10.6 Solved Problems

1. Consider two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$, both defined on X, given as follows:

$\mu(x_i;X)$	x_1	x_2	x_3	x_4	x_5
$\underset{\sim}{A}$	0.2	0.3	0.4	0.7	0.1
$\underset{\sim}{B}$	0.4	0.5	0.6	0.8	0.9

Express the following λ-cut sets using Zadeh's notation:

(a) $(\overline{\underset{\sim}{A}})_{0.7}$; (b) $(\underset{\sim}{B})_{0.2}$; (c) $(\underset{\sim}{A} \cup \underset{\sim}{B})_{0.6}$;
(d) $(\underset{\sim}{A} \cap \underset{\sim}{B})_{0.5}$; (e) $(\underset{\sim}{A} \cup \overline{\underset{\sim}{A}})_{0.7}$; (f) $(\underset{\sim}{B} \cap \overline{\underset{\sim}{B}})_{0.3}$;
(g) $(\overline{\underset{\sim}{A} \cap \underset{\sim}{B}})_{0.6}$; (h) $(\overline{\underset{\sim}{A} \cup \underset{\sim}{B}})_{0.8}$

Solution: The two fuzzy sets given are

$$\underset{\sim}{A} = \left\{\frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.1}{x_5}\right\}$$

$$\underset{\sim}{B} = \left\{\frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{0.9}{x_5}\right\}$$

We now find the λ-cut set:

$$A_\lambda = \{x \mid \mu_{\underset{\sim}{A}}(x) \geq \lambda\}$$

(a) $(\overline{\underset{\sim}{A}})_{0.7} = 1 - \mu_{\underset{\sim}{A}}(x)$

$$= \left\{\frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.3}{x_4} + \frac{0.9}{x_5}\right\}$$

$$(\overline{A})_{0.7} = \{x_1, x_2, x_5\}$$

(b) $$\underset{\sim}{B} = \left\{\frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{0.9}{x_5}\right\}$$

$$(B)_{0.2} = \{x_1, x_2, x_3, x_4, x_5\}$$

(c) $(\underset{\sim}{A} \cup \underset{\sim}{B}) = \max[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)]$

$$= \left\{\frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{0.9}{x_5}\right\}$$

$$(A \cup B)_{0.6} = \{x_3, x_4, x_5\}$$

(d) $(\underset{\sim}{A} \cap \underset{\sim}{B}) = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)]$

$$= \left\{\frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.1}{x_5}\right\}$$

$$(A \cap B)_{0.5} = \{x_4\}$$

(e) $(\underset{\sim}{A} \cup \overline{\underset{\sim}{A}}) = \max[\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)]$

$$= \left\{\frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.7}{x_4} + \frac{0.9}{x_5}\right\}$$

$$(A \cup \overline{A})_{0.7} = \{x_1, x_2, x_4, x_5\}$$

(f) $(\underset{\sim}{B} \cap \overline{\underset{\sim}{B}}) = \min[\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)]$

$$= \left\{ \frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.4}{x_3} + \frac{0.2}{x_4} + \frac{0.1}{x_5} \right\}$$

$(B \cap \overline{B})_{0.3} = \{x_1, x_2, x_3\}$

(g) $(\overline{\underset{\sim}{A} \cap \underset{\sim}{B}}) = 1 - \mu_{(\underset{\sim}{A} \cap \underset{\sim}{B})}$

$$= \left\{ \frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.3}{x_4} + \frac{0.9}{x_5} \right\}$$

$(\overline{A \cap B})_{0.6} = \{x_1, x_2, x_3, x_5\}$

(h) $(\overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}}) = \max[\mu_{\overline{\underset{\sim}{A}}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)]$

$$= \left\{ \frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.3}{x_4} + \frac{0.9}{x_5} \right\}$$

