Hyperbolic Functions

Definitions

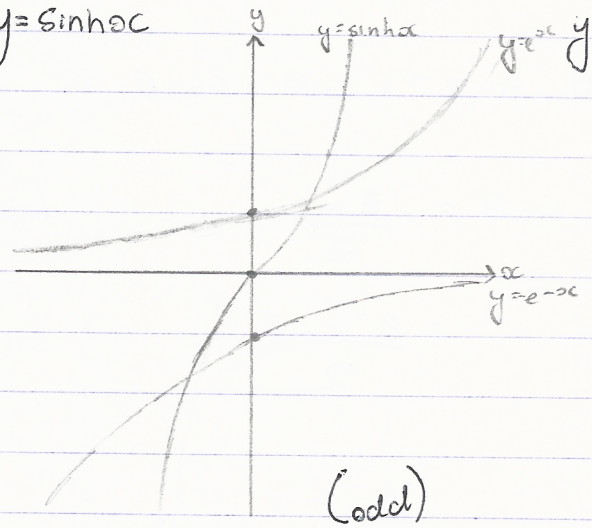
$\sinh x = \dfrac{e^{x} - e^{-x}}{2}$ (avg. of 2 functions)

$\cosh x = \dfrac{e^{x} + e^{-x}}{2}$ (avg. of 2 functions)

$\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{\sinh x}{\cosh x}$

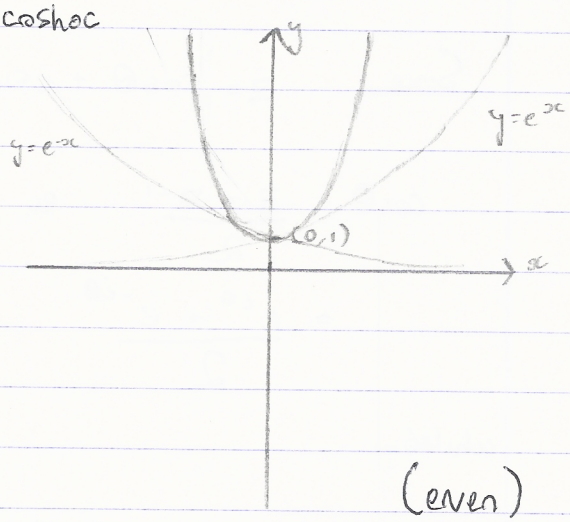
Graphs

$y = \sinh x$



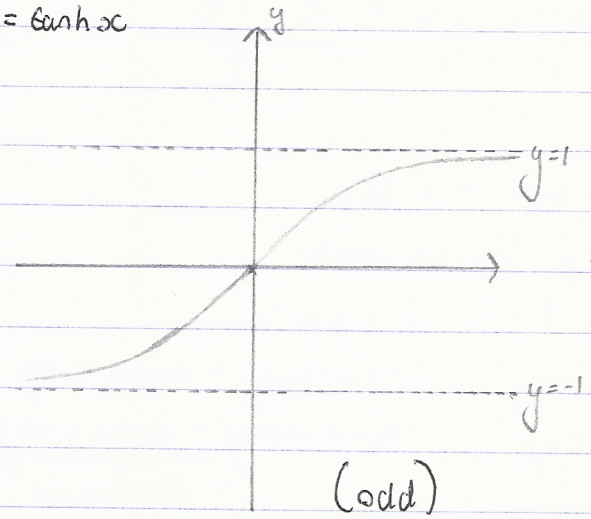
(odd)

$y = \cosh x$



(even)

$y = \tanh x$



(odd)

asymptotes: $y = 1$, $y = -1$

$\because$ as $x \to \infty$, $e^{-x} \to 0$

so $\dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \to \dfrac{e^{x}}{e^{x}} \to 1$

Inverses

$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

$\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1 + x}{1 - x}\right)$

positive value taken by convention

Osborne's Rule

- "Given the product of 2 sine functions in a trigonometric identity, the corresponding hyperbolic identity is replaced by the positive product of 2 sinh functions"

Thus from formula book identities:

$$\cosh^2\alpha + \sinh^2\alpha \equiv \cosh 2\alpha \quad \& \quad \cos^2\alpha - \sin^2\alpha \equiv \cos 2\alpha$$

