Inequalities

Worked Example

$$\frac{3x-5}{1-x} \geq x-3$$

must multiply by $(1-x)^2$ to ensure the no. is positive & sign doesn't have to be flipped

$$(3x-5)(1-x) \geq (x-3)(1-x)^2$$

$$\Rightarrow 3x - 3x^2 - 5 + 5x \geq (x-3)(1-2x+x^2)$$

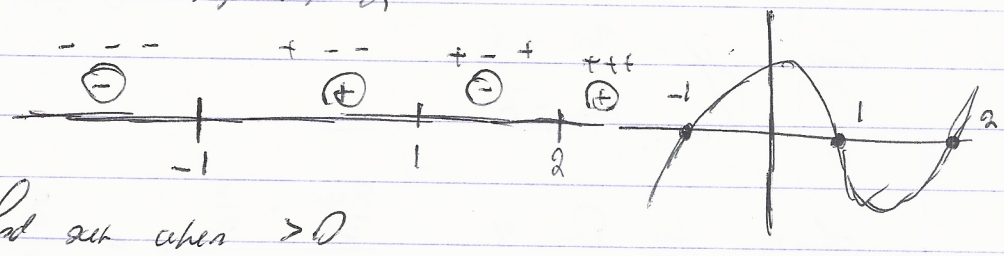
$$-3x^2 + 8x - 5 \geq x - 2x^2 + x^3 - 3 + 6x - 3x^2$$

$$\geq x^3 - 5x^2 + 7x - 3$$

$$0 \geq x^3 - 5x^2 + 7x - 3$$

$$0 \geq (x+1)(x-2)(x-1)$$

$\Rightarrow$ critical values $= -1, 1, 2$



to find out when > 0

$$\Rightarrow x \leq -1 \text{ or } 1 \leq x \leq 2$$

now check solutions: not valid for $x = 1$ $\because$ of fraction $\Rightarrow$ asymptote at $x = 1$ & solutions are...

$$x \leq -1 \text{ or } 1 < x \leq 2$$

OR simply take away from both sides

An asymptote is a line that a curve approaches for large values of $|x|$ or $|y|$

Vertical Asymptotes

E.g., $y = 1/x - 2 \Rightarrow$ vertical asymptote of $x = 2$

- assessentially $f(x) = 1/x$ translated by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ to $f(x-2) = 1/x - 2$
- essentially when denominator $= 0$
- however if denominator has imaginary roots then no vertical asymptote $\Rightarrow y = \dfrac{x^2 + x - 7}{x^2 + 8x + 5}$

Horizontal Asymptotes

E.g., $y = \dfrac{3x + 5}{2x - 7}$

divide by $2 \Rightarrow \dfrac{\frac{3x}{x} + \frac{5}{x}}{\frac{2x}{x} - \frac{7}{x}} = \dfrac{3 + \frac{5}{x}}{2 - \frac{7}{x}}$

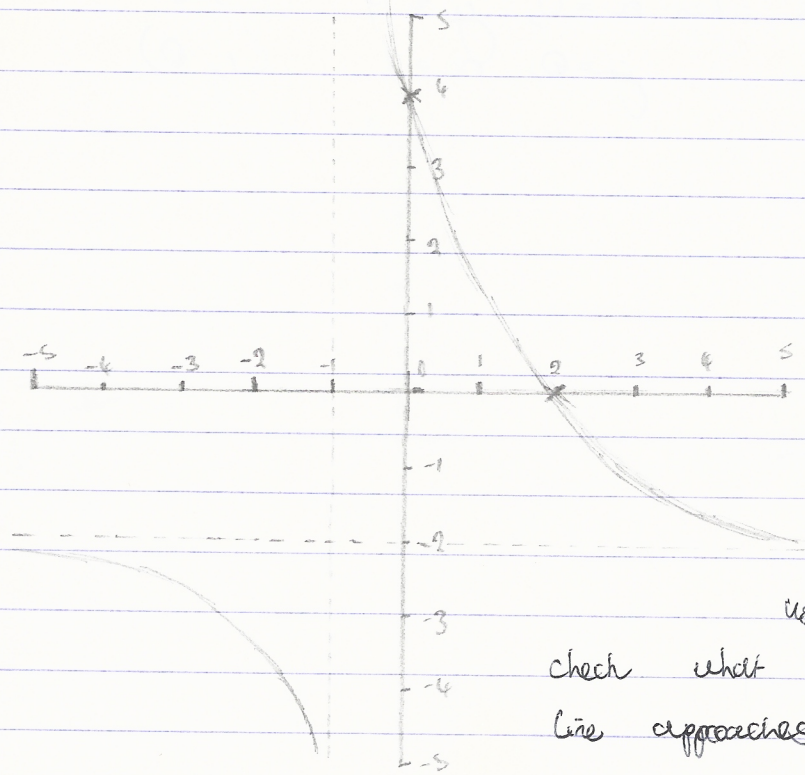
$\Rightarrow$ for 'large' values of x, or as $x \longrightarrow \pm\infty$ or $|x| \rightarrow \infty$ then $5/x$ & $-7/x$ approach 0 & can have no effect on $y = 3/2$ horizontal asymptote

$\Rightarrow x = 8.6$ & $y = 3/2$

$$y = \frac{4-2x}{x+1}$$, sketch

(horizontal ∥ $y = -2$ & vertical ∥ $x = -1$

crosses x-axis & $y = 0 \Rightarrow x = 2 \Rightarrow (2, 0)$ &
" y-axis & $x = 0 \Rightarrow y = 4 \Rightarrow (0, 4)$



using the calculator, check what happens as the line approaches)

Rational functions of this form

• A rational function is essentially an algebraic fraction in which the numerator and denominator are both polynomials.