$(\overline{A} \cup \overline{B})_{0.8} = \{x_1, x_5\}$

2. Using Zadeh's notation, determine the λ-cut sets for the given fuzzy sets:

$$\underset{\sim}{S_1} = \left\{ \frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} + \frac{1.0}{80} + \frac{1.0}{100} \right\}$$

$$\underset{\sim}{S_2} = \left\{ \frac{0}{0} + \frac{0.45}{20} + \frac{0.6}{40} + \frac{0.8}{60} + \frac{.95}{80} + \frac{1.0}{100} \right\}$$

Express the following for $\lambda = 0.5$:

(a) $(\underset{\sim}{S_1} \cup \underset{\sim}{S_2})$; (b) $(\underset{\sim}{S_1} \cap \underset{\sim}{S_2})$; (c) $\overline{\underset{\sim}{S_1}}$; (d) $\overline{\underset{\sim}{S_2}}$;

(e) $(\overline{\underset{\sim}{S_1} \cup \underset{\sim}{S_2}})$; (f) $(\overline{\underset{\sim}{S_1} \cap \underset{\sim}{S_2}})$

Solution: The two fuzzy sets given are

$$\underset{\sim}{S_1} = \left\{ \frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} + \frac{1.0}{80} + \frac{1.0}{100} \right\}$$

$$\underset{\sim}{S_2} = \left\{ \frac{0}{0} + \frac{0.45}{20} + \frac{0.6}{40} + \frac{0.8}{60} + \frac{.95}{80} + \frac{1.0}{100} \right\}$$

The λ-cut set is obtained using

$$A_\lambda = \{x \mid \mu_{\underset{\sim}{A}}(x) \geq \lambda\}$$

Here $\lambda = 0.5$.

(a) $(\underset{\sim}{S_1} \cup \underset{\sim}{S_2}) = \max[\mu_{\underset{\sim}{S_1}}(x), \mu_{\underset{\sim}{S_2}}(x)]$

$$= \left\{ \frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} + \frac{1.0}{80} + \frac{1.0}{100} \right\}$$

$(S_1 \cup S_2)_{0.5} = \{20, 40, 60, 80, 100\}$

(b) $(\underset{\sim}{S_1} \cap \underset{\sim}{S_2}) = \min[\mu_{\underset{\sim}{S_1}}(x), \mu_{\underset{\sim}{S_2}}(x)]$

$$= \left\{ \frac{0}{0} + \frac{0.45}{20} + \frac{0.6}{40} + \frac{0.8}{60} + \frac{.95}{80} + \frac{1.0}{100} \right\}$$

$(S_1 \cap S_2)_{0.5} = \{40, 60, 80, 100\}$

(c) $\overline{\underset{\sim}{S_1}} = 1 - \mu_{\underset{\sim}{S_1}}(x)$

$$= \left\{ \frac{1}{0} + \frac{0.5}{20} + \frac{0.35}{40} + \frac{0.15}{60} + \frac{0}{80} + \frac{0}{100} \right\}$$

$(\overline{S_1})_{0.5} = \{0, 20\}$

(d) $\overline{\underset{\sim}{S_2}} = 1 - \mu_{\underset{\sim}{S_2}}(x)$

$$= \left\{ \frac{1}{0} + \frac{0.55}{20} + \frac{0.4}{40} + \frac{0.2}{60} + \frac{0.05}{80} + \frac{0}{100} \right\}$$

$(\overline{S_2})_{0.5} = \{0, 20\}$

(e) $(\overline{\underset{\sim}{S_1} \cup \underset{\sim}{S_2}}) = 1 - \mu_{\underset{\sim}{S_1} \cup \underset{\sim}{S_2}(x)}$

$$= \left\{ \frac{1}{0} + \frac{0.5}{20} + \frac{0.35}{40} + \frac{0.15}{60} + \frac{0}{80} + \frac{0}{100} \right\}$$