Since... $z = \cos\theta + i\sin\theta$ & $z^{-1} = \cos\theta - i\sin\theta$

$$\cos\theta = \frac{z + z^{-1}}{2} \quad \& \quad \sin\theta = \frac{z - z^{-1}}{2}$$

$$= \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad\qquad = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

whilst...

$$\cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2} \quad \& \quad \sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2}$$

More Identities

- Formula book: $\cosh^2\alpha - \sinh^2\alpha \equiv 1$

  $\sinh 2\alpha \equiv 2\sinh\alpha\cosh\alpha$

  $\cosh 2\alpha \equiv \cosh^2\alpha + \sinh^2\alpha$

- non-Formula book: $\sinh(\alpha + y) \equiv \sinh\alpha\cosh y + \cosh\alpha\sinh y$

  $\cosh(\alpha + y) \equiv \cosh\alpha\cosh y \underbrace{+ \sinh\alpha\sinh y}_{\text{Osborne's Rule}}$

& using odd / even functions:

$$\sinh(\alpha - y) \equiv \sinh\alpha\cosh y - \cosh\alpha\sinh y$$

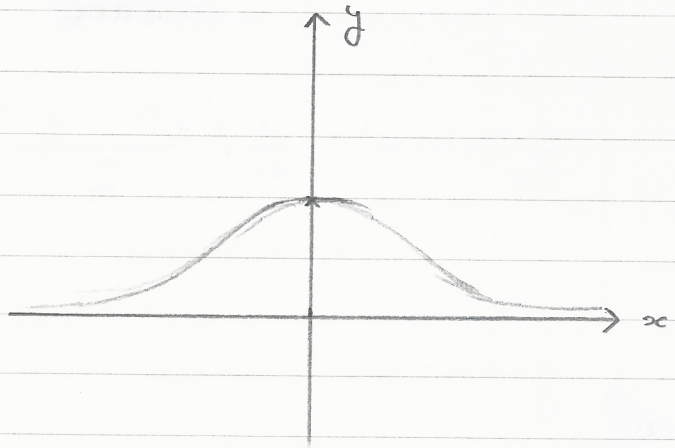
$$\cosh(\alpha - y) \equiv \cosh\alpha\cosh y - \sinh\alpha\sinh y$$

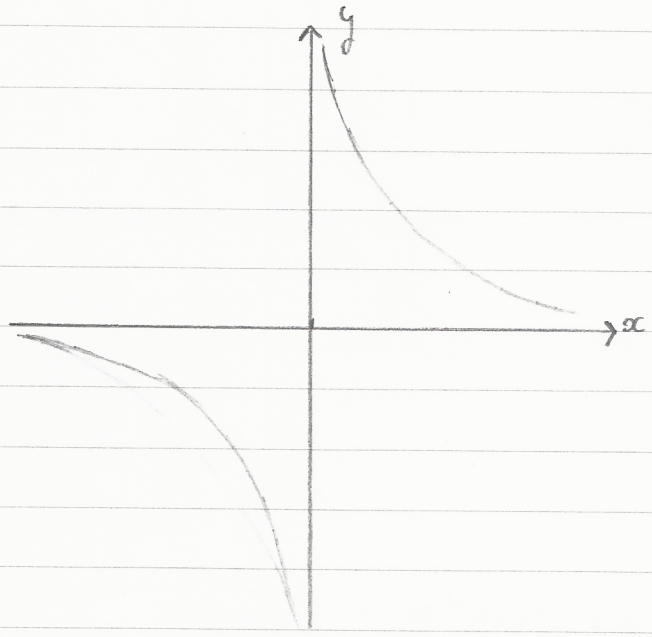
Reciprocal Hyperbolic Functions

- $\operatorname{sech} x \equiv 1/\cosh x$
- $\operatorname{cosech} x \equiv 1/\sinh x$
- $\coth x \equiv 1/\tanh x$

