Roots of Polynomials

Derivation

$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + c = 0$

Let's factorise to give $(x-\alpha)(x-\beta) = 0$ where $x = \alpha, \beta$

$$(x-\alpha)(x-\beta) = x^2 - \alpha x - \beta x + \alpha\beta$$
$$= x^2 - (\alpha+\beta)x + \alpha\beta$$

OR

$$x^2 + b/a\, x + c/a$$

Gnn

∵ equating parts $\Rightarrow -(\alpha+\beta) = b/a$ OR $\alpha+\beta = -b/a$

& $\alpha\beta = c/a$

$x^2 - (\text{sum of roots}) + (\text{product of roots}) = 0$

Forming Related Roots

$2x^2 - 5x - 6 = 0$ w/ roots α β $\Rightarrow$ find eq. w/ roots $1/\alpha$ & $1/\beta$

$\alpha+\beta = \frac{-b}{a} = \frac{--5}{2} = 5/2$ & $\alpha\beta = c/a = -6/2 = -3$

Sum new roots $\Rightarrow$ $1/\alpha + 1/\beta = \alpha+\beta/\alpha\beta = \frac{5/2}{-3} = -5/6$

Product new roots $\Rightarrow$ $1/\alpha \times 1/\beta = 1/\alpha\beta = 1/-3 = -1/3$

$x^2 - (\text{Sum of new roots}) + (\text{product of new roots}) = 0$

$x^2 + 5/6\, x - 1/3 = 0 \Rightarrow 6x^2 + 5x - 2 = 0$

Expansion of square

$\alpha^2 + \beta^2 = (\alpha+\beta) - 2\alpha\beta$

∵ don't want $2\alpha\beta$

Expansion of Cubic

0				1			
1			1		1		
2		1		2		1	
3	1		3		3		1
4	1	4		6		4	1

$$(\alpha+\beta)^3 = \underline{1\alpha^3\beta^0} + 3\alpha^2\beta + 3\alpha\beta^2 + \underline{1\alpha^0\beta^3}$$
$$= \alpha^3 + \beta^3 + 3\alpha^2\beta + 3\alpha\beta^2$$

$$\Rightarrow \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha^2\beta + 3\alpha\beta^2$$
$$= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$$

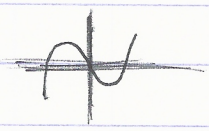
Expansion of Quartic

$$(\alpha+\beta)^4 = \alpha^4 + 4\alpha^3\beta^1 + 6\alpha^2 + \beta^2 + 4\alpha^1\beta^3 + \beta^4$$

$$\Rightarrow \alpha^4 + \beta^4 = (\alpha+\beta)^4 - 4\alpha^3\beta^1 - 4\alpha\beta^3 - 6\alpha^2\beta^2$$
$$= (\alpha+\beta)^4 - 4\alpha^3\beta - 4\alpha\beta^3 - 6(\alpha\beta)^2$$
$$= (\alpha+\beta)^4 - 4\alpha\beta(\alpha^2+\beta)^2 - 6(\alpha\beta)^2$$
$$= (\alpha+\beta)^4 - 4\alpha\beta((\alpha+\beta)^2 - 2\alpha\beta) - 6(\alpha\beta)^2$$
$$= (\alpha+\beta)^4 - 4\alpha\beta(\alpha+\beta)^2 + 8(\alpha\beta)^2 - 6(\alpha\beta)^2$$
$$= (\alpha+\beta)^4 - 4\alpha\beta(\alpha+\beta)^2 + 2(\alpha\beta)^2$$
$$= (\alpha+\beta)^4 - 2\alpha\beta(2(\alpha+\beta)^2 - \alpha\beta)$$

Roots of Higher Degree Polynomials

- All cubics have at least 1 real root.



- Cubics:

$\Sigma\alpha$ (sum of roots) $= \alpha + \beta + \gamma = -b/a$

$\Sigma\alpha\beta$ (sum of pairs of roots) $= \alpha\beta + \alpha\gamma + \beta\gamma = c/a$

$\Sigma\alpha\beta\gamma$ (product of roots) $= \alpha\beta\gamma = -d/a$

- Quartics

$$\Sigma\alpha = \alpha + \beta + \gamma + \delta = -b/a$$

$$\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c/a$$

$$\Sigma\alpha\beta\gamma = \alpha\beta\gamma + \alpha\gamma\delta + \beta\gamma\delta + \alpha\beta\delta = -d/a$$

$$\Sigma\alpha\beta\gamma\delta = e/a$$

Follows $\binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4}$ & alternates positive & negative.

Substitution

E.g., $x^3 - 8x^2 + 4 = 0$ has roots α, β, γ. Find a related cubic with roots $2\alpha, 2\beta, 2\gamma$.

Let $y = 2\alpha \Rightarrow \alpha = \frac{1}{2}y$

SOMETIMES FASTER TO USE THIS

$$\left(\tfrac{1}{2}y\right)^3 - 8\left(\tfrac{1}{2}y\right)^2 + 4 = 0$$

$$\frac{1}{8}y^3 - \frac{3}{4}y^2 + 4 = 0$$

$\times 8$

$$y^3 - 6y^2 + 32 = 0$$

essentially subbing in $\frac{1}{2}x$

Let $f(x) = x^3 - 8x^2 + 4$, $f(\frac{1}{2}x)$ will have double roots $\therefore$ stretch in x-direction s.f. 2

Adaptation from Eddie Woo's videos

$$ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$$
$$= x^3 - \alpha x^2 - \beta x^2 - \gamma x^2 + \alpha\beta x + \beta\gamma x + \alpha\gamma x - \alpha\beta\gamma$$
$$= x^3 + b/a\, x^2 + c/a\, x + d/a$$

To ensure the no. of terms you've expanded is correct, use Pascal's triangle:

${}^3C_2$ gives no. of x^2 terms ∵ from picking 3 terms, you want 2 x's

Eg ${}^3C_3$ gives no. of x^3 terms ∵ picking 3 x terms from 3 terms

		1			0
	1		1		1
1		2		1	2
1	3		3	1	3