Game Theory

Starter

A \ B	Stone	Paper	Scissors
Stone	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

(all from player A's prespective)

(the idea is neither player knows what the other will do)

→ zero sum ∴ one's loses + other's wins = 0

→ Called a play-off matrix : shows all outcomes

This is a zero-sum game in which one person's win is another's losses.

Content

- Play-safe strategy is to choose the option with best worst outcome RATHER than gambling. AKA highest minimum payoff. (can be + or −)
- A stable solution is when neither player can improve their payoff by taking a different strategy.

From A's pov

A \ B	Y	Z
W	2	-2
X	5	-4

E.g., if B adopts Y then A picking W will gain 2 from B or by picking X will gain 5 from B.

Use rows for A & columns for B.

from B's pov

A \ B	Y	Z
W	-2	2
X	-5	4

A \ B	1	2	3	Row min
1	-1	0	1	-1
2	-2	1	3	-2
3	-3	0	-4	-4
Col max	-1	1	3	

'best' worst case for A / best case for B

smallest values

'best' worst case for A ∵ losing 1 is better than losing 2 or 4

(look at other mins)

'best' worst case for B

B gaining one is better than losing 1 & 3

largest values

now ∵ $-1 = -1$, there is a stable solution ∵ cannot change

to improve on strategy.

Dominance: • column can be ignored if each element is greater than or equal to ($\geq$) corresponding element of another column

E.g.,

A \ B	1	2
1	6	-2
2	-3	-4

⇒ 2 has dominance over 1 for B ∵ either lose 6 or gain 3 OR gain 2 or gain 4

⇒ 2 is inferior to 1 for A ∵ either lose 3 or lose 4 OR gain 6 or lose 2

Simple mixed strategies

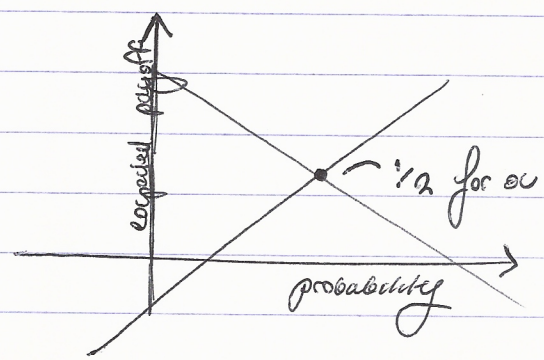
A \ B	1	2
1	3	0
2	-1	2

- Let probability of A picking 1 be p & ∴ A picking 2 be $1-p$

⇒ if B choses 1, A can expect to gain $3p + -1(1-p)$
$= 4p - 1$

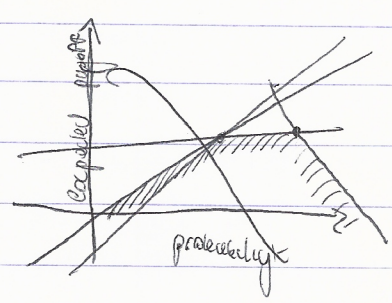
OR if B chases 2, A can expect to gain $0p + 2(1-p) = 2 - 2p$

Solving simultaneously: $4p - 1 = 2 - 2p \Rightarrow 6p = 3 \Rightarrow p = 1/2$

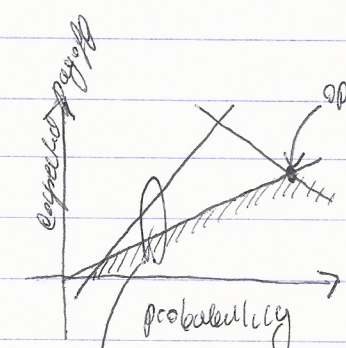


ALLOWS TO SOLVE FOR PROBABILITY

Re... for multiple:



you make them equal to find the probability of where they intersect



optimal strategy ∴ shade under line & highest shaded region

you can have either payoff here ⇒ not definite ∴ depends on what other picks.

Terms

- Play-safe strategy is what each player should pick.
- Value of the game is who the game favors or disadvantages. E.g., for rock paper scissors, it's 0.
- A pure strategy is a definite choice a player makes.
- An optional mixed strategy is probability-based.

REVISION SHEET – DECISION MATHS

GAME THEORY

The main ideas are covered in

AQA	Edexcel	MEI	OCR
D2	D2		D2

The main ideas in this topic are:

- Finding out whether a two person zero sum game has a stable solution.
- If a stable solution does not exist, using a graphical method to find an optimal mixed strategy.

Before the exam you should know:

- What is meant by a two person, zero sum game.
- How to interpret a payoff matrix.
- How to find a playsafe strategy.
- What is meant by a stable solution.
- How to find the value of a game.
- How to simplify games using dominance.
- How to find an optimal mixed strategy for a game with no stable solution.
- How to convert a two person zero sum game into a linear programming problem.

In a two person, **zero sum game**, one player's gain is the same as the other player's loss.

A **pay-off matrix** represents the gain for one of the players for each combination of strategies for the two players in a two-person game.