$(\overline{S_1 \cup S_2})_{0.5} = \{0, 20\}$

(f) $(\overline{\underset{\sim}{S_1} \cap \underset{\sim}{S_2}}) = 1 - \mu_{\underset{\sim}{S_1} \cap \underset{\sim}{S_2}(x)}$

$$= \left\{ \frac{1}{0} + \frac{0.55}{20} + \frac{0.4}{40} + \frac{0.2}{60} + \frac{0.05}{80} + \frac{0}{100} \right\}$$

$(\overline{S_1 \cap S_2})_{0.5} = \{0, 20\}$

3. Consider the two fuzzy sets

$$\underset{\sim}{A} = \left\{ \frac{0}{0.2} + \frac{0.8}{0.4} + \frac{1}{0.6} \right\}$$

$$\text{and} \quad \underset{\sim}{B} = \left\{ \frac{0.9}{0.2} + \frac{0.7}{0.4} + \frac{0.3}{0.6} \right\}$$

Using Zadeh's notations, express the fuzzy sets into λ-cut sets for $\lambda = 0.4$ and $\lambda = 0.7$ for the following operations:

(a) $\overline{\underset{\sim}{A}}$; (b) $\overline{\underset{\sim}{B}}$: (c) $\underset{\sim}{A} \cup \underset{\sim}{B}$;

(d) $\underset{\sim}{A} \cap \underset{\sim}{B}$; (e) $\overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}}$; (f) $\overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}}$

Solution: The two fuzzy sets given are

$$\underset{\sim}{A} = \left\{ \frac{0}{0.2} + \frac{0.8}{0.4} + \frac{1}{0.6} \right\}$$

$$\text{and} \quad \underset{\sim}{B} = \left\{ \frac{0.9}{0.2} + \frac{0.7}{0.4} + \frac{0.3}{0.6} \right\}$$

Case (i): $\lambda = 0.4$

(a) $$\overline{\underset{\sim}{A}} = 1 - \mu_{\underset{\sim}{A}}(x) = \left\{ \frac{1}{0.2} + \frac{0.2}{0.4} + \frac{0}{0.6} \right\}$$

$(\overline{A})_{0.4} = \{0.2\}$

(b) $$\overline{\underset{\sim}{B}} = 1 - \mu_{\underset{\sim}{B}}(y) = \left\{ \frac{0.1}{0.2} + \frac{0.3}{0.4} + \frac{0.7}{0.6} \right\}$$

$(\overline{B})_{0.4} = \{0.6\}$

(c) $$\underset{\sim}{A} \cup \underset{\sim}{B} = \max[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)]$$

$$= \left\{ \frac{0.9}{0.2} + \frac{0.8}{0.4} + \frac{1}{0.6} \right\}$$

$(A \cup B)_{0.4} = \{0.2, 0.4, 0.6\}$

(d) $$\underset{\sim}{A} \cap \underset{\sim}{B} = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)]$$

$$= \left\{ \frac{0}{0.2} + \frac{0.7}{0.4} + \frac{0.3}{0.6} \right\}$$

$(A \cap B)_{0.4} = \{0.4\}$

(e) $$\overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}} = \max[\mu_{\overline{\underset{\sim}{A}}}(x), \mu_{\overline{\underset{\sim}{B}}}(y)]$$

$$= \left\{ \frac{1}{0.2} + \frac{0.3}{0.4} + \frac{0.7}{0.6} \right\}$$

$(\overline{A} \cup \overline{B})_{0.4} = \{0.2, 0.6\}$

(f) $$\overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}} = \min[\mu_{\overline{\underset{\sim}{A}}}(x), \mu_{\overline{\underset{\sim}{B}}}(y)]$$

$$= \left\{ \frac{0.1}{0.2} + \frac{0.2}{0.4} + \frac{0}{0.6} \right\}$$

$(\overline{A} \cap \overline{B})_{0.4} = \{\phi\}$

Case (ii): $\lambda = 0.7$

(a) $$\overline{\underset{\sim}{A}} = 1 - \mu_{\underset{\sim}{A}}(x) = \left\{ \frac{1}{0.2} + \frac{0.2}{0.4} + \frac{0}{0.6} \right\}$$

