Vectors

Scalar/dot Product

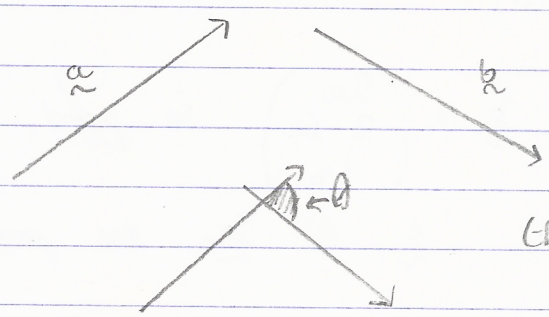
$$a \cdot b = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

(Note: position vectors refers to the bracket form as above & are relative to the origin)

Length of a vector

$$|a| = \left| \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Angle between 2 vectors



Finding it only involves using the direction vectors.

the angle coming out of the direction

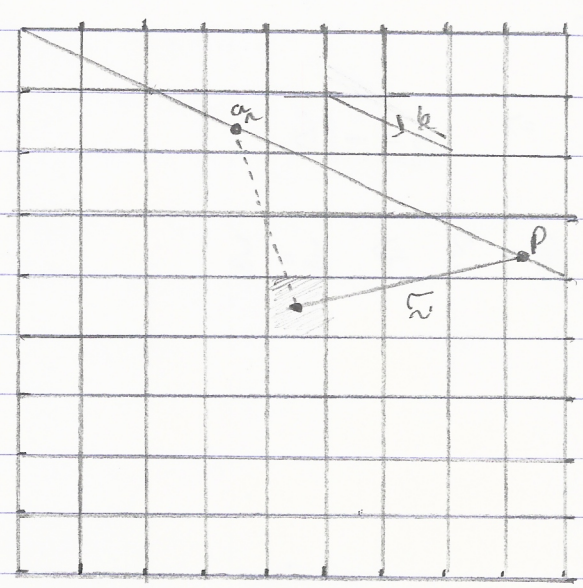
$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

dot product of the vectors

multiply length of each

Perpendicular when $\cos\theta = 0 \therefore \theta = 90° \Rightarrow$ when $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0 \therefore |\underset{\sim}{a}||\underset{\sim}{b}| \neq 0$ (never ever)

Vector eqn of a line



You can get to any point on a line by using $\underset{\sim}{a}$ (a fixed point on the line) & travelling a scalar multiple of $\underset{\sim}{b}$ (a vector parallel to the line)

direction vector

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$$

(Remember to sketch even 3D vectors)

Vector line joining 2 points

$A(-1, 5, 2)$ & $B(2, 1, -3)$

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -\begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

direction vector

$$= \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{r} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$$

Ratios

Let $\underset{\sim}{b_1} = \begin{pmatrix} a-1 \\ -a-1 \\ b-1 \end{pmatrix}$ & $\underset{\sim}{b_2} = \begin{pmatrix} 2a \\ 3-5a \\ 15 \end{pmatrix}$ so find the values of a & b for which $\underset{\sim}{b_1}$ & $\underset{\sim}{b_2}$ are parallel.

ratios must equal

$$\frac{a-1}{-a-1} = \frac{2a}{3-5a} \Rightarrow (a-1)(3-5a) = 2a(-a-1)$$

$$\Rightarrow 3a^2 - 10a + 3 = 0$$

$$\Rightarrow a = 1/3,\ 3$$

$\Rightarrow$ Let β = some scalar multiple

$\beta(a-1) = 2a \Rightarrow$ when $a = 1/3$ $\beta(-2/3) = 2/3$

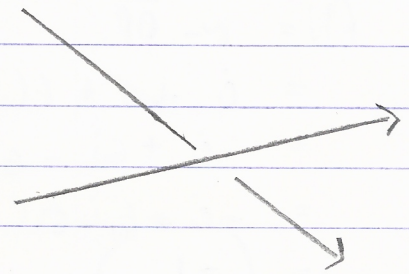
& $\Rightarrow \beta = -1$ so...

$$\beta(\beta(-b)) = 15 \Rightarrow \underline{b = -15}$$

& when $a = 3$, $\beta = 3 \Rightarrow b = 5$

Skew lines

- When lines aren't parallel or perpendicular. E.g.