$$\frac{px + q}{ax^2 + bx + c}$$ if $b^2 - 4ac > 0$ then 2 vertical asymptotes, if $b^2 - 4ac = 0$ 1, if $b^2 - 4ac < 0 \Rightarrow 0$

$y = 0$ will be one here ∵

$$y = \frac{\frac{p}{x} + \frac{q}{x^2}}{a + \frac{b}{x} + \frac{c}{x^2}} \Rightarrow \text{as } |x| \to \infty,\ y \to 0$$

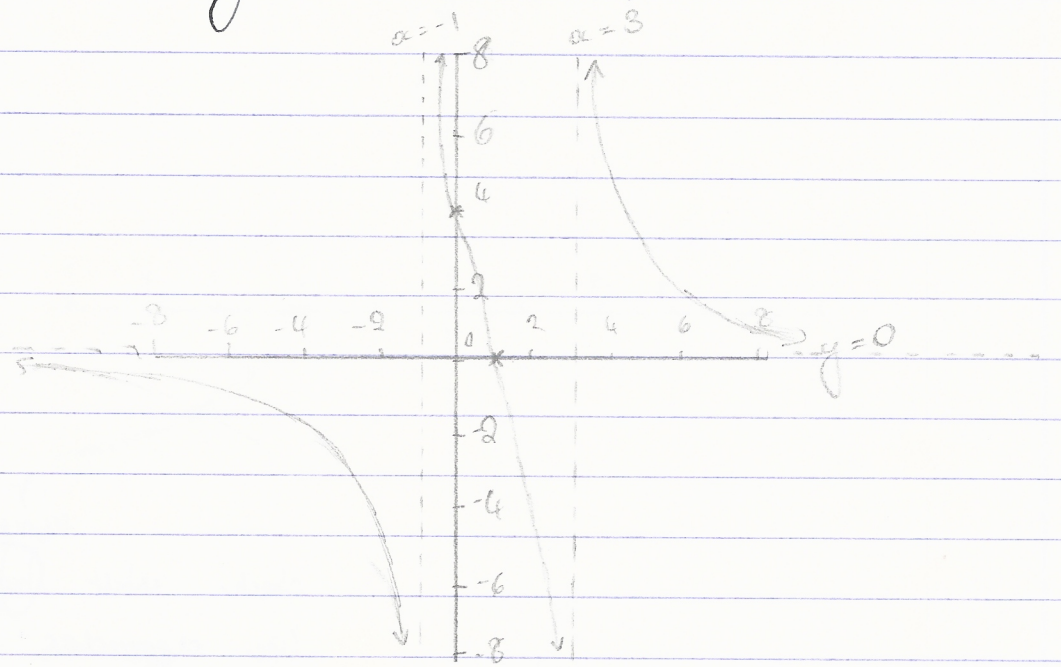
Example $y = \dfrac{12x - 12}{(x+1)(x-3)}$

vertical asymptotes: $x = -1$ & $x = 3$

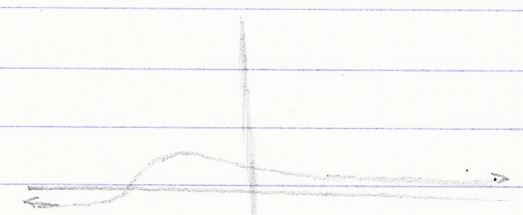
horizontal " $y = 0$

crosses when $x = 0$, $y = 4$ $(0, 4)$

$y = 0$, $x = 1$ $(1, 0)$



Maximum & minimum points $y = \dfrac{x + 4}{x^2 - 3x + 10}$ $\Rightarrow$



you can use $y = k$ where

repeated root $\Rightarrow$ tangent $\Rightarrow$ max/min pt

2 distinct $\Rightarrow$ intersection or none $\Rightarrow$ no intersection.

$\Rightarrow k = \dfrac{x + 4}{x^2 - 3x + 10}$ & solve for k with $b^2 - 4ac = 0$

$x \geq 0$ to find critical values

Example of max/min

$$k = \frac{x+5}{x^2-x+6} \Rightarrow kx^2 - kx + 6k = x+5$$

$$kx^2 + (-k-1)x + (6k-5) = 0$$

$$a = k, \quad b = -k-1, \quad c = 6k-5$$

$b^2 - 4ac \Rightarrow (-k-1)^2 - 4(k)(6k-5) \geq 0$ for all valid points

$$\Rightarrow (-k-1)^2 - 4(k)(6k-5) \geq 0$$

$$k^2 + 2k + 1 - 4k(6k-5) \geq 0$$

$$k^2 + 2k + 1 - 24k^2 + 20k \geq 0$$

$$-23k^2 + 22k + 1 \geq 0 \qquad a-1$$

$$23k^3 - 22k - 1 \leq 0$$

$$(23k+1)(k-1) \leq 0 \Rightarrow \text{critical values} = -1/23, \ 1$$

$\Rightarrow$ $\ominus\ominus$ $\oplus$ $-1/23$ $+\ -$ $\ominus$ 1 $+\ +$ $\oplus$

$\Rightarrow$ less than one for y $-1/23 \leq k \leq 1$ so exists for these y-values

Sub y back in to find stationary points

$$\frac{-1}{23} = \frac{x+5}{x^2-x+6} \Rightarrow -x^2 + x - 6 = 23x + 115$$

$$x^2 + 22x + 121 = 0$$

$$\Rightarrow x = -11$$

$$\Rightarrow \underline{\underline{(-11, -1/23)}}$$

$$1 = \frac{x+5}{x^2-x+6} \Rightarrow x^2 - x + 6 = x + 5 \Rightarrow x^2 + 2x + 1 = 0$$

$$\Rightarrow x = -1$$

$$\Rightarrow \underline{\underline{(-1, 1)}}$$