$(\overline{A})_{0.7} = \{0.2\}$

(b) $$\overline{\underset{\sim}{B}} = 1 - \mu_{\underset{\sim}{B}}(y) = \left\{ \frac{0.1}{0.2} + \frac{0.3}{0.4} + \frac{0.7}{0.6} \right\}$$

$(\overline{B})_{0.7} = \{0.6\}$

(c) $$\underset{\sim}{A} \cup \underset{\sim}{B} = \max[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)]$$

$$= \left\{ \frac{0.9}{0.2} + \frac{0.8}{0.4} + \frac{1}{0.6} \right\}$$

$(A \cup B)_{0.7} = \{0.2, 0.4, 0.6\}$

(d) $$\underset{\sim}{A} \cap \underset{\sim}{B} = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)]$$

$$= \left\{ \frac{0}{0.2} + \frac{0.7}{0.4} + \frac{0.3}{0.6} \right\}$$

$(A \cap B)_{0.7} = \{0.4\}$

(e) $$\overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}} = \max[\mu_{\overline{\underset{\sim}{A}}}(x), \mu_{\overline{\underset{\sim}{B}}}(y)]$$

$$= \left\{ \frac{1}{0.2} + \frac{0.3}{0.4} + \frac{0.7}{0.6} \right\}$$

$(\overline{A} \cup \overline{B})_{0.7} = \{0.2, 0.6\}$

(f) $$\overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}} = \min[\mu_{\overline{\underset{\sim}{A}}}(x), \mu_{\overline{\underset{\sim}{B}}}(y)]$$

$$= \left\{ \frac{0.1}{0.2} + \frac{0.2}{0.4} + \frac{0}{0.6} \right\}$$

$(\overline{A} \cap \overline{B})_{0.7} = \{\phi\}$

4. Consider the discrete fuzzy set defined on the universe $X = \{a, b, c, d, e\}$ as

$$\underset{\sim}{A} = \left\{ \frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0}{e} \right\}$$

Using Zadeh's notation, find the λ-cut sets for $\lambda = 1, 0.9, 0.6, 0.3, 0^+$ and 0.

Solution: The fuzzy set given on the universe of discourse is

$$\underset{\sim}{A} = \left\{\frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0}{e}\right\}$$

The λ-cut set is given as

$$A_\lambda = \{x \mid \mu_{\underset{\sim}{A}}(x) \geq \lambda\}$$

It should be noted that the sets present in λ-cut set will have unity membership and the sets not in λ-cut set have zero membership. Hence λ-cut sets for different values of λ can be expressed as follows.

(a) $\lambda = 1$, $\quad A_1 = \left\{\frac{1}{a} + \frac{0}{b} + \frac{0}{c} + \frac{0}{d} + \frac{0}{e}\right\}$

(b) $\lambda = 0.9$, $\quad A_{0.9} = \left\{\frac{1}{a} + \frac{1}{b} + \frac{0}{c} + \frac{0}{d} + \frac{0}{e}\right\}$

(c) $\lambda = 0.6$, $\quad A_{0.6} = \left\{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{0}{d} + \frac{0}{e}\right\}$

(d) $\lambda = 0.3$, $\quad A_{0.3} = \left\{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{0}{e}\right\}$

(e) $\lambda = 0^+$, $\quad A_{0^+} = \left\{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{0}{e}\right\}$

(f) $\lambda = 0$, $\quad A_0 = \left\{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right\}$

5. Determine the crisp λ-cut relation when $\lambda = 0.1, 0^+, 0.3$ and 0.9 for the following relation $\underset{\sim}{R}$:

$$\underset{\sim}{R} = \begin{bmatrix} 0 & 0.2 & 0.4 \\ 0.3 & 0.7 & 0.1 \\ 0.8 & 0.9 & 1.0 \end{bmatrix}$$

