptance / fail to rejection of H_0 region (eg, if 10% level where P(X <= 31) = 0.08552, acceptance is X >= 32 under new mean)
3. Find the probability of lying within the acceptance region under the new mean.

203. What assumptions are required for the validity of a t-test?

1.The sample is random.
2.The sample is taken from a normally distributed population.

204. What is the correct interpretation of a p%-confidence interval? What is it generated from?

●The population mean, although unknown is fixed so it's meaningless to attach a probability to it.
●**Correct Interpretation:** If you take repeated samples and form many confidence intervals, you expect p% of them to contain μ.
●It's generated from a sample.

205. What does the central limit theorem state?

The distribution of your sample means tends to the normal distribution (yet with a lower standard deviation than the population) as the sample size increases.

206. How can you calculate the test statistic for a t-test?

$$\frac{(\bar{x} - \mu)}{\left(\frac{S}{\sqrt{n}}\right)}$$

Where S^2 is an unbiased estimator of σ^2 which is found using…

$$S^2 = \frac{\sum (x_i - \bar{x})^2}{n - 1}$$

---

207. What is the equation for the expectation of a discrete uniform distribution?

$$E(T) = \frac{n + 1}{2}$$

---

208. What 4 conditions are needed for something to follow a Poisson distribution?

- Events occur at a constant average rate.
- Events occur randomly and independently.
- There can only be a discrete number of events.
- Events occur singly.

Events occurring singly means that the probability of two events occurring in the same narrow interval is negligible.
Furthermore, independent and random would mean no queues present (eg, cars on a motorway must be able to overtake freely and calls to a switch board must be dealt with instantly / no queuing).

---

209. How can t-distributions be used to generate a p% confidence interval? And when?

$$\bar{x} - t \times \frac{s}{\sqrt{n}} < \mu < \bar{x} + t \times \frac{s}{\sqrt{n}}$$

Where t is from a t-distribution of n-1 degrees of freedom:

$$t = t_{n-1}^{-1}\left(\frac{1+p}{2}\right),$$

When sample size is small and population variance is unknown.
Example:

A sample of size 16 is taken, whose mean is 13.6 and whose standard deviation is 20.4, in order to generate a 95% confidence interval for the mean of the population. Find the confidence interval.

For $p = 95\%$ and with $16 - 1 = 15$ degrees of freedom, $t_{15}^{-1}(0.975) = 2.13$
The interval $13.6 - 2.13 \times \frac{20.4}{\sqrt{16}} < \mu < 13.6 + 2.13 \times \frac{20.4}{\sqrt{16}}$
simplifies to $2.74 < \mu < 24.46$

210. How can the mean and variance for exponential distribution be derived?

$$E(X) = \int_0^\infty x f(x) dx = \lambda \int_0^\infty x e^{-\lambda x} dx$$
$$= \lambda \left\{ \left[x \frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda} dx \right\}$$
$$= \lambda \left\{ 0 - \left[\frac{e^{-\lambda x}}{\lambda^2} \right]_0^\infty \right\} = \lambda \left(\frac{1}{\lambda^2} \right)$$
$$= \frac{1}{\lambda}$$

$$E(X^2) = \int_0^\infty x^2 f(x) dx = \lambda \int_0^\infty x^2 e^{-\lambda x} dx$$
$$= \lambda \left\{ \left[x^2 \frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty + \int_0^\infty \frac{e^{-\lambda x}}{\lambda} 2x dx \right\}$$
$$= \lambda \left\{ 0 - 2 \left(\left[x \frac{e^{-\lambda x}}{\lambda^2} \right]_0^\infty - \int_0^\infty \frac{e^{-\lambda x}}{\lambda^2} dx \right) \right\}$$
$$= \lambda \left\{ 0 - 2 \left(0 + \left[\frac{e^{-\lambda x}}{\lambda^3} \right]_0^\infty \right) \right\} = \lambda \frac{2}{\lambda^3} = \frac{2}{\lambda^2}$$
$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 = \frac{1}{\lambda^2}$$

The derivation for E(X) requires integration by parts once. For $E(X^2)$, it's required twice.

