Series, Summations, & Methods of Differences

Fundamentals - $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ & $\sum_{r=1}^{n} k = nk$ $\because \sum_{r=1}^{n} 1 = \underbrace{1+1+1+\dots+1}_{n \text{ times}}$

$\Rightarrow \sum_{r=1}^{n} (ar+b) = a\sum_{r=1}^{n} r + nb$

Examples $\sum_{r=1}^{3} r = 1+2+3 = 6$ $\Rightarrow$ this is an arithmetic series $\because$ each term differs from the least by a constant

Arithmetic Series Sum

$$\underbrace{S_n}_{\text{"sum of first n terms"}} = \frac{n}{2}(\underbrace{a}_{\text{first term}} + \underbrace{l}_{\text{last term}})$$

E.g., if $a_1 = 5$ & $a_{20} = 62$

$\Rightarrow \frac{20}{2} \times (5+62) = 670$

QR find sum of first 40 terms of 2, 5, 8, 11

$a_{40} = a_1 + \overbrace{(n-1)}^{3n-1}\overbrace{d}^{\text{common difference}}$ } nth term

$= 2 + 39(3) = 119$

$\Rightarrow S_n = \frac{40}{2} \times (2+119) = 2420$

Deriving first n terms

$S_n = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$

$\Rightarrow S_n = n + (n-1) + (n-2) + \dots + 3 + 2 + 1$

$2S_n = (n+1) + (n+1) + (n+1) + \dots (n+1) + (n+1) + (n+1)$ (n times)

$\Rightarrow 2S_n = n(n+1)$

$\Rightarrow S_n = n(n+1)/2$

Deriving first n^2 terms

n	0	1	2	3	4
$\sum_{r=0}^{n} r^2$	0	1	5	14	30

1 4 9 16

3 5 7

2 2

$\Rightarrow$ cubic

$$\sum_{r=0}^{n} r^2 = An^3 + Bn^2 + Cn + D$$

when $n=0$, $D=0$

$n=1$, $A+B+C=1$

$n=2$, $8A+4B+2C=5$

$n=3$, $27A+9B+3C=14$

using calculator $\Rightarrow$ $A = 1/3$, $B = 1/2$, $C = 1/6$

$$\Rightarrow \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n = \tfrac{1}{6}(2n^3 + 3n^2 + n)$$

$$= \tfrac{1}{6}n(2n^2 + 3n + 1) = \frac{n}{6}(2n+1)(n+1)$$

Alternating - If you wish to alternate + & - $\Rightarrow$ $\sum_{r=1}^{n} (-1)^{r} r = -1 + 2 - 3 + 2 + \dots$

Methods of Differences

$$\sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+1} = \frac{1}{1} - \cancel{\frac{1}{2}}$$

$$+ \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$$

$$+ \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$$

$$+ \cancel{\cdots} - \cancel{\cdots}$$

$$+ \cancel{\frac{1}{r}} - \frac{1}{r+1} = 1 - \frac{1}{r+1}$$

$\Rightarrow$ as $n \to \infty$, $\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) \to 1$

Partial Fractions

$$\frac{18}{(3r-2)(3r+1)(3r+4)} = \frac{A}{3r-2} + \frac{B}{3r+1} + \frac{C}{3r+4}$$

$$\Rightarrow 18 = A(3r+1)(3r+4) + B(3r-2)(3r+4) + C(3r-2)(3r+1)$$

(with everything that isn't above the number)

when $r = -1/3 \Rightarrow A(0) + B(-3)(3) + C(0)$

$-9B = 18 \Rightarrow B = -2$

eliminate 2 & equate

$r = -4/3 \Rightarrow A(0) + B(0) + C(-6)(-3)$

$18C = 18 \Rightarrow C = 1$

$r = 2/3 \Rightarrow A(3)(6) + B(0) + C(0) = 18$

$18A = 18$

$A = 1$

More on Partial Sums

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$$ nth term

Very generalised → It's more of a way of finding the X_{th} term of a sequence

nth term

$a_n = S_n - S_{n-1}$ ⇒ if $S_n \frac{n+1}{n+10}$ ⇒ $\frac{n+1}{n+10} - \frac{n}{n+9}$

Imagine doing $a_3 = S_3 - S_2$ YET generalising it.