Solution: For the given fuzzy relation, the λ-cut relation is given by

$$R_\lambda = \{(x,y) \mid \mu_{\underset{\sim}{R}(x,y)} \geq \lambda\}$$
$$= \{1 \mid \mu_{\underset{\sim}{R}(x,y)} \geq \lambda ; 0 \mid \mu_{\underset{\sim}{R}(x,y)} < \lambda\}$$

(a) $\lambda = 0.1$,

$$R_{0.1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) $\lambda = 0^+$,

$$R_{0^+} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(c) $\lambda = 0.3$,

$$R_{0.3} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(d) $\lambda = 0.9$,

$$R_{0.9} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

6. For the fuzzy relation $\underset{\sim}{R}$,

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 0.1 & 0 & 0.5 & 0.3 \\ 0.02 & 0.1 & 0.55 & 1 & 0.6 \\ 0.2 & 1 & 0.6 & 1 & 0 \\ 0.03 & 0.5 & 1 & 0.3 & 0 \end{bmatrix}$$

find the λ-cut relation for $\lambda = 0^+, 0.1, 0.4$ and 0.8.

Solution: For the given fuzzy relation, the λ-cut relation can be obtained by the following relation:

$$R_\lambda = \begin{cases} 1, & \mu_{\underset{\sim}{R}(x,y)} \geq \lambda \\ 0, & \mu_{\underset{\sim}{R}(x,y)} < \lambda \end{cases}$$

(a) $\lambda = 0^+$,

$$R_{0^+} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(b) $\lambda = 0.1$,

$$R_{0.1} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(c) $\lambda = 0.4$,

$$R_{0.4} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

(d) $\lambda = 0.8$,

$$R_{0.8} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

7. For the fuzzy relation $\underset{\sim}{R}$,

$$\underset{\sim}{R} = \begin{bmatrix} 0.2 & 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.5 & 0.7 & 1 & 0.8 \\ 0.4 & 0.6 & 0.8 & 0.9 & 0.4 \\ 0.9 & 1 & 0.8 & 0.6 & 0.4 \end{bmatrix}$$

find the λ-cut relation for $\lambda = 0.2, 0.4, 0.7$ and 0.9

Solution: For the given fuzzy relation, the λ-cut relation is given by

$$R_\lambda = \begin{cases} 1, & \mu_{\underset{\sim}{R}(x,y)} \geq \lambda \\ 0, & \mu_{\underset{\sim}{R}(x,y)} < \lambda \end{cases}$$

(a) $\lambda = 0.2$,

$$R_{0.2} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(b) $\lambda = 0.4$,

$$R_{0.4} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(c) $\lambda = 0.7$,

$$R_{0.7} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

(d) $\lambda = 0.9$,

$$R_{0.9} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

8. Show that any λ-cut relation of a fuzzy tolerance relation results in a crisp tolerance relation.

Solution: Consider the fuzzy relation

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

It is a fuzzy tolerance relation because it does not satisfy transitive property, i.e.,

$$\mu_{\underset{\sim}{R}}(x_1, x_2) = 0.8, \quad \mu_{\underset{\sim}{R}}(x_2, x_5) = 0.9$$

From the relation $\underset{\sim}{R}$, we have

$$\mu_{\underset{\sim}{R}}(x_1, x_5) = 0.2 \tag{1}$$

But on calculating we obtain

$$\mu_{\underset{\sim}{R}}(x_1, x_5) = \min\left[\mu_{\underset{\sim}{R}}(x_1, x_2), \mu_{\underset{\sim}{R}}(x_2, x_5)\right] = \min[0.8, 0.9] = 0.8 \tag{2}$$

As (1) $\neq$ (2), therefore transitive property is not satisfied. Now assume $\lambda = 0.8$. Then the crisp relation formed is

$$R_{0.8} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Now $(x_1, x_2) \in R$, $(x_2, x_5) \in R$, but $(x_1, x_5) \notin R (x_1, x_5) \notin R$. Hence $R_{0.8}$ is a crisp tolerance relation. Thus λ-cut relation for a fuzzy tolerance relation is a crisp tolerance relation.