---

211. What is a Type I and Type II error?

- Type I - rejecting H_0 when in fact it is true (false positive)
- Type II - failing to reject H_0 when it is in fact false (false negative)

Outcomes	Underlying truth	
	H_0 false	H_0 true
Reject H_0	Correct	Type I error
Don't reject H_0	Type II error	Correct

---

212. What should you do under a $\chi 2$ test if any expected frequencies ≤ 5?

1. Group data **appropriately**.
2. Find the new degrees of freedom.
3. Find the test statistic (perhaps using Yates' Continuity Correction).

NOTE THIS IS LESS THAN OR EQUAL TO 5.

---

213. How is a p%-confidence interval generated by a sample size n (where n is large)?

$$\bar{x} - z \times \frac{s}{\sqrt{n}} < \mu < \bar{x} + z \times \frac{s}{\sqrt{n}}$$

Where z is calculated from p.

---

214. What is a probability mass function?

A function giving exact values for discrete random variables.

---

215. What happens as the sample size of a t-test increase?

The t-distribution approaches a normal distribution.

---

216. What two conditions must hold true for all pdfs (probability density functions)?

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad f(x) \geq 0, \forall\, x$$

---

217. What are the two outcomes under exponential distribution? (ie, the cdf)

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Where F(x) is the probability of an event occurring in the time x.

Where λ is the mean no. of occurence in a unit time.

---

218. What is expected value?

The theoretical mean of a random variable.

Hence it's not exactly based on sample data. The more tests you perform, the closer the mean of all your outcome will become to the expected value.

---

219. When is a t-test suitable?

1.If the population variance is unknown, t-test is suitable (if it is then use normal distribution).

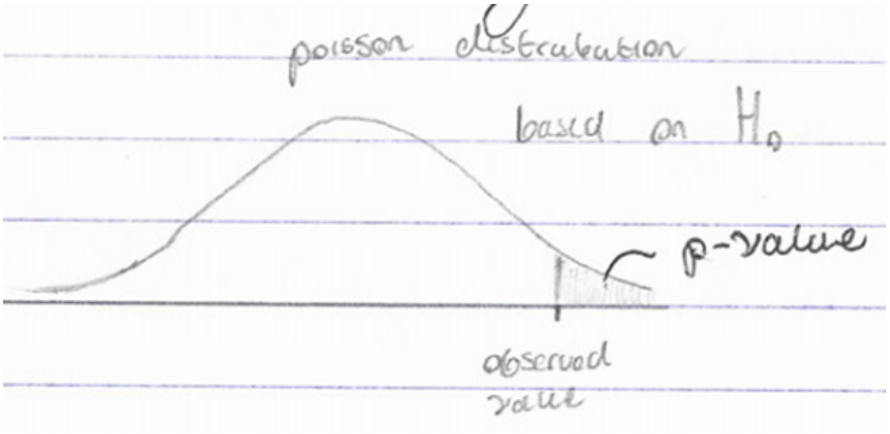
2.If the sample size is small (ie, $n \leq 30$).

Unless explicitly stated otherwise, you can always use a t-test. It's just a lack of technology that made it the case of using z-distribution at this cut-off as convention early on.

---

220. What is the p-value?

The probability of getting ≥ or ≤ a value given the null hypothesis is true (in other words, the probability of the data is at least as extreme as observed).



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221. What is a continuous uniform distribution / rectangular distribution?

The area is (b - a)h = 1 so h = 1/(b-a).

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & otherwise \end{cases}$$

---

222. What is the pdf of the exponential distribution?

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

This is by differentiating 1 - $e^{-\lambda t}$.

