Binary Operations

Key Idea
- Have two inputs and give one output.

Sets
- It can be applied to any set, some examples include:
  - $\mathbb{N}$: $\{1, 2, 3, \ldots\}$ which are the counting numbers.
  - $\mathbb{Z}$: $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$ which are all integers
    $\mathbb{Z}^+$ for positive and $\mathbb{Z}^-$ for negative.
  - $\mathbb{Q}$ is the set of rational no.'s

(Note, zero is considered to be even)

Closure

A set is said to be closed under a binary operation $*$ if for any two no.'s from the set, the result $*$ returns a member from the set.

Eg., $\mathbb{R}$ is closed under addition $(3 + 5 = 8)$

OR

$S = \{1, 2, 8\}$

$*$	1	2	3
1	1	0	-1
2	3	2	1
3	5	4	3

Use this Cayley table to check all the permutations

These two are not in $S \Rightarrow$ it is closed under $*$

OR

$a * b = a - b$ is closed under $\mathbb{Z}$

whereas $a * b = a/b$ isn't

Commutative A binary operation, $*$ is commutative if $a * b = b * a$

E.g., $a \triangleright b = 2a + b$ OR addition (as its order doesn't matter)

$a * b = 2a + b$ & $b * a = 2b + a$

$\Rightarrow$ not commutative (or using real values)

In a table, this will have a line of symmetry:

$*$	a	b	c
a	b	a	c
b	a	c	b
c	c	b	a

Associative A binary operation, $*$, is associative if $(a*b)*c = a*(b*c)$

E.g., $a \triangleright b = 2a + b$

$(a * b) * c = (2a + b) * c = 2(2a + b) + c = 4a + 2b + c$

$a * (b * c) = a + (2b + c) = 2a + 2b + c$

$\Rightarrow$ not associative

OR

$(3 \times 4) \times 5 = 3(4 \times 5) = 60$

Identity Elements

- $e * a = a * e = a$ under $a, e \in \mathbb{R}$
- e is the identity element under operation $*$ leaves the output unchanged from the input
- For addition, $e = 0$, for multiplication, $e = 1$.

- You can prove that there will only be 1 element.
  - Begin by assuming you have 2, e & f
    $\Rightarrow e * f = e$ & $e * f = f$
    $\Rightarrow e = f \Rightarrow$ only one

- They can be found using a table:

*	a	b	c	d
a	c	a	d	b
b	a	b	c	d
c	d	c	d	a
d	b	d	a	c

b is the identity element as the rows and coloumns match

OR using operations:

$a * b = a + b + 3$ where $a, b \in \mathbb{R}$

now find it using both ways to make sure

$a * e = a$ & $a * e = a + e + 3 = a$

$e + 3 = 0$

$e = \underline{-3}$

$e * a = a$ & $e * a = e + a + 3 = a$

$e + 3 = 0$

$e = -3$

Inverse Elements

This is NOT $1/a$

- If $a * a^{-1} = a^{-1} * e = e$
- For addition, $e = 0$
  $\Rightarrow$ for $a = 5$, $5 + a^{-1} = 0 \Rightarrow a^{-1} = -5$ as under the operation $a + b$, this gives e
- For multiplication, $e = 1$
  $\Rightarrow$ for $a = 5$, $5 \times a^{-1} = 1 \Rightarrow a^{-1} = 1/5$ as under the operation $a \times b$, this gives e

Finding inverses:

*	a	b	c	d
a	c	a	d	b
b	a	b	c	d
c	d	c	b	a
d	b	d	a	c

① find the identity element
② see where else this element is found

E.g., $a * d = b \Rightarrow a^{-1} = d$
OR $d^{-1} = a$

$\boxed{b^{-1} = b}$ Remember the identity element is always self-inverse

$c^{-1} = c$ also self-inverse

OR form an operation

$$a * b = a + b + ab \qquad a, b \in \mathbb{R}$$

① find identity element
$$a * e = a + e + ae = a$$
$$e + ae = 0$$
$$e(1+a) = 0 \Rightarrow e = 0$$

② find inverse
$$a * a^{-1} = 0$$
$$\Rightarrow a + a^{-1} + aa^{-1} = 0$$
$$a^{-1} + aa^{-1} = -a$$
$$a^{-1}(1+a) = -a$$
$$a^{-1} = -a / 1+a \qquad \text{where } a \neq -1$$

$$\Rightarrow a * \frac{-a}{1+a} \text{ under } a * b = e = 0$$