One travelling under the other.

Example:

$$L_1 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \& \quad L_2 = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

When $L_1 = L_2 \Rightarrow$ ① $1 + 2\lambda = \mu \quad \Rightarrow \mu = 5$

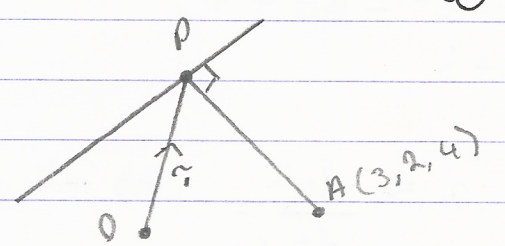
② $5 - \lambda = 3 \quad \Rightarrow \lambda = 2$ sub into ①

③ $4 + \lambda = 5 + \mu$

Check with ③ $\Rightarrow 6 \neq 10$

$\Rightarrow \because$ it doesn't satisfy the 3rd, it's skew

Distance to a point



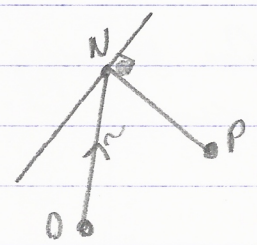
A line is closest to the point in question when the vector from the point A to some point P on the line is perpendicular to the line's direction.

$$\Rightarrow \overrightarrow{AP} = \overrightarrow{AO} + r_1$$
$$= -\overrightarrow{OA} + r_1$$

Now $\overrightarrow{AP} \cdot$ direction vector $= 0 \Rightarrow$ use to find the value of λ and use this to find the point P & finally distance to OG.

Example of distance from point to line (closest)

$\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + t(\underset{\sim}{j} - \underset{\sim}{k})$ & $P(2, 1, -2)$



$$\overline{PN} = \underset{\sim}{r} - \overline{OP}$$
$$= \underset{\sim}{i} + \cancel{\underset{\sim}{j}} + t(\underset{\sim}{j} - \underset{\sim}{k}) - 2\underset{\sim}{i} - \cancel{\underset{\sim}{j}} + 2\underset{\sim}{k}$$
$$= -\underset{\sim}{i} + t\underset{\sim}{j} - t\underset{\sim}{k} + 2\underset{\sim}{k}$$
$$= -\underset{\sim}{i} + t\underset{\sim}{j} + (t-2)\underset{\sim}{k}$$
$$= \begin{pmatrix} -1 \\ t \\ -t+2 \end{pmatrix}$$

$$\Rightarrow \overline{PN} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ t \\ -t+2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = +t + t - 2 = 0$$
$$\Rightarrow 2t - 2 = 0$$
$$\underline{\underline{t = 1}}$$

$$\Rightarrow \overline{PN} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow |\overline{PN}| = \sqrt{(-1)^2 + 1^2 + 1^2} = \underline{\underline{\sqrt{3}}}$$

<u>OR</u> let a = length

$$a^2 = (-1)^2 + t^2 + \cancel{(-1)^2}(t-2)^2$$
$$= 1 + t^2 + t^2 - 4t + 4$$
$$= 2t^2 - 4t + 5$$

minimising a^2 will minimise $a \Rightarrow \dfrac{da^2}{dt} = 4t - 4$

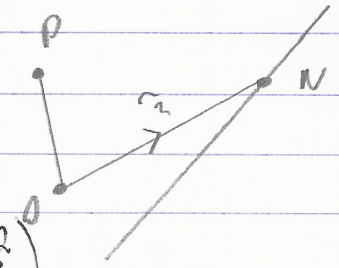
$4t - 4 = 0 \Rightarrow \underline{\underline{t = 1}}$

sub into $\sqrt{2t^2 - 4t + 5} = \sqrt{2 - 4 + 5} = \sqrt{7 - 4} = \underline{\underline{\sqrt{3}}}$