---

223. How is the expected value calculated under DRVs?

$$E(X) = \sum_{\forall x} (x \cdot P(X = x))$$

-Multiply each value by its probability of occurring.

$$E[g(X)] = \sum_{\forall x} (g(x) \cdot P(X = g(x)))$$

This is how $E(X^2)$ or $E(X^{-1})$ works

---

224. What is the variance of… X ± Y? When is this the case?

Var(X) + Var(Y)
This is as long as they're independent.

---

225. What is the equation for the variance of a discrete uniform distribution?

$$Var(T) = \frac{n^2 - 1}{12}$$

---

226. Prove the equation for the variance of discrete uniform distribution

$$\begin{aligned} Var(T) &= E(T^2) - \left(E(T)\right)^2 \\ &= \frac{1}{n} \cdot \sum_{k=1}^{n} k^2 - \left(\frac{1}{n} \cdot \sum_{k=1}^{n} k\right)^2 \\ &= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} - \left(\frac{1}{n} \cdot \frac{n(n+1)}{2}\right)^2 \\ &= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 \\ &= (n+1)\left[\frac{2n+1}{6} - \frac{n+1}{4}\right] \\ &= (n+1)\left[\frac{n-1}{12}\right] = \frac{n^2-1}{12} \end{aligned}$$

---

227. What is H_0 and H_1?

-H_0 - null hypothesis.
-H_1 - alternative hypothesis (what you're testing for).

---

228. What is the expectation of… aX ± b?

E(aX + b) = aE(X) ± b

---

229. What is the expectation of… X ± Y? When is this the case?

E(X ± Y) = E(X) ± E(Y)
This is as long as they're independent.

---

230. What is the formula for variance (σ^2)?

$$\sigma^2 = E(X^2) - (E(X))^2$$

"The mean of the squares minus the square of the mean."

---

231. What is the variance of… aX+b?

a^2Var(X)
Adding a constant doesn't affect the variability of X.

---

232. Prove the equation for the expectation of discrete uniform distribution

$$\begin{aligned} E(T) &= \left(1 \cdot \frac{1}{n}\right) + \left(2 \cdot \frac{1}{n}\right) + \cdots + \left(n \cdot \frac{1}{n}\right) \\ &= \frac{1}{n}(1 + 2 + \cdots + n) \\ &= \frac{1}{n}\sum_{k=1}^{n} k = \frac{1}{n} \cdot \frac{n(n+1)}{2} \\ &= \frac{n+1}{2} \end{aligned}$$

---

233. What is a cdf (cumulative distribution function)?

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)\, dx$$

This is useful for finding medians and quartiles (e.g., F(x) = 0.5).

---

234. How is the evidence against H_0 measured?

-By using the p-value.
-The smaller it is, the greater the evidence **UNTIL** it's smaller than the significance value at which you reject H_0.
The greater the sample size, the more power the p-value has.

---

235. What is P(Type I Error) equal to in the case of continuous and the case of discrete?

- Continuous - α (the significance value).
- Discrete - the probability of something occur just below the significance value.
Reducing this value does increase the chances of a Type II error so you need a balance.

---

236. How are expected values calculated for a $\chi 2$ test?

$$\frac{\text{Row Total} \cdot \text{Col Total}}{\text{Grand Total}}$$

---

237. What is Yates' Continuity Correction? And when is it applied?

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

$$\chi^2 = \sum \frac{(|O_i - E_i| - 0.5)^2}{E_i}$$

When there is 1 degree of freedom (meaning 2x2 table).

---

238. What does $\chi 2$ test tell you?

How far apart the observed and expected values are from each other.
If it is above or below the critical value for given significance level and degrees of freedom, then you can say that there must be some association.

---

239. How are degrees of freedom for $\chi 2$ calculated?

(m - 1)(n - 1) where m and n are the number of rows and columns respectively.

---

240. How do you work out the power of a test?

1 - P(Type II Error)
This is, obviously, the probability of not making a Type II error.