$y = \operatorname{sech} x$

Domain: $\{x \in \mathbb{R}\}$
Range: $\{y \in \mathbb{R} : 0 < y \leq 1\}$



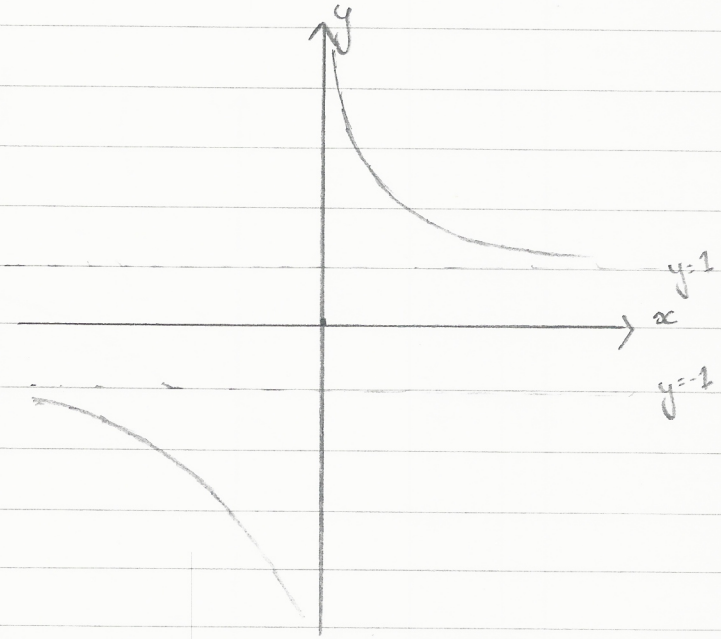


$y = \operatorname{cosech} x$

Domain: $\{x \in \mathbb{R},\ x \neq 0\}$
Range: $\{y \in \mathbb{R},\ y \neq 0\}$

$y = \coth x$

Domain: $\{x \in \mathbb{R},\ x \neq 0\}$
Range: $\{y \in \mathbb{R} : y > 1,\ y < -1\}$



Differentiating hyperbolics

$$\frac{d}{dx}(\cosh ax) = \frac{d}{dx}\left(\frac{e^{ax} + e^{-ax}}{2}\right) = a\left(\frac{e^{ax} - e^{-ax}}{2}\right)$$

$$= a \sinh ax$$

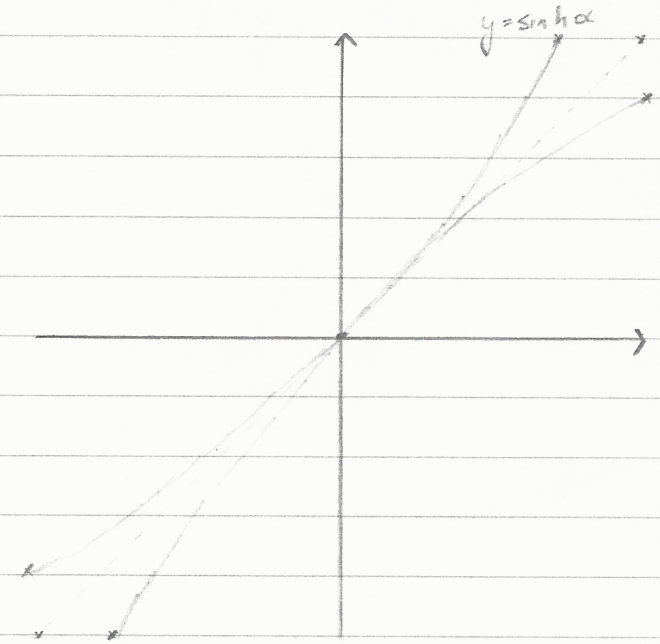
$$\frac{d}{dx}(\sinh ax) = \frac{d}{dx}\left(\frac{e^{ax} - e^{-ax}}{2}\right) = a\left(\frac{e^{ax} + e^{-ax}}{2}\right)$$