Perpendicular Distance Between 2 Lines (parallel)

$$r_1 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} \quad \& \quad r_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

Pick some point on r_1, e.g., $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$



$$\Rightarrow \overline{PN} = \overline{PO} + \overline{ON}\,(r_2)$$

$$= r_2 - \overline{OP}$$

$$= \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 - 2\lambda \\ 2 + 3\lambda \\ -1 - \lambda \end{pmatrix}$$

$$\Rightarrow x = (-1)(2\lambda + 1), \quad y = (3\lambda + 2), \quad z = (-1)(\lambda + 1)$$

$$a^2 = (2\lambda + 1)^2 + (3\lambda + 2)^2 + (\lambda + 1)^2$$

$$= 4\lambda^2 + 4\lambda + 1 + 3^2\lambda + \ldots$$

$$= 14\lambda^2 + 18\lambda + 6$$

CALCULUS ONLY WORKS IF PARALLEL

$$\frac{da^2}{d\lambda} = 28\lambda + 18 \quad \Rightarrow \quad \lambda = \frac{-18}{28} = -\frac{9}{14}$$

$$\text{sub into } a^2 = \frac{3}{14} \quad \Rightarrow \quad a = \frac{\sqrt{42}}{14}$$

OR

$$\begin{pmatrix} -2\lambda - 1 \\ 3\lambda + 2 \\ -\lambda - 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = 4\lambda + 2 + 9\lambda + 6 + \lambda + 1 = 0$$

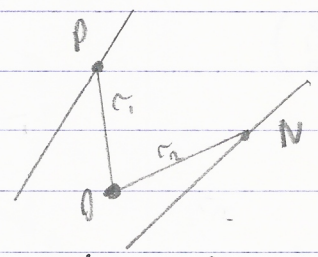
$$\Rightarrow 14\lambda + 9 = 0$$

$$\lambda = -9/14$$

$$\Rightarrow \overline{PN} = \begin{pmatrix} 2/7 \\ 1/14 \\ -5/14 \end{pmatrix} \quad \Rightarrow \quad |\overline{PN}| = \frac{\sqrt{42}}{14}$$

Shortest Distance Between 2 Skew Lines

$$r_1 = \begin{pmatrix} 5 \\ 0 \\ -4 \end{pmatrix} + m\begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} + n\begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$$



$$\overline{PN} = \overline{PO} + \overline{ON}$$
$$= -\underset{\sim}{P} + \underset{\sim}{N}$$
$$= \begin{pmatrix} 5n + 3m - 5 \\ -3m - 4n - 3 \\ -n + 6 \end{pmatrix}$$

$$\because \underset{\sim}{P} = \begin{pmatrix} 5 - 3m \\ 3m \\ -4 \end{pmatrix} \text{ \& } \underset{\sim}{N} = \begin{pmatrix} 5n \\ -3 - 4n \\ 2 - n \end{pmatrix}$$

Perpendicular to both r_1 & r_2

$$\begin{pmatrix} 5n + 3m - 5 \\ -3m - 4n - 3 \\ -n + 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix} = -18m - 27n + 6 = 0$$

$$\text{"} \cdot \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} = 42n + 27m = 19$$

$$m = \frac{-29}{3} \text{ \& } n = 20/3$$

$$\Rightarrow \text{Vector} = \begin{pmatrix} -2/3 \\ -2/3 \\ -2/3 \end{pmatrix} \qquad |\overline{PN}| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$$

$$= \sqrt{4/3} = \boxed{\frac{2\sqrt{3}}{3}}$$