Finding a play-safe strategy for a zero-sum game

In the pay-off matrix for A, find the minimum entry in each row; A will use the strategy that involves the greatnesses of these values (maximin).Then find the maximum entry in each column, B will use the strategy that involves the lowest of these values (minimax).

A **stable solution** occurs if and only if the maximum(row minimum) = the minimum(column maximum).
A assumes B uses his play-safe strategy and cannot do better by using an alternative strategy and B assumes A uses his play-safe strategy and cannot do better by using an alternative strategy. A stable solution is sometimes called a **saddle point**.

Example: Anna and Barry play a zero sum game. The game can be represented by the following payoff matrix for Anna

	B_1	B_2	B_3
A_1	3	2	5
A_2	-1	4	-2
A_3	2	1	3

(a) Show that there is not stable solution

Solution

ignore this

	B_1	B_2	B_3	Row max	minimax
A_1	3	2	5	**5**	
A_2	-1	4	-2	**4**	**3**
A_3	2	1	3	**3**	
Col min	**-1**	**1**	**-2**		
maximin	**1**				

$1 \neq 3$ so solution is not stable

Dominance: A row/column can be eliminated if all the entries in that row/column are less than or equal to the corresponding entries of another row/column because the player would never choose that row/column.

Finding a mixed strategy using a graph: If player A has two possible strategies, assume he adopts the first strategy with probability p and the second with probability $1-p$. The expected payoff for A will depend on which strategy B chooses so plot A's expected payoff for each of B's strategies as lines on a graph. A's best strategy is to play with the value of p which gives the highest point at the intersection of the lines. This will maximise A's minimum return (maximin). The value of the game can be calculated using this value of p.

	B_1	B_2	B_3
A_1	3	2	5
A_2	-1	4	-2
A_3	2	1	3

Example (cont)

(b) Explain why Anna should never play strategy A_3

(c) Find the best mixed strategy for Anna and give the value of the game for her.

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Solution

(b) Anna should not play A_3 since row A_3 is dominated by row A_1 since $3 > 2$, $2 > 1$ and $5 > 3$.

(c)

	B_1	B_2	B_3
A_1	3	2	5
A_2	-1	4	-2

Let Anna play
A_1 with probability p
A_2 with probability $(1-p)$

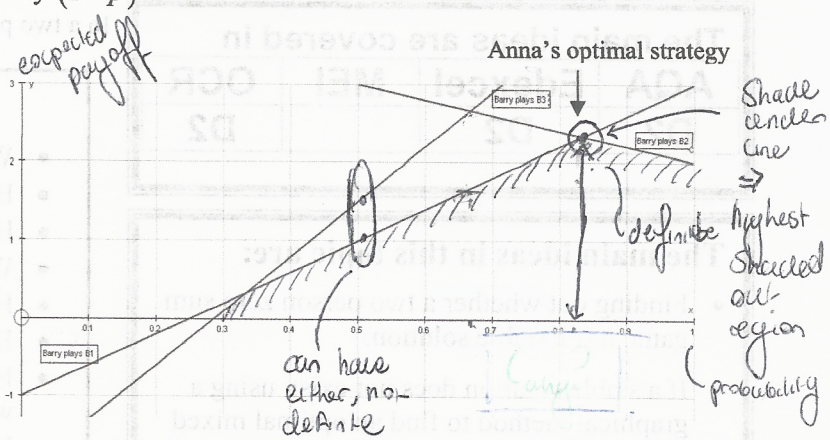
Anna's expected payoff if Barry plays strategy:

B_1: $3p-1(1-p)=4p-1$
B_2: $2p+4(1-p)=4-2p$
B_3: $5p-2(1-p)=7p-2$

Solving for p:
$4p-1=4-2p$
$\Leftrightarrow 6p=5$
$\Leftrightarrow p=\frac{5}{6}$

why?

∴ to find probability of where they intersect



From the graph you can see that the optimal expected return for Anna occurs at the intersection of the lines for Barry playing strategies B_1 and B_2.

So Anna should play A_1 with probability $\frac{5}{6}$ and A_2 with probability $\frac{1}{6}$

Value of the game: $4\left(\frac{5}{6}\right)-1=\frac{20}{6}-1=\frac{14}{6}=2\frac{1}{3}$

Finding a mixed strategy using the Simplex algorithm

The graphical method cannot be used if player A has more than two strategies so we must use the simplex method.

Let the probabilities of each of A's strategies be p_1, p_2, p_3 etc.

If there are any negative elements in the payoff matrix, add k to make them all positive.

Model the situation as a linear programming problem:

Maximise V-k (this is the expected payoff from the original game)
Subject to $V \le$ expected payoff from B playing strategy 1
$V \le$ expected payoff from B playing strategy 2 *etc*
$p_1 + p_2 + p_3 + \ldots \le 1$ (because the sum of the probabilities cannot exceed 1)
$p_1,\ p_2,\ p_3 \ge 0$ (no negative probabilities)

This information can now be put into an initial simplex tableau and solved in the usual way.

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