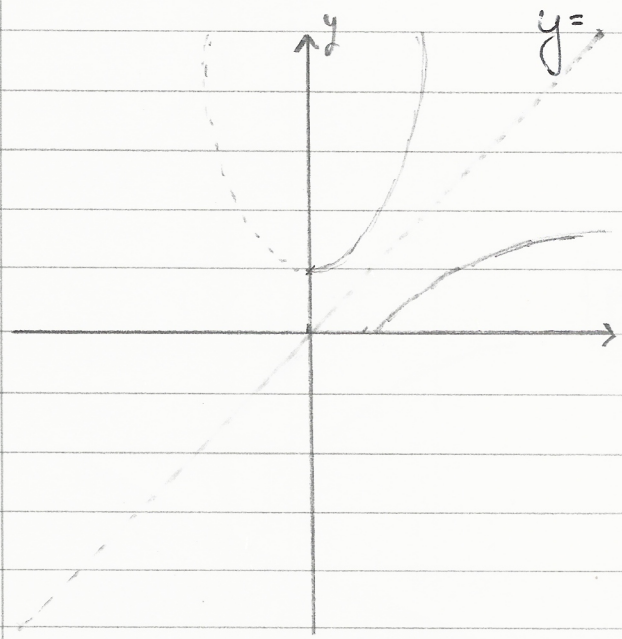
$$= a \cosh ax$$

Inverse Hyperbolic Functions

$y = \sinh^{-1} x = \operatorname{arcsinh} x$

Domain: $\{x \in \mathbb{R}\}$
Range: $\{y \in \mathbb{R}\}$



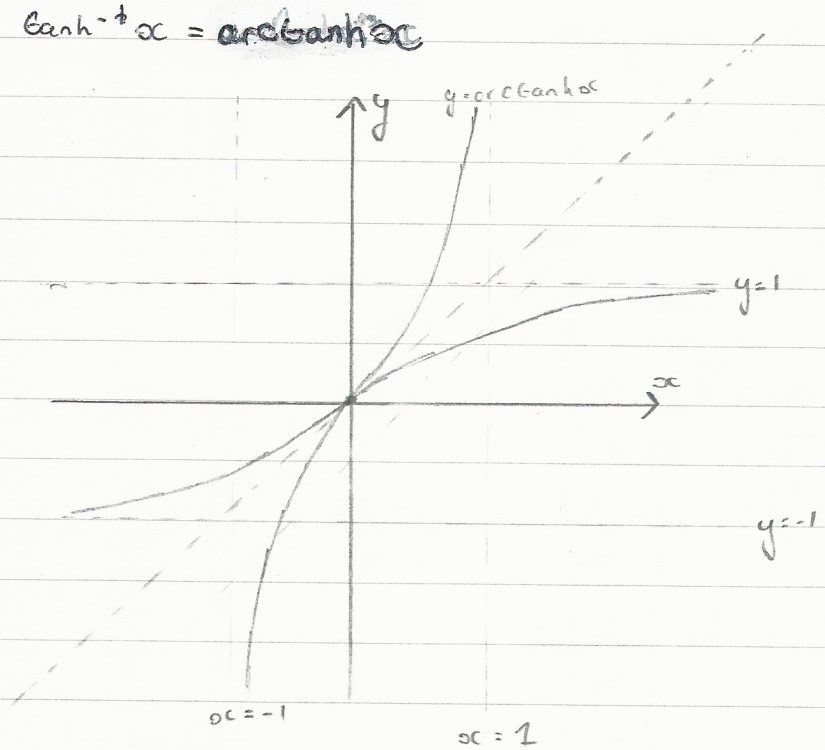


$y = \cosh^{-1} x = \operatorname{arccosh} x$

$y = \cosh x \Rightarrow$ Domain: $\{x \in \mathbb{R} : x > 0\}$
Range: $\{y \in \mathbb{R} : y > 1\}$

$y = \operatorname{arccosh} x \Rightarrow$ Domain: $\{x \in \mathbb{R} : x > 1\}$
Range: $\{y \in \mathbb{R} \quad x > 0\}$

$y = \tanh^{-1} x = \operatorname{arctanh} x$



Domain: $\{x \in \mathbb{R}\}$
Range: $\{-1 < y < 1\}$

Logarithmic Form

$y = \operatorname{arcsinh} x \Rightarrow \sinh y = x$

$$\frac{e^y - e^{-y}}{2} = x$$

$$e^y - e^{-y} = 2x$$

$$e^{2y} - 1 = 2x e^y$$

$$e^{2y} - 2x e^y - 1 = 0$$

$$(e^y - x)^2 - x^2 - 1 = 0$$

$$e^y - x = \pm\sqrt{x^2+1}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$y = \ln\left(x + \sqrt{x^2+1}\right)$$

since $e^y > 0$ for all $y \in \mathbb{R}$,
$e^y \neq x - \sqrt{x^2+1}$ $\because$
$\sqrt{x^2+1} > x \Rightarrow x - \sqrt{x^2-1} < 0$

$$\Rightarrow \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$y = \operatorname{arctanh} x \Rightarrow \tanh y = x$