9. Show that λ-cut relation of a fuzzy equivalence relation results in a crisp equivalence relation.

Solution: Consider the following fuzzy equivalence relation:

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 0.8 & 004 & 0.5 & 0.8 \\ 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ 0.4 & 0.4 & 1 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0.4 & 1 & 0.5 \\ 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{bmatrix}$$

The relation $\underset{\sim}{R}$ satisfies transitive property, i.e.,

$$\mu_{\underset{\sim}{R}}(x_1, x_2) = 0.8, \quad \mu_{\underset{\sim}{R}}(x_2, x_5) = 0.9$$

From the relation $\underset{\sim}{R}$, we have

$$\mu_{\underset{\sim}{R}}(x_1, x_5) = 0.8 \tag{1}$$

On calculating we obtain

$$\mu_{\underset{\sim}{R}}(x_1, x_5) = \min[\mu_{\underset{\sim}{R}}(x_1, x_2), \mu_{\underset{\sim}{R}}(x_2, x_5)]$$
$$= \min[0.8, 0.9] = 0.8 \tag{2}$$

As (1)-= (2), therefore transitive property satisfied; hence it forms an equivalence relation. Now assume $\lambda = 0.8$. Then the crisp relation formed is

$$R_{0.8} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Now $(x_1, x_2) \in R$, $(x_2, x_5) \in R$ and $(x_1, x_5) \in R$. Hence, λ-cut relation of a fuzzy equivalence relation results in a crisp equivalence relation.

10. For the given membership function as shown in Figure 1 below, determine the defuzzified output value by seven methods.

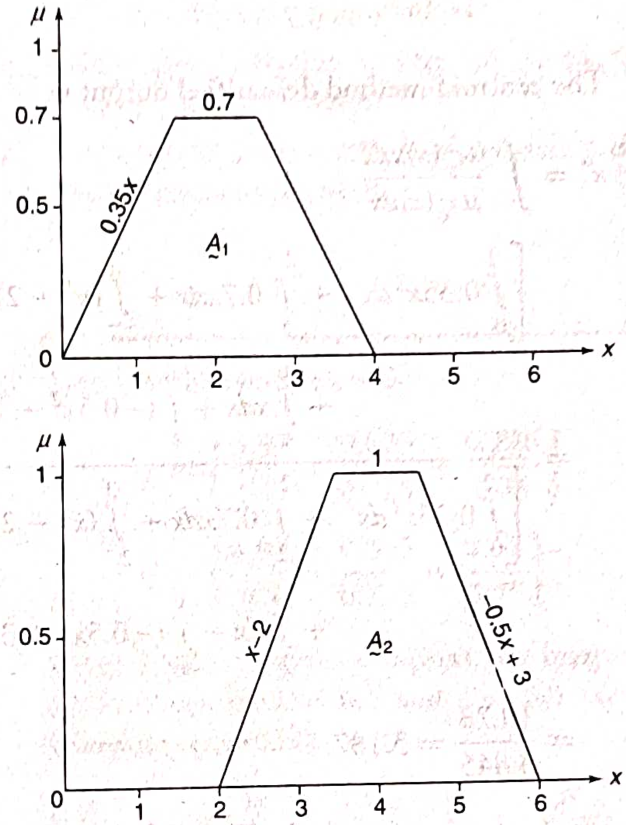

Figure 1 Membership functions.

Solution: The defuzzified output value can be obtained by the following methods.

