Matrices

Basics/ Introduction

- The order of a matrix is denoted as $n \times m$ where n = row no. & m = col. no. (can be the other way around sometimes)
- When adding or subtracting, the matrices have to be of the same order.

- A zero matrix is a matrix where all elements = 0.
- For matrix multiplication, of $n_1 \times m_1$ & $n_2 \times m_2$. $m_1 = n_1$ & resulting matrix order is $n_1 \times m_2$.
- There is no such thing as matrix division:

$$4 \times 5 = 20 \Rightarrow 20 \div 4 = 5 \text{ or } 20 \times \tfrac{1}{4} = 5$$

- You have to use inverses, $4^{-1} = 1/4$.
- Dividing is essentially multiply by an inverse.

Multiplication

- By convention, matrix multiplication is done as...

$$\begin{bmatrix} 2 & -2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 7 & -6 \end{bmatrix} = \begin{bmatrix} 2 \times -1 + -2 \times 7 & \dots \\ \dots & \dots \end{bmatrix}$$

First row × first column = first row, first column.
first row × second column = first row, second column
...

- In general, $AB \neq BA$ (where A & B are matrices) unless it involves the identities or in other cases.
- Matrix multiplication is associative ∴ $A(BC) = (AB)C$ as transformations are applied in reverse order.
- It is also distributive where $A(B+C) = AB + AC$.

Transposing

- Transposing works as...

$$\begin{bmatrix} 3 & 4 & 2 \\ 1 & 7 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 \\ 4 & 7 \\ 2 & 5 \end{bmatrix}$$

Determinant & Inverses

- Written as $|A|$, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = ad - bc$.
- Multiplying a matrix by its inverse gives the identity matrix:

$$\begin{bmatrix} 5 & 2 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be found by...

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

make these negative go this

flip these to this

- If a matrix A has $|A| = 0$, there is no inverse. In simultaneous equations... no solutions!

Basic Transformations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ stretch in x-direction by s.f. k.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ stretch in y-direction by s.f. k.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ enlargement centre $(0, 0)$ s.f. k.

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$ rotation form.

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$ reflection form.

Examples of Rotations

① $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\cos\theta = 0 \Rightarrow \theta = 90^\circ$ & $\sin\theta = 1 \Rightarrow \theta = 90^\circ$

$\Rightarrow$ anticlockwise rotation of 90° centre $(0,0)$

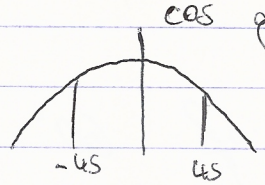
Reflections (Examples)

① $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

$\cos 2\theta = 0 \Rightarrow 2\theta = 90 \Rightarrow \theta = 45$

$\sin 2\theta = -1 \Rightarrow 2\theta = -90 \Rightarrow \theta = -45$

cos graph:



$\Rightarrow y = \tan(-45)x = -x$

② $y = 3x \Rightarrow \tan\theta = 3 \Rightarrow \theta = 71.565\ldots \Rightarrow 2\theta = 143.13\ldots$

$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} -4/5 & 6/10 \\ -6/10 & -4/5 \end{bmatrix}$

Composite Transformations

- If A represents a 2×2 transformation & B is another. $B \cdot A$ represents 'A followed by B' OR $C \cdot B \cdot A$ is 'A followed by B followed by C'

$\because f(g(x))$ is also read backwards $\Rightarrow$ matrix compositions are too.

Matrix Composition Transformation Example

Q A // $y = 1/\sqrt{3}\,x \Rightarrow \theta = 30$

$$A = \begin{bmatrix} 0.5 & \sqrt{3}/2 \\ \sqrt{3}/2 & -0.5 \end{bmatrix}$$

B // $y = \sqrt{3}\,x \Rightarrow \theta = 60$

$$B = \begin{bmatrix} -0.5 & \sqrt{3}/2 \\ \sqrt{3}/2 & 0.5 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} 0.5 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 0.5 \end{bmatrix}$$ of A followed by B

Which is also a rotation of 60° anticlockwise centre (0, 0)

Q Enlargement & rotation $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$

$h\cos\theta = -1$ ~~then~~ $-h\sin\theta = -1$

$\Rightarrow h\sin\theta = 1$

$\tan\theta = \dfrac{h\sin\theta}{h\cos\theta} = \dfrac{-1}{1} \Rightarrow \theta = -45°$

$h\sin -45 = 1 \Rightarrow h = 1/\sin -45 = -\sqrt{2}$

enlargement of s.f. $-\sqrt{2}$ & rotation 45° <u>clockwise</u>

Faster Method of Transforming

$$\begin{bmatrix} 3 & -2 \\ 0 & 6 \end{bmatrix} \begin{array}{c} \begin{array}{cccc} O & F & D & E \end{array} \\ \begin{bmatrix} 0 & 2 & 3 & 5 \\ 0 & 3 & 0 & 3 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \begin{bmatrix} 0 & 0 & 9 & 9 \\ 0 & 18 & 0 & 18 \end{bmatrix} \\ \begin{array}{cccc} O & F' & D' & E' \end{array} \end{array}$$

Determinants

- area of image = area of original × $|\det(T)|$
- If $\det(T) > 0$, the orientation is preserved.

- A 3×3 determinant can be found as follows:

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

watch out!

Delete all the rows & columns associated with a then multiply by the minors

- An example is below:

$$\det\begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{pmatrix} = 0\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 0 - 0 + 1(2+1)$$

$$= 3$$

- If $\det(A) = 0$, A is a singular matrix.