---

241. What is exponential distribution used to do?

Describe the waiting times between Poisson events.

---

242. What is the memoryless function of the exponential distribution?

Say an event hasn't occurred after 30 seconds after t = 0. The probability of an event occurring in at least 10 seconds equals observing the event 10 seconds after t = 0.

---

243. How can 2 Poisson Distributions be combined?

Key point

If X and Y are two independent random variables that can be modelled using the Poisson distribution with distributions given by $X \sim \mathrm{Po}(\lambda_x)$ and $Y \sim \mathrm{Po}(\lambda_y)$, then $X + Y \sim \mathrm{Po}(\lambda_x + \lambda_x)$

This leads to result that m x X = Po(m x λ) where m is a positive rational number.

---

244. How can you find the mode of a CRV?

1.Differentiate the **PROBABILITY DENSITY FUNCTION**.
2.Find where it equals 0.
While the probability of getting this exact value is negligible, a lot of values will occur around here.

---

245. What is the mean of a continuous uniform distribution and its derivation?

$$E(X) = \int_a^b x f(x)\ dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$
$$= \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} = \frac{a+b}{2}$$

---

246. How are degrees of freedom calculated for t-tests?

It has (n - 1) degrees of freedom where n is number of datapoints.
The intuitive understanding is, if we have 6 numbers (of which we know 5) and know the mean then we can easily find the 6th. Ultimately, only 5 of these numbers contribute to the standard error.

---

247. What is a standard error?

Key point

The standard deviation of the sample mean, $\frac{s}{\sqrt{n}}$, is called the **standard error**.

Which is…

$$\bar{X} \approx N\left(\mu, \frac{\sigma^2}{n}\right)$$

---

248. What is the Var(g(x)) and E(g(x)) of a CRV?

Key point

If X is a random variable with probability density function $f(x)$, then the mean value and variance of $g(X)$ are given by

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

$$\begin{aligned} Var(g(X)) &= E(\{g(X)\}^2) - \{E(g(X))\}^2 \\ &= \int_{-\infty}^{\infty} \{g(x)^2\}f(x)dx - \{E(g(X))\}^2 \end{aligned}$$

Essentially, it's each value multiplied by its probability of occurring.

Example:

A continuous random variable X has probability density function $f(x) = \frac{x+1}{6}$ for x between 1 and 3
Find the expected value of

a $2X^2$

b $X + \frac{15}{X^3}$

a $E(2X^2) = \int_1^3 2x^2\left(\frac{x+1}{6}\right)dx$

$= \frac{1}{3}\int_1^3 (x^3 + x^2)dx$

$= \frac{1}{3}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_1^3$

$= 9.56$ (2 dp)

b $E\left(X + \frac{15}{X^3}\right) = \int_1^3 \left(x + \frac{15}{x^3}\right)\left(\frac{x+1}{6}\right)dx$

$= \frac{1}{6}\int_1^3 (x^2 + x + 15x^{-2} + 15x^{-3})dx$

$= \frac{1}{6}\left[\frac{x^3}{3} + \frac{x^2}{2} - 15x^{-1} - \frac{15}{2}x^{-2}\right]_1^3$

$= \frac{44}{9}$

Or use $E\left(X + \frac{15}{X^3}\right) = E(X) + E\left(\frac{15}{X^3}\right)$

---

249. What is equal to the mean under Poisson?

Mean (λ) = median = mode = variance.

λ is average rate of occurrence in a given interval (eg, average number of events an hour).

---

250. What is the variance of a continuous uniform distribution and its derivation?

$$E(X^2) = \int_a^b x^2 f(x)\ dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$
$$= \frac{1}{b-a} \cdot \frac{b^3 - a^3}{3}$$
$$= \frac{1}{b-a} \cdot \frac{(b-a)(b^2+ab+a^2)}{3}$$
$$= \frac{b^2+ab+a^2}{3}$$

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4}$$
$$= \frac{(a-b)^2}{12}$$