- Centroid method

  The two points are (0, 0) and (2, 0.7). The straight line is given by $(y - y_1) = m(x - x_1)$. Hence.

$$y - 0 = \frac{0.7}{2}(x - 0)$$

$A_{11} \Rightarrow y = 0.35x$

$A_{12} \Rightarrow y = 0.7$

$A_{13} \Rightarrow$ not necessary

$A_{21} \Rightarrow$ the two points are (2, 0), (3, 1)

$y = x - 2$

$A_{22} \Rightarrow y = 1$

$A_{23} \Rightarrow$ the two points are $(4, 1)$, $(6, 0)$
we get $y = -0.5x + 3$

(A) From A_{12} we obtain $y = 0.7$.
(B) From A_{21} we obtain $y = x-2$. On substituting the value $y = 0.7$ in (B), we obtain

$$x - 2 = 0.7 \Rightarrow x = 2.7$$
$$y = 0.7$$

The centroid method defuzzified output is

$$x^* = \int \frac{\mu_{C}(x)x dx}{\mu_{C}(x)dx}$$

$$= \frac{\left[\int_0^2 0.35x^2 dx + \int_2^{2.7} 0.7x dx + \int_{2.7}^3 (x^2 - 2)dx + \int_3^4 x dx + \int_4^6 (-0.5x^2 + 3x)dx\right]}{\left[\int_0^2 0.35x^2 dx + \int_2^{2.7} 0.7x dx + \int_{2.7}^3 (x^2 - 2)dx + \int_3^4 dx + \int_4^6 (-0.5x^2 + 3x)dx\right]}$$

$$= \frac{10.78}{3.445} = 3.187$$

- *Weighted average method:* The defuzzified value here is given by

$$x^* = \frac{2(0.7) + 4(1)}{0.7 + 1} = 3.176$$

- *Mean-max method:* The crisp output value here is given by

$$x^* = \frac{a+b}{2} = \frac{2.5 + 3.5}{2} = 3$$

- *Center of sums method:* The defuzzified value x^* is given by

$$x^* = \frac{\int_x x \sum_{i=1}^{n} \mu_{C_i}(x)dx}{\int_x \sum_{i=1}^{n} \mu_{C_i}(x)dx}$$

$$= \frac{\left[\int_0^6 [\frac{1}{2} \times 0.7 \times (3+2) \times 2 + \frac{1}{2} \times 1 \times (2+4) \times 4]dx\right]}{\left[\int_0^6 [\frac{1}{2} \times 0.7 \times (3+2) + \frac{1}{2} \times 1 \times (2+4) \times 4]dx\right]}$$

$$= \frac{\int_0^6 (3.5 + 12)dx}{\int_0^6 (1.75 + 3)dx} = 2.84$$

- *Center of largest area:*

$$\text{Area of I} = \frac{1}{2} \times 0.7 \times (2.7 + 0.7) = 1.19$$

$$\text{Area of II} = \frac{1}{2} \times 1 \times (2+3) \times \frac{1}{2} \times 0.7 = 2.255$$

Area of II is found to be larger; therefore the defuzzified output value is given by

$$x^* = \frac{\int \mu_{C_j}(x)x dx}{\int \mu_{C_j}(x)dx}$$

$$= \frac{\left[\int_{2.7}^3 \frac{1}{2} \times 0.3 \times 0.3 \times 2.85 dx + \int_3^4 1 \times 1 \times 3.5 dx + \int_4^6 \frac{1}{2} \times 2 \times 1 dx\right]}{\left[\int_{2.7}^3 \frac{1}{2} \times 0.3 \times 0.3 dx + \int_3^4 1 \times 1 dx + \int_4^6 \frac{1}{2} \times 2 \times 1 dx\right]}$$

$$= \frac{\int_{2.7}^3 0.12825 dx + \int_3^4 3.5 dx + \int_4^6 5 dx}{\int_{2.7}^3 0.045 dx + \int_3^4 dx + \int_4^6 dx} = 4.49$$

- *First of maxima:* The defuzzified output value is

$$x^* = 3$$

- *Last of maxima:* The defuzzified output value is

$$x^* = 4$$