Inverses & Equations

- Inverse of A is $A^{-1} \Rightarrow A^{-1}A = AA^{-1} = I$
- Useful for simultaneous equations

Inverses of 3×3

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 1 & -2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\det A = 2\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} + -3\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= 2(-1 - -4) - 1(-3) + -3(6)$$

$$= 2(3) + 3 - 18 = 9 - 18 = -9$$

eliminate the element & find the minors

$$\begin{bmatrix} +\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 6 \\ -5 & -2 & -4 \\ 1 & -5 & -1 \end{bmatrix}$$

THEN TRANSPOSE

$$\Rightarrow \begin{bmatrix} 3 & -5 & 1 \\ 3 & -2 & -5 \\ 6 & -4 & 1 \end{bmatrix}$$

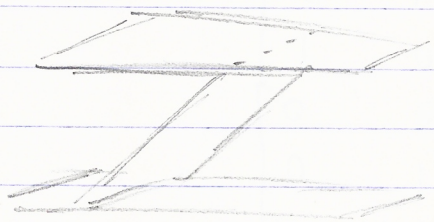
$$A^{-1} = \frac{-1}{9}\begin{bmatrix} 3 & -5 & 1 \\ 3 & -2 & -5 \\ 6 & -4 & -1 \end{bmatrix} = \begin{bmatrix} -1/3 & 5/9 & -1/9 \\ -1/3 & 2/9 & 5/9 \\ -2/3 & 4/9 & 1/9 \end{bmatrix}$$

Systems of Equations

$$\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1}\begin{pmatrix} 11 \\ 6 \end{pmatrix} = \frac{-1}{14}\begin{pmatrix} -1 & -3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 11 \\ -6 \end{pmatrix}$$

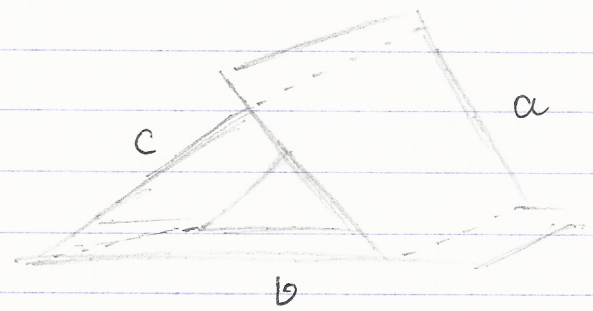
$$= \begin{pmatrix} -1/2 \\ -16/7 \end{pmatrix} \quad \text{where } M = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$$

If 3 planes don't have a single point of intersection, they are <u>inconsistent</u>.



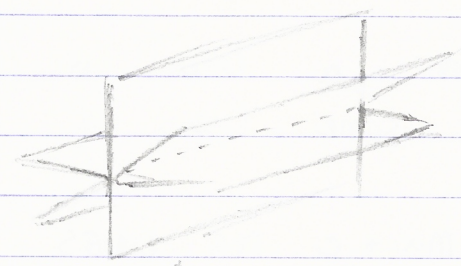
Two parallel planes

Triangular prism (where 2 planes add like vectors to the other)

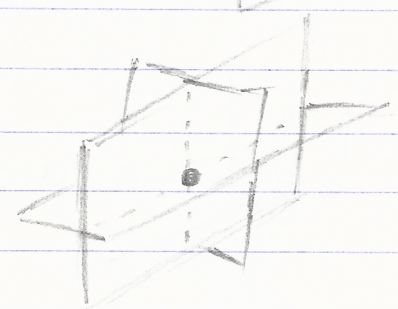


$a + b = c$

<u>OTHERWISE</u>



a sheaf along which are infinitely many solutions



a single point of intersection where there is <u>exactly</u> one solution.

(Ignore what the equation equals when checking if two are parallel)

If the det = 0, there are ∞ solutions or infinitely many solutions (a sheaf).

3D transformations

Reflections in $x=0$ $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

" " $y=0$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

" " $z=0$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(These are not in the formula book)

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + y\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + z\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

Invariant Lines & Points

- If every point on a line is mapped to another point on the same line, it is an invariant line.

YOU USE $y = mx + c$ FOR LINES

$$\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ mx + c \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$ OR USE $mx' + c$

$x + mx + c = x'$

$3(mx + c) = y'$

THEN WRITE IN $y' = mx' + c$ FORM

$$3(mx + c) = m(x + mx + c) + c$$
$$3mx + 3c = mx + m^2x + mc + c$$
$$2mx - m^2x - mc + 2c = 0$$
$$(2m - m^2)x - mc + 2c = 0$$

EQUAT BOTH SIDES

$$2m - m^2 = 0 \Rightarrow m = 0 \text{ or } 2$$
$$2c - mc = 0 \Rightarrow m = 0,\ c = 0. \ \underline{\underline{\text{OR}}} \quad m = 2,\ c = \text{any value}$$

∴ Substituting 2 into m gives $2c - 2c = 0$

$\Rightarrow$ Line 1 $y = 0$ & $y = 2x + c$

- If T (transformation) & x (vector), $Tx = x$ then x is an invariant point.
- Example:

$$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + 3y = x \Rightarrow x - 3y$$
$$-x - 2y = y \Rightarrow x = -3y$$

WHICH gives a line of invariant points of $x = -3y$

<u>Or</u>
$$\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{matrix} 2x + 2y = 0 \\ y - x = 0 \end{matrix}$$

$$\begin{aligned} 2x + y &= 0 \\ -2x + 2y &= 0 \ + \\ \hline 3y &= 0 \end{aligned} \Rightarrow y = 0 \Rightarrow x = 0$$