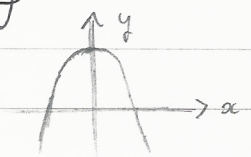
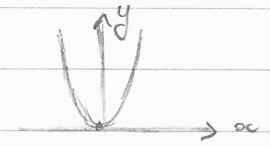
A2 Differentiation & Curve Sketching

- If $\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave



- If $\frac{d^2y}{dx^2} > 0 \Rightarrow$ convex

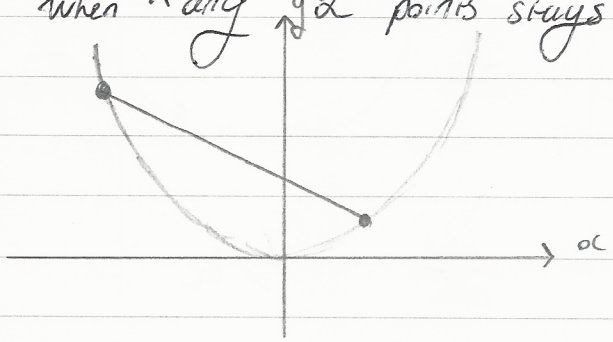


- If $\frac{d^2y}{dx^2} = 0 \Rightarrow$ point of inflection

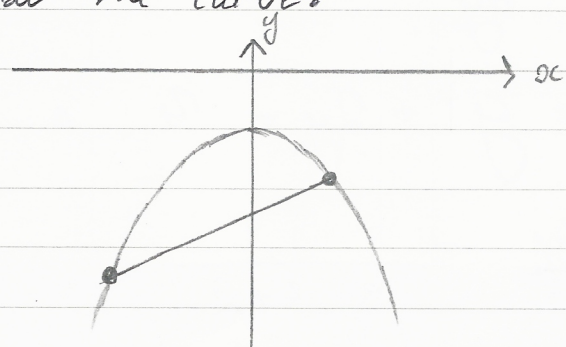
- A point of inflection is when a graph changes from concave to convex or vice versa (i.e., a change of concavity).

Note:
- a point of inflection $\Rightarrow \frac{d^2y}{dx^2} = 0$ YET not vice versa
- If $f''(x) = 0$ and $f'(x) \neq 0$ at some value x then it is definitely a point of inflection
- If $f''(x) = 0$ and $f'(x) = 0$ ~~then~~ at some value x then check the second derivative either side for a change in concavity

- Convex is when a line connecting any 2 points stays above above the curve:



- Concave is when a line connecting any 2 points stays below the curve:



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Product Rule

$$y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= uv' + vu'$$

Quotient Rule

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{vu' - uv'}{v^2}$$

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

Differentiating common functions

This only works in radians as the derivation is based on small angle approximation.

$$\begin{tikzcd}
 & \sin x \arrow[dr, "\frac{d}{dx}"] & \\
-\cos x \arrow[ur, "\frac{d}{dx}"] & & \cos x \arrow[dl, "\frac{d}{dx}"] \\
 & -\sin x \arrow[ul, "\frac{d}{dx}"] &
\end{tikzcd}$$

$$y = \ln x, \quad \frac{dy}{dx} = 1/x$$

Inverse Differentiation

$$\left(\frac{dy}{dx}\right)^{-1} = \frac{dx}{dy}$$

E.g., $x = y^3 + 2y + 4 \quad \therefore \quad \frac{dx}{dy} = 3y^2 + 2$

$$\therefore \frac{dy}{dx} = \frac{1}{3y^2+2}$$

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Differentiating $\arcsin x$

$$y = \arcsin x$$

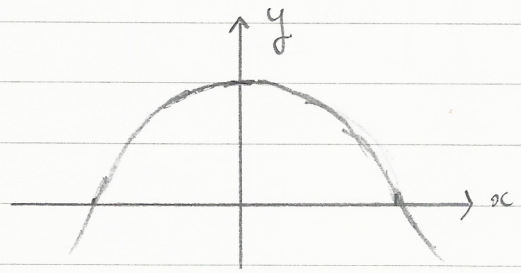
$$\Rightarrow x = \sin y$$

$$\Rightarrow \frac{dx}{dy} = \cos y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\Rightarrow \frac{dy}{dx} \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

We take the positive root $\because$ $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$



$$-\frac{\pi}{2} \leq y \leq \pi/2$$

$0 \leq \cos y \leq 1$ $\therefore$ always positive

Differentiating $y = a^x$

$$y = a^x$$

$$\ln y = x \ln a$$

$$y = e^{x \ln a}, \text{ let } u = \ln a \Rightarrow y = e^{ux}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = ue^{ux}$$

$$= \ln a \, e^{\ln a \, x}$$

A2

Differentiating $\ln x$

- $\frac{d}{dx}(\ln x) = \frac{1}{x} \Leftrightarrow \int \frac{1}{x}\,dx = \ln|x| + C$

Derivation for integration by parts

$$y = uv \Rightarrow y' = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int \frac{d}{dx}(uv)\,dx = \int\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)dx$$

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\therefore \int uv'\,dx = uv - \int vu'\,dx$$

$$\underline{\underline{\text{IR}}} \int_a^b uv'\,dx = \left[uv\right]_a^b - \int_a^b vu'\,dx$$

Special case?

- $\int e^{\overset{u}{x}}\cos(\overset{v'}{x})\,dx$ $\qquad u = e^x \quad v = \sin(x)$ $\qquad u' = e^x \quad v' = \cos(x)$

$$= e^x\sin(x) - \int e^{\overset{u}{x}}\sin(\overset{v'}{x})\,dx \qquad u = e^x \quad v = -\cos x \qquad u' = e^x \quad v' = \sin x$$

$$= e^x\sin(x) - \left[e^x \cdot -\cos(x) - \int -\cos(x)e^x\,dx\right.$$

$$= e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)\,dx$$

$$\therefore 2\int e^x\cos(x)\,dx = e^x\left[\sin(x) + \cos(x)\right]$$

$$\int e^x\cos(x)\,dx = \frac{1}{2}e^x\left[\sin(x) + \cos(x)\right]$$

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Implicit differentiation example

$$x^2 + y^2 + 2x - 7 = 0$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 + \frac{d}{dx}2x + \frac{d}{dx}(-7) = \frac{d}{dx}0$$

$$2x + \frac{d}{dy}(y^2)\frac{dy}{dx} + 2 = 0$$

$$2x + 2y\frac{dy}{dx} + 2 = 0$$

$$x + y\frac{dy}{dx} + 1 = 0$$

$$\frac{dy}{dx} = \frac{-x-1}{y}$$ (function of both x & y)

For a tangent with gradient 1 $\therefore$ $\frac{-x-1}{y} = 1$ $\therefore$ $y = -x-1$
(a relationship between the 2 variables)