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x$$

$$\frac{e^{2y} - 1}{e^{2y} + 1} = x$$

$$e^{2y} - 1 = x e^{2y} + x$$

$$(1 - x) e^{2y} = x + 1$$

$$e^{2y} = \frac{x+1}{1-x} = \frac{1+x}{1-x}$$

$$2y = \ln \left| \frac{x+1}{x-1} \right|$$

$$y = \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right|$$

$$= \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right)$$

$-1 < x < 1$

$1 > -x > -1$

$2 > 1 - x > 0$

$\therefore$ no need for $|\ |$ signs.

$y = \operatorname{arccosh} x \Rightarrow \cosh y = x$

$$e^y + e^{-y} = 2x$$

$$e^{2y} - 2x e^y + 1 = 0$$

$$(e^y - x)^2 - x^2 + 1 = 0$$

$$e^y - x = \pm\sqrt{x^2 - 1}$$

$$e^y = x \pm \sqrt{x^2 - 1}$$

for $x > 1$

yet since $\left(x + \sqrt{x^2-1}\right)\left(x - \sqrt{x^2-1}\right) = 1$

$$\Rightarrow x + \sqrt{x^2-1} = \left(x - \sqrt{x^2-1}\right)^{-1}$$

$$\Rightarrow x - \sqrt{x^2-1} = \left(x + \sqrt{x^2-1}\right)^{-1}$$

$\therefore e^y = x + \sqrt{x^2-1}, \left(x + \sqrt{x^2-1}\right)^{-1}$ yet $\operatorname{arccosh} x \geq 0$

Differentiating Inverse Hyperbolics

$$y = \tanh^{-1}\left(\frac{x}{a}\right)$$

$$\tanh y = \frac{x}{a}$$

$$\frac{d}{dx}\tanh y = \frac{d}{dx}\frac{x}{a}$$

$$\frac{d}{dy}(\tanh y)\frac{dy}{dx} = \frac{1}{a}$$

$$\operatorname{sech}^2 y \frac{dy}{dx} = \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{a\operatorname{sech}^2 y}$$

$$\cosh^2 y - \sinh^2 y = 1$$

$$1 - \tanh^2 y = \operatorname{sech}^2 y$$

$$= \frac{1}{a(1-\tanh^2 y)}$$

$$= \frac{1}{a\left(1-\frac{x^2}{a^2}\right)}$$

$$= \frac{1}{a - \frac{x^2}{a}}$$

$$\frac{dy}{dx} = \frac{a}{a^2 - x^2}$$

$$\therefore \int \frac{a}{a^2 - x^2}\,dx = \tanh^{-1}\left(\frac{x}{a}\right) + C$$

$$y = \sinh^{-1}\left(\frac{x}{a}\right) \Rightarrow \sinh y = \frac{x}{a}$$

$$\frac{d\sinh y}{dx} = \frac{1}{a}$$

positive root $\because$ positive graph

$$\cosh y \frac{dy}{dx} = \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{\cosh y \times a} = \frac{1}{\sqrt{1+\sinh^2 y} \times a}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2+x^2}}$$

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$y = \operatorname{arccosh}\left(\frac{x}{a}\right) \Rightarrow \cosh y = \frac{x}{a}$$

$$\frac{d}{dx}(\cosh y) = \frac{d}{dx}\frac{x}{a}$$

$$\frac{d}{dy}\cosh y \frac{dy}{dx} = \frac{1}{a}$$

$$\sinh y \frac{dy}{dx} = \frac{1}{a} \qquad \cosh^2 y - \sinh^2 y = 1$$

$$\frac{dy}{dx} = \frac{1}{a \sinh y}$$

$$= \frac{1}{a\sqrt{\cosh^2 y - 1}} \qquad \text{use positive root} \because > 0$$

$$= \frac{1}{a\sqrt{\frac{x^2}{a^2} - 1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-